Module 6 self assessment

Question 1

Find the flux of $\mathbf{f}(x, y) = xy\hat{\mathbf{i}} + 5y\hat{\mathbf{j}}$ through the arc of the unit circle centred at the origin between the lines $y = x$ and $y = -x$ with anticlockwise direction. The arc is above the x axis. Use polar coordinates.

Solution:

The two lines, defining the start and end of the arc are respectively $y = x$ and $y = -x$ respectively, according to the anticlockwise direction. These cut through the circle at $\theta = \pi/4$ and $\theta = 3\pi/4$ hence we can convert the path in polar coordinates as

$$x = \cos\theta, \quad y = \sin\theta, \quad \text{for} \quad \theta : \frac{\pi}{4} \to \frac{3\pi}{4}$$

From $\mathrm{d}x = -\sin\theta \mathrm{d}\theta$ and $\mathrm{d}y = \cos\theta \mathrm{d}\theta$, and $\hat{\mathbf{n}}\mathrm{d}s = \mathrm{d}y\hat{\mathbf{i}} - \mathrm{d}x\hat{\mathbf{j}}$ we have

$$\begin{aligned}
\int_c \mathbf{f}\cdot\hat{\mathbf{n}}\mathrm{d}s &= \int_c xy\mathrm{d}y - \int_c 5y\mathrm{d}x \\
&= \int_c \cos^2\theta\sin\theta\mathrm{d}\theta + \int_c 5\sin^2\theta\mathrm{d}\theta \\
&= \int_{\pi/4}^{3\pi/4} \cos^2\theta\sin\theta\mathrm{d}\theta + 5\int_{\pi/4}^{3\pi/4} \sin^2\theta\mathrm{d}\theta \\
&= \left[-\frac{1}{3}\cos^3\theta\right]_{\pi/4}^{3\pi/4} + 5\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right]_{\pi/4}^{3\pi/4} = \frac{1}{3\sqrt{2}} + \frac{5}{4}(\pi + 2).
\end{aligned}$$

Question 2

Find the flux of $\mathbf{f}(x, y) = y\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ through the ellipse

$$\frac{x^2}{4} + y^2 = 1$$

with anticlockwise direction, using polar coordinates.

Solution:

Recall the general form of the equation of the ellipse (centred at the origin) in Cartesian coordinates as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are the half-axes of the ellipse on the x and y axes respectively. Clearly the special case for $a = b$ is the circle. In our case $a = 2$ and $b = 1$. In polar coordinates, the equation of this ellipse is, different to the circle, as

$$x = a\cos\theta, \quad y = b\sin\theta, \quad \text{for} \quad \theta : 0 \to 2\pi.$$

Differentiating we have

$$\mathrm{d}x = -a \sin\theta \mathrm{d}\theta, \quad \mathrm{d}y = b \cos\theta \mathrm{d}\theta$$

and thus forming the flux integral we get

$$\begin{aligned}
\int_c \mathbf{f} \cdot \hat{\mathbf{n}} \mathrm{d}s &= \int_c y \mathrm{d}y - \int_c y \mathrm{d}x \\
&= \int_c \sin\theta \cos\theta \mathrm{d}\theta + 2 \int_c \sin^2\theta \mathrm{d}\theta \\
&= \int_0^{2\pi} \sin\theta \cos\theta \mathrm{d}\theta + 2 \int_0^{2\pi} \sin^2\theta \mathrm{d}\theta \\
&= \Big[-\frac{1}{2} \cos^2\theta \Big]_0^{2\pi} + 2 \Big[\frac{\theta}{2} - \frac{1}{4} \sin(2\theta) \Big]_0^{2\pi} = 2\pi.
\end{aligned}$$