Engineering Mathematics 2B

Module 4: Line integration

Nick Polydorides

School of Engineering

THE UNIVERSITY of EDINBURGH

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Outcomes

Motivation: Work in motion

Work is the energy transferred to or from an object as it moves due to the application of a force.

In cars this force comes from the thrust of an engine, and the motion depends on the steering. When is the car ‘more responsive’ to the engine’s force, on a straight patch or a corner?

Motivation: Work in gravity

Cranes pull up a lot of weight, but how much work do they do?

Loads are moved vertically or horizontally. The gravitational force is purely vertical (downward pointing).

Introducing the line integral

Two special line integrals that you already know

$$\int_a^b f(x)\mathrm{d}x \quad \text{and} \quad \int_c^d f(y)\mathrm{d}y$$

What's special about them?

1. They are 1D,
2. The integration is over straight lines,
3. In both cases the integrand[1] and the integration path 'live' on the same axes.

I am expecting that all of you know how to solve the above.

[1] the function to be integrated

Line integrals

What happens when the path of integration is not on one of the axes?

Path may not be straight and thus span in 2 or 3 dimensions.

The general form of line integral is

$$\int_{c(x,y)} f(x,y)\mathrm{d}s \quad \text{and} \quad \int_{c(x,y,z)} f(x,y,z)\mathrm{d}s$$

c is the path (trajectory) of integration embedded in 2D or 3D space

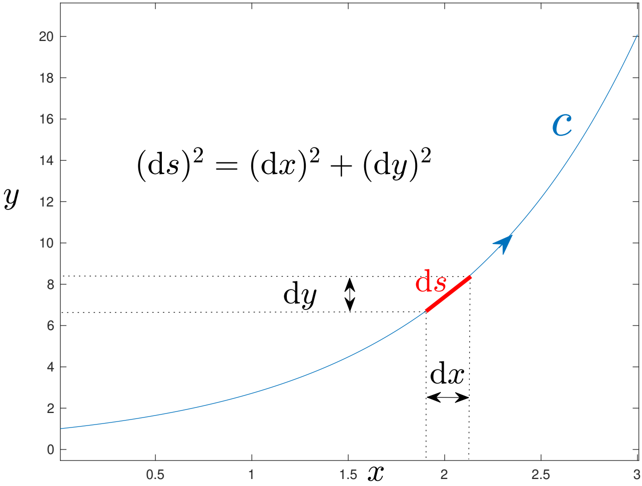
c can be open ended or closed (loop), and

c always has a direction, e.g. from start to end.

What is $\mathrm{d}s$?

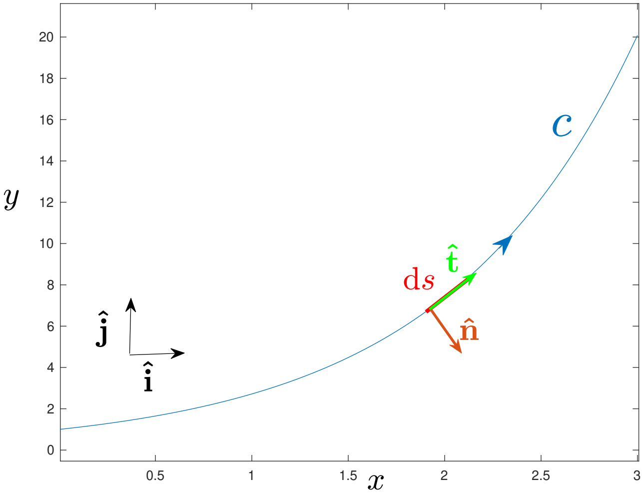
Line integrals in 2D

Consider the blue $c : y - e^x = 0$ on the xy plane



Line integrals in 2D

Two unit vectors on c: the tangential $\hat{\mathbf{t}}$ in the direction of c and the normal $\hat{\mathbf{n}}$ conventionally rotated $\pi/2$ clockwise from $\hat{\mathbf{t}}$.



