

lecture 2: Quantum computation review

Nature behaves in non-local manner.

I think that a particle must have a separate reality independent of measurements. That is, an electron has spin, location and so forth even when it is not being measured. I like to think the moon is there even if I am not looking at it

— Albert Einstein

... experiments have now shown that what bothered Einstein is not a debatable point but the observed behavior of the real world

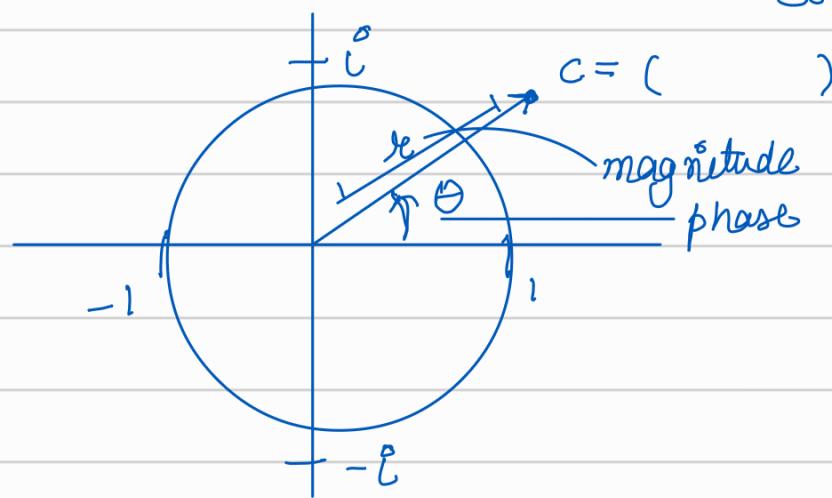
— N David Mermin

1) Linear Algebra

$$\text{ket } |\Psi\rangle = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix} \in \mathbb{C}^d \quad a+ib \in \mathbb{C}$$

$$re^{i\theta} \in \mathbb{C}$$

$$a, b, r, \theta \in \mathbb{R}$$



complex conjugate c^*

$$\begin{aligned} &\Rightarrow a - ib \\ &\Rightarrow re^{-i\theta} \end{aligned}$$

$$c = r(\cos\theta + i\sin\theta) \quad \text{Euler's formula}$$

$$\rightarrow |c| = \sqrt{c^* c}$$

$$\text{or } |re^{i\theta}| = |r| |e^{i\theta}|$$

$$\Rightarrow |r| |e^{-i\theta} e^{i\theta}|$$

$$c + c^* = 2\operatorname{Re}(c)$$

$$c - c^* = 2\operatorname{Im}(c)$$

$$cc^* = |c|^2$$

"bra ψ "

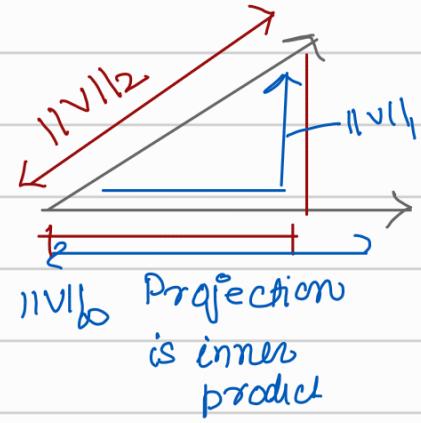
$$\langle \psi | = (\psi_1^* \ \psi_2^* \ \dots \ \psi_d^*)$$

Inner product

$$\langle \psi_1 | \psi \rangle = \langle \psi_1 | \psi \rangle \in \mathbb{C}$$

$d \times 1$

$$\Rightarrow \sum_{i=1}^d \psi_i^* \psi_i$$



$$\text{Euclidean Norm} \quad \| |\psi\rangle \|_2 = \sqrt{\langle \psi | \psi \rangle}$$

length of the vector:

$$\| |\psi\rangle \|_1 = \sum_i |\psi_i| \quad \text{one norm}$$

$$\| |\psi\rangle \|_\infty = \max_i |\psi_i| \quad \text{infinity norm}$$

Orthonormal Basis set $\{|\psi_i\rangle\} \subseteq \mathbb{C}^d$ s.t. $\langle \psi_i | \psi_j \rangle = \delta_{ij}$

