Algebraic Dynamics

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Author: Tianle Yang

Email: loveandjustice@88.com

Homepage: https://www.tianleyang.com

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Preface

This document provides an introduction to algebraic dynamics, focusing on the study of dynamical systems defined by algebraic maps on algebraic varieties. The main interesting objects are (X, f), where X is an algebraic variety over a finitely generated field $\mathbf{k}$ of characteristic zero and $f : X \to X$ is a dominant rational self-map defined over $\mathbf{k}$.

Contents

Chapter 1

The first introduction

1.1 Some conjectures in algebraic dynamics

Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$ and $\mathbb{k}$ be its algebraic closure.

1.1.1 Three orbit conjectures

Conjecture 1.1.1. Let X be a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational self-map defined over $\mathbf{k}$. Let $x \in X(\mathbb{k})$ be a point such that its orbit $O_f(x) := \{f^n(x) : n \geq 0\}$ is well-defined and Zariski dense in X. Let $V \subseteq X$ be a subvariety defined over $\mathbf{k}$. Then the set $\{n \geq 0 : f^n(x) \in V\}$ is a union of finitely many arithmetic progressions. Yang:

Conjecture 1.1.2. Let X be a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational self-map defined over $\mathbf{k}$. If there is no nonconstant rational function $g \in \mathbf{k}(X)$ such that $g \circ f = g$, then there exists a point $x \in X(\mathbb{k})$ whose orbit $O_f(x) := \{f^n(x) : n \geq 0\}$ is well-defined and Zariski dense in X. Yang:

Conjecture 1.1.3. Let X be a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational self-map defined over $\mathbf{k}$. Let $x \in X(\mathbb{k})$ be a point whose orbit $O_f(x) := \{f^n(x) : n \geq 0\}$ is well-defined. If the orbit $O_f(x)$ is Zariski dense in X, then the arithmetic degree $\alpha_f(x)$ exists and equals to the first dynamical degree $\lambda_1(f)$. Yang:

1.2 Examples of dynamics

1.2.1 On product of elliptic curves

In this subsection, we consider the dynamics induced by matrices on the product of elliptic curves.

Let E be an elliptic curve without complex multiplication. Consider the abelian variety $X = E \times E$.

Let $f_A : X \to X$ be the endomorphism defined by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $[F_1], [F_2], [\Delta]$ be the classes of the fibers of the two projections and the diagonal in $\mathrm{NS}(X)$. It is well-known that they span $\mathrm{NS}(X)$ and the intersection numbers are given by

$$[F_1]^2 = [F_2]^2 = [\Delta]^2 = 0, \quad [F_1] \cdot [F_2] = [F_1] \cdot [\Delta] = [F_2] \cdot [\Delta] = 1;$$

see [Laz04, Section 1.5.B].

We have that $f_A^*[F_1]$ is given by $[a]_E(x) + [b]_E(y) = 0$. Then

$$f_A^*[F_1].[F_1] = b^2, \quad f_A^*[F_1].[F_2] = a^2, \quad f_A^*[F_1].[\Delta] = (a+b)^2.$$

Hence

$$f_A^*[F_1] = (a^2 + ab)[F_1] + (b^2 + ab)[F_2] - ab[\Delta].$$

Similarly, we have

$$\begin{aligned} f_A^*[F_2] &= (c^2 + cd)[F_1] + (d^2 + cd)[F_2] - cd[\Delta], \\ f_A^*[\Delta] &= (a-c)(a+b-c-d)[F_1] + (b-d)(a+b-c-d)[F_2] - (a-c)(b-d)[\Delta]. \end{aligned}$$

Thus, the matrix representation of f_A^* on $\mathrm{NS}(X)$ with respect to the basis $\{[F_1], [F_2], [\Delta]\}$ is

$$\begin{pmatrix} a^2 + ab & c^2 + cd & (a-c)(a+b-c-d) \\ b^2 + ab & d^2 + cd & (b-d)(a+b-c-d) \\ -ab & -cd & -(a-c)(b-d) \end{pmatrix}.$$

