Lec - 5

1) Two - Point Boundary value Problem (BVP)

consider

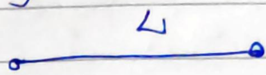
$$y''(x) = f(x, y, y')$$
$$y(a) = \alpha \qquad y(b) = \beta \qquad \text{on } a \le x \le b$$

Many application

i) spring mass

$$mu'' + \gamma u' + ku = f(t)$$

ii) vibration of a Elastic string



$$y'' = ky + mx(x-L)$$
B.C $y(0) = 0, \qquad y'(L) = 0.5$ LINEAR EXAMPLES

NON - LINEAR

$$-(y')^2 - 2b(x)y + 2y \cdot y'' = 0$$
$$y(0) = 1 \qquad y(1) = 2$$

SHOOTING METHOD

IVP w/one IC as parameter adjust to reach the target $y(b) = \beta$

LINEAR SHOOTING: General Problem

$$y'' = u(x) + v(x)y + w(x)y', \qquad y(a) = \alpha, \quad y(b) = \beta \quad -(1)$$

Let $\bar{y}$ be:

$$\bar{y}'' = u + v\bar{y} + w\bar{y}', \qquad y(a) = \alpha, \quad y'(a) = 0 \quad -(2)$$

Let $\tilde{y}$ be:

$$\tilde{y}'' = u + v\tilde{y} + w\tilde{y}', \qquad \tilde{y}(a) = \alpha, \quad \tilde{y}'(a) = 1 \quad (3)$$

solve (II) and (III) to obtain $\bar{y}(x)$, $\tilde{y}(x)$ on $a \le x \le b$

Let $y(x) = \lambda \bar{y}(x) + (1-\lambda)\tilde{y}(x)$ — (*)
where λ : TBD s.t y solves (I)

check DE

$$② \times \lambda + ③ \times (1-\lambda)$$

$$\Rightarrow \lambda u + \lambda v \bar{y} + \lambda w \bar{y}' + u + v\tilde{y} + w\tilde{y}' - \lambda u - \lambda v \tilde{y} - \lambda w \tilde{y}'$$

$$[\lambda \bar{y} + (1-\lambda)\tilde{y}]'' = \cancel{u + v\tilde{y} + w\tilde{y}'}$$
$$u + v \cdot y + w y'$$

$$\boxed{y'' = u + vy + wy'}$$ — Holds for all.

Boundary cond.

$$y(a) = \lambda \cdot \bar{y}(a) + (1-\lambda)\tilde{y}(a)$$
$$= \alpha$$

$$y(b) = \boxed{\lambda \bar{y}(b) + (1-\lambda)\tilde{y}(b) = \beta}$$

solve this for λ

$$\lambda(\bar{y}(b) - \tilde{y}(b)) = \beta - \tilde{y}(b)$$

$$\lambda = \frac{\beta - \tilde{y}(b)}{\bar{y}(b) - \tilde{y}(b)}$$ — (*)(*)

PRACTICAL

$x_1 = \bar{y}$ $x_2 = \bar{y}'$ $x_3 = \tilde{y}$ $x_4 = \tilde{y}'$

$x_1' = x_2$	$x_1(a) = \alpha$
$x_2' = u + vx_1 + wx_2$	$x_2(a) = 0$
$x_3' = x_4$	$x_3(a) = \alpha$
$x_4' = u + vx_3 + wx_4$	$x_4(a) = 1$

NON LINEAR SHOOTING

Consider

$$y'' = f(x, y, y') \qquad a \le x \le b.$$

B.Cs $\quad y(a) = \alpha \quad y(b) = \beta$

f: non-Linear func in y & y'

Let $\tilde{y}$ be some IVP.

$$\tilde{y}'' = f(x, \tilde{y}, \tilde{y}') \quad \text{— (A)}$$

$\tilde{y}(a) = \alpha$

$\tilde{y}'(a)\ z$

Goal: Find z s.t $\tilde{y}(b) = \beta$

(A) can be solved $\Rightarrow \tilde{y}(b)$

Obs. $\tilde{y}(b)$ depends on z

Let $\tilde{y}(b) = \phi(z)$. $\leftarrow$ non Linear func.

GOAL: choose z st. $\phi(z) - \beta = 0$

Chose z_1, z_2 Solve (A). $\phi_1 = \phi(z_1) \quad \phi_2 = \phi(z_2)$

FINITE DIFF METHOD

→ Two-point BVP.

