Module 15 self-assessment

Question 1

Let X_1 and X_2 be independent standard normal random variables, and define $Y_1 = 2X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find

1. the expectations of Y_1 and Y_2,
2. the covariance between Y_1 and Y_2, and
3. the joint $p_{Y_1,Y_2}(y_1, y_2)$

Solution:

(1). The expectations are easy to see that are equal to zero since we are given that $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$,

$$\mathbb{E}[Y_1] = \mathbb{E}[2X_1 + X_2] = 2\mathbb{E}[X_1] + \mathbb{E}[X_2] = 0 + 0 = 0,$$

and

$$\mathbb{E}[Y_2] = \mathbb{E}[X_1 - X_2] = \mathbb{E}[X_1] - \mathbb{E}[X_2] = 0 - 0 = 0.$$

(2). The covariance is

$$\begin{aligned}
\mathrm{Cov}(Y_1, Y_2) &= \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_1]\mathbb{E}[Y_2] \\
&= \mathbb{E}[Y_1 Y_2] \\
&= \mathbb{E}[(2X_1 + X_2)(X_1 - X_2)] \\
&= \mathbb{E}[2X_1^2 - X_1 X_2 - X_2^2] \\
&= 2\mathbb{E}[X_1^2] - \mathbb{E}[X_1 X_2] - \mathbb{E}[X_2^2] \\
&= 2\big(\mathrm{Var}(X_1) - \mathbb{E}[X_1]^2\big) - \mathbb{E}[X_1]\mathbb{E}[X_2] - \big(\mathrm{Var}(X_2) - \mathbb{E}[X_2]^2\big) \\
&= 2 - 0 - 1 = 1,
\end{aligned}$$

where we have expressed the expectation of the product of X_1 and X_2 as the product of their respective expectations since we are given that the variables are independent.

(3). The joint $p_{Y_1,Y_2}(y_1, y_2)$ is clearly a bivariate normal and it is determined by the means, covariance and correlation of Y_1 and Y_2, of which we have everything apart from the correlation coefficient

$$\rho(Y_1, Y_2) = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_{Y_1} \cdot \sigma_{Y_2}},$$

where the standard deviations of Y_1 and Y_2 can be found by the square roots of their respective variances. As X_1 and X_2 are independent

$$\mathrm{Var}(Y_1) = \mathrm{Var}(2X_1 + X_2) = 4\mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 5 \quad \Rightarrow \quad \sigma_{Y_1} = \sqrt{5},$$

and

$$\mathrm{Var}(Y_2) = \mathrm{Var}(X_1 - X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 2 \quad \Rightarrow \quad \sigma_{Y_2} = \sqrt{2},$$

hence

$$\rho(Y_1, Y_2) = \frac{1}{\sqrt{5} \cdot \sqrt{2}} = \frac{1}{\sqrt{10}}$$

and

$$p_{Y_1,Y_2}(y_1, y_2) = \frac{1}{\sqrt{2\pi |\Sigma|}} \exp\Big\{-\frac{1}{2} \begin{pmatrix} y_1 & y_2 \end{pmatrix} \Sigma^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\Big\}$$

with $\Sigma = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$, $|\Sigma| = 9$ and $\Sigma^{-1} = \frac{1}{9}\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$, thus in its simplest form the bivariate is

$$p_{Y_1,Y_2}(y_1, y_2) = \frac{1}{9\sqrt{2\pi}} \exp\Big\{-\frac{1}{9} y_1^2 + \frac{1}{9} y_1 y_2 - \frac{5}{9} y_2^2\Big\}.$$

Question 2

Let X and Y be continuous random variables with joint density $p_{X,Y}(x, y) = x + y$ defined on the square $[0, 1] \times [0, 1]$. Compute $F_{X,Y}(x, y)$ their respective CDF, and the marginal densities of X and Y. Are X and Y independent? What is the covariance $\mathrm{Cov}(X, Y)$?

Solution:

To compute the cumulative density function let us take a point (a, b) such as $0 \le x \le 1$ and $0 \le y \le 1$, and consider the CDF as the integral of the density function

$$\mathbb{P}(X \le a, Y \le b) = \int_0^a \int_0^b p_{X,Y}(x, y) \mathrm{d}x\mathrm{d}y = \int_0^a \int_0^b (x + y) \mathrm{d}y\mathrm{d}x.$$

This gives an inner integral

$$\int_0^b (x + y)\mathrm{d}y = \Big[xy + \frac{y^2}{2}\Big]_0^b = xb + \frac{b^2}{2},$$

and an outer

$$\int_0^a (xb + \frac{b^2}{2})\mathrm{d}x = \Big[\frac{x^2}{2}b + \frac{b^2}{2}x\Big]_0^a = \frac{a^2 b + ab^2}{2}.$$

Generalising for all (a, b) in $[0, 1] \times [0, 1]$ we get

$$F_{X,Y}(x, y) = \frac{x^2 y + y^2 x}{2}, \quad 0 \le x, y \le 1.$$

The marginals are then obtained by integrating the joint PDF as

$$p_X(x) = \int_0^1 (x + y)\mathrm{d}y = \Big[xy + \frac{y^2}{2}\Big]_0^1 = x + \frac{1}{2},$$

and

$$p_Y(y) = \int_0^1 (x + y)\mathrm{d}x = \Big[\frac{x^2}{2} + xy\Big]_0^1 = y + \frac{1}{2}.$$

Joint variables X and Y are independent if their joint PDF is separable, i.e. if

$$p_{X,Y}(x,y) = p_X(x)p_Y(y).$$

Here is it clear that

$$x + y \neq (x + \frac{1}{2})(y + \frac{1}{2}),$$

so X and Y are not independent.

For the covariance we have

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y],$$

where

$$\mathbb{E}[X] = \int_0^1 x(x + \frac{1}{2})\mathrm{d}x = 7/12$$

and similarly $\mathbb{E}[Y] = 7/12$, while

$$\mathbb{E}[f(X,Y)] = \mathbb{E}[XY] = \int_0^1 \int_0^1 xy(x+y)\mathrm{d}x\mathrm{d}y = \frac{1}{3}.$$

Putting these into the formula for $\mathrm{Cov}(X,Y)$ gives -144^{-1}.