High Order

1. $y'' \cdot y^3 = 1$, $y\left(\frac{1}{2}\right) = 1$, $y'\left(\frac{1}{2}\right) = 1$

$\Rightarrow y'' = \frac{1}{y^3}$ — x - is missing

$y' = p /' \Rightarrow y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \underbrace{\frac{dy}{dx}}_{p} = \frac{dp}{dy} \cdot p$

$p \cdot \frac{dp}{dy} = \frac{1}{y^3} \Rightarrow p \cdot dp = \frac{dy}{y^3}$ — Separable eq $/\int \Rightarrow$

$\Rightarrow \frac{p^2}{2} = \frac{1}{-2y^2} + C \Rightarrow \frac{(y')^2}{2} = \frac{1}{-2y^2} + C$

$\frac{\left(y'\left(\frac{1}{2}\right)\right)^2}{2} = -\frac{1}{2y^2\left(\frac{1}{2}\right)} + C \Rightarrow \frac{1}{2} = -\frac{1}{2} + C \Rightarrow C = 1$

$\frac{(y')^2}{2} = 1 - \frac{1}{2y^2} \Rightarrow y' = \pm\sqrt{2 - \frac{1}{y^2}} = \pm\sqrt{\frac{2y^2-1}{y^2}} = \pm\frac{\sqrt{2y^2-1}}{y}$

$\Rightarrow \pm\frac{y}{\sqrt{2y^2-1}} \cdot \frac{dy}{dx} = 1 /\cdot dx \Rightarrow \frac{1}{2} \cdot \frac{2y}{\sqrt{2y^2-1}} \cdot dy = \pm 1 \cdot dx \;/\int \Rightarrow$

$t = \sqrt{2y^2-1} /' \Rightarrow dt = \frac{1}{\cancel{2}\sqrt{2y^2-1}} \cdot \cancel{4}^{2} y \, dy$

$\Rightarrow \frac{1}{2}\int 1 \, dt = \pm x + C \Rightarrow \frac{\sqrt{2y^2-1}}{2} = \pm x + C$

② $y'' + y = e^{-x} \neq 0 \Rightarrow$ non-homogeneous equation
$(= 0 \Rightarrow$ homogeneous equation$)$

General solution: $y = y_h + y_o$

Step 1: $y_h = ?$

Characteristic polynomial: $\lambda^2 + 1 = 0 \Rightarrow \left\{ \begin{array}{l} \lambda_1 = i \\ \underbrace{\lambda_2 = -i}_{2 \text{ solutions}} \end{array} \right. \in \mathbb{C}$

$\Rightarrow$ The fundamental system of solutions is:
$S = \{ e^{0 \cdot x} \cdot \cos(1 \cdot x),\ e^{0 \cdot x} \cdot \sin(1 \cdot x) \} \Rightarrow S = \underbrace{\{ \cos x, \sin x \}}_{2 \text{ solutions}}$

real part

imaginary part

$$\boxed{\Rightarrow y_h = c_1 \cdot \cos x + c_2 \cdot \sin x}$$

Step 2: Undetermined Coefficients Method:

$$e^{-x} = e^{\alpha x} [P_n(x) \cdot \cos(\beta x) + Q_m(x) \cdot \sin(\beta x)]$$

$$\Rightarrow \left\{ \begin{array}{l} \alpha = -1 \\ \beta = 0 \end{array} \right. \quad \begin{array}{l} P_n(x) = 1 \Rightarrow n = 0 \\ Q_m(x) = 0 \Rightarrow m = 0 \end{array} \Bigg| \Rightarrow \max\{m, n\} = 0$$

$\Downarrow$

$\cos(x) = 1$

$\sin(x) = 0 \Rightarrow Q_m(x)$ can have any value

Verify: $\alpha \pm i\beta = -1 \pm 0 = -1 \Rightarrow k = 0$ (because $-1 \neq \lambda_1$, $-1 \neq \lambda_2$)

$$\Rightarrow y_o = x^k \cdot e^{\alpha x} [T_p(x) \cdot \cos(\beta x) + R_p(x) \cdot \sin(\beta x)]$$

$p = \max\{m, n\} = 0$

B

$$\Rightarrow y_0 = x^0\cdot e^{-x}\left[a\cdot \underbrace{\cos(0\cdot x)}_{=1} + b\cdot \underbrace{\sin(0\cdot x)}_{=0}\right] = e^{-x}\cdot a$$

$$\Rightarrow y_0' = -ae^{-x} \Rightarrow y_0'' = ae^{-x}$$

$$\Rightarrow a\cdot e^{-x} + a\cdot e^{-x} = e^{-x} \;/\cdot\frac{1}{e^{-x}} \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2} \Rightarrow$$

$$\boxed{\Rightarrow y_0 = \frac{e^{-x}}{2}} \Rightarrow \boxed{y = C_1\cdot\cos x + C_2\cdot\sin x + \frac{e^{-x}}{2}}$$

When we have $e^{-x}\cdot\ln x$ or $\frac{e^{-x}}{x^2}$, Undetermined Coefficients Method won't work!

