Engineering Mathematics 2B

Module 10: Triple integrals

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Module 10 contents

Motivation

Theory
- Triple integrals
- Applications
- Methodology
- Cylindrical coordinates

Outcomes

Motivation:

Triple integrals occur when trying to compute:

1. The volume and mass of arbitrary 3D bodies.
2. The average value of a function over a 3D region.
3. The centre of mass and geometric centre of a 3D body.
4. The moment of inertia.

Triple integral

Consider R to be a 3D body of finite dimensions, and let $f(x, y, z)$ a function that is defined everywhere in R, then a triple (volume) integral has the form

$$I = \iiint\limits_R f(x, y, z)\, \mathrm{d}V$$

where $\mathrm{d}V$ is the volume element. In Cartesian coordinates this is a tiny cube

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = \mathrm{d}x\mathrm{d}z\mathrm{d}y = \mathrm{d}y\mathrm{d}z\mathrm{d}x$$

Geometrically, they correspond to volumes of arbitrary, closed 3D regions[1].

[1] Recall that the double integrals are volumes between a function and the horizontal plane.

Application 1: Volume and mass of a body

If R is a 3D body then the volume of R is given by

$$|R| = \iiint\limits_R \mathrm{d}V$$

Assuming that R has a density function $\rho(x, y, z)$ then

$$\rho = \frac{\Delta M}{\Delta V} \quad \Rightarrow \quad \mathrm{d}M = \rho \mathrm{d}V$$

The mass of R is a volume integral

$$M = \iiint\limits_R \rho \, \mathrm{d}V.$$

Application 2: Average value and centre of mass

If f is a continuous function then its average value over R is

$$\bar{f} = \frac{1}{|R|} \iiint_R f \, \mathrm{d}V,$$

while the ρ-weighted average of f is

$$\bar{f}_\rho = \frac{1}{M} \iiint_R \rho \, f \, \mathrm{d}V,$$

where M is the integral of the weight function ρ over R and $|R|$ the volume.

The centre of mass of R with density ρ has coordinates $(\bar{x}, \bar{y}, \bar{z})$ given by

$$\bar{x} = \frac{1}{M} \iiint_R \rho \, x \, \mathrm{d}V, \quad \bar{y} = \frac{1}{M} \iiint_R \rho \, y \, \mathrm{d}V, \quad \bar{z} = \frac{1}{M} \iiint_R \rho \, z \, \mathrm{d}V.$$

The geometric centre is given as above with $\rho = 1$ (or any positive constant).

Application 3: Moment of inertia

A body's moment of inertia with respect to an axis is defined as its 'resistance to rotate about the axis' under the influence of a force acting at a point on the body. Let z-axis be the axis of rotation then the MoI is

$$I_z = \iiint\limits_R \underbrace{(x^2 + y^2)}_{\text{in Cartesian}} \rho \, \mathrm{d}V = \iiint\limits_R \underbrace{r^2}_{\text{in cylindrical}} \rho \, \mathrm{d}V,$$

For the remaining two axes we have

$$I_x = \iiint\limits_R (y^2 + z^2) \rho \, \mathrm{d}V, \quad I_y = \iiint\limits_R (x^2 + z^2) \rho \, \mathrm{d}V$$

Methodology

To solve triple integrals

(1) Draw a schematic of R

(2) Choose the integration order and write the triple integral as three iterated integrals in the respective coordinates.

(3) Starting from the inner find the limits of R for that coordinate.

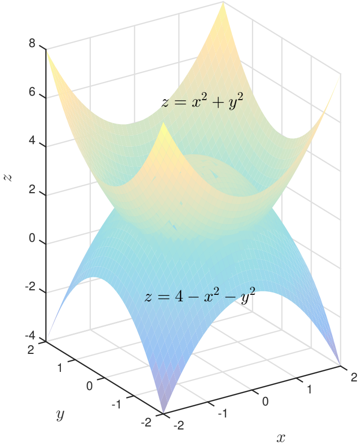
(4) Project R onto the plane of the remaining two coordinates to see its 2D shadow.

(5) Setup the limits for the middle and outer integrals by considering a double integral for the shadow.

(6) Solve the three integrals from inside out.

Example

Find the volume of a 3D region R between the paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$, casting it as a triple integral.



Example

The triple integral for the volume of R is

$$|R| = \iiint_R \mathrm{d}V,$$

and we have to set it up as an iterated integral.

In Cartesian coordinates

$$|R| = \int_?^? \int_?^? \int_?^? \mathrm{d}z\mathrm{d}y\mathrm{d}x$$

opting for $\mathrm{d}V = \mathrm{d}z\mathrm{d}y\mathrm{d}x$, so the innermost integral is for $\mathrm{d}z$.

