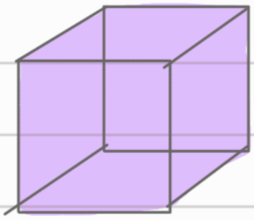
1. Basics of Quantum Information



In simplified description quantum states are represented by vectors; operators are represented by unitary matrices, but we can't model noise with it. Hence we shift to density matrices.

1.1 Classical Information

Assume a physical system X that stores information, and this can be in one of a finite number of classical states at each moment.

If X is a bit, then its classical set is $\Sigma = \{0, 1\}$

There may be uncertainty about the classical state of a system, where each classical state has some probability associated with it.

Eg: Probabilistic State

$P_r(X=0) = \frac{3}{4}$ $\qquad$ $P_r(X=1) = \frac{1}{4}$

or by a column vector/ probability vector: $\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$

all entries are $\mathbb{R}^+$

1.1.1 Dirac Notation (part 1)

Σ be any classical state set.

We denote by $|a\rangle$ the column vector having an 1 in the entry corresponding to $a \in \Sigma$ and 0 for other.

Eg:

$|0\rangle = \begin{bmatrix} 1 \end{bmatrix}$ $\qquad$ $|1\rangle = \begin{bmatrix} 0 \end{bmatrix}$

$\begin{bmatrix} \vdots \\ 0 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} \vdots \\ 1 \end{bmatrix}$

Vectors of this form are called standard basis vector.
Every vector can be expressed uniquely as linear combination of standard basis vector.

Eg: $\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{3}{4}|0\rangle + \frac{1}{4}|1\rangle$

1.1.2 Measuring Probabilistic state

What happens if we measure a system X while it is in some probabilistic state?
We see a classical state, chosen at random according to the probabilities

Suppose we see the classical state $a \in \Sigma$.

$$\Pr(X = a) = 1$$

Eg: $\Pr(X=0) = \frac{3}{4}$ $\qquad$ $\Pr(X=1) = \frac{1}{4}$

$$\begin{tikzcd}[column sep=small] & \frac{3}{4}|0\rangle + \frac{1}{4}|1\rangle \arrow[dl, "\text{prob} = \frac{3}{4}"'] \arrow[dr, "\text{prob} = \frac{1}{4}"] & \\ |0\rangle & & |1\rangle \end{tikzcd}$$

1.1.3 Deterministic operations

(no randomness or uncertainty)

Every function $f : \Sigma \rightarrow \Sigma$ describes a deterministic operation that transforms the classical state 'a' into $f(a)$ $\forall\, a \in \Sigma$.

Given any function $f : \Sigma \to \Sigma$, there is a unique matrix M satisfying

$$M|a\rangle = |f(a)\rangle \qquad \text{for every } a \in \Sigma$$

This matrix has exactly one 1 in each column and 0 for others

$$\underset{\text{row} \quad \text{column}}{M(b,a)} = \begin{cases} 1, & b = f(a) \\ 0, & b \neq f(a) \end{cases}$$

The action of this operation is described by matrix-vector multiplication

$$v \longmapsto Mv$$

Ex: For $\Sigma = \{0,1\}$, there are four function $f : \Sigma \to \Sigma$

a	$f_1(a)$
0	0
1	0

constant-0

a	$f_2(a)$
0	0
1	1

a	$f_3(a)$
0	1
1	0

a	$f_4(a)$
0	1
1	1

Constant-1

and the matrices

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\underset{\text{row} \quad \text{column}}{M(b,a)} = \begin{cases} 1, & b = f(a) \\ 0, & b \neq f(a) \end{cases}$$

1.1.4 Dirac Notation (part 2)

We denote $\langle a|$ the row vector having a 1 in the entry corresponding to $a \in \Sigma$ with 0 for all the other entries.

Eg: $\Sigma = \{0,1\}$

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Multiplying a row vector to a column vector yields a scalar.

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$\langle a|b\rangle = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$ inner prod.

Multiply a column vector to row vector yields a matrix.

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & \\ \cdot & 1 & \cdot & \cdot \\ \vdots & \vdots & & \end{pmatrix}$$

Eg:

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ and so on

$|a\rangle\langle b|$ has 1 in the (a,b) entry and 0 for other.

