Exp# 07: Constrained Optimization using the method of Lagrange's Multipliers

Aim:

Finding Maxima/Minima of a function of several variables under a given constraint by using Lagrange's method of multipliers.

Introduction:

In many practical and theoretical applications, it is required to find the maximum and minimum of a function of several variables, where the variables are connected by some relation or condition known as constraint.

If $f(x, y, z)$ is a function of three independent variables where x, y, z are related by a known constraint $g(x, y, z) = k$, then the problem is to find extreme values of $f(x, y, z)$ subject to $g(x, y, z) = k$.

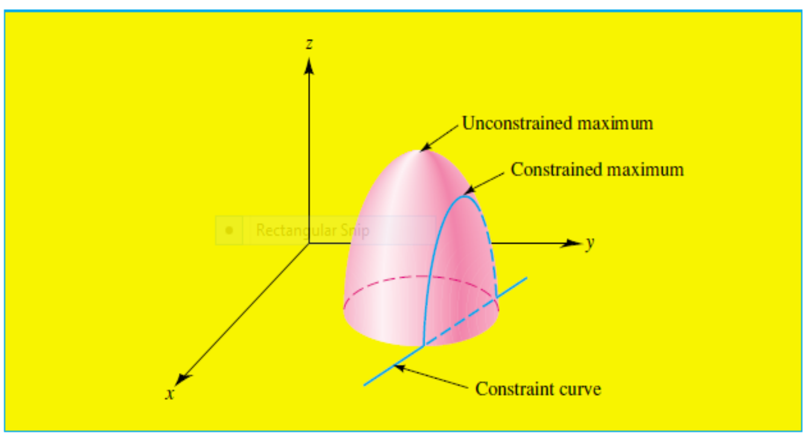
Lagrange's method of multipliers:

Lagrange's method of multiplier's introduces an additional unknown constant known as Lagrange multiplier. This method involves in the following steps:
If $f(x, y, z)$ is a function subject to the constraint $g(x, y, z) = k$:

Step1: Form the auxiliary equation $F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$.

Step2: Partially differentiate F with respect to x, y, z respectively.

Step3: Solve the four equations $F_x = 0$, $F_y = 0$, $F_z = 0$ and $g(x, y, z) = k$ for the Lagrange's Multiplier λ and the stationary values x, y, z.



Example. Find the minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 10$.

Given: $f(x, y) = x^2 + y^2$.
Let $g(x, y) : x + y - 10 = 0$.
The auxiliary equation is

$$\begin{aligned} F(x, y) &= f(x, y) + \lambda g(x, y) \\ &= (x^2 + y^2) + \lambda(x + y - 10) \\ \implies F_x &= 2x + \lambda, \quad F_y = 2y + \lambda \end{aligned}$$

Solving $F_x = 0$, $F_y = 0$, and $g(x, y) = 0$, we get: $x = -\frac{\lambda}{2}$, $y = -\frac{\lambda}{2}$; $\lambda = -10$.
Hence, $\text{Min} f(x, y) = f(5, 5) = 5^2 + 5^2 = 50$.

MATLAB commands required:

Following are the new MatLab commands one needs to know for the present experiment.

`jacobian(f,v)`	Computes the Jacobian of the scalar or vector f with respect to the vector v. The (i,j)-th entry of the result is $df(i)/dv(j)$. Note that when f is scalar, the Jacobian of f is the gradient of f. Also, note that scalar v is allowed, although this is just $diff(f,v)$.
`ezmesh(f)`	plots a graph of f(x,y) using MESH. Works similar to `ezsurf` without `colormap`.
`get(H,'X')`	returns the value of the specified property (X) for the graphics object with handle H.

