

# *Algebraic Dynamics*

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# Algebraic Dynamics

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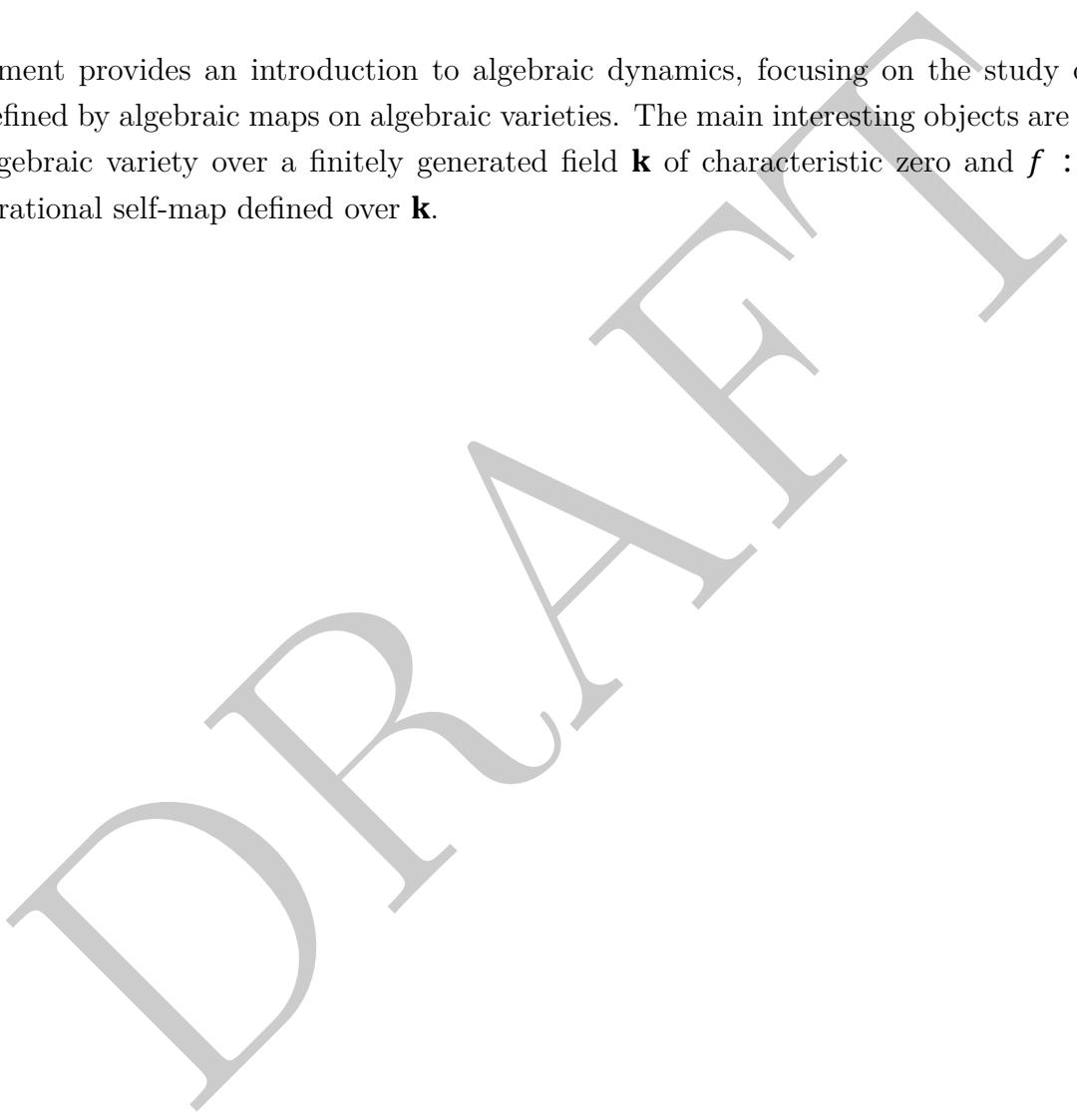
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# Preface

This document provides an introduction to algebraic dynamics, focusing on the study of dynamical systems defined by algebraic maps on algebraic varieties. The main interesting objects are  $(X, f)$ , where  $X$  is an algebraic variety over a finitely generated field  $\mathbf{k}$  of characteristic zero and  $f : X \rightarrow X$  is a dominant rational self-map defined over  $\mathbf{k}$ .