If we take $e_1 = [F_1], e_2 = [F_2], e_3 = [\Delta] - [F_1] - [F_2]$ as a new basis of $\mathrm{NS}(X)$, then the matrix representation of f_A^* on $\mathrm{NS}(X)$ with respect to the basis $\{e_1, e_2, e_3\}$ is

$$M = \begin{pmatrix} a^2 & c^2 & -2ac \\ b^2 & d^2 & -2bd \\ -ab & -cd & ad + bc \end{pmatrix}.$$

The characteristic polynomial of M is given by

$$\chi_{f_A^*}(T) = (T - (ad - bc))(T^2 - (a^2 + d^2 + 2bc)T + (ad - bc)^2).$$

Suppose that the eigenvalues of A are λ, μ. Then the eigenvalues of f_A^* on $\mathrm{NS}(X)$ are given by $\lambda^2, \mu^2, \lambda\nu$. When $a - d, b, c$ are not all zero, $\mathrm{NS}(X)$ has two invariant subspaces of dimension 1 and 2 respectively. They are given by

$$V_1 = \mathbb{Q} \cdot \begin{pmatrix} 2c \\ -2b \\ a-d \end{pmatrix}, \quad V_2 = \mathbb{Q} \cdot \begin{pmatrix} 0 \\ a-d \\ c \end{pmatrix} \oplus \mathbb{Q} \cdot \begin{pmatrix} d-a \\ 0 \\ b \end{pmatrix} = \{(p,q,r) \mid bp - cq + (a-d)r = 0\}.$$

with respect to the basis $\{e_i\}$. One can use **??** to check this in SageMathCell.

With respect to the basis $\{e_i\}$, the cones are given by

$$\mathrm{Nef}(X) = \mathrm{Psef}(X) = \{pe_1 + qe_2 + re_3 \mid p, q \geq 0, \quad pq \geq r^2\}.$$

We analyze the intersection of V_1, V_2 with $\mathrm{Psef}(X)$. On the plane $P : p+q = 2$, fix a coordinate system (s,r) with $s = (p-q)/2 = p-1 = 1-q$. The cone $\mathrm{Psef}(X)$ is given by the disk $\{(s,r) \mid s^2 + r^2 \le 1\}$. The plane V_2 is given by the equation $b(1+s) - c(1-s) + (a-d)r = (b-c) + (b+c)s + (a-d)r = 0$. If $b = c$, then the line V_1 does not intersect the plane P, hence does not intersect the interior of $\mathrm{Psef}(X)$. Otherwise, $V_1 \cap P$ is given by the point $(s,r) = \left(\frac{c+b}{c-b}, \frac{a-d}{c-b}\right)$. There are three cases:

(a) $(a+d)^2 < 4(ad-bc)$: V_1 intersects the interior of $\mathrm{Psef}(X)$, V_2 intersects $\mathrm{Psef}(X)$ at only the origin.

(b) $(a+d)^2 = 4(ad-bc)$: V_1 intersects the boundary of $\mathrm{Psef}(X)$ at a ray, V_2 intersects the boundary of $\mathrm{Psef}(X)$ at a ray.

(c) $(a+d)^2 > 4(ad-bc)$: V_1 intersects $\mathrm{Psef}(X)$ at only the origin, V_2 intersects the interior of $\mathrm{Psef}(X)$.

Note that if $b = c$, we always have $(a+d)^2 > 4(ad-bc)$ under the assumption that $a-d, b, c$ are not all zero. And note that $(a+d)^2 - 4(ad-bc) = \mathrm{disc}(\chi_A)$. Hence, we have the conclusion in table 1.1.

Case	$V_1 \cap \mathrm{Psef}(X)$	$V_2 \cap \mathrm{Psef}(X)$	dimension of minimal invariant subspace intersecting $\mathrm{Psef}(X)^\circ$
A is scalar			
A has only one eigenvalue	Boundary (ray)	Boundary (ray)	3
A has complex eigenvalues	Interior	Origin only	1
A has distinct real eigenvalues	Origin only	Interior	2

Table 1.1: Intersection of invariant subspaces V_1, V_2 with $\mathrm{Psef}(X)$ in different cases.