MatLab code for Lagrange's multiplier method for two variables:

```
clc
clearvars
syms x y L
f = input('Enter the function f(x,y): ');
g = input('Enter the constraint function g(x,y): ');
F = f + L*g;
gradF = jacobian(F,[x,y]);
[L,x1,y1] = solve(g,gradF(1),gradF(2),'Real',true); % Solving only for Real x and y
x1 = double(x1); y1 = double(y1);
xmx = max(x1); xmn = min(x1); % Finding max and min of x-coordinates for plot range
ymx = max(y1); ymn = min(y1); % Finding max and min of y-coordinates for plot range
range = [xmn-3 xmx+3 ymn-3 ymx+3]; % Setting plot range
ezmesh(f,range);hold on; grid on;
h = ezplot(g,range); set(h,'LineWidth',2);
tmp = get(h,'contourMatrix');
xdt = tmp(1,2:end); % Avoiding first x-data point
ydt = tmp(2,2:end); % Avoiding first y-data point
zdt = double(subs(f,{x,y},{xdt,ydt}));
plot3(xdt,ydt,zdt,'-r','LineWidth',2);axis(range);
for i = 1:numel(x1)
   G(i) = subs(f,[x,y],[x1(i),y1(i)])
   plot3(x1(i),y1(i),G(i),'*k','MarkerSize',20);
end
title('Constrained Maxima/Minima')
```

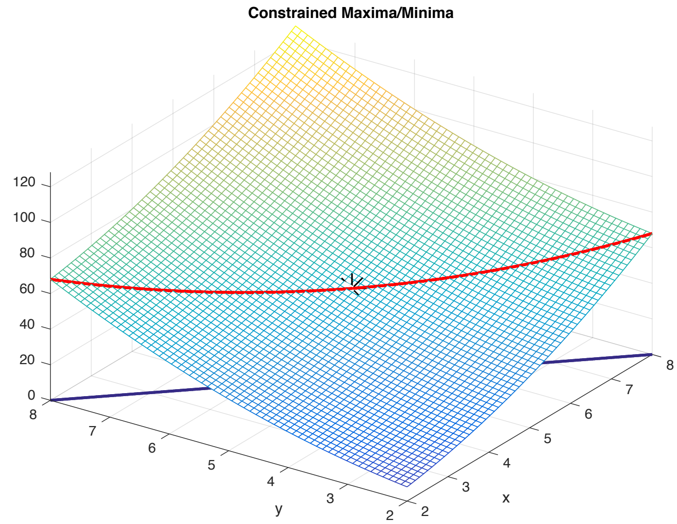
Note. The option `'Real'` in the `solve` command may not work in the older versions of MatLab. Moreover, the visualization of the constraint curve on the surface is provided only for an aid to understand the problem.

Sample Output #1:

Example. Find the minimum of $f(x,y) = x^2 + y^2$ subject to the constraint $x + y = 10$.

```
Enter the function f(x,y): x^2+y^2
Enter the constraint function g(x,y): x+y-10
G =
50
```

Figure Window



MatLab code for Lagrange's multiplier method for three variables:

```
clc
clearvars
syms x y z L
f = input('Enter the function f(x,y,z): ');
g = input('Enter the constraint function g(x,y,z): ');
F = f + L*g;
gradF = jacobian(F,[x,y,z]);
[L,x1,y1,z1] = solve(g,gradF(1),gradF(2),gradF(3));
Z = [x1 y1 z1];
disp('[x y z]=')
disp(Z)
```

Sample Output #1:

Example. 1. Find the maximum and minimum distances from the origin to the curve $3x^2+4xy+6y^2-140$.

```
Enter the function f(x,y,z): x^2+y^2+z^2
Enter the constraint function g(x,y,z): 3*x^2+4*x*y+6*y^2-140
[x y z]=
[        -2,      -4, 0]
[         2,       4, 0]
[ -2*14^(1/2), 14^(1/2), 0]
[  2*14^(1/2), -14^(1/2), 0]
```

Exercise Question

SAQ # 1. The highway department is planning to build a picnic area for motorists along a major highway. It is to be rectangular with an area of 5,000 square yards and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing that will be needed to complete the job?