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# Contents

|   |   |
|---|---|
| Preface   | i |
| <b>1 Dynamics on Neron Severi groups</b>                | 1 |
| <b>2 Examples of dynamics</b>                           | 3 |
| 2.1 Examples of dynamic systems on ruled surfaces ..... | 3 |
| 2.1.1 Automorphism groups .....                         | 3 |
| 2.1.2 Polarized induced by base .....                   | 3 |
| 2.2 Examples of dynamics on abelian varieties .....     | 4 |
| 2.2.1 Product of elliptic curves .....                  | 4 |
| <b>3 Dynamical Iitaka theory</b>                        | 7 |
| References  | 9 |

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# Chapter 1

## Dynamics on Neron Severi groups

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# Chapter 2

## Examples of dynamics

### 2.1 Examples of dynamic systems on ruled surfaces

In this section, fix an algebraically closed field  $\mathbb{k}$  of characteristic zero.

#### 2.1.1 Automorphism groups

The main reference is [Mar71].

$$\begin{aligned} H_r &= \left\{ \left( \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & \alpha \end{pmatrix}, \dots, \begin{pmatrix} \alpha & t_r \\ 0 & \alpha \end{pmatrix} \right) \in GL(2, k) \times \dots \times GL(2, k) \mid \begin{array}{l} \alpha \in G_m \\ t_i \in k \end{array} \right\}, \\ H'_r &= \left\{ \left( \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & \beta \end{pmatrix}, \dots, \begin{pmatrix} \alpha & t_r \\ 0 & \beta \end{pmatrix} \right) \in GL(2, k) \times \dots \times GL(2, k) \mid \begin{array}{l} \alpha, \beta \in G_m \\ t_i \in k \end{array} \right\}, \\ \bar{H}'_r &= \left\{ \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha & t_r \\ 0 & 1 \end{pmatrix} \right) \in GL(2, k) \times \dots \times GL(2, k) \mid \begin{array}{l} \alpha \in G_m \\ t_i \in k \end{array} \right\}. \end{aligned}$$

**Theorem 2.1.1.** Let  $\pi : X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e)) \rightarrow C = \mathbb{P}^1$  be a rational ruled surface. We have an exact sequence of algebraic groups

$$1 \rightarrow \bar{H}'_{e+1} \rightarrow \text{Aut}(X) \rightarrow \text{PGL}(2, \mathbb{k}) \rightarrow 1.$$

#### 2.1.2 Polarized induced by base

Let  $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$  be a rank 2 vector bundle on a smooth projective curve  $C$  of genus  $g \leq 1$ , where  $\mathcal{L}$  is a line bundle of degree  $-e$  on  $C$ . Let  $\pi : X = \mathbb{P}_C(\mathcal{E}) \rightarrow C$  be the associated ruled surface. Let  $g : C \rightarrow C$  be an endomorphism of degree  $q$  such that  $g^*\mathcal{L} \cong \mathcal{L}^q$ . Fix the isomorphism  $g^*\mathcal{L} \cong \mathcal{L}^q$ , we have

$$g^* : \mathcal{E} \rightarrow \mathcal{O}_C \oplus \mathcal{L}^q \hookrightarrow \text{Sym}^q \mathcal{E}.$$

## 2.2 Examples of dynamics on abelian varieties

On this section, fix an algebraically closed field  $\mathbb{k}$  of characteristic zero. Everything is defined over  $\mathbb{k}$  unless otherwise specified.

### 2.2.1 Product of elliptic curves

In this subsection, we consider the dynamics induced by matrices on the product of elliptic curves.

