Module 14 self-assessment

Question 1

Consider a continuous random variable X with a probability density function

$$p_X(x) = \begin{cases} x/2 & 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}.$$

If $Y = 1 - \frac{\sqrt{4-X^2}}{2}$ what is the PDF of Y?

Solution:

Following the transformation rule, we must deduce $x(y)$ from the given relation between the two variables

$$y = 1 - \frac{\sqrt{4-x^2}}{2}.$$

Rearranging and getting rid of the square root yields

$$x = \pm 2\sqrt{y(2-y)},$$

from where we pick the positive formula since the probability of x being negative is zero, from $p_X(x)$. Evaluating

$$p_X(x(y)) = \sqrt{y(2-y)}, \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2(1-y)}{\sqrt{y(2-y)}}$$

which yields

$$p_Y(y) = p_X(x(y))\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = 2(1-y), \quad 0 \le y \le 1,$$

and the limits for y are deduced from those of x using the transformation.

Question 2

The Mollytech company manufactures laser pointers, which are packaged in boxes of 20 for shipment. Tests have shown that 4% of their light products are defective.

- What is the probability that a box, ready for shipment, contains exactly 3 defective pointers?
- What is the probability that the box contains 3 or more defective pointers?
- Compute the average number of defective laser pointers per box and the standard deviation.

Solution:

The probability of a single laser pointer being defective is 0.04. Since such a device is either defective or not, then we can model the 'defectiveness' of a laser pointer as a Bernoulli random variable with probability of success $p = 0.04$. Since these are packaged, independently, in sets of 20, then we can cast the number of defective units in a pack as a Binomial random variable X with $n = 20$ and $p = 0.04$,

$$p_X(x) = \mathbb{P}(X = x) = \binom{20}{x} 0.04^x \, 0.96^{20-x}, \quad x = 0, 1, \ldots, 20$$

- $\mathbb{P}(X = 3) = \binom{20}{3} 0.04^3 \cdot 0.96^{17} \approx 0.036$
- $\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X < 3) = 1 - \binom{20}{2} 0.04^2 \cdot 0.96^{18} - \binom{20}{1} 0.04^1 \cdot 0.96^{19} - \binom{20}{0} 0.04^0 \cdot 0.96^{20} \approx 0.044$
- $\mathbb{E}[X] = np = 20 \cdot 0.04 = 0.8$, so less than one unit per pack, and $\sigma_X^2 = np(1-p) = 20 \cdot 0.04 \cdot 0.96 \approx 0.768$, hence $\sigma_X \approx 0.876$.