Engineering Mathematics 2B

Module 9: Green's theorems

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Module 9 contents

Motivation

Theory

- Conditions for Green's theorems
- Green's theorem for work
- Green's theorem for flux

Outcomes

Motivation:

Heat flow

Conservation of mass

Circulation of fluids

Boundary value problems:

"What's going on in the interior of a region when we observe the phenomena on its boundary?"

Setting

Green's theorems apply when integrating a 2D vector field $\mathbf{F}$ that is continuous inside a closed loop path c with anticlockwise direction.

A vector field $\mathbf{F}$ is continuous inside a region R enclosed by c, if it has a finite magnitude without jump discontinuities at every point (x, y) in R.

Before you choose to apply them you must check that these conditions apply.

There is however a work-around solution for when c has the 'wrong' (clockwise) direction.

Green's theorem for work

If

$$\mathbf{F}(x, y) = f(x, y)\hat{\mathbf{i}} + g(x, y)\hat{\mathbf{j}}$$

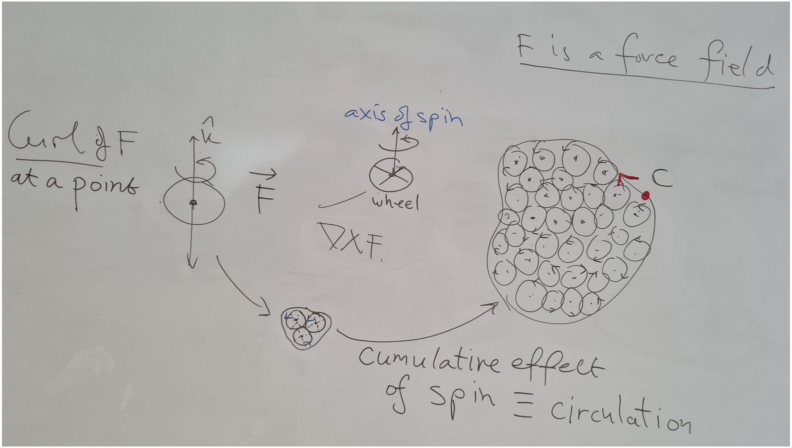
is a 2D continuous vector field then the work of $\mathbf{F}$ on a 2D c with anticlockwise direction is equal in value to the double integral of the magnitude of the curl of $\mathbf{F}$ in the enclosing region R

$$\oint_c \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_c (f\mathrm{d}x + g\mathrm{d}y) = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}A$$

Remember that the curl of a 2D vector field is always aligned to the $\hat{\mathbf{k}}$ direction.

Green's theorem for work

Physical interpretation: From microscopic circulation (curl) to macroscopic circulation (work).



Green's theorem for work

Use: In essence, Green's theorem allows you to choose between a line and a double integral to solve,... whichever you find easier.

Caveat: If c is clockwise then this reverses the sign of the work integral[1], hence

$$\oint_c \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\iint_R \nabla \times \mathbf{F} \, \mathrm{d}A$$

Conservative fields: Notice that the theorem conforms with our definition of conservative fields[2].

$$\text{If } \nabla \times \mathbf{F} = 0 \quad \Longrightarrow \quad \mathbf{F} \text{ is conservative} \quad \Longrightarrow \quad \oint_c \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$$

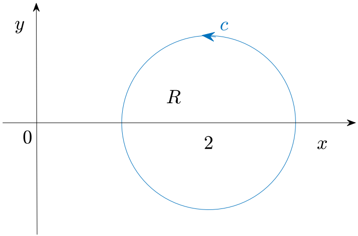
[1] By virtue of the 'reversing the direction of the path' property of line integrals in module 4.

[2] Work in conservative fields in module 4.

Example

Let c be a circle of unit radius centred at $(x, y) = (2, 0)$ with counterclockwise direction. Compute the work integral for $\mathbf{F} = y\mathrm{e}^{-x}\hat{\mathbf{i}} + (\frac{1}{2}x^2 - \mathrm{e}^{-x})\hat{\mathbf{j}}$.

$$\oint_c \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_c y\mathrm{e}^{-x}\mathrm{d}x + \int_c (\frac{1}{2}x^2 - \mathrm{e}^{-x})\mathrm{d}y.$$



Example

Suppose we approach this first as a line integral, ignoring Green's theorem.

