Module 5 self assessment

Question 1

Find the flux of $\mathbf{f}(x, y) = xy^2\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ through the semicircle

$$c : x^2 + y^2 = 1, \quad y \leq 0$$

with clockwise direction.

Solution:

The path c is the half of the unit circle centred at the origin where $y \leq 0$, and we can thus assume that it starts at $(-1, 0)$ and runs straight on the x axis until $(1, 0)$, before taking on the negative arc and loop over back to $(-1, 0)$. It is natural to split the close loop path c into the first segment, say c_1 and the 180 degrees arc c_2. The flux is

$$\begin{aligned}\int_c \mathbf{f} \cdot \hat{\mathbf{n}}\mathrm{d}s &= \int_{c_1} \mathbf{f} \cdot \hat{\mathbf{n}}\mathrm{d}s + \int_{c_2} \mathbf{f} \cdot \hat{\mathbf{n}}\mathrm{d}s \\ &= \int_{c_1} xy^2\mathrm{d}y - \int_{c_1} x\mathrm{d}x + \int_{c_2} xy^2\mathrm{d}y - \int_{c_2} x\mathrm{d}x\end{aligned}$$

Noting that on c_1 $\mathrm{d}y = 0$ (as well as $y = 0$) the first integral above vanishes. Moreover the second and forth integrals for the direction of c become

$$\int_{c_1} x\mathrm{d}x = \int_{-1}^{1} x\mathrm{d}x, \quad \text{and} \quad \int_{c_2} x\mathrm{d}x = \int_{1}^{-1} x\mathrm{d}x,$$

and thus they cancel each other out leaving

$$\begin{aligned}\int_c \mathbf{f} \cdot \hat{\mathbf{n}}\mathrm{d}s &= \int_{c_2} xy^2\mathrm{d}y \\ &= \int_{c_2 : x \geq 0} xy^2\mathrm{d}y + \int_{c_2 : x < 0} xy^2\mathrm{d}y \\ &= \int_{c_2 : x \geq 0} \sqrt{1 - y^2}y^2\mathrm{d}y + \int_{c_2 : x < 0} -\sqrt{1 - y^2}y^2\mathrm{d}y \\ &= \int_{0}^{-1} \sqrt{1 - y^2}y^2\mathrm{d}y + \int_{-1}^{0} -\sqrt{1 - y^2}y^2\mathrm{d}y \\ &= 2\int_{0}^{-1} \sqrt{1 - y^2}y^2\mathrm{d}y = -\frac{\pi}{8}.\end{aligned}$$

Question 2

Find the flux of $\mathbf{f}(x,y) = y\sin x\hat{\mathbf{i}} + \sin y\hat{\mathbf{j}}$ through a square with vertices $(0,0)$, $(\frac{\pi}{2},0)$, $(\frac{\pi}{2},\frac{\pi}{2})$ and $(0,\frac{\pi}{2})$ in anticlockwise direction.

Solution:

If you draw the schematic for the path c you will see that this forms a square of side $\pi/2$, with its bottom left corner at the origin. This is one of those cases where one has to make some observations before starting to integrate. Remember that in the video I mentioned these 'special' paths that are parallel to the axes. As c runs anticlockwise, beginning from the origin let us call its four segments as: c_1 the line from $(0,0)$ to $(\pi/2,0)$; c_2 the line from $(\pi/2,0)$ to $(\pi/2,\pi/2)$; c_3 that from $(\pi/2,\pi/2)$ to $(0,\pi/2)$, and finally c_4 the one from $(0,\pi/2)$ to $(0,0)$.

The observations we need to do are:

- On c_1, $y=0$ and $\mathrm{d}y=0$,
- On c_2, $x=\pi/2$ and $\mathrm{d}x=0$,
- On c_3, $y=\pi/2$ and $\mathrm{d}y=0$, and
- On c_4, $x=0$ and $\mathrm{d}x=0$

The flux integral is thus

$$\begin{aligned}
\int_c \mathbf{f}\cdot\hat{\mathbf{n}}\mathrm{d}s &= \int_{c_1}\mathbf{f}\cdot\hat{\mathbf{n}}\mathrm{d}s + \int_{c_2}\mathbf{f}\cdot\hat{\mathbf{n}}\mathrm{d}s + \int_{c_3}\mathbf{f}\cdot\hat{\mathbf{n}}\mathrm{d}s + \int_{c_4}\mathbf{f}\cdot\hat{\mathbf{n}}\mathrm{d}s \\
&= \sum_{i=1}^{4}\int_{c_i} y\sin x\mathrm{d}y - \int_{c_i}\sin y\mathrm{d}x \\
&= \int_{c_2} y\mathrm{d}y - \int_{c_3}\mathrm{d}x = \int_0^{\frac{\pi}{2}} y\mathrm{d}y - \int_{\frac{\pi}{2}}^{0}\mathrm{d}x = \frac{\pi^2}{8} + \frac{\pi}{2}.
\end{aligned}$$

Notice that both scalar integrals on c_1 are zero since $y=\sin y=0$ there; and so are those on c_4 where $\sin x=0$ and $\mathrm{d}x=0$. On c_2 we loose the $\mathrm{d}x$ integral and on c_3 the $\mathrm{d}y$.

Question 3

Solve the integral

$$\int_c x(x+y)\mathrm{d}s,$$

on a c given parametrically in terms of the position vector $\mathbf{r}(t) = (1+t)\hat{\mathbf{i}} + (2+3t)\hat{\mathbf{j}}$ for $0\le t\le 1$.

Hint: Recall that the definition of the position vector in Cartesian coordinates is $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

Solution:

Sketching this path will reveal that it is a straight line segment from $(1,2)$ to $(2,5)$, but since the exercise gives us a parameterisation of the path in t we can take this directly,

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = (1+t)\hat{\mathbf{i}} + (2+3t)\hat{\mathbf{j}}$$

hence $x = 1+t$, and $y = 2+3t$ for $0 \le t \le 1$. These definitions imply that $\mathrm{d}x = \mathrm{d}t$ and $\mathrm{d}y = 3\mathrm{d}t$ therefore

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \sqrt{1+9}\mathrm{d}t = \sqrt{10}\mathrm{d}t.$$

Substituting into the line integral we get

$$\begin{aligned}\int_c x(x+y)\mathrm{d}s &= \int_c (1+t)(1+t+2+3t)\sqrt{10}\mathrm{d}t \\ &= \sqrt{10}\int_0^1 (4t^2+7t+3)\mathrm{d}t \\ &= \sqrt{10}\left[\frac{4}{3}t^3 + \frac{7}{2}t^2 + 3t\right]_0^1 = \sqrt{10}\frac{47}{6} \approx 24.77\end{aligned}$$