δ_{ij} Kronecker delta

standard basis / computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad |d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Any $|\psi\rangle \in \mathbb{C}^d$ as $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle \in \mathbb{C}^d$

$$\text{if } \| |\psi\rangle \|_2 = 1 \Rightarrow \sum_i |\alpha_i|^2 = 1$$

normalized

Linear Maps and Matrices

Map $\Phi: \mathbb{C}^d \rightarrow \mathbb{C}^d$ is linear if

$$\Phi \left(\sum_i \alpha_i |\psi_i\rangle \right) = \sum_i \alpha_i \Phi (|\psi_i\rangle) \quad (\text{linearity})$$

$L(x, y) \rightarrow$ set of linear maps from $x \mapsto y$

$L(x) \rightarrow \Phi : x \mapsto x$

matrix representation of $\Phi : \mathbb{C}^d \mapsto \mathbb{C}^d$, A_Φ is a $d \times d$ matrix

$$A = \begin{bmatrix} & & & & \\ & \vdots & & & \\ \Phi(10\rangle) & \Phi(11\rangle) & \cdots & \Phi(d-1)\rangle \\ & \vdots & & \\ & & & \end{bmatrix}_{d \times d} \in \mathbb{C}^{d \times d}$$

EXERCISE $\Phi(10\rangle = |1\rangle \quad \Phi(11\rangle = |0\rangle$

$$A_\Phi = \begin{pmatrix} |1\rangle & |0\rangle \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ X gate}$$

$A|1\rangle = i^{\text{th}}$ column of A .

$$|10\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow |1\rangle$$

Matrix Multiplication

$$AB_{(i,j)} = \sum_{k=1}^d A(i,k) B(k,j)$$

Recall! matrix multiplication $AB \neq BA$ in general
(this keeps matter stable)

if $AB = BA$, say A and B commute.

$$[A, B] = AB - BA = 0$$

EXERCISE

$xz \neq zx$ ($xz = -zx$) They anti-commute.

Spaces i) Image of A : $\text{Im}(A) = \{ |\psi\rangle \in \mathbb{C}^d \mid \exists |\phi\rangle \in \mathbb{C}^d \text{ such that } A|\phi\rangle = |\psi\rangle\}$

ii) Null space of A : $\text{Null}(A) = \{ |\psi\rangle \in \mathbb{C}^d \mid A|\psi\rangle = 0\}$

$$\text{rank}(A) = \dim(\text{Im}(A))$$

Rank-nullity theorem

$$\text{rank}(A) + \dim(\text{null}(A)) = d \quad (\text{full dimension})$$

Matrix operations a) $A^+ = (A^\top)^*$ conjugate transpose, adjoint, dagger.

recall $(AB)^+ = B^+ A^+$ (flips the order)

b) Trace $(A) = \sum_{i=1}^d A_{i,i}$
Trace is linear

Tr: $L(\mathbb{C}^d) \mapsto \mathbb{C}$

take a matrix

gives a single number.

and is cyclic

$$\text{Tr}(\overbrace{ABC}) = \text{Tr}(BCA)$$

the implication is $\text{Tr}(AB) = \text{Tr}(BA)$ always

Outer Product

$|v\rangle, |w\rangle \in \mathbb{C}^d \rightarrow$ outer product
 $|v\rangle \otimes |w\rangle \in L(\mathbb{C}^d)$
 $d \times 1 \quad 1 \times d$

E.g. $|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$|i\rangle \otimes |j\rangle \rightarrow 1 \text{ in position } (i,j), 0 \text{ otherwise}$

Trick

$$\langle e_i | A | j \rangle = \underbrace{\langle e_i | \left(\sum_{i,j} A(e_i, j) | e_i \rangle \otimes | j \rangle \right) | j \rangle}_{d \times d \quad d \times d \quad d \times 1}$$

std. basis are orthonormal

$$\Rightarrow \sum_{i,j} A(e_i, j) \underbrace{\langle i | i \rangle}_{\delta_{ii}} \underbrace{\langle j | j \rangle}_{\delta_{jj}}$$

$$\Rightarrow A(i, j)$$

(outer product take vectors together and make matrices out of it)