Now we focus on the case when A has only one eigenvalue, i.e. A is similar to a Jordan block. Assume that A is not a scalar matrix, i.e. $a-d, b, c$ are not both zero. Then easily see that $V_1 \subset V_2$ iff $(a+d)^2 = 4(ad-bc)$ iff A has the Jordan normal form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{Z}$. Hence in this case, there is a finite equivariant cover $(X, f) \to (X, g)$ such that g is induced by the Jordan block.

Example 1.2.1. Let $f : X \to X$ be the endomorphism defined by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then the matrix representation of f^* on $\mathrm{NS}(X)$ with respect to the basis $\{e_1, e_2, e_3\}$ is

$$M = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & -4 \\ -2 & 0 & 4 \end{pmatrix}.$$

The Jordan normal form of M is

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

The invariant subspaces of M are given by

$$V_1 = \mathbb{R} \cdot \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \quad V_2 = \{(p,q,r) \mid p = 0\}.$$

We have $V_2 \cap \mathrm{Psef}(X) = V_1 \cap \mathrm{Psef}(X) = \mathbb{R}_{\geq 0} \cdot (0, 1, 0)^T$. The action of f^* on $\mathrm{NS}(X)$ induces a dynamics on the plane $P : p + q = 2$; see **??**.

1.2.2 On projective bundles

Chapter 2

Dynamics on Neron-Severi groups

Chapter 3

p-adic method

3.1 Adelic topology

The main reference of this section is [Xie25]. Fix a finitely generated field $\Bbbk$ over $\overline{\mathbb{Q}}$. Let X be variety over $\Bbbk$.

Let L/K be a field extension. We denote by $M_{L/K}$ (resp. $M_{L,K}$) the set of places of L which restrict to K is trivial (resp. non-trivial).

3.1.1 Adelic subset

Notation 3.1.1. Let $\mathbf{k}$ be a subfield of $\Bbbk$ which is finitely generated over $\mathbb{Q}$ and over which X is defined (such $\mathbf{k}$ is called a *defined field* of X). Denote by $I_{\mathbf{k}}$ the set of all embeddings $\sigma : \mathbf{k} \to \mathbb{C}_\sigma$ over $\mathbf{k}$, where $\mathbb{C}_\sigma = \mathbb{C}_p$ or $\mathbb{C}$. Every such σ corresponds to a place $v \in M_{\mathbf{k},\mathbb{Q}}$ by pulling back the standard absolute value on $\mathbb{C}_p$ or $\mathbb{C}$. For each $\sigma \in I_{\mathbf{k}}$, set $E_\sigma = \{\tau : \Bbbk \to \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$. For every $\tau \in I_{\mathbf{k}}$, we have an inclusion map $\phi_\tau : X(\Bbbk) \to X(\mathbb{C}_\tau)$.

On $X(\mathbb{C}_\tau)$, we have the analytic topology induced from $\mathbb{C}_\tau$.

Definition 3.1.2. Let $\mathbf{k}$ be a defined field of X. Let $\sigma, \sigma_i \in I_{\mathbf{k}}$ and $U \subset X(\mathbb{C}_\sigma), U_i \subseteq X(\mathbb{C}_{\sigma_i})$ be an open subsets in the analytic topology. We define

$$X_{\mathbf{k}}(\sigma, U) \coloneqq \bigcup_{\tau \in E_\sigma} \phi_\tau^{-1}(U) \subseteq X(\Bbbk)$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) \coloneqq \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\Bbbk).$$

The subset of form $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is called a *basic adelic open subset* of $X(\Bbbk)$.

Definition 3.1.3. A *general adelic subset* of $X(\Bbbk)$ is defined to a subset of the form $\pi(B)$ with $\pi : Y \to X$ a flat morphism of varieties and B a basic adelic open subset of $Y(\Bbbk)$.

Remark 3.1.4. To define a general adelic subset of $X(\mathbb{k})$, there are two fields involved: the field $\mathbf{k}$ over which the basic adelic subset is defined, and the field $\mathbf{l}$ over which the morphism $\pi : Y \to X$ is defined.

If we fix a defined field $\mathbf{k}_0$ of X, by [Xie25, Proposition 3.15 (ii) and (v)], we can always choose $\mathbf{l} = \mathbf{k}_0$ and $\mathbf{k}$ is a finite extension of $\mathbf{k}_0$.

