Finite Difference Method

$$y''(x) = u(x) + v(x)\,y + w(x)\,y'$$

BCs $y(a) = \alpha$
$y(b) = \beta$

Discretization : Uniform Grid (h)

$$h = \frac{b-a}{n}, \quad x_i = a + ih \qquad i = 0, 1, \dots, n$$

Seek $y_i \cong y(x_i)$

central Finite Diff.

$$y'(x_i) \approx \frac{y(x_{i-1}) - y(x_{i+1})}{x_{i-1} - x_{i+1}} \approx \frac{y_{i-1} - y_{i+1}}{-2h}$$

$$y''(x_i) \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}.$$

Plug in DE

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = u_i + v_i y_i + w_i \left(\frac{y_{i-1} - y_{i+1}}{-2h}\right)$$

$$y_{i-1} - 2y_i + y_{i+1} = h^2 u_i + h^2 v_i y_i + h w_i \frac{(y_{i+1} - y_{i-1})}{2}$$

$$y_{i-1}\underbrace{\left(1 + \frac{h w_i}{2}\right)}_{a_i} - \underbrace{(2 + h^2 v_i)}_{d_i} y_i + \underbrace{\left(1 - \frac{h w_i}{2}\right)}_{c_i} y_{i+1} = h^2 u_i$$

$$a_i y_{i-1} + d_i y_i + c_i y_{i+1} = h^2 u_i$$

$(n-1)$ eqs

$i = 1, 2, \dots, n-1$

BCs $y_0 = y(x_0) = \alpha$.
$y_n = y(x_n) = \beta$

Unknowns
$y_1, y_2, \dots y_{n-1}$
$n-1$ unknowns

Example: setup FD for BVP. $[0,1]$

$y''(x) = -4(y-x)$ $y(0)=1$ $y(1)=2$ 04:00pm

Uniform grid. $x_i = ih$ $h = \frac{1}{n} \Rightarrow \frac{b-a}{n}$

$$\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = -4y_i + 4x_i$$

$$y_{i-1} - (2-4h^2)y_i + y_{i+1} = 4h^2 x_i$$

Put the Boundary condition

$y_0 = 1$ $y_n = 2$

$i=1$

$$\cancel{y_0} - (2-4h^2)y_1 + y_2 = 4h^2 x_1 - 1$$

1

$i = n-1$ $y_{n-2} - (2-4h^2)y_{n-1} = 4h^2 x_{n-1} - 2$

FINITE DIFF FOR PDE

$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = f(x, y, \phi, \phi_x, \phi_y)$$

$\partial_x = \frac{\partial}{\partial x}$ | $\partial_{xx} = \frac{\partial^2}{\partial x^2}$ | $\partial_{xy} = \frac{\partial}{\partial x \partial y}$.

→ Quasilinear

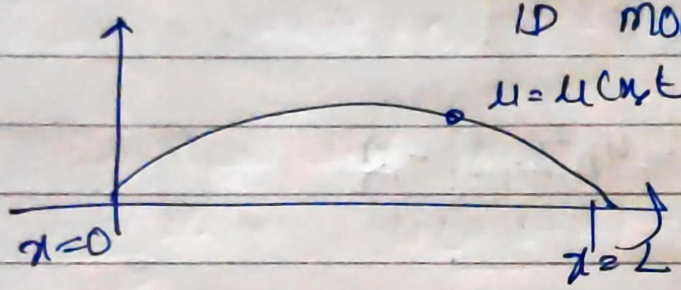
$B^2 - 4AC < 0$ ELLIPTIC

$B^2 - 4AC = 0$ PARABOLIC

$B^2 - 4AC > 0$ HYPERBOLIC.

i) HYPERBOLIC.

1D model of VIBRATING STRING



Displacement governed by wave eq.

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = T \frac{\partial^2 u(x,t)}{\partial x^2}$$

$u(x,0) = f(x)$ Initial pos.

$\frac{\partial u}{\partial t}(x,0) = g(x)$ Velocity

$u(0,t) = 0$

$u(L,t) = 0$ Boundary vals

ρ = mass of string per unit length

T = Tension in the string

(ii) PARABOLIC. 1D model for the heat flow in an insulated rod of length L.

