The p-adic Method

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Contents

1 Adelic topology

The main reference of this section is [Xie25].

1.1 Basic adelic subset

Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$ and $\Bbbk$ be its algebraic closure.

Let $\mathcal{J}_{\mathbf{k}}$ be the set of all embeddings $\sigma : \mathbf{k} \to \mathbb{C}_\sigma$ over $\mathbf{k}$, where $\mathbb{C}_\sigma = \mathbb{C}_p$ or $\mathbb{C}$ according to whether v is non-archimedean or archimedean. For each $\sigma \in \mathcal{J}_{\mathbf{k}}$, set $\mathcal{J}_\sigma = \{\tau : \Bbbk \to \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$. For every $\tau \in \mathcal{J}_{\Bbbk}$, we have an induced map $\phi_\tau : X(\Bbbk) \to X(\mathbb{C}_\tau)$.

On $X(\mathbb{C}_\tau)$, we have the analytic topology induced from $\mathbb{C}_\tau$.

Definition 1.1. Let $\sigma, \sigma_i \in \mathcal{J}_{\mathbf{k}}$ and let $U, U_i \subseteq X(\mathbb{C}_\sigma)$ be an open subset in the analytic topology. We define

$$X_{\mathbf{k}}(\sigma, U) := \bigcup_{\tau \in \mathcal{J}_\sigma} \phi_\tau^{-1}(U) \subseteq X(\Bbbk)$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) := \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\Bbbk).$$

The subset of form $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is called a *basic adelic open subset* of $X(\Bbbk)$, where $U \subseteq X(\mathbb{C}_\sigma)$ is an open subset in the analytic topology.

Date: November 24, 2025, **Author**: Tianle Yang, My Homepage

Theorem 1.2 (ref. [Xie25, Proposition 3.9] cf. [**<empty citation>**]). Suppose that X is irreducible. Then for any finite collection of places $\sigma_1, \ldots, \sigma_n \in M_{\mathbf{k}}$ and any basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$ for $i = 1, \ldots, n$, the intersection

$$\bigcap_{i=1}^{n} X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

1.2 General adelic subset

1.3 Adelic topology

Yang: Is $\mathbb{A}^1(\mathbb{Q})$ adelic closed in $\mathbb{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?

Yang: Describe all adelic closed subset in $\mathbb{A}^1(\overline{\mathbb{Q}})$.

2 Interpolation

The main reference of this section is [Ame11; BGT10; Poo14]. Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$ and $\Bbbk$ be its algebraic closure. Let X be a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$. We want to show that after iteration, we can interpolate the iterates of f on an analytic open subset of $X(\mathbb{C}_p)$ for some prime p.

2.1 Interpolation of analytic maps

In this subsection, we find the interpolation of analytic maps on an analytic disk. Fix a complete non-archimedean field $\mathbf{k}$ of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$ for some prime p. We use the method of difference operators given in [Poo14].

Let $E = E(0,1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$ be the closed unit ball in $\mathbf{k}^d$ and $\Phi : E \to E$ be an analytic map, i.e., $\Phi \in \mathbf{k}^\circ\{\underline{T}\}^d$. Here the norm on $\mathbf{k}^d$ or $\mathbf{k}^\circ\{\underline{T}\}^d$ is the supremum norm, i.e., $|x| = \max_{1\leq i\leq d} |x_i|$. For every analytic map h from E to E, we define

$$\Delta(h) := h \circ \Phi - h, \quad \Delta^n(h) := \Delta(\Delta^{n-1}(h)) \text{ for } n \geq 1,$$

and $\Delta^0(h) = h$. Note that $\Delta^n(h)$ is still an analytic map from E to E by the strong triangle inequality.