As c is a circle, then we better convert to polar coordinates. Every point (x, y) on c can be expressed in terms of (r, θ) using

$$x = 2 + \cos\theta, \quad y = \sin\theta$$

thus $\mathrm{d}x = -\sin\theta \mathrm{d}\theta$, $\mathrm{d}y = \cos\theta \mathrm{d}\theta$, and the line integral becomes

$$\int_0^{2\pi} \frac{1}{2}(2 + \cos\theta)^2 \cos\theta - \mathrm{e}^{-(2+\cos\theta)}(\cos\theta + \sin^2\theta)\mathrm{d}\theta = \ldots ??$$

Example

Converting it into a double integral using Green's theorem for work, since c is closed counterclockwise and $\mathbf{F}$ is continuous everywhere on the plane, then

$$\nabla \times \mathbf{F} = \frac{\partial}{\partial x}(\frac{1}{2}x^2 - \mathrm{e}^{-x}) - \frac{\partial}{\partial y} y\mathrm{e}^{-x} = x + \mathrm{e}^{-x} - \mathrm{e}^{-x} = x\hat{\mathbf{k}}.$$

Effectively, the double integral becomes

$$\iint\limits_R \nabla \times \mathbf{F} \cdot \hat{\mathbf{k}}\, \mathrm{d}A = \iint\limits_R x \mathrm{d}A.$$

Recall that the x coordinate of the geometric centre as

$$\bar{x} = \frac{1}{|R|} \iint\limits_R x \mathrm{d}A$$

then using geometry alone we have $\bar{x} = 2$ and thus

$$\iint\limits_R x \mathrm{d}A = |R|\bar{x} = 2\pi.$$

Another example

Show that the value of the integral

$$\oint_c xy^2 \mathrm{d}x + (x^2 y + 2x)\mathrm{d}y,$$

around any square c on the xy plane depends only on the area of the square and not on its actual position.

Let c run anticlockwise. Since $\mathbf{F}$ is continuous everywhere on the plane then, if R denotes the square within c then by Green's theorem

$$\oint_c xy^2 \mathrm{d}x + (x^2 y + 2x)\mathrm{d}y = \iint_R 2xy + 2 - 2xy \, \mathrm{d}A = 2|R|.$$

Green's theorem for flux

If $\mathbf{F}$ is a continuous vector field and c a closed path with anti-clockwise direction then the flux of $\mathbf{F}$ outwards c is equal in value to the double integral of the divergence of $\mathbf{F}$ in the enclosing region

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s = \iint_R \nabla \cdot \mathbf{F} \, \mathrm{d}A$$

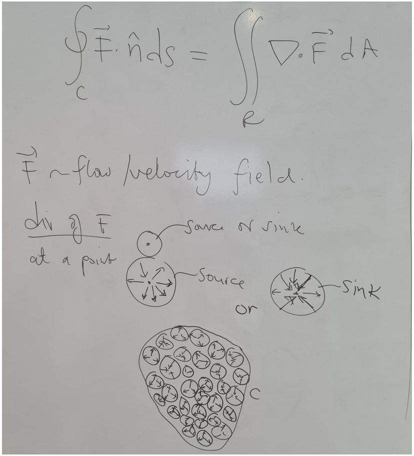
For $\mathbf{F} = f(x, y)\hat{\mathbf{i}} + g(x, y)\hat{\mathbf{j}}$ this can be expressed in scalar form as

$$\oint_c (f \mathrm{d}y - g \mathrm{d}x) = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \mathrm{d}A.$$

Green's theorem for flux is also known as the normal form of Green's theorem, or the divergence theorem in 2D.

Green's theorem for flux

Physical interpretation: From microscopic source/sink (divergence) to macroscopic flux.



Green's theorem for flux

Use: In essence, Green's theorem allows you to choose between a line and a double integral to solve, ... whichever you find easier.

Caveat: If c is clockwise then this reverses the sign of the flux integral[3], hence the flux is now towards the interior of c

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s = - \iint_R \nabla \cdot \mathbf{F} \, \mathrm{d}A$$

Solenoidal fields: This theorem asserts that solenoidal fields have zero flux through closed paths.

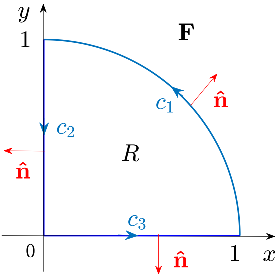
$$\text{If } \nabla \cdot \mathbf{F} = 0 \quad \Longrightarrow \quad \mathbf{F} \text{ is solenoidal} \quad \Longrightarrow \quad \oint_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s = 0.$$

[3] By the reversing the direction of the path property of line integrals in module 4.

