Engineering Mathematics 2B

Module 3: Differentiation

Nick Polydorides

School of Engineering

THE UNIVERSITY of EDINBURGH

Module 3 contents

Motivation:

There are some universal principles underpinning phenomena in electromagnetics, heat transfer, and other fields.

We have already seen that in electrostatics $\mathbf{E} = -\nabla\phi$, and this implies that $\mathbf{E}$ is conservative. In module 2 however we said that if $\mathbf{E}$ is conservative then $\nabla \times \mathbf{E} = 0$.

We also saw that the magnetic field is always solenoidal, that is $\nabla \cdot \mathbf{H} = 0$. Maxwell has shown that $\nabla \times \mathbf{E} = -\mathrm{i}\omega\mu\mathbf{H}$, that is the magnetic field is the curl of the electric field multiplied by a constant. (The reverse also holds).

These aren't mere coincidences. In fact, the curl of a gradient field is always a zero field, and the divergence of a curl field is always zero.

Harmonic fields

Laplace's equation is ubiquitous in engineering and physics. It describes potential in electrostatics, temperature in steady-state heat transfer, and flow in porous media (among others)

$$\nabla \cdot \nabla f = 0$$

The notation

$$\nabla \cdot \nabla = \nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

allows to write this as $\nabla^2 f = 0$, with ∇^2 known as the Laplace operator. Any scalar field f that satisfies this equation is called harmonic.

Recall that we have defined the vector operator[1] $\nabla \doteq (\partial/\partial x, \partial/\partial y)$ so you can verify the above.

[1] In 2D. The extension to 3D is trivial.

Harmonic fields

Consider an infinitely large 2D (resp. 3D) object Ω that we heat using a high power, high precision laser beam at point $\mathbf{r}'$. The temperature f in the object is a scalar field given by

$$\nabla^2 f = -\delta(\mathbf{r} - \mathbf{r}'), \quad \forall \mathbf{r} \in \Omega.$$

thus at any point other than $\mathbf{r}'$ it satisfies Laplace's equation

$$\nabla^2 f = 0, \quad \mathbf{r} \in \Omega \setminus \mathbf{r}'.$$

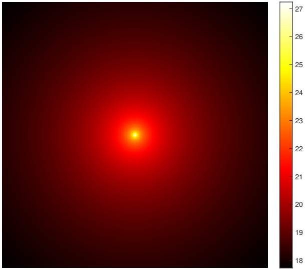
If Ω is 2D then the temperature in the object is given by the harmonic field

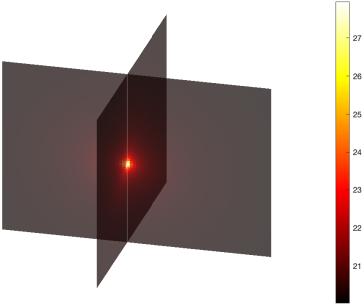
$$f_{\text{2D}} = -\frac{1}{2\pi} \log |\mathbf{r} - \mathbf{r}'|,$$

while in 3D

$$f_{\text{3D}} = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

Temperature harmonic fields





Plots of f_{2D} and f_{3D} assuming $\mathbf{r}'$ is the origin and the object's initial temperature was 17 C.

Harmonic fields

A graphical plot of the harmonic field is reassuring, but we still need to verify $\nabla^2 f = 0$ mathematically.

To compute the Laplacian of $f = -\frac{1}{2\pi}\log(\sqrt{x^2+y^2})$ recall the partial derivatives of the 2D distance function $\frac{\partial}{\partial x}\log(\sqrt{x^2+y^2}) = \frac{x}{x^2+y^2}$, and $\frac{\partial}{\partial y}\log(\sqrt{x^2+y^2}) = \frac{y}{x^2+y^2}$.

$$\begin{aligned}
\nabla^2 f = \nabla \cdot \nabla f &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2} \\
&= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = 0
\end{aligned}$$

This holds everywhere apart from $(0, 0)$.

Try the 3D case for $f_{3\mathrm{D}}$ as an exercise.