Let  $E$  be an elliptic curve without complex multiplication. Consider the abelian variety  $X = E \times E$ . Let  $f_A : X \rightarrow X$  be the endomorphism defined by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let  $[F_1], [F_2], [\Delta]$  be the classes of the fibers of the two projections and the diagonal in  $\text{NS}(X)$ . It is well-known that they span  $\text{NS}(X)$  and the intersection numbers are given by

$$[F_1]^2 = [F_2]^2 = [\Delta]^2 = 0, \quad [F_1] \cdot [F_2] = [F_1] \cdot [\Delta] = [F_2] \cdot [\Delta] = 1;$$

see [Laz04, Section 1.5.B].

We have that  $f_A^*[F_1]$  is given by  $[a]_E(x) + [b]_E(y) = 0$ . Then

$$f_A^*[F_1] \cdot [F_1] = b^2, \quad f_A^*[F_1] \cdot [F_2] = a^2, \quad f_A^*[F_1] \cdot [\Delta] = (a+b)^2.$$

Hence

$$f_A^*[F_1] = (a^2 + ab)[F_1] + (b^2 + ab)[F_2] - ab[\Delta].$$

Similarly, we have

$$f_A^*[F_2] = (c^2 + cd)[F_1] + (d^2 + cd)[F_2] - cd[\Delta],$$

$$f_A^*[\Delta] = (a - c)(a + b - c - d)[F_1] + (b - d)(a + b - c - d)[F_2] - (a - c)(b - d)[\Delta].$$

Thus, the matrix representation of  $f_A^*$  on  $\text{NS}(X)$  with respect to the basis  $\{[F_1], [F_2], [\Delta]\}$  is

$$\begin{pmatrix} a^2 + ab & c^2 + cd & (a - c)(a + b - c - d) \\ b^2 + ab & d^2 + cd & (b - d)(a + b - c - d) \\ -ab & -cd & -(a - c)(b - d) \end{pmatrix}.$$

If we take  $e_1 = [F_1], e_2 = [F_2], e_3 = [\Delta] - [F_1] - [F_2]$  as a new basis of  $\text{NS}(X)$ , then the matrix representation of  $f_A^*$  on  $\text{NS}(X)$  with respect to the basis  $\{e_1, e_2, e_3\}$  is

$$M = \begin{pmatrix} a^2 & c^2 & -2ac \\ b^2 & d^2 & -2bd \\ -ab & -cd & ad + bc \end{pmatrix}.$$

The characteristic polynomial of  $M$  is given by

$$\chi_{f_A^*}(T) = (T - (ad - bc))(T^2 - (a^2 + d^2 + 2bc)T + (ad - bc)^2).$$

Suppose that the eigenvalues of  $A$  are  $\lambda, \mu$ . Then the eigenvalues of  $f_A^*$  on  $\text{NS}(X)$  are given by  $\lambda^2, \mu^2, \lambda\nu$ . When  $a-d, b, c$  are not all zero,  $\text{NS}(X)$  has two invariant subspaces of dimension 1 and 2 respectively. They are given by

$$V_1 = \mathbb{Q} \cdot \begin{pmatrix} 2c \\ -2b \\ a-d \end{pmatrix}, \quad V_2 = \mathbb{Q} \cdot \begin{pmatrix} 0 \\ a-d \\ c \end{pmatrix} \oplus \mathbb{Q} \cdot \begin{pmatrix} d-a \\ 0 \\ b \end{pmatrix} = \{(p, q, r) \mid bp - cq + (a-d)r = 0\}.$$

with respect to the basis  $\{e_i\}$ . One can use ?? to check this in [SageMathCell](#).