Step 2: Variation of constants method :

$$y_0 = C_1(x)\cdot\cos(x) + C_2(x)\cdot\sin x \;/\,' \Rightarrow$$

$$\Rightarrow y_0' = C_1'(x)\cdot\cos(x) - C_1(x)\cdot\sin x + C_2'(x)\cdot\sin x + C_2(x)\cdot\cos x$$

$$y_0' = \underbrace{C_1'(x)\cdot\cos(x) + C_2'(x)\cdot\sin x}_{=0} - C_1(x)\cdot\sin x + C_2(x)\cdot\cos x \;/\,'$$

$$\Rightarrow y_0'' = -C_1'(x)\cdot\sin(x) - C_1(x)\cdot\cos(x) + C_2'(x)\cdot\cos x - C_2(x)\cdot\sin(x)$$

$$y_0'' = \underbrace{-C_1'(x)\cdot\sin(x) + C_2'(x)\cdot\cos(x)}_{=f(x)=e^{-x}} - C_1(x)\cdot\cos(x) - C_2(x)\cdot\sin(x)$$

$$\begin{cases} C_1'(x)\cdot\cos(x) + C_2'(x)\cdot\sin x = 0 \\ -C_1'(x)\cdot\sin(x) + C_2'(x)\cdot\cos(x) = e^{-x} \end{cases}$$

$$\Delta = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\Delta c_1' = \begin{vmatrix} 0 & \sin x \\ e^{-x} & \cos x \end{vmatrix} = -e^{-x} \cdot \sin x$$

$$\Delta c_2' = \begin{vmatrix} \cos x & 0 \\ -\sin x & e^{-x} \end{vmatrix} = e^{-x} \cdot \cos x$$

$$c_1'(x) = \frac{\Delta c_1}{\Delta} = -e^{-x} \cdot \sin x \Rightarrow c_1(x) = \int -e^{-x} \cdot \sin x = \frac{e^{-x}(\sin x + \cos x)}{2}$$

$$c_2'(x) = \frac{\Delta c_2}{\Delta} = e^{x} \cdot \cos x \Rightarrow c_2(x) = \int e^{x} \cdot \cos x = \frac{e^{-x}(\cos x - \sin x)}{2}$$

$$i = \int e^{-x} \cdot \cos x = \ldots$$

$$f(x) = \cos x \qquad f'(x) = -\sin x$$

$$g'(x) = e^{-x} \qquad g(x) = -e^{-x}$$

$$\Rightarrow y_0 = \frac{e^{-x}(\sin x + \cos x)}{2} \cdot \cos x + \frac{e^{-x}(\cos x - \sin x)}{2} \cdot \sin x$$

$$y_0 = \frac{1}{2} \cdot e^{-x}$$

Pairs of Continuous r.v.

X, Y - continuous r.v.

Joint CDF $\Rightarrow F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

Properties:

- $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} F_{X,Y}(x,y) = 1$
- $\lim\limits_{x \to \infty} F_{X,Y}(x,y) = \lim\limits_{y \to \infty} F_{X,Y}(x,y) = 0$
- $F \nearrow$
- $F \in [0,1]$
- $F_X(x) = \lim\limits_{y \to \infty} F_{X,Y}(x,y)$, $\quad F_Y(y) = \lim\limits_{x \to \infty} F_{X,Y}(x,y)$
- $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, du\, dv$

Joint PDF $\Rightarrow f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$

1. $f_{X,Y}(x,y) = \begin{cases} \dfrac{4x+2y}{3}, & x,y \in [0,1] \\ \text{zero}, & \text{otherwise} \end{cases}$

