Maximum Cut Problem using Quantum Machine Learning

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(Dated: April 20, 2021)

The Maximum cut problem or the Max-Cut problem has its uses in Network design[1], Machine Learning [2] and Statistical physics. Max-Cut problem partition the vertices of a graph ($G = (V, E)$) into two sets (S, T), so the weights joined by the vertices should be maximized. Finding the Max-Cut is an NP-complete problem [3] and was among the first problems found to be NP-complete. This term paper will look at a computationally tractable case of the Max-Cut problem and solve it in Classical and Quantum Computer.

I. INTRODUCTION

A. Cut

A **Cut** is a partition of a given graph's vertices, separated in two disjoint subsets.

Formally, A Cut $C = (S, T)$ is a partition of a graph $G = (V, E)$ in two subsets S and T, such that $u \in S$, $v \in T$ and $(u, v) \in E$.

1. *Minimum Cut*

A minimum cut is a partition of a graph in two disjoint subsets such that the cut is minimal in some metric(Eg Weight).

In optimization theory, the **max-flow min-cut theorem**[4] asserts that in any flow network, the maximum amount of flow travelling from the source to the sink is equal to the total weight of the edges in a minimum cut.

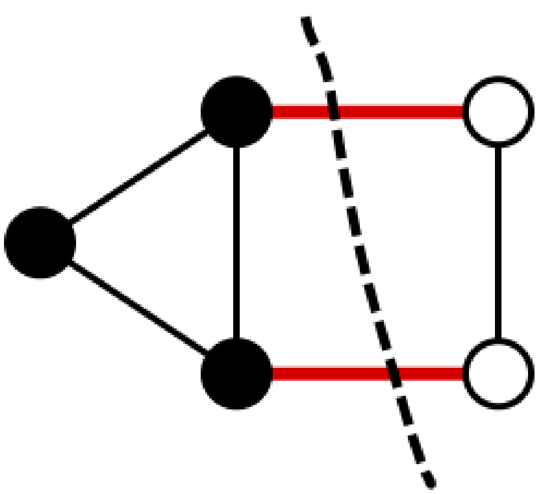

FIG. 1. A Minimum Cut

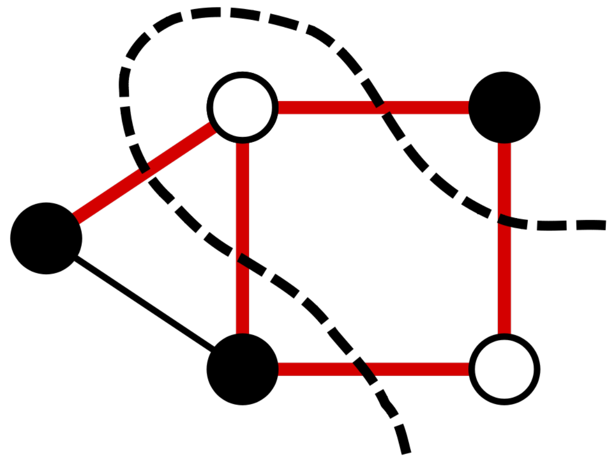

FIG. 2. A Maximum Cut

2. *Maximum Cut*

A maximum cut is at least the size of any other cut.

It is a separation of the vertices of a graph into two disjoint sets, S and T, so the number of edges between the set S and the set T is as large as possible.

The problem for finding such a maximum cut in a given graph is known as the **Max-Cut Problem**.

In this term paper, we will focus mainly on the Maximum cut and the Max-Cut problem.

II. MAXIMUM CUT PROBLEM

As the name suggests, we want to get the cut that maximizes the number of crossing edges in an Undirected Graph [5].

A. Complexity Class

In Computational Complexity theory, the problem to find the Maximum Cut belongs to the NP-Complete problem. A complexity class is just the set of problems based on their resource usage complexity. Resources being **Time** and **Space** or **Memory**.

To simply put the problem, If we are given a graph, and we wanted to know whether or not there exists a cut that cuts a certain number of edges, it is an NP-complete problem. Hence no polynomial-time algorithm is known. But like other NP-complete problems, there are some computationally feasible exceptional cases, like the bipartite graph example above.

