Engineering Mathematics 2B

Module 7: Double integration

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Module 7 contents

Motivation:

Double integrals occur when trying to compute:

1. The area of arbitrary regions on the plane.
2. The average value of a function over a 2D region.
3. The centre of mass of an object.
4. The geometric centre of an object.

Definition

Double integrals on the xy frame are defined as

$$\iint\limits_{R} f(x,y)\mathrm{d}A$$

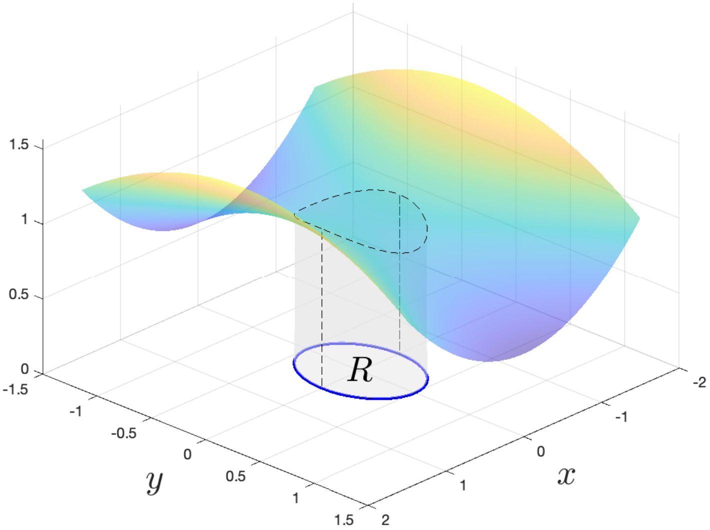
where $f(x,y)$ is a scalar field, R is a closed region on the xy plane, and $\mathrm{d}A$ is the area integration element.

In Cartesian coordinates, on the xy plane

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = \mathrm{d}y\mathrm{d}x$$

is a tiny square element with area $\mathrm{d}A$.

The double integral



Approach

To solve a double integral we **replace it with two single variable integrals**, the so-called inner and outer integrals, and solve from inside out.

The main challenge is finding the appropriate limits for the resulting inner and outer single variable integrals.

Some hints on finding the limits:

1. The outer integral limits are always constant,
2. The inner integral limits are typically variable,
3. The special case where all integral limits are constants is when integrating in Cartesian coordinates over a rectangular region R or when integrating in polar with R circular.

Double integral as an integral of line integrals

The double integral of $f(x, y)$ on over region R in $\mathrm{d}y\mathrm{d}x$

$$\iint\limits_R f(x,y)\mathrm{d}A = \int_{x_{min}}^{x_{max}} S(x)\mathrm{d}x$$

$$= \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\mathrm{d}y\ \mathrm{d}x$$

where $S(x)$ is a line integral on a path <u>normal to the x axis</u>.

Double integral as an integral of line integrals

Alternatively, the same integral in $\mathrm{d}x\mathrm{d}y$

$$\iint\limits_R f(x,y)\mathrm{d}A = \int_{y_{min}}^{y_{max}} S(y)\mathrm{d}y$$

$$= \int_{y_{min}}^{y_{max}} \int_{x_{min}(y)}^{x_{max}(y)} f(x,y)\mathrm{d}x\ \mathrm{d}y$$

where $S(y)$ is a line integral on a path normal to the y axis[1].

[1] Check out my animations on Learn.

Methodology

(1) Draw the region R on the xy plane. (do not draw $f(x, y)$)

(2) Choose the integration order, i.e. $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ or $\mathrm{d}A = \mathrm{d}y\mathrm{d}x$.

(3) Fix the limits of the outer integral based on the boundaries of R on the respective axis (e.g. the y axis if $\mathrm{d}x\mathrm{d}y$)

(4) Pick an *arbitrary* point in between the outer limits and draw a path c transcending R in the direction of the axis in the inner integral.

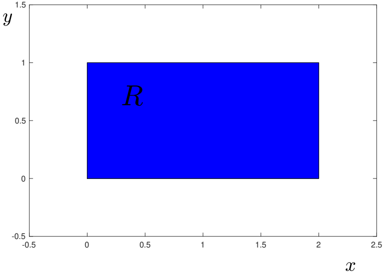
(5) Fix the inner integral limits based on the definition of the curves from where the path c starts and ends.

(6) We perform the inner integration first and then the outer. When integrating with respect to one variable (e.g. x or y) we treat the other one as a constant.

