Module 2 self assessment

Question 1

Find the potential of the conservative field

$$\mathbf{C}(x, y) = (2y - \mathrm{e}^y)\hat{\mathbf{i}} + (2x - x\mathrm{e}^y)\hat{\mathbf{j}}$$

Solution:

Since we are told that $\mathbf{C}$ is conservative we can assume that $\nabla \times \mathbf{C} = 0$ which can be verified using the definition of the 2D curl

$$\nabla \times \mathbf{C} = \frac{\partial}{\partial x}(2x - x\mathrm{e}^y) - \frac{\partial}{\partial y}(2y - \mathrm{e}^y) = 2 - \mathrm{e}^y - (2 - \mathrm{e}^y) = 0.$$

To find the potential we set $\mathbf{C} = \nabla f$ and thus

$$(i) \quad \frac{\partial f}{\partial x} = 2y - \mathrm{e}^y, \qquad (ii) \frac{\partial f}{\partial y} = 2x - x\mathrm{e}^y$$

Following the standard procedure we begin by rearranging (i) and integrating with respect to x

$$f = \int 2y - \mathrm{e}^y \mathrm{d}x = 2xy - x\mathrm{e}^y + g(y),$$

for some function $g(y)$. Differentiating the above with respect to y and setting it equal to (ii) gives

$$\frac{\partial f}{\partial y} = 2x - x\mathrm{e}^y + \frac{\partial g}{\partial y} = 2x - x\mathrm{e}^y.$$

Since the above match for $\partial g/\partial y = 0$ we conclude that g is a constant value and does not depend on y (or x) so

$$f(x, y) = 2xy - x\mathrm{e}^y + c,$$

for some constant c.

Question 2

A point mass m positioned at coordinates (x, y, z) is attracted towards the origin $(0, 0, 0)$ with a force whose magnitude is gmr^{-2} where g is an acceleration-related constant and r is the distance between the point mass and the origin.

(i) Assuming that both m and g are known, derive an expression for the force field $\mathbf{f}$ at any point in the 3D frame apart from the origin, justifying your reasoning.

(ii) Show that, aside the origin, the force field is conservative <u>and</u> derive its potential function ϕ, such that $\mathbf{f} = \nabla\phi$. Hint, you may find the vector calculus identity

$$\nabla \times (\phi\mathbf{a}) = \phi\nabla \times \mathbf{a} + \nabla\phi \times \mathbf{a}$$

useful. Here ϕ is a scalar field, and $\mathbf{a}$ a vector field.

(5)

Solution:

Part (i) Firstly recall the *standard* definitions of the position vector and distance from the origin function in Cartesian coordinates, namely

$$\mathbf{r}(x,y,z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \quad r(x,y,z) = |\mathbf{r}| = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$$

Now since the force attracts the mass from its position towards the origin, hence pointing exactly in the opposite direction of $\mathbf{r}$ then it is not difficult to see that

$$\mathbf{f}(x,y,z) = -\frac{gm}{r^3}\mathbf{r}, \qquad r \neq 0$$

whose magnitude $|\mathbf{f}|$ matches the expression given by the exercise.

Part (ii) To verify $\nabla \times \mathbf{f} = 0$ we can use the vector calculus identity

$$\nabla \times (\phi\mathbf{a}) = \phi\nabla \times \mathbf{a} + \nabla\phi \times \mathbf{a}$$

with $\phi = 1/r^3$, $r \neq 0$ and $\mathbf{a} = \mathbf{r}$ to get

$$\begin{aligned}
\nabla \times \mathbf{f} &= -gm\Big(\frac{1}{r^3}\nabla \times \mathbf{r} + \nabla\frac{1}{r^3} \times \mathbf{r}\Big) \\
&= -gm\Big(\frac{1}{r^3}\nabla \times \mathbf{r} - \frac{3}{r^5}\mathbf{r} \times \mathbf{r}\Big) \\
&= -gm\Big(\frac{1}{r^3}\nabla \times \mathbf{r}\Big)
\end{aligned}$$

since for any nonzero vector field $\mathbf{a}$, we have $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ (this is the zeros vector). Further the curl of the position vector vanishes since

$$\nabla \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \Big(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\Big)\hat{\mathbf{i}} + \Big(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\Big)\hat{\mathbf{j}} + \Big(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\Big)\hat{\mathbf{k}} = \mathbf{0},$$

since x, y and z are independent variables (their respective axes are orthogonal to each other) and hence $\mathbf{f}$ is conservative or irrotational or gradient field everywhere apart from the origin, where the field is not defined.

To derive its potential we can make use of the vector identity $\nabla(r^n) = nr^{n-2}\mathbf{r}$ with $n = -1$ and thus

$$\mathbf{f} = \nabla\phi, \quad \phi = gm\frac{1}{r}$$