Engineering Mathematics 2B

Module 12: Introduction to Probability

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Module 12 contents

Motivation for studying random event independence

An electronic chip your company develops has been installed in a new model of an autonomous vehicle. The vehicle manufacturer is suing your company claiming that due to a bug in the controller three, preventable, car accidents have occurred.

You need to investigate if:

The chips involved in these accidents passed all internal checks whilst still having a bug, or

The incidents of chip malfunction were caused by external factors, e.g. the car manufacturer not following the installation instructions

If the 3 accidents occurred independently from each other, your company maybe in serious trouble...

Event Independence

Recall that for A and B random events, the intersection rule asserts

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A) = \mathbb{P}(B)\mathbb{P}(A|B),$$

where the second equality holds since $\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A)$.

If A and B are independent then, and only then,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B),$$

since $\mathbb{P}(B|A) = \mathbb{P}(B)$ and $\mathbb{P}(A|B) = \mathbb{P}(A)$.

If A and B are independent conditioned on an event C occurring the above formulas change trivially to

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C)\,\mathbb{P}(B|C)$$

Conditional probability

Let the sample space have three outcomes as $\Omega = \{a, b, c\}$, and define events

$$A: \text{ `}a\text{ occurs'}, \quad B: \text{`}b\text{ occurs'}, \quad C: \text{`}c\text{ occurs'}.$$

The probability of event A conditioned on event B occurring is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad \text{subject to } \mathbb{P}(B) > 0$$

The conditional probability $\mathbb{P}(A|B)$ inherits all the probability axioms. In effect, if A and C disjoint then

$$\begin{aligned} \text{non-negativity} \quad & \mathbb{P}(A|B) \geq 0, \\ \text{additivity} \quad & \mathbb{P}(A \cup C|B) = \mathbb{P}(A|B) + \mathbb{P}(C|B), \\ \text{normalisation} \quad & \mathbb{P}(A|B) + \mathbb{P}(C|B) = 1 \end{aligned}$$

Total probability

Let n **disjoint events** $A_1, \ldots, A_n$ forming a partition of the sample space,

$$\Omega = \{a_1, \ldots, a_n\},$$

where A_i : is the event of outcome a_i occurring and due to the normalisation axiom we have

$$\sum_{i=1}^{n} \mathbb{P}(A_i) = 1.$$

Now define a new event B that **depends** in some way **on** $A_1, \ldots, A_n$.

The **total probability** of event B is

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i \cap B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B|A_i).$$

where $\mathbb{P}(B|A_i)$ is the probability of event B conditioned on A_i occurring.

Total probability example

We roll a fair dice. If the outcome is less than 4 we roll once more otherwise we stop. What is the probability of two even outcomes?

Looking at the possible outcomes yields

$$\begin{aligned}\Omega = \big\{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\&(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\&(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\&(4,\,),(5,\,),(6,\,)\big\}\end{aligned}$$

Notice that having a second roll depends on the outcome of the first roll, hence we choose the events

$$A_i : \text{`roll outcome is equal to } i\text{'}, \quad i = 1, \dots, 6$$

and the desired event B : 'two even outcomes'.

Total probability example

Recall that we roll twice if the first outcome is less than 4, i.e. one of 1, 2, 3, then the probability of two even rolls is

$$\mathbb{P}(B) = \sum_{i=1}^{6} \mathbb{P}(A_i)\mathbb{P}(B|A_i).$$

$\mathbb{P}(A_i) = \frac{1}{6}$ for all i since the dice is fair.

$\mathbb{P}(B|A_4) = \mathbb{P}(B|A_5) = \mathbb{P}(B|A_6) = 0$ since if the outcome of the first roll is 4 or higher there's no second roll.

$\mathbb{P}(B|A_1) = \mathbb{P}(B|A_3) = 0$ since if the outcome of the first roll is odd, there can't be two even outcomes.

