UNIT-3 » RELATIONS, PARTIAL ORDERING AND RECURSION

PART-I RELATIONS

❖ INTRODUCTION

- ✓ Much of mathematics is about finding a pattern – a recognizable link between quantities that change. In our daily life, we come across many patterns that characterize relations such as brother and sister, father and son, teacher and student.
- ✓ In mathematics, also we come across many relations such as number m is less than number n, line l is parallel to line m, set A is a subset of set B. In all these, we notice that a relation involves pairs of objects in certain order.
- ✓ In this chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair.

❖ RELATION

- ✓ Let A and B be non-empty sets. A binary relation or simply relation from A to B is a subset of $A \times B$.
- ✓ Suppose that R is a relation from A to B. Then, R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair, $x \in A$ and $y \in B$, exactly one of the following is true.
  - ➢ $(x, y) \in R$, then we say x is R-related to y, written as $x\, R\, y$.
  - ➢ $(x, y) \notin R$, then we say x is not R-related to y.
- ✓ Note: If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation on A.

❖ DOMAIN

- ✓ Let R be a binary relation from A to B. Then, the domain is denoted and defined as,

$$D_R = \{x \in A : (x, y) \in R\}$$

❖ RANGE

- ✓ Let R be a binary relation from A to B. Then, the range is denoted and defined as,

$$R_R = \{y \in B : (x, y) \in R\}$$

- Examples
  - A familiar example is relation "greater than" for real numbers. This relation is denoted by " $>$ ". In fact, " $>$ " should be considered as the name of a set whose elements are ordered pairs.

$$>= \{(x, y) : x \text{ and } y \text{ are real numbers and } x > y\}$$

  - The definition of relation permits any set of ordered pairs to define a relation like

$$S = \{(2,4), (1,3), (x, 6), (\text{veer}, *)\}$$

Here, $D_R = \{2,1, x, \text{veer}\}$ & $R_R = \{4, 3, 6, *\}$

UNIVERSAL RELATION

- Let A and B be two non-empty sets. Then, $A \times B$, subset of itself, is called universal relation from A to B.
- Example: Let $A = \{1,2\}$ and $B = \{x, y\}$.

$$A \times B = \{\{1, x\}, \{1, y\}, \{2, x\}, \{2, y\}\}$$

VOID RELATION

- Let A and B be two non-empty sets. Then, the empty set $\{\phi\} \subset A \times B$ is called void (null) relation from A to B.
- Example: Let $A = \{1,2\}$ and $B = \{x, y\}$. Then, the relation presented by the set $\{\emptyset\}$ is called void (null) relation.

UNION, INTERSECTION AND COMPLEMENT OPERATIONS ON RELATIONS

- Let $A = \{1,2,3,4\}$ and

$$R = \{(x, y) : \text{for } x, y \in A, x - y \text{ is an integral multiple of } 2\} = \{(1,3), (3,1), (2,4), (4,2)\}$$

$$S = \{(x, y) : \text{for } x, y \in A, x - y \text{ is an integral multiple of } 3\} = \{(1,4), (4,1)\}$$

  - $R \cup S = \{(1,3), (3,1), (2,4), (4,2), (1,4), (4,1)\}$
  - $R \cap S = \emptyset$
- For complement, let $A = \{a, b\}$ and $B = \{1,2\}$, then $A \times B = \{\{a, 1\}, \{b, 1\}, \{a, 2\}, \{b, 2\}\}$. Also, let $R = \{\{a, 1\}, \{a, 2\}, \{b, 2\}\}$.
  - Then, complement relation for R is $\{\{b, 1\}\}$.

❖ PROPERTIES OF BINARY RELATIONS IN A SET

✓ Reflexive

- A binary relation R in a set A is said to be reflexive if, for every $x \in A$,

$$(x, x) \in R$$

✓ Irreflexive

- A binary relation R in a set A is said to be irreflexive if, for every $x \in A$,

$$(x, x) \notin R$$

✓ Symmetric

- A binary relation R in a set A is said to be symmetric if, for every $x, y \in A$,

$$\text{whenever } (x, y) \in R, \text{then } (y, x) \in R$$

✓ Antisymmetric

- A binary relation R in a set A is said to be antisymmetric if, for every $x, y \in A$,

$$\text{whenever } (x, y) \in R \ \& \ (y, x) \in R, \text{then } x = y$$

✓ Transitive

- A binary relation R in a set A is said to be transitive if, for every $x, y, z \in A$,

$$\text{whenever } (x, y) \in R \ \& \ (y, z) \in R, \text{then } (x, z) \in R$$

✓ Notes

- If relations R and S both are reflexive, then $R \cup S$ and $R \cap S$ are also reflexive.
- If relations R and S are symmetric and transitive, then $R \cap S$ is also symmetric and transitive.

METHOD-1: BASIC EXAMPLES ON RELATION

H	1	Define the following terms with example. Binary relation, Domain, Range, Universal relation, Void relation, Union, Antisymmetric, Intersection, Complement relation, Reflexive, Irreflexive, Symmetric, Transitive.	

H	2	Let $A = \{(1,2),(2,4),(3,3)\}$ and $B = \{(1,3),(2,4),(4,2)\}$. Find $A \cup B, A \cap B, D(A), D(B), D(A \cup B), R(A), R(B)$ and $R(A \cap B)$. **Answer :** $\mathbf{A \cup B = \{(1,2),(2,4),(3,3),(1,3),(4,2)\}, A \cap B = \{(2,4)\},}$ $\mathbf{D(A) = \{1,2,3\}, D(B) = \{1,2,4\}, D(A \cup B) = \{1,2,3,4\},}$ $\mathbf{R(A) = \{2,4,3\}, R(B) = \{3,4,2\}, R(A \cap B) = \{4\}}$	
C	3	What are the ranges of the relations $S = \{(x, x^2) : x \in N\}$ and $T = \{(x, 2x) : x \in N\}$,where $N = \{0,1,2,3, \dots\}$? Also, find $S \cap T$ and $S \cup T$. **Answer :** $\mathbf{R(S) = \{x^2 : x \in N\}, R(T) = \{2x : x \in N\}, S \cap T = \{(0,0),(2,4)\},}$ $\mathbf{S \cup T = \{(0,0),(1,1),(1,2),(2,4),(3,9),(3,6),(4,16),(4,8), \dots \dots\}}$	
C	4	Let L denotes the relation "less than or equal to" and D denotes the relation "divides", where x D y means "x divides y", defined on a set $\{1,2,3,6\}$. Write L and D as a sets, and find $L \cap D$. **Answer :** $\mathbf{D = \{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),(3,6),(6,6)\}}$ $\mathbf{L = \{(1,1),(1,2),(1,3),(1,6),(2,2),(2,3),(2,6),(3,3),(3,6),(6,6)\}}$ $\mathbf{L \cap D = \{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),(3,6),(6,6)\}}$	
H	5	Give an example of a relation which is a) neither reflexive nor irreflexive? b) both symmetric and antisymmetric? c) reflexive but not symmetric? d) symmetric but not reflexive or transitive? e) transitive but not reflexive or symmetric? **Answer :** $\mathbf{A = \{(1,1),(2,3),(3,2),(3,3)\}, B = \{(x,y) : x, y \in \mathbb{N}, x = y\},}$ $\mathbf{C = \{(1,1),(1,2),(2,2)\}, D = \{(1,2),(2,1)\},}$ $\mathbf{E = \{(1,2),(2,3),(1,3)\}}$	
H	6	Check whether the following relations are transitive or not. $R_1 = \{(1,1)\}$, $R_2 = \{(1,2),(2,2)\}$, $R_3 = \{(1,2),(2,3),(1,3),(2,1)\}$ **Answer : yes, yes, no**	
C	7	Let L denotes the relation "less than or equal to" and D denotes the relation "divides", where x D y means "x divides y", defined on a set $\{1,2,3,6\}$. Show that both L and D are reflexive, antisymmetric and transitive.	

