Module 4 self assessment

Question 1

Let c be the path consisting of the straight line from $(0,0)$ to $(5\sqrt{2},0)$ followed by the arc from $(5\sqrt{2},0)$ to $(0,5\sqrt{2})$ that is part of the circle of radius $5\sqrt{2}$ centred at the origin. Compute the work integral

$$\int_c \mathbf{F}\cdot \mathrm{d}\mathbf{r},$$

(i) for the field $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, and (ii) $\mathbf{F} = x\hat{\mathbf{j}}$.

Solution:

(i) The integration path has two segments, c_1 a straight line on the x axis and c_2 an arc in the positive quadrant. In this case, there is no work over c_2 since the vector field is orthogonal to that curve (with zero tangential component), hence

$$\int_c (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\cdot \mathrm{d}\mathbf{r} = \int_{c_1} (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\cdot \mathrm{d}\mathbf{r}.$$

On c_1 we have $y = 0$ and $\mathrm{d}y = 0$, and x runs from 0 to $5\sqrt{2}$, so

$$\int_{c_1} (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\cdot \mathrm{d}\mathbf{r} = \int_{c_1} (x\mathrm{d}x + y\mathrm{d}y) = \int_0^{5\sqrt{2}} x\mathrm{d}x = 25.$$

(ii) In this case $\mathbf{F}$ is orthogonal to the c_1 path segment hence

$$\int_c \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{c_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{c_2} x\mathrm{d}y$$

Exploiting the shape of c_2 (part of a circle with radius $5\sqrt{2}$) we may choose to parameterise it as $x = 5\sqrt{2}\cos\theta$, $y = 5\sqrt{2}\sin\theta$ with $\theta : 0 \to \frac{\pi}{2}$. Substituting in the integral gives

$$\int_{c_2} x\mathrm{d}y = \int_0^{\frac{\pi}{2}} 5\sqrt{2}\cos\theta\, 5\sqrt{2}\cos\theta \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 50\Big(\frac{1+\cos 2\theta}{2}\Big)\mathrm{d}\theta = \frac{25\pi}{2}.$$

Question 2

If $\mathbf{A} = (2y+3)\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + (yz-x)\hat{\mathbf{k}}$, evaluate $\int_c \mathbf{A}\cdot \mathrm{d}\mathbf{r}$ along c if this is made of the straight lines from (0,0,0) to (0,0,1), then to (0,1,1) and ending at (2,1,1).

Solution:

Let $P = (0,0,0)$, $Q = (0,0,1)$, $R = (0,1,1)$, and $S = (2,1,1)$ then by the linearity of integration we can split the integral for the whole path into three line integrals over straight lines

$$\int_c \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_P^Q \mathbf{A}\cdot \mathrm{d}\mathbf{r} + \int_Q^R \mathbf{A}\cdot \mathrm{d}\mathbf{r} + \int_R^S \mathbf{A}\cdot \mathrm{d}\mathbf{r}.$$

A graph will show that the segment PQ lies on the z axis, so over this path the integral has

$$x = 0,\ y = 0,\ \mathrm{d}x = 0,\ \mathrm{d}y = 0,$$

and thus

$$\int_P^Q \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_P^Q (yz-x)\mathrm{d}z = \int_0^1 0\mathrm{d}z = 0.$$

Another way to verify this result is to notice that on the z-axis, where $x = y = 0$, the vector field is $\mathbf{A} = (3,0,0)$, i.e. it is constant and normal to both the z and y axis. Since $\mathrm{d}\mathbf{r}$ is tangent to z axis, then we see that the two vectors are orthogonal to each other. Then notice that QR lies horizontally on the zy plane, thus z is fixed on this curve and

$$x = 0,\ \mathrm{d}x = 0,\ z = 1,\ \mathrm{d}z = 0$$

thus

$$\int_Q^R \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_Q^R xz\mathrm{d}y = \int_0^1 0\mathrm{d}y = 0$$

while in RS the curve moves parallel to the x axis, $x : 0 \to 2$ without variation in its y and z coordinates

$$y = 1,\ \mathrm{d}y = 0,\ z = 1,\ \mathrm{d}z = 0$$

yielding

$$\int_R^S \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_R^S (2y+3)\mathrm{d}x = \int_0^2 5\mathrm{d}x = 10.$$