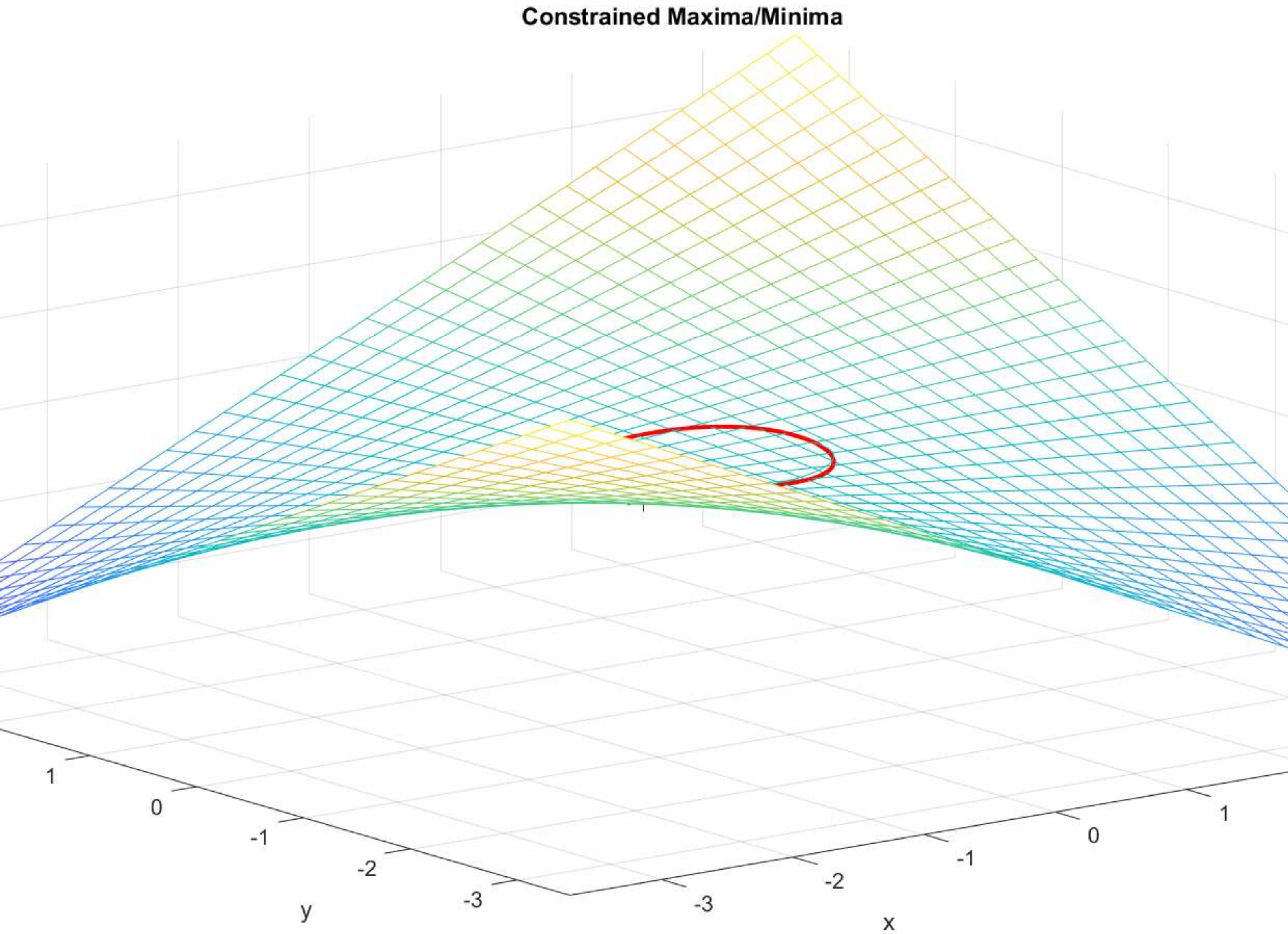
SAQ # 2. Let $f(x, y) = x^2y^2$ represents the utility function or customer satisfaction derived by a consumer from the consumption of a certain amount of product x and certain amount of product y. Maximize the utility function subject to the constraint $2x + 4y = 40$.

SAQ # 3. Find the dimension of rectangular box with the largest possible volume with an open top and one portion to be constructed from 162 sq inches of cardboard. (Note: The amount of the material used in construction of box is $xy + 2xz + 2yz = 162$).