EXERCISE q: $\text{Tr}(A) = \sum_i A(i, i)$

$$\sum_i \langle i | A | i \rangle$$

Extracting out the diagonal entries

Eigenvalues and Eigen vectors

A matrix $A \in L(\mathbb{C}^d)$ have eigenvector $|v\rangle$ if

$$A|v\rangle = \lambda|v\rangle \quad \xrightarrow{\text{eigenvalues}}$$

An operator A is normal iff $AA^+ = A^+A$ iff A has a spectral decomposition

$$A = \sum_i \lambda_i | \lambda_i \rangle \otimes | \lambda_i \rangle^\dagger \quad \begin{matrix} \nearrow \text{eigen} \\ \text{vectors} \\ \downarrow \\ \text{eigenvalues} \end{matrix}$$

If A is normal then the eigenvectors can be chosen as orthonormal basis.

if A is normal then, (because spectral decomposition)

$$\text{Tr}(A) = \text{Tr}\left(\sum_i \lambda_i^0 |\lambda_i^0\rangle\langle\lambda_i^0| \right)$$

(+trace is linear)

$$\Rightarrow \sum_i \lambda_i^0 \langle \lambda_i^0 | \lambda_i^0 \rangle$$

(+trace is cyclic)

Trace of any operator is always the sum of its eigenvalues.

EXERCISE: If A is diagonalizable (normal) then prove that

$\text{rank}(A) = \text{number of non-zero eigenvalues of } A$

Rank of a diagonalizable matrix is equal to the number of non-zero eigenvalues

If A is diagonalizable then $\exists P$ invertible matrix such that $P^{-1}AP = D$ where D is diagonal matrix having eigenvalues of A on its diagonal and each eigenvalue appears as many times as its multiplicity.

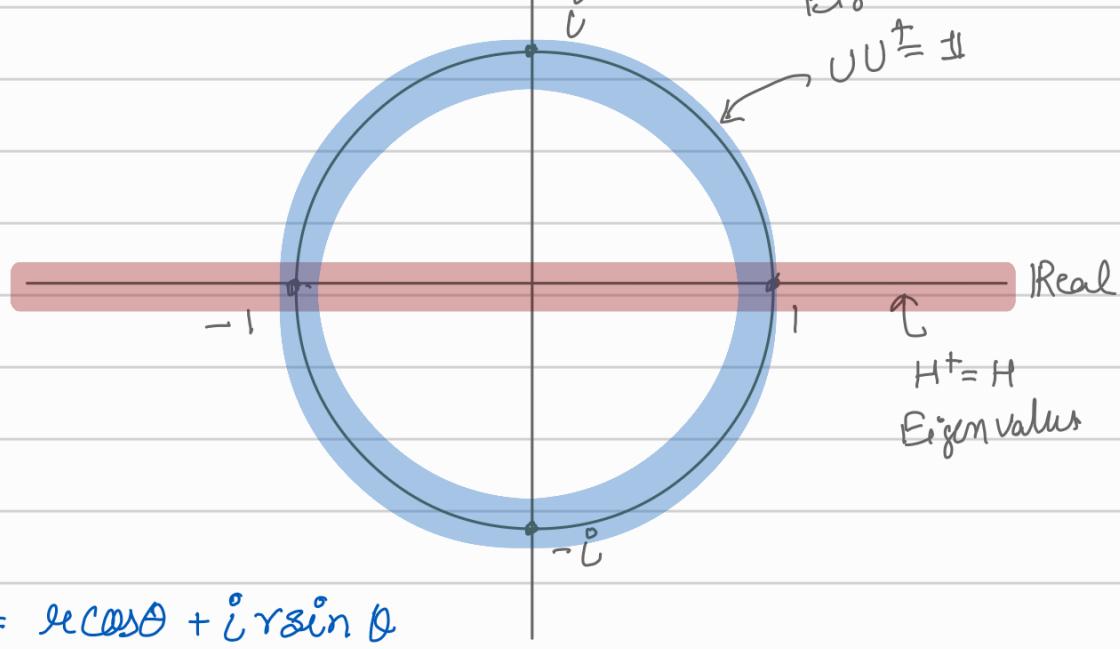
$$\text{rk}(A) = \text{rk}(P^{-1}AP) = \text{rk}(D)$$

Important classes of Matrices

i) Unitary matrices : $U \in L(C^n)$ if $UU^T = I = U^T U$

EXERCISE: Eigenvalues of unitary operator are always on the unit circle.