This leaves $\mathbb{P}(B) = \mathbb{P}(A_2)\mathbb{P}(B|A_2) = \frac{1}{6}\,\frac{1}{2} = \frac{1}{12}$.

Viewing the solution

Since $\mathbb{P}(B) = \mathbb{P}(A_2)\mathbb{P}(B|A_2)$ then the probability of A_2 is 1 in 6 as per the first column of Ω (arranged as a matrix), and that of $B|A_2$ is 1 in 2 as per the red entries in the second row of Ω.

$$\begin{aligned}\Omega = \Big\{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\ &\boxed{(2,1),\color{red}{(2,2)},(2,3),\color{red}{(2,4)},(2,5),\color{red}{(2,6)}}\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\ &(4,\,)\\ &(5,\,)\\ &(6,\,)\Big\}\end{aligned}$$

Alternative solution

A bit more thinking, a bit less writing.

Define events Z_1 : 'roll outcome is one of $\{1,2,3\}$' and $Z_2 = Z_1^c$: 'roll outcome is one of $\{4,5,6\}$'. As before B : 'two even rolls'.

$$\begin{aligned}
\mathbb{P}(B) &= \sum_{i=1}^{2} \mathbb{P}(Z_i)\mathbb{P}(B|Z_i) \\
&= \mathbb{P}(Z_1)\mathbb{P}(B|Z_1) + \mathbb{P}(Z_2)\mathbb{P}(B|Z_2) \\
&= \mathbb{P}(Z_1)\mathbb{P}(B|Z_1) \\
&= \mathbb{P}(Z_1)\mathbb{P}(\text{roll 1 is even} \,|\, Z_1 \ \cap \ \text{roll 2 is even} \,|\, Z_1) \\
&= \frac{1}{2}\,\frac{1}{3}\,\frac{1}{2} = \frac{1}{12}
\end{aligned}$$

where we used the conditional independence to simplify

$$\mathbb{P}(B|Z_1) = \mathbb{P}(\text{roll 1 is even} \,|\, Z_1 \ \cap \ \text{roll 2 is even} \,|\, Z_1).$$

Example extended

Same rules. Find the probability that the sum of the outcomes is at least 5. Set B : 'sum total of roll(s) is at least 5'. Recall the sample space

$$\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\ (4,),(5,),(6,)\}$$

Like before we have

$$\mathbb{P}(B) = \sum_{i=1}^{6} \mathbb{P}(A_i)\mathbb{P}(B|A_i) = \frac{1}{6}\sum_{i=1}^{6} \mathbb{P}(B|A_i)$$

but now $\mathbb{P}(B|A_1) = \frac{1}{2}$, $\mathbb{P}(B|A_2) = \frac{2}{3}$, $\mathbb{P}(B|A_3) = \frac{5}{6}$, $\mathbb{P}(B|A_4) = 0$, $\mathbb{P}(B|A_5) = 1$, and $\mathbb{P}(B|A_6) = 1$, yielding $\mathbb{P}(B) = \frac{2}{3}$

Conditional probability example: Cancer diagnostics

Tumours in humans are detectable by tomographic imaging with a probability 0.99. Once a tumour is reconstructed the scanner triggers an alarm. The alarm however, may be triggered even when there is no tumour (i.e. false alarm) at a probability 0.10. According to scientific evidence adults over 50 have a probability of cancer equal to 0.05.

Find the probability of false positive, and that of false negative?

Define the following events

$$T : \text{`Tumour present'}, \quad A : \text{`Alarm sound'}$$

We can approach this question using formulas, as well as graphically using a tree diagram.

