Engineering Mathematics 2B

Module 5: Line integration

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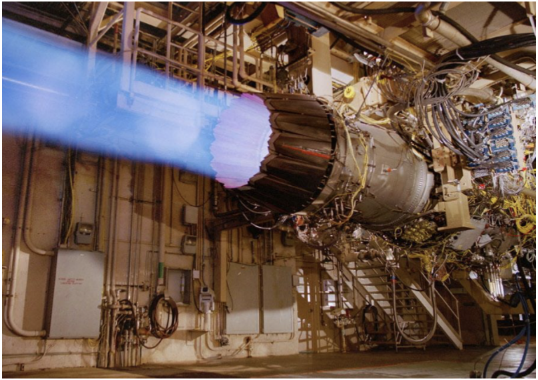
Module 5 contents

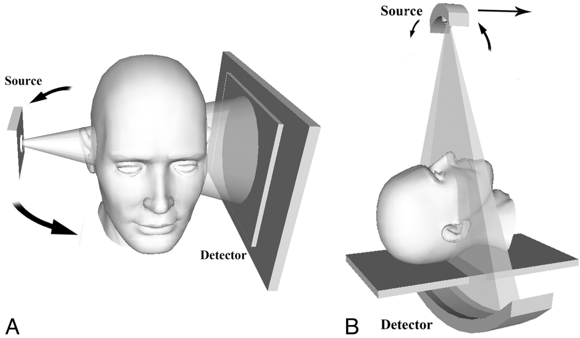
Motivation: Flux of emissions

Flux is the rate at which mass or energy (or both) passes through a surface, per unit time.



How much CO_2 comes out of the exhaust? The concentration and velocity of the gases are critical but so is the aperture.

Motivation: Line integrals in radiology



In X-ray Computed Tomography, the data are line integrals (thousands within the cone) corresponding to the attenuation of the X-ray beam intensity along the particular paths, originating from a source and ending up at different points on the detector.

Introducing the flux integral

In module 4 we saw that the work integral

$$\int_c \mathbf{F}(\mathbf{r}) \cdot \, \mathrm{d}\mathbf{r} \equiv \int_c \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{t}} \, \mathrm{d}s$$

computes the total component of the vector field **F** that is **tangential** to and in the direction of the curve c.

The **flux integral**

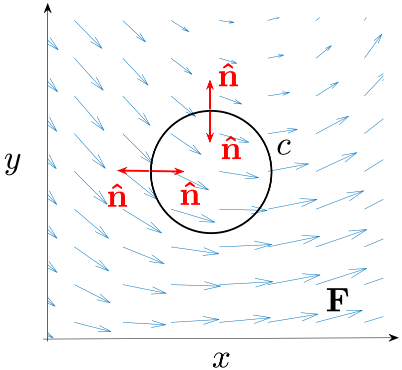
$$\int_c \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{n}} \, \mathrm{d}s \equiv \int_c \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{S}$$

computes the component of the vector field **F** that is **normal to the curve** c. This is the net amount of **F** *flowing through* c in the direction of $\hat{\mathbf{n}}$ per unit time.

From a physical point of view work is typically associated with forces, while flux is encountered in velocities and fluids.

Flux integral sign

When explaining the physical meaning of the divergence of $\mathbf{F}$ in module 2, we saw a schematic like this



and said $\nabla \cdot \mathbf{F} = \lim_{\Omega \to 0} \frac{1}{\Omega} \int_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s$ is the flux of $\mathbf{F}$ through circle c when its radius becomes tiny. If c is pointing clockwise $\hat{\mathbf{n}}$ points inside the circle. Flow towards $\hat{\mathbf{n}}$ contributes positive flux, and otherwise negative flux.

Flux integrals in 2D

To compute

$$\int_c \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{n}} \mathrm{d}s,$$

when given $\mathbf{F}$ and c we need

1. $\hat{\mathbf{n}}$ and its direction on c,
2. the direction of c, and
3. to find $\mathrm{d}s$

If c is on the xy plane, then $y = f_c(x)$ so let's write

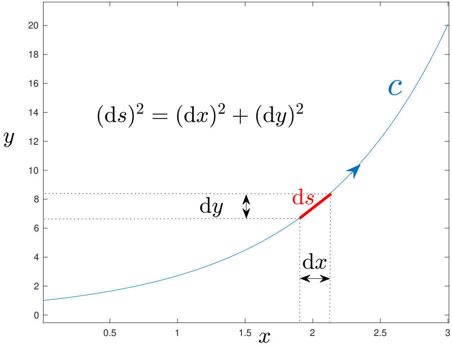
$$c : y - f_c(x) = 0$$

Recall from module 1 'normal on a surface' that

$$\mathbf{n} = \nabla c = -\frac{\partial}{\partial x} f_c(x)\hat{\mathbf{i}} + \hat{\mathbf{j}} \quad \Rightarrow \quad \hat{\mathbf{n}} = \frac{1}{|\nabla c|} \nabla c.$$

Flux integrals in 2D

Having got $\mathbf{n}$ from c and normalised it to $\hat{\mathbf{n}}$ we are left to find $\mathrm{d}s$.