Eigenvalues



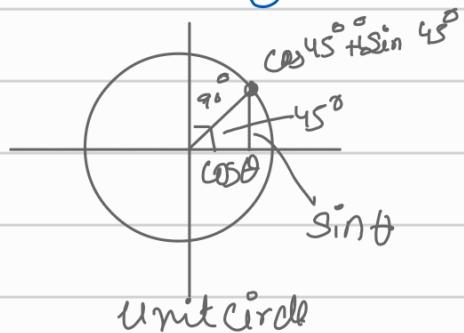
$$x + iy = r \cos \theta + ir \sin \theta \\ = re^{i\theta}$$

$$r = |x + iy| = \sqrt{(x+iy)(x+iy)} \Rightarrow \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Let's see how can we write complex numbers using these notations.

$$\cos 45^\circ + i \sin 45^\circ \Rightarrow \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \\ \Rightarrow e^{i\frac{\pi}{4}}$$



Unitary matrices preserves norms of a vector.

eigenvalues λ eigenvectors $|\psi\rangle$

$$U|\psi\rangle = \lambda|\psi\rangle \quad \sim \text{Eigenvalue equation}$$

Unitary matrices preserve norms

$$\| |\psi\rangle \| = \| U|\psi\rangle \| \\ \| \psi\rangle \| = \| \lambda|\psi\rangle \|$$

$$\begin{aligned} \sqrt{\langle \psi | \psi \rangle} &= \sqrt{(\lambda |\psi\rangle)^* (\lambda |\psi\rangle)} \\ &= \sqrt{\langle \psi | \lambda^* \lambda |\psi\rangle} \\ &= \sqrt{\lambda^* \lambda \langle \psi | \psi \rangle} \\ \sqrt{\langle \psi | \psi \rangle} &\Rightarrow |\lambda| \sqrt{\langle \psi | \psi \rangle} \end{aligned}$$

absolute value of complex number, how far is it from the center of the circle

$$|\lambda| = 1$$

~ hence the eigenvalues of a unitary operator lies on the complex numbers

Unitary matrices are high dimensional generalization of unit complex numbers.

Set of unitary : $U(\mathbb{C})$

ii) Hermitian $H \in L(\mathbb{C}^d)$ if $H = H^\dagger$

Set of Hermitian operator $\text{Herm } L(\mathbb{C})$

Their eigen values are always IR

and they generalize real numbers:

iii) Positive (semi-) definite : Hermitian + non-negative or strictly positive

Pos $L(\mathbb{C})$

If one of your eigenvalues is 0, you're not definite
invertible. $H \geq 0$ $H > 0$
semi-definite

generalize set of non-negative and positive reals.

iv) Orthogonal projections : $\Pi \in L(\mathbb{C}^d)$ s.t.
 Π is Hermitian and $\Pi^2 = \Pi$

$\Pi^2 = \Pi \Rightarrow$ eigenvalues $\in \{0, 1\}$

Spectral decomp: $\Pi = \sum_i (\psi_i \otimes \psi_i^\dagger)$ for $\{\psi_i\}$ orthonormal basis,

rank-1 proj : $|\psi \otimes \psi^\dagger|$

only one - nonneg

Some Revision and Notes

Maps $\Phi: \mathbb{C}^d \rightarrow \mathbb{C}^d$ are linear, meaning.

$\sum_i \alpha_i |\psi_i\rangle \in \mathbb{C}^d$ that

$$\Phi\left(\sum_i \alpha_i |\psi_i\rangle\right) = \sum_i \alpha_i \Phi|\psi_i\rangle$$

hence linear maps.

$L(\mathbb{C}^d)$ they have matrix representation

$$A_\Phi = [\Phi|0\rangle, \Phi|1\rangle, \dots, \Phi|d-1\rangle]$$

we define the i^{th} column of A_Φ as $\Phi|i\rangle$ for $\{|i\rangle\}$ the standard basis for \mathbb{C}^d .