$$K \frac{\partial^2}{\partial x^2} u(x,t) = \sigma \rho \frac{\partial u}{\partial t}(x,t)$$

temperature

WAVE EQUATION. $0 < x < a \mid 0 < t < b$

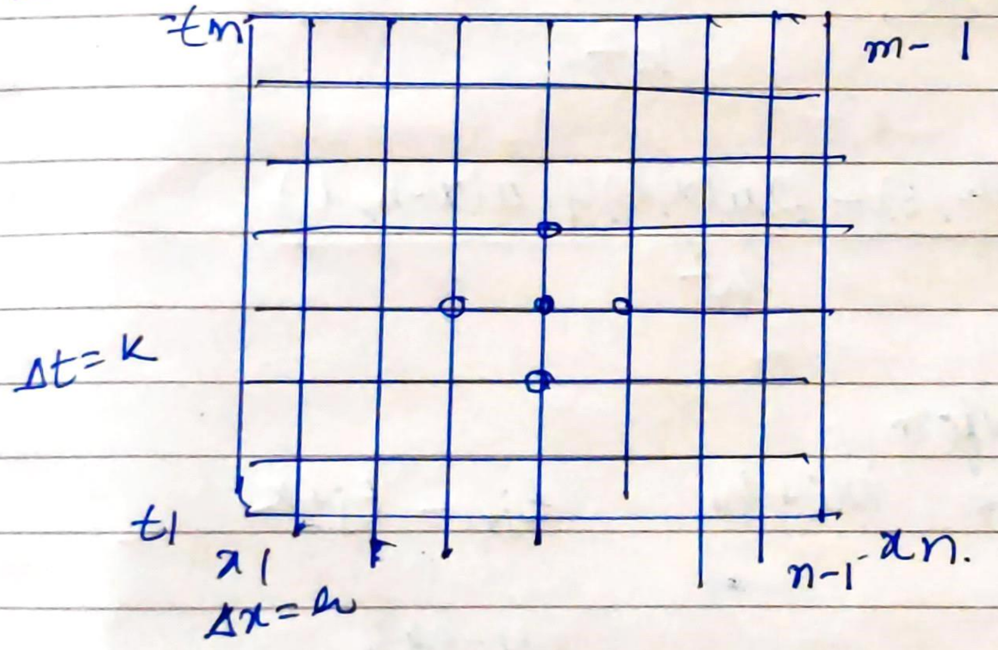
$$c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}.$$

BC's. $u(0,t) = 0$ $u(0,t) = 0$

$u(x,0) = f(x)$ $u_t(x,0) = g(x)$

analytical soln obtained by Fourier series

Derivation of Diff eq. $R = \{(x,t) : 0 \le x \le a,\ 0 \le t < b\}$



Start at bottom row.

$t = t_1 = 0$

$u(x_i, t_1) = f(x_i)$

$0 < x < a$
$0 < t < b$

Wave Equation

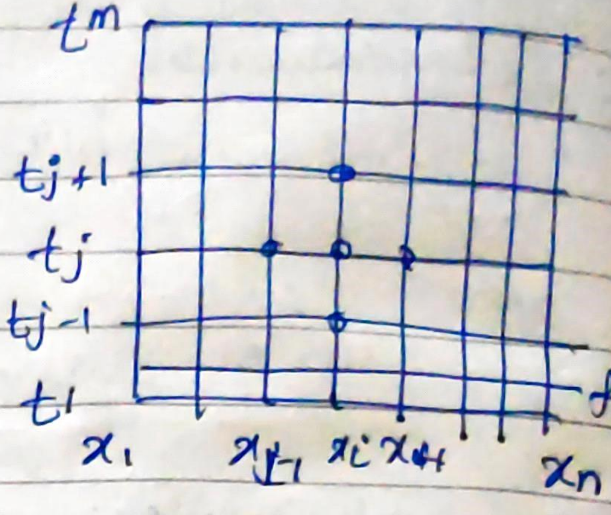
$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

Boundary condition

$u(0,t) = 0$ $\qquad u(a,t) = 0$

$u(x,0) = f(x)$

$\frac{\partial u(x,0)}{\partial t} = g(x)$



Rectangle

Bottom row

$t = t_1 = 0$

$u(x_i, t_1) = f(x_i)$

central diff form

$$\frac{\partial^2 u}{\partial t^2} = \frac{u(x, t+k) - 2u(x,t) + u(x, t-k)}{k^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h, t) - 2u(x,t) + u(x-h, t)}{h^2}$$

grid spacing uniform

$x_{i+1} = x_i + h \qquad t_{j+1} = t_j + k$

Equat $\qquad r = ck/h$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$u = u(x,y)$ $\quad \dfrac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \dfrac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$\frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial y^2} + u = 1$$

Linear second Order PDE

$$A \cdot \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0.$$

$A = A(x,y)$ $\quad B = B(x,y)$ $\quad C = C(x,y)$ $\quad D = D(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u)$

$B^2 - 4AC < 0$	ELLIPTIC	$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$	Laplace Eq.
$B^2 - 4AC = 0$	PARABOLIC	$\frac{\partial T}{\partial t} = k \partial^2 T / \partial x^2$	Heat conduction
$B^2 - 4AC > 0$	HYPERBOLIC	$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 u}{\partial t^2}$	wave

