Engineering Mathematics 2B

Module 14: Introduction to Probability

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Decision under uncertainty

SpaceX's 2nd Starship Test Has a 60% Chance of Success

Stage separation will be the biggest risk.

By Sissi Cao • 06/26/23 12:38pm

The SpaceX Starship lifts off from the launchpad during a flight test from Starbase in Boca Chica, Texas, on April 20, 2023. PATRICK T. FALLON/AFP via Getty Images

SpaceX is implementing "well over a thousand changes" to its next Starship prototype and fixing its launch pad in Boca Chica, Texas ahead of the rocket's second orbital test this summer, said CEO Elon Musk. During a Twitter Spaces discussion on June 24 with his biographer Ashlee Vance, Musk said, with all the modifications, the chance of the second Starship flight reaching Earth's orbit will go up to 60 percent from his previous estimate of 50 percent.

"I think the probability of this next flight getting to orbit is much higher than the last one. Maybe it's like 60 percent," Musk said. "It depends on how well we do at stage separation."

The Bernoulli random variable

Bernoulli with parameter $0 \le p \le 1$, taking outcomes in $\{0, 1\}$. Describes the toss of a 'numerical coin', i.e. success (=wanted outcome) or failure (=the other outcome) in a single trial (toss).

The parameter p is called probability of success. If the coin is fair ("1" and "0" equally likely), then $p = 0.5$.

$$p_X(x) = \mathbb{P}(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}, \quad \mathbb{E}[X] = p.$$

$$\mathrm{Var}(X) = p(1-p), \quad F_X(x) = \mathbb{P}(X \le x) = \begin{cases} 0 & x < 0 \\ 1 - p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Bernoulli example: Betting strategies

Consider the following betting games involving rolling a fair six-sided dice once. If you were the agency how much money would you put on each award?

X : Roll outcome is 3. X is Bernoulli with $p = \frac{1}{6}$, hence

$$\mathbb{E}[X] = \frac{1}{6}(17\%), \quad \mathrm{Var}(X) = \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}\ (14\%)$$

Y : Roll outcome is odd. Y is Bernoulli with $p = \frac{1}{2}$, hence

$$\mathbb{E}[Y] = \frac{1}{2}(50\%), \quad \mathrm{Var}(Y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}\ (25\%)$$

Z: Roll outcome is < 6. Z is Bernoulli with $p = \frac{5}{6}$, hence

$$\mathbb{E}[Z] = \frac{5}{6}(83\%), \quad \mathrm{Var}(Z) = \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}\ (14\%)$$

Bernoulli example

Other classical Bernoulli variables: A student passing or failing an exam, a soccer player scoring from the penalty spot.

If we are only interested in whether something happens or not (or something is or isn't) then Bernoulli is the right way to express our beliefs.

It is interesting to see some extreme cases: Let X_1, X_2 and X_3 Bernoulli with $p = 0.01$, $p = 0.5$ and $p = 0.99$ respectively.

So X_1 is highly unlikely (to be =1), X_3 is highly likely (to be =1), and X_2 is equally likely to be as is not to be.

How strong can we be about these beliefs: Note that $\text{Var}(X_1) = \text{Var}(X_3) \approx 0.01$, but $\text{Var}(X_2) = 0.25$.

The Binomial random variable

Binomial with parameters $0 \leq p \leq 1$ and n: Describes the number of successes x in n **independent** Bernoulli trials.

The possible outcomes are all integers in $\{0, n\}$. Abrev. $\mathrm{Bin}(p, n)$,

$$p_X(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

$$F_X(k) = \mathbb{P}(X \leq k) = \sum_{x=0}^{\lfloor k \rfloor} \binom{n}{x} p^x (1-p)^{n-x}$$

$$\mathbb{E}[X] = np, \quad \mathrm{Var}(X) = np(1-p),$$

The binomial function is defined as $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ and $\lfloor k \rfloor$ is the 'floor' under k, the greatest integer less than or equal to k.

Example: Quality control by sampling

Bernoulli and Binomial random variables can be used to model the presence or absence of defects in high volume manufacturing.

Each outcome of the manufacturing process (= a part) can be classified as either good (= success) or defective (= failure).

If the process is *perfect* then they are all good. If it is not, then the quality of the part is modelled as a Bernoulli random variable with parameter p.

If it is not practical to test them all then we test a sample and extrapolate the results.

Binomial example cont

Suppose we pick at random 10 out of a possible 100 parts, and test them individually.

By counting $p = \frac{\text{number of acceptable parts among the 10}}{10}$.

Say for example, $p = \frac{8}{10} = 0.8$

Scaling to 100 parts, we expect to have about 20 defective parts.