Lemma 2.1. We have the following binomial theorem:

$$\sum_{m=0}^{n} \binom{n}{m} \Delta^m(\mathrm{id}_E) = \Phi^n.$$

Proof. By induction, we have

$$\begin{aligned}\Delta^m(\mathrm{id}_E) &= \Delta\left(\sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^{m-1-k}\Phi^k\right)\\ &= \sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^{m-1-k}\Phi^{k+1} - \sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^{m-1-k}\Phi^k\\ &= \sum_{k=0}^{m}\left(\binom{m-1}{k-1}(-1)^{m-k} - \binom{m-1}{k}(-1)^{m-1-k}\right)\Phi^k\\ &= \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\Phi^k.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}\Delta^m(\mathrm{id}_E) &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}(-1)^{m-k}\Phi^k\\ &= \sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{k}\binom{n-k}{m-k}(-1)^{m-k}\Phi^k\\ &= \sum_{k=0}^{n}\binom{n}{k}\Phi^k\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\\ &= \binom{n}{n}\Phi^n + \sum_{k=0}^{n-1}\binom{n}{k}\Phi^k\cdot(1-1)^{n-k}\\ &= \Phi^n.\end{aligned}$$

We finish the proof. □

Lemma 2.2. Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$ satisfies $\|\Phi - \mathrm{id}_E\| \le r_p$. Then Yang: To be added.

Proof. Yang: To be added. □

Theorem 2.3 (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3])**.** Let $\mathbf{k}$ be a complete non-archimedean field of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$. Set $r_p = p^{-1/(p-1)}$.
Then there exists a function $F \in \mathbf{k}\{\underline{T}, S\}^d$ such that for each $n \in \mathbb{Z}_{\ge 0}$ and each $x \in E$,

$$F(x, n) = \Phi^n(x).$$

Proof. Yang: To be added. □

Yang: If f is invertible, can we see that g is unique? Yang: It seems right.

Example 2.4. Let $\mathbf{k} = \mathbb{Q}_p$ with $p \ge 3$, and let $\Phi : E \to E$ be the analytic map defined by $\Phi(x) = px^2 + x$. Then we have $\|\Phi - \mathrm{id}_E\| = \|pT^2\| = 1/p < r_p$. Yang: To be checked.

2.2 Pick integral models

Lemma 2.5. Yang: There is a good model for (X, f).

2.3 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

Theorem 2.6 (ref.[Xie25, Proposition 3.24])**.** Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$.
There exists an iteration $g = f^m$ of f, an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \geq 3$, an analytic open subset $U \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \to U$ such that

(1) $U \cong (\mathbb{C}_p^\circ)^d$ analytically, where $d = \dim X$;

(2) g is well-defined on U, U is invariant under g and $\|g|_U - \mathrm{id}_U\| < 1/p$;

(3) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

Example 2.7. Let $X = E \times E$ with E an elliptic curve without complex multiplication defined over a number field $\mathbf{k}$, and let $f : X \to X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. Yang: To be continued.

3 Applications

3.1 Existence of non-preperiodic points

Theorem 3.1 (ref.[Ame11, Corollary 9])**.** Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$. Then there exists a basic adelic subset $U \subset X(\mathbf{k})$ such that the forward orbit $O_f(x) = \{f^n(x) : n \geq 0\}$ is well-defined and infinite for every $x \in U$.

3.2 DML conjecture for étale morphisms

Theorem 3.2 (ref.[BGT10, Theorem 1.3])**.** Let $\mathbf{k}$ be a field of characteristic 0, X a variety defined over $\mathbf{k}$, and $f : X \to X$ an étale morphism defined over $\mathbf{k}$. The DML conjecture holds for (X, f).

3.3 DML conjecture for adelic general points

References

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[Poo14] Bjorn Poonen. "p-adic interpolation of iterates". In: *Bulletin of the London Mathematical Society* 46.3 (2014), pp. 525–527 (cit. on pp. 2, 3).

[Xie25] Junyi Xie. "The existence of Zariski dense orbits for endomorphisms of projective surfaces". In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on pp. 1, 2, 4).