Examples of dynamics

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2 Examples of dynamic systems on ruled surfaces

In this section, fix an algebraically closed field $\Bbbk$ of characteristic zero.

2.1 Automorphism groups

The main reference is [**Mar71**].

$$H_r = \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & \alpha \end{pmatrix}, \cdots, \begin{pmatrix} \alpha & t_r \\ 0 & \alpha \end{pmatrix} \right) \in GL(2,k) \times \cdots \times GL(2,k) \;\middle|\; \begin{matrix} \alpha \in G_m \\ t_i \in k \end{matrix} \right\},$$
$$H_r' = \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & \beta \end{pmatrix}, \cdots, \begin{pmatrix} \alpha & t_r \\ 0 & \beta \end{pmatrix} \right) \in GL(2,k) \times \cdots \times GL(2,k) \;\middle|\; \begin{matrix} \alpha, \beta \in G_m \\ t_i \in k \end{matrix} \right\},$$
$$\overline{H}_r' = \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} \alpha & t_r \\ 0 & 1 \end{pmatrix} \right) \in GL(2,k) \times \cdots \times GL(2,k) \;\middle|\; \begin{matrix} \alpha \in G_m \\ t_i \in k \end{matrix} \right\}.$$

Theorem 2.1. Let $\pi : X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e)) \to C = \mathbb{P}^1$ be a rational ruled surface. We have an exact sequence of algebraic groups

$$1 \to \overline{H}_{e+1}' \to \mathrm{Aut}(X) \to \mathrm{PGL}(2, \Bbbk) \to 1.$$

2.2 Polarized induced by base

Let $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ be a rank 2 vector bundle on a smooth projective curve C of genus $g \leq 1$, where $\mathcal{L}$ is a line bundle of degree $-e$ on C. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to C$ be the associated ruled surface. Let $g : C \to C$ be an endomorphism of degree q such that $g^*\mathcal{L} \cong \mathcal{L}^q$. Fix the isomorphism $g^*\mathcal{L} \cong \mathcal{L}^q$, we have

$$g^* : \mathcal{E} \to \mathcal{O}_C \oplus \mathcal{L}^q \hookrightarrow \mathrm{Sym}^q \mathcal{E}.$$

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3 Examples of dynamics on abelian varieties

On this section, fix an algebraically closed field $\Bbbk$ of characteristic zero. Everything is defined over $\Bbbk$ unless otherwise specified.

3.1 Product of elliptic curves

In this subsection, we consider the dynamics induced by matrices on the product of elliptic curves.

Let E be an elliptic curve without complex multiplication. Consider the abelian variety $X = E \times E$. Let $f_A : X \to X$ be the endomorphism defined by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $[F_1], [F_2], [\Delta]$ be the classes of the fibers of the two projections and the diagonal in $\mathrm{NS}(X)$. It is well-known that they span $\mathrm{NS}(X)$ and the intersection numbers are given by

$$[F_1]^2 = [F_2]^2 = [\Delta]^2 = 0, \quad [F_1] \cdot [F_2] = [F_1] \cdot [\Delta] = [F_2] \cdot [\Delta] = 1;$$

see [**Laz04a**].

We have that $f_A^*[F_1]$ is given by $[a]_E(x) + [b]_E(y) = 0$. Then

$$f_A^*[F_1].[F_1] = b^2, \quad f_A^*[F_1].[F_2] = a^2, \quad f_A^*[F_1].[\Delta] = (a+b)^2.$$

Hence

$$f_A^*[F_1] = (a^2 + ab)[F_1] + (b^2 + ab)[F_2] - ab[\Delta].$$

Similarly, we have

$$\begin{aligned} f_A^*[F_2] &= (c^2 + cd)[F_1] + (d^2 + cd)[F_2] - cd[\Delta], \\ f_A^*[\Delta] &= (a-c)(a+b-c-d)[F_1] + (b-d)(a+b-c-d)[F_2] - (a-c)(b-d)[\Delta]. \end{aligned}$$

Thus, the matrix representation of f_A^* on $\mathrm{NS}(X)$ with respect to the basis $\{[F_1], [F_2], [\Delta]\}$ is

$$\begin{pmatrix} a^2 + ab & c^2 + cd & (a-c)(a+b-c-d) \\ b^2 + ab & d^2 + cd & (b-d)(a+b-c-d) \\ -ab & -cd & -(a-c)(b-d) \end{pmatrix}.$$

If we take $e_1 = [F_1], e_2 = [F_2], e_3 = [\Delta] - [F_1] - [F_2]$ as a new basis of $\mathrm{NS}(X)$, then the matrix representation of f_A^* on $\mathrm{NS}(X)$ with respect to the basis $\{e_1, e_2, e_3\}$ is

$$M = \begin{pmatrix} a^2 & c^2 & -2ac \\ b^2 & d^2 & -2bd \\ -ab & -cd & ad + bc \end{pmatrix}.$$