Proposition 3.1.5. The finite union and intersection of general adelic subsets are still general adelic subsets.

By Proposition 3.1.5, the general adelic subsets form a basis of topology on $X(\mathbb{k})$. Hence we have the following definition.

Definition 3.1.6. The *adelic topology* on $X(\mathbb{k})$ is defined to be the topology generated by all general adelic subsets of $X(\mathbb{k})$, i.e., an adelic open subset is an arbitrary union of general adelic subsets.

Proposition 3.1.7. We have the following properties of adelic topology:

(a) adelic topology is finer than Zariski topology;

(b) morphisms of varieties are continuous with respect to adelic topology;

(c) flat morphisms of varieties are open with respect to adelic topology.

Proposition 3.1.8. The action of $\mathsf{Gal}(\mathbb{k}/\mathbf{k})$ on $X(\mathbb{k})$, namely $(\sigma, x) \mapsto \sigma(x)$, is continuous with respect to the adelic topology on $X(\mathbb{k})$ and the profinite topology on $\mathsf{Gal}(\mathbb{k}/\mathbf{k})$.

Recall that on $\mathbb{P}^1_{\mathbb{Q}}$, the Artin-Whaples Approximation Theorem says that for any finite collection of places $v_1, \dots, v_n$ of $\mathbb{Q}$ and any open subsets $U_i \subseteq \mathbb{P}^1(\mathbb{Q}_{v_i})$, the intersection $\bigcap_{i=1}^n (U_i \cap \mathbb{P}^1(\mathbb{Q}))$ is non-empty. The following lemma is a generalization and it is the motivation of the definition of adelic subsets.

Theorem 3.1.9. Adelic topology preserves the irreducibility of varieties. Explicitly, on a variety, the intersection of any finite collection of non-empty adelic open subsets is non-empty.

Lemma 3.1.10 (ref. [Xie25, Proposition 3.9], cf. [MZ14, Theorem 1.2])**.** For any finite collection of basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$, the intersection

$$\bigcap_{i=1}^{n} X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

Slogan *On a variety, the small open ball in one place will be dense with respect to other places.*

Yang: Is $\mathbb{A}^1(\mathbb{Q})$ adelic closed in $\mathbb{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?

Yang: Describe all adelic closed subset in $\mathbb{A}^1(\overline{\mathbb{Q}})$.

3.2 Interpolation

The main reference of this section is [Ame11; BGT10; Poo14]. Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$ and $\bar{\mathbb{k}}$ be its algebraic closure. Let X be a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$. We want to show that after iteration, we can interpolate the iterates of f on an analytic open subset of $X(\mathbb{C}_p)$ for some prime p.

3.2.1 Interpolation of analytic maps

In this subsection, we find the interpolation of analytic maps on an analytic disk. Fix a complete non-archimedean field $\mathbf{k}$ of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$ for some prime p. We use the method of difference operators given in [Poo14].

Let $E = E(0,1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$ be the closed unit ball in $\mathbf{k}^d$ and $\Phi : E \to E$ be an analytic map, i.e., $\Phi \in \mathbf{k}^\circ\{\underline{T}\}^d$. Here the norm on $\mathbf{k}^d$ or $\mathbf{k}^\circ\{\underline{T}\}^d$ is the supremum norm, i.e., $|x| = \max_{1\leq i\leq d} |x_i|$. For every analytic map h from E to E, we define

$$\Delta(h) := h \circ \Phi - h, \quad \Delta^n(h) := \Delta(\Delta^{n-1}(h)) \text{ for } n \geq 1,$$

and $\Delta^0(h) = h$. Note that $\Delta^n(h)$ is still an analytic map from E to E by the strong triangle inequality.