C	8	Given $S = \{1,2,3,4,\ldots,10\}$ and a relation R on S, where $R = \{(x,y) : x + y = 10\}$, what are the properties of the relation R? **Answer : symmetric but not reflexive, irreflexive, antisymmetric or transitive.**	

❖ RELATION MATRIX

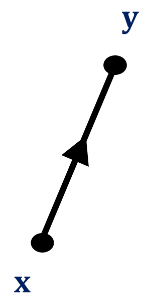
- ✓ A relation R from a set A to a set B can be represented by a matrix called relation matrix of R.
- ✓ The relation matrix of R can be represented by constructing a table whose columns are successive elements of B & rows are successive elements of A. i.e. if $(x_i, y_j) \in R$, then we enter 1 in i^{th} row and j^{th} column. Similarly, if $(x_i, y_j) \notin R$, then we enter 0 in i^{th} row and j^{th} column.
- ✓ Let $A = \{x_1, x_2, x_3\}$ and $B = \{y_1, y_2\}$. Also, let $R = \{(x_1, y_1), (x_2, y_1), (x_3, y_2), (x_2, y_2)\}$. Then, the table representation looks like,

	y_1	y_2
x_1	1	0
x_2	1	1
x_3	0	1

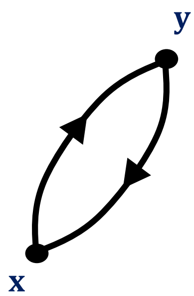
- ✓ Hence, the relation matrix is $M_R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

❖ GRAPH OF A RELATION

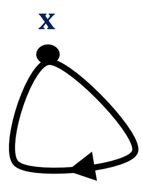
- ✓ A relation can also be represented pictorially by drawing its graph.
- ✓ Let R be a relation in $A = \{x_1, x_2, \ldots, x_m\}$. The elements of A are represented by points or circles called nodes. The nodes may also be called vertices.
- ✓ Now, if $(x_i, x_j) \in R$, then we connect nodes x_i and x_j by an arc and put an arrow on the arc in the direction from x_i to x_j. Thus, when all the nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows, we get a graph of the relation R.
- ✓ If $(x_i, x_j) \in R$ and $(x_j, x_i) \in R$, then we draw two arcs between x_i and x_j, which is called a loop.



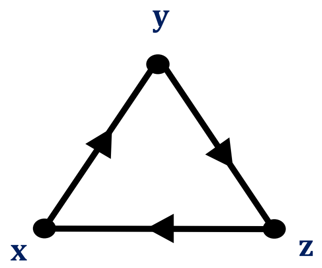
x R y



x R y and y R x

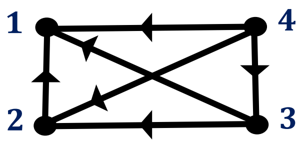
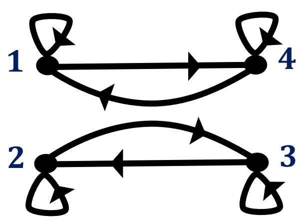


x R x



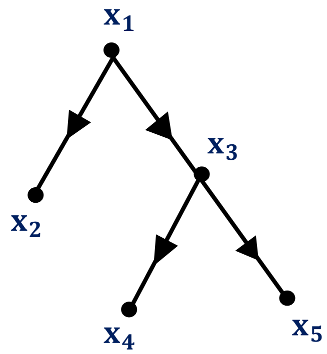
x R y , y R z and z R x

METHOD-2: EXAMPLES ON RELATION MATRIX AND GRAPH OF A RELATION

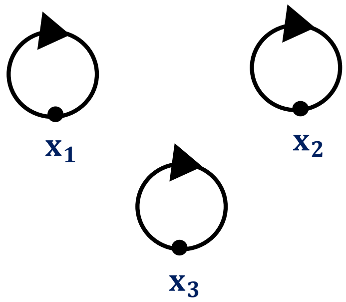
C	1	Let $X = \{1,2,3,4\}$ and $R = \{(x,y) : x > y\}$. Draw the graph of R and give its matrix. **Answer :** $M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ 	
H	2	Let $A = \{a, b, c\}$ and denote the subsets of A by $B_0, B_1, \dots, B_7$ given as $B_0 = \phi$, $B_1 = \{c\}$, $B_2 = \{b\}$, $B_3 = \{b, c\}$, $B_4 = \{a\}$, $B_5 = \{a, c\}$, $B_6 = \{a, b\}$, $B_7 = \{a, b, c\}$. If R is the relation of proper subset on these subsets, then give the matrix of the relation. **Answer :** $M_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	
H	3	Let $X = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3)\}$. Write matrix of R and sketch its graph. **Answer :** $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ 	

C **4** Determine the properties of the relations given by the graphs as given below. Also, write corresponding relation matrices.

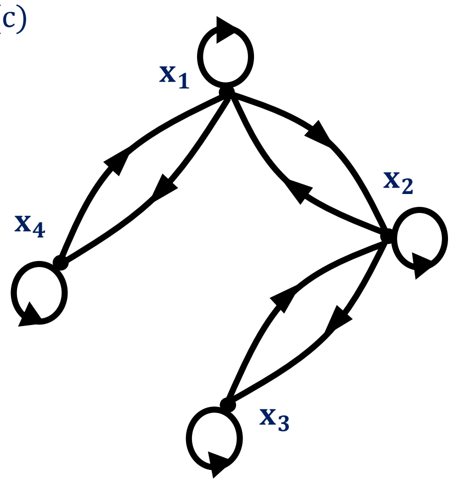
(a)



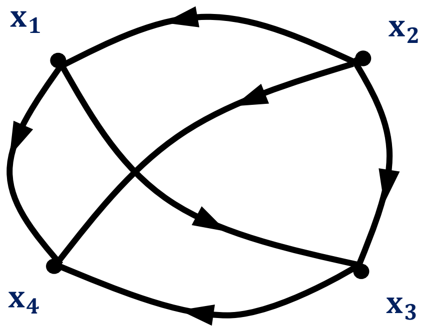
(b)



(c)



(d)



Answer : relation (a) is antisymmetric, relation (b) is reflexive, relation (c) is reflexive and symmetric and relation (d) is transitive.

$$M_a = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, M_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_c = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_d = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

❖ PARTITION AND COVERING OF A SET

- ✓ Let S be a given set and $A = \{A_1, A_2, A_3, \dots, A_m\}$ where each A_i ; $i = 1,2, \dots, m$ is a subset of S and

$$\bigcup_{i=1}^{m} A_i = S$$

then, the set A is called a covering of S and the sets $A_1, A_2, \dots, A_m$ are said to be covers of S.