The work integral

The **work integral** of vector field $\mathbf{F}$ in the direction of path c

$$\int_c \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_c \mathbf{F}(\mathbf{r}) \cdot \underbrace{\hat{\mathbf{t}}\,\mathrm{d}s}_{\mathrm{d}\mathbf{r}}$$

computes the total component of the vector field $\mathbf{F}$ that is tangential to and in the direction of the curve c.

Recall $\hat{\mathbf{t}}$ is the unit tangent to c and if c is on the xy plane then

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}, \quad \mathrm{d}\mathbf{r} = \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}}, \quad \mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}.$$

Note that $\mathbf{F}$ is defined on the whole plane, but only the part on c is involved in the integral. $\mathrm{d}\mathbf{r}$ is only defined on c.

Notice: $\mathbf{F}(\mathbf{r})$ is an alternative notation for $\mathbf{F}(x, y, z)$ that uses the position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

The work integral

$$W = \int_c \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

if $\mathbf{F} = f(x,y)\hat{\mathbf{i}} + g(x,y)\hat{\mathbf{j}}$ then executing the inner product within the integral yields two scalar integrals[2],

$$W = \int_c f(x,y)\mathrm{d}x + \int_c g(x,y)\mathrm{d}y$$

To solve them, we use c, a relation between x and y, and substitute for the 'other' variable in each of the integrands just so they become functions of a single variable.

We then find the limits in the respective variable of integration by looking at the shadow of c on that axis.

[2] In 3D where $\mathbf{F}(f,g,h)$ and $\mathrm{d}\mathbf{r} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ this yields three integrals.

Methodology

As an example consider the integral

$$I = \int_c xy \, \mathrm{d}x,$$

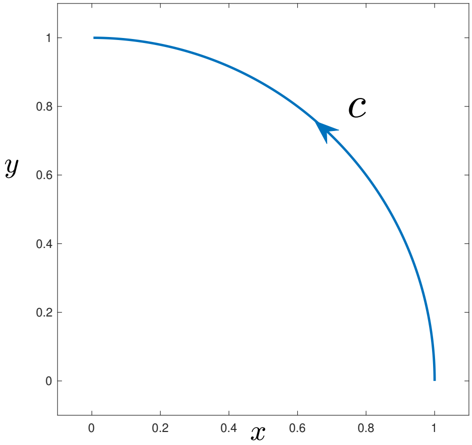
along the curve $c : x^2 + y^2 = 1$ starting from point $(1,0)$ and ending at $(0,1)$, where c resides in the first quadrant.

(1) Draw a sketch of c: This is an arc of the unit circle centred at the origin. Do not draw the integrand!

(2) From the definition of c, replace one of the variables of the integrand
From $c : x^2 + y^2 = 1$ we get $y = \sqrt{1 - x^2}$

$$I = \int_c xy \, \mathrm{d}x = \int_{\color{red}c} x\sqrt{1 - x^2} \, \mathrm{d}{\color{red}x}.$$

Methodology



Note the direction of c starting at $(1, 0)$ and ending at $(0, 1)$. Every point (x, y) on c satisfies $x^2 + y^2 = 1$.

Methodology

(3) Find the integral limits as the bounds of the integrated variable on c

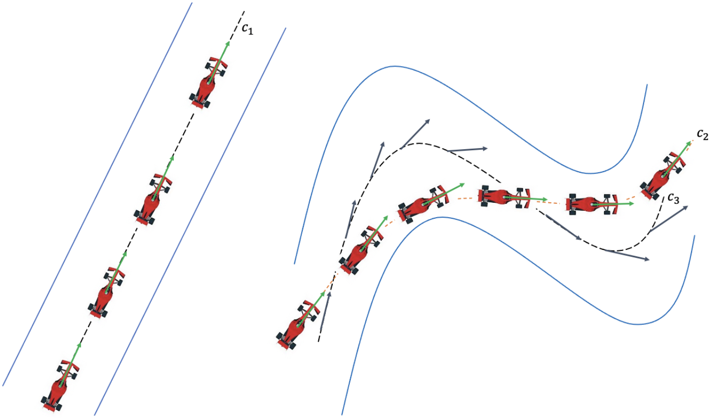
To eliminate c and replace it with simple bounds on x, recall that c starts from $(1, 0)$ and ends at $(0, 1)$. Hence we must integrate from 1 to 0 (order is important)

