Engineering Mathematics 2B

Module 8: Double integration

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Module 8 contents

Motivation:

Double integrals occur when trying to compute:

1. The area of arbitrary regions on the plane.
2. The average value of a function over a 2D region.
3. The centre of mass of an object.
4. The geometric centre of an object.

Application 1: Area of region R on the xy plane

The **area of a closed region** R on the xy plane, denoted as $|R|$, is the double integral

$$|R| = \iint_R \mathrm{d}A,$$

which is equivalent to the volume of solid with base R and height $f(x, y) = 1$

$$|R| = \iint_R 1 \, \mathrm{d}A$$

To evaluate this double integral we convert it to inner and outer integrals following the methodology in module 7.

Application 2: Average of function over a region

The **average value of a function** f over a closed region R

$$\bar{f} = \frac{1}{|R|} \iint\limits_R f \mathrm{d}A,$$

while the **total amount** of f in R is

$$f_t = \iint\limits_R f \mathrm{d}A.$$

If $\rho(x, y)$ is a weight function defined on R, then a ρ-**weighted average of** f is

$$\bar{f}_\rho = \frac{1}{\rho_t} \iint\limits_R f \rho \, \mathrm{d}A, \quad \text{where} \quad \rho_t = \iint\limits_R \rho \mathrm{d}A.$$

Application 3: The centre of mass of an object

The **centre of mass** of a 2D object with mass M and density profile $\rho(x, y)$ has coordinates $(\bar{x}, \bar{y})$ as

$$\bar{x} = \frac{1}{M} \iint\limits_R x\rho \, \mathrm{d}A, \quad \bar{y} = \frac{1}{M} \iint\limits_R y\rho \, \mathrm{d}A.$$

A tiny piece of the object with mass ΔM and area ΔA satisfies

$$\Delta M = \rho \Delta A, \quad \text{(recall: density = mass over area)}$$

Taking the double integral over R on both sides above yields the mass of R as

$$M = \iint\limits_R \rho \mathrm{d}A$$

A little help from geometry

A homogeneous object has ρ constant over R and thus $(\bar{x}, \bar{y})$ coincide with its geometric centre.

For a closed region R of area $|R|$

$$\bar{x} = \frac{1}{|R|} \iint_R x \mathrm{d}A \quad \Longleftrightarrow \quad \iint_R x \mathrm{d}A = |R|\, \bar{x},$$

and

$$\bar{y} = \frac{1}{|R|} \iint_R y \mathrm{d}A \quad \Longleftrightarrow \quad \iint_R y \mathrm{d}A = |R|\, \bar{y}.$$

If R has a regular shape and we can find its centre $(\bar{x}, \bar{y})$ and area $|R|$ from geometry then we don't have to integrate to get the value of the 'red' integrals.

Variable transforms

Aim: Sometimes it may be convenient to introduce new variables in order to simplify the integration.

As an example consider finding the area of the ellipse[1]

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

on the xy plane.

Since the points inside the ellipse satisfy $R : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$ then we have to solve

$$|R| = \iint_R \mathrm{d}x\mathrm{d}y$$

[1] Although ellipse is a regular shape, its area is not as well known as that of the circle.

Variable transforms

If we **change variables**, i.e. rescale the axes, using

$$u = \frac{x}{a}, \quad \text{and} \quad v = \frac{y}{b},$$

then on the uv plane the ellipse is expressed as 'a unit circle'

$$u^2 + v^2 = 1$$

This transform expresses the ellipse R on the xy plane, as a unit circle R' on the uv plane.
Assembling the integral in uv coordinates, remembering that for $u = \frac{x}{a}$, $v = \frac{y}{b}$ we have $\mathrm{d}u = \frac{1}{a}\mathrm{d}x, \quad \mathrm{d}v = \frac{1}{b}\mathrm{d}y$,

$$\iint\limits_{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1} \mathrm{d}x\mathrm{d}y = \iint\limits_{u^2+v^2\leq 1} ab\, \mathrm{d}u\mathrm{d}v = ab\pi,$$

since on the uv plane R' is a unit disk of area π.

Variable transforms

In changing of variables we must express $\mathrm{d}A$ in terms of the new variables of integration, e.g. in the ellipse example the square $\mathrm{d}x\mathrm{d}y$ on xy became a rectangle $ab\,\mathrm{d}u\mathrm{d}v$ in uv.