- ✓ Also, if the elements of A, which are subsets of S, are mutually disjoint, then A is called a partition of S and sets $A_1, A_2, \dots, A_m$ are called the blocks of the partition.
- ✓ Example: Let $S = \{x, y, z\}$. Then
  - ➢ $A = \{\{x, y\}, \{y, z\}\}$ is a covering of S.
  - ➢ $B = \{\{x\}, \{y, z\}\}$ is a partition of S.
  - ➢ $C = \{\{x, y, z\}\}$ is a partition of S.
  - ➢ $D = \{\{x\}, \{y\}, \{z\}\}$ is a partition of S.
  - ➢ $E = \{\{x\}, \{x, y\}, \{x, z\}\}$ is a covering of S.
- ✓ Note: Every partition is a cover, but a cover may not be a partition.

❖ EQUIVALENCE RELATION

- ✓ A relation R on A is called an equivalence relation if it is reflexive, symmetric and transitive.
- ✓ Example: Let $A = \{1, 2, \dots, 7\}$ and $R = \{(x, y) : x - y \text{ is divisible by } 3\}$.
  - ➢ For any $a \in A$, $a - a$ is divisible by 3. Hence, $a\,R\,a$. So, R is reflexive.
  - ➢ For any $a, b \in A$, let $a\,R\,b$. i.e. $a - b$ is divisible by 3. So, $b - a$ is also divisible by 3. Hence, $b\,R\,a$. So, R is symmetric.
  - ➢ For any $a, b, c \in A$, let $a\,R\,b$ and $b\,R\,c$. i.e. $a - b$ and $b - c$ are divisible by 3. So, $a - c = (a - b) + (b - c)$ is also divisible by 3. Hence, $a\,R\,c$. So, R is transitive.
  - ➢ Hence, R is an equivalence relation on A.

❖ EQUIVALENCE CLASS

- ✓ Let R be an equivalence relation on a set A. For any $x \in A$, the set $[x]_R \subseteq A$ given by

$$[x]_R = \{y : y \in A \text{ and } x\,R\,y\}$$

is called an R-equivalence class generated by $x \in A$.

❖ PROPERTIES OF EQUIVALENCE CLASS

- ✓ For any $x \in A$, we have $x\ R\ x$ because R is reflexive, therefore $x \in [x]_R$.
- ✓ Let $y \in A$ be any element such that $x\ R\ y$, then we have $[x]_R = [y]_R$.
- ✓ If $(x, y) \notin R$, then, $[x]_R \neq [y]_R$. Because, if $[x]_R = [y]_R$, then there exist at least one $z \in [x]_R$ and $z \in [y]_R$, that gives $x\ R\ z$ and $y\ R\ z$, i.e. $x\ R\ y$, which contradicts to $(x, y) \notin R$.

❖ COMPATIBILITY RELATION

- ✓ A relation R in A is said to be compatibility relation if it is reflexive and symmetric.
- ✓ Note: Every equivalence relation is a compatibility relation. But, reverse may not be true.

❖ MAXIMAL COMPATIBILITY BLOCK

- ✓ Let A be a set and R be a compatibility relation on A. Then, a subset $C \subseteq A$ is called a maximal compatibility block if any element of C is compatible to every other element of C and no element of $A - C$ is compatible to all the elements of C.
- ✓ Let $R = \{(x, y) : x, y \in X \text{ and } x \text{ and } y \text{ contain some common letter}\}$ be a relation on $X = \{\text{ball, bed, dog, let, egg}\}$. Then, R is a compatibility relation denoted by " $\approx$ ". Also, note that R is not equivalence relation. If we denote the elements of X by x_1, x_2, x_3, x_4, x_5 then the graph is as shown here in figure (1).

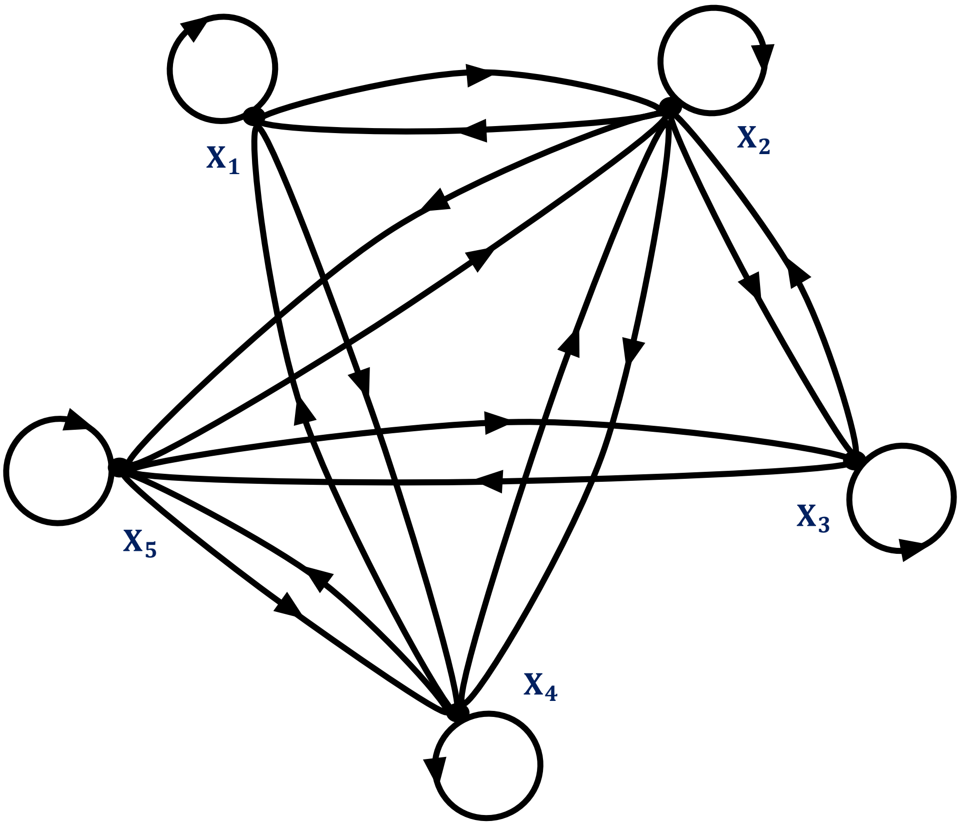


Figure (1)

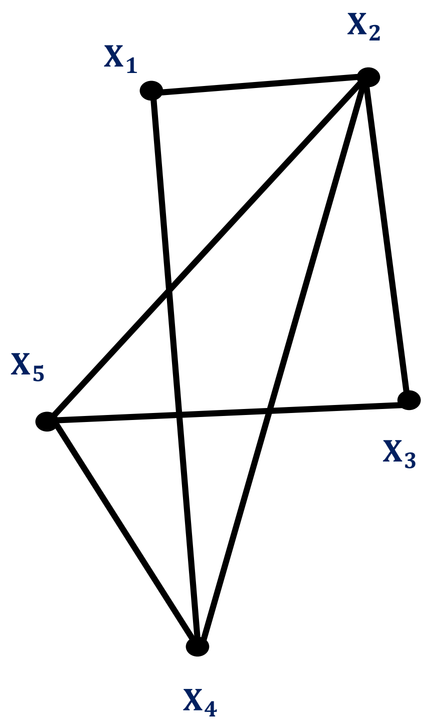


Figure (2)

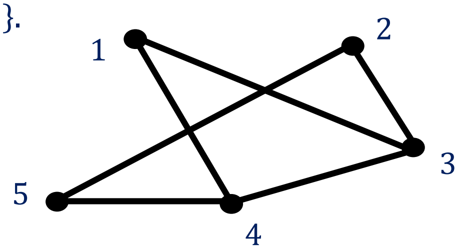
- ✓ Since, relation is compatibility relation, it is not necessary to draw loops at each element nor it is necessary to draw both x R y and y R x. So, we can simplify graph as shown in figure (2).
- ✓ The relation matrix here is symmetric and has its diagonal elements unity. Therefore, it is sufficient to give only the elements of the lower triangular part only as shown as below.

	x_1	x_2	x_3	x_4
x_2	1			
x_3	0	1		
x_4	1	1	0	
x_5	0	1	1	1

- ✓ It is clear that the subsets $\{x_1, x_2, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}$ are maximal compatibility blocks.

