# The p -adic Method



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# 1 Adelic topology

The main reference of this section is [Xie25]. Fix a finitely generated field $\Bbbk$ over $\overline{\mathbb{Q}}$. Let $X$ be variety over $\Bbbk$.

Let $L/K$ be a field extension. We denote by $M_{L/K}$ (resp. $M_{L,K}$) the set of places of $L$ which restrict to $K$ is trivial (resp. non-trivial).

## 1.1 Adelic subset

**Notation 1.1.** Let $\mathbf{k}$ be a subfield of $\Bbbk$ which is finitely generated over $\mathbb{Q}$ and over which $X$ is defined (such $\mathbf{k}$ is called a *defined field* of $X$). Denote by $I_{\mathbf{k}}$ the set of all embeddings $\sigma : \mathbf{k} \to \mathbb{C}_\sigma$ over $\mathbf{k}$, where $\mathbb{C}_\sigma = \mathbb{C}_p$ or $\mathbb{C}$. Every such $\sigma$ corresponds to a place $v \in M_{\mathbf{k},\mathbb{Q}}$ by pulling back the standard absolute value on $\mathbb{C}_p$ or $\mathbb{C}$. For each $\sigma \in I_{\mathbf{k}}$, set $E_\sigma = \{\tau : \Bbbk \to \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$. For every $\tau \in I_{\mathbf{k}}$, we have an inclusion map $\phi_\tau : X(\Bbbk) \to X(\mathbb{C}_\tau)$.

On $X(\mathbb{C}_\tau)$, we have the analytic topology induced from $\mathbb{C}_\tau$.

**Definition 1.2.** Let $\mathbf{k}$ be a defined field of $X$. Let $\sigma, \sigma_i \in I_{\mathbf{k}}$ and $U \subset X(\mathbb{C}_\sigma), U_i \subseteq X(\mathbb{C}_{\sigma_i})$ be an open subsets in the analytic topology. We define

$$X_{\mathbf{k}}(\sigma, U) \coloneqq \bigcup_{\tau \in E_\sigma} \phi_\tau^{-1}(U) \subseteq X(\Bbbk)$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) \coloneqq \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\Bbbk).$$

**Date**: February 12, 2026, **Author**: Tianle Yang, My Homepage

The subset of form $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is called a *basic adelic open subset* of $X(\Bbbk)$.

**Definition 1.3.** A *general adelic subset* of $X(\Bbbk)$ is defined to a subset of the form $\pi(B)$ with $\pi : Y \to X$ a flat morphism of varieties and $B$ a basic adelic open subset of $Y(\Bbbk)$.

**Remark 1.4.** To define a general adelic subset of $X(\Bbbk)$, there are two fields involved: the field $\mathbf{k}$ over which the basic adelic subset is defined, and the field $\mathbf{l}$ over which the morphism $\pi : Y \to X$ is defined.

If we fix a defined field $\mathbf{k}_0$ of $X$, by [Xie25, Proposition 3.15 (ii) and (v)], we can always choose $\mathbf{l} = \mathbf{k}_0$ and $\mathbf{k}$ is a finite extension of $\mathbf{k}_0$.

**Proposition 1.5.** The finite union and intersection of general adelic subsets are still general adelic subsets.

By Proposition 1.5, the general adelic subsets form a basis of topology on $X(\Bbbk)$. Hence we have the following definition.

**Definition 1.6.** The *adelic topology* on $X(\Bbbk)$ is defined to be the topology generated by all general adelic subsets of $X(\Bbbk)$, i.e., an adelic open subset is an arbitrary union of general adelic subsets.

**Proposition 1.7.** We have the following properties of adelic topology:

(a) adelic topology is finer than Zariski topology;

(b) morphisms of varieties are continuous with respect to adelic topology;

(c) flat morphisms of varieties are open with respect to adelic topology.

**Proposition 1.8.** The action of $\mathsf{Gal}(\Bbbk/\mathbf{k})$ on $X(\Bbbk)$, namely $(\sigma, x) \mapsto \sigma(x)$, is continuous with respect to the adelic topology on $X(\Bbbk)$ and the profinite topology on $\mathsf{Gal}(\Bbbk/\mathbf{k})$.

Recall that on $\mathbb{P}^1_{\mathbb{Q}}$, the Artin-Whaples Approximation Theorem says that for any finite collection of places $v_1, \dots, v_n$ of $\mathbb{Q}$ and any open subsets $U_i \subseteq \mathbb{P}^1(\mathbb{Q}_{v_i})$, the intersection $\bigcap_{i=1}^n (U_i \cap \mathbb{P}^1(\mathbb{Q}))$ is non-empty. The following lemma is a generalization and it is the motivation of the definition of adelic subsets.

**Theorem 1.9.** Adelic topology preserves the irreducibility of varieties. Explicitly, on a variety, the intersection of any finite collection of non-empty adelic open subsets is non-empty.

**Lemma 1.10** (ref. [Xie25, Proposition 3.9], cf. [MZ14, Theorem 1.2])**.** For any finite collection of basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$, the intersection

$$\bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

**Slogan** *On a variety, the small open ball in one place will be dense with respect to other places.*

Yang: Is $\mathbb{A}^1(\mathbb{Q})$ adelic closed in $\mathbb{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?
Yang: Describe all adelic closed subset in $\mathbb{A}^1(\overline{\mathbb{Q}})$.