Laplace Eq: Rectangular plate

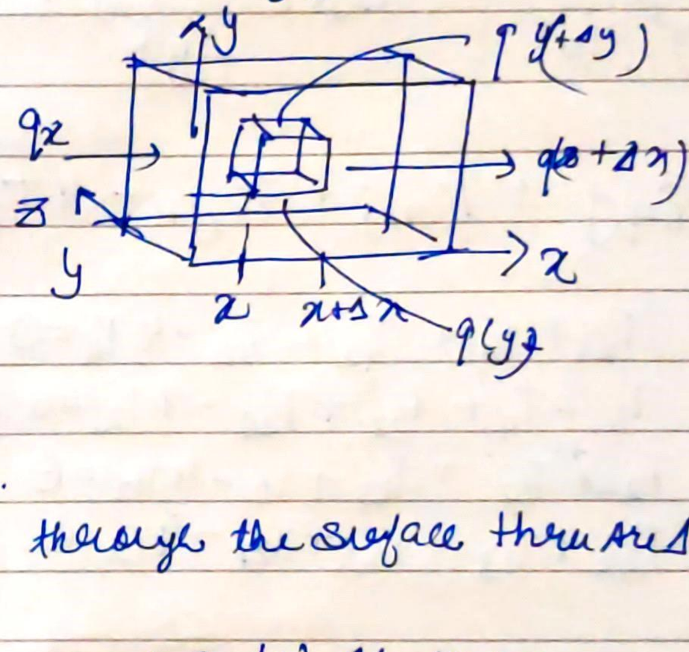


Plate element has volume

$V = \Delta x \Delta y \Delta z$

TEMP NOT CHANGE

$q_x(x)$ = heat flux at point x.

Total heat transferred in Δt through the surface thru Area ΔA.

$\theta = q_x(x) \Delta A \cdot \Delta t$

$\Rightarrow q_x(x) \Delta t \cdot \Delta y \Delta z + q_y(y) \Delta t \Delta x \Delta z$

$= q_x(x+\Delta x) \Delta t \cdot \Delta y \Delta z + q_y(y+\Delta y) \Delta t \Delta x \Delta z$

$[q_x(x) - q_x(x+\Delta x)] \Delta y = [q_y(y+\Delta y) - q_y(y)] \Delta x$

devide $\Delta x \Delta y$ $\quad \lim \Delta x\ \Delta y = 0$

$$\frac{[q_x(x) - q_x(x+\Delta x)]}{\Delta x} = \frac{q_y(y+\Delta y) - q_y(y)}{\Delta y}$$

$$\boxed{\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0}$$

$q_x = k' \frac{\partial T}{\partial x}$ Fourier law for heat conduct

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0$$

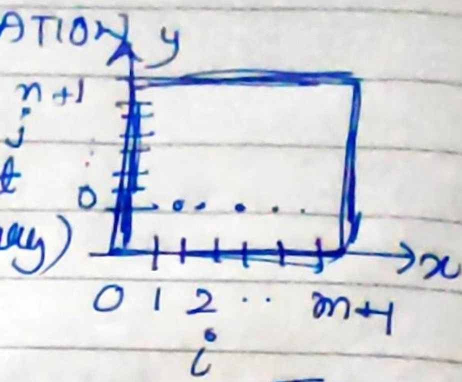
$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

HENCE THE LAPLACE EQUATION.

Poisson EQUATION

SOLVE THIS EQ.

Fixed temp at bound. (Dirichlet Boundary)



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

FINITE DIFFERENCE

T_{ij}

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2}$$

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} - 4T_{i,j} + T_{i,j-1} = 0$$

$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$

$$T_{31} + T_{11} + T_{22} + T_{20} - 4T_{21} = 0$$

$$T_{22} + T_{02} + T_{13} + T_{0} - 4T_{12} = 0$$

$$T_{32} + T_{13} + T_{23} + T_{21} - 4T_{22} = 0$$

$$T_{i,j} = \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right)$$

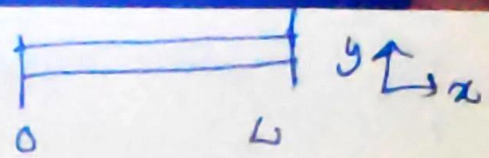
Gauss Siedel Iteration

~~LEAP~~. LIEBMANN METHOD

PARABOLIC PDE

HEAT CONDUCTION

$K \cdot \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

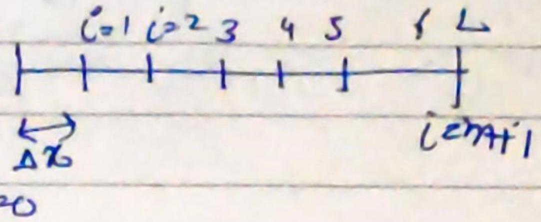


$T(x,t) = ?$

1) Initial cond. $T(x,0) = T^0(x)$

2). Boundary cond. $T(0,t) = T_0(t)$, $T(L,t) = T_L(t)$

FINITE DIFF.



