I. Modele matematice intrare-ieșire (MM-ii)

1. MM-ii în timp continuu

$$a_m y^{(m)}(t) + a_{m-1} y^{(m-1)}(t) + \dots + a_1 y^{(1)}(t) + a_0 y(t) = b_m u^{(m)}(t) + \dots + b_1 u^{(1)}(t) + b_0 u(t) \quad / \mathcal{L}$$

$$\Rightarrow a_m s^m y(s) + a_{m-1} s^{m-1} y(s) + \dots + a_1 s\, y(s) + a_0 y(s) = b_m s^m u(s) + \dots + b_1 s\, u(s) + b_0 u(s)$$

$$\Rightarrow y(s)(a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0) = u(s)(b_m s^m + \dots + b_1 s + b_0) \Rightarrow$$

$$\boxed{H(s) = \frac{y(s)}{u(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}}$$

2. MM-ii în timp discret

$$a_m y(k+m) + a_{m-1} y(k+m-1) + \dots + a_1 y(k+1) + a_0 y(k) = b_m u(k+m) + \dots + b_1 u(k+1) + b_0 u(k) \quad / \mathcal{Z}$$

$$a_m z^m y(z) + a_{m-1} z^{m-1} y(z) + \dots + a_1 z\, y(z) + a_0 y(z) = b_m z^m u(z) + \dots + b_1 z\, u(z) + b_0 u(z)$$

$$y(z)(a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0) = u(z)(b_m z^m + \dots + b_1 z + b_0) \Rightarrow$$

$$\boxed{H(z) = \frac{y(z)}{u(z)} = \frac{b_m z^m + \dots + b_1 z + b_0}{a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0}}$$

II. Modele matematice intrare-stare-ieșire (MM-isi)

1. MM-isi în timp continuu

$$\begin{cases} \dot{x}(t) = A x(t) + B u(t) \\ y(t) = C x(t) + D u(t) \end{cases} \quad / \mathcal{L}$$

, pt. sisteme fizic realizabile $D = 0$

$$\Rightarrow \begin{cases} s X(s) = A X(s) + B u(s) \Rightarrow s X(s) - A X(s) = B u(s) \Rightarrow X(s)(s I - A) = B u(s) \\ y(s) = C X(s) \end{cases}$$

$$\Rightarrow X(s) = (s I - A)^{-1} B u(s)$$

$$y(s) = C (s I - A)^{-1} B u(s) \Rightarrow \boxed{H(s) = \frac{y(s)}{u(s)} = C (s I - A)^{-1} B}$$

2. MM-ISI în timp discret

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \Big/ \mathcal{Z} \Rightarrow \begin{aligned} zX(z) &= AX(z) + BU(z) \\ Y(z) &= CX(z) \end{aligned} \Rightarrow$$

$$X(z)(zI-A)^{-1} = BU(z)$$

$$Y(z) = C(zI-A)^{-1}BU(z) \Rightarrow \boxed{H(z) = \frac{Y(z)}{U(z)} = C(zI-A)^{-1}B}$$

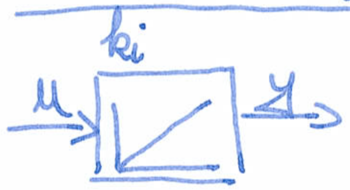
III Elemente de transfer (ET)

1. Elementul de transfer de tip proporțional (ET-P)

MM-II: $y(t) = k\,u(t) \;/\; \mathcal{L}$

$$Y(s) = k\,U(s) \Rightarrow \boxed{H(s) = \frac{Y(s)}{U(s)} = k}$$

2. Elementul de transfer de tip integrator (ET-i)

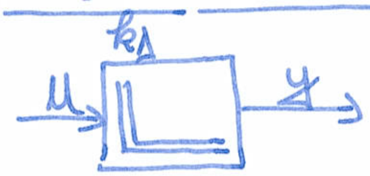


MM-II: $y(t) = k_i \int_0^t u(\tau)d\tau$ / derivăm

$$y'(t) = k_i u(t) \;/\; \mathcal{L} \Rightarrow sY(s) = k_i U(s) \Rightarrow$$

$$Y(s) = \frac{k_i}{s} U(s) \Rightarrow \boxed{H(s) = \frac{Y(s)}{U(s)} = \frac{k_i}{s}}$$