Notice that the z coordinate of any point within R is **bounded from below** by $z = x^2 + y^2$ and **above** by $z = 4 - x^2 - y^2$, hence the inner set of limits are readily available.

Example

Effectively, within R then $x^2 + y^2 \leq z \leq 4 - x^2 - y^2$,

$$|R| = \int_{?}^{?} \int_{?}^{?} \int_{x^2+y^2}^{4-x^2-y^2} \mathrm{d}z\mathrm{d}y\mathrm{d}x$$

To find the other two sets of limits (for x and y) we must project R onto the xy plane. This projection is also known as the **shadow** of R.

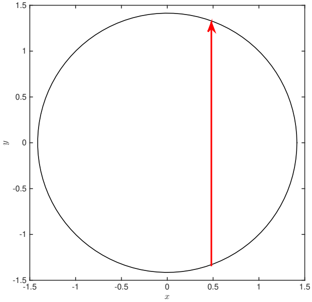
The shadow of R on xy is a disk, centred at the origin. Setting $z = 0$ in the definition of the paraboloids yields the circle at the boundary of the shadow[2]

$$x^2 + y^2 = 4 - x^2 - y^2 \;\Rightarrow\; x^2 + y^2 = 2.$$

[2] In polar coordinates the sought circle is $r^2 = 2$.

Example

For the shadow of R we set up a **double integral** on the xy plane.



For the remaining integration order dydx this yields

$$|R| = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} \mathrm{d}z\mathrm{d}y\mathrm{d}x$$

Looks complicated to evaluate...

Cylindrical coordinates

Equations of circles in the limits is a sign that polar coordinates might be better suited to the integration.

On the other hand we have a simple (given) bound for z, e.g. the definitions of the two paraboloids forming R

Key idea: Keep the z Cartesian coordinate and switch to polar coordinates on the xy plane.

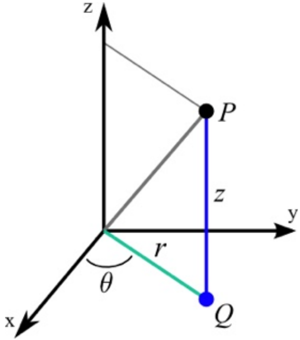
This arrangement yields a new set of coordinates (r, θ, z) known as **cylindrical coordinates**, that relate to Cartesian by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

thus in cylindrical coordinates the volume element is

$$\boxed{\mathrm{d}V = r\,\mathrm{d}z\mathrm{d}r\mathrm{d}\theta = \mathrm{d}x\mathrm{d}y\mathrm{d}z}$$

Cylindrical coordinates



The plane above extends also to negative z. Note that since the radius is always non-negative $r \geq 0$, the expression $\theta = \theta_0$ defines a half-plane normal to the xy plane.

Example cont.

In switching from Cartesian to Cylindrical the integral of the last example

$$|R| = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

changes to

$$|R| = \int_{0}^{2\pi} \int_{0}^{\sqrt{2}} \int_{r^2}^{4-r^2} r \, \mathrm{d}z \mathrm{d}r \mathrm{d}\theta$$

which is much easier to compute

$$\int_{r^2}^{4-r^2} r \mathrm{d}z = r \Big[z \Big]_{r^2}^{4-r^2} = 4r - 2r^3,$$

$$\int_0^{\sqrt{2}} 4r - 2r^3 \mathrm{d}r = 4\left[\frac{r^2}{2}\right]_0^{\sqrt{2}} - 2\left[\frac{r^4}{4}\right]_0^{\sqrt{2}} = 2, \quad \int_0^{2\pi} 2\mathrm{d}\theta = 4\pi = |R|.$$

Graphs in cylindrical coordinates

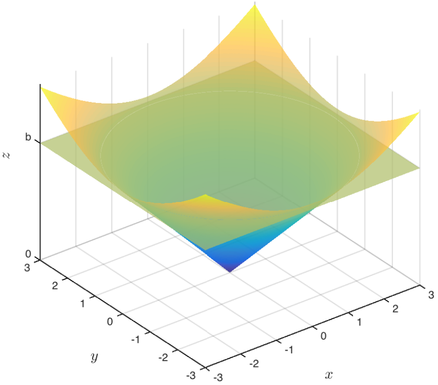
A few points to consider about **cylindrical coordinates**:

The expression $r = a$ in polar coords is a circle on the xy plane, while in cylindrical is a cylinder of radius a and infinite length centred at the origin aligned to the z axes.

The expression $\theta = \theta_0$ in polar coords is a straight line starting at the origin, while in cylindrical it is a half-plane normal to xy plane.