❖ HOW TO FIND MAXIMAL COMPATIBILITY BLOCK AND MATRIX

- ✓ For this first we draw a simplified graph of the compatibility relation and pick from this graph the largest complete polygons i.e. a polygon in which any vertex is connected to every other vertex. In addition to this any element which is related only to itself forms a maximal compatibility block. Similarly, any two elements which are compatible to one another but to no other elements also form a maximal compatibility block. For example, triangle.
- ✓ Example: The maximal compatibility blocks of a compatibility relation R with simple graph as given below are $\{1,3,4\}, \{2,3\}, \{4,5\}, \{2,5\}$.



  - ➢ Also, the matrix for this compatibility relation is as given below.

	1	2	3	4
2	0			
3	1	1		
4	1	0	1	
5	0	1	0	1

❖ COMPOSITE RELATION

- ✓ Let R be a relation from A to B and S be a relation from B to C. Then, a relation written as $R \circ S$ is called a composite relation of R and S, defined by

$$R \circ S = \{(x, z) : \text{for } x \in A \text{ and } z \in C \text{ there exist } y \in B \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

- ✓ The operation of obtaining $R \circ S$ is called composition of relations.

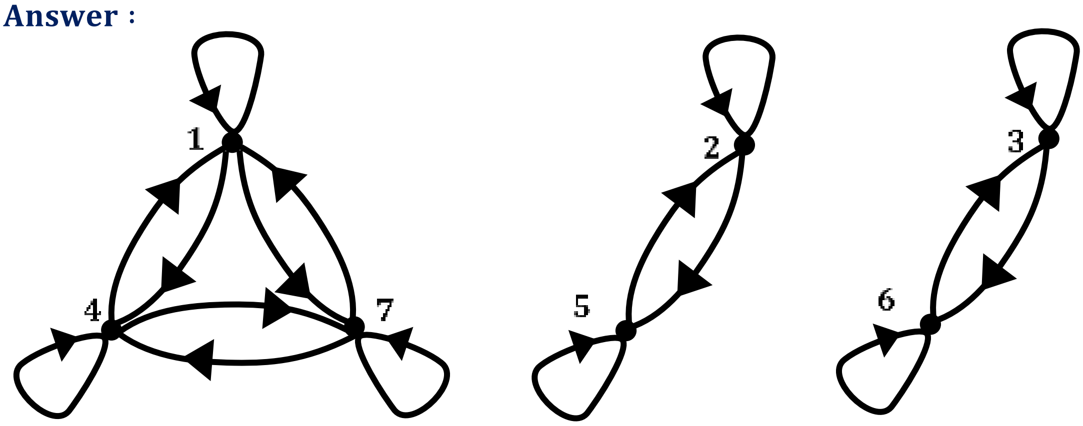
❖ CONVERSE OF A RELATION

- ✓ Given a relation R from A to B, a relation $\widetilde{R}$ from B to A is called the converse of R, where the ordered pairs of $\widetilde{R}$ are obtained by interchanging the members in each of the ordered pairs of R. This means, for $x \in A$ and $y \in B$, $x\,R\,y \iff y\,\widetilde{R}\,x$.

❖ TRANSITIVE CLOSURE OF A RELATION

- ✓ Let R be relation in a finite set A. The relation $R^+ = R \cup R^2 \cup R^3 \cup \ldots$ in A is called the transitive closure of R in A.

METHOD-3: EXAMPLES ON EQUIVALENCE AND COMPATIBILITY RELATION

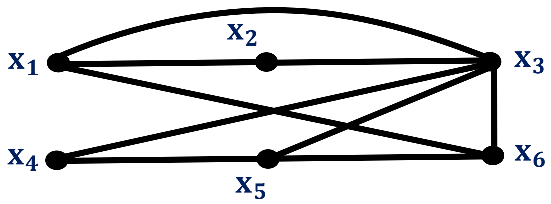
H	1	Define with examples: Partition, Covering, Equivalence relation, Equivalence class, Compatibility relation, Maximal compatibility block, Composite relation, Converse of a relation, Transitive closure.	
C	2	Let $X = \{1,2, \ldots \ldots, 7\}$ and $R = \{(x, y) : x - y \text{ is divisible by } 3\}$. Show that R is an equivalence relation. Also, draw the graph of R. **Answer :** 	
H	3	Let $X = \{1,2, \ldots \ldots, 7\}$ and $R = \{(x, y) : x - y \text{ is even}\}$. Show that R is an equivalence relation.	

H	4	Let $Z = A_1 \cup A_2 \cup A_3$. Where $A_1 = \{\dots, 1, 4, 7, \dots\}, A_2 = \{\dots, 2, 5, 8, \dots\}$ and $A_3 = \{\dots, 3, 6, 9, \dots\}$. Then, define equivalence relation whose equivalence classes are A_1, A_2, A_3. **Answer :** $\mathbf{R = \{(x, y) : x - y}$ **is devisible by 3**$\}$ **over the set of integers**	
C	5	Let $\mathbb{Z}$ be the set of integers and R be the relation called "congruence modulo 3" defined by $R = \{(x, y) : x - y \text{ is divisible by } 3\}$. Determine equivalence classes generated by the elements of $\mathbb{Z}$. **Answer :** $\mathbb{Z}/\mathbf{R} = \{[\mathbf{0}]_\mathbf{R}, [\mathbf{1}]_\mathbf{R}, [\mathbf{2}]_\mathbf{R}\}$	
H	6	Prove that relation "congruence modulo m" given by $R = \{(x, y) : x - y \text{ is divisible by } m\}$, over the set of positive integers, is an equivalence relation.	
H	7	Let S be the set of all statement functions in n variables and let R be the relation given by $R = \{(x, y) ; x \Leftrightarrow y\}$. Discuss the equivalence classes generated by the elements of S. **Answer : there are** $\mathbf{2^{2^n}}$ **R-equivalence classes**	
H	8	Let $X = \{a, b, c, d, e\}$ and let $C = \{\{a, b\}, \{c\}, \{d, e\}\}$. Show that the partition C defines an equivalence relation on X.	
C	9	Let R denote a relation on the set of ordered pairs of positive integers such that $(x, y)\ R\ (u, v)$ iff $xv = yu$. Show that R is an equivalence relation.	
C	10	Let $R = \{(x, y) : x, y \in X \text{ and } x \text{ and } y \text{ contain some common letter}\}$ be relation on X where $X = \{ball, bed, dog, let, egg\}$. Then, show that R is a compatibility relation.	