2nd order Linear ODE

$$y''(x) = u(x) + v(x)y + w(x)y' \qquad a \le x \le b$$

BCs $\quad y(a) = \alpha \quad , y(b) = \beta \qquad$ Dirchlet

DISCRETIZATION: Uniform grid h

$h = \frac{b-a}{n}$, $x_i = a + ih$ $\quad i = 0, 1 \ldots n$

Seek $\quad y_i \cong y(x_i)$

Finite diff approx for derivatives

Central F.D

$$y'(x_i) \cong \frac{y(x_{i-1}) - y(x_{i+1})}{x_{i-1} - x_{i+1}} \approx \frac{y_{i-1} - y_{i+1}}{-2h}$$

$$y''(x_i) \cong \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Plug in DE at x_i

Notatn

$u_i = u(x_i)$, $v_i = v(x_i)$, $w_i = w(x_i)$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = u_i + v_i y_i + w_i y'$$

$$= u_i + v_i y_i + w_i \times \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y_{i-1} - 2y_i + y_{i+1} = h^2 u_i + h^2 u_i y_i + h w_i \frac{y_{i+1} - y_{i-1}}{2}$$

$$y_{i-1} \underbrace{\left(1 + \frac{h}{2} w_i\right)}_{a_i} - y_i \underbrace{(2 + h^2 v_i)}_{d_i}$$

$$+ y_{i+1} \underbrace{\left(1 - \frac{h}{2} w_i\right)}_{c_i} = h^2 u_i$$

$$a_i y_{i-1} + d_i y_i + c_i y_{i+1} = h^2 u_i$$

$i = 1, 2, \ldots, n-1 \qquad (n-1)$ eqs.

BCs: $y_0 = y(a) = \alpha \qquad y_n = y(b) = \beta$

unknown $y_1, \dots y_2, \dots y_{n-1}$

i) $i=1$

$$a_1 y_0 + d_1 y_1 + c_1 y_2 = h^2 u_1$$

$$\Rightarrow d_1 y_1 + c_1 y_2 = h^2 u_1 - a_1 \alpha$$

$i = n-1$

$$a_{n-1} y_{n-2} + d_{n-1} y_{n-1} + c_{n-1} y_n = h^2 u_{n-1}$$

$$a_{n-1} y_{n-2} + d_{n-1} y_{n-1} = h^2 u_{n-1} - c_{n-1}\beta$$

In matrix vector

$$A \cdot \vec{y} = \vec{b_1}$$

$$\begin{pmatrix} d_1 & c_1 & & & \\ a_2 & d_2 & c_2 & & \\ & a_3 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-2} \\ & & & a_{n-1} & d_{n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} h^2 u_1 - a_1\alpha \\ h^2 u_2 \\ \vdots \\ h^2 u_{n-1} - c_{n-1}\beta \end{pmatrix}$$

Gaussian Elimination

Diagonal Dominant

$$|d_i| \geq |a_i| + |c_i|$$

$$|2 + h^2 v_i| > \left|1 + \frac{h}{2} w_i\right| + \left|1 - \frac{h}{2} w_i\right|$$

Assume $v_i \geq 0, \quad \left|\frac{h}{2} w_i\right| \leq 1$

i.e $h \leq \dfrac{2}{|w_i|} \qquad$ chose $h = \dfrac{2}{\max |w(x)|}$

$a \leq x \leq b.$

$\Rightarrow$ A is diagonal dominance

Eg.

$$y''(x) = -4(y-x) \qquad y(0)=1, \; y(1)=2$$

Ans: Uniform grid

$$x_i = ih \qquad , h = \frac{1}{n}$$

$$\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = -4y_i + 4x_i$$

$$y_{i-1}(1) \quad -(2-4h^2)\, y_i + y_{i+1} = 4h^2 x_i$$

$$y(0) = 1 \qquad y_n = 2$$

$i=1$: $\quad -(2-4h^2)y_1 + y_2 = 4h^2 x_1 - 1 \quad$ — (1)

$i=n-1 \quad y_{n-2} - (2-4h^2)y_{n-1} = 4h^2 x_{n-1} - 2 \quad$ — (2)

] BCs

matrix vector form.

$$\begin{bmatrix} 1 & -2+4h^2 & 1 & \cdot & \cdot \\ & 1 & -2+4h^2 & 1 & \cdot \\ & & 1 & \ddots & \\ \cdot & & & & \\ & \vdots & & 1 & -2+4h^2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 4h^2 x_1 - 1 \\ 4h^2 x_2 \\ \vdots \\ \vdots \\ 4h^2 x_{n-2} \\ 4h^2 x_{n-1} - 2 \end{bmatrix}$$

BC. change

$$y(0) = 1 \qquad , y'(1) = 3$$

y_n is unknown $\Rightarrow$ $y_1, y_2, \cdots y_n$ # $= n$.

$y'(1) = 3$ discretize it

$$\frac{y_n - y_{n-1}}{h} = 3 \Rightarrow \boxed{-y_{n-1} + y_n = 3h}$$