



# Fourth Order RK-Method

---

The most commonly used method is Runge-Kutta fourth order method.

The fourth order RK-method is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$



where

$$k_1 = hf(x_i, y_i),$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right),$$

$$k_4 = hf(x_i + h, y_i + k_3).$$



# Example 1

---

Find the approximate solution of the initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t}, \quad 1 \leq t \leq 3$$

with the initial condition

$$x(1) = 1,$$

using the Runge-Kutta second order and fourth order with step size of  $h = 1$ .



# Solution

---

**RK 2nd order method. The formula is**

$$y_{i+1} = y_i + \frac{h}{2} (k_1 + k_2) ,$$

where  $k_1 = f(x_i, t_i)$ ,  $k_2 = f(x_i + h, t_i + hk_1)$ .

Here,  $h = 1$  and  $t_0 = x_0 = 1$ . Therefore,

$$k_1 = f(t_0, x_0) = 1 + \frac{x_0}{t_0} = 2,$$

$$k_2 = f(t_0 + h, x_0 + hk_1) = f(2, 3) = 1 + \frac{3}{2} = 2.5.$$



Thus,

$$y_1 = 1 + \frac{1}{2}(2 + 2.5) = 3.25.$$

Similary, we calculate  $y_2$ .



**RK 4th order method. The formula is**

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

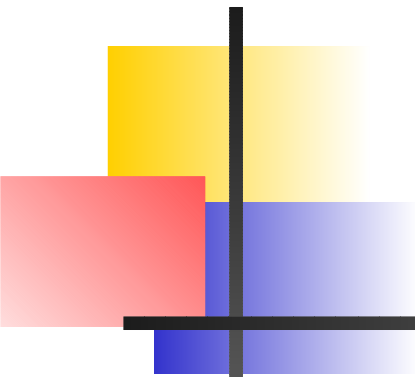
where

$$k_1 = hf(t_i, x_i),$$

$$k_2 = hf\left(t_i + \frac{h}{2}, x_i + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(t_i + \frac{h}{2}, x_i + \frac{k_2}{2}\right),$$

$$k_4 = hf(t_i + h, x_i + k_3).$$


$$k_1 = hf(t_0, x_0) = f(1, 1) = 2,$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right) = f(1.5, 2) = 2.333333,$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right) = 2.444444,$$

$$k_4 = hf(t_0 + h, x_0 + k_3) = 2.722222.$$



We find

$$\begin{aligned} y_1 &= 1 + \frac{1}{6}(2 + 2 * 2.3333333 \\ &\quad + 2 * 2.4444444 + 2.7222222) \\ &= 3.379630. \end{aligned}$$

Similarly, we calculate  $y_2$ .





# System of First Order ODE's

We consider the  $n$  system of first order equations

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n), \quad y_1(x_0) = y_{10},$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n), \quad y_2(x_0) = y_{20},$$

...

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n), \quad y_n(x_0) = y_{n0}.$$

With the help of single step methods, we find the approximate solution for the system of  $n$  equations.



We now consider the two first order equations of the form

$$\begin{aligned}\frac{dy_1}{dx} &= f_1(x, y_1, y_2), \quad y_1(x_0) = y_{10}, \\ \frac{dy_2}{dx} &= f_2(x, y_1, y_2), \quad y_2(x_0) = y_{20}.\end{aligned}$$

We use RK second order and fourth order method to solve the system of equations.



# 2nd Order RK Method

The formula for second order RK method with step size  $h$  is

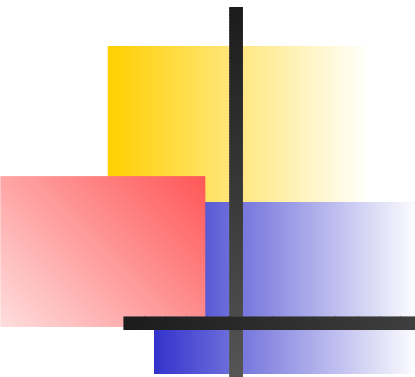
$$y_{1,n+1} = y_{1,n} + \frac{1}{2}(s_1 + k_1),$$

$$y_{2,n+1} = y_{2,n} + \frac{1}{2}(s_2 + k_2),$$

where

$$k_1 = hf_1(x_n, y_{1n}, y_{2n}),$$

$$k_2 = hf_2(x_n, y_{1n}, y_{2n}),$$


$$\begin{aligned}s_1 &= hf_1(x_n + h, y_{1n} + k_1, y_{2n} + k_2), \\s_2 &= hf_2(x_n + h, y_{1n} + k_1, y_{2n} + k_2).\end{aligned}$$

