Engineering Mathematics 2B

Module 6: Polar coordinates

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Module 6 contents

Mathematical convenience

When integrating on curved, non-straight paths polar coordinates lead to indegrands that are simpler, which in turn simplifies the integration.

Take for example the equation of a circle of radius a centred at the origin.

In Cartesian coordinates this is

$$x^2 + y^2 = a^2,$$

while in polar it becomes

$$r = a$$

Reversely, the equation of a line through the origin in Cartesian is

$$y = ax,$$

and there's no expression for such a line in polar coordinates.

Cartesian coordinates

So far we have seen:

1. Field definitions in x, y (and z for 3D)

2. Have defined $\nabla \doteq \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}}$, and the differential operators for gradient, divergence and curl with respect to x, y and z

3. Defined path c on the xy plane in x, y

4. Gave definitions for work and flux integrals in $\mathrm{d}\mathbf{r} = \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}}$, and $\hat{\mathbf{n}}\mathrm{d}s = \mathrm{d}y\hat{\mathbf{i}} - \mathrm{d}x\hat{\mathbf{j}}$.

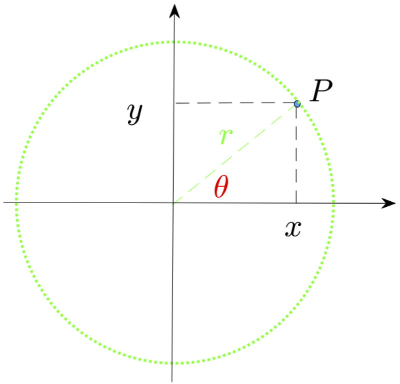
Cartesian coordinates seem to be a *natural* choice, at least in terms of human perception and image interpretation.

Cartesian

Our surroundings are dominated by straight horizontal and vertical lines, 90 degrees angles... and much fewer curves

Polar coordinates

Every point P on the xy plane admits two different sets of coordinates, both with 2 degrees of freedom:



Cartesian (x, y) or polar (r, θ)? Note that despite how we call these **two** coordinates the position of the point is one and the same! Moreover, the xy plane is the $r\theta$ plane.

Polar coordinates

From the Cartesian (x, y) where $-\infty \leq x,\, y \leq +\infty$ we get the polar by[1]

$$r = \sqrt{x^2 + y^2}, \qquad 0 \leq r \leq +\infty$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

From the polar (r, θ) we can get the Cartesian by

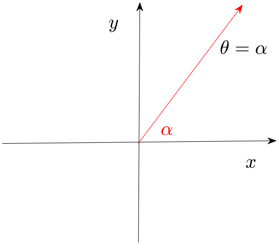
$$x = r\cos\theta, \quad y = r\sin\theta.$$

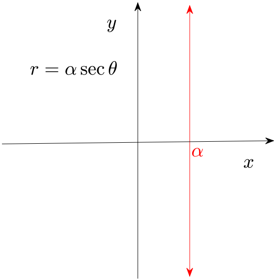
For a constant α:

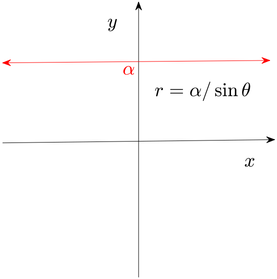
1. $x = \alpha$ and $y = \alpha$ are infinitely long lines normal to x and y axes at α respectively.
2. $r = \alpha$ is a circle of radius α centred at the origin.
3. $\theta = \alpha$ is a line starting at the origin and extending to $r = +\infty$ with an angle α.

[1] $\tan^{-1}(\phi) = \arctan(\phi)$.