SAQ # 4. An editor has been allotted $60,000 to spend on the development and promotion of a new book. It is estimated that if x thousand dollars is spent on development and y thousand on promotion, approximately $f(x, y) = 20x^{3/2}y$ copies of the book will be sold. How much money should the editor allocate to development and how much to promotion in order to maximize sales?

```
%Modified program for Lagranges Multiplier Method
clear
clc
syms x y z L
f=input('Enter the function f(x,y):');
g=input('Enter the constrained funtion g(x,y):');
F=f+L*g;
gradF=jacobian(F,[x y]);
S=solve(g,gradF(1),gradF(2),'Real',true);%Solving only for real L,x,y
St_pts=double([S.x,S.y]);%Matrix with stationary points in rows.
h1=St_pts(:,1);
h2=St_pts(:,2);
X=[h1];
Y=[h2];
disp('Stationary points are:')
STP=table(X,Y)
range=double([min(S.x)-3 max(S.x)+3 min(S.y)-3 max(S.y)+3]);%setting plot range
[n,m]=size(St_pts);
for i = 1:n
F(i)=subs(f,{x,y},{S.x(i),S.y(i)});%Finding values of f at all the stationary points
end
if n>1
F_max=max(F);
disp(['The maximum value of f under the given constraint g is:' char(F_max)]);
F_min=min(F);
disp(['The minimum value of f under the given constraint g is:' char(F_min)]);
else
disp(['The extremum value of f under the given constraint g is:' char(F)]);
end
fmesh(f,range);
hold on
h=ezplot(g,range);
tmp = get(h,'contourMatrix');
xdt = tmp(1,2:end);% Avoiding first x-data point
ydt = tmp(2,2:end);% Avoiding first y-data point
zdt = double(subs(f,{x,y},{xdt,ydt}));
plot3(xdt,ydt,zdt,'-r','LineWidth',2);
axis(range);
for i = 1:numel(x)
plot3(S.x(i),S.y(i),F(i),'*k','MarkerSize',20);
end
hold off
title('Constrained Maxima/Minima')
```

```
Enter the function f(x,y):x*y
Enter the constrained funtion g(x,y):x^2+2*y^2-1
Stationary points are:

STP =

  4×2 table

       X          Y
    ________    ____

    -0.70711    -0.5
     0.70711    -0.5
    -0.70711     0.5
     0.70711     0.5

The maximum value of f under the given constraint g is:2^(1/2)/4
The minimum value of f under the given constraint g is:-2^(1/2)/4
>>
```


Constrained Maxima/Minima
1
0
-1
-2
-3
y
-3
-2
-1
0
1
x

```
clear all
clc
syms x y z L
f = input('Enter the function f(x,y,z): ');
g = input('Enter the constraint function g(x,y,z): ');
F = f + L*g;
gradF = jacobian(F,[x,y,z]);
S = solve(g,gradF(1),gradF(2),gradF(3));
st_pts=double([S.x,S.y,S.z]);
h1=st_pts(:,1);
h2=st_pts(:,2);
h3=st_pts(:,3);
X=[h1];
Y=[h2];
Z=[h3];
disp('Stationary points are:')
STP=table(X,Y,Z)
[n,m]=size(st_pts);
for i = 1:n
F(i)=subs(f,{x,y,z},{S.x(i),S.y(i),S.z(i)});%Finding values of f at all the stationary
points
end
if n>1
F_max=max(F);
disp(['The maximum value of f under the given constraint g is:' char(F_max)]);
F_min=min(F);
disp(['The minimum value of f under the given constraint g is:' char(F_min)]);
else
disp(['The extremum value of f under the given constraint g is:' char(F)]);
end
```

```
Enter the function f(x,y,z): x^2+y^2+z^2
Enter the constraint function g(x,y,z): 3*x^2+4*x*y+6*y^2-140
Stationary points are:

STP =

  4×3 table

       X          Y       Z
    _______    _______    _

         -2         -4    0
          2          4    0
    -7.4833     3.7417    0
     7.4833    -3.7417    0

The maximum value of f under the given constraint g is:70
The minimum value of f under the given constraint g is:20
>>
```