Similarly, the formula for fourth order RK-method is

$$\begin{aligned}y_{1,n+1} &= y_{1,n} + \frac{1}{6}(k_1 + 2s_1 + 2l_1 + p_1), \\y_{2,n+1} &= y_{2,n} + \frac{1}{6}(k_2 + 2s_2 + 2l_2 + p_2),\end{aligned}$$



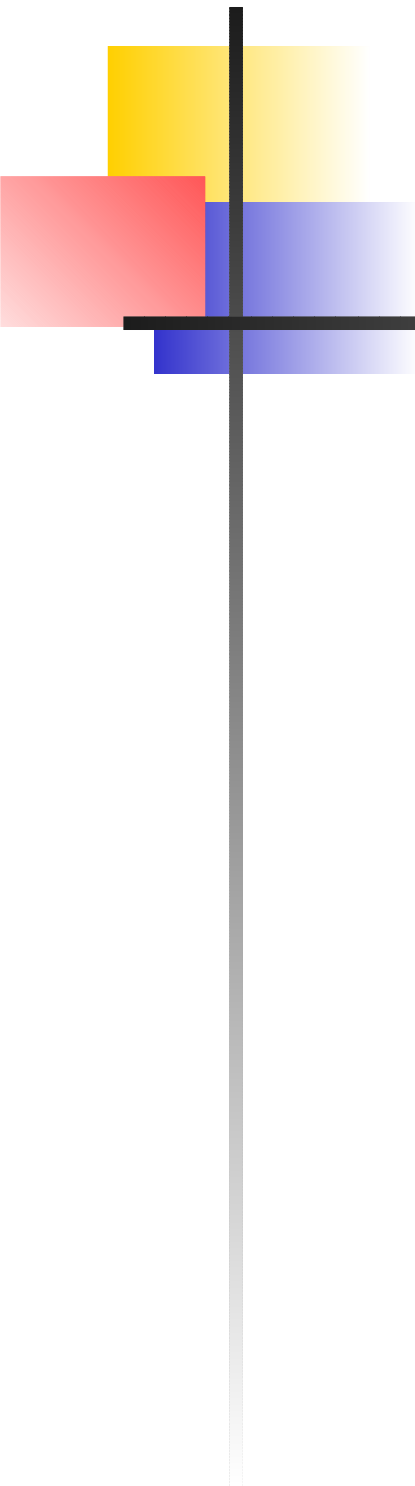
where

$$k_1 = hf_1(x_n, y_{1n}, y_{2n}),$$

$$k_2 = hf_2(x_n, y_{1n}, y_{2n}),$$

$$s_1 = hf_1\left(x_n + \frac{h}{2}, y_{1n} + \frac{k_1}{2}, y_{2n} + \frac{k_2}{2}\right),$$

$$s_2 = hf_2\left(x_n + \frac{h}{2}, y_{1n} + \frac{k_1}{2}, y_{2n} + \frac{k_2}{2}\right)$$


$$\begin{aligned}l_1 &= hf_1\left(x_n + \frac{h}{2}, y_{1n} + \frac{s_1}{2}, y_{2n} + \frac{s_2}{2}\right), \\l_2 &= hf_2\left(x_n + \frac{h}{2}, y_{1n} + \frac{s_1}{2}, y_{2n} + \frac{s_2}{2}\right), \\p_1 &= hf_1(x_n + h, y_{1n} + l_1, y_{2n} + l_2), \\p_2 &= hf_2(x_n + h, y_{1n} + l_1, y_{2n} + l_2).\end{aligned}$$



## Example 2

Use RK-method 2nd order and 4th order to find the approximate solution of  $y(0.1)$  and  $z(0.1)$  as a solution of pair of equations

$$\begin{aligned}\frac{dy}{dx} &= x + z, \\ \frac{dz}{dx} &= y - x\end{aligned}$$

with the initial conditions

$$y(0) = 1, \quad z(0) = -1.$$

Take step size  $h = 0.1$ .



# Multistep Methods

The general form of a linear  $m$ -step multistep method is

$$\frac{y_{n+1} - a_1 y_n - a_2 y_{n-1} - \cdots - a_m y_{n+1-m}}{h_i} \\ = b_0 f(x_{n+1}, y_{n+1}) + b_1 f(x_n, y_n) \\ + \cdots + b_m f(x_{n+1-m}, y_{n+1-m}).$$

When  $b_0 = 0$ , the method is called explicit, otherwise, it is said to be implicit.