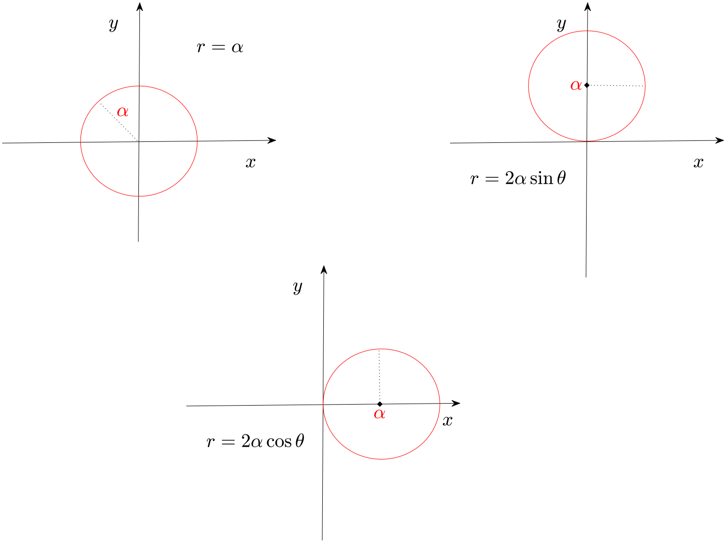
Useful curves in polar coordinates







Useful curves in polar coordinates



Fields in polar

Unlike the Cartesian frame whose x and y axes are fixed in the directions of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, those of the polar frame are not. In polar we have: $\hat{\boldsymbol{\rho}}$ the unit position vector and $\hat{\boldsymbol{\phi}}$ defining the $r\theta$ plane (between them are orthogonal like $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$)

$$\hat{\boldsymbol{\rho}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{\sqrt{x^2+y^2}}\,, \qquad \hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{\sqrt{x^2+y^2}}, \qquad \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}} = 0.$$

If $f(r,\theta)$ and $\mathbf{a}(r,\theta) = f(r,\theta)\hat{\boldsymbol{\rho}} + g(r,\theta)\hat{\boldsymbol{\phi}}$ are continuous fields on the $r\theta$ plane, the gradient and the divergence in polar coordinates are

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{\rho}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\phi}}\,, \qquad \nabla \cdot \mathbf{a} = \frac{1}{r}\frac{\partial (rf)}{\partial r} + \frac{1}{r}\frac{\partial g}{\partial \theta}.$$

We don't have to use these expressions in order to work out line integrals in polar coordinates.

Work in polar

Find the work of $\mathbf{a} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$ on the circle $c : x^2 + y^2 = 4$, in anticlockwise direction.

$$\int_c \mathbf{a} \cdot \mathrm{d}\mathbf{r} = \int_c y \mathrm{d}x - \int_c x \mathrm{d}y$$

From c we have $y = \pm\sqrt{4 - x^2}$ and $x = \pm\sqrt{4 - y^2}$. To take account of the sign changes we must split the first integral into "above" and "below" the x axes as

$$\int_c y \mathrm{d}x = \int_2^{-2} \sqrt{4 - x^2} \mathrm{d}x + \int_{-2}^{2} -\sqrt{4 - x^2} \mathrm{d}x,$$

and work similarly for the second, this time splitting into "right" and "left" the y axes as

$$\int_c x \mathrm{d}y = \int_{-2}^{2} \sqrt{4 - y^2} \mathrm{d}y + \int_2^{-2} -\sqrt{4 - y^2} \mathrm{d}y.$$

Work in polar

Exploiting the symmetry of the limits and the integrands we arrive at

$$\begin{aligned}\int_c y\mathrm{d}x - \int_c x\mathrm{d}y &= 2\Big(\int_2^{-2} \sqrt{4-x^2}\mathrm{d}x - \int_{-2}^{2} \sqrt{4-y^2}\mathrm{d}y\Big) \\ &= 2\Big(\int_2^{-2} \sqrt{4-x^2}\mathrm{d}x + \int_{2}^{-2} \sqrt{4-y^2}\mathrm{d}y\Big) \\ &= 4\int_2^{-2} \sqrt{4-x^2}\mathrm{d}x = 4\,(-2\pi) = -8\pi.\end{aligned}$$

On the other hand, switching to polar we have that on c, $x = 2\cos\theta$ and $y = 2\sin\theta$, hence $\mathrm{d}x = -2\sin\theta\mathrm{d}\theta$ and $\mathrm{d}y = 2\cos\theta\mathrm{d}\theta$ hence

$$\begin{aligned}\int_c \mathbf{a}\cdot\mathrm{d}\mathbf{r} &= \int_c 2\sin\theta(-2\sin\theta)\mathrm{d}\theta - \int_c 2\cos\theta(2\cos\theta)\mathrm{d}\theta \\ &= -4\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\mathrm{d}\theta = -8\pi.\end{aligned}$$

Flux in polar

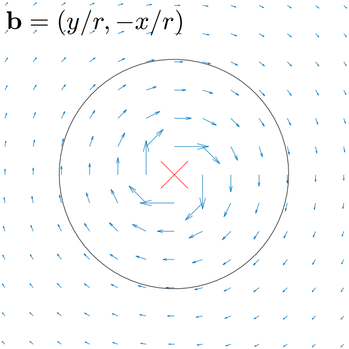
Find the flux of $\mathbf{a} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$ on the circle $c : x^2 + y^2 = 4$, in anticlockwise direction.