The image of a matrix A is the set of all possible output vectors under the action of A .

$$\text{Im}(A) := \{|\psi\rangle \in \mathbb{C}^d \mid |\psi\rangle = A|\phi\rangle \text{ for some } \phi \in \mathbb{C}^d\}$$

The rank of A is the dimension of its image. i.e $\dim(\text{Im}(A))$

$$\text{Null}(A) := \{|\psi\rangle \in \mathbb{C}^d \mid A|\psi\rangle = 0\}$$

Rank- Nullity Theorem

$$\dim(\text{Null}(A)) + \dim(\text{Im}(A)) = d$$

A trace is a linear map $\text{Tr}: L(\mathbb{C}^d) \rightarrow \mathbb{C}$ summing the entries on the diagonal of A .

$$\text{Tr}(A) = \sum_{i=1}^d A(i,i)$$

Trace is cyclic

$$\text{Tr}(ABC) = \text{Tr}(CAB)$$

Operator Functions

How to apply $f: \mathbb{R} \mapsto \mathbb{R}$ to matrices. $f(x) = \sqrt{x}$
 for H being a Hermitian operator $H \in \text{Herm}(\mathbb{C})$ with
 spectral decompos. $H = \sum \lambda_i^0 | \lambda_i^0 \rangle \langle \lambda_i^0 |$

$f: \text{Herm}(\mathbb{C}^d) \mapsto \text{Herm}(\mathbb{C}^d)$
 ↗ apply to scalar.

$$\Rightarrow f(H) = \sum_i f(\lambda_i^0) | \lambda_i^0 \rangle \langle \lambda_i^0 |$$

EXERCISE $f(x) = e^x$. Prove $f(H) = e^H := I + H + \frac{H^2}{2!} + \frac{H^3}{3!} + \dots$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Since H is a Hermitian matrix, we can diagonalize as
 $H = U \Lambda U^+$ where U is a unitary matrix and
 Λ is a diagonal matrix containing the eigenvalues of H .

$$f(H) = f(U \Lambda U^+) = U f(\Lambda) U^+$$

$$\text{Now for } f(\Lambda) = e^\Lambda = \sum_{n=0}^{\infty} \frac{\Lambda^n}{n!}$$

Since Λ is a diagonal matrix, Λ^n is also a diagonal matrix
 with the i -th diagonal element raised to power of n , denoted
 as λ_i^n , where λ_i^0 is the i -th eigenvalue of H . \therefore

$$e^\Lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}$$

Now, substitute e^H back into the expression

$$f(H) = U f(\lambda) U^+ = U \left(\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix} \right) U^+$$

By regrouping,

$$f(H) = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} Ue^{\lambda_1}U^+ & 0 & \cdots & 0 \\ 0 & Ue^{\lambda_2}U^+ & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Ue^{\lambda_n}U^+ \end{pmatrix} = e^H$$

EXERCISE $f(x) = \sqrt{x}$, $\Rightarrow \sqrt{I} = \sqrt{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{1} \end{bmatrix}$

