Module 1 self-assessment

Advice: Do NOT use $\cdot$ or $\times$ for scalar multiplication.

Question 1

Compute the gradient and of the function $f(x, y, z) = z - xy$ and then the directional derivative of f in the direction $\mathbf{q} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$. Evaluate the directional derivative at the point $(1, 1, 1)$.

Solution:

The gradient field of $f = z - xy$ is

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} = -y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}} \equiv (-y, -x, 1).$$

Clearly ∇f is a three-dimensional vector field. To find the directional derivative of f in the direction of the vector $\mathbf{q}$ we have $\nabla f \cdot \hat{\mathbf{q}}$ where

$$\hat{\mathbf{q}} = \frac{\mathbf{q}}{|\mathbf{q}|} = \frac{1}{\sqrt{2}}\mathbf{q}.$$

Since $\mathbf{q} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ is a 2D field, embedded onto the xy plane, we must express it in an equivalent 3D format as $\mathbf{q} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 0\hat{\mathbf{k}} = (1, 1, 0)$ to be able to perform the inner product operation with the gradient as

$$\nabla_{\mathbf{q}} f = \nabla f \cdot \hat{\mathbf{q}} = \frac{-1}{\sqrt{2}}(x + y).$$

Notice that the directional derivative is invariant in the z direction (same for all z), hence at the point $(1, 1, 1)$ it evaluates at

$$\nabla_{\mathbf{q}} f(1, 1, 1) = -\sqrt{2}.$$

Question 2

Dielectrophoresis is a phenomenon where forces are exerted on a particle that is subjected to a non-uniform electric field. In such a case, three force fields are acting simultaneously on a particle positioned at the origin. The forces fields can be described as:

- $\mathbf{F}_1 = \big(xy + 1, y - 2, yz^3 + 3\big)$
- $\mathbf{F}_2$ has constant magnitude 9 and acts in direction $(-2, 2, 1)$
- $\mathbf{F}_3 = 4\hat{\mathbf{i}} - 8\hat{\mathbf{j}}$,

Find the resultant force that acts on the particle. What additional force must be imposed to reduce the resultant to zero?

Solution:

- At the origin $\mathbf{F}_1 = (1, -2, 3)$.
- $\mathbf{F}_2$ has to be converted. $|(-2, 2, 1)| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$, so the unit vector in the direction of $\mathbf{F}_2$ is given by $\frac{1}{3}(-2, 2, 1) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$. $\mathbf{F}_2$ works in this direction and has magnitude 9, so $\mathbf{F}_2 = 9\left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = (-6, 6, 3)$.
- $\mathbf{F}_3$, in the same notation, is equal to $(4, -8, 0)$.

The resultant force $\mathbf{F}$ is then given by

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} = (1, -2, 3) + (-6, 6, 3) + (4, -8, 0) = (-1, -4, 6)\,.$$

The additional force that must be imposed to reduce the resultant force to zero is

$$-\mathbf{F} = (1, 4, -6)\,.$$