Lemma 3.2.1. We have the following binomial theorem:

$$\sum_{m=0}^{n} \binom{n}{m} \Delta^m(\mathrm{id}_E) = \Phi^n.$$

Proof. By induction, we have

$$\begin{aligned}
\Delta^m(\mathrm{id}_E) &= \Delta\left(\sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k\right) \\
&= \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^{k+1} - \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k \\
&= \sum_{k=0}^{m} \left(\binom{m-1}{k-1} (-1)^{m-k} - \binom{m-1}{k} (-1)^{m-1-k}\right) \Phi^k \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Phi^k.
\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}\Delta^m(\mathrm{id}_E) &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}(-1)^{m-k}\Phi^k \\ &= \sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{k}\binom{n-k}{m-k}(-1)^{m-k}\Phi^k \\ &= \sum_{k=0}^{n}\binom{n}{k}\Phi^k\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l \\ &= \binom{n}{n}\Phi^n + \sum_{k=0}^{n-1}\binom{n}{k}\Phi^k\cdot(1-1)^{n-k} \\ &= \Phi^n.\end{aligned}$$

We finish the proof. □

Lemma 3.2.2. Suppose that $\Phi = (\Phi_1,\cdots,\Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$ satisfies $\|\Phi - \mathrm{id}_E\| \le r_p$. Then Yang: To be added.

Proof. Yang: To be added. □

Theorem 3.2.3 (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3])**.** Let $\mathbf{k}$ be a complete non-archimedean field of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$. Set $r_p = p^{-1/(p-1)}$.
Suppose that $\Phi = (\Phi_1,\cdots,\Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$ satisfies $r := r_p/\|\Phi - \mathrm{id}_E\| > 1$. Then there exists a function $F \in \mathbf{k}\{\underline{T}, S/s\}^d$, $1 < s < r$, such that for each $n \in \mathbb{Z}_{\ge 0}$ and each $x \in E$,

$$F(x,n) = \Phi^n(x).$$

Proof. Consider the formal series

$$F(\underline{T},S) := \sum_{n=0}^{\infty}\binom{S}{n}\Delta^n(\mathrm{id}_E)(\underline{T}) = \sum_{n=0}^{\infty} f_n.$$

We only need to show that $f_n \in \mathbf{k}\{\underline{T}, S/s\}^d$ and $\|f_n\| \to 0$ as $n \to \infty$.
Yang: To be added. □

Yang: If f is invertible, can we see that g is unique? Yang: It seems right.

Example 3.2.4. Let $\mathbf{k} = \mathbb{Q}_p$ with $p \ge 3$, and let $\Phi : E \to E$ be the analytic map defined by $\Phi(x) = px^2 + x$. Then we have $\|\Phi - \mathrm{id}_E\| = \|pT^2\| = 1/p < r_p$. Yang: To be checked.

3.2.2 Pick integral models

Lemma 3.2.5. Let $f : X \dashrightarrow X$ be a dominant rational self-map of a projective variety defined over a finitely generated field $\mathbf{k}$ over $\mathbb{Q}$.
Then there exists a subring $R \subseteq \mathbf{k}$ of finite type over $\mathbb{Z}$, a projective scheme $\mathcal{X}$ over $\operatorname{Spec} R$ with generic fiber X, and a rational self-map $\mathcal{f} : \mathcal{X} \dashrightarrow \mathcal{X}$ over $\operatorname{Spec} R$ with generic fiber f such that

(a) for every prime ideal $\mathfrak{p}$ of R, the special fiber $\mathcal{X}_{\mathfrak{p}}$ is geometrically integral and of the same dimension as X;

(b) the union of non-smooth locus of $\mathcal{X}$ and indeterminacy locus, non-étale locus of $\mathcal{f}$ does not contain any entire special fiber $\mathcal{X}_{\mathfrak{p}}$;

Moreover, if X is smooth (resp. f is a morphism, resp. f is étale), then we can further require that $\mathcal{X}$ is smooth over $\operatorname{Spec} R$ (resp. $\mathcal{f}$ is a morphism, resp. $\mathcal{f}$ is étale over $\operatorname{Spec} R$).

Yang: We can embedd R into $\mathbb{C}_p$ for some p.

3.2.3 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

Theorem 3.2.6 (ref.[Xie25, Proposition 3.24])**.** Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$.
There exists an iteration $g = f^m$ of f, an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \geq 3$, an analytic open subset $U \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \to U$ such that

(1) $U \cong (\mathbb{C}_p^\circ)^d$ analytically, where $d = \dim X$;

(2) g is well-defined on U, U is invariant under g and $\|g|_U - \mathrm{id}_U\| < 1/p$;

(3) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

Example 3.2.7. Let $X = E \times E$ with E an elliptic curve without complex multiplication defined over a number field $\mathbf{k}$, and let $f : X \to X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. Yang: To be continued.

