Module 10 self-assessment

Question 1

Elastic energy is the potential mechanical energy stored in the volume of a material when that volume is compressed or stretched. The elastic energy W corresponding to a volume R of a certain material is $q^2R/(2EI)$, where q is its stress and E and I are constants. Find the elastic energy of a cylindrical volume of radius a and length l in which the stress varies directly as the distance from its axis, being zero at the axis and q_0 at the outer surface. Justify your answer by commenting on the influence of a and l on this energy.

Solution:

Taking a very small sample of that material, say of volume ΔR then the elastic energy required to deform it is

$$\Delta W = \frac{q^2}{2EI}\Delta R,$$

where ΔW is a function of the stress q within ΔR and of ΔR itself. To get to the elastic energy of the cylinder we simply have to integrate W over the volume of the cylinder, say V

$$W_{\text{cyl}} = \iiint_V \frac{q^2}{2EI}\,\mathrm{d}V$$

The exercise also provides a relation for q on the cylinder, just so that can be expressed in terms of the distance from the axis of the cylinder, starting at zero when $r = 0$ and growing linearly to an upper bound q_0 occurring at $r = a$. This linear relation can be expressed as

$$q = q_0\frac{r}{a},$$

hence the volume integral simplifies to

$$W_{\text{cyl}} = \iiint_V \frac{(q_0^2\frac{r^2}{a^2})}{2EI}\,\mathrm{d}V$$

To setup the iterated integral on the cylinder we use cylindrical coordinates for $\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$, thus the definition of the cylinder is $r = a$ and its height is bounded by $0 \le z \le l$, yielding

$$W_{\text{cyl}} = \frac{q_0^2}{2EIa^2}\int_0^l\int_0^{2\pi}\int_0^a r^3\mathrm{d}r\mathrm{d}\theta\mathrm{d}z,$$

with an inner integral

$$\int_0^a r^3\mathrm{d}r = \frac{a^4}{4},$$

and outer integral

$$W_{\text{cyl}} = \frac{q_0^2}{2EIa^2}\frac{a^4}{4}2\pi l = \frac{a^2 l\pi q_0^2}{4EI}.$$

From the above notice that the energy W_{cyl} depends quadratically on the cylinder's radius a and linearly on its length l, with the stored energy increasing as the radius and length increase. This is a reasonable result since the capacity of the cylinder to store energy relates to the energy that is required to deform its shape.

Question 2

Compute the integral

$$\iiint\limits_R z\,\mathrm{d}V,$$

for a region R at the intersection of two unit spheres centred at $(0,0,0)$ and $(0,0,1)$ respectively.

Solution:

By the definition of the geometric centre of R

$$\iiint\limits_R z\,\mathrm{d}V = \bar{z}\iiint\limits_R \mathrm{d}V = \bar{z}|R|$$

where $|R|$ is the volume of the region R and $\bar{z}$ is the z coordinate of the region's geometric centre $(0,0,\frac{1}{2})$.

There are several ways to solve this problem. Setting up the integral for the region using the equations of the two spheres is perhaps the most complicated. A simpler way is through geometry: Noticing that R is symmetric about the plane $z=\frac{1}{2}$ we can set $|R| = 2|R_1|$ where $|R_1|$ is the part of R above the plane and below the surface of the lower sphere. Setting up the triple integral in cylindrical coordinates for $\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$ yields

$$|R_1| = \int_{\frac{1}{2}}^{1}\int_0^{2\pi}\int_0^{\sqrt{1-z^2}} r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

with an inner integral

$$\int_0^{\sqrt{1-z^2}} r\mathrm{d}r = \left[\frac{r^2}{2}\right]_0^{\sqrt{1-z^2}} = \frac{1}{2}(1-z^2)$$

a middle integral

$$\int_0^{2\pi}\frac{1}{2}(1-z^2)\mathrm{d}\theta = \pi(1-z^2)$$

and an outer

$$|R_1| = \int_{\frac{1}{2}}^{1}\pi(1-z^2)\mathrm{d}z = \frac{\pi}{2} - \frac{\pi}{3}\frac{7}{8} = \frac{5}{24}\pi.$$

Therefore the required volume is $|R| = \frac{5}{12}\pi$ and the integral

$$\iiint\limits_R z\,\mathrm{d}V = \bar{z}|R| = \frac{1}{2}\frac{5}{12}\pi = \frac{5}{24}\pi.$$

Question 3

Let R be the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. Setup an iterated integral in cylindrical coordinates for the volume of R. You are not required to solve it.

Solution:

Converting from Cartesian to cylindrical, the paraboloids are

$$z = r^2, \quad \text{and} \quad z = 8 - r^2.$$

and they can be found to intersect at $z = 4$ marking a 2D shadow of a disk with radius 2.

Setting the integral in $\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$ order allows to exploit the fact that the two paraboloids are the upper and lower bounds for the inner integral for z. Hence

$$|R| = \iiint_R \mathrm{d}V = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$$

where the inner integral limits are for the shadow of R on the xy plane, which is a disk of radius 2 centred at the origin. The outer limits for the angle follow trivially by the symmetry of the disk with respect to the angle θ.