C **11** Let the compatibility relation on a set $\{x_1, x_2, x_3, \dots, x_6\}$ be given by the following matrix. Draw the graph. Also, find maximal compatibility blocks of the relation.

x_2	1				
x_3	1	1			
x_4	0	0	1		
x_5	0	0	1	1	
x_6	1	0	1	0	1
	x_1	x_2	x_3	x_4	x_5

Answer : $\{\mathbf{x_1, x_2, x_3}\}, \{\mathbf{x_1, x_3, x_6}\}$,
$\{\mathbf{x_3, x_4, x_5}\}, \{\mathbf{x_3, x_5, x_6}\}$



H **12** Let $R = \{(1,2), (3,4), (2,2)\}$ and $S = \{(4,2), (2,5), (3,1), (1,3)\}$. Find $R \circ S, S \circ R$, $R \circ (S \circ R), (R \circ S) \circ R, R \circ R, S \circ S$, and $R \circ R \circ R$.

Answer : $\mathbf{R \circ S = \{(1,5), (3,2), (2,5)\}, S \circ R = \{(4,2), (3,2), (1,4)\}}$

$\mathbf{R \circ R = \{(1,2), (2,2)\}, S \circ S = \{(4,5), (3,3), (1,1)\}}$

$\mathbf{R \circ R \circ R = \{(1,2), (2,2)\}, R \circ (S \circ R) = (R \circ S) \circ R = \{(3,2)\}}$

C **13** Let R and S be two relations on a set of positive integers A, where $R = \{(x, 2x) : x \in A\}$ and $S = \{(x, 7x) : x \in A\}$. Find $R \circ S, R \circ R, R \circ R \circ R$ and $R \circ S \circ R$.

Answer : $\mathbf{R \circ S = \{(x, 14x) : x \in A\}, R \circ R = \{(x, 4x) : x \in A\}}$

$\mathbf{R \circ R \circ R = \{(x, 8x) : x \in A\}, R \circ S \circ R = \{(x, 28x) : x \in A\}}$

C **14** Let $R = \{(1,2), (3,4), (2,2)\}$ and $S = \{(4,2), (2,5), (3,1), (1,3)\}$ be relations on a set $A = \{1, 2, 3, 4, 5\}$. Obtain relation matrices for $R \circ S$ and $S \circ R$.

Answer : $\mathbf{M_{R \circ S}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbf{M_{S \circ R}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

H	**15**	Given the relation matrix $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ of a relation R on a set $\{a, b, c\}$, find the relation matrices of $\widetilde{R}, R^2 = R \circ R, R^3 = R \circ R \circ R$, and $R \circ \widetilde{R}$. **Answer** : $\mathbf{M}_{\widetilde{R}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \mathbf{M}_{R^2} = \mathbf{M}_{R^3} = \mathbf{M}_{R \circ \widetilde{R}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$	
C	**16**	Given the relation matrices M_R and M_S, find $M_{R \circ S}, M_{\widetilde{R}}, M_{\widetilde{S}}, M_{R \tilde{\circ} S}$, and show that $M_{R \tilde{\circ} S} = M_{\widetilde{S} \circ \widetilde{R}}$. $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ **Answer** : $\mathbf{M}_{R \circ S} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}, \mathbf{M}_{\widetilde{R}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \mathbf{M}_{\widetilde{S}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix},$ $\mathbf{M}_{R \tilde{\circ} S} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$	

PART-II PARTIAL ORDERING

❖ PARTIAL ORDERING

- ✓ A binary relation R in a set P is called a partial order relation or partial ordering in P iff R is reflexive, antisymmetric, and transitive. Also, it is denoted by $\leq$ and the ordered pair $(P, \leq)$ is called a partially ordered set or a poset.
- ✓ Examples
  - ➢ Let P be the set of real numbers. The relation $\leq$ (less than or equal to) is a partial ordering on P.
  - ➢ The relation inclusion $\subseteq$ on $P = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ is a partial ordering.
  - ➢ Let $P = \{2, 3, 6, 8\}$ and $\leq$ be a relation "divides", then $(P, \leq)$ is a poset.

❖ TOTALLY ORDERED SET

- ✓ Let $(P, \leq)$ be a partially ordered set. If for every $x, y \in P$ we have either $x \leq y$ or $y \leq x$, then $\leq$ is called a simple ordering or linear ordering on P, and $(P, \leq)$ is called a totally ordered or simply ordered set or a chain.
- ✓ Example: Set $I_n = \{1,2,3 \dots, n\}$ with natural ordering "less than or equal to".

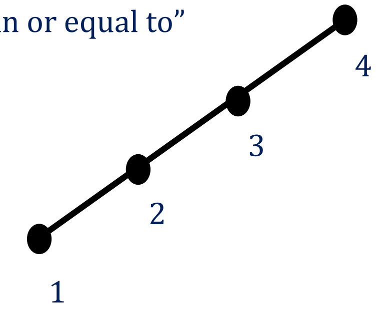
❖ COVER

- ✓ In a poset $(P, \leq)$ an element $y \in P$ is said to cover an element $x \in P$ if $x \leq y$ and if there does not exist an element $z \in P$ such that $x \leq z$ and $z \leq y$.

❖ HASSE DIAGRAM

- ✓ A poset $(P, \leq)$ can be represented by means of a diagram known as a Hasse diagram or partial ordered set diagram. In such a diagram, each element is represented by a small circle or a dot.
- ✓ The circle for $x \in P$ drawn below the circle for $y \in P$ if $x \leq y$, and a line is drawn between x and y.
- ✓ If $x \leq y$, but y does not cover x, then x and y are not connected directly by a single line. However, they are connected through one or more elements of P.

- ✓ Example: Let $P = \{1, 2, 3, 4\}$ and $\leq$ be a relation "less than or equal to" then the Hasse diagram is as follows.

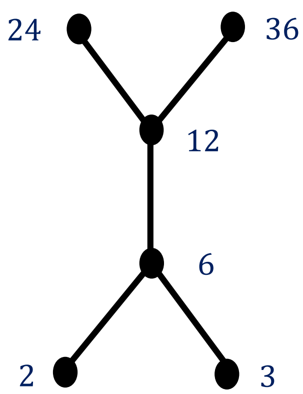


❖ LEAST AND GREATEST MEMBER

- ✓ Let $(P, \leq)$ be a poset. If there exist an element $y \in P$ such that $y \leq x$ for all $x \in P$, then y is called the least member in P relative to the partial ordering $\leq$. Similarly, if there exists an element $y \in P$ such that $x \leq y$ for all $x \in P$, then y is called greatest member in P relative to $\leq$.
- ✓ The least member is usually denoted by 0 and greatest member by 1.
- ✓ Example: Let $P = \{1, 2, 3, 4\}$ and $\leq$ be a relation "less than or equal to".
  - ➢ Here, least member is 1 and greatest member is 4.
- ✓ Note: For any poset least and greatest member, if exists, are unique. It may happen that the least or the greatest member does not exist.

❖ MINIMAL AND MAXIMAL MEMBER

- ✓ Let $(P, \leq)$ be a poset. If there does not exist an element $x \in P$ such that $x \leq y$ for $y \in P$, then y is called the minimal member in P relative to the partial ordering $\leq$. If there does not exist an element $x \in P$ such that $y \leq x$ for $y \in P$, then y is called the maximal member in P relative to the partial ordering $\leq$.
- ✓ Example: Let $P = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be for divide.



  - ➢ Here, there are two minimal members 2 and 3.
  - ➢ Also, there are two maximal members 24 and 36.

❖ UPPER BOUND AND LOWER BOUND

- ✓ Let $(P, \leq)$ be a poset and let $A \subseteq P$. Any element $x \in P$ is an upper bound for A if for all $a \in A$, $a \leq x$. Similarly, any element $x \in P$ is a lower bound for A if for all $a \in A$, $x \leq a$.

