Engineering Mathematics 2B

Module 11: Introduction to Probability

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Module 11 contents

Probability in nature & engineering

Probability theory is the principled framework for studying uncertain or inherently random phenomena.

Quantum physics, such as the decay of radioactive atoms, the emission of photons from an atom, and the tunneling of particles through a barrier. These phenomena are governed by the laws of quantum mechanics, which are probabilistic in nature.

Chemical processes, the random movement of small particles suspended in a fluid is known as *Brownian motion*. This motion is caused by the collisions of the particles with the fluid molecules, for which we have no deterministic observations or models.

Probability in nature & engineering

Genetics & epidemiology are deeply intertwined with probability, both in the inheritance of genes and in modelling random mutations. Making models for the spread of viruses is also an intrinsically, data-driven, probabilistic process.

Computer science: Randomised algorithms make random choices while they are run, and in many important applications they are simpler and more efficient compared to deterministic ones. Probability is the foundation of machine learning and artificial intelligence.

Environmental science: Weather forecasts (future prediction) are computed and expressed in terms of probability.

Randomness

Randomness manifests itself in Engineering in one of two ways:

1. The intrinsic nature of the phenomenon is random, or
2. Our knowledge about the phenomenon is incomplete.

Our incomplete knowledge about something introduces uncertainty in our understanding.

A phenomenon that has more than one possible outcomes is called a random experiment or trial.

That is to say that the trial's outcome is uncertain. e.g. taking a measurement with an instrument in the lab.

A very brief introduction to Sets

Definition: A set is a collection of different objects, typically called elements or members.

$x \in A$ means element x is a member of (belongs to the) set A.

$A \subseteq B$: Set A is a subset of B meaning all members of set A are also members of B.

$\bar{A}$: Let $A \subseteq B$. The complement of A are the elements of B that do not belong to A. Sometimes denoted as A^c.

$A \cup B$: The union of sets A and B is the set of all objects that belong to either or both of the two sets.

$A \cap B$: The intersection of sets A and B is the set of all objects that are members of both sets.

$A \setminus B$: The set difference of A and B is the set of all members of A that are not members of B.

Sets A and B are disjoint if $A \cap B = \emptyset$.

Basic probability concepts

Consider we have a trial or experiment. We can then define various outcomes, random events and an associated sample space.

the **outcome** is the actual result of the trial or experiment.

a **sample space** Ω is the set of **all possible** outcomes of the trial.

a **random event** is the occurrence of a **subset** of Ω, i.e. the occurrence of a set of possible outcomes.

An event 'outcome A has occurred' is allotted a non-negative **probability**, denoted as $\mathbb{P}(A)$ that captures our belief about the likelihood of A occurring.

Notation: $\mathbb{P}(\text{outcome} \in A) = x$ reads the probability of event A occurring is equal to x.

Basic probability concepts - an example

Consider rolling a standard, six-sided dice. Let A be the random event,

$$A := \text{roll outcome is odd} \equiv \{1, 3, 5\},$$

In this case the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. If the dice is fair, then

$$\mathbb{P}(\text{outcome is } n) = \frac{1}{6}, \quad \text{for} \quad n = 1, 2, 3, 4, 5, 6.$$

In effect,

$$\mathbb{P}(A) = \frac{1}{2}$$

The definition of Ω is not unique. For example one can define $\Omega = \{\text{odd}, \text{even}\}$, through which we have as above

$$\mathbb{P}(A) = \frac{1}{2}.$$

Probability axioms

For a probability law $\mathbb{P}$ the following three axioms **always** hold:

1. **Non-negativity**: $\mathbb{P}(A) \geq 0$, for every event A.

2. **Additivity**: If A and B are **disjoint** events in Ω then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B),$$

   where $A \cup B$ reads "event A or B occurring".

   A and B are said to be **disjoint** if they are **mutually exclusive**, i.e. $A \cap B = \emptyset$.

3. **Normalisation**: The probability of the entire sample space Ω (= cumulatively for all possible outcomes) is equal to 1

$$\mathbb{P}(\Omega) = 1.$$

Probability axioms

Non-negativity: The smallest probability an event could take is zero, i.e. $0 \leq \mathbb{P}(A)$.

If $\mathbb{P}(A) = 0$ then event A is impossible. On the contrary, if $\mathbb{P}(A) = 1$, then A is said to be certain.

Additivity: The probability of the union of two disjoint events is the sum of the probabilities of the individual events.

If A and B are two distinct outcomes (say in a dice roll) then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B),$$

because roll outcomes are mutually exclusive.

Normalisation: The sum of probabilities of all events in a sample space equals 1.

Discrete uniform probabilities: Counting

If the sample space consists of n outcomes of equal probability then the probability of an event A is

$$\mathbb{P}(A) = \frac{\text{number of occurrences of } A \text{ in } \Omega}{n}$$

A dice or a coin is called 'fair' if all outcomes have equal probability.