Consider an auxiliary variable t and a displacement $\mathrm{d}t$, then

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

can be derived from Pythagoras' theorem.

Methodology

The aim of the "trick" of bringing in t is to write the whole flux integral with respect to t.

(1) Find $\hat{\mathbf{n}}$ from c.

(2) Write x as a function of t, e.g. $x = t$ or $x = g(t)$. Find y as a function of t from $y = f_c(x)$.

(3) Compute $\mathrm{d}x/\mathrm{d}t$ by differentiating the definition in **(2)**, e.g. $\mathrm{d}x = \mathrm{d}t$ if $x = t$.

(4) Compute $\mathrm{d}y/\mathrm{d}t$ by differentiating the definition in **(3)** using the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

e.g. $\mathrm{d}y/\mathrm{d}t = \mathrm{d}f_c/\mathrm{d}t$ if $x = t$.

(5) Substitute **(2)**-**(4)** into the integrand to express x, y and c in terms of t and $\mathrm{d}s$ in terms of $\mathrm{d}t$.

An alternative methodology

There is a faster way of computing $\hat{\mathbf{n}}\mathrm{d}s$ without finding $\hat{\mathbf{n}}$.

Since $\hat{\mathbf{n}}$ is normal to c and therefore normal to $\mathrm{d}\mathbf{r}$, which is

$$\mathrm{d}\mathbf{r} = \hat{\mathbf{t}}\mathrm{d}s = \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}}$$

then

$$\hat{\mathbf{n}}\mathrm{d}s \cdot \mathrm{d}\mathbf{r} = 0$$

We can rotate $\mathrm{d}\mathbf{r}$ by $\pi/2$ clockwise to align with $\hat{\mathbf{n}}$ using the CW-rotation-by-θ matrix

$$R_{\mathrm{cw}}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

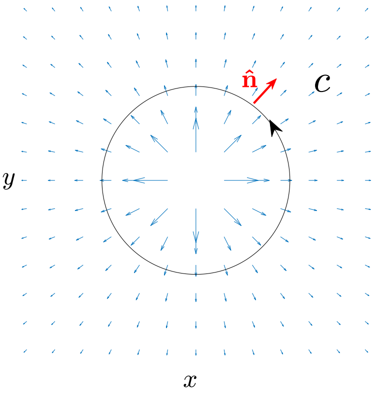
$$\hat{\mathbf{n}}\mathrm{d}s = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \mathrm{d}y\hat{\mathbf{i}} - \mathrm{d}x\hat{\mathbf{j}}$$

Hence if $\mathbf{F} = f(x,y)\hat{\mathbf{i}} + g(x,y)\hat{\mathbf{j}}$ then

$$\int_c \mathbf{F} \cdot \hat{\mathbf{n}}\mathrm{d}s = \int_c f(x,y)\mathrm{d}y - \int_c g(x,y)\mathrm{d}x.$$

Equivalence in methodologies

The flux of $\mathbf{F} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})/(x^2+y^2) \equiv \frac{1}{x^2+y^2}\big(x, y\big)$ through the circle $c : x^2 + y^2 = a^2$, anticlockwise.



Do we expect the flux to be positive, negative or zero?

Equivalence in methodologies

Since c is a circle of radius a centred at the origin then

$$\nabla c = \big(2x, 2y\big), \quad |\nabla c| = 2\sqrt{x^2 + y^2} = 2a,$$

thus

$$\int_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s = \frac{1}{2a} \int_c \frac{1}{x^2 + y^2} \big(x, y\big) \cdot \big(2x, 2y\big) \mathrm{d}s = \frac{1}{a} \int_c \mathrm{d}s = 2\pi.$$

