
The p -adic Method

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1 Adelic topology

The main reference of this section is [\[Xie25\]](#).

1.1 Basic adelic subset

Let \mathbf{k} be a finitely generated field over \mathbb{Q} and \mathbb{k} be its algebraic closure.

Let $\mathcal{J}_{\mathbf{k}}$ be the set of all embeddings $\sigma : \mathbf{k} \rightarrow \mathbb{C}_{\sigma}$ over \mathbf{k} , where $\mathbb{C}_{\sigma} = \mathbb{C}_p$ or \mathbb{C} according to whether v is non-archimedean or archimedean. For each $\sigma \in \mathcal{J}_{\mathbf{k}}$, set $\mathcal{J}_{\sigma} = \{\tau : \mathbb{k} \rightarrow \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$. For every $\tau \in \mathcal{J}_{\mathbf{k}}$, we have an induced map $\phi_{\tau} : X(\mathbb{k}) \rightarrow X(\mathbb{C}_{\tau})$.

On $X(\mathbb{C}_{\tau})$, we have the analytic topology induced from \mathbb{C}_{τ} .

Definition 1.1. Let $\sigma, \sigma_i \in \mathcal{J}_{\mathbf{k}}$ and let $U, U_i \subseteq X(\mathbb{C}_{\sigma})$ be an open subset in the analytic topology. We define

$$X_{\mathbf{k}}(\sigma, U) := \bigcup_{\tau \in \mathcal{J}_{\sigma}} \phi_{\tau}^{-1}(U) \subseteq X(\mathbb{k})$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) := \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\mathbb{k}).$$

The subset of form $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is called a *basic adelic open subset* of $X(\mathbb{k})$, where $U \subseteq X(\mathbb{C}_{\sigma})$ is an open subset in the analytic topology.

Theorem 1.2 (ref. [Xie25, Proposition 3.9] cf. []). Suppose that X is irreducible. Then for any finite collection of places $\sigma_1, \dots, \sigma_n \in M_{\mathbf{k}}$ and any basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$ for $i = 1, \dots, n$, the intersection

$$\bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

1.2 General adelic subset

1.3 Adelic topology

Yang: Is $\mathbf{A}^1(\mathbb{Q})$ adelic closed in $\mathbf{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?

Yang: Describe all adelic closed subset in $\mathbf{A}^1(\overline{\mathbb{Q}})$.

2 Interpolation

2.1 Pick integral model

2.2 Interpolation of analytic maps

The main reference of this section is [Ame11; BGT10; Poo14]. We first state the main theorem of this section.

Theorem 2.1 (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3]). Let \mathbf{k} be a complete non-archimedean field of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$. Set $r_p = p^{-1/(p-1)}$. Let $E = E(0, 1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$ be the closed unit ball in \mathbf{k}^d . Here the norm on \mathbf{k}^d or $\mathbf{k}^{\circ}\{T\}^d$ is the supremum norm, i.e., $|x| = \max_{1 \leq i \leq d} |x_i|$.

Suppose that $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathbf{k}^{\circ}\{T\}^d$ satisfies $\|\Phi - \text{id}_E\| \leq r_p$. Then there exists a function $F \in \mathbf{k}\{T, S\}^d$ such that for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in E$,

$$F(x, n) = \Phi^n(x).$$

Proof. Yang: To be added. □

Yang: If f is invertible, can we see that g is unique? Yang: It seems right.

Example 2.2. Let $\mathbf{k} = \mathbb{Q}_p$ with $p \geq 3$, and let $\Phi: E \rightarrow E$ be the analytic map defined by $\Phi(x) = px^2 + x$. Then we have $\|\Phi - \text{id}_E\| = \|pT^2\| = 1/p < r_p$. Yang: To be checked.

2.3 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

Theorem 2.3 (ref.[Xie25, Proposition 3.24]). Let \mathbf{k} be a finitely generated field over \mathbb{Q} , X a projective variety defined over \mathbf{k} , and $f: X \dashrightarrow X$ a dominant rational self-map defined over \mathbf{k} .

There exists an iteration $g = f^m$ of f , an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \geq 3$, an analytic open subset $U \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \rightarrow U$ such that

- (1) $U \cong (\mathbb{C}_p^\circ)^d$ analytically, where $d = \dim X$;
- (2) g is well-defined on U , U is invariant under g and $\|g|_U - \text{id}_U\| < 1/p$;
- (3) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

Lemma 2.4. **Yang:** There is a good model for (X, f) .

Example 2.5. Let $X = E \times E$ with E an elliptic curve without complex multiplication defined over a number field \mathbf{k} , and let $f : X \rightarrow X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. **Yang:** To be continued.

3 Applications

3.1 Existence of non-preperiodic points

Theorem 3.1 (ref.[Ame11, Corollary 9]). Let \mathbf{k} be an algebraically closed field of characteristic 0, X a projective variety defined over \mathbf{k} , and $f : X \dashrightarrow X$ a dominant rational self-map defined over \mathbf{k} . Then there exists a basic adelic subset $U \subset X(\mathbf{k})$ such that the forward orbit $O_f(x) = \{f^n(x) : n \geq 0\}$ is well-defined and infinite for every $x \in U$.

3.2 DML conjecture for étale morphisms

Theorem 3.2 (ref.[BGT10, Theorem 1.3]). Let \mathbf{k} be a field of characteristic 0, X a variety defined over \mathbf{k} , and $f : X \rightarrow X$ an étale morphism defined over \mathbf{k} . The DML conjecture holds for (X, f) .

3.3 DML conjecture for adelic general points

References

- [Ame11] E Amerik. “Existence of non-preperiodic algebraic points for a rational self-map of infinite order”. In: *Mathematical Research Letters* 18.2 (2011), pp. 251–256 (cit. on pp. 2, 3).
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- [Xie25] Junyi Xie. “The existence of Zariski dense orbits for endomorphisms of projective surfaces”. In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on pp. 1, 2).