$x_i = i \cdot \Delta x$

$i = 1, \ldots, m$

are the interior points

$t^l = l \cdot \Delta t \qquad l = 0, \ldots, n$

EXPLICIT EULER

$T(x_i, t^l)$

$\left.\frac{\partial T}{\partial t}\right|_i^l = \frac{T_i^{l+1} - T_i^l}{\Delta t}$

$k \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

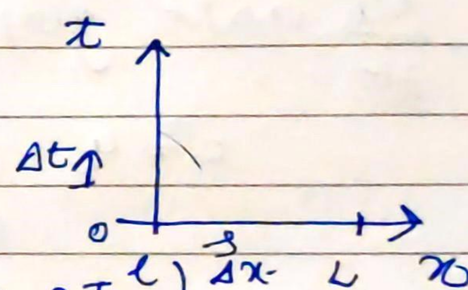
METHOD OF LINES

$\frac{T_i^{l+1} - T_i^l}{\Delta t} = k \left.\frac{\partial^2 T}{\partial x^2}\right|_i^l$

$$\boxed{T_i^{l+1} = T_i^l + k \left.\frac{\partial^2 T}{\partial x^2}\right|_i^l \Delta t}$$

$\left.\frac{\partial^2 T}{\partial x^2}\right|_i^l = \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2}$

Put in the Eq



$T_i^{l+1} = T_i^l + \frac{k\Delta t}{\Delta x^2}\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right)$

$\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{k}$? stable

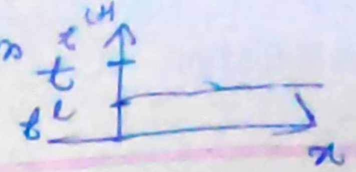
$\lambda = \frac{k\Delta t}{\Delta x^2}$

IMPLICIT

$$\boxed{-\lambda T_{i-1}^{l+1} + (1+2\lambda) T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^l}$$

TRIDIAGONAL.

$$K\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$ Crank-Nicolson



T_i^l

$$\left.\frac{\partial T}{\partial t}\right|_i^{l+1/2} = \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_i^{l+1/2} = \frac{1}{2}\left[\left(\frac{T_{i+1}^{l} - 2T_i^l + T_{i-1}^l}{\Delta x^2}\right) + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x^2}\right]$$

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda) T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^l + 2(1-\lambda) T_i^l + \lambda T_{i+1}^l$$

WAVE

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)$$

$$u_{tt}(x,t) = 4u_{xx}(x,t) \qquad 0 < x < 1 \ (a) \qquad 0 < t < 0.5 \ (b)$$

$u(0,t) = 0 \qquad u(1,t) = 0$

$u(x,0) = \sin(\pi x) + \sin(2\pi x)$

$\frac{\partial u}{\partial t}(x,0) = g(x) = 0$

$c^2 = 4 \quad | \quad c = 2$

$a = 1 \qquad b = 0.5$

Stable $\quad \mu = \frac{ck}{h} \leq 1$ $\qquad \mu = 1.$

$h = 0.1$

$g(x) = 0$ and $\mu = 1$

Second Row

$$u_{i,2} = (1-\mu^2) f_i + k g_i + \frac{\mu^2}{2}(f_{i+1} + f_{i-1})$$

$$\Rightarrow \frac{1}{2}(f_{i+1} + f_{i-1}) \qquad i = 2, \ldots$$

$$u_{i,j+1} = (2-2a^2)\,u_{i,j} + a^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x,0) = I(x)$

$u(0,t) = u(L,t) = 0$

$\frac{\partial}{\partial t} u(x,0) = 0$

$c = \sqrt{T/\rho}$ velocity of wave

$T \rightarrow$ Tension on string

ρ - density

Discrete time and space

$x_i = i\Delta x$ $\quad t_i = n\Delta t$

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_n) \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2}$$

$$\frac{\partial^2 u}{\partial x^2}\, u(x_i, t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

substitute

$u_i^0 = I(x_i)$

$u_i^{n-1} = u_i^{n+1}$

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + C^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

C = Courant number $\boxed{C = \frac{c\,\Delta t}{\Delta x}}$

$n=1$ $\quad u_i^{-1}$ but $t = -\Delta t$ diff.

outside mesh

use $u_t = 0$ to elim u_i^{-1}