With respect to the basis  $\{e_i\}$ , the cones are given by

$$\text{Nef}(X) = \text{Psef}(X) = \{pe_1 + qe_2 + re_3 \mid p, q \geq 0, \quad pq \geq r^2\}.$$

We analyze the intersection of  $V_1, V_2$  with  $\text{Psef}(X)$ . On the plane  $P : p+q=2$ , fix a coordinate system  $(s, r)$  with  $s=(p-q)/2=p-1=1-q$ . The cone  $\text{Psef}(X)$  is given by the disk  $\{(s, r) \mid s^2 + r^2 \leq 1\}$ . The plane  $V_2$  is given by the equation  $b(1+s)-c(1-s)+(a-d)r=(b-c)+(b+c)s+(a-d)r=0$ . If  $b=c$ , then the line  $V_1$  does not intersect the plane  $P$ , hence does not intersect the interior of  $\text{Psef}(X)$ . Otherwise,  $V_1 \cap P$  is given by the point  $(s, r) = \left(\frac{c+b}{c-b}, \frac{a-d}{c-b}\right)$ . There are three cases:

- (a)  $(a+d)^2 < 4(ad-bc)$ :  $V_1$  intersects the interior of  $\text{Psef}(X)$ ,  $V_2$  intersects  $\text{Psef}(X)$  at only the origin.
- (b)  $(a+d)^2 = 4(ad-bc)$ :  $V_1$  intersects the boundary of  $\text{Psef}(X)$  at a ray,  $V_2$  intersects the boundary of  $\text{Psef}(X)$  at a ray.
- (c)  $(a+d)^2 > 4(ad-bc)$ :  $V_1$  intersects  $\text{Psef}(X)$  at only the origin,  $V_2$  intersects the interior of  $\text{Psef}(X)$ .

Note that if  $b=c$ , we always have  $(a+d)^2 > 4(ad-bc)$  under the assumption that  $a-d, b, c$  are not all zero. And note that  $(a+d)^2 - 4(ad-bc) = \text{disc}(\chi_A)$ . Hence, we have the conclusion in [table 2.1](#).

| Case                              | $V_1 \cap \text{Psef}(X)$ | $V_2 \cap \text{Psef}(X)$ | dimension of minimal invariant subspace intersecting $\text{Psef}(X)^\circ$ |
|-----------------------------------|---------------------------|---------------------------|---|
| $A$ is scalar                     |                           |                           |   |
| $A$ has only one eigenvalue       | Boundary (ray)            | Boundary (ray)            | 3   |
| $A$ has complex eigenvalues       | Interior                  | Origin only               | 1   |
| $A$ has distinct real eigenvalues | Origin only               | Interior                  | 2   |

Table 2.1: Intersection of invariant subspaces  $V_1, V_2$  with  $\text{Psef}(X)$  in different cases.

Now we focus on the case when  $A$  has only one eigenvalue, i.e.  $A$  is similar to a Jordan block. Assume that  $A$  is not a scalar matrix, i.e.  $a-d, b, c$  are not both zero. Then easily see that  $V_1 \subset V_2$  iff  $(a+d)^2 = 4(ad-bc)$  iff  $A$  has the Jordan normal form  $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$  for some  $\lambda \in \mathbb{Z}$ . Hence in this case, there is a finite equivariant cover  $(X, f) \rightarrow (X, g)$  such that  $g$  is induced by the Jordan block.

**Example 2.2.1.** Let  $f : X \rightarrow X$  be the endomorphism defined by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then the matrix representation of  $f^*$  on  $\text{NS}(X)$  with respect to the basis  $\{e_1, e_2, e_3\}$  is

$$M = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & -4 \\ -2 & 0 & 4 \end{pmatrix}.$$

The Jordan normal form of  $M$  is

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

The invariant subspaces of  $M$  are given by

$$V_1 = \mathbb{R} \cdot \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \quad V_2 = \{(p, q, r) \mid p = 0\}.$$

We have  $V_2 \cap \text{Psef}(X) = V_1 \cap \text{Psef}(X) = \mathbb{R}_{\geq 0} \cdot (0, 1, 0)^T$ . The action of  $f^*$  on  $\text{NS}(X)$  induces a dynamics on the plane  $P : p + q = 2$ ; see ??.

## Chapter 3

# Dynamical Iitaka theory

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# References

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