To estimate the probability of having 7 good parts in a new sample of 10, we need the binomial Bin(0.8,10). With $x = 7$ and $n = 10$ this leads to an estimated probability of 20%

$$p_X(7) = \binom{10}{7} 0.8^7 (1 - 0.8)^{10-7} \approx 0.2$$

The Uniform random variable

Uniform distribution with parameters a, b: Assigns equal likelihood to all (infinite) outcomes of a random variable over the interval $-\infty < a \le x \le b < +\infty$. Often abbreviated as $\mathcal{U}(a, b)$

$$p_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}, \quad \mathbb{E}[X] = \frac{1}{2}(a+b)$$

$$\mathrm{Var}(X) = \frac{1}{12}(b-a)^2, \quad F_X(x) = \begin{cases} 0 & a < x \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$$

Recalling the definitions

$$F_X(x) = \mathbb{P}(X \le x), \quad \text{and} \quad \mathrm{Var}(X) = \int_{-\infty}^{+\infty} p_X(x)(x - \mathbb{E}[X])^2 \mathrm{d}x$$

then you can verify the formulas above using $p_X(x)$.

Uniform example

If we assign a uniform distribution to a random variable, that means that the most precise information we have is that its value could be anywhere between two bounds. Any value within these bounds is equally likely.

Manufacturing tolerances: The diameter of a cylinder on a design is 5 cm. The diameter of a manufactured cylinder is given as $x = 5 \pm 10\%$ cm then the actual size of the diameter could be anywhere between 4.9 and 5.1 cm.

Flight departure times: Flight departure times are given in slots, depending on air traffic and weather conditions. The flight time on your ticket is the earliest time in the slot, but the actual flight can depart anytime within that slot with equal probability.

Uniform example cont

A brand of light bulb advertises that their product can take between 1000 and 1160 hours of operation before needing replacement. If you need a bulb that must be able to operate for 1100 h would you buy this?

According to the manufacturer, X : 'lifetime of bulb' in hours, then $X = \mathcal{U}(1000, 1160)$.

The expected lifetime is $\mathbb{E}[X] = \frac{1}{2}(1000 + 1160) = 1080$ h, and the variance is $\mathrm{Var}(X) = \frac{1}{12}(1160 - 1000)^2 \approx 2133$ h^2. (What are we meant to make out of this?)

$$\mathbb{P}(X \geq 1100) = 1 - F_X(1100) = 1 - \frac{100}{160} = 0.357$$

The Gaussian (aka Normal) random variable

Normal distribution with parameters μ and σ: Describes random numbers that can take any real value, but their likelihood is concentrated (=peaks) at the mean μ. Abbreviated as $\mathcal{N}(\mu, \sigma^2)$.

$$\boxed{p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

where

$$\mathbb{E}[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2, \quad F_X(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

The positive parameter $\sigma^2 > 0$ is the variance, and $\sigma > 0$ is known as the standard deviation.

The special case of the normal with $\mu = 0$ and $\sigma = 1$ is known as the **standard normal** distribution.

Gaussian example

If a random variable is Gaussian, this means that its samples are more likely to be close to its expected value than otherwise.

Why is this important?
All measurements are random variables, whose expectation (mean value) is the impossible to get 'true', or 'noiseless' value.

Consider the scales (weighing) machine one finds in supermarkets for measuring mass. Let X: measured mass, $\mathbb{E}[X]$ the true mass, and x a scales reading. Would X be a case of a normally or a uniformly distributed variable?

Gaussian example: Instrument calibration

In instrumentation science to calibrate an instrument requires an underpinning model of instrumentation noise

$$x = x^* + \eta, \quad \eta \sim \mathcal{N}(\mu_\eta, \sigma_\eta^2)$$

where x is the observable (measured) value, x^* the unobservable true value, and η a Gaussian noise variable with mean μ_η and variance σ_η^2.

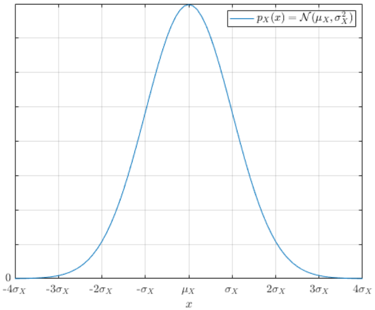
x^* is not a random variable, but x is due to η. This η distorts the information in x^* so ideally we want to get rid of it.