Example

Integrate $f(x,y) = 1 - x^2 - y^2$ over the region $R : 0 \leq x \leq 2,\ 0 \leq y \leq 1$.

$$\iint_R 1 - x^2 - y^2 \mathrm{d}A =$$



Example

R is the *whole* rectangle. Integrating in $\mathrm{d}y\mathrm{d}x$ yields

$$\iint_R 1 - x^2 - y^2 \mathrm{d}A = \int_0^2 \int_0^1 1 - x^2 - y^2 \mathrm{d}y \mathrm{d}x$$

The inner integral yields

$$\int_0^1 1 - x^2 - y^2 \mathrm{d}y = \left[y - x^2 y - \frac{y^3}{3} \right]_0^1 = \frac{2}{3} - x^2,$$

while the outer integral becomes

$$\int_0^2 \frac{2}{3} - x^2 \mathrm{d}x = \left[\frac{2}{3}x - \frac{x^3}{3} \right]_0^2 = -\frac{4}{3}.$$

Example

Repeating in the reverse integration order yields

$$\iint_R 1 - x^2 - y^2 \mathrm{d}A = \int_0^1 \int_0^2 1 - x^2 - y^2 \mathrm{d}x\mathrm{d}y$$

which gives an inner integral

$$\int_0^2 1 - x^2 - y^2 \mathrm{d}x = \left[x - \frac{x^3}{3} - y^2 x\right]_0^2 = -\frac{2}{3} - 2y^2$$

and thus after computing the outer integral we arrive at the same result

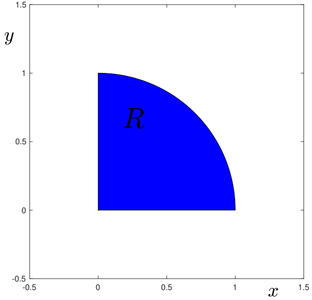
$$\int_0^1 -\frac{2}{3} - 2y^2 \mathrm{d}y = \left[-\frac{2}{3}y - 2\frac{y^3}{3}\right]_0^1 = -\frac{4}{3}$$

Integrating over rectangular regions in Cartesian coordinates yields double integrals with all limits constant.

Another example

Integrate $f(x, y) = 1 - x^2 - y^2$ over the region $R : x^2 + y^2 \leq 1,\ x \geq 0, y \geq 0$.

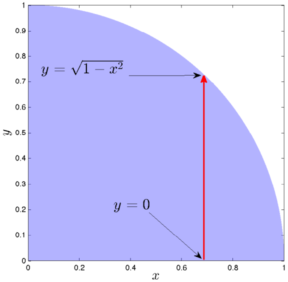
$$\iint_R 1 - x^2 - y^2 \mathrm{d}A =$$

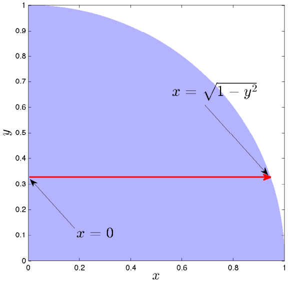


R is a quarter of the unit disk centred at the origin.

Another example cont.

Viewing the region R from above





Left, integrals of $S(x)$ in $\mathrm{d}y\mathrm{d}x$: For an arbitrary x (start of red arrow), the path of the line integral within R, starts at $y = 0$ and ends up at $y = \sqrt{1 - x^2}$.

Right, integrals of $S(y)$ in $\mathrm{d}x\mathrm{d}y$: For an arbitrary y (start of red arrow), the path of the line integral within R starts at $x = 0$ and ends up at $x = \sqrt{1 - y^2}$.

Another example cont.

Suppose we want to setup the inner integral with respect to y and the outer with respect to x, hence $\mathrm{d}A = \mathrm{d}y\mathrm{d}x$ (left figure)

$$\iint_R 1 - x^2 - y^2 \mathrm{d}A = \int_0^1 \int_0^{\sqrt{1-x^2}} 1 - x^2 - y^2 \mathrm{d}y\mathrm{d}x.$$

Alternatively (right figure) we set $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ and

$$\iint_R 1 - x^2 - y^2 \mathrm{d}A = \int_0^1 \int_0^{\sqrt{1-y^2}} 1 - x^2 - y^2 \mathrm{d}x\mathrm{d}y.$$

From the symmetry between x and y in the integrand and the limits is it easy to see that the two integrals above have the same value.

Another example cont.

Attempting the $\mathrm{d}A = \mathrm{d}y\mathrm{d}x$ version gives

$$\begin{aligned}\int_0^{\sqrt{1-x^2}} 1 - x^2 - y^2 \mathrm{d}y &= \left[y(1-x^2) - \frac{y^3}{3} \right]_0^{\sqrt{1-x^2}} \\ &= \frac{2}{3}(1-x^2)^{3/2}\end{aligned}$$

To solve the outer integral that involves square roots we need some trigonometry. Using the transformation $x = \sin\theta$ yields

$$\mathrm{d}x = \cos\theta \mathrm{d}\theta \quad \Rightarrow \quad \sqrt{1-x^2} = \cos\theta,$$

$$\int_0^1 \frac{2}{3}(1-x^2)^{3/2}\mathrm{d}x = \frac{2}{3}\int_0^{\pi/2} \cos^4\theta \mathrm{d}\theta$$

Using the double angle formula $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ twice, then after a page of trigonometry we arrive at the double integral result $\frac{\pi}{8}$.