Solenoidal fields have zero flux across closed loops

Consider computing the flux of $\mathbf{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ across the closed path $c = c_1 \cup c_2 \cup c_3$

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s$$



Example

Since $\mathbf{F} = (y, x)$ is continuous everywhere on the plane and c is anticlockwise (with $\hat{\mathbf{n}}$ pointing outwards) Green's theorem applies

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s = \iint_R \nabla \cdot \mathbf{F} \mathrm{d}A = \iint_R \left(\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} \right) \mathrm{d}A = 0$$

To confirm this we perform the flux integral anyway, on each segment of the path, and then adding them together

$$\begin{aligned} \oint_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s &= \int_{c_1} \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s + \int_{c_2} \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s + \int_{c_3} \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s \\ &= \sum_{i=1}^{3} \int_{c_i} y \mathrm{d}y - \int_{c_i} x \mathrm{d}x, \end{aligned}$$

since $\hat{\mathbf{n}} \mathrm{d}s = (\mathrm{d}y, -\mathrm{d}x)$.

Example

On c_1 we have $x = \cos\theta$, $y = \sin\theta$

$$\begin{aligned}\int_{c_1} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}s &= \int_{c_1} y\mathrm{d}y - \int_{c_1} x\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\mathrm{d}\theta - \int_0^{\frac{\pi}{2}} \cos\theta(-\sin\theta\mathrm{d}\theta) = 1\end{aligned}$$

On c_2 we have $\mathrm{d}x = 0$ and $x = 0$

$$\int_{c_1} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}s = \int_1^0 y\mathrm{d}y = -\frac{1}{2},$$

and on c_3 where $\mathrm{d}y = 0$ and $y = 0$

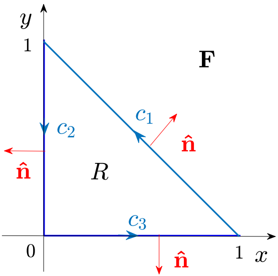
$$\int_{c_1} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}s = -\int_0^1 x\mathrm{d}x = -\frac{1}{2}$$

giving a total flux of zero across the path c, confirming Green's theorem for flux.

Another example

The 'convenience' of Green's for flux is when the double integral is simpler than the line integral for flux.

Find for example the flux of $\mathbf{F} = (x^2 - y^2, 2xy)$ on a path c formed by three segments making a triangle between $(0,0)$, $(1,0)$ and $(0,1)$ anticlockwise.



Since $\mathbf{F}$ is continuous and $c = c_1 \cup c_2 \cup c_3$ is closed, anticlockwise then Green's for flux applies.

Another example cont.

It is not difficult to see that $\nabla \cdot \mathbf{F} = 4x$, and thus the double integral becomes appealing

$$\oint \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s = \iint_R 4x \mathrm{d}A = 4\bar{x}|R|,$$

with $\bar{x} = 1/3$ at the geometric centre of the triangle R, and $|R| = 1/2$.

If we don't have $\bar{x}$ handy we must compute the double integral. To get the limits we need the equation of the line on c_1 which is $c_1 : y + x - 1 = 0$. Thus

$$\iint_R 4x \mathrm{d}A = 4 \int_0^1 \int_0^{1-y} x \mathrm{d}x \mathrm{d}y = 2 \int_0^1 \left[x^2\right]_0^{1-y} \mathrm{d}y = \frac{2}{3}.$$

Formulas

Let $\mathbf{F}(x,y) = f(x,y)\hat{\mathbf{i}} + g(x,y)\hat{\mathbf{j}}$ be continuous in a region R encircled by c that has anticlockwise direction.

- Green's for work:

$$\oint_c \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint\limits_R \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)\mathrm{d}A$$

- Green's for flux:

$$\oint_c \mathbf{F}\cdot \hat{\mathbf{n}}\mathrm{d}s = \iint\limits_R \Big(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\Big)\mathrm{d}A$$

Main outcomes of module 9

You MUST know:

1. Green's theorem for work, its physical interpretation and necessary conditions.
2. Green's theorem for flux, its physical interpretation and necessary conditions.
3. The implications of Greens theorems for conservative and solenoidal fields.

Good to know:

That Green's theorem for flux in 3D domains is known as Gauss's divergence theorem.

That Green's theorem for work in 3D domains is known as Stokes' theorem.

Who was Green? `https://en.wikipedia.org/wiki/George_Green_(mathematician)`