Example

Find the MoI I_z of the homogeneous cone $z = ar$ bounded by the plane $z = b$, with density $\rho = 1$. (a, b are given constants)



Let us use cylindrical coordinates. (cone is a type of cylinder)

Example

Given the bounds $ar \leq z \leq b$ from the equation of the cone, choose $\mathrm{d}V = r\,\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$, and set $\rho = 1$ for homogeneity to get

$$I_z = \iiint_R r^2 \rho \mathrm{d}V = \int_?^? \int_?^? \int_{ar}^{b} r^2\, r\ \mathrm{d}z\mathrm{d}r\mathrm{d}\theta$$

The shadow of the cone R on xy is a disk of some radius r, formed where the cone $z = ar$ meets the horizontal plane $z = b$, hence the middle limits are

$$I_z = \int_?^? \int_0^{\frac{b}{a}} \int_{ar}^{b} r^2\, r\ \mathrm{d}z\mathrm{d}r\mathrm{d}\theta$$

and by the symmetry of the shadow (and R) with respect to the rotation around the z axes, the bounds for θ are

$$I_z = \int_0^{2\pi} \int_0^{\frac{b}{a}} \int_{ar}^{b} r^3\ \mathrm{d}z\mathrm{d}r\mathrm{d}\theta = \ldots \quad = \frac{\pi b^5}{10a^4}.$$

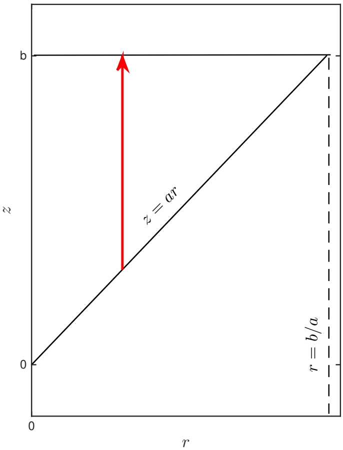
Example cont.

As an extension to this example let us change the order of integration to

$$I_z = \iiint\limits_R r^2 \rho \mathrm{d}V = \int_?^? \int_?^? \int_?^? r^3 \, \mathrm{d}\theta \mathrm{d}z \mathrm{d}r.$$

By symmetry, at any angle θ within $[0, 2\pi]$ the **shadow** of R on the zr half-plane is a **right angle triangle**.

Example cont.



On the zr plane, r is the horizontal axis and z the vertical one.

Example cont.

From the figure we see that by fixing the inner limits at $\theta : 0 \to 2\pi$ gives

$$I_z = \int_?^? \int_?^? \int_0^{2\pi} r^3 \, \mathrm{d}\theta \mathrm{d}z \mathrm{d}r,$$

while looking at the shadow triangle, we can opt to fix the range of r and bound z. At a given r, z varies like $z : ar \to b$, then

$$I_z = \int_0^{\frac{b}{a}} \int_{ar}^{b} \int_0^{2\pi} r^3 \, \mathrm{d}\theta \mathrm{d}z \mathrm{d}r$$

which gives inner and middle integrals as

$$\int_0^{2\pi} r^3 \mathrm{d}\theta = 2\pi r^3, \qquad 2\pi \int_{ar}^{b} r^3 \mathrm{d}z = 2\pi r^3 (b - ar),$$

and an outer integral for the MoI as

$$I_z = 2\pi \int_0^{\frac{b}{a}} r^3 (b - ar) \mathrm{d}r = 2\pi \left[b\frac{r^4}{4} - a\frac{r^5}{5} \right]_0^{\frac{b}{a}} = \frac{\pi b^5}{10a^4}.$$

Formulas

- Cylindrical to Cartesian transform

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

- The volume integration element

$$\mathrm{d}V = r\,\mathrm{d}z\mathrm{d}r\mathrm{d}\theta = \mathrm{d}x\mathrm{d}y\mathrm{d}z \quad \text{(and permutations)}$$

- The centre of mass of R with density ρ has coordinates

$$\bar{x} = \frac{1}{M}\iiint\limits_R \rho\,x\,\mathrm{d}V, \quad \bar{y} = \frac{1}{M}\iiint\limits_R \rho\,y\,\mathrm{d}V, \quad \bar{z} = \frac{1}{M}\iiint\limits_R \rho\,z\,\mathrm{d}V.$$

- Moment of inertia of 3D volume R wrt z axis

$$I_z = \iiint\limits_R (x^2+y^2)\rho\,\mathrm{d}V = \iiint\limits_R r^2\rho\,\mathrm{d}V,$$

Main outcomes of module 10

You MUST know:

1. The applications of triple integrals.
2. To setup and solve triple integrals in Cartesian and cylindrical coordinates.

Good to know:

Equations of the surfaces of some regular shapes like cones, spheres, cylinders and planes.