

# Interpolation

The main reference of this section is [Ame11; BGT10; Poo14]. Let  $\mathbf{k}$  be a finitely generated field over  $\mathbb{Q}$  and  $\bar{\mathbf{k}}$  be its algebraic closure. Let  $X$  be a variety over  $\mathbf{k}$  and  $f : X \dashrightarrow X$  a dominant rational self-map defined over  $\mathbf{k}$ . We want to show that after iteration, we can interpolate the iterates of  $f$  on an analytic open subset of  $X(\mathbb{C}_p)$  for some prime  $p$ .

## 1 Interpolation of analytic maps

In this subsection, we find the interpolation of analytic maps on an analytic disk. Fix a complete non-archimedean field  $\mathbf{k}$  of characteristic 0 with  $|p|_{\mathbf{k}} = 1/p$  for some prime  $p$ . We use the method of difference operators given in [Poo14].

Let  $E = E(0, 1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$  be the closed unit ball in  $\mathbf{k}^d$  and  $\Phi : E \rightarrow E$  be an analytic map, i.e.,  $\Phi \in \mathbf{k}^\circ\{T\}^d$ . Here the norm on  $\mathbf{k}^d$  or  $\mathbf{k}^\circ\{T\}^d$  is the supremum norm, i.e.,  $|x| = \max_{1 \leq i \leq d} |x_i|$ . For every analytic map  $h$  from  $E$  to  $E$ , we define

$$\Delta(h) := h \circ \Phi - h, \quad \Delta^n(h) := \Delta(\Delta^{n-1}(h)) \text{ for } n \geq 1,$$

and  $\Delta^0(h) = h$ . Note that  $\Delta^n(h)$  is still an analytic map from  $E$  to  $E$  by the strong triangle inequality.

**Lemma 1.** We have the following binomial theorem:

$$\sum_{m=0}^n \binom{n}{m} \Delta^m(\text{id}_E) = \Phi^n.$$

*Proof.* By induction, we have

$$\begin{aligned} \Delta^m(\text{id}_E) &= \Delta \left( \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k \right) \\ &= \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^{k+1} - \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k \\ &= \sum_{k=0}^m \left( \binom{m-1}{k-1} (-1)^{m-k} - \binom{m-1}{k} (-1)^{m-1-k} \right) \Phi^k \\ &= \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \Phi^k. \end{aligned}$$

It follows that

$$\begin{aligned}
 \sum_{m=0}^n \binom{n}{m} \Delta^m(\text{id}_E) &= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} (-1)^{m-k} \Phi^k \\
 &= \sum_{k=0}^n \sum_{m=k}^n \binom{n}{k} \binom{n-k}{m-k} (-1)^{m-k} \Phi^k \\
 &= \sum_{k=0}^n \binom{n}{k} \Phi^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \\
 &= \binom{n}{n} \Phi^n + \sum_{k=0}^{n-1} \binom{n}{k} \Phi^k \cdot (1-1)^{n-k} \\
 &= \Phi^n.
 \end{aligned}$$

We finish the proof. □

**Lemma 2.** Suppose that  $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$  satisfies  $\|\Phi - \text{id}_E\| \leq r_p$ . Then **Yang: To be added.**

*Proof.* **Yang: To be added.** □

**Theorem 3** (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3]). Let  $\mathbf{k}$  be a complete non-archimedean field of characteristic 0 with  $|p|_{\mathbf{k}} = 1/p$ . Set  $r_p = p^{-1/(p-1)}$ .

Suppose that  $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$  satisfies  $r := r_p / \|\Phi - \text{id}_E\| > 1$ . Then there exists a function  $F \in \mathbf{k}\{\underline{T}, S/S\}^d$ ,  $1 < s < r$ , such that for each  $n \in \mathbb{Z}_{\geq 0}$  and each  $x \in E$ ,

$$F(x, n) = \Phi^n(x).$$

*Proof.* Consider the formal series

$$F(\underline{T}, S) := \sum_{n=0}^{\infty} \binom{S}{n} \Delta^n(\text{id}_E)(\underline{T}) = \sum_{n=0}^{\infty} f_n.$$

We only need to show that  $f_n \in \mathbf{k}\{\underline{T}, S/S\}^d$  and  $\|f_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Yang: To be added.** □

