Computational methods II

Initial Value Problem: For the soln to be defined uniquely the number of independent boundary condition should equal to the order of differential eq. This is the simplest form of differential eq, where the required number of boundary conditions are specified at one point.

Problems arising in ODE can be usually reduced to that for a system of first-order diff eq.

$$\frac{d^m y}{dt^m} = f\left(t, y, \frac{dy}{dt}, \cdots \frac{d^{m-1}}{dt^{m-1}}\right)$$

can be reduced to system of 'm' first order eq by def $y_1 = y$

$$y_{j+1} = \frac{d^j y}{dt^j} \quad \forall\ j = 1, \ldots m-1$$

$$\frac{dy_j}{dt} = y_{j+1} \qquad \frac{dy_m}{dt} = f(t, y_1, y_2, \cdots y_m)$$

independent variable is time - t

" general system of differential equations can be reduced to a system of first-order differential equation ".

$$\frac{dy_j}{dt} = f(t, y, y_1, \cdots y_n) \qquad (j = (1, 2, \cdots n))$$

to complete the specification for an IVP, we need 'n' boundary conditions

$$g_j(y_1(t_0), y_2(t_0), \cdots y_n(t_0)) = 0$$

$$j = 1, 2, \ldots n.$$

Solution of Single First order differential eqs.

Taylor Series methods for ODEs

initial condition

$$x'(t) = f(t, x(t)) \qquad x(t_0) = x_0$$

Let $t_1 = t_0 + h$ Compute $x_1(t_1) = x(t_0 + h)$

Taylor Series at t_0: Assume Taylor Series Exists

$$x(t_1) = x(t_0) + h\,x'(t_0) + \frac{h^2}{2} x''(t_0) + \cdots + \frac{h^m}{m!} x^{(m)}(t_0)$$

Main idea: Use partial sum to approximate

$$x_1 = x(t_0) + h x'(t_0) + \ldots \frac{h^m x^{(m)}(t_0)}{m!}.$$

Error in each step

$$x(t_0+h) - x_1 = \sum_{k=m+1}^{\infty} \frac{h^k}{k!} x^{(k)}(t_0)$$

Use Taylor theorem

If you have a sum like this and it converges, then the sum is dominated by the leading term.

Error

$$\Rightarrow \frac{h^{m+1}}{(m+1)!} x^{(m+1)}(\xi) \qquad \text{for some } \xi$$

$$t_0 \le \xi \le t_0 + h$$

Simple case:

For $m = 1$

$$x_1 = x_0 + h x'(t_0) \Rightarrow x_0 + h \cdot f(t_0, x_0)$$

$\vdots$

Iteration

$$x_{k+1} = x_k + h \cdot f(t_k, x_k), \qquad k = 0, 1, 2, \ldots, N$$

Forward Euler's method.

For $m = 2$

$$x_1 = x_0 + h\,x'(t_0) + \frac{h^2}{2} x''(t_0)$$

$$\Rightarrow x_0 + h \cdot f(t_0, x_0) + \frac{h^2}{2} x''(t_0)$$

$$x''(t) = \left(f(t, x(t))\right)' = f_t + f_x x'$$

$$x''(t) \Rightarrow f_t + f_x \cdot f$$

so, we get

$$x_1 = x_0 + h \cdot f(t_0, x_0) + \frac{h^2}{2}\left(f_t(t_0, x_0) + f_x(t_0, x_0) \cdot f(x_0, x_0)\right)$$

