Module 9 self-assessment

Question 1

Using only line integration derive the area of the circle with radius a.

Solution:

The area of the circle as a double integral is

$$|R| = \iint\limits_R \mathrm{d}A, \quad \text{where} \quad R : x^2 + y^2 \leq a^2.$$

where of course $|R| = \pi a^2$. Through Green's theorem on the plane, this double integral can be expressed as a line integral over the closed loop $c : x^2 + y^2 = a^2$, if its direction is taken counterclockwise. Since

$$\oint_c \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint\limits_R \operatorname{curl} \mathbf{F} \mathrm{d}A,$$

then to match

$$|R| = \iint\limits_R \operatorname{curl} \mathbf{F} \mathrm{d}A,$$

we require $\operatorname{curl} \mathbf{F} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 1$. Let for example $\mathbf{F} = x\hat{\mathbf{j}}$, (although many other options are possible)

$$\begin{aligned}
|R| &= \oint_c \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= \oint x \mathrm{d}y \\
&= \int_0^{2\pi} a^2 \cos^2 \theta \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{a^2}{2} (\cos 2\theta + 1) \mathrm{d}\theta \\
&= \frac{a^2}{2} \Big[\sin\theta \cos\theta + \theta \Big]_0^{2\pi} = a^2 \pi.
\end{aligned}$$

Question 2

Consider two unit circles centred at $(0,0)$ and $(0,1)$ respectively. Let us denote the first one by L for 'lower' and the later as H for 'higher'. A path c is formed by the arc of the L circle inside H connected with the arc of H inside L, in anticlockwise direction. Setup two iterated integrals for the flux of $\mathbf{f}(x,y) = 3x^2y\hat{\mathbf{i}} + xy\hat{\mathbf{j}}$ through this c and find their appropriate limits. You do not need to solve it.

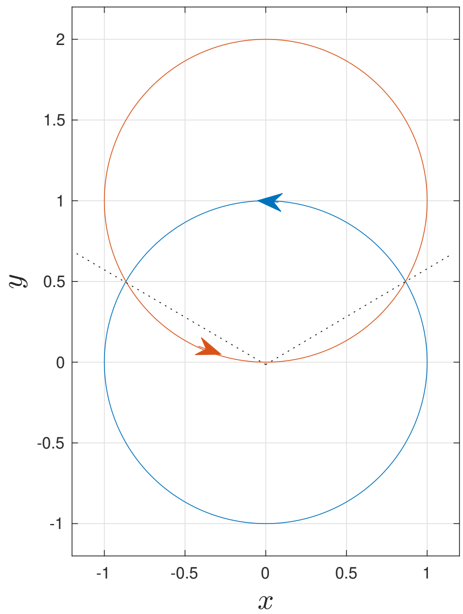

Figure: Integration path for question 2.

Solution:

As $\mathbf{f}$ is continuous, c closed and anticlockwise Green's for flux gives

$$\oint_c \mathbf{f}\cdot\hat{\mathbf{n}}\mathrm{d}s = \iint_R \nabla\cdot\mathbf{f}\mathrm{d}A,$$

where

$$\nabla\cdot\mathbf{f} = 6xy + x$$

Effectively, we have

$$\iint_R (6xy+x)\mathrm{d}A = 6\iint_R xy\mathrm{d}A + \iint_R x\mathrm{d}A.$$

From geometry the second integral on the right hand side above can be shown to be *zero* since $\bar{x} = 0$ in R, the region enclosed by c.

Since we are dealing with circles it is better to use polar coordinates, but take care because some of the quantities we will change are lines/curves and some are areas. That is

- The L circle is $r = 1$ because we are only changing the points points on L (not inside it) from Cartesian to polar.
- The H circle is $r = 2\sin\theta$, again we fix the radius to 1, as we are only looking at the surface of the H circle (not its interior region).
- To change x and y in the integrand for the double integration inside the whole region R we must use $x = r\cos\theta$ and $y = r\sin\theta$ as r and θ both vary in the interior of that region.

There is also a key observation when it comes to find the limits for r and θ inside R: One has to spit the double integral at the dotted lines of the figure, as the upper bound for r switches from H before the left dotted line, and then to L until the right dotted line, and then back to H. In effect the double integrals above become

$$\begin{aligned}\iint_R (6xy + x)\mathrm{d}A = &\int_0^{\frac{\pi}{6}} \int_0^{2\sin\theta} (6r^2 \sin\theta\cos\theta + r\cos\theta) r\mathrm{d}r\mathrm{d}\theta \\ &+ \int_{\frac{5\pi}{6}}^{\pi} \int_0^{2\sin\theta} (6r^2 \sin\theta\cos\theta + r\cos\theta) r\mathrm{d}r\mathrm{d}\theta \\ &+ \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_0^{1} (6r^2 \sin\theta\cos\theta + r\cos\theta) r\mathrm{d}r\mathrm{d}\theta\end{aligned}$$

where the limits for $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ at the dotted lines can be found by equating the equations of H and L as $1 = 2\cos\theta$ and solve for $\theta = \arcsin\frac{1}{2} = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$.