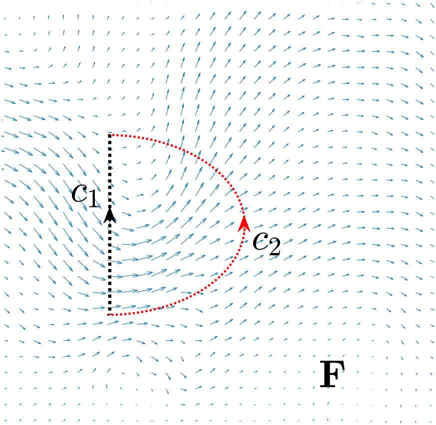
Alternatively, by $\hat{\mathbf{n}} \mathrm{d}s = (\mathrm{d}y, -\mathrm{d}x)$ we have

$$\begin{aligned} \int_c \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s &= \int_c \frac{1}{x^2 + y^2} \big(x, y\big) \cdot \big(\mathrm{d}y, -\mathrm{d}x\big) = \int_c \frac{x}{a^2} \mathrm{d}y - \int_c \frac{y}{a^2} \mathrm{d}x \\ &= \int_c \cos t \cos t \, \mathrm{d}t - \int_c \sin t (-\sin t) \, \mathrm{d}t = \int_0^{2\pi} \mathrm{d}t = 2\pi. \end{aligned}$$

by setting $x = a \cos t$, $y = a \sin t$ on c (to avoid integrating roots!)

Comparing fluxes

Consider the flux integrals of $\mathbf{F}$ through c_1 and c_2 (direction of flux is from west to east)



How do the fluxes through c_1 and c_2 interfaces compare?

Scalar line integrals

Work and flux integrals 'automatically' induce scalar integrals with respect to $\mathrm{d}x$, $\mathrm{d}y$ (and $\mathrm{d}z$ in 3D)

How do we compute scalar 2D integrals

$$\int_c f(x, y)\mathrm{d}s$$

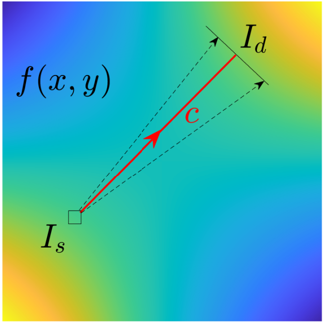
We have touched upon this when outlining the first methodology for flux integrals where $\hat{\mathbf{n}}$ was calculated explicitly.

We will now address these scalar line integrals on their own merit as they arise in many applications, from biomedical imaging to atmospheric monitoring etc.

Scalar integrals

In X-ray attenuation measurements for example

$$I_d = I_s\, \mathrm{e}^{-\int_c f(x,y)\mathrm{d}s} \quad \Leftrightarrow \quad \log \frac{I_s}{I_d} = \int_c f(x,y)\mathrm{d}s$$



I_s/ I_d is the intensity of light at the source/ detector, and $f(x,y)$ is the medium that absorbs X-rays and attenuates the beam as it travels through it.

Scalar integration

(1) We bring in an auxiliary variable t, by an ad hoc assignment $x = g(t)$ or $y = g(t)$.

If we set $x = g(t)$ then we work out y as a function of t from c .

(2) The differential element $\mathrm{d}s$ transforms into $\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t$

(3) Substitution into the line integral

$$\int_c f(x,y)\mathrm{d}s = \int_{t=a}^{t=b} f\big(x(t), y(t)\big)\sqrt{\Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big)^2 + \Big(\frac{\mathrm{d}y}{\mathrm{d}t}\Big)^2}\,\mathrm{d}t$$

where a and b are the bounds of c with respect to t.

In 3D

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big)^2 + \Big(\frac{\mathrm{d}y}{\mathrm{d}t}\Big)^2 + \Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)^2}$$

Formulas

Let $\mathbf{F}(\mathbf{r}) = \big(f, g\big)$.

- The flux of $\mathbf{F}$ through the interface c in the direction $\hat{\mathbf{n}}$ is

$$\int_c \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{n}} \mathrm{d}s$$

- $\hat{\mathbf{n}} \mathrm{d}s = \mathrm{d}y \hat{\mathbf{i}} - \mathrm{d}x \hat{\mathbf{j}}$
- $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$

Main outcomes of module 5

You MUST know:

1. How to pose the flux integral for $\mathbf{f}$ through a path c.
2. The two methodologies for solving flux integrals.
3. How to find the unit normal on circles, parabolas, rectangles.
4. Flux is maximised when the field is normal to the path (no tangential component) and vanishes if the field is aligned to it (no normal component).
5. How to compute line integrals of scalar functions with respect to $\mathrm{d}s$.

Good to know:

Special cases of straight paths c: $\mathrm{d}s$ reduces to $\mathrm{d}y$ if c is perpendicular to the x axis, and x is constant on c. $\mathrm{d}s$ reduces to $\mathrm{d}x$ if c is perpendicular to the y axis, and y is constant on the path.