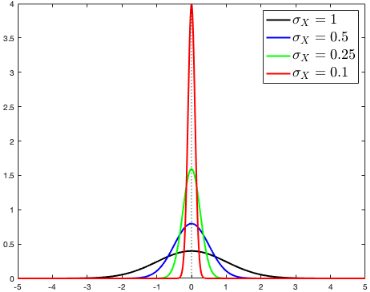
η **cannot** be eradicated, but it can be suppressed.

1. Calibration pushes μ_η close to zero (sometimes too close to zero, i.e. precision electronics £££)
2. Signal processing (averaging) pushes σ^2 to smaller values.

The famous bell curve

The normal (Gaussian) PDF with mean μ_X and variance σ_X^2.





The larger the variance, the flatter the $p_X(x)$ near the mean, hence the larger the uncertainty (range) in the outcome.

Calculating probabilities for the normal

To evaluate $F_X(x) = \mathbb{P}(X \leq x)$ of $X \sim \mathcal{N}(\mu, \sigma^2)$

Standardisation: Compute probability integrals for X in terms of those for the standard normal variable $Z \sim \mathcal{N}(0, 1)$.

Obtain $F_X(x) = \mathbb{P}(X \leq x)$ from that of the standard normal $\Phi(z) = F_Z(z) = \mathbb{P}(Z \leq z)$ as

$$\begin{aligned}
\mathbb{P}(X \leq x) &= \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \\
&= \mathbb{P}\left(Z \leq \frac{x - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{x - \mu}{\sigma}\right) \quad \text{where } \Phi \text{ is a table}
\end{aligned}$$

To get $\mathbb{P}(X \geq x)$ use $\mathbb{P}(X \geq x) = 1 - \mathbb{P}(X \leq x)$.

The cumulative standard normal table $\Phi(z)$

z	−0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.09
-4.0	0.00003	0.00003	0.00003	0.00003	0.00003	0.00003	0.00002	0.00002	0.00002	0.00002
-3.9	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003
-3.8	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.7	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008
-3.6	0.00016	0.00015	0.00015	0.00014	0.00014	0.00013	0.00013	0.00012	0.00012	0.00011
-3.5	0.00023	0.00022	0.00022	0.00021	0.00020	0.00019	0.00019	0.00018	0.00017	0.00017
−3.4	0.00034	0.00032	0.00031	0.00030	0.00029	0.00028	0.00027	0.00026	0.00025	0.00024
−3.3	0.00048	0.00047	0.00045	0.00043	0.00042	0.00040	0.00039	0.00038	0.00036	0.00035
−3.2	0.00069	0.00066	0.00064	0.00062	0.00060	0.00058	0.00056	0.00054	0.00052	0.00050
−3.1	0.00097	0.00094	0.00090	0.00087	0.00084	0.00082	0.00079	0.00076	0.00074	0.00071
−3.0	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.00100
−2.9	0.00187	0.00181	0.00175	0.00169	0.00164	0.00159	0.00154	0.00149	0.00144	0.00139
−2.8	0.00256	0.00248	0.00240	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
−2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.00280	0.00272	0.00264
−2.6	0.00466	0.00453	0.00440	0.00427	0.00415	0.00402	0.00391	0.00379	0.00368	0.00357
−2.5	0.00621	0.00604	0.00587	0.00570	0.00554	0.00539	0.00523	0.00508	0.00494	0.00480
−2.4	0.00820	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
−2.3	0.01072	0.01044	0.01017	0.00990	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
−2.2	0.01390	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.01160	0.01130	0.01101
−2.1	0.01786	0.01743	0.01700	0.01659	0.01618	0.01578	0.01539	0.01500	0.01463	0.01426
−2.0	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.01970	0.01923	0.01876	0.01831
−1.9	0.02872	0.02807	0.02743	0.02680	0.02619	0.02559	0.02500	0.02442	0.02385	0.02330
−1.8	0.03593	0.03515	0.03438	0.03362	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
−1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.03920	0.03836	0.03754	0.03673
−1.6	0.05480	0.05370	0.05262	0.05155	0.05050	0.04947	0.04846	0.04746	0.04648	0.04551
−1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
−1.4	0.08076	0.07927	0.07780	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
−1.3	0.09680	0.09510	0.09342	0.09176	0.09012	0.08851	0.08692	0.08534	0.08379	0.08226
−1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10383	0.10204	0.10027	0.09853
−1.1	0.13567	0.13350	0.13136	0.12924	0.12714	0.12507	0.12302	0.12100	0.11900	0.11702
−1.0	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
−0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
−0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
−0.7	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
−0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510
−0.5	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
−0.4	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
−0.3	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
−0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
−0.1	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
−0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998

For $-4.00 \leq z \leq 4.00$. Source: Wikipedia

Computing probabilities and quantiles in Python

Let $Z \sim \mathcal{N}(0,1)$.

