Module 3 self assessment

Question 1

Show, using the definition of the curl, the identity

$$\nabla \times f\mathbf{a} = f\nabla \times \mathbf{a} + \nabla f \times \mathbf{a}$$

Solution:

Let $\mathbf{a} = a_i\hat{\mathbf{i}} + a_j\hat{\mathbf{j}} + a_k\hat{\mathbf{k}}$ for some scalar fields a_i, a_j and a_k. Straight from the definition of the curl we have

$$\begin{aligned}
\nabla \times f\mathbf{a} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fa_i & fa_j & fa_k \end{vmatrix} \\
&= \Big(\frac{\partial}{\partial y} fa_k - \frac{\partial}{\partial z} fa_j\Big)\hat{\mathbf{i}} + \Big(\frac{\partial}{\partial z} fa_i - \frac{\partial}{\partial x} fa_k\Big)\hat{\mathbf{j}} + \Big(\frac{\partial}{\partial x} fa_j - \frac{\partial}{\partial y} fa_i\Big)\hat{\mathbf{k}} \\
&= \Big(\frac{\partial f}{\partial y} a_k + \frac{\partial a_k}{\partial y} f - \frac{\partial f}{\partial z} a_j - \frac{\partial a_j}{\partial z} f\Big)\hat{\mathbf{i}} \\
&\quad + \Big(\frac{\partial f}{\partial z} a_i + \frac{\partial a_i}{\partial z} f - \frac{\partial f}{\partial x} a_k - \frac{\partial a_k}{\partial x} f\Big)\hat{\mathbf{j}} \\
&\quad + \Big(\frac{\partial f}{\partial x} a_j + \frac{\partial a_j}{\partial x} f - \frac{\partial f}{\partial y} a_i - \frac{\partial a_i}{\partial y} f\Big)\hat{\mathbf{k}},
\end{aligned}$$

where the last equation is due to the chain differentiation rule. Recalling that the curl of $\mathbf{a}$ is

$$\nabla \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_i & a_j & a_k \end{vmatrix} = \Big(\frac{\partial a_k}{\partial y} - \frac{\partial a_j}{\partial z}\Big)\hat{\mathbf{i}} + \Big(\frac{\partial a_i}{\partial z} - \frac{\partial a_k}{\partial x}\Big)\hat{\mathbf{j}} + \Big(\frac{\partial a_j}{\partial x} - \frac{\partial a_i}{\partial y}\Big)\hat{\mathbf{k}}$$

notice that the even terms in the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ components in the previous expression are simply $f\ \nabla \times \mathbf{a}$. Therefore it remains to show that the odd terms in these parentheses are $\nabla f \times \mathbf{a}$, which in determinant form becomes

$$\nabla f \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ a_i & a_j & a_k \end{vmatrix}.$$

Question 2

Compute the Laplacian of the squared distance function in 3D.

Solution:

Directly from the expression of r we obtain

$$r^2 = x^2 + y^2 + z^2,$$

that has a gradient field

$$\nabla r^2 = 2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$

Taking the divergence of the above gradient gives

$$\nabla \cdot \nabla r^2 = 2\nabla \cdot (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = 2\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 6.$$