$$\sqrt{I} \leftarrow \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$$

Basic quantum compilation

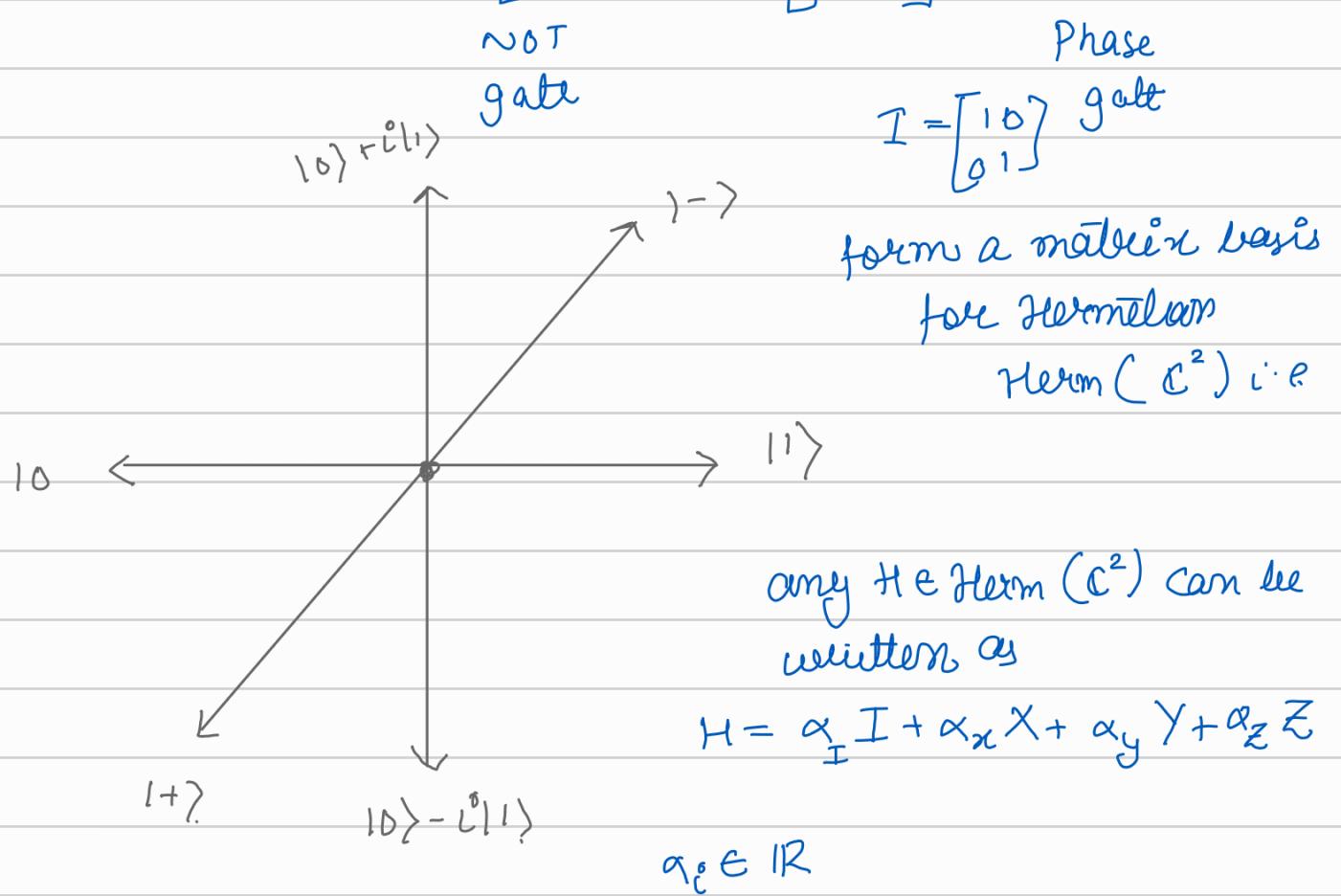
Pure state compilation - d -dimensional pure state $| \psi \rangle = \sum_{i=0}^{d-1} \alpha_i | i \rangle \in \mathbb{C}^d$

$$\| |\psi\rangle \|_2 = 1$$

$$\text{d-dim} = \frac{\text{qudit}}{\text{qubit}}$$

Quantum gates - set of allowable operators are $U \in \mathcal{U}$, such that $UU^+ = I$. The gates are all reversible.

$$\text{for } d=3, \text{ we see } X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Hadamard $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ introduces Superposition

 $H|0\rangle = |+\rangle \Rightarrow \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$
 $H|1\rangle = |- \rangle \Rightarrow \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$

Relativistic Phase $Z|0\rangle = |0\rangle$ $Z|1\rangle = -|1\rangle$ $Z|+\rangle = \frac{1}{\sqrt{2}} Z|0\rangle + \frac{1}{\sqrt{2}} Z|1\rangle = |+\rangle$

Phase $Z|-\rangle = |-\rangle$

NOT gate

Global phase $e^{i\theta}|0\rangle + e^{i\theta}|1\rangle \Rightarrow e^{i\theta}(|0\rangle + |1\rangle)$

We cannot detect global phase in measurement.

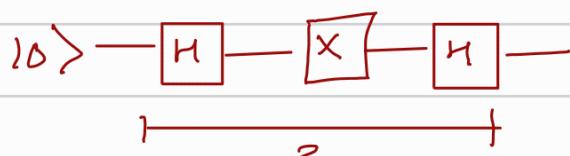
Quantum Circuit Model



$HX|\Psi\rangle$

EXERCISE What gate does this circuit simulate?

