Engineering Mathematics 2B

Module 2: Differentiation

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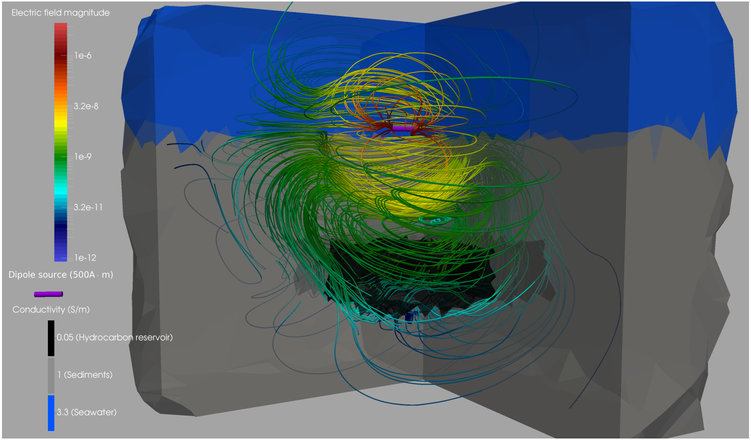
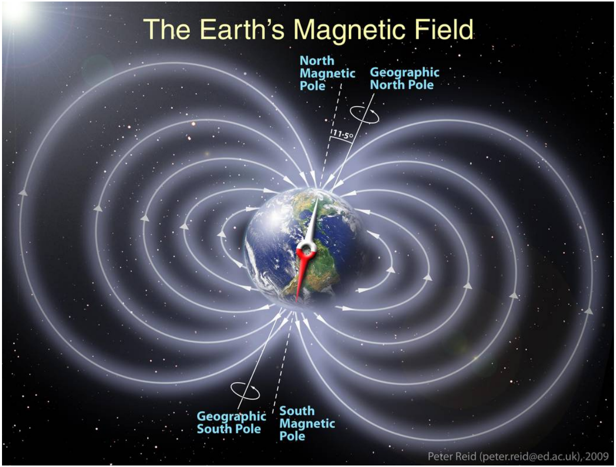
Module 2 contents

Motivation: Electric Dipole field



Offshore geophysical exploration.

Motivation: Magnetic Dipole fields



Earth's magnetic field is solenoidal. [Credits. NASA]

Motivation

Not every vector field describes a natural phenomenon, many of them exist for the sake of learning vector calculus.

Today I will look into some special fields like the electric and magnetic field of a dipole source.

Such dipoles could be negatively and positively charged particles, (batteries) or permanent magnets with North and South poles.

What's so special about them? They are induced from **potentials** by taking their **gradients**. The magnetic field of a permanent magnet doesn't generate or consume energy.

To explain these concepts we need to introduce the divergence and curl of a vector field. Like the gradient of a scalar field, these are differential operators too.

Gradient fields - Conservative

Not all vector fields are gradients of a scalar field. But how would we know?

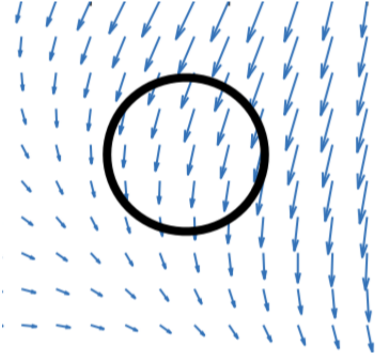
Gradient fields also known as **conservative** vector fields have zero curl.

In case we know that a given $\mathbf{F} = \nabla f$, then how do we find its potential f? Going from f to $\mathbf{F}$ is easy by taking the gradient. But how can we go backwards?

We do that by a trivial process called anti-differentiation! It involves taking derivatives and single variable integrals. See my worked example 1 in this module.

The divergence

Consider an arbitrary vector field $\mathbf{F}$, say it describes the velocity of a fluid on the plane. Let's zoom in to see it at a "point".



The divergence $\nabla \cdot \mathbf{F}$ is the net amount of flux through a small volume around a point. The diameter of the circle is tiny.

The divergence

The divergence of a vector field is a scalar field.

If $\mathbf{F} = f(x,y)\hat{\mathbf{i}} + g(x,y)\hat{\mathbf{j}}$ where f, g are continuous then

$$\nabla\cdot\mathbf{F}(x,y) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}, \qquad \left(\text{resp.} \quad \nabla\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \quad \text{in 3D}\right)$$

The divergence of $\mathbf{F}$ at a point tells us whether $\mathbf{F}$ is generated or stored at that point.

If there's more flux going in than coming out, then divergence is negative, and $\mathbf{F}$ is stored there.

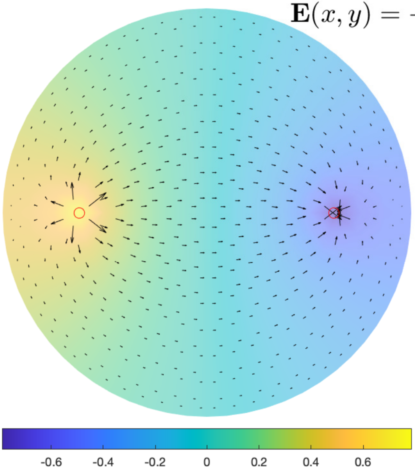
If there's more flux coming out than going in, then divergence is positive, and $\mathbf{F}$ is generated there.