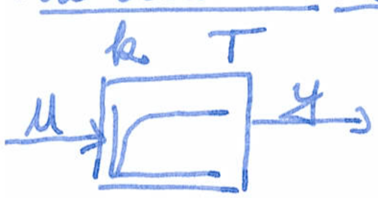
3. Elementul de transfer de tip derivator (ET-D)



MM-II: $y(t) = k_D u'(t) \;/\; \mathcal{L} \Rightarrow$

$$Y(s) = k_D s\,U(s) \Rightarrow \boxed{H(s) = \frac{Y(s)}{U(s)} = k_D s}$$

4. Elementul de transfer de tip proporțional cu temporizare de ordinul 1 (ET-PT1) – filtru trece-jos

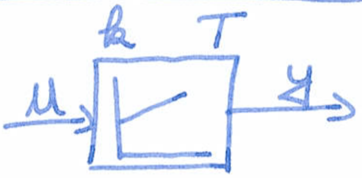


MM-II: $Ty'(t) + y(t) = k\,u(t) \;/\; \mathcal{L}$

$$TsY(s) + Y(s) = kU(s) \Rightarrow Y(s)(Ts+1) = kU(s)$$

$$\Rightarrow \boxed{H(s) = \frac{Y(s)}{U(s)} = \frac{k}{1+sT}}$$

5. Elementul de transfer de tip proporțional-integrator (ET-PI)



MM-ii: $y(t) = k\left[u(t) + \frac{1}{T}\int_0^t u(\tau)d\tau\right] \Rightarrow$

$y(t) = k u(t) + \frac{k}{T}\int_0^t u(\tau)d\tau$ / derivăm

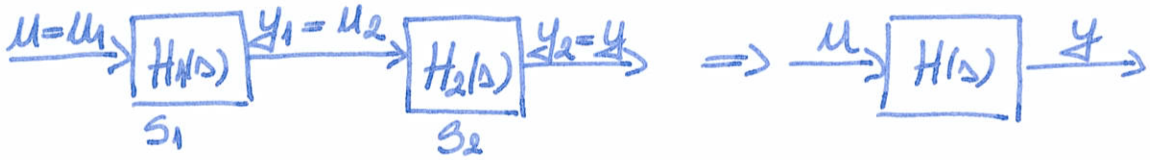
$y'(t) = k u'(t) + \frac{k}{T} u(t) \,/\, \mathcal{L} \Rightarrow$

$s y(s) = k s u(s) + \frac{k}{T} u(s) \Rightarrow s y(s) = u(s)\left(k s + \frac{k}{T}\right) = u(s)\frac{k(sT+1)}{T} \Rightarrow$

$$\boxed{H(s) = \frac{y(s)}{u(s)} = \frac{k(sT+1)}{sT}}$$

IV. Principalele conexiuni de sisteme

1. Conexiunea serie



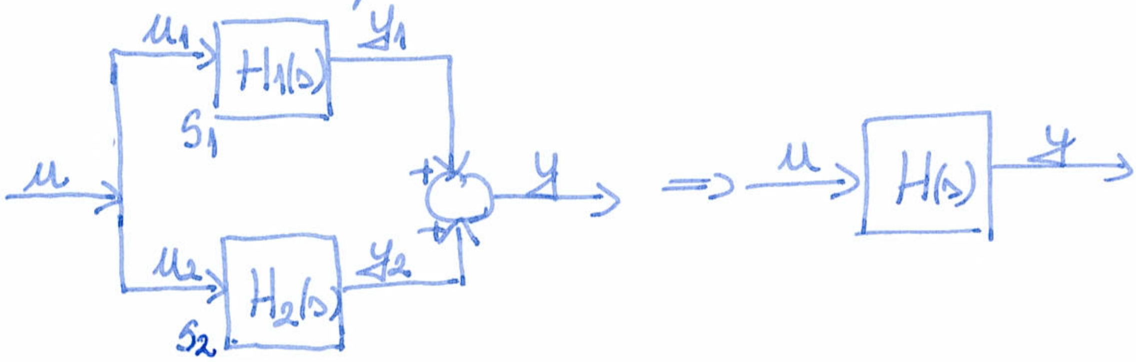
S_1: $\left.\begin{array}{l} y_1(s) = H_1(s)u_1(s) \\ y_1(s) = u_2(s) \\ u_1(s) = u(s) \end{array}\right\} \Rightarrow u_2(s) = H_1(s)u(s)$