1.1.5 Deterministic operation (contd.)

Given a function $f : \Sigma \to \Sigma$ there is a unique matrix M satisfy-

$$M|a\rangle = |f(a)\rangle$$

The matrix may be expressed as

$$M = \sum_{b \in \Sigma} |f(b)\rangle\langle b|$$

Eg:

$$M|a\rangle = \left(\sum_{b\in\Sigma} |f(b)\rangle\langle b|\right)|a\rangle = \sum_{b\in\Sigma} |f(b)\rangle\langle b|a\rangle$$

$$\Rightarrow |f(a)\rangle$$

1.1.6 Probabilistic Operations

Probabilistic Operations are classical operations that may introduce randomness or uncertainty.

Eg: If classical state is 0, then do nothing
If classical state is 1, then flip the bit with prob 1/2

Probabilistic operations are described as stochastic matrices

- All entries $\in \mathbb{R}^+$ and $\sum_i = 1$

Eg

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1.1.7 composing operations

Suppose X is a system and $M_1, \ldots, M_n$ are stochastic matrices representing probabilistic operations on X.

Applying the first probabilistic operation to the probability vector, then applying the second probabilistic operation to the results, get

$$M_2(M_1 v) = (M_2 M_1) v$$

$M_2(M_1 v) = (M_2 M_1) v$

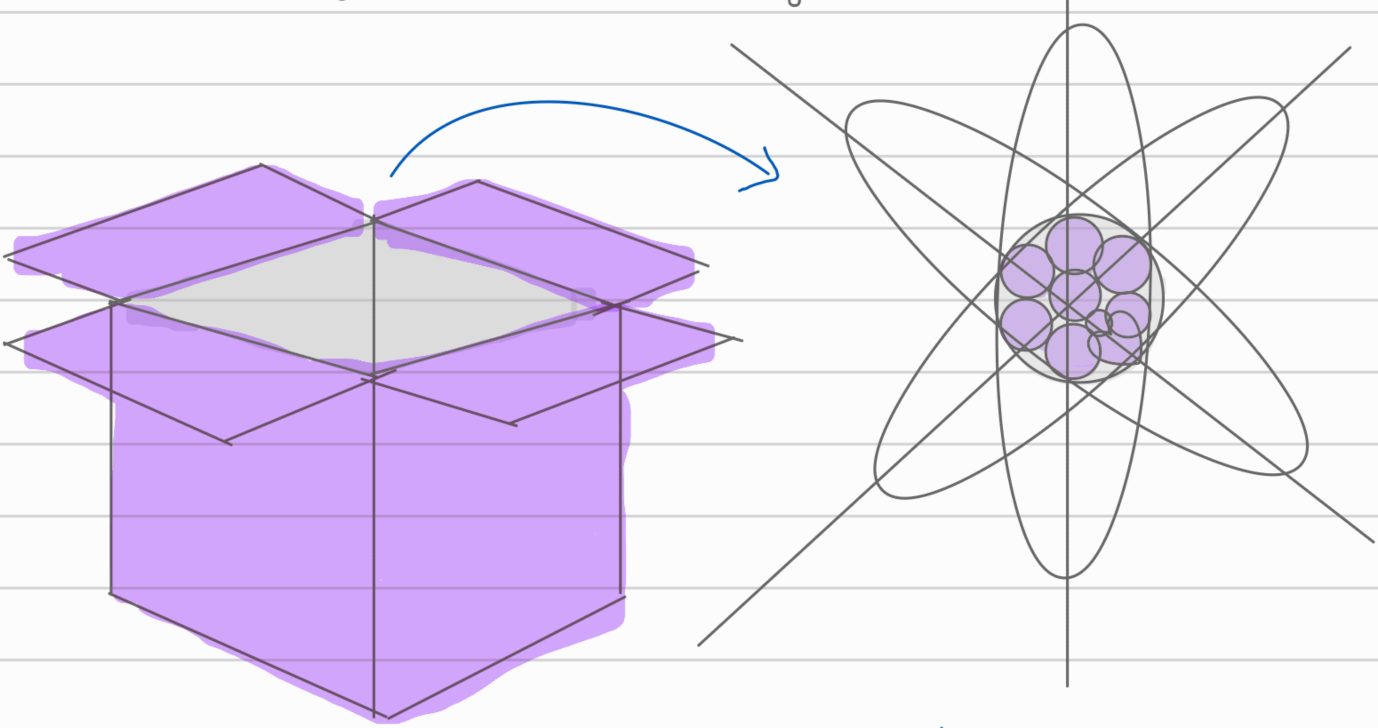
composing ... matrix product

The order is important: matrix multiplication is not commutative

Eg: $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$M_2 M_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $M_1 M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

1.2 Quantum Information



A quantum state of a system is represented by a column vector whose indices are placed in correspondence with the classical states of that system.