Swapping the order of integration

Switching the order of integration may simplify a lot the resulting double integral.

If the region of integration R is rectangular, changing between $\mathrm{d}x\mathrm{d}y$ and $\mathrm{d}y\mathrm{d}x$ is trivial

$$\int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} f(x,y)\mathrm{d}y\mathrm{d}x$$

In the general case however to reverse the order requires drawing the region R and then working out the integral limits.

Changing the order of integration example

Consider the double integral

$$\int_0^1 \int_x^{\sqrt{x}} \frac{\mathrm{e}^y}{y} \mathrm{d}y \mathrm{d}x.$$

Taking the inner integral

$$\int_x^{\sqrt{x}} \frac{\mathrm{e}^y}{y} \mathrm{d}y = ????$$

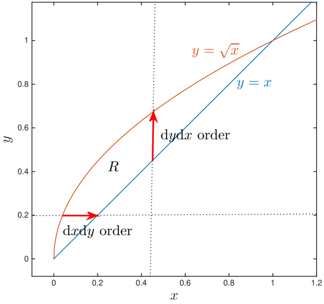
Changing the parameterisation with $x = r\cos\theta$, $y = r\sin\theta$ does not improve matters, it rather complicates them.... e.g. $\mathrm{e}^{r\sin\theta}/r\sin\theta \mathrm{d}r$.

Hence a possible way out is reversing the order of integration.

Example

To swap the order we must understand the graph of R from the limits

$$\int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} \frac{\mathrm{e}^{y}}{y} \mathrm{d}y \mathrm{d}x.$$



As the inner integral will now be with respect to x, we fix the value of y (see horizontal red arrow) and look for the values of $x : x_{min} \rightarrow x_{max}$ within R at that y.

Example

From the figure it is easy to see that for any fixed y such that $0 \leq y \leq 1$ the bounds for x are $x = y^2 \to y$. Also note that R's upper point is at $(1,1)$ where the two curves meet. Effectively,

$$\int_0^1 \int_x^{\sqrt{x}} \frac{\mathrm{e}^y}{y} \mathrm{d}y \mathrm{d}x = \int_0^1 \int_{y^2}^{y} \frac{\mathrm{e}^y}{y} \mathrm{d}x \mathrm{d}y$$

which gives a tractable inner integral

$$\int_{y^2}^{y} \frac{\mathrm{e}^y}{y} \mathrm{d}x = \frac{\mathrm{e}^y}{y} \left[x\right]_{y^2}^{y} = \mathrm{e}^y(1-y),$$

setting up a simple outer integral

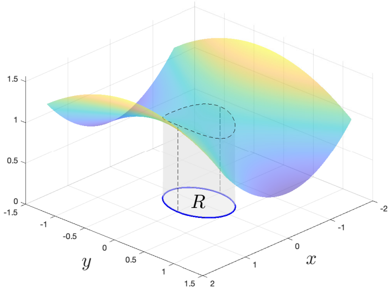
$$\int_0^1 \mathrm{e}^y(1-y)\mathrm{d}y = \Big[\mathrm{e}^y - \mathrm{e}^y(y-1)\Big]_0^1 = \mathrm{e} - 2.$$

Geometric interpretation

Geometrically, the value of the double integral

$$\iint_R f(x,y)\mathrm{d}A$$

equals to the signed **volume** between the graph of $f(x,y)$ and the region R on the xy plane.

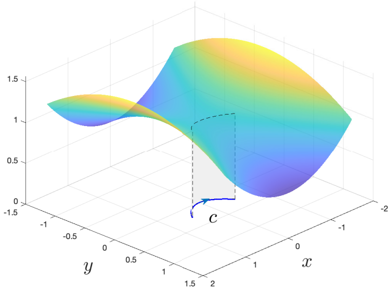


Geometric interpretation

By contrast the value of the line integral

$$\int_c f(x,y)\mathrm{d}s$$

equals to the signed **area** between the graph of $f(x,y)$ and the path c on the xy plane.



Formulas

In 2D the area elements in Cartesian and polar are

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = \mathrm{d}y\mathrm{d}x = r\mathrm{d}r\mathrm{d}\theta = r\mathrm{d}\theta\mathrm{d}r$$

Main outcomes of module 7

You MUST know:

1. How to pose double integrals in Cartesian coordinates.
2. How to find the limits of the inner and outer integrals.
3. How to switch between integration orders.
4. How to solve double integrals.
5. The geometric interpretation of double and line integrals.

Good to know:

The graphs of some simple (regular) 2D shapes, e.g. circles, ellipses, etc.