$$I = \int_{c} x\sqrt{1 - x^2}\mathrm{d}x = \int_{1}^{0} x\sqrt{1 - x^2}\mathrm{d}x$$

From this point on the integral is converted into a single variable integral, (EM1B syllabus)

$$I = -\frac{1}{3}\left[(1 - x^2)^{\frac{3}{2}}\right]_1^0 = -\frac{1}{3}.$$

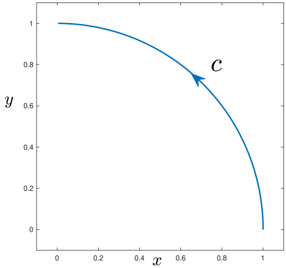
When does work pay off?

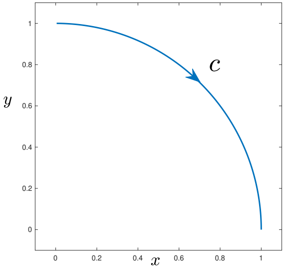


Work done on c_1, c_2 and c_3 paths. Work is maximised when the velocity (motion) is *aligned* to the displacement. The car cannot bend to take the shape of the track exactly. Green/gray arrows for the velocity.

Basic properties of line integrals (not just work)

1. Reversing the direction of c reverses the sign of the line integral.





Basic properties of line integrals (not just work)

1. Reversing the direction of c reverses the sign of the line integral.

Going back to the example $I = \int_c xy\mathrm{d}x$ with $c : x^2 + y^2 = 1$ in the first quadrant, but this time starting at $(0, 1)$ and ending at $(1, 0)$.

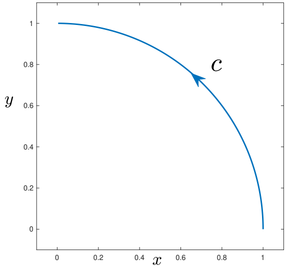
It is easy to see that after the substitution $y = \sqrt{1 - x^2}$ the integral with respect to x becomes

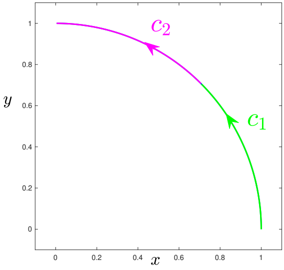
$$\int_0^1 x\sqrt{1 - x^2}\mathrm{d}x = -\int_1^0 x\sqrt{1 - x^2}\mathrm{d}x = -I.$$

using the property of definite single variable integrals.

Basic properties of line integrals (not just work)

2. Splitting c into segments does not alter the value of the line integral.





Basic properties of line integrals (not just work)

2. Splitting c into segments does not alter the value of the line integral.

Consider splitting c in two disjoint arcs such that $c = c_1 \cup c_2$, where[3]

$$c = \begin{cases} c_1 : x^2 + y^2 = 1 & \text{from } (1{,}0) \text{ to } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ c_2 : x^2 + y^2 = 1 & \text{from } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ to } (0, 1) \end{cases}$$

By the additivity of single variable integration

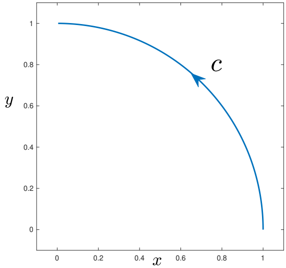
$$\int_{c_1} x\sqrt{1 - x^2}\mathrm{d}x + \int_{c_2} x\sqrt{1 - x^2}\mathrm{d}x$$

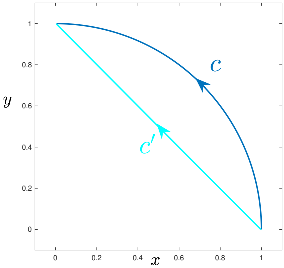
$$= \int_1^{\frac{1}{\sqrt{2}}} x\sqrt{1 - x^2}\mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^0 x\sqrt{1 - x^2}\mathrm{d}x = \int_1^0 x\sqrt{1 - x^2}\mathrm{d}x = I$$

[3] We can split into more if needed.