- The entries $\in \mathbb{C}$
- The sum of absolute value squared of the entries must $= 1$

Euclidean Norm for vectors with complex number entries is defined

$$v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \Rightarrow \|v\| = \sqrt{\sum_{k=1}^{n} |\alpha_k|^2}$$

Quantum state vectors are therefore unit vectors with respect to this norm.

Example of qubit states

- standard basis states : $|0\rangle$ and $|1\rangle$
- Plus / minus state

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

- A state without a special name

$$\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle \qquad \begin{bmatrix} \frac{1+2i}{3} \\ \frac{2}{3} \end{bmatrix}$$

1.2.1 Dirac Notation (Part 3)

The dirac notation can be used for arbitrary vectors: any name can be used in place of classical state. Kets are column vectors, bras are row vectors.

Eg $|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle \qquad \langle\psi| = \frac{1-2i}{3}\langle 0| - \frac{2}{3}\langle 1|$

For any column vector $|\psi\rangle$, the row vector $\langle\psi|$ is conjugate transpose of $|\psi\rangle$

$$\langle\psi| = |\psi\rangle^{\dagger}$$

$$|\psi\rangle = \begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix} \qquad \langle\psi| = \begin{pmatrix} \frac{1-2i}{3} & -\frac{2}{3} \end{pmatrix}$$

1.2.2 Measuring Quantum States

Standard Basis Measurement

The possible outcomes are the classical states. The probability for each classical state to be the outcome is the absolute value squared of the corresponding quantum state vector entry.

Eg a) $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

yields an outcome as follows.

$$\Pr(\text{outcome is } 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \qquad \Pr(\text{outcome is } 1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

b) $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ gives the same result.

Measuring a system changes its quantum state: if we obtain the classical state a, the new quantum state becomes $|a\rangle$

Eg:
$$\begin{tikzcd} & \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle \arrow[dl, "\frac{5}{9}"'] \arrow[dr, "4/9"] & \\ |0\rangle & & |1\rangle \end{tikzcd}$$

collapse of quantum state

1.2.3 Unitary Operations

The set of allowable operations that can be performed on a quantum state is different that it is for classical information.

Operations on quantum states are represented by unitary matrices

A square matrix $\mathcal{U}$ having complex number entries is unitary if it satisfies

$$\boxed{\mathcal{U}^{-1} = \mathcal{U}^{\dagger}}$$

$$\mathcal{U}^{\dagger}\mathcal{U} = \mathbb{1} = \mathcal{U}\mathcal{U}^{\dagger}$$

it should never change the Euclidean norm of the vector.

$$\|\mathcal{U}v\| = \|v\|$$

1.2.4 Qubit Unitary Operations

1. Pauli operation

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

bit flip (for σ_x) — Phase flip (for σ_z)

$$\sigma_x|0\rangle = |1\rangle \qquad \sigma_z|0\rangle = |0\rangle$$

$$\sigma_x|1\rangle = |0\rangle \qquad \sigma_z|1\rangle = -|1\rangle$$

2. Hadamard operation

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\boxed{\begin{aligned} H|+\rangle &= |0\rangle \\ H|-\rangle &= |1\rangle \end{aligned}}$$

$$\boxed{H|0\rangle = |+\rangle} \quad \boxed{H|1\rangle = |-\rangle}$$

3. Phase Operation

$$P_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \qquad \theta \in \mathbb{R}$$

$$S = P_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad T = P_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

$$T|0\rangle = |0\rangle$$
$$T|1\rangle = \frac{1+i}{\sqrt{2}}|1\rangle$$

$$T|+\rangle = T\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$
$$\Rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$$

1.2.5 Composing unitary operation

composing unitary operations are represented by matrix multiplication

Eg: square of NOT

Unitary matrices are closed under multiplication

$$HSH = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$$

$$(HSH)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$