Module 13 self-assessment

Question 1

A civil engineer is studying an "exit-right-only" lane ending on a set of traffic lights. The lane is long enough to fit a queue of 7 cars. Let X be the number of cars in the lane at the end of a randomly chosen red light. From some observations she did, she thinks that the the probability $\mathbb{P}(X = x) \propto (x+1)(8-x)$ for $x \in \{0, 1, 2, 3, 4, 5, 6, 7\}$. Find the PMF of X and the probability of $X \geq 5$.

Solution:

From the wording, X is a discrete random variable with eight possible outcomes. As we are given the probability of each outcome up to a constant of proportionality then we can rely on the normalisation axiom to find that scaling constant. Effectively, evaluating for $x = 0, 1, 2, \ldots, 7$ the discrete function

$$f(x) = (x+1)(8-x) = -x^2 + 7x + 8$$

gives

$$\{f(x)\}_{x=0}^{7} := \{8, 14, 18, 20, 20, 18, 14, 8\},$$

which is all positive and sums to 120. Dividing gives

$$p_X(x) = \frac{1}{120} f(x), \quad x \in \{0, 1, 2, 3, 4, 5, 6, 7\}.$$

To find $\mathbb{P}(X \geq 5)$ we have

$$\mathbb{P}(X \geq 5) = 1 - \mathbb{P}(X \leq 4) = 1 - F_X(4) = 1 - \sum_{x=0}^{4} p_X(x) = 1 - \frac{1}{120}(8 + 14 + 18 + 20 + 20),$$

thus

$$\mathbb{P}(X \geq 5) = 1 - \frac{80}{120} = \frac{1}{3}.$$

Question 2

Consider a continuous random variable X with a probability density function

$$p_X(x) = \frac{2}{a^2} x, \quad 0 \leq x \leq a$$

Find its cumulative $F_X(x)$, the expectations $\mathbb{E}[X]$, and $\mathbb{E}[X^2 - a^2]$, and the variance $\mathrm{Var}(X)$.

Solution:

The cumulative is the integral of the density function, hence

$$F_X(x) = \int_0^x p_X(x)\mathrm{d}x = \frac{2}{a^2}\int_0^x x\mathrm{d}x = \frac{2}{a^2}\frac{x^2}{2} = \frac{x^2}{a^2}, \quad 0 \le x \le a.$$

$$\mathbb{E}[X] = \int_0^a p_X(x)x\mathrm{d}x = \frac{2}{a^2}\int_0^a x^2\mathrm{d}x = \frac{2}{a^2}\Big[\frac{x^3}{3}\Big]_0^a = \frac{2}{a^2}\frac{a^3}{3} = \frac{2a}{3}.$$

From the property of the expectation, "$\mathbb{E}[X \pm \text{constant}] = \mathbb{E}[X] \pm \text{constant}$" we have

$$\mathbb{E}[X^2 - a^2] = \mathbb{E}[X^2] - a^2 = \int_0^a p_X(x)x^2\mathrm{d}x - a^2 = \frac{2}{a^2}\int_0^a x^3\mathrm{d}x - a^2 = \frac{2}{a^2}\frac{a^4}{4} - a^2 = -\frac{a^2}{2}.$$

To compute the variance we can use

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{a^2}{2} - \frac{4a^2}{9} = \frac{a^2}{18}$$

using the results readily available above. We can also verify it by integrating directly as

$$\begin{aligned}
\mathrm{Var}(X) &= \int_0^a p_X(x)(x - \mathbb{E}[X])^2\mathrm{d}x \\
&= \frac{2}{a^2}\int_0^a x\big(x - \frac{2a}{3}\big)^2\mathrm{d}x \\
&= \frac{2}{a^2}\int_0^a \Big(x^3 - \frac{4a}{3}x^2 + \frac{4a^2}{9}x\Big)\mathrm{d}x \\
&= \frac{2}{a^2}\Big(\frac{1}{4}a^4 - \frac{4}{9}a^4 + \frac{2}{9}a^4\Big) = \frac{1}{18}a^2.
\end{aligned}$$