S_2: $\left.\begin{array}{l} y_2(s) = H_2(s)u_2(s) \\ y_2(s) = y(s) \end{array}\right\} \Rightarrow y(s) = H_2(s)u_2(s)$

$\Rightarrow y(s) = H_1(s)H_2(s)\cdot u(s)$

$\Rightarrow \boxed{H(s) = \frac{y(s)}{u(s)} = H_1(s)H_2(s)}$

2. Conexiunea paralel



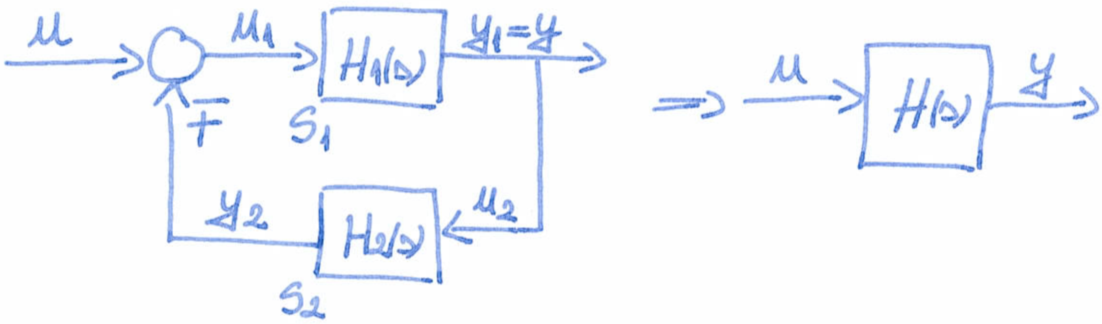
S_1: $\left.\begin{array}{l} y_1(s) = H_1(s)u_1(s) \\ u_1(s) = u(s) \end{array}\right\} \Rightarrow y_1(s) = H_1(s)u(s)$

S_2: $\left.\begin{array}{l} y_2(s) = H_2(s)u_2(s) \\ u_2(s) = u(s) \end{array}\right\} \Rightarrow y_2(s) = H_2(s)u(s)$

$\Rightarrow y(s) = y_1(s) + y_2(s) = [H_1(s) + H_2(s)]\cdot u(s)$

$\Rightarrow \boxed{H(s) = \frac{y(s)}{u(s)} = H_1(s) + H_2(s)}$

3. Conexiunea cu reacție



$$S_1: \left.\begin{aligned} y_1(s) &= H_1(s)u_1(s) \\ y_1(s) &= y(s) \\ u_1(s) &= u(s) \mp y_2(s) \end{aligned}\right\} \Rightarrow y(s) = H_1(s)[u(s) \mp y_2(s)]$$

$$S_2: \left.\begin{aligned} y_2(s) &= H_2(s)u_2(s) \\ u_2(s) &= y(s) \end{aligned}\right\} \Rightarrow y_2(s) = H_2(s)y(s)$$

$\Rightarrow$

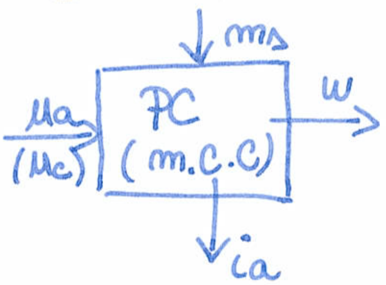
$$y(s) = H_1(s)[u(s) \mp H_2(s)y(s)] = H_1(s)u(s) \mp H_1(s)H_2(s)y(s) \Rightarrow$$

$$y(s) \pm H_1(s)H_2(s)y(s) = H_1(s)u(s) \Rightarrow y(s)[1 \pm H_1(s)H_2(s)] = H_1(s)u(s)$$

$$\Rightarrow \boxed{H(s) = \frac{y(s)}{u(s)} = \frac{H_1(s)}{1 \pm H_1(s)H_2(s)}}$$

Aplicație: Modelarea unui motor de curent continuu (m.c.c)

1. Să se găsească MM-isi-ul aferent PC
2. Să se calculeze matricea de transfer folosind MM-isi-ul
3. Să se calculeze funcțiile de transfer utilizând SBI din fig.1.5



Ecuațiile primare aferente PC:

$$\begin{cases} \frac{L_a}{R_a}\frac{di_a(t)}{dt} + i_a(t) = \frac{1}{R_a}\left[u_a(t) - e_w(t)\right] \\ J\frac{dw(t)}{dt} = m_a(t) - m_f(t) - m_s(t) \end{cases}$$