Consider solving

$$\iint_R (3x - 2y)(x + y)\mathrm{d}x\mathrm{d}y,$$

over a region R with a change $u = 3x - 2y$ and $v = x + y$.

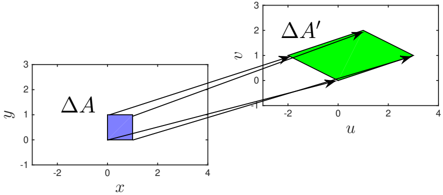
How does ΔA on R gets mapped to $\Delta A'$ on R'? Are they the same, smaller, bigger, different shape, ...?

What's the relation between $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ and $\mathrm{d}A' = \mathrm{d}u\mathrm{d}v$?

Example

Let's pick a ΔA with vertices $(x, y) = \{(0,0),(0,1),(1,0),(1,1)\}$.

Mapping these vertices on the uv plane using $u = 3x - 2y$ and $v = x + y$, ΔA becomes a parallelogram with area $|\Delta A'| = 5$.



The transformed integral should be

$$\iint_R (3x-2y)(x+y)\boxed{\mathrm{d}x\mathrm{d}y} = \iint_{R'} uv \boxed{\frac{1}{5}\mathrm{d}u\mathrm{d}v}$$

Changing from (x, y) to (u, v)

Given (u, v) as functions of (x, y) we can express ΔA in terms of Δx and Δy and so the mapping **from** (x, y) **to** (u, v) satisfies

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \approx \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

from where we can deduce that

$$\mathrm{d}A' = \mathrm{d}u\mathrm{d}v \approx \text{abs} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \mathrm{d}x\mathrm{d}y = \text{abs} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \mathrm{d}A$$

or simply

$$\mathrm{d}u\mathrm{d}v \approx ||\mathbf{J}||\mathrm{d}x\mathrm{d}y,$$

with $||\mathbf{J}||$ the absolute value of the determinant of the Jacobian matrix of the mapping from (x, y) to (u, v).

Changing from (u, v) to (x, y)

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or simply

$$\mathrm{d}x\mathrm{d}y \approx ||\mathbf{J}||\mathrm{d}u\mathrm{d}v,$$

with $||\mathbf{J}||$ the absolute value of the determinant of the Jacobian matrix of the mapping from (u, v) to (x, y).

Example

Compute the integral

$$\iint_R x^2 y \,\mathrm{d}x\mathrm{d}y, \quad \text{for } R : 0 \leq (x, y) \leq 1,$$

by changing variables through the mapping $u = x$ and $v = xy$.

This mapping gives (u, v) in terms of (x, y) thus

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = |x|$$

Since x is positive everywhere in R apart from a single point, then $\mathrm{d}x\mathrm{d}y = \frac{1}{x}\mathrm{d}u\mathrm{d}v$. To setup the integrand in terms of u and v we have

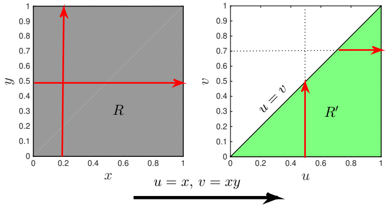
$$x^2 y \mathrm{d}x\mathrm{d}y = x^2 y \frac{1}{x} \mathrm{d}u\mathrm{d}v = v\mathrm{d}u\mathrm{d}v$$

Example

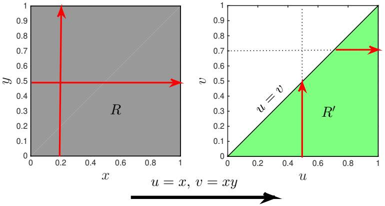
To work out the bounds for u and v in the transformed integral we need to find how does the square R changes to R'. Recall that for $R : 0 \leq (x, y) \leq 1$

$$x : 0 \to 1 \text{ (inner)}, \quad y : 0 \to 1 \text{ (outer)}.$$

Mapping the vertices of R on the uv plane the **square becomes … a triangle**



Example



Effectively the integral over R' works out as

$$\int_0^1 \int_v^1 v\,\mathrm{d}u\mathrm{d}v = \int_0^1 \Big[vu\Big]_v^1 \mathrm{d}v = \frac{1}{6}$$