Example 1: Setup Taylor Series with $m = 1$, $m = 2$ for

a) $x' = -x + e^{-t}$ $\quad x(0) = 0$

Exact soln. $\quad x(t) = t \cdot e^{-t}$

Apply TSM

$t_0 = 0$, $x_0 = 0$ $\quad f(t, x) = -x + e^{-t}$

For $m = 1$

$$x_{k+1} = x_k + h\,f(t_k, x_k)$$

$$= x_k + h(-x_k + e^{-t_k})$$

For $m = 2$

$$x'' = f(t, x(t))' \Rightarrow (-x(t) + e^{-t})'$$

$$\Rightarrow -x' - e^{-t} \Rightarrow x - e^{-t} - e^{-t}$$

$$\Rightarrow x - 2e^{-t}$$

$$x_{k+1} = x_k + h(-x_k + e^{-t_k}) + \frac{h^2}{2}(x_k - 2e^{-t_k})$$

Example 2: Set up TSM w/ $m = 1, 2, 3, 4$ for

$$x' = x \qquad x(0) = 1$$

Exact soln. $x(t) = e^t$

Ans) $x' = x$; $x'' = x$

$\Rightarrow x^{(m)} = x$.

For any m:

$$x_{k+1} = x_k + h\,x'(t_k) + \frac{h^2}{2}.x''(t_k) + \cdots + \frac{h^m}{m!} x^{(m)}(t_k)$$

$$x_1 = x_k + h x_k + \frac{h^2}{2} x_k + \frac{h^3}{3!} x_k + \cdots$$

$$= x_k\left(1 + h + \frac{h^2}{2} + \cdots + \frac{h^m}{m!}\right)$$

$m=1$: $x_{k+1} = x_k(1+h)$

$m=2$: $x_{k+1} = x_k\left(1+h+\frac{h^2}{2}\right)$

$m=3$: $x_{k+1} = x_k\left(1+h+\frac{h^2}{2}+\frac{h^3}{6}\right)$.

ERROR ANALYSIS

Given IVP $\quad x' = f(t,x)$, $x(t_0) = x_0 \quad$ - ⊛

Local Error and Truncation Error

generated at each step / at each iteration

$$e_k = |x_{k+1} - x(t_k + h)|$$

approx solution $\qquad$ exact soln

$$= \frac{h^{m+1}}{(m+1)!}\left|x^{(m+1)}(\xi)\right| \Rightarrow \frac{h^{m+1}}{(m+1)!}\left|\frac{d^m}{dt^m} f(t, x(t))\right|$$

Assume $\left|\frac{d^m f}{dt^m}\right|$ bounded. Let $M = \frac{1}{(m+1)!} \max \left|\frac{d^m f}{dt^m}\right|$

$e^{k} \le M \cdot h^{m+1}$

Total Error $T =$ final computing time from to $\to T$

$h =$ grid size, $N = \frac{T}{h}$ (# of steps)

$\Rightarrow N \cdot h = T$

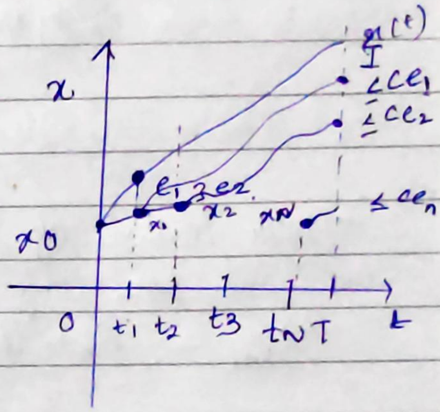
Assume (*) is well posed i.e solution is stable w.r.t perturbation on the Initial Condition

Let $x_0, \bar{x}_0$ be two ICs, let $x(t), \bar{x}(t)$ be two soln, then

Lipschitz cond

$$|x(t) - \bar{x}(t)| \le C|x_0 - \bar{x}_0| \qquad 0 \le t \le T$$

Accumulate discrete Error, control



Total Error

$$E \le \sum_{k=1}^{N} C \cdot e_k$$

$$e_k \le M h^{m+1}$$

$$E \le \sum_{k=1}^{N} C \cdot M \cdot h^{m+1}$$

$$E = N.C.M\, h^{m+1}$$

$$\Rightarrow (N \cdot h)\, C.M.h^m \Rightarrow \underbrace{\boxed{T \cdot C \cdot M}}_{\text{Constants} = C'} \cdot h^m$$

$E \le C' \cdot h^m$ $\qquad m^{th}$ order.

Total Error is 1 order less than Local Error.