The characteristic polynomial of M is given by

$$\chi_{f_A^*}(T) = (T - (ad - bc))(T^2 - (a^2 + d^2 + 2bc)T + (ad - bc)^2).$$

Suppose that the eigenvalues of A are λ, μ. Then the eigenvalues of f_A^* on $\mathrm{NS}(X)$ are given by $\lambda^2, \mu^2, \lambda\nu$. When $a-d, b, c$ are not all zero, $\mathrm{NS}(X)$ has two invariant subspaces of dimension 1 and 2 respectively. They are given by

$$V_1 = \mathbb{Q} \cdot \begin{pmatrix} 2c \\ -2b \\ a-d \end{pmatrix}, \quad V_2 = \mathbb{Q} \cdot \begin{pmatrix} 0 \\ a-d \\ c \end{pmatrix} \oplus \mathbb{Q} \cdot \begin{pmatrix} d-a \\ 0 \\ b \end{pmatrix} = \{(p,q,r) \mid bp - cq + (a-d)r = 0\}.$$

with respect to the basis $\{e_i\}$. One can use ?? to check this in SageMathCell.

With respect to the basis $\{e_i\}$, the cones are given by

$$\mathrm{Nef}(X) = \mathrm{Psef}(X) = \{pe_1 + qe_2 + re_3 \mid p, q \geq 0, \quad pq \geq r^2\}.$$

We analyze the intersection of V_1, V_2 with $\mathrm{Psef}(X)$. On the plane $P : p + q = 2$, fix a coordinate system (s, r) with $s = (p-q)/2 = p - 1 = 1 - q$. The cone $\mathrm{Psef}(X)$ is given by the disk $\{(s,r) \mid s^2 + r^2 \leq 1\}$. The plane V_2 is given by the equation $b(1+s) - c(1-s) + (a-d)r = (b-c) + (b+c)s + (a-d)r = 0$. If $b = c$, then the line V_1 does not intersect the plane P, hence does not intersect the interior of $\mathrm{Psef}(X)$. Otherwise, $V_1 \cap P$ is given by the point $(s,r) = \left(\frac{c+b}{c-b}, \frac{a-d}{c-b}\right)$. There are three cases:

(a) $(a+d)^2 < 4(ad-bc)$: V_1 intersects the interior of $\mathrm{Psef}(X)$, V_2 intersects $\mathrm{Psef}(X)$ at only the origin.

(b) $(a+d)^2 = 4(ad-bc)$: V_1 intersects the boundary of $\mathrm{Psef}(X)$ at a ray, V_2 intersects the boundary of $\mathrm{Psef}(X)$ at a ray.

(c) $(a+d)^2 > 4(ad-bc)$: V_1 intersects $\mathrm{Psef}(X)$ at only the origin, V_2 intersects the interior of $\mathrm{Psef}(X)$.

Note that if $b = c$, we always have $(a+d)^2 > 4(ad-bc)$ under the assumption that $a-d, b, c$ are not all zero. And note that $(a+d)^2 - 4(ad-bc) = \mathrm{disc}(\chi_A)$. Hence, we have the conclusion in table 1.

Case	$V_1 \cap \mathrm{Psef}(X)$	$V_2 \cap \mathrm{Psef}(X)$	dimension of minimal invariant subspace intersecting $\mathrm{Psef}(X)^\circ$
A is scalar			
A has only one eigenvalue	Boundary (ray)	Boundary (ray)	3
A has complex eigenvalues	Interior	Origin only	1
A has distinct real eigenvalues	Origin only	Interior	2

Table 1: Intersection of invariant subspaces V_1, V_2 with $\mathrm{Psef}(X)$ in different cases.

Now we focus on the case when A has only one eigenvalue, i.e. A is similar to a Jordan block. Assume that A is not a scalar matrix, i.e. $a-d, b, c$ are not both zero. Then easily see that $V_1 \subset V_2$ iff $(a+d)^2 = 4(ad-bc)$ iff A has the Jordan normal form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{Z}$. Hence in this case, there is a finite equivariant cover $(X, f) \to (X, g)$ such that g is induced by the Jordan block.

Example 3.1. Let $f : X \to X$ be the endomorphism defined by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then the matrix representation of f^* on $\mathrm{NS}(X)$ with respect to the basis $\{e_1, e_2, e_3\}$ is

$$M = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & -4 \\ -2 & 0 & 4 \end{pmatrix}.$$

The Jordan normal form of M is

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

The invariant subspaces of M are given by

$$V_1 = \mathbb{R} \cdot \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \quad V_2 = \{(p, q, r) \mid p = 0\}.$$

We have $V_2 \cap \mathrm{Psef}(X) = V_1 \cap \mathrm{Psef}(X) = \mathbb{R}_{\geq 0} \cdot (0, 1, 0)^T$. The action of f^* on $\mathrm{NS}(X)$ induces a dynamics on the plane $P : p + q = 2$; see ??.