B. Definition

Consider an n-node undirected graph $G = (V, E)$ where $|V| = n$ with edge weights $w_{ij} > 0$, $w_{ij} = w_{ji}$, for $(i, j) \in E$. A cut is defined as a partition of the original set V into two subsets. The cost function to be optimized is in this case the sum of weights of edges connecting points in the two different subsets, crossing the cut. By assigning $x_i = 0$ or $x_i = 1$ to each node i, one tries to maximize the global profit function (here and in the following summations run over indices $0, 1, \ldots n-1$)

$$\tilde{C}(x) = \sum_{i,j} w_{ij} x_i (1 - x_j)$$

An extension to this model has the nodes themselves carry weights. With this additional information in our model, the objective function to maximize becomes:

$$C(x) = \sum_{i,j} w_{i,j} x_i (1 - x_j) + \sum_i w_i x_i$$

C. Mapping it to a Quantum Computer

To find solution for the problem on a quantum computer, we need to map it to the **Ising Hamiltonian** We assign $x_i \to (1 - Z_i)/2$, where Z_i is the Pauli Z operator with the eigen values ± 1.

$$C(Z) = \sum_{i,j} \frac{w_{i,j}}{4} (1 - Z_i)(1 + Z_i) + \sum_i \frac{w_i}{2} (1 - Z_i)$$

$$= -\frac{1}{2} (\sum_{i<j} w_{ij} Z_i Z_j + \sum_i w_i Z_i) + const$$

Here the const $= \sum_{i<j} w_{ij}/2 + \sum_i w_i/2$. So, now all we have to do is minimize the Ising Hamiltonian:

$$H = \sum_i w_i Z_i + \sum_{i<j} w_{ij} Z_i Z_j$$

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D. How Does the Algorithm Works?

The algorithm works as follows:

- First, chose the w_i and the w_{ij} in the given target Ising problem.
- Then, chose the depth of the quantum circuit m, and can be modified according to the problem.
- Chose the set of gates and controls θ and a trial wave function $\psi(\theta)$, using the quantum circuit made.
- Evaluate:

$$C(\theta) = <\psi(\theta)|H|\psi(\theta)>$$

$$= \sum_i w_i <\psi(\theta)|Z_i|\psi(\theta) + \sum_{i<j} w_{ij} <\psi(\theta)|Z_i Z_j|\psi(\theta)>$$

  by sampling the outcome of the circuit in the Z-basis and then adding the expectation values of the Individual terms together.
- Then, use a classical optimizer to set a new set for controls.
- Repeat this step, until $C(\theta)$ reaches minimun, hence the cost function minimizes.
- Use the last θ parameter, and then generate a final set of samples from $|<z_i|\psi(\theta)>|^2$ to get the answer.

Getting a good heuristic algorithms, depends on the choice of the trial wavefunction. Suppose, one uses a wavefunction whose entanglement aligns with our problems target. Or we can also make the entanglement a variable in our function. So, we make the trial wavefunction as:

$$|\psi(\theta)> = [U_{single}(\theta) U_{entangler}]^m | + >$$

Here, $U_{entagler}$ is collection of the fully entangling gates, and $U_{single}(\theta) = \prod_{i=1}^{n} Y(\theta_i)$, where n is the number of qubits and m is the depth of the quantum circuit.

III. CODE AND JUPYTER NOTEBOOKS

Max-Cut Problem.
Max-Cut Problem QAOA
Max-Cut and Traveling Salesman Problem
Solving Max-Cut problem using QAOA

[1] R. Ahuja, T. Magnanti and J. Orlin, ”Network Flows: theory, algorithms and applications,” Prentice-Hall, Inc. Englewood Cliffs, New Jersey. 1993
[2] A Deep Learning Algorithm for the Max-Cut Problem Based on Pointer Network Structure with Supervised Learning and Reinforcement Learning Strategies by Shenshen Gu and Yue Yang
[3] Reducibility Among Combinatorial Problems by Richard Karp used Stephen Cook
[4] Dantzig, G.B.; Fulkerson, D.R. (9 September 1964). ”On the max-flow min-cut theorem of networks
[5] Undirected Graph