Hint: spectral decompos~~x~~
 X, Z



Soln: $X = |0\rangle\langle 1| + |1\rangle\langle 0|$

$$H = \frac{1}{\sqrt{2}} (|0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| - |0\rangle\langle 0|)$$

$$HXH|\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| - |0\rangle\langle 0|) (|0\rangle + |1\rangle) \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

we get

$$HXH|\Psi\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |0\rangle) = -Z|\Psi\rangle$$

Hence, the circuit HXH simulates a Z gate.

Composite Quantum Systems

we make use tensor product, $\otimes : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \rightarrow \mathbb{C}^{d_1 \times d_2}$

formally if we have.

$$(|a\rangle \otimes |c\rangle) (|i,j\rangle) = \underbrace{\Psi_{ij}|a\rangle_j}_{n \text{ copies}}$$

$$\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \cong \mathbb{C}^{2^n}$$

Eg. $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$

Properties of tensor product

$$(|a\rangle + |b\rangle) \otimes |c\rangle = |a\rangle \otimes |c\rangle + |b\rangle \otimes |c\rangle$$

$$|a\rangle \otimes (|c\rangle + |d\rangle) = |a\rangle \otimes |c\rangle + |a\rangle \otimes |d\rangle$$

$$c(|a\rangle \otimes |c\rangle) = (c|a\rangle) \otimes |c\rangle = |a\rangle \otimes (c|c\rangle)$$

$$(|a\rangle \otimes |c\rangle)^+ = |a\rangle^+ \otimes |c\rangle^+ \Rightarrow \langle a| \otimes \langle c|$$

$$(\langle a| \otimes \langle c|)(|b\rangle \otimes |d\rangle) = \langle ab\rangle \langle cd|$$

Quantum Entanglement:

$$|\psi\rangle, |\phi\rangle \rightarrow \mathbb{C}^d \mapsto |\psi\rangle \otimes |\phi\rangle \rightarrow \mathbb{C}^{d^2}$$

but the converse is not generally true

$$\exists |\eta\rangle \in \mathbb{C}^{d^2} \text{ s.t. } \nexists |\psi\rangle, |\phi\rangle \rightarrow \mathbb{C}^d \text{ s.t. } |\psi\rangle \otimes |\phi\rangle = |\eta\rangle$$

↳ entangled states.

Schmidt decomposition: $\forall |\eta\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, we can always

$$|\eta\rangle = \sum_{i=0}^{\min(d_1, d_2)} \alpha_i^* |a_i\rangle |b_i\rangle \leftarrow \begin{array}{l} \text{Schmidt vectors (orthonormal)} \\ \text{Schmidt coefficient, } \alpha_i^* \geq 0 \end{array}$$

Schmidt rank: # of non-zero Schmidt coefficients.

$|\eta\rangle$ is entangled iff Schmidt rank > 1.

E.g.: Bell Basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |11\rangle) \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |11\rangle)$$

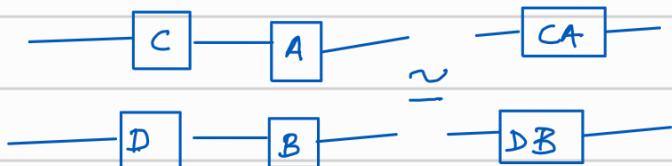
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$$

Two Qubit Gates

tensor product of 1-qubit gate
 $\times \otimes \in U(\mathbb{C}^4)$
 genuine 2-qubit gate

i) Tensor product of 1-qubit gate

$$(A \otimes B)(C \otimes D) = \overbrace{AC}^{\substack{| \\ q_1}} \otimes \overbrace{BD}^{\substack{| \\ q_2}}$$



$$\text{Tr}((C \otimes I)(X \otimes X)) = \text{Tr}(X^2 \otimes X^2) \Rightarrow \text{Tr}(I \otimes I) \neq \text{Tr}(I) \text{Tr}(I) \\ \Rightarrow 2 \times 2 = 4$$

$$\text{Tr}(A \otimes B) = \text{Tr}(A) \cdot \text{Tr}(B)$$



ii) Genuine 2-qubit gate (e.g. CNOT)

Controlled - NOT (CNOT)

$$CNOT|00\rangle = |00\rangle$$

$$CNOT|01\rangle = |01\rangle$$

$$CNOT|10\rangle = |11\rangle$$

$$CNOT|11\rangle = |10\rangle$$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$CNOT = \begin{bmatrix} [I] & 0 \\ 0 & [X] \end{bmatrix}$$

→ Measurement.