❖ LEAST UPPER BOUND AND GREATEST LOWER BOUND

- ✓ Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. An element $x \in P$ is a least upper bound or supremum for A if x is an upper bound for A and $x \leq y$ where y is any upper bound for A. Similarly, an element $x \in P$ is a greatest lower bound or infimum for A if x is an lower bound for A and $y \leq x$ where y is any lower bound for A.
- ✓ Least upper bound is denoted by "LUB" or "SUP", and greatest lower bound is denoted by "GLB" or "INF".
- ✓ Example: Let $P = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be for divide. Also, let $A = \{6, 12\}$.
  - ➢ Here, lower bounds are 2, 3 and 6. But, the greatest lower bound is 6.
  - ➢ Similarly, upper bounds are 12, 24 and 36. But, the least upper bound is 12.
- ✓ Note: Both "GLB" and "LUB" are unique if exists.

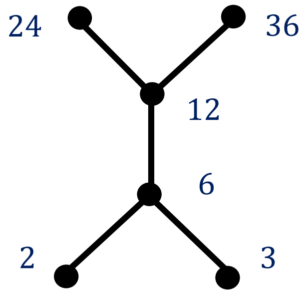
❖ WELL-ORDERED SET

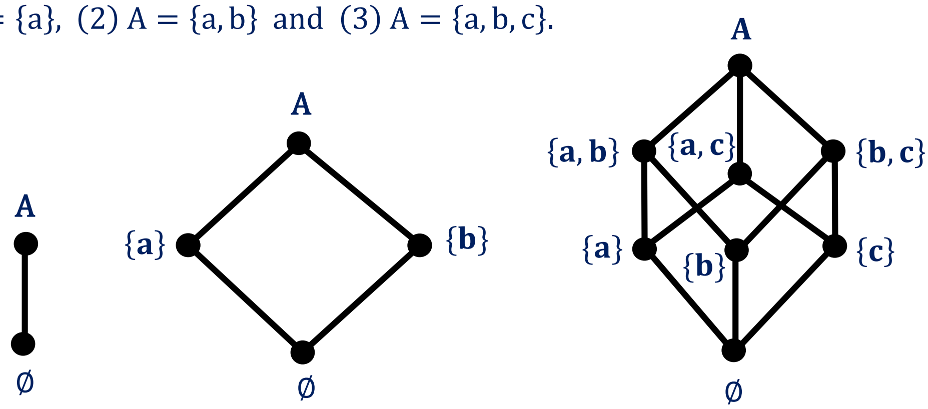
- ✓ A partially ordered set is called well-ordered if every non-empty subset of it has a least member.
- ✓ Example: A simplest well-ordered set is $I_n = \{1, 2, \dots, n\}$ with natural ordering "less than or equal to".

METHOD-4: EXAMPLES ON POSET AND HASSE DIAGRAM

H	**1**	Define the following terms with example: Partial order relation, Partially ordered set (poset), Simple (linear) ordering, Totally ordered set (chain or simply ordered set), Cover, Least member, Greatest member, Minimal member, Maximal member, Upper bound, Lower bound, Least upper bound (supremum), Greatest lower bound (infimum), Well-ordered set.	
C	**2**	Show that the relation $\subseteq$(inclusion) on a set $P(A)$, i. e. power set of $A = \{a, b, c\}$, is a partial ordering.	
H	**3**	Show that $(P(A), \subseteq)$ is a poset.	

<table>
<tr><td>C</td><td>4</td><td>Let $A = \{2, 3, 6, 12, 24, 36\}$ and the relation $\le$ be such that $x \le y$ if x divides y. Draw the Hasse diagram of $(A, \le)$.

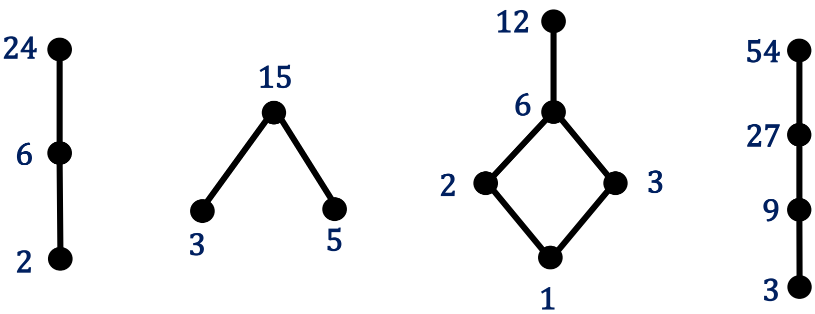
Answer :

</td><td></td></tr>
<tr><td>H</td><td>5</td><td>Let $\subseteq$ be a relation on a set $P(A)$, i.e. power set of A. Then, draw Hasse diagram for (1) $A = \{a\}$, (2) $A = \{a, b\}$ and (3) $A = \{a, b, c\}$.

Answer :


(1) (2) (3)</td><td></td></tr>
<tr><td>C</td><td>6</td><td>Give an example of a set A such that $(P(A), \subseteq)$ is a totally ordered set.

Answer : $A = \{a\}$</td><td></td></tr>
<tr><td>C</td><td>7</td><td>Draw the Hasse diagram of the following sets under the partial order relation “divides” and indicate those which are totally ordered sets.

(a) $\{2, 6, 24\}$, (b) $\{3, 5, 15\}$, (c) $\{1, 2, 3, 6, 12\}$ and (d) $\{3, 9, 27, 54\}$

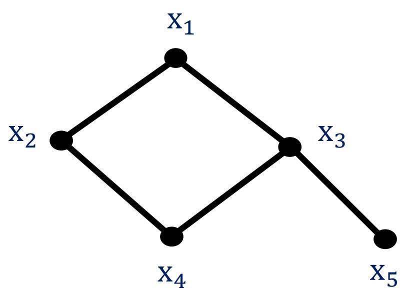
Answer : sets (a) and (d) are totally ordered sets.


(a) (b) (c) (d)</td><td></td></tr>
</table>

H **8** Give a relation which is both a partial order relation and an equivalence relation on a set.

Answer : $\mathbf{R = \{(x, y) : x = y\}}$

C **9** Hasse diagram of a poset (P, R), where $P = \{x_1, x_2, x_3, x_4, x_5\}$, is given below. Find out which of the followings are true?



a) $x_1 \, R \, x_2$ e) $x_4 \, R \, x_1$

b) $x_3 \, R \, x_5$ f) $x_2 \, R \, x_5$

c) $x_1 \, R \, x_1$ g) $x_2 \, R \, x_3$

d) $x_4 \, R \, x_5$

Answer : **false, false, true, false, true, false, false**

C **10** For above poset given in example 9. Find least and greatest member in P if exists. Also, find minimal and maximal elements. Find upper and lower bounds. Find LUB and GLB if exists.

Answer : **least member does not exist, greatest member is $\mathbf{x_1}$**

minimal elements are $\mathbf{x_4}$ & $\mathbf{x_5}$, maximal element is $\mathbf{x_1}$

lower bound does not exist, upper bound & LUB is $\mathbf{x_1}$

C **11** Let $P = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y. Then, find least and greatest member in P if exists. Also, find minimal and maximal elements. Find upper bounds, lower bounds, LUB and GLB if exists for sets (a) $\{2, 3, 6\}$, (b) $\{2, 3\}$, (c) $\{12, 6\}$, (d) $\{24, 36\}$ and (e) $\{3, 12, 24\}$.

