Interpolation

The main reference of this section is [Ame11; BGT10; Poo14; Xie25].

1 Interpolation of analytic maps

In this subsection, we find the interpolation of analytic maps on an analytic disk. Fix a complete non-archimedean field $\mathbf{k}$ of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$ for some prime p. Set $r_p = p^{-1/(p-1)}$. We use the method of difference operators given in [Poo14].

Let $E = E(0,1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$ be the closed unit ball in $\mathbf{k}^d$ and $\Phi : E \to E$ be an analytic map, i.e., $\Phi \in \mathbf{k}^\circ\{\underline{X}\}^d$. Here the norm on $\mathbf{k}^d$ or $\mathbf{k}^\circ\{\underline{X}\}^d$ is the supremum norm, i.e., $|x| = \max_{1\leq i\leq d} |x_i|$. For every analytic map h from E to E, we define

$$\Delta(h) := h \circ \Phi - h, \quad \Delta^n(h) := \Delta(\Delta^{n-1}(h)) \text{ for } n \geq 1,$$

and $\Delta^0(h) = h$. Note that $\Delta^n(h)$ is still an analytic map from E to E by the strong triangle inequality.

Lemma 1. We have the following binomial theorem:

$$\sum_{m=0}^{n} \binom{n}{m} \Delta^m(\mathrm{id}_E) = \Phi^n.$$

Proof. By induction, we have

$$\begin{aligned}
\Delta^m(\mathrm{id}_E) &= \Delta\left(\sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k\right) \\
&= \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^{k+1} - \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k \\
&= \sum_{k=0}^{m} \left(\binom{m-1}{k-1} (-1)^{m-k} - \binom{m-1}{k} (-1)^{m-1-k}\right) \Phi^k \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Phi^k.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{m=0}^{n} \binom{n}{m} \Delta^m(\mathrm{id}_E) &= \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} \binom{m}{k} (-1)^{m-k} \Phi^k \\
&= \sum_{k=0}^{n} \sum_{m=k}^{n} \binom{n}{k} \binom{n-k}{m-k} (-1)^{m-k} \Phi^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \Phi^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \\
&= \binom{n}{n} \Phi^n + \sum_{k=0}^{n-1} \binom{n}{k} \Phi^k \cdot (1-1)^{n-k} \\
&= \Phi^n.
\end{aligned}$$

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We finish the proof. □

Lemma 2. Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{X}\}^d$ satisfies $\|\Phi - \mathrm{id}_E\| \le r$. Then

$$\|\Delta^n(\mathrm{id}_E)\| \le r^n$$

Proof. By Yang: ref, we have

$$\|\Delta(h)\| = \|h \circ \Phi - h\| \le \|h\| \cdot \|\Phi - \mathrm{id}_E\| \le r\|h\|.$$

Hence by induction, the result follows. □

Theorem 3 (ref.[Poo14, Theorem 1], cf.[BGT10, Theorem 3.3])**.** Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{X}\}^d$ satisfies $r := \|\Phi - \mathrm{id}_E\| < r_p$. Then there exists a unique function $F \in \mathbf{k}\{\underline{X}, T/s\}^d$, such that for each $n \in \mathbb{Z}_{\ge 0}$ and each $x \in E$,

$$F(x, n) = \Phi^n(x).$$

Here s is any real number with $1 < s < r_p/r$.

Proof. Consider the formal series

$$F(\underline{X}, T) := \sum_{n=0}^{\infty} \binom{T}{n} \Delta^n(\mathrm{id}_E)(\underline{X}).$$

Recall the Newton's binomial function

$$\binom{T}{n} := \frac{T(T-1)\cdots(T-n+1)}{n!} \in \mathbf{k}[T].$$

Since $\binom{T}{n}$ is a polynomial in T and $\Delta^n(\mathrm{id}_E)(\underline{X}) \in \mathbf{k}^\circ\{\underline{X}\}^d$, we have $f_n = \binom{T}{n}\Delta^n(\mathrm{id}_E)(\underline{X}) \in \mathbf{k}\{\underline{X}, T\}^d$. Note that for each $n \in \mathbb{Z}_{\ge 0}$, then $|n!|_{\mathbf{k}} \ge r_p^n$. Hence we have

$$\left\|\binom{T}{n}\right\| \le s^n r_p^{-n}$$

since $s > 1$. By Lemma 2, we have

$$\|f_n\| \le \left\|\binom{T}{n}\right\| \cdot \|\Delta^n(\mathrm{id}_E)\| \le s^n r_p^{-n} r^n = (sr/r_p)^n.$$

Since $sr/r_p < 1$, the series $F(\underline{X}, T) = \sum_{n=0}^{\infty} f_n$ converges in $\mathbf{k}\{\underline{X}, T/s\}^d$. By Lemma 1, we have $F(x, n) = \Phi^n(x)$ for each $n \in \mathbb{Z}_{\ge 0}$ and each $x \in E$. The uniqueness of F follows from Yang: ref. □

Yang: If f is invertible, can we see that g is unique? Yang: It seems right.

Example 4. Let $\mathbf{k} = \mathbb{Q}_p$ with $p \ge 3$, and let $\Phi : E \to E$ be the analytic map defined by $\Phi(x) = px^2 + x$. Then we have $\|\Phi - \mathrm{id}_E\| = \|pT^2\| = 1/p < r_p$. Yang: To be checked.

2 Interpolation on an analytic open subset of morphisms

Let $f : X \dashrightarrow X$ be a dominant rational self-map of a projective variety of dimension d defined over a finitely generated field $\mathbf{k}$ over $\mathbb{Q}$. We try to find some analytic local interpolation of the iterates of f

on $X(\mathbb{C}_p)$ for some prime p.

Lemma 5. There exists a subring $R \subseteq \mathbf{k}$ of finite type over $\mathbb{Z}$, a projective scheme $\mathcal{X}$ over $\operatorname{Spec} R$ with generic fiber X, and a rational self-map $\mathcal{f} : \mathcal{X} \dashrightarrow \mathcal{X}$ over $\operatorname{Spec} R$ with generic fiber f such that

(a) for every prime ideal $\mathfrak{p}$ of R, the special fiber $\mathcal{X}_{\mathfrak{p}}$ is geometrically integral and of the same dimension as X;

(b) the union of non-smooth locus of $\mathcal{X}$ and indeterminacy locus, non-étale locus of $\mathcal{f}$ does not contain any entire special fiber $\mathcal{X}_{\mathfrak{p}}$;

Moreover, if X is smooth (resp. f is a morphism, resp. f is étale), then we can further require that $\mathcal{X}$ is smooth over $\operatorname{Spec} R$ (resp. $\mathcal{f}$ is a morphism, resp. $\mathcal{f}$ is étale over $\operatorname{Spec} R$).

Yang: We can embedd R into $\mathbb{C}_p$ for some p.

Theorem 6 (ref.[Xie25, Proposition 3.24])**.** There exists an iteration $g = f^m$ of f, an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \geq 3$, an analytic open subset $U \cong (\mathbb{C}_p^\circ)^d \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \to U$ such that

(a) g is well-defined on U, U is invariant under g and $\|g|_U - \mathrm{id}_U\| < 1/p$;

(b) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

Example 7. Let $X = E \times E$ with E an elliptic curve without complex multiplication defined over a number field $\mathbf{k}$, and let $f : X \to X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. Yang: To be continued.

References

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