In multistep method, we require multiple starting values.





# Adams-Bashforth Methods

---

We assume a uniform discretization in the  $x$ -domain, i.e., we define

$$x_i = a + ih,$$

where  $h = \frac{b-a}{N}$ , for some positive integer  $N$ .

Let the given differential equation is of the form

$$y'(x) = f(x, y).$$

We now integrate the given differential equation from  $x = x_i$  to  $x = x_{i+1}$ .



This yields the equation

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} f(x, y(x)) dx.$$

Next, we write

$$f(x, y(x)) = P_{m-1}(x) + R_{m-1}(x),$$



where

$$P_{m-1}(x) = \sum_{j=1}^m L_{m-1,j}(x) f(x_{i+1-j}, y(x_{i+1-j})),$$

is the Lagrange form of the polynomial of degree at most  $m - 1$  that interpolates  $f$  at the  $m$  points  $x_i, x_{i-1}, x_{i-2}, \dots, x_{i+1-m}$ .



And

$$R_{m-1}(x) = \frac{f^{(m)}(\xi, y(\xi))}{m!} \prod_{j=1}^m (x - x_{i+1-j})$$

is the corresponding remainder term.

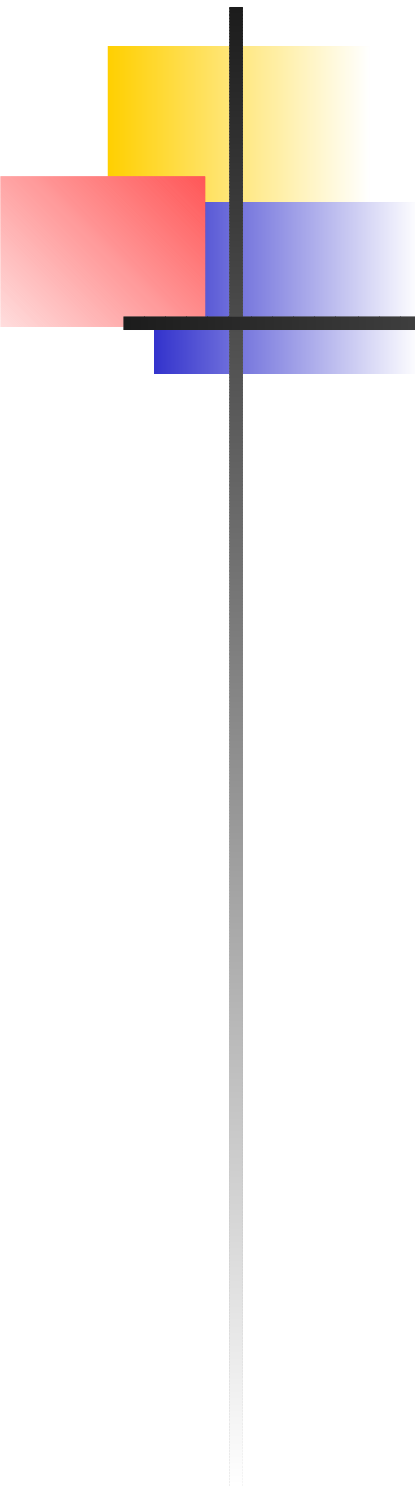


# Two-Step Adams Bashforth Method

Since  $m = 2$ , we write

$$\begin{aligned} f(x, y(x)) &= \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i, y_i) \\ &+ \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}, y_{i-1}) \\ &+ \frac{f''(\xi, y(\xi))}{2} (x - x_i)(x - x_{i-1}). \end{aligned}$$

Integrating the Lagrange polynomials yields


$$b_1 = \int_{x_i}^{x_{i+1}} \frac{x - x_{i-1}}{x_i - x_{i-1}} dx = \frac{3h}{2},$$

$$b_2 = \int_{x_i}^{x_{i+1}} \frac{x - x_i}{x_{i-1} - x_i} dx = -\frac{h}{2}.$$



Therefore, the two-step Adams-Bashforth method is

$$y_{n+1} = y_n + \frac{h}{2} [3f(x_n, y_n) - f(x_{n-1}, y_{n-1})] .$$

Proceeding in a similar manner, the three step Adams-bashforth method is

$$\begin{aligned} y_{n+1} = & y_n + \frac{h}{12} [23f(x_n, y_n) - 16f(x_{n-1}, y_{n-1}) \\ & + 5f(x_{n-2}, y_{n-2})] . \end{aligned}$$