Basic properties of line integrals (not just work)

3. Changing the shape of c changes the integrand, hence the value of the line integral.





Basic properties of line integrals (not just work)

3. Changing the shape of c changes the integrand, hence the value of the line integral.

Consider now that the path c changes from an arc to a straight line from $(1,0)$ to $(0,1)$. While the endpoints and the direction of c are unchanged, its definition is now $c' : y = 1 - x$.

$$\begin{aligned}\int_{c'} xy\mathrm{d}x &= \int_{c'} x(1-x)\mathrm{d}x \\ &= \int_1^0 x - x^2\mathrm{d}x \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_1^0 = -\frac{1}{6} \neq I\end{aligned}$$

Basic properties of line integrals (not just work)

4. Changing the parameterisation of c does not change the value of the line integral.

Recall

$$I = \int_c xy\mathrm{d}x, \quad c : x^2 + y^2 = 1, \quad x, y \geq 0$$

If we now chose to parameterise as $x = \cos t$ then this yields $y = \sin t$ based on the given c. In turn, $\mathrm{d}x = -\sin t\mathrm{d}t$ and we must change the limits from $x : 1 \to 0$ to $t : \cos^{-1} 1 \to \cos^{-1} 0$.

$$\begin{aligned} \int_c xy\mathrm{d}x &= -\int_c \cos t \sin^2 t\mathrm{d}t \\ &= \left[-\frac{1}{3}\sin^3 t\right]_0^{\frac{\pi}{2}} = -\frac{1}{3} = I \end{aligned}$$

Work in conservative fields

A notable exception to the rule: for a conservative vector field $\mathbf{F}$ the work integral

$$\int_c \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

is **independent** of the particular path c and depends only on the endpoints of c. In effect for $\mathbf{F} = \nabla f$

$$\int_c \nabla f \cdot \mathrm{d}\mathbf{r} = f(c_{\text{end}}) - f(c_{\text{start}}).$$

Consequently, for $\mathbf{F} = \nabla f$ the work integral around any closed loop c is zero[4]

$$\oint_c \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

since on such a c every point is both a start and an end.

[4] The circle on the integral sign denotes c is a closed loop.

Formulas

Let $\mathbf{F}(\mathbf{r}) = \big(f(\mathbf{r}), g(\mathbf{r}), h(\mathbf{r})\big)$

- The position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
- The displacement vector $\mathrm{d}\mathbf{r} = \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}} + \mathrm{d}z\hat{\mathbf{k}}$
- The work integral of $\mathbf{F}(\mathbf{r})$ on path c is

$$W = \int_c \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_c f \mathrm{d}x + \int_c f \mathrm{d}y + \int_c f \mathrm{d}z$$

- If f is defined everywhere then

$$\oint_c \nabla f \cdot \mathrm{d}\mathbf{r} = 0$$

Main outcomes of module 4

You MUST know:

1. Work is typically associated with force fields.
2. How to pose the work integral for an $\mathbf{f}$ and a path c.
3. The procedure for solving line integrals, including drawing the graphs of 'simple' c curves in 2D.
4. The four basic properties of line integrals, and the exception for conservative fields.
5. Work is maximised when the field is aligned to the path c (no normal component) and vanishes if the field is normal to it (no tangential component).

Good to know:

The simplified formula of the work done by a force on a straight line is: magnitude of the force times distance. Can you see how this generalises to the integral above for non straight lines?