The curl of a gradient

In module 2 we have established that if the curl of a continuous vector field $\mathbf{a}$ is zero, then there must exist a scalar potential function f such that $\mathbf{a} = \nabla f$. This can be read in two ways:

$$\text{If} \quad \mathbf{a} = \nabla f \quad \text{then} \quad \nabla \times \mathbf{a} = 0^2$$

$$\text{If} \quad \nabla \times \mathbf{a} = 0 \quad \text{then there must exists } f \text{ such that} \quad \mathbf{a} = \nabla f$$

This isn't just a piece of mathematical jargon. It asserts for example that in electrostatics the curl of the electric field has to be zero, since we have electric potential.

If we magnetise a non-metallic body (e.g. a human inside an MRI scanner) then there are no currents flowing through. Since Maxwell gives $\mathbf{J} = \nabla \times \mathbf{H}$, then $\mathbf{J} = 0$ means that there must exist a magnetic potential, $\mathbf{H} = -\nabla\phi$.

[2] This is a vector field $(0, 0, 0)$.

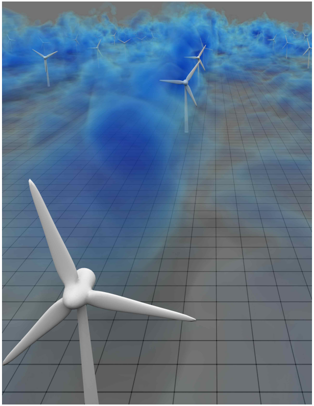
The curl of a gradient proof

The identify $\nabla \times \nabla f = 0$ holds true for any continuous f.

$$\begin{aligned}\nabla \times \nabla f &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)\hat{\mathbf{k}} \\ &= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}.\end{aligned}$$

All conservative, irrotational, gradient fields have zero curl.

The curl of the wind



Simulation of a wind velocity field courtesy of John Hopkins University.

Wind fields are hardly ever conservative, and that's a good thing when aiming to spin turbine blades.

The divergence of the curl

The divergence of the curl of a vector field is always zero.

This statement can be posed in two ways:

$$\text{If} \quad \mathbf{a} = \nabla \times \mathbf{b} \quad \text{then} \quad \nabla \cdot \mathbf{a} = 0$$

$$\text{If} \quad \nabla \cdot \mathbf{a} = 0 \quad \text{then there must exists } \mathbf{b} \text{ such that} \quad \mathbf{a} = \nabla \times \mathbf{b}$$

Maxwell's state that $\nabla \cdot \mathbf{B} = 0$, where $\mathbf{B}$ is the magnetic flux density, related to the magnetic field by $\mathbf{B} = \mu\mathbf{H}$. This in turn justifies the co-existence of magnetic and electric fields, since

$$\mathbf{H} = \frac{\mathrm{i}}{\omega\mu} \nabla \times \mathbf{E}.$$

The divergence of a curl proof

The identify $\nabla \cdot \nabla \times \mathbf{a} = 0$ holds true for any continuous $\mathbf{a}$.

Let $\mathbf{a} = f(x,y,z)\hat{\mathbf{i}} + g(x,y,z)\hat{\mathbf{j}} + h(x,y,z)\hat{\mathbf{k}}$, then

$$\begin{aligned}
\nabla \cdot \nabla \times \mathbf{a} &= \nabla \cdot \left(\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}} \right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \\
&= \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 h}{\partial y \partial x} - \frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 g}{\partial z \partial x} + \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} = 0
\end{aligned}$$

since for any function $w(x,y,z)$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}, \quad \frac{\partial^2 w}{\partial x \partial z} = \frac{\partial^2 w}{\partial z \partial x}, \quad \frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 w}{\partial z \partial y}$$

Formulas

Let $\mathbf{F} = (f, g, h)$ a continuous vector field.

- The Laplacian of f is

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Harmonic fields have $\Delta f = 0$
- $\nabla \times \nabla f = 0$
- $\nabla \cdot \nabla \times \mathbf{F} = 0$

Main outcomes of module 3

You MUST know:

1. What are the scalar harmonic fields and how they are related to conservative vector fields.
2. The curl of a gradient field is always zero.
3. The divergence of a curl field is always zero.
4. To compute second partial derivatives.
5. The notation $\operatorname{curl} \mathbf{a} \equiv \nabla \times \mathbf{a}$, $\operatorname{div} \mathbf{a} \equiv \nabla \cdot \mathbf{a}$, $\operatorname{grad} f \equiv \nabla f$.

Good to know:

Try to find examples of irrotational and solenoidal vector fields relevant to your own discipline.