The four-step Adams-Bashforth method is

$$\begin{aligned} y_{n+1} = & y_n + \frac{h}{24} [55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) \\ & + 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})]. \end{aligned}$$





# Example 5

---

Use two-step Adams-Bashforth method to find the approximate solution of

$$\begin{aligned}\frac{dx}{dt} &= 1 + \frac{x}{t}, \\ x(1) &= 1,\end{aligned}$$

near  $x(1.5)$ . Take step size  $h = 0.5$ .



# Solution

---

The two-step Adams-Bashforth formula is

$$x_{n+1} = x_n + \frac{h}{2} [3f(t_n, x_n) - f(t_{n-1}, x_{n-1})] .$$

We note that for finding  $x_2$ , we require  $x_1$  and  $x_0$ .

We calculate  $x_1$  with the help of second order Taylor's method

$$x_1 = 2.125.$$



Now,

$$\begin{aligned}x_2 &= x_1 + \frac{h}{2} [3f(t_1, x_1) - f(t_0, x_0)] , \\ &= 3.4375.\end{aligned}$$



# Adams-Moulton Methods

---

The derivation of Adams-Moulton methods follows exactly the same procedure as the derivation of the Adams-Bashforth method with one exception.

In addition to interpolating  $f$  at  $x_i, x_{i-1}, x_{i-2}, \dots, x_{i+1-m}$ , we also interpolate at  $x_{i+1}$ .



Hence, we write

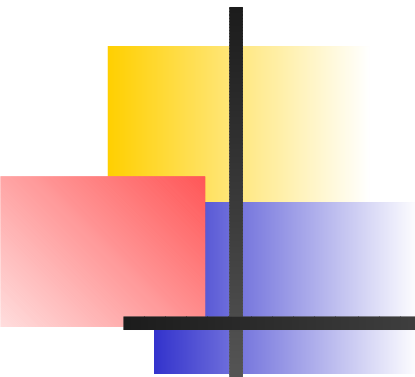
$$f(x, y(x)) = P_m(x) + R_m(x),$$

where

$$P_m(x) = \sum_{j=0}^m L_{m,j}(x) f(x_{i+1-j}, y(x_{i+1-j})),$$

and

$$R_m(x) = \frac{f^{(m+1)}(\xi, y(\xi))}{(m+1)!} \prod_{j=0}^m (x - x_{i+1-j}).$$



Since  $P_m(x)$  contains a term involving  $f(x_{i+1}, y(x_{i+1}))$ , the resulting method will be implicit.

Furthermore, by using an additional point in the interpolating polynomial the degree of the remainder term is increased by one over the Adams-Bashforth case.

## 2-Step Adams-Moulton Method

Since  $m = 2$ , we write

$$\begin{aligned} f(x, y(x)) = & \frac{(x - x_i)(x - x_{i-1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})} f(x_{i+1}, y_{i+1}) \\ & + \frac{(x - x_{i+1})(x - x_{i-1})}{(x_i - x_{i+1})(x_i - x_{i-1})} f(x_i, y_i) \\ & + \frac{(x - x_{i+1})(x - x_i)}{(x_{i-1} - x_{i+1})(x_{i-1} - x_i)} f(x_{i-1}, y_{i-1}) \\ & + \frac{f'''(\xi, y(\xi))}{6} (x - x_{i+1})(x - x_i)(x - x_{i-1}). \end{aligned}$$



Integrating the Lagrange polynomials yields

$$b_0 = \int_{x_i}^{x_{i+1}} \frac{(x - x_i)(x - x_{i-1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})} dt = \frac{5h}{12},$$

$$b_1 = \int_{x_i}^{x_{i+1}} \frac{(x - x_{i+1})(x - x_{i-1})}{(x_i - x_{i+1})(x_i - x_{i-1})} dt = \frac{2h}{3},$$

$$b_2 = \int_{x_i}^{x_{i+1}} \frac{(x - x_{i+1})(x - x_i)}{(x_{i-1} - x_{i+1})(x_{i-1} - x_i)} dt = -\frac{h}{12}.$$





Therefore, the two-step Adams-Moulton method is

$$\frac{y_{i+1} - y_i}{h} = \frac{5}{12}f(x_{i+1}, y_{i+1}) + \frac{2}{3}f(x_i, y_i) - \frac{1}{12}f(x_{i-1}, y_{i-1}).$$

To find the approximate solution, we required  $y_0$  and  $y_1$ .

