Module 8 self-assessment

Question 1

Evaluate the double integral

$$\iint\limits_R e^{\frac{y-x}{y+x}} \mathrm{d}A$$

over the region R within a triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$.

Solution:

Changing to better coordinates using the transformation $u = y - x$ and $v = y + x$ the integration element on the uv plane is

$$\mathrm{d}u\mathrm{d}v \approx |\mathbf{J}|\mathrm{d}A = |\mathbf{J}|\mathrm{d}x\mathrm{d}y,$$

where the absolute value of the determinant of the mapping from (x, y) to (u, v) is

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = |-2| = 2,$$

hence substituting into the integral yields

$$\iint\limits_R e^{\frac{y-x}{y+x}} \mathrm{d}x\mathrm{d}y = \frac{1}{2} \iint\limits_{R'} e^{\frac{u}{v}} \mathrm{d}u\mathrm{d}v$$

where R' is the mapping of the triangle R on the uv plane which takes the form of another triangle (of double the area!), since the vertices of R are mapped as $(0,0) \to (0,0)$, $(0,1) \to (1,1)$ and $(1,0) \to (-1,1)$. From the drawing of R' it is easy to see that its non horizontal sides are on the lines $u = v$ and $u = -v$, thus the inner integral is

$$\frac{1}{2} \int_{-v}^{v} e^{\frac{u}{v}} \mathrm{d}u = \frac{1}{2} \left[v e^{\frac{u}{v}} \right]_{-v}^{v} = \frac{1}{2} v(e - e^{-1}),$$

and therefore the outer integral is

$$\frac{1}{2} \int_0^1 v(e - e^{-1}) \mathrm{d}v = \frac{1}{4}(e - e^{-1}).$$

Question 2

Using double integrals show that the geometric centre of a triangle with vertices $(0,0)$, $(1,3)$ and $(3,-1)$ has coordinates $\left(\frac{4}{3}, \frac{2}{3}\right)$.

Solution:

The geometric centre of the triangle R can be expressed as

$$\bar{x} = \frac{1}{|R|}\iint\limits_R x\mathrm{d}A, \quad \bar{y} = \frac{1}{|R|}\iint\limits_R y\mathrm{d}A,$$

as $\bar{x}$ and $\bar{y}$ are the average values of the x and y values of the points in the closure of R. The area of the triangle can be computed in a number of ways, such as the absolute value of the determinant

$$|R| = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1 & 3 & -1 \end{vmatrix} = |-5| = 5.$$

Notice that the three sides of the triangle R have equations $y = 3x$, $y = 5 - 2x$ and $y = -\frac{1}{3}x$, so we must setup the double integral in two parts: one for $0 \leq x \leq 1$ where y in R is bounded by the two sides $y = 3x$ (above) and $y = -\frac{1}{3}x$ (below) and then for $1 \leq x \leq 3$ where y in R is bounded by the pair of lines $y = 5 - 2x$ (above) and $y = -\frac{1}{3}x$ (below).

The geometric centres are thus

$$\begin{aligned}
\bar{x} &= \frac{1}{|R|}\int_0^1 \int_{-\frac{1}{3}x}^{3x} x\mathrm{d}y\mathrm{d}x + \frac{1}{|R|}\int_1^3 \int_{-\frac{1}{3}x}^{5-2x} x\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{5}\int_0^1 x(3x + \frac{1}{3}x)\mathrm{d}x + \frac{1}{5}\int_1^3 (5x - \frac{5}{3}x^2)\mathrm{d}x \\
&= \frac{1}{5}\Big(\frac{10}{3}\Big[\frac{x^3}{3}\Big]_0^1 + \Big[\frac{5}{2}x^2 - \frac{5}{3}\frac{x^3}{3}\Big]_1^3\Big) = \frac{4}{3},
\end{aligned}$$

and

$$\begin{aligned}
\bar{y} &= \frac{1}{|R|}\int_0^1 \int_{-\frac{1}{3}x}^{3x} y\mathrm{d}y\mathrm{d}x + \frac{1}{|R|}\int_1^3 \int_{-\frac{1}{3}x}^{5-2x} y\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{5}\int_0^1 \frac{1}{2}(9x^2 - \frac{x^2}{9})\mathrm{d}x + \frac{1}{5}\int_1^3 \frac{1}{2}(4x^2 - \frac{x^2}{9} - 20x + 25)\mathrm{d}x \\
&= \frac{1}{10}\Big(\Big[3x^3 - \frac{1}{9}\frac{x^3}{3}\Big]_0^1 + \Big[\frac{4}{3}x^3 - \frac{x^3}{27} - 10x^2 + 25x\Big]_1^3\Big) = \frac{2}{3}.
\end{aligned}$$

Question 3

Find the centre of mass of a unit disk centred at the origin, if its right half is twice as dense as the left half. Give your answer parametrically in terms of the density ρ, taking for example the density on the left as ρ and on the right as 2ρ.

Solution:

By symmetry, it is clear that the centre of mass will be somewhere on the x-axis, and so $\bar{y} = 0$. Since the right half is twice as dense as the left, we can take the density profile over the disk as

$$\rho(x, y) = \begin{cases} \rho & x \leq 0 \\ 2\rho & x > 0 \end{cases},$$

where ρ is some positive constant. Using the centre of mass formula

$$\bar{x} = \frac{1}{m} \iint\limits_R x\rho(x, y)\mathrm{d}A,$$

where the mass is the integral of the density function over the area of the disk

$$m = \iint\limits_R \rho(x, y)\mathrm{d}A.$$

To work out these integrals we split the domain in two homogeneous semi-disks $R = R_l \cup R_r$ where R_l denotes the left part of the disk with density ρ, and R_r the right, denser part whose density is 2ρ. Effectively the integral of the mass becomes

$$\begin{aligned} m = \iint\limits_R \rho(x, y)\mathrm{d}A &= \iint\limits_{R_l} \rho(x, y)\mathrm{d}A + \iint\limits_{R_r} \rho(x, y)\mathrm{d}A \\ &= \rho \iint\limits_{R_l} \mathrm{d}A + 2\rho \iint\limits_{R_r} \mathrm{d}A \\ &= \rho|R_l| + 2\rho|R_r| = \rho\frac{|R|}{2} + 2\rho\frac{|R|}{2} = \frac{3}{2}\rho|R| = \frac{3}{2}\pi\rho, \end{aligned}$$

where $|R|$ is the area of R. Similarly the other double integral involved in $\bar{x}$ can be computed by splitting over the two regions as

$$\begin{aligned} \iint\limits_R \rho(x, y)\mathrm{d}A &= \iint\limits_{R_l} x\rho(x, y)\mathrm{d}A + \iint\limits_{R_r} x\rho(x, y)\mathrm{d}A \\ &= \rho \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^1 r^2 \cos\theta \,\mathrm{d}r\mathrm{d}\theta + 2\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^2 \cos\theta \,\mathrm{d}r\mathrm{d}\theta \\ &= \rho \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\frac{r^3}{3}\right]_0^1 \cos\theta\mathrm{d}\theta + 2\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^1 \cos\theta\mathrm{d}\theta \\ &= \frac{\rho}{3}\Big[\sin\theta\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \frac{2\rho}{3}\Big[\sin\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{2\rho}{3} + \frac{4\rho}{3} = \frac{2\rho}{3}. \end{aligned}$$

Substituting into the definition of $\bar{x}$ yields

$$\bar{x} = \frac{2}{3\pi\rho}\frac{2\rho}{3} = \frac{4}{9\pi}.$$