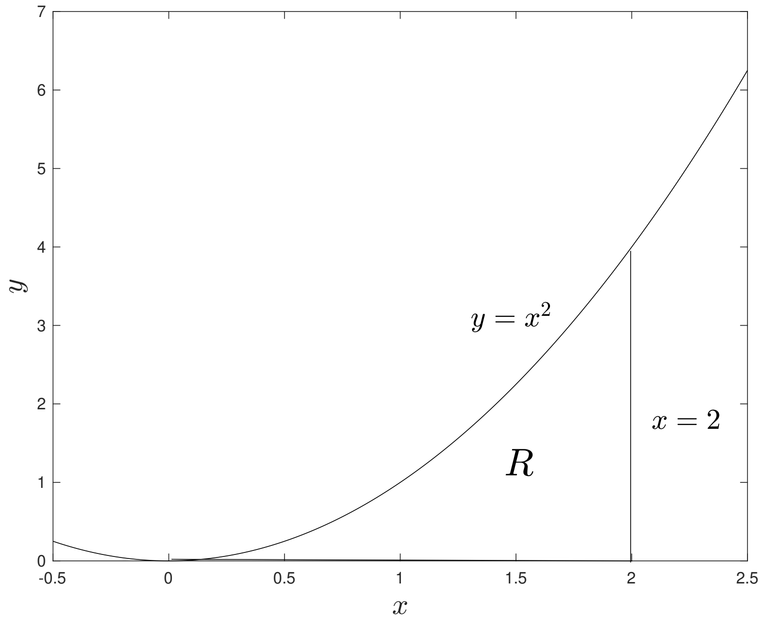
Module 7 self-assessment

Question 1

Compute

$$\iint_R xy\mathrm{d}A$$

when R is the region under the curve $y = x^2$ from $(0,0)$ to $(2,4)$.



The region of integration for question 1.

Solution:

This is a straightforward double integral where the particular choice of integration order is not that critical. Suppose we go for

$$\iint_R xy\mathrm{d}A = \iint_R xy\mathrm{d}y\mathrm{d}x,$$

slicing normally to the x axis, beginning at the leftmost of R: the origin where $x = 0$ up until the rightmost at $x = 2$. All these vertical line integrals in between these two limits will start at the horizontal axis and will extend upwards in increasing y direction, until the curve $y = x^2$ is reached. Therefore the limits of the iterated integral are

$$\begin{aligned}\iint_R xy\mathrm{d}y\mathrm{d}x &= \int_0^2 \int_0^{x^2} xy\mathrm{d}y\mathrm{d}x \\ &= \int_0^2 x\left[\frac{y^2}{2}\right]_0^{x^2} \mathrm{d}x \\ &= \frac{1}{2}\int_0^2 x^5\mathrm{d}x = \frac{1}{2}\left[\frac{x^6}{6}\right]_0^2 = \frac{64}{12}\end{aligned}$$

Question 2

Evaluate the integral

$$\int_0^1 \int_{3y}^3 \mathrm{e}^{2x}\mathrm{d}x\mathrm{d}y$$

after reversing the order of integration.

Hint: You may use the single variable integral

$$\int x\mathrm{e}^{ax}\mathrm{d}x = \frac{\mathrm{e}^{ax}}{2}(x - \frac{1}{a}).$$

Solution:

From the definitions of the limits it is easy to see that R is the region where

$$R:\ 0 \le y \le 1, \quad 3y \le x \le 3.$$

That is R is the triangular region below the line $x = 3y$ (which we can also write as $y = x/3$) going through the origin and the vertical line $x = 3$. Visually, this gives a triangle with vertices at $(0, 0)$, $(3, 0)$ and $(3, 1)$. In the existing integration order we are slicing normally to the y axes, so reversing that we get