Evaluating Φ at an argument z, i.e. $\Phi(z)$ gives a probability.

Evaluating the inverse of Φ at a probability p gives the quantile point z such that $\mathbb{P}(Z \leq z) = p$.

```
from scipy.stats import norm
import numpy as np


Phi_at_half = norm(loc = 0, scale = 1).cdf(0.5)
Phi_at_zero = norm(loc = 0, scale = 1).cdf(0)
prob_z_positive = 1 - Phi_at_zero
quantile_95 = norm.ppf(0.95, loc = 0, scale = 1)
```

PDFs of functions of random variables

Let X be continuous with $p_X(x)$ for $a \leq x \leq b$. If $Y = f(X)$ and f is invertible what is $p_Y(y)$? Recall that

$$\mathbb{P}(a \leq X \leq b) = \int_{x=a}^{x=b} p_X(x)\mathrm{d}x = 1.$$

Substituting $x = f^{-1}(y)$, and changing the integral into one for y yields

$$\int_{x=a}^{x=b} p_X(x)\mathrm{d}x = \int_{f(a)}^{f(b)} p_X\left(f^{-1}(y)\right)\frac{\mathrm{d}x}{\mathrm{d}y}\mathrm{d}y = \int_{f(a)}^{f(b)} p_Y(y)\mathrm{d}y,$$

The PDF of Y is thus

$$p_Y(y) = p_X\left(f^{-1}(y)\right)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right|, \quad \text{for} \quad y_{\min} \leq y \leq y_{\max}.$$

where $y_{\min} = \min\{f(a), f(b)\}$, $y_{\max} = \max\{f(a), f(b)\}$

Example

Let X be continuous with

$$p_X(x) = 2x\cos x^2, \quad 0 \le x \le \sqrt{\frac{\pi}{2}}.$$

If $Y = X^2$, find (i) the probability density function $p_Y(y)$, and (ii) its support, i.e. the values of y where this is not zero.

The expression for the PDF of Y is

$$p_Y(y) = p_X\left(f^{-1}(y)\right)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right|$$

For $f : y = x^2$, we have $f^{-1} : x = \sqrt{y}$ and thus $\mathrm{d}x = \frac{\mathrm{d}y}{2\sqrt{y}}$ hence

$$p_Y(y) = 2\sqrt{y}\ \cos y\ \frac{1}{2\sqrt{y}} = \cos y.$$

Example cont

To find the support of Y we use

$$y_{\min} = \min\{f(a), f(b)\}, \quad y_{\max} = \max\{f(a), f(b)\}$$

where from $p_X(x)$ we have $a = 0$, and $b = \sqrt{\frac{\pi}{2}}$. With $f : y = x^2$ we get

$$y_{\min} = \min\{0, \frac{\pi}{2}\} = 0, \quad y_{\min} = \max\{0, \frac{\pi}{2}\} = \frac{\pi}{2}$$

hence the complete answer is

$$p_Y(y) = \cos y, \quad 0 \leq y \leq \frac{\pi}{2}$$

You can easily verify that

$$\int_0^{\frac{\pi}{2}} p_Y(y)\mathrm{d}y = \Big[\sin y\Big]_0^{\frac{\pi}{2}} = 1.$$

Formulas

- $X \sim \text{Bern}(p)$, $\mathbb{E}[X] = p$, $\text{Var}(X) = p(1-p)$
- $X \sim \text{Bin}(p, n)$, $\mathbb{E}[X] = np$, $\text{Var}(X) = np(1-p)$
- $X \sim \mathcal{U}(a, b)$, $\mathbb{E}[X] = \frac{1}{2}(a+b)$, $\text{Var}(X) = \frac{1}{12}(b-a)^2$
- $X \sim \mathcal{N}(\mu, \sigma^2)$, $\mathbb{E}[X] = \mu$, $\text{Var}(X) = \sigma^2$
- If $X \sim \mathcal{N}(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$
- If $Z \sim \mathcal{N}(0, 1)$ then $\mathbb{P}(Z \leq z) = \Phi(z)$
- If $X \sim p_X(x)$ and $Y = f(X)$ with f invertible then $p_Y(y) = p_X\left(f^{-1}(y)\right)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| \quad \text{where} \quad y_{\min} \leq y \leq y_{\max}$

Main outcomes of module 14

You MUST know:

1. The properties of the Bernoulli and Binomial random variables.
2. The properties of the uniform and Gaussian random variables.
3. To read probabilities and quantiles from the table $\Phi(z)$.
4. To work out the PDF of functions of random variables.

Good to know:

Python functions that compute the cumulative and inverse cumulative of a normal distribution.