The initial condition give  $y_0$  and  $y_1$  can be obtained using any third order one-step method.



Similarly, we derive the three-step  
Adams-Moulton method is given by

$$y_{i+1} = y_i + \frac{h}{24} [9f(x_{i+1}, y_{i+1}) \\ + 19f(x_i, y_i) - 5f(x_{i-1}, y_{i-1}) \\ + f(x_{i-2}, y_{i-2})].$$

The required starting values for the three step  
method should be obtained using a fourth-order  
one-step method.



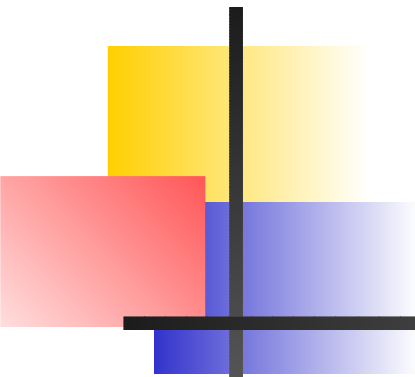
# Predictor-Corrector Method

The most popular of the predictor-corrector scheme is the Adams fourth-order predictor-corrector method.

This uses the four-step, fourth-order Adams-Bashforth method

$$y_{n+1}^{(0)} = y_n + \frac{h}{24} [55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) + 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})],$$

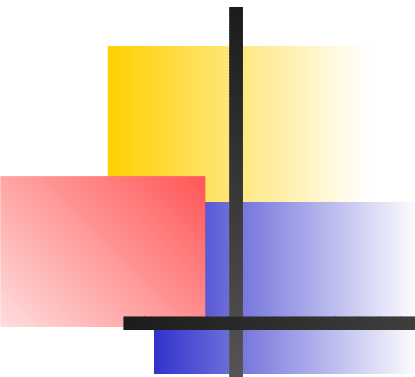
as a predictor



followed by the three-step, fourth-order  
Adams-Moulton method

$$y_{n+1}^{(1)} = y_n + \frac{h}{24} [9f(x_{n+1}, y_{n+1}^{(0)}) + 19f(x_n, y_n) \\ - 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})],$$

as a corrector.



Note that the computation of  $y_4^{(0)}$ , we require the values  $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$ .

These values should be given with the given equation or should be computed using single step methods like, Euler's method, Taylor's method or Runge-Kutta methods.

To compute the value  $y_4^{(1)}$ , we require the values

$(x_1, y_1), (x_2, y_2), (x_3, y_3)$  and  $(x_4, y_4^{(0)})$ .



# Modifier

Let  $y(x_{n+1})$  denotes the true solution of  $y$  at  $x_{n+1}$ .  
Then

$$y(x_{n+1}) = y_{n+1}^{(0)} + \frac{251}{720}h^5 f^{(iv)}(\xi_1), \quad x_{n-3} < \xi_1 < x_{n+1},$$

$$y(x_{n+1}) = y_{n+1}^{(1)} - \frac{19}{720}h^5 f^{(iv)}(\xi_2), \quad x_{n-2} < \xi_2 < x_{n+1}.$$

On subtracting, we get

$$0 = y_{n+1}^{(0)} - y_{n+1}^{(1)} + \frac{270}{720}h^5 f^{(iv)}(\xi).$$



This implies that

$$y_{n+1}^{(1)} - y_{n+1}^{(0)} = \frac{270}{720} h^5 f^{(iv)}(\xi).$$

We rewrite as

$$\frac{h^5}{720} f^{(iv)}(\xi) = \frac{1}{270} (y_{n+1}^{(1)} - y_{n+1}^{(0)}).$$



Finally, we find the modifier as

$$y(x_{n+1}) - y_{n+1}^{(1)} = -\frac{19}{270}(y_{n+1}^{(1)} - y_{n+1}^{(0)}) = D_{n+1}.$$

Here,  $D_{n+1}$  is called as modifier.





# Example

---

Find  $y(1.4)$  by Adams Moulton 4th order predictor-corrector pair with modifier as a solution of

$$\frac{dy}{dx} = x^3 + xy,$$

$y(1) = 2$ ,  $y(1.1) = 1.6$ ,  $y(1.2) = 0.34$  and  $y(1.3) = 0.594$  with spacing  $h = 0.1$ .