$$\begin{aligned}\int_0^1 \int_{3y}^3 \mathrm{e}^{2x}\mathrm{d}x\mathrm{d}y &= \int_0^3 \int_0^{x/3} \mathrm{e}^{2x}\mathrm{d}y\mathrm{d}x \\ &= \int_0^3 \mathrm{e}^{2x}\left[y\right]_0^{x/3}\mathrm{d}x \\ &= \frac{1}{3}\int_0^3 x\mathrm{e}^{2x}\mathrm{d}x \\ &= \frac{1}{3}\left[\frac{\mathrm{e}^{2x}}{2}(x - \frac{1}{2})\right]_0^3 = \frac{1}{3}\left(\frac{5\mathrm{e}^6}{4} + \frac{1}{4}\right)\end{aligned}$$

Question 3

Compute the integral

$$\iint_R y \mathrm{d}y \mathrm{d}x,$$

where R is the region bounded from below by the straight line $y = 2x$ and above the parabola $y = 3 - x^2$.

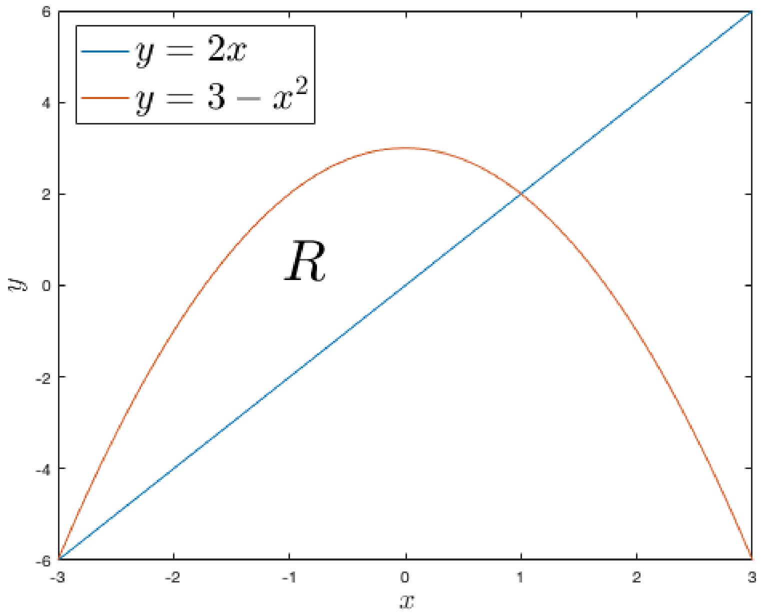


Figure 1: The region of integration in question 3.

Solution:

A sketch of the region R will show that this is bounded from above by the parabola and from below by the straight line. The outer limits of R on the x axis are given by the two points where the two curves intersect,

$$2x = 3 - x^2 \implies x^2 + 2x - 3 = 0 = (x + 3)(x - 1)$$

so the leftmost point of R in x is $x = -3$ and $x = 1$ giving the outer limits. For the inner limits for $\mathrm{d}y$ we note that since we are slicing normally to the x axis, these slices will start from the line $y = 2x$ (the lower bound of R) and will extend up to the parabola

$y = 3 - x^2$ (the upper bound of R). The iterated integral is therefore

$$\begin{aligned}
\iint_R y \mathrm{d}y \mathrm{d}x &= \int_{-3}^{1} \int_{2x}^{3-x^2} y \mathrm{d}y \mathrm{d}x \\
&= \int_{-3}^{1} \frac{1}{2} \Big[y^2 \Big]_{2x}^{3-x^2} \mathrm{d}x \\
&= \frac{1}{2} \int_{-3}^{1} \Big((3 - x^2)^2 - 4x^2 \Big) \mathrm{d}x \\
&= \frac{1}{2} \int_{-3}^{1} (9 - 10x^2 + x^4) \mathrm{d}x \\
&= \frac{1}{2} \Big[9x - \frac{10}{3} x^3 + \frac{x^5}{5} \Big]_{-3}^{1} = -\frac{64}{15}.
\end{aligned}$$