For a fair dice where $\Omega = \{1, 2, 3, 4, 5, 6\}$ then it is easy to see that

$$\mathbb{P}(A = \text{odd}) = \frac{3}{6} = \frac{1}{2}, \qquad \mathbb{P}(A > 4) = \frac{2}{6} = \frac{1}{3}$$

Discrete uniform probabilities: Counting

An urn has 10 balls of which 5 are red, 3 are green and 2 are blue. Suppose we pick one without watching so define

$$A = \text{blue ball is picked}$$

The sample space involves the selection of each of the 10 coloured balls. Making sure that the elements of the sample space (set) are uniquely represented we have

$$\Omega = \{\text{red}_1, \text{red}_2, \text{red}_3, \text{red}_4, \text{red}_5, \text{green}_1, \text{green}_2, \text{green}_3, \text{blue}_1, \text{blue}_2\},$$

so

$$\mathbb{P}(A) = \frac{2}{10}.$$

Now if we let $B = \text{ball picked not blue}$, then

$$\mathbb{P}(B) = \frac{8}{10}.$$

Probability laws: The complement

The **complement** of an event A is the **non-occurrence** of that event, and it is denoted as A^c or $\bar{A}$.

From the last dice example notice that

$$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(A^c) = 1.$$

A more standard paradigm for complement is the toss of a coin with 'head' and 'tail' sides. We have a sample space $\Omega = \{\text{head}, \text{tail}\}$, and we can define events

$$H = \text{outcome is head}, \quad T = \text{outcome is tail}$$

and thus $H = T^c$ and $T = H^c$.

If $\mathbb{P}(H) = a$ then from the normalisation axiom

$$\mathbb{P}(T) = \mathbb{P}(H^c) = 1 - \mathbb{P}(H) = 1 - a.$$

Probability laws: The union of two events

A **union** of two events denotes the logical disjunction (OR) of the events.

Important fact: $\mathbb{P}(A \cup A^c) = 1$.

The **addition rule**, that generalises the additivity axiom, reads 'the probability of event A or B occurring is'

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B),$$

where $\mathbb{P}(A \cap B)$ is 'the probability of A and B occurring'.

If A and B are disjoint then $\mathbb{P}(A \cap B) = 0$, the addition rule simplifies to the additivity axiom

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

Probability laws: The intersection of two events

The **intersection** of two events denotes the logical conjunction (AND) of the events.

The **product rule** of probabilities asserts that 'the probability of A and B occurring equals to the product of the probability of A times the probability of B conditioned on A'

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$$

Notation: $\mathbb{P}(B|A)$ is the probability of B occurring given that A occurs. This is referred to as conditional probability.

Important: Typically $\mathbb{P}(X|Y) \neq \mathbb{P}(Y|X)$ for any distinct events X and Y.

If events A and B are **independent** then

$$\mathbb{P}(B|A) = \mathbb{P}(B) \quad \Rightarrow \quad \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Union versus Intersection example

Your EV is running out of battery and you need to charge at the nearest station, that only has 2 charging points. You only have 1 hour to spare, but you don't know if the charging points will be free to use once you are there. What are your chances of charging up in time?

Define events C_1 : charger one available to use, and C_2 : charger two available to use. Using the union rule,

$$\mathbb{P}(\text{EV charged in time}) = \mathbb{P}(C_1 \cup C_2) = \mathbb{P}(C_1) + \mathbb{P}(C_2)$$

If another car arrives at the station the same time as you then using the intersection rule the probability of charging in time decreases

$$\mathbb{P}(\text{EV charged in time}) = \mathbb{P}\big(C_1 \cap C_2\big) = \mathbb{P}(C_1)\mathbb{P}(C_2)$$

Formulas

- $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$,
- If $\Omega = \{A, B, C\}$ then $\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = 1$
- If $\Omega = \{A, B, C\}$ then $\mathbb{P}(A^c) = \mathbb{P}(B \cup C)$
- For $\mathbb{P}(A \cup B)$ check if A and B are *mutually exclusive*:

$$\mathbb{P}(A \cup B) = \begin{cases} \mathbb{P}(A) + \mathbb{P}(B) & \text{if they are} \\ \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) & \text{if they are not} \end{cases}$$

- For $\mathbb{P}(A \cap B)$ check if A and B are *independent*:

$$\mathbb{P}(A \cap B) = \begin{cases} \mathbb{P}(A)\,\mathbb{P}(B) & \text{if they are} \\ \mathbb{P}(A)\,\mathbb{P}(B|A) & \text{if they are not} \end{cases}$$

- $\mathbb{P}(A \cup B) = \mathbb{P}(B \cup A)$, $\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A)$ but $\mathbb{P}(B|A) \neq \mathbb{P}(A|B)$ unless A and B are independent.

Main outcomes of module 11

You MUST know:

1. The three probability axioms.
2. The complement, union and intersection laws for two events.
3. How to compute probabilities by counting.
4. The meaning of disjoint and independent events.