If the net flux is zero, then the divergence is zero, and $\mathbf{F}$ is said to be **solenoidal** (or *incompressible*).

The divergence

Recall the example of the electrostatic potential - electric field we saw in module 1. What's the divergence of $\mathbf{E}$ like?

$$\mathbf{E}(x,y) = -\nabla\phi(x,y)$$



The curl

The triad of differential operators is competed by the curl.

The curl of a vector field $\mathbf{F}$, denoted as $\nabla \times \mathbf{F}$, is a vector field describing the rotation $\mathbf{F}$ causes to a particle positioned within its domain.

Its direction is that of the axis of rotation and its magnitude equal to the speed of rotation.

Any continuous vector field that is *not a gradient* of a potential will have a nonzero curl.

Fields with zero curl are called **irrotational**.

The curl

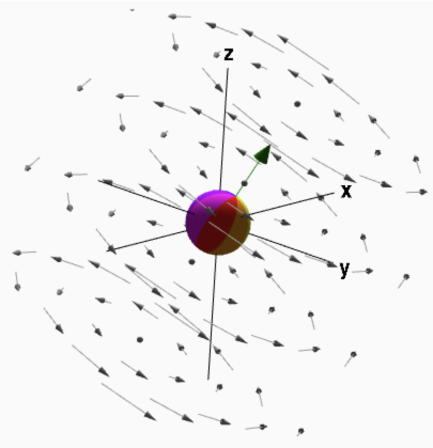
In 3D, for $\mathbf{F} = (f, g, h)$ continuous, the curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}}\end{aligned}$$

In 2D, for $\mathbf{F} = (f, g)$ continuous (on the xy plane), the curl is

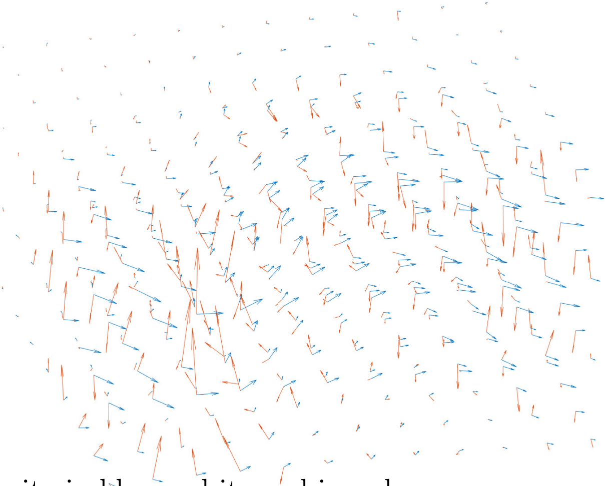
$$\nabla \times \mathbf{F} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}}$$

The curl



Arrows of $\mathbf{F}$ can spin any particle in its way. The one at the origin for example spins around the axis $\nabla \times \mathbf{F}(0,0)$ shown by the green arrow.

The curl of a turbulent wind field



Wind velocity in blue and its curl in red.

The triad of differential operators

We have learned about the gradient, the divergence and the curl.

To rationalise their notation consider this 'del' vector operator

$$\nabla \doteq \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}, \quad \text{(in 3D)}$$

a scalar field f and a vector field $\mathbf{F}$. Then

∇f: the **gradient** of f (scalar multiplication) takes us from a scalar field to a vector field

$\nabla \cdot \mathbf{F}$: the **divergence** of $\mathbf{F}$ (inner product) takes us from a vector field to a scalar field

$\nabla \times \mathbf{F}$: the **curl** of $\mathbf{F}$ (cross product) takes us from a vector field to another vector field

Formulas

Let $\mathbf{F}(x,y,z) = f(x,y,z)\hat{\mathbf{i}} + g(x,y,z)\hat{\mathbf{j}} + h(x,y,z)\hat{\mathbf{k}}$ and $\mathbf{a}(x,y) = f(x,y)\hat{\mathbf{i}} + g(x,y)\hat{\mathbf{j}}$ where f, g and h are continuous everywhere.

- The divergence $\nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$
- The curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}}\end{aligned}$$

- The 2D ‘curl’ is

$$\nabla \times \mathbf{F} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}}$$

Main outcomes of module 2

You MUST know:

1. The physical meaning and how we compute the divergence and the curl.
2. What does it mean for a vector field to be conservative, solenoidal or irrotational.
3. How to establish whether a vector field is conservative and how to compute its potential.

Good to know:

There's a long list of vector calculus identities. Some of them have a great scientific merit and others are useful in solving exercises. Section 3.3.5 of the book. Wikipedia has a list too `https://en.wikipedia.org/wiki/Vector_calculus_identities`