știind că $m_a(t) = k_m i_a(t)$, $m_f(t) = k_f w(t)$, $e_w(t) = k_e w(t)$, $T_a = \frac{L_a}{R_a}$

1. Notăm $\begin{aligned} x_1(t) &= i_a(t) \\ x_2(t) &= w(t) \end{aligned} \Rightarrow \begin{cases} \frac{L_a}{R_a}\dot{x}_1(t) + x_1(t) = \frac{1}{R_a}[u_a(t) - e_w(t)] \\ J\dot{x}_2(t) = m_a(t) - m_f(t) - m_s(t) \end{cases}$

$$\boxed{k_1 \approx 0} \Rightarrow \begin{cases} L_a \dot{x}_1(t) + R_a x_1(t) = u_a(t) - k_e x_2(t) \\ J \dot{x}_2(t) = k_m x_1(t) - m_s(t) \end{cases} \Rightarrow$$

$$\begin{cases} L_a \dot{x}_1(t) = -R_a x_1(t) - k_e x_2(t) + u_a(t) \\ J \dot{x}_2(t) = k_m x_1(t) - m_s(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = -\overset{1/T_a}{\overset{\shortparallel}{\dfrac{R_a}{L_a}}} x_1(t) - \dfrac{k_e}{L_a} x_2(t) + \dfrac{1}{L_a} u_a \\ \dot{x}_2(t) = \dfrac{k_m}{J} x_1(t) - \dfrac{1}{J} m_s(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_a} & -\frac{k_e}{L_a} \\ \frac{k_m}{J} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{J} \end{bmatrix} \begin{bmatrix} u_a(t) \\ m_s(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

2) $A = \begin{bmatrix} -\frac{1}{T_a} & -\frac{k_e}{L_a} \\ \frac{k_m}{J} & 0 \end{bmatrix}$; $B = \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{J} \end{bmatrix}$; $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$H(s) = C \underbrace{(sI - A)}_{M}{}^{-1} B = \begin{bmatrix} H_{i_a u_a}(s) & H_{i_a m_s}(s) \\ H_{\omega u_a}(s) & H_{\omega m_s}(s) \end{bmatrix}$$

$$M = sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{1}{T_a} & -\frac{k_e}{L_a} \\ \frac{k_m}{J} & 0 \end{bmatrix} = \begin{bmatrix} s + \frac{1}{T_a} & \frac{k_e}{L_a} \\ -\frac{k_m}{J} & s \end{bmatrix}$$

$$M^t = \begin{bmatrix} s + \frac{1}{T_a} & -\frac{k_m}{J} \\ \frac{k_e}{L_a} & s \end{bmatrix}, \quad \boxed{M^{-1} = \frac{1}{\det M} \cdot M^*}$$

$$M^* = \begin{bmatrix} s & -\frac{k_e}{L_a} \\ \frac{k_m}{J} & s + \frac{1}{T_a} \end{bmatrix}; \quad \det M = s\left(s + \frac{1}{T_a}\right) + \frac{k_m k_e}{J L_a} = s^2 + \frac{s}{T_a} + \frac{k_m k_e}{J L_a}$$

$$J = \frac{T_m k_m k_e}{R_a}$$

$$\Rightarrow \det M = s^2 + \frac{s}{T_a} + \frac{\cancel{k_m k_e}}{\frac{T_m \cancel{k_m k_e}}{R_a} L_a} = s^2 + \frac{s}{T_a} + \frac{1}{T_m T_a} = \frac{T_a T_m s^2 + s T_m + 1}{T_a T_m}$$

$$M^{-1} = \frac{T_aT_m}{T_aT_m s^2 + T_m s + 1} \cdot \begin{bmatrix} s & -\frac{k_e}{L_a} \\ \frac{k_m}{J} & s+\frac{1}{T_a} \end{bmatrix} \Rightarrow$$

$$H(s) = C M^{-1} B = \frac{T_aT_m}{T_aT_m s^2 + T_m s + 1} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s & -\frac{k_e}{L_a} \\ \frac{k_m}{J} & s+\frac{1}{T_a} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{J} \end{bmatrix} \Rightarrow$$