Answer : **least and greatest member does not exists**

minimal elements are 2 & 3, maximal elements are 24 and 36

	Upper bounds	Lower bounds	LUB	GLB
(a)	6,12, 24,36	Not exist	6	Not exist
(b)	6,12, 24,36	Not exist	6	Not exist
(c)	12,24, 36	2,3, 6	12	6
(d)	Not exist	2,3, 6,12	Not exist	12
(e)	24	3	24	3

❖ LATTICE

- ✓ A lattice is a poset $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound.
- ✓ The greatest lower bound of a subset $\{a, b\} \subseteq L$ is called meet and denoted by GLB $\{a, b\}$ or $a * b$ or $a \wedge b$ or $a \cdot b$. The least upper bound is called join and denoted by LUB $\{a, b\}$ or $a \oplus b$ or $a \vee b$ or $a + b$.
- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively.

❖ PROPERTIES OF LATTICES

- ✓ Let $(L, \leq)$ be a lattice and $*$ and $\oplus$ be two binary operations meet and join. Then, for

$$a, b, c \in L$$

  - ➢ $a * a = a$ and $a \oplus a = a$ (Idempotent)
  - ➢ $a * b = b * a$ and $a \oplus b = b \oplus a$ (Commutative)
  - ➢ $(a * b) * c = a * (b * c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (Associative)
  - ➢ $a * (a \oplus b) = a$ and $a \oplus (a * b) = a$ (Absorption)

❖ COMPLETE LATTICE

- ✓ A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.
- ✓ Note: Every finite lattice must be complete.
- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively.

❖ DISTRIBUTIVE LATTICE

- ✓ A lattice $(L, *, \oplus)$ is called a distributive lattice if for any $a, b, c \in L$,

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively.

MODULAR LATTICE

- A lattice $(L,*,\oplus)$ is said to be modular if $a \le c \Rightarrow a\oplus(b*c) = (a\oplus b)*c$.
- Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S),\subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively.

BOUNDED LATTICE

- A lattice is said to be bounded if it has both least and greatest elements, i.e. 0 and 1.
- Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S),\subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively, and least element is ϕ and greatest element is S.

COMPLEMENT

- In a bounded Lattice $(L,*,\oplus,0,1)$ an element $b \in L$ is called a complement of an element $a \in L$ if $a*b = 0$ and $a\oplus b = 1$.

COMPLEMENTED LATTICE

- A lattice $(L,*,\oplus,0,1)$ is said to be a complemented lattice if every element of L has at least one complement.
- Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S),\subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively, and least element is ϕ and greatest element is S.

BOOLEAN LATTICE

- A Boolean lattice (Boolean algebra) is a complemented, distributive lattice.
- Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S),\subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively, and least element is ϕ and greatest element is S.

PSEUDO BOOLEAN LATTICE

- A bounded lattice $(L,\le)$ is called pseudo Boolean lattice if for all $a,b \in L$, there exists $c \in L$ such that $a*x \le b \Leftrightarrow x \le c, \forall x \in L$.
- If such element c exists, then it is unique and will be denoted by $b : a$.

- ✓ Example: Let S be any set and P(S) be its power set. The partially ordered set $(P(S), \subseteq)$ is a lattice in which the meet and join are the same as the operations $\cap$ and $\cup$ respectively, and least element is ϕ and greatest element is S.

METHOD-5: BASIC EXAMPLES ON LATTICE

H	1	Define With example: Lattice, Complete Lattice, Distributive lattice, Modular lattice, Bounded lattice, Complement element, Complemented lattice, Boolean lattice(algebra), Pseudo Boolean lattice(algebra).	
C	2	Let $A = \{a, b, c\}$. Then, show that the poset $(p(A), \subseteq)$ is a lattice.	
H	3	Let $A = \{1,2,3\}$. Check whether the poset $(p(A), \subseteq)$ is a lattice. $\textbf{Answer : yes, } (\mathbf{p(A)}, \subseteq) \textbf{ is a lattice}$	
C	4	Let $A = \{2,3,4,6,8,12,24,36\}$. Check whether the poset $(A, \mid)$ is a lattice. $\textbf{Answer : no, } (\mathbf{A}, \mid) \textbf{ is not a lattice}$	
H	5	Let S_n be the set of factors(divisors) of positive integer n. Draw Hasse diagram for lattice $(S_n, \mid)$ for $n = 6,24,30,45$.	
C	6	Check whether $([0,1], \leq)$ is a lattice. If yes, what are meet and join? $\textbf{Answer : yes, } ([\mathbf{0}, \mathbf{1}], \leq) \textbf{ is a lattice with meet } \{\mathbf{a}, \mathbf{b}\} = \textbf{min } \{\mathbf{a}, \mathbf{b}\}$ $\textbf{and join } \{\mathbf{a}, \mathbf{b}\} = \textbf{max } \{\mathbf{a}, \mathbf{b}\}$	
H	7	Define lattice. Determine whether poset $(\{1,2,3,4,5\}, \mid)$ is a lattice. $\textbf{Answer : no, } (\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}, \mid) \textbf{ is not a lattice}$	

PART-III RECURSION

❖ RECURSION

- ✓ Suppose that n is a natural number. We often defines n! as $n! = n \times (n-1) \times \ldots \times 2 \times 1$. Sometimes, it is difficult to define a computation explicitly and it is easy to define it in terms of itself, that is, recursively.
- ✓ Recursion is an elegant and powerful problem solving technique, used extensively in both discrete mathematics and computer science. We can use recursion to define sequence, functions, sets, algorithms and many more.

❖ RECURRENCE RELATION

- ✓ A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more preceding terms of the sequence like $a_0, a_1, \ldots, a_{n-1}$.
- ✓ Example: Consider the following instructions for generating a sequence.
  - ➢ Start with 1.
  - ➢ Given a term of sequence, get the next term by adding two to it.
  - ➢ If we generate terms of sequence with these two rules, we obtain

$$1, 3, 5, 7, \ldots$$

  - ➢ If we denote the sequence as $a_1, a_2, a_3, a_4, \ldots$
  - ➢ Then, we can rephrase instruction as, $a_n = a_{n-1} + 2$; $n \geq 2$ and $a_1 = 1$.
  - ➢ The equation $a_n = a_{n-1} + 2$; $n \geq 2$ is an recurrence relation and $a_1 = 1$ is an initial condition.

❖ SOLVING RECURRENCE RELATION

- ✓ Solving recurrence relation means finding an explicit formula for a_n (as $f(n)$).
- ✓ The following are methods for this.
  - ➢ Generating function
  - ➢ Undetermined coefficients

❖ GENERATING FUNCTION METHOD

✓ Example: Let $a_n - 3a_{n-1} = 2 \,; n \geq 1$ with $a_0 = 1$.