Alternatively, in reversing the order of integration

$$\int_0^1 \int_0^u v\,\mathrm{d}v\mathrm{d}u = \int_0^1 \Big[\frac{v^2}{2}\Big]_0^u \mathrm{d}u = \frac{1}{6}$$

Double integrals in polar

The special case where we switch between polar to Cartesian,

$$x = r\cos\theta, \quad y = r\sin\theta \qquad \text{from } (r,\theta) \text{ to } (x,y)$$

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x) \qquad \text{from } (x,y) \text{ to } (r,\theta)$$

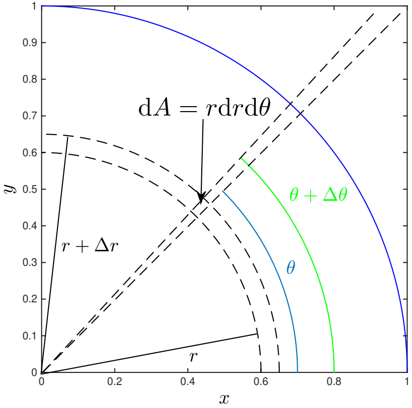
Changing from polar to Cartesian coordinates using $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = |r(\cos^2\theta + \sin^2\theta)| = r,$$

thus the integration element in polar coordinates satisfies

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta = r\mathrm{d}A'.$$

The shape of $\mathrm{d}A$ on the $r\theta$ plane



Example

Recall one of the module 7 examples $f(x,y) = 1-x^2-y^2 = 1-r^2$ for $R : x^2+y^2 \leq 1,\ x, y \geq 0$

$$\iint_R 1-x^2-y^2\,\mathrm{d}A = \int_?^? \int_?^? (1-r^2)\,r\mathrm{d}r\mathrm{d}\theta$$

From the quarter disk geometry of R it is clear that within the first quadrant: $0 \leq \theta \leq \frac{\pi}{2}$ hence

$$\iint_R 1-x^2-y^2\,\mathrm{d}A = \int_0^{\frac{\pi}{2}} \int_?^? (1-r^2)\,r\mathrm{d}r\mathrm{d}\theta$$

but note however that at for all possible θ the values of r within the quarter disk are fixed to $0 \leq r \leq 1$, yielding

$$\iint_R 1-x^2-y^2\,\mathrm{d}A = \int_0^{\frac{\pi}{2}} \int_0^1 (1-r^2)\,r\mathrm{d}r\mathrm{d}\theta.$$

Example

Effectively, this yields a **simple inner integral**

$$\int_0^1 r - r^3 \mathrm{d}r \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{1}{4}$$

and an even simpler outer integral

$$\int_0^{\frac{\pi}{2}} \frac{1}{4} \mathrm{d}\theta = \frac{\pi}{8},$$

leading to the same result without the laborious integration in Cartesian coordinates.

Formulas

Let R a closed region on the xy plane and $f(x,y)$ a function that's continuous therein.

- The area of R is $|R| = \iint_R \mathrm{d}A$
- The average of f over R is $\bar{f} = \frac{1}{|R|} \iint_R f \mathrm{d}A$
- The centre of mass of R if it has density ρ and mass M has coordinates $\bar{x} = \frac{1}{M} \iint_R x\rho \mathrm{d}A$, $\bar{y} = \frac{1}{M} \iint_R y\rho \mathrm{d}A$
- The geometric centre of R has coordinates as above with $\rho = 1$ and $M = |R|$.
- Transforms $u(x,y)$, $v(x,y)$ yield $\mathrm{d}u\mathrm{d}v \approx ||J||\mathrm{d}x\mathrm{d}y$ with $J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$
- Transforms $x(u,v)$, $y(u,v)$ yield $\mathrm{d}x\mathrm{d}y \approx ||J||\mathrm{d}u\mathrm{d}v$ with $J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

Main outcomes of module 8

You MUST know:

1. The three applications of double integrals.
2. How to change integration variables using the variable transforms.
3. How to pose and solve double integrals in polar coordinates.
4. How to use geometry to solve some simple double integrals.

Good to know:

Geometric centre of the circle, ellipse, square, triangles etc.