$$H(s) = \frac{T_aT_m}{T_aT_m s^2 + T_m s + 1} \begin{bmatrix} s & -\frac{k_e}{L_a} \\ \frac{k_m}{J} & s+\frac{1}{T_a} \end{bmatrix} \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{J} \end{bmatrix} \Rightarrow$$

$$H(s) = \frac{T_aT_m}{T_aT_m s^2 + T_m s + 1} \begin{bmatrix} \frac{s}{L_a} & \frac{k_e}{JL_a} \\ \frac{k_m}{JL_a} & -\frac{1}{J}\left(s+\frac{1}{T_a}\right) \end{bmatrix} = \begin{bmatrix} H_{iaua}(s) & H_{iams}(s) \\ H_{\omega ua}(s) & H_{\omega ms}(s) \end{bmatrix}$$

$$H_{iaua}(s) = \frac{T_aT_m}{T_aT_m s^2 + T_m s + 1} \cdot \frac{s}{L_a} = \frac{\frac{\cancel{L_a}}{R_a} T_m}{T_aT_m s^2 + T_m s + 1} \cdot \frac{s}{\cancel{L_a}} \Rightarrow$$

$$\boxed{H_{iaua}(s) = \frac{(T_m/R_a)\cdot s}{T_aT_m s^2 + T_m s + 1}} \Rightarrow \boxed{H_{iauc}(s) = \frac{((k_E T_m)/R_a) s}{T_aT_m s^2 + T_m s + 1}}$$

$$H_{iams}(s) = \frac{T_aT_m}{T_aT_m s^2 + T_m s + 1} \cdot \frac{k_e}{JL_a} = \frac{\cancel{T_aT_m}}{T_aT_m s^2 + T_m s + 1} \cdot \frac{\cancel{k_e}}{\frac{\cancel{T_m} k_m \cancel{k_e}}{R_a} \cdot L_a}$$

$$\Rightarrow \boxed{H_{iams}(s) = \frac{1/k_m}{T_aT_m s^2 + T_m s + 1}}$$

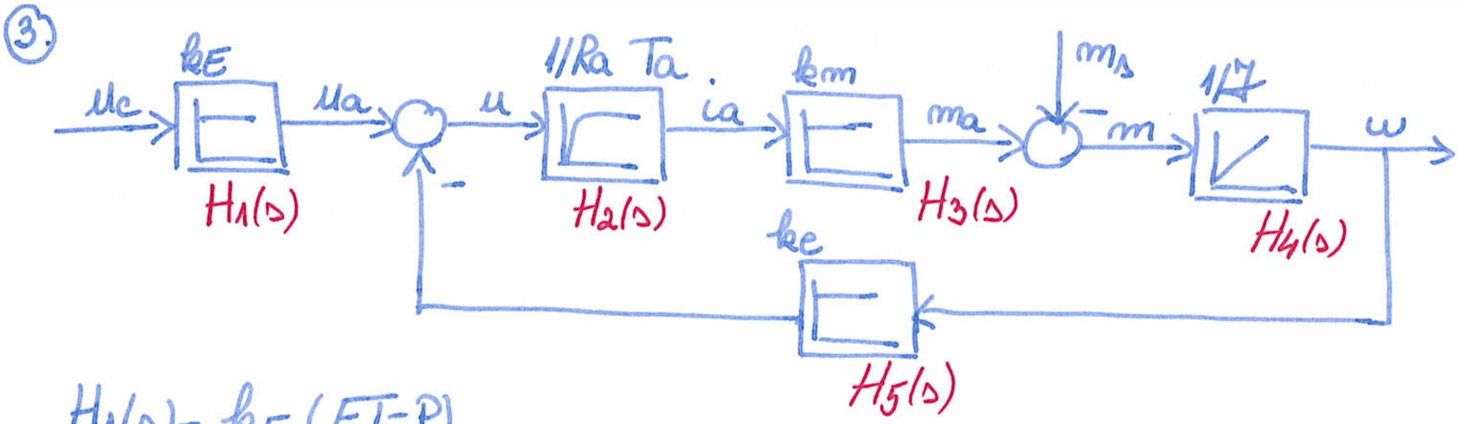
$$H_{\omega ua}(s) = \frac{T_aT_m}{T_aT_m s^2 + T_m s + 1} \cdot \frac{k_m}{JL_a} = \frac{\cancel{T_aT_m}}{T_aT_m s^2 + T_m s + 1} \cdot \frac{\cancel{k_m}}{\frac{\cancel{T_m k_m} k_e}{R_a} L_a}$$