3.3 Applications

3.3.1 Existence of non-preperiodic points

Theorem 3.3.1 (ref.[Ame11, Corollary 9])**.** Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$.
Then there exists a basic adelic subset $U \subset X(\mathbf{k})$ such that the forward orbit $O_f(x) = \{f^n(x) : n \geq 0\}$ is well-defined and infinite for every $x \in U$.

3.3.2 DML conjecture for étale morphisms

Theorem 3.3.2 (ref.[BGT10, Theorem 1.3])**.** Let $\mathbf{k}$ be a field of characteristic 0, X a variety defined over $\mathbf{k}$, and $f : X \to X$ an étale morphism defined over $\mathbf{k}$. The DML conjecture holds for (X, f).

3.3.3 DML conjecture for adelic general points

Chapter 4

Dynamics on Berkovich spaces

4.1 A theorem for attractor on k-affinoid spaces

In this section, we copy [Xie25, Appendix A].

Fix an algebraically closed non-Archimedean complete field $\Bbbk$ of characteristic zero and with non-trivial valuation, typically $\Bbbk = \mathbb{C}_p$.

Setup Let A be a strictly and reduced $\Bbbk$-affinoid algebra, and $X = \mathscr{M}(A)$ be its Berkovich spectrum. Let $f : X \to X$ be a finite morphism of Berkovich spaces over $\Bbbk$. We further assume that there exists a subvariety $\tilde{Z} \subseteq \tilde{X}$ such that $\tilde{f}(\tilde{X}) = \tilde{Z}$ and $\tilde{f}|_{\tilde{Z}}$ is an automorphism of $\tilde{Z}$.

Definition 4.1.1. Let ρ_A be the spectral radius on A. For any $g \in A$, we define

$$\rho_f(g) \coloneqq \lim_{n\to\infty} \rho_A((f^*)^n(g))$$

where $f^* : A \to A$ is the induced endomorphism on A. Denote by

$$\mathrm{Qnil}_f \coloneqq \{g \in A \mid \rho_f(g) = 0\}.$$

Proposition 4.1.2. There exists a constant $0 < c < 1$ and $m \geq 1$ such that for any $g \in \mathrm{Qnil}_f$, we have

$$\rho_A((f^*)^m(g)) \leq c\rho_A(g).$$

Proposition 4.1.3. Set $Y = V(\mathrm{Qnil}_f) \subseteq X$ be the closed analytic subspace defined by the ideal Qnil_f. We have $\tilde{Y} = \tilde{Z}$. Yang: To be checked

The main result is the following theorem.

Theorem 4.1.4 (ref. [Xie25, Theorem 8.3])**.** We have

(a) $f|_Y : Y \to Y$ is an automorphism;

(b) there exists a unique $\psi : X \to Y$ making $Y \hookrightarrow X$ a section of ψ, such that $f|_Y \circ \psi = \psi \circ f$;

Moreover, there exists $C > 0$, $0 < \beta < 1$ such that for any $x \in X$ and any $h \in A$, we have

$$|h(f^n(\psi(x))) - h(f^n(x))| \leq C\beta^n \rho_A(h), \quad \forall n \geq 0.$$

Yang: To be revised.

4.1.1 Zariski dense orbit in dimension two

Zariski dense orbit in dimension two Let $X = \mathscr{M}(\Bbbk x, y)$ be the closed unit polydisc of dimension two over $\Bbbk$. Suppose that $f : X \to X$ satisfies

$$\tilde{f} : (x, y) \mapsto (ax + b, 0).$$

In this case, the attractor Y is the line defined by $y = 0$, and $f|_Y$ is an automorphism of Y.Yang: To be revised.

Proposition 4.1.5. Suppose that $f|_Y$ is not of finite order and $f^{-1}(Y) \neq X$. Then there exists an affinoid subdomain $U \subseteq X$ such that for every point $x \in U$, the orbit $\{f^n(x)\}_{n\geq 0}$ is Zariski dense in X. Yang:

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