Adelic topology

The main reference of this section is [Xie25]. Fix a finitely generated field $\mathbf{k}$ over $\mathbb{Q}$ and its algebraic closure $\Bbbk$. Let X be a (geometrically integral) variety over $\mathbf{k}$.

Notation 1. Let L/K be a field extension. We denote by $M_{L/K}$ (resp. $M_{L,K}$) the set of places of L which restrict to K is trivial (resp. non-trivial).

1 Adelic subset

Notations Let $I_{\mathbf{k}}$ be the set of all embeddings $\sigma : \mathbf{k} \to \mathbb{C}_\sigma$ over $\mathbf{k}$, where $\mathbb{C}_\sigma = \mathbb{C}_p$ or $\mathbb{C}$. Every such σ corresponds to a place $v \in M_{\mathbf{k},\mathbb{Q}}$ by pulling back the standard absolute value on $\mathbb{C}_p$ or $\mathbb{C}$. For each $\sigma \in I_{\mathbf{k}}$, set $E_\sigma = \{\tau : \Bbbk \to \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$. For every $\tau \in I_{\mathbf{k}}$, we have an induced map $\phi_\tau : X(\Bbbk) \to X(\mathbb{C}_\tau)$.

On $X(\mathbb{C}_\tau)$, we have the analytic topology induced from $\mathbb{C}_\tau$.

Definition 2. Let $\sigma, \sigma_i \in I_{\mathbf{k}}$ and let $U, U_i \subseteq X(\mathbb{C}_\sigma)$ be an open subset in the analytic topology. We define

$$X_{\mathbf{k}}(\sigma, U) := \bigcup_{\tau \in E_\sigma} \phi_\tau^{-1}(U) \subseteq X(\Bbbk)$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) := \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\Bbbk).$$

The subset of form $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is called a *basic adelic open subset* of $X(\Bbbk)$.

Definition 3. A *general adelic subset* of $X(\Bbbk)$ is defined to a subset of the form $\pi(B)$ with $\pi : Y \to X$ a flat morphism of varieties over $\mathbf{k}$ and B a basic adelic open subset of $Y(\Bbbk)$.

Recall that on $\mathbb{A}^1_{\mathbb{Q}}$, the Artin-Whaples Approximation Theorem says that for any finite collection of places $v_1, \ldots, v_n$ of $\mathbb{Q}$ and any open subsets $U_i \subseteq \mathbb{Q}_{v_i}$, the intersection $\bigcap_{i=1}^n (U_i \cap \mathbb{A}^1(\mathbb{Q}))$ is non-empty. The following lemma is a generalization and it is the motivation of the definition of adelic subsets.

Lemma 4 (ref. [Xie25, Proposition 3.9], cf. [MZ14, Theorem 1.2])**.** For any finite collection of basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$, the intersection

$$\bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

Slogan *On a variety, the small open ball in one place will be dense with respect to other places.*

Proposition 5. Let X be a variety over $\mathbf{k}$ and $\mathbf{l}$ a finite field extension of $\mathbf{k}$. Then the following properties hold:

(a) let $f : Y \to X_{\mathbf{l}}$ be a morphism and $A \subset X_{\mathbf{l}}(\Bbbk)$ a general adelic subset, then $f^{-1}(A)$ is a general

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adelic subset of Y;

Remark 6. Although adelic subsets are subsets of $X(\mathbb{k})$, they depend on the field $\mathbf{k}$ over which X is defined. Yang: For example

2 Adelic topology

Definition 7. The *adelic topology* on $X(\mathbb{k})$ is defined to be the topology generated by all general adelic subsets of $X(\mathbb{k})$.

Yang: Is $\mathbb{A}^1(\mathbb{Q})$ adelic closed in $\mathbb{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?
Yang: Describe all adelic closed subset in $\mathbb{A}^1(\overline{\mathbb{Q}})$.

Proposition 8. We have the following properties of adelic topology:

(a) adelic topology is finer than Zariski topology (in other words, every Zariski open subset is adelic open);

(b) morphisms of varieties are continuous with respect to adelic topology;

(c) flat morphisms of varieties are open with respect to adelic topology.

Proposition 9. The action of $\mathsf{Gal}(\mathbb{k}/\mathbf{k})$ on $X(\mathbb{k})$, namely $(\sigma, x) \mapsto \sigma(x)$, is continuous with respect to the adelic topology on $X(\mathbb{k})$ and the profinite topology on $\mathsf{Gal}(\mathbb{k}/\mathbf{k})$.

Theorem 10. Adelic topology preserves the irreducibility of varieties. Explicitly, on a variety, the intersection of any finite collection of non-empty adelic open subsets is non-empty.

Appendix

References

[MZ14] Vincenzo Mantova and Umberto Zannier. "Artin-Whaples approximations of bounded degree in algebraic varieties". In: *Proceedings of the American Mathematical Society* 142.9 (2014), pp. 2953–2964 (cit. on p. 1).

[Xie25] Junyi Xie. "The existence of Zariski dense orbits for endomorphisms of projective surfaces". In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on p. 1).