$$\Rightarrow \boxed{H_{\omega ua}(s) = \frac{1/k_e}{T_aT_m s^2 + T_m s + 1}} \Rightarrow \boxed{H_{\omega uc}(s) = \frac{k_E/k_e}{T_aT_m s^2 + T_m s + 1}}$$

$$H_{\omega ms}(s) = \frac{T_aT_m}{T_aT_m s^2 + T_m s + 1} \cdot \frac{-(1+sT_a)}{JT_a} = \frac{\cancel{T_aT_m}}{T_aT_m s^2 + T_m s + 1} \cdot \frac{-(1+sT_a)}{\frac{\cancel{T_m} k_m k_e}{R_a} \cancel{T_a}}$$

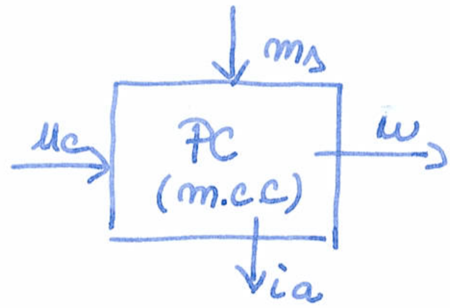
$$\Rightarrow \boxed{H_{\omega ms}(s) = -\frac{R_a}{k_m k_e} \cdot \frac{(1+sT_a)}{T_aT_m s^2 + T_m s + 1}}$$

3.



$H_1(s) = k_E \; (ET\text{-}P)$

$H_2(s) = \dfrac{1/R_a}{1+sT_a} \; (ET\text{-}PT_1)$

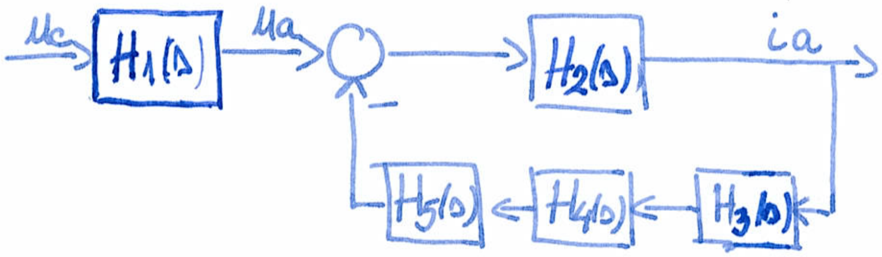


$H_3(s) = k_m \; (ET\text{-}P)$

$H_4(s) = \dfrac{1/J}{s} = \dfrac{1}{Js} \; (ET\text{-}I)$

$H_5(s) = k_e \; (ET\text{-}P)$

$$H_{iau_c}(s) = \left.\frac{i_a(s)}{u_c(s)}\right|_{m_s=0} = H_1(s) \cdot \frac{H_2(s)}{1+H_2(s)\left[H_3(s)H_4(s)H_5(s)\right]}$$

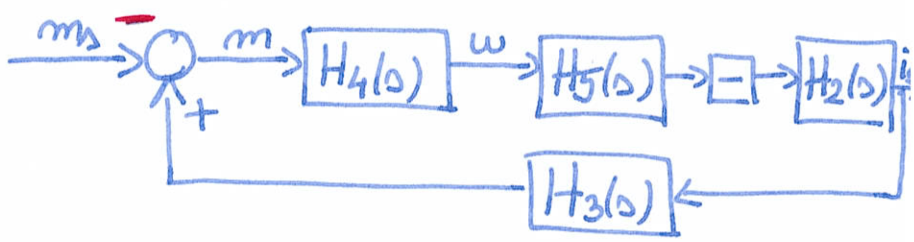


$$H_{iau_c}(s) = k_E \cdot \frac{\dfrac{1/R_a}{1+sT_a}}{1+\dfrac{1/R_a}{1+sT_a}\cdot\dfrac{k_m k_e}{Js}} = k_E \cdot \frac{\dfrac{1/R_a}{\cancel{1+sT_a}}}{\dfrac{Js(1+sT_a)+(k_m k_e)/R_a}{Js\,(\cancel{1+sT_a})}} =$$