Measurement is done to extract information from the system.

→ basic type : projective / von-Neumann measurement

they are just set of projectors $B = \sum_{i=0}^m \Pi_i$

$$\sum_{i=0}^m \Pi_i = I \quad \text{(completeness relation)}$$

If each $\Pi_i = |\Psi_i\rangle\langle\Psi_i|$ i.e. $\text{rank}(\Pi_i) = 1$
say measure in the basis $\{|\Psi_i\rangle\}$

Given state $|\Psi\rangle \in \mathbb{C}^d$ and measurement $B = \sum \Pi_i \}_{i=0}^m \subseteq \mathbb{C}^d$

$$\begin{aligned} i) \quad P(\text{outcome } i | |\Psi\rangle) &= \text{Tr} (\Pi_i |\Psi\rangle \langle \Psi| \Pi_i^\dagger) \\ &= \text{Tr} (\Pi_i^\dagger |\Psi\rangle \langle \Psi| \Pi_i) \Rightarrow \text{Tr} (\Pi_i^\dagger |\Psi\rangle \langle \Psi|) \end{aligned}$$

ii) Post measurement state

$$|\Psi'\rangle = \frac{\Pi_i^\dagger |\Psi\rangle}{\sqrt{\text{Tr} (\Pi_i^\dagger |\Psi\rangle \langle \Psi| \Pi_i)}}$$

Mixed state Quantum Computation

A mixed state represented by a density operator $\rho \in \mathcal{L}(\mathbb{C}^d)$

s.t. ① $\rho_i \geq 0$ (PSD)

$$\textcircled{II} \quad \text{Tr}(\rho) = 1$$

Quantum uncertainty - entanglement
classical uncertainty - randomness

Spectral decomposition $\rho = \sum_i p_i^\dagger |\psi_i\rangle \langle \psi_i|$

$$\sum_i p_i^\dagger = 1 \text{ by } \textcircled{II}$$

Maximally Mixed state $\rho \in \mathcal{L}(\mathbb{C}^d)$ s.t. $\rho = \frac{I}{d}$ - state of no information

\nexists orthonormal basis $\{|\psi_i\rangle\}_{i=1}^d$, $\sum_{i=1}^d |\psi_i\rangle \langle \psi_i| = I$

Partial Trace Operation

$$\text{Tr}_B : \mathcal{L}(\mathbb{C}^d) \otimes \mathcal{L}(\mathbb{C}^d) \xrightarrow{\quad} \mathcal{L}(\mathbb{C}^d)$$

$$\text{Tr}_B(\rho_{AB}) = \sum_{i=1}^d \left(I_A \otimes \langle i | \right) \rho_{AB} \left(I_A \otimes |i\rangle \right) \quad \text{+ tracing out the second system.}$$

EXERCISE : Prove that $\text{Tr}_B(\rho_B \otimes \rho_A) = \rho_A \cdot \text{Tr}(\rho_B) = \rho_A$

Given ρ_A and ρ_B are density matrices

we know $\text{Tr}(\rho_A) = 1$ $\text{Tr}(\rho_B) = 1$

$$\text{Tr}_B(\rho_A \otimes \rho_B) = \sum_i (\rho_A \otimes \rho_B)_i^i$$

$$\Rightarrow \sum_i \sum_j \rho_A_{ij} \rho_B_{ij}$$

$$\Rightarrow \sum_i \rho_A_i$$

Separable States Pure states $|\Psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ are not entangled,
and are separable iff $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$

$$\begin{matrix} \in \\ \mathbb{C}^{d_1} \end{matrix} \quad \begin{matrix} \in \\ \mathbb{C}^{d_2} \end{matrix}$$

Extending this idea for mixed states.

A bipartite density matrix $\rho \in \mathcal{L}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ is separable if

$$\rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i| \otimes |\varphi_i\rangle \langle \varphi_i|$$