**Yang:** If  $f$  is invertible, can we see that  $g$  is unique? **Yang:** It seems right.

**Example 4.** Let  $\mathbf{k} = \mathbb{Q}_p$  with  $p \geq 3$ , and let  $\Phi : E \rightarrow E$  be the analytic map defined by  $\Phi(x) = px^2 + x$ . Then we have  $\|\Phi - \text{id}_E\| = \|pT^2\| = 1/p < r_p$ . **Yang: To be checked.**

## 2 Pick integral models

**Lemma 5.** Let  $f : X \dashrightarrow X$  be a dominant rational self-map of a projective variety defined over a finitely generated field  $\mathbf{k}$  over  $\mathbb{Q}$ .

Then there exists a subring  $R \subseteq \mathbf{k}$  of finite type over  $\mathbb{Z}$ , a projective scheme  $\mathcal{X}$  over  $\text{Spec } R$  with generic fiber  $X$ , and a rational self-map  $f : \mathcal{X} \dashrightarrow \mathcal{X}$  over  $\text{Spec } R$  with generic fiber  $f$  such that

- (a) for every prime ideal  $\mathfrak{p}$  of  $R$ , the special fiber  $\mathcal{X}_{\mathfrak{p}}$  is geometrically integral and of the same

dimension as  $X$ ;

- (b) the union of non-smooth locus of  $\mathcal{X}$  and indeterminacy locus, non-étale locus of  $f$  does not contain any entire special fiber  $\mathcal{X}_p$ ;

Moreover, if  $X$  is smooth (resp.  $f$  is a morphism, resp.  $f$  is étale), then we can further require that  $\mathcal{X}$  is smooth over  $\mathrm{Spec} R$  (resp.  $f$  is a morphism, resp.  $f$  is étale over  $\mathrm{Spec} R$ ).

Yang: We can embed  $R$  into  $\mathbb{C}_p$  for some  $p$ .

### 3 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

**Theorem 6** (ref. [Xie25, Proposition 3.24]). Let  $\mathbf{k}$  be a finitely generated field over  $\mathbb{Q}$ ,  $X$  a projective variety defined over  $\mathbf{k}$ , and  $f : X \dashrightarrow X$  a dominant rational self-map defined over  $\mathbf{k}$ .

There exists an iteration  $g = f^m$  of  $f$ , an embedding  $\mathbf{k} \hookrightarrow \mathbb{C}_p$  for some prime  $p \geq 3$ , an analytic open subset  $U \subseteq X(\mathbb{C}_p)$  and an analytic map  $\Phi : \mathbb{C}_p^\circ \times U \rightarrow U$  such that

- (1)  $U \cong (\mathbb{C}_p^\circ)^d$  analytically, where  $d = \dim X$ ;
- (2)  $g$  is well-defined on  $U$ ,  $U$  is invariant under  $g$  and  $\|g|_U - \mathrm{id}_U\| < 1/p$ ;
- (3)  $\Phi(n, x) = g^n(x)$  for each  $n \in \mathbb{Z}_{\geq 0}$  and each  $x \in U$ ;

**Example 7.** Let  $X = E \times E$  with  $E$  an elliptic curve without complex multiplication defined over a number field  $\mathbf{k}$ , and let  $f : X \rightarrow X$  be the endomorphism defined by  $(a, b) \mapsto (a + b, b)$ . Yang: To be continued.

## References

- [Ame11] E Amerik. “Existence of non-preperiodic algebraic points for a rational self-map of infinite order”. In: *Mathematical Research Letters* 18.2 (2011), pp. 251–256 (cit. on p. 1).
- [BGT10] Jason P Bell, Dragos Ghioca, and Thomas J Tucker. “The dynamical Mordell-Lang problem for étale maps”. In: *American journal of mathematics* 132.6 (2010), pp. 1655–1675 (cit. on pp. 1, 2).
- [Poo14] Bjorn Poonen. “p-adic interpolation of iterates”. In: *Bulletin of the London Mathematical Society* 46.3 (2014), pp. 525–527 (cit. on pp. 1, 2).
- [Xie25] Junyi Xie. “The existence of Zariski dense orbits for endomorphisms of projective surfaces”. In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on p. 3).