- Given recurrence relation is $a_n - 3a_{n-1} = 2$.
- Multiplying both sides by z^n, we obtain

$$a_n z^n - 3a_{n-1}z^n = 2z^n$$

- Since $n \geq 1$, summing for all n. we get,

$$\sum_{n=1}^{\infty} a_n z^n - 3\sum_{n=1}^{\infty} a_{n-1}z^n = 2\sum_{n=1}^{\infty} z^n \quad \ldots\ldots \quad (1)$$

- Let $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

$$\sum_{n=1}^{\infty} a_n z^n = a_1 z + a_2 z^2 + a_3 z^3 + \cdots = A(z) - a_0$$

$$\sum_{n=1}^{\infty} a_{n-1} z^n = z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} = z(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots) = zA(z)$$

$$\sum_{n=1}^{\infty} z^n = z + z^2 + z^3 + \cdots = \frac{z}{1-z}$$

- Substituting values in equation (1). We get,

$$[A(z) - a_0] - 3zA(z) = 2\frac{z}{1-z}$$

$$\Longrightarrow (1-3z)A(z) = \frac{2z}{1-z} + a_0$$

$$\Longrightarrow (1-3z)A(z) = \frac{2z}{1-z} + 1 \qquad (\because \text{given that } a_0 = 1)$$

$$\Longrightarrow A(z) = \frac{2z}{(1-z)(1-3z)} + \frac{1}{(1-3z)}$$

$$\Longrightarrow A(z) = \frac{1+z}{(1-z)(1-3z)}$$

$$\Longrightarrow A(z) = \frac{2}{(1-3z)} - \frac{1}{(1-z)}$$

$$\Longrightarrow A(z) = 2[1 + (3z) + (3z)^2 + (3z)^3 + \cdots] - [1 + z + z^2 + z^3 + \cdots]$$

$$\Longrightarrow A(z) = 2\sum_{n=0}^{\infty}(3)^n z^n - \sum_{n=0}^{\infty} z^n$$

$$\Longrightarrow a_n = 2(3)^n - 1 \,;\, n \geq 0$$

- Which is required solution of the given recurrence relation.

❖ UNDETERMINED COEFFICIENTS METHOD

✓ Example: Let $a_n - 3a_{n-1} = 2\,; n \geq 1$ with $a_0 = 1$.

- Here, the total solution a_n is given by $a_n = a_n^{(h)} + a_n^{(P)}$.

Part-I $\left(\text{how to find } a_n^{(h)}\right)$

- The characteristic equation is $(\lambda - 3) = 0$.

$$\Longrightarrow \lambda = 3$$

$$\Longrightarrow a_n^{(h)} = C_1(3)^n$$

$a_n^{(h)}$ according to λ	
λ	$a_n^{(h)}$
$\lambda_1 \neq \lambda_2 \neq \lambda_3$	$C_1(\lambda_1)^n + C_2(\lambda_2)^n + C_3(\lambda_3)^n$
$\lambda_1 = \lambda_2 = \lambda_3$	$(C_1 + C_2 n + C_3 n^2)(\lambda_1)^n$

Part-II $\left(\text{how to find } a_n^{(P)}\right)$

- Here, $f_n = 2$.
- Therefore, particular solution is of the type $f_n =$ constant.
- Hence, we will consider $a_n^{(P)} = P_0$.

$$\Longrightarrow a_{n-1} = P_0$$

- Substituting these values in given recurrence relation. We get,

$$P_0 - 3P_0 = 2$$

$$\Longrightarrow -2P_0 = 2$$

$$\Longrightarrow P_0 = -1$$

- So, $a_n^{(P)} = P_0 = -1$.
- Hence, required solution is $a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = C_1(3)^n - 1$$

$a_n^{(P)}$ according to f_n	
f_n	$a_n^{(P)}$
constant	P_0
$a + bn + cn^2 + dn^3 + \cdots$	$P_0 + P_1 n + P_2 n^2 + P_3 n^3 + \cdots$
ab^n $(b \neq \lambda)$	$P_0 b^n$
ab^n $(b = \lambda$ with multiplicity m$)$	$P_0 n^m b^n$

Part-III (how to find constants using given conditions)

- We are given that $a_0 = 1$.
- So, $a_n = C_1(3)^n - 1$

  $\Longrightarrow a_0 = C_1(3)^0 - 1$

  $\Longrightarrow 1 = C_1 - 1$

  $\Longrightarrow C_1 = 2$
- Hence, required solution is $a_n = 2(3)^n - 1$.

METHOD-6: EXAMPLES ON RECURRENCE RELATION

H	1	Solve the recurrence relation using the method of generating function $$a_n = 3a_{n-1}\ ;n \geq 1.$$ **Answer :** $\mathbf{a_n = (3)^n a_0\ ;n \geq 0}$	
H	2	Solve the recurrence relation using the method of generating function $$a_n + 2a_{n-1} - 15a_{n-2} = 0\ ;n \geq 2, a_0 = 0, a_1 = 1.$$ **Answer :** $\mathbf{a_n = \frac{1}{8}(3)^n - \frac{1}{8}(-5)^n\ ;n \geq 0}$	
C	3	Solve the recurrence relation using the method of generating function $$a_n - 3a_{n-1} = 2\ ;n \geq 1, a_0 = 1.$$ **Answer :** $\mathbf{a_n = 2(3)^n - 1\ ;n \geq 0}$	
C	4	Solve the recurrence relation using the method of generating function $$a_n - 5a_{n-1} + 6a_{n-2} = 3^n\ ;n \geq 2, a_0 = 0, a_1 = 2.$$ **Answer :** $\mathbf{a_n = (3)^n\left(\frac{1}{2} - (2)^n + \frac{1}{2}(3)^n\right) + 2((3)^n - (2)^n)\ ;n \geq 0}$	
C	5	Solve the following recurrence relation using the method of undetermined coefficients. a) $a_n = 2a_{n-1} - a_{n-2}\ ;a_1 = 1.5, a_2 = 3$ b) $a_n = 3a_{n-1} - 2a_{n-2}\ ;a_1 = -2, a_2 = 4$ c) $a_n - 7a_{n-1} + 10a_{n-2} = 0\ ;a_0 = 0, a_1 = 3$ d) $a_n - 4a_{n-1} + 4a_{n-2} = 0\ ;a_0 = 1, a_1 = 6$ e) $a_n + 2a_{n-1} - 15a_{n-2} = 0\ ;a_0 = 0, a_1 = 1$ **Answer :** $\mathbf{a_n = (1.5n)(1)^n}$, $\mathbf{a_n = (-8)(1)^n + 3(2)^n}$, $\mathbf{a_n = (5)^n - (2)^n}$, $\mathbf{a_n = (1 + 2n)(2)^n}$, $\mathbf{a_n = \frac{1}{8}(3)^n - \frac{1}{8}(-5)^n}$	
C	6	Solve the recurrence relation $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 2^n\ ;n \geq 3, a_0 = a_1 = 0, a_2 = 2$ using the method of undetermined coefficients. **Answer :** $\mathbf{a_n = \left(-\frac{1}{8} + \frac{1}{4}n\right)(-2)^n + (2)^{(n-3)}}$	

H	7	Solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2} + 5^n\,; n \geq 2, a_0 = -2, a_1 = 1$ using the method of undetermined coefficients. **Answer :** $\mathbf{a_n = -\frac{17}{24}(-1)^n - \frac{27}{8}(3)^n + \frac{25}{12}(5)^n}$	
C	8	Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$ using the method of undetermined coefficients. **Answer :** $\mathbf{a_n = C_1(-2)^n + C_2(-3)^n + \frac{115}{228} + \frac{17}{24}n + \frac{1}{4}n^2}$	
C	9	Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = n + 2^n\,; n \geq 2, a_0 = 1, a_1 = 1$ using the method of undetermined coefficients. **Answer :** $\mathbf{a_n = \frac{17}{4}(3)^n - 5(2)^n + \frac{7}{4} + \frac{1}{2}n - 2n(2)^n}$	
H	10	Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = n + 3^n$ using undetermined coefficients method. **Answer :** $\mathbf{a_n = (C_1 + C_2 n)(2)^n + 4 + n + 9(3)^n}$	

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