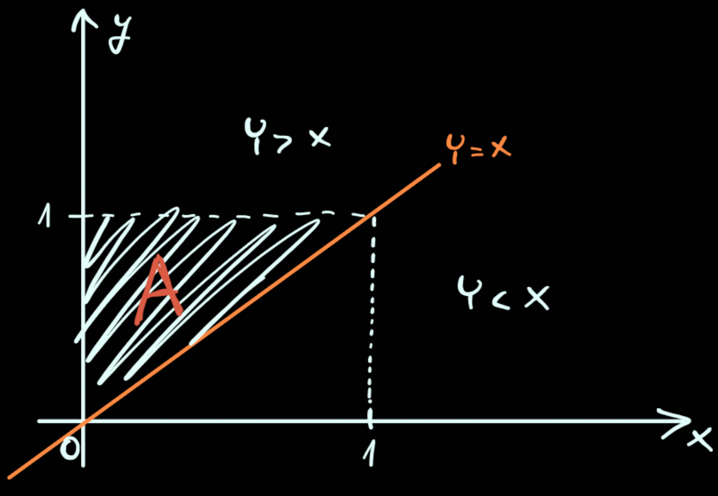
a. $P(Y > X)$

b. $P(Y > 0{,}5)$

c. CDF of $W = \dfrac{X}{Y}$

d. $E(W)$

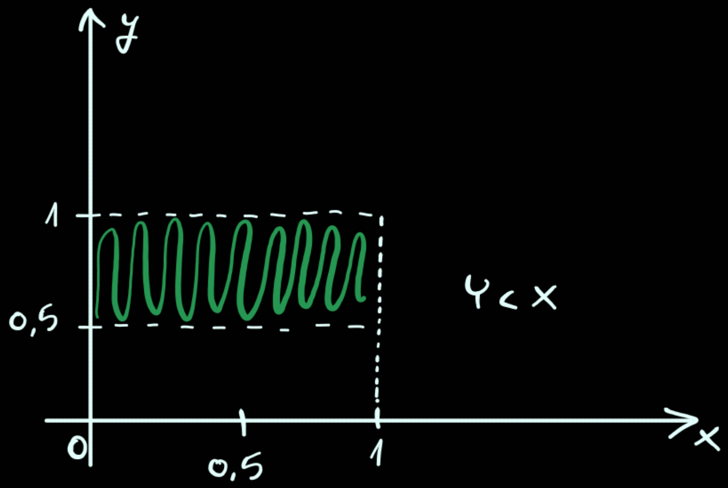
a. $P(Y>X) = \iint\limits_{A} f_{X,Y}(x,y)\,dx\,dy = \int_0^1 \int_x^1 \left(\frac{4x+2y}{3}\,dy \right) dx$



$\mathrm{Proj}_{Ox} A = [0,1]$

$y \in [x,1]$

b. $P(Y>0,5) = \int_0^1 \int_{0,5}^1 \frac{4x+2y}{3}\,dy\,dx$



c. $W = \frac{X}{Y}$

$F_W(w) = P(W \le w) = P\left(\frac{X}{Y} \le w\right)$

Linear Systems

1. $\dot{X} = A \cdot X$

$$\begin{cases} \dot{x}_1 = x_2 + 2x_3 \quad /' \\ \dot{x}_2 = x_1 + 2x_3 \\ \dot{x}_3 = \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{cases} \Rightarrow \quad x_2 = \dot{x}_1 - 2x_3$$

$$\Rightarrow \ddot{x}_1 = \dot{x}_2 + 2\dot{x}_3 = x_1 + 2x_3 + 2\cdot\frac{1}{2}x_1 + 2\cdot\frac{1}{2}x_2$$

$$\ddot{x}_1 = 2x_1 + \cancel{2x_3} + \dot{x}_1 - \cancel{2x_3}$$

$$\ddot{x}_1 = \dot{x}_1 + 2x_1 \Rightarrow \ddot{x}_1 - \dot{x}_1 - 2x_1 = 0$$ – homogeneous Higher Order DE

Characteristic polynomial: $\lambda^2 - \lambda - 2 = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 2 \end{cases}$

The fundamental system of solutions is:

$$S = \{e^{-x}, e^{2x}\}$$

$$\Rightarrow x_1 = c_1 \cdot e^{-x} + c_2 \cdot e^{2x}$$

$$\begin{cases} \dot{x}_2 = 2x_3 + c_1 \cdot e^{-t} + c_2 \cdot e^{2t} \Rightarrow x_3 = \frac{1}{2}\dot{x}_2 - c_1 \cdot e^{-t} - c_2 \cdot e^{2t} \\ \dot{x}_3 = \frac{1}{2}x_2 + \frac{c_1 \cdot e^{-t}}{2} + \frac{c_2 \cdot e^{2t}}{2} \end{cases}$$

$$\Rightarrow \frac{1}{2}\ddot{x}_2 + c_1 \cdot e^{-t} - 2c_2 \cdot e^{2t} = \frac{1}{2}x_2 + \frac{c_1 \cdot e^{-t}}{2} + \frac{c_2 \cdot e^{2t}}{2}$$

$$\ddot{x}_2 + 2c_1 \cdot e^{-t} - 4c_2 \cdot e^{2t} = x_2 + c_1 \cdot e^{-t} + c_2 \cdot e^{2t}$$

$$\ddot{x}_2 - x_2 = -c_1 \cdot e^{-t} + 5c_2 \cdot e^{2t}$$