Example of conditional probability: Cancer screening

From the wording we are given

$$\mathbb{P}(A|T) = 0.99, \quad \mathbb{P}(A|T^c) = 0.10, \quad \mathbb{P}(T) = 0.05$$

and from this information we want:

$$\mathbb{P}(A^c \cap T) \text{ (false negative)} \quad \text{and} \quad \mathbb{P}(A \cap T^c) \text{ (false positive)}.$$

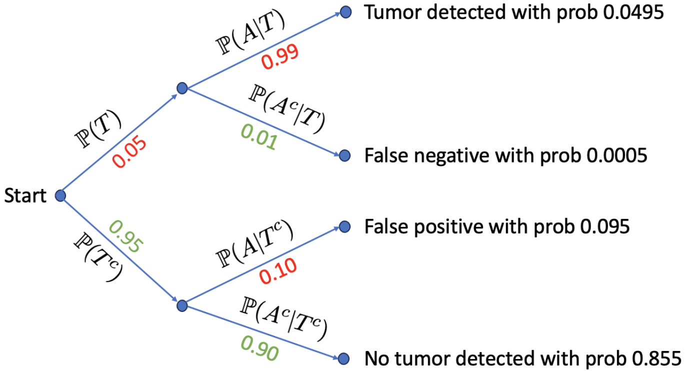
Rearranging the conditional probability formula yields

$$\mathbb{P}(A^c \cap T) = \mathbb{P}(A^c|T)\mathbb{P}(T) = (1 - 0.99) \cdot 0.05 = 0.0005,$$

and similarly

$$\mathbb{P}(A \cap T^c) = \mathbb{P}(A|T^c)\mathbb{P}(T^c) = 0.1 \cdot 0.95 = 0.095$$

Example of conditional probability: Cancer screening



Decision tree diagram: To find the false positive and false negative probabilities we multiply the probabilities of the branches connecting the several events leading to them. The probabilities of the branches leaving a node should sum up to 1.

Bayes' theorem

In the previous example we were looking for probabilities of two events co-occurring, e.g. 'no alarm & tumor' or 'alarm & no tumor'.

Suppose we now pose a different question: 'What is the probability of a tumour being present **if there is** an alarm?'

What we are looking for here is the conditional probability $\mathbb{P}(T|A)$.

Note that this is different to asking 'What is the probability of a tumour **and** an alarm' $\mathbb{P}(T \cap A)$, i.e. detection.

Bayes' theorem

The celebrated **Bayes' theorem** relates the conditional probabilities $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$ with the total probability $\mathbb{P}(B) > 0$ as

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(A)\mathbb{P}(B|A)}{\mathbb{P}(B)}}$$

Typically, the probability $\mathbb{P}(B)$ at the denominator is computed using the total probability formula

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$

Example of Bayes' theorem: Cancer screening

By direct application of Bayes' theorem we have

$$
\begin{aligned}
\mathbb{P}(T|A) &= \frac{\mathbb{P}(T)\mathbb{P}(A|T)}{\mathbb{P}(A)} \\
&= \frac{\mathbb{P}(T)\mathbb{P}(A|T)}{\mathbb{P}(T)\mathbb{P}(A|T) + \mathbb{P}(T^c)\mathbb{P}(A|T^c)} \\
&= \frac{0.05 \cdot 0.99}{0.05 \cdot 0.99 + 0.95 \times 0.10} \\
&\approx 0.3425
\end{aligned}
$$

Notice that $\mathbb{P}(T) = 0.05$ and $\mathbb{P}(A|T) = 0.99$ are directly available from the question wording (or the decision tree).

Care is needed in working the total probability $\mathbb{P}(A)$ at the denominator. The alarm may sound either when there is a tumour or when there is not.

Formulas

- $\mathbb{P}(A|B) \geq 0$
- $\mathbb{P}(A \cup B|C) = \mathbb{P}(A|C) + \mathbb{P}(B|C) - \mathbb{P}(A \cap B|C)$
- $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B|A_i)$
- $$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$
- $$\mathbb{P}(A|B) = \frac{\mathbb{P}(A)\mathbb{P}(B|A)}{\mathbb{P}(B)}$$
- If A and B are conditionally independent

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C)\,\mathbb{P}(B|C)$$

Main outcomes of module 12

You MUST know:

1. The conditional probability of an event.
2. The total probability and Bayes' theorem.
3. To use tree diagrams for finding conditional probabilities.