

Interpolation

The main reference of this section is [Ame11; BGT10; Poo14]. Let \mathbf{k} be a finitely generated field over \mathbb{Q} and \mathbb{k} be its algebraic closure. Let X be a variety over \mathbf{k} and $f : X \dashrightarrow X$ a dominant rational self-map defined over \mathbf{k} . We want to show that after iteration, we can interpolate the iterates of f on an analytic open subset of $X(\mathbb{C}_p)$ for some prime p .

1 Interpolation of analytic maps

In this subsection, we find the interpolation of analytic maps on an analytic disk. Fix a complete non-archimedean field \mathbf{k} of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$ for some prime p . We use the method of difference operators given in [Poo14].

Let $E = E(0, 1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$ be the closed unit ball in \mathbf{k}^d and $\Phi : E \rightarrow E$ be an analytic map, i.e., $\Phi \in \mathbf{k}^\circ[\underline{T}]^d$. Here the norm on \mathbf{k}^d or $\mathbf{k}^\circ[\underline{T}]^d$ is the supremum norm, i.e., $|x| = \max_{1 \leq i \leq d} |x_i|$. For every analytic map h from E to E , we define

$$\Delta(h) := h \circ \Phi - h, \quad \Delta^n(h) := \Delta(\Delta^{n-1}(h)) \text{ for } n \geq 1,$$

and $\Delta^0(h) = h$. Note that $\Delta^n(h)$ is still an analytic map from E to E by the strong triangle inequality.

Lemma 1. We have the following binomial theorem:

$$\sum_{m=0}^n \binom{n}{m} \Delta^m(\text{id}_E) = \Phi^n.$$

Proof. By induction, we have

$$\begin{aligned} \Delta^m(\text{id}_E) &= \Delta \left(\sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k \right) \\ &= \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^{k+1} - \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k \\ &= \sum_{k=0}^m \left(\binom{m-1}{k-1} (-1)^{m-k} - \binom{m-1}{k} (-1)^{m-1-k} \right) \Phi^k \\ &= \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \Phi^k. \end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{m=0}^n \binom{n}{m} \Delta^m(\text{id}_E) &= \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} (-1)^{m-k} \Phi^k \\
&= \sum_{k=0}^n \sum_{m=k}^n \binom{n}{k} \binom{n-k}{m-k} (-1)^{m-k} \Phi^k \\
&= \sum_{k=0}^n \binom{n}{k} \Phi^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \\
&= \binom{n}{n} \Phi^n + \sum_{k=0}^{n-1} \binom{n}{k} \Phi^k \cdot (1-1)^{n-k} \\
&= \Phi^n.
\end{aligned}$$

We finish the proof. □

Lemma 2. Suppose that $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathbf{k}^{\circ}\{\underline{T}\}^d$ satisfies $\|\Phi - \text{id}_E\| \leq r_p$. Then **Yang: To be added.**

Proof. **Yang: To be added.** □

Theorem 3 (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3]). Let \mathbf{k} be a complete non-archimedean field of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$. Set $r_p = p^{-1/(p-1)}$.

Suppose that $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathbf{k}^{\circ}\{\underline{T}\}^d$ satisfies $r := r_p/\|\Phi - \text{id}_E\| > 1$. Then there exists a function $F \in \mathbf{k}\{\underline{T}, S/s\}^d$, $1 < s < r$, such that for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in E$,

$$F(x, n) = \Phi^n(x).$$

Proof. Consider the formal series

$$F(\underline{T}, S) := \sum_{n=0}^{\infty} \binom{S}{n} \Delta^n(\text{id}_E)(\underline{T}) = \sum_{n=0}^{\infty} f_n.$$

We only need to show that $f_n \in \mathbf{k}\{\underline{T}, S/s\}^d$ and $\|f_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Yang: To be added. □

Yang: If f is invertible, can we see that g is unique? **Yang:** It seems right.

Example 4. Let $\mathbf{k} = \mathbb{Q}_p$ with $p \geq 3$, and let $\Phi : E \rightarrow E$ be the analytic map defined by $\Phi(x) = px^2 + x$. Then we have $\|\Phi - \text{id}_E\| = \|pT^2\| = 1/p < r_p$. **Yang: To be checked.**

2 Pick integral models

Lemma 5. Let $f : X \dashrightarrow X$ be a dominant rational self-map of a projective variety defined over a finitely generated field \mathbf{k} over \mathbb{Q} .

Then there exists a subring $R \subseteq \mathbf{k}$ of finite type over \mathbb{Z} , a projective scheme \mathcal{X} over $\text{Spec } R$ with generic fiber X , and a rational self-map $f : \mathcal{X} \dashrightarrow \mathcal{X}$ over $\text{Spec } R$ with generic fiber f such that

- (a) for every prime ideal \mathfrak{p} of R , the special fiber $\mathcal{X}_{\mathfrak{p}}$ is geometrically integral and of the same

dimension as X ;

- (b) the union of non-smooth locus of \mathcal{X} and indeterminacy locus, non-étale locus of f does not contain any entire special fiber $\mathcal{X}_{\mathfrak{p}}$;

Moreover, if X is smooth (resp. f is a morphism, resp. f is étale), then we can further require that \mathcal{X} is smooth over $\text{Spec } R$ (resp. f is a morphism, resp. f is étale over $\text{Spec } R$).

Yang: We can embed R into \mathbb{C}_p for some p .

3 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

Theorem 6 (ref.[Xie25, Proposition 3.24]). Let \mathbf{k} be a finitely generated field over \mathbb{Q} , X a projective variety defined over \mathbf{k} , and $f : X \dashrightarrow X$ a dominant rational self-map defined over \mathbf{k} .

There exists an iteration $g = f^m$ of f , an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \geq 3$, an analytic open subset $U \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \rightarrow U$ such that

- (1) $U \cong (\mathbb{C}_p^\circ)^d$ analytically, where $d = \dim X$;
- (2) g is well-defined on U , U is invariant under g and $\|g|_U - \text{id}_U\| < 1/p$;
- (3) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

Example 7. Let $X = E \times E$ with E an elliptic curve without complex multiplication defined over a number field \mathbf{k} , and let $f : X \rightarrow X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. Yang: To be continued.

References

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- [Xie25] Junyi Xie. “The existence of Zariski dense orbits for endomorphisms of projective surfaces”. In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on p. 3).