Invoking the 'Cartesian to polar' change in coordinates on c, $x = 2\cos\theta$ and $y = 2\sin\theta$, hence $\mathrm{d}x = -2\sin\theta\mathrm{d}\theta$ and $\mathrm{d}y = 2\cos\theta\mathrm{d}\theta$

$$\begin{aligned}
\int_c \mathbf{a}\cdot\hat{\mathbf{n}}\,\mathrm{d}s &= \int_c \mathbf{a}\cdot\left(\mathrm{d}y\hat{\mathbf{i}} - \mathrm{d}x\hat{\mathbf{j}}\right)\\
&= \int_c y\,\mathrm{d}y + \int_c x\,\mathrm{d}x\\
&= \int_0^{2\pi} 2\sin\theta(2\cos\theta)\mathrm{d}\theta + \int_0^{2\pi} 2\cos\theta(-2\sin\theta)\mathrm{d}\theta\\
&= \int_0^{2\pi} 4\sin\theta\cos\theta\mathrm{d}\theta - \int_0^{2\pi} 4\sin\theta\cos\theta\mathrm{d}\theta = 0
\end{aligned}$$

The mixing velocity

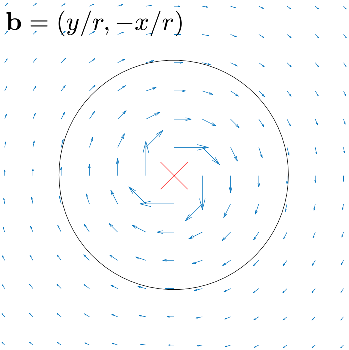
Now consider its scaled version $\mathbf{b} = \mathbf{a}/r$ where $r = \sqrt{x^2 + y^2}$ emulating the flow (velocity) of a liquid during mixing.



The velocity of the mixer is clockwise and decreases in magnitude away from the spinning blade which is at the centre of the circle.

The mixing velocity

Suppose that the velocity is **b**.



This particular choice of velocity **b** makes the flow tangential to the container walls.

The mixing velocity

(1) The curl of the velocity field is called vorticity and for the given $\mathbf{b}$ it is equal to

$$\nabla \times \mathbf{b} = \Big(\frac{\partial}{\partial x}\,(-\frac{x}{r}) - \frac{\partial}{\partial y}(\frac{y}{r})\Big)\hat{\mathbf{k}} = -\frac{1}{r}\hat{\mathbf{k}}$$

pointing downwards (or inwards to the slide) due to the negative $\hat{\mathbf{k}}$ direction. The magnitude of the vorticity diminishes as we move away from the spinning blade.

(2) The liquid does not 'attempt to escape' the container walls, as there is no normal component on that surface.

Assuming the container is circular in shape notice that the velocity is aligned to the circle (any circle centred at the origin), which indicates that the flux of $\mathbf{b}$ through that circle is zero.

$$\int_c \mathbf{b}\cdot\hat{\mathbf{n}}\mathrm{d}s = \int_c \frac{y}{r}\,\mathrm{d}y - \int_c \frac{x}{r}\,\mathrm{d}x = r - r = 0.$$

Formulas

To switch between Cartesian (x, y) where $-\infty \leq x,\ y \leq +\infty$ and polar (r, θ) coordinates we use

$$r = \sqrt{x^2 + y^2}, \qquad 0 \leq r \leq +\infty$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

and

$$x = r\cos\theta, \quad y = r\sin\theta.$$

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Main outcomes of module 6

You **MUST** know:

1. How to pose the work and flux integrals in polar coordinates.
2. The transformation from polar to Cartesian and vice versa.
3. The equations of basic graphs in polar coordinates.
4. How to solve single variable integrals with trigonometric functions.

Good to know:

Velocity for mixing.