$$= \frac{k_E}{R_a} \cdot \frac{Js}{Js(1+sT_a)+\dfrac{k_m k_e}{R_a}} = \frac{k_E}{R_a} \cdot \frac{T_m \dfrac{\cancel{k_m k_e}}{\cancel{R_a}}\, s}{T_m \dfrac{\cancel{k_m k_e}}{\cancel{R_a}}\, s(1+sT_a)+\dfrac{\cancel{k_m k_e}}{\cancel{R_a}}} \Rightarrow$$

$$\boxed{H_{iau_c}(s) = \frac{\left((k_E T_m)/R_a\right) s}{T_a T_m s^2 + T_m s + 1}}$$

$$H_{iam_s}(s) = \left.\frac{i_a(s)}{m_s(s)}\right|_{u_c=0}$$



$$H_{iams}(s) = -\frac{H_4(s)H_5(s)(-)H_2(s)}{1 - H_4(s)H_5(s)(-)H_2(s)\cdot H_3(s)} = \frac{H_2(s)H_4(s)H_5(s)}{1 + H_2(s)H_3(s)H_4(s)H_5(s)}$$

de la semnul perturbației

$$H_{iams}(s) = \frac{\frac{1/R_a}{\cancel{1+sT_a}}\cdot\frac{1}{\cancel{Js}}\cdot k_e}{\frac{Js(1+sT_a)+(k_m k_e)/R_a}{\cancel{Js(1+sT_a)}}} = \frac{\frac{\cancel{k_e}}{\cancel{R_a}}}{\frac{k_m \cancel{k_e}}{\cancel{R_a}}(T_a T_m s^2 + T_m s + 1)} \Rightarrow$$

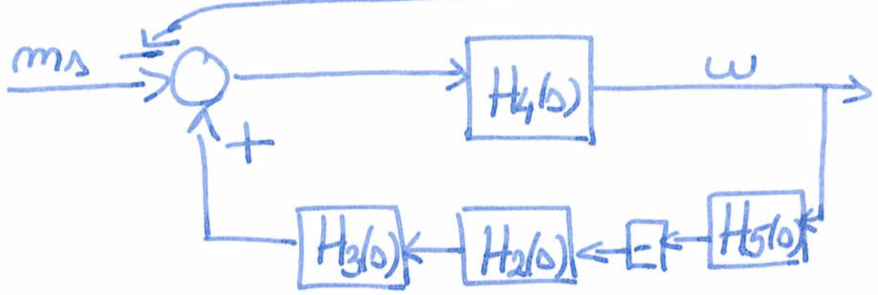
$$\boxed{H_{iams}(s) = \frac{1/k_m}{T_a T_m s^2 + T_m s + 1}}$$

$$H_{\omega uc}(s) = \frac{\omega(s)}{u_c(s)}\bigg|_{m_s=0} = H_1(s)\cdot\frac{H_2(s)H_3(s)H_4(s)}{1 + H_2(s)H_3(s)H_4(s)H_5(s)} =$$

$$= k_E\cdot\frac{\frac{1/R_a}{\cancel{1+sT_a}}\cdot\frac{k_m}{\cancel{Js}}}{\frac{Js(1+sT_a)+(k_m k_e)/R_a}{\cancel{Js(1+sT_a)}}} = k_E\,\frac{\frac{\cancel{k_m}}{\cancel{R_a}}}{\frac{\cancel{k_m}k_e}{\cancel{R_a}}(T_a T_m s^2 + T_m s + 1)} \Rightarrow$$

$$\boxed{H_{\omega uc}(s) = \frac{k_E/k_e}{T_a T_m s^2 + T_m s + 1}}$$

$$H_{\omega ms}(s) = \frac{\omega(s)}{m_s(s)}\bigg|_{u_c=0} = -\frac{H_4(s)}{1 - H_4(s)\left[(-)H_5(s)H_2(s)H_3(s)\right]} \Rightarrow$$



$$H_{\omega ms}(s) = -\frac{H_4(s)}{1 + H_2(s)H_3(s)H_4(s)H_5(s)} = -\frac{\frac{1}{\cancel{Js}}}{\frac{Js(1+sT_a)+(k_m k_e)/R_a}{\cancel{Js}(1+sT_a)}} =$$

$$= -\frac{1+sT_a}{\frac{k_m k_e}{R_a}(T_a T_m s^2 + T_m s + 1)} \Rightarrow \boxed{H_{\omega ms}(s) = -\frac{R_a}{k_m k_e}\cdot\frac{1+sT_a}{T_a T_m s^2 + T_m s + 1}}$$