New Idea: No computing $\frac{d^m f}{dt^m}$ $\qquad x' = f(t,x)$

Use only $= f(t,x)$

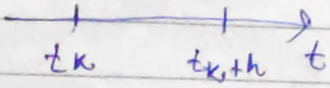
Runge-Kutta Method

1st order $\quad x_{k+1} = x_k + h\cdot f(t_k, x_k)$

2nd order $\quad x_{k+1} = x_k + \frac{1}{2}K_1 + \frac{1}{2}K_2$

$$\text{where } \begin{cases} K_1 = h\cdot f(t_k, x_k) \\ K_2 = h\cdot f(t_k + h, x_k + K_1) \end{cases} \qquad x' = f$$

Heun's method.



Theorem: Heun's Method is 2nd order.

Pf: It suffices to show local error is $O(h^3)$

Taylor series for $f(t_k + h, x_k + K_1)$ expanded at (t_k, x_k)

$$f(t_k + h, x_k + K_1) = f(t_k, x_k) + h\cdot f_t(t_k, x_k) + K_1 f_x(t_k, x_k) + \underbrace{O(h^2, K_1^2)}_{O(h^2)}$$

$$\Rightarrow K_2 = h\cdot\left(f + h f_t + h\cdot f\cdot f_x + O(h^2)\right)$$

Multi-variable Taylor series

$f(t_k + h, x_k + K_1)$ expanded at (t_k, x_k)

$$f(t_k + h, x_k + K_1) = f(t_k, x_k) + h f_t(t_k, x_k) + K_1 f_x(t_k, x_k) + \underbrace{O(h^2, K_1^2)}_{O(h^2)}$$

$$\Rightarrow K_2 = h(f + h f_t + h f\cdot f_x + O(h^2))$$

$$x_{k+1} = x_k + \frac{1}{2}h\cdot f + \frac{1}{2}h(f + h f_t + h f\cdot f_x + O(h^2))$$

$$\Rightarrow x_k + \frac{1}{2}h\cdot f + \frac{1}{2}h\cdot f + \frac{h^2}{2}f_t + \frac{h^2}{2}f f_x + O(h^3)$$

$$\Rightarrow x_k + h\cdot f + \frac{h^2}{2}(f_t + f\cdot f_x) + O(h^3)$$

Exact soln. $\quad x(t) \cdot \quad x'(t) = f(x,t) \quad x(t_k) = x$

Exact soln at $x(t_k+h)$

using Taylor Series $\Rightarrow x(t_k) + h \cdot x'(t_k) + \frac{h^2}{2} x''(t_k) + O(h^3)$

$\Rightarrow x(t_k) + h \cdot f + \frac{h^2}{2}\left[f_t + f_x \cdot x'\right] + O(h^3)$

$\Rightarrow x_k + hf + \frac{h^2}{2}\left[f_t + f_x \cdot f\right] + O(h^3)$

Local Errors $\Rightarrow |x_{k+1} - x(t_k+h)| = O(h^3)$

Local Errors is third order.

general form of mth order Runge Kutta

$x_{k+1} = x_k + w_1 k_1 + w_2 k_2 + \dots + w_m k_m$

where $k_1 = hf(t_k, x_k)$

$k_2 = h \cdot f(t_k + a_2 \cdot h,\ x_k + b_2 k_1)$

$k_3 = h \cdot f(t_k + a_3 h,\ x_k + b_3 k_1 + c_3 k_2)$

$\vdots$

Classical Runge-Kutta 4

$x_{k+1} = x_k + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$k_1 = h \cdot f(t_k, x_k)$ $\quad k_2 = h \cdot f(t_k + \frac{1}{2}h,\ x_k + \frac{1}{2}k_1)$

$k_3 = h \cdot f(t_k + \frac{1}{2}h,\ x_k + \frac{1}{2}k_2)$ $\quad k_4 = h \cdot f(t_k + h,\ x_k + k_3)$

simpson's method

→ Adaptive methods

Observation

1. Small h → smaller errors
2. Higher order → smaller error
3. Uniform grid, gives varying local errors

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Use only $= f(t,x)$

Runge - Kutta Method

1st order $\quad x_{k+1} = x_k + h \cdot f(t_k, x_k)$

2nd order $\quad x_{k+1} = x_k + \frac{1}{2} K_1 + \frac{1}{2} K_2$

where $\begin{cases} K_1 = h \cdot f(t_k, x_k) \\ K_2 = h \cdot f(t_k + h, x_k + k_1) \end{cases}$ $\qquad x' = f$

Heun's method. $\qquad t_k \quad t_k + h \quad t$

Theorem: Heun's Method is 2nd order.

Pf: It suffices to show local error is $O(h^3)$

Taylor series for $f(t_k + h, x_k + K_1)$ expanded at (t_k, x_k)

$$f(t_k + h, x_k + K_1) = f(t_k, x_k) + h \cdot f_t(t_k, x_k) + K_1 f_x(t_k, x_k) + \underbrace{O(h^2, k_1^2)}_{O(h^2)}$$

$$\Rightarrow K_2 = h \cdot \left(f + h f_t + h \cdot f \cdot f_x + O(h^2)\right)$$

Multi- variable Taylor series

$f(t_k + h, x_k + K_1)$ expanded at (t_k, x_k)

$$f(t_k + h, x_k + k_1) = f(t_k, x_k) + h f_t(t_k, x_k) + k_1 f_x(t_k, x_k) + \underbrace{O(h^2, k_1^2)}_{O(h^2)}$$

$$\Rightarrow K_2 = h(f + h f_t + h f \cdot f_x + O(h^2))$$

$$x_{k+1} = x_k + \frac{1}{2} h \cdot f + \frac{1}{2} h (f + h f_t + h f \cdot f_x + O(h^2))$$

$$\Rightarrow x_k + \frac{1}{2} h \cdot f + \frac{1}{2} h \cdot f + \frac{h^2}{2} f_t + \frac{h^2}{2} f f_x + O(h^3)$$

$$\Rightarrow x_k + h \cdot f + \frac{h^2}{2}(f_t + f \cdot f_x) + O(h^3)$$

Exact soln. $\quad x(t) \cdot \quad x'(t) = f(x,t) \quad x(t_k) = x$

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$$\Rightarrow x_k + hf + \frac{h^2}{2}\left[f_t + f_x \cdot f\right] + O(h^3)$$

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⋮

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$$x_{k+1} = x_k + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

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simpson's method

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③ Uniform grid, gives varying local errors

Idea: $\to x_{k+1}$
$\searrow \tilde{x}_{k+1}$ (better method)

$|x_{k+1} - \tilde{x}_{k+1}|$: error measurement

- if error >> tolerance, reject, half the step size (h)
- if error << tolerance, accept, double the step size (h)
- if error $\approx$ tolerance, accept., continue

① Method 1 : take RK4, one step w/ h
Method 2 : take RK4, two steps w/ $h/2$

Use RK4, RK5 $\to$ Runge-Kutta-Fehlberg
$\underset{m_1}{\text{RK4}}$, $\underset{m_2}{\text{RK5}}$