# 2 Interpolation

The main reference of this section is [Ame11; BGT10; Poo14]. Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$ and $\Bbbk$ be its algebraic closure. Let $X$ be a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$. We want to show that after iteration, we can interpolate the iterates of $f$ on an analytic open subset of $X(\mathbb{C}_p)$ for some prime $p$.

## 2.1 Interpolation of analytic maps

In this subsection, we find the interpolation of analytic maps on an analytic disk. Fix a complete non-archimedean field $\mathbf{k}$ of characteristic $0$ with $|p|_{\mathbf{k}} = 1/p$ for some prime $p$. We use the method of difference operators given in [Poo14].

Let $E = E(0,1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$ be the closed unit ball in $\mathbf{k}^d$ and $\Phi : E \to E$ be an analytic map, i.e., $\Phi \in \mathbf{k}^\circ\{\underline{T}\}^d$. Here the norm on $\mathbf{k}^d$ or $\mathbf{k}^\circ\{\underline{T}\}^d$ is the supremum norm, i.e., $|x| = \max_{1\leq i\leq d} |x_i|$. For every analytic map $h$ from $E$ to $E$, we define

$$\Delta(h) := h \circ \Phi - h, \quad \Delta^n(h) := \Delta(\Delta^{n-1}(h)) \text{ for } n \geq 1,$$

and $\Delta^0(h) = h$. Note that $\Delta^n(h)$ is still an analytic map from $E$ to $E$ by the strong triangle inequality.

**Lemma 2.1.** We have the following binomial theorem:

$$\sum_{m=0}^{n} \binom{n}{m} \Delta^m(\mathrm{id}_E) = \Phi^n.$$

*Proof.* By induction, we have

$$\begin{aligned}
\Delta^m(\mathrm{id}_E) &= \Delta\left(\sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k\right) \\
&= \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^{k+1} - \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k \\
&= \sum_{k=0}^{m} \left(\binom{m-1}{k-1} (-1)^{m-k} - \binom{m-1}{k} (-1)^{m-1-k}\right) \Phi^k \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Phi^k.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{m=0}^{n}\binom{n}{m}\Delta^m(\mathrm{id}_E) &= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}(-1)^{m-k}\Phi^k \\
&= \sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{k}\binom{n-k}{m-k}(-1)^{m-k}\Phi^k \\
&= \sum_{k=0}^{n}\binom{n}{k}\Phi^k\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l \\
&= \binom{n}{n}\Phi^n + \sum_{k=0}^{n-1}\binom{n}{k}\Phi^k\cdot(1-1)^{n-k} \\
&= \Phi^n.
\end{aligned}$$

We finish the proof. □

**Lemma 2.2.** Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$ satisfies $\|\Phi - \mathrm{id}_E\| \leq r_p$. Then Yang: To be added.

*Proof.* Yang: To be added. □

**Theorem 2.3** (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3])**.** Let $\mathbf{k}$ be a complete non-archimedean field of characteristic $0$ with $|p|_{\mathbf{k}} = 1/p$. Set $r_p = p^{-1/(p-1)}$.
Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$ satisfies $r := r_p/\|\Phi - \mathrm{id}_E\| > 1$. Then there exists a function $F \in \mathbf{k}\{\underline{T}, S/s\}^d$, $1 < s < r$, such that for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in E$,

$$F(x, n) = \Phi^n(x).$$

*Proof.* Consider the formal series

$$F(\underline{T}, S) := \sum_{n=0}^{\infty}\binom{S}{n}\Delta^n(\mathrm{id}_E)(\underline{T}) = \sum_{n=0}^{\infty} f_n.$$

We only need to show that $f_n \in \mathbf{k}\{\underline{T}, S/s\}^d$ and $\|f_n\| \to 0$ as $n \to \infty$.
Yang: To be added. □

Yang: If $f$ is invertible, can we see that $g$ is unique? Yang: It seems right.

**Example 2.4.** Let $\mathbf{k} = \mathbb{Q}_p$ with $p \geq 3$, and let $\Phi : E \to E$ be the analytic map defined by $\Phi(x) = px^2 + x$. Then we have $\|\Phi - \mathrm{id}_E\| = \|pT^2\| = 1/p < r_p$. Yang: To be checked.

## 2.2 Pick integral models

**Lemma 2.5.** Let $f : X \dashrightarrow X$ be a dominant rational self-map of a projective variety defined over a finitely generated field $\mathbf{k}$ over $\mathbb{Q}$.
Then there exists a subring $R \subseteq \mathbf{k}$ of finite type over $\mathbb{Z}$, a projective scheme $\mathcal{X}$ over $\operatorname{Spec} R$ with generic fiber $X$, and a rational self-map $\mathcal{f} : \mathcal{X} \dashrightarrow \mathcal{X}$ over $\operatorname{Spec} R$ with generic fiber $f$ such that

(a) for every prime ideal $\mathfrak{p}$ of $R$, the special fiber $\mathcal{X}_{\mathfrak{p}}$ is geometrically integral and of the same dimension as $X$;

(b) the union of non-smooth locus of $\mathcal{X}$ and indeterminacy locus, non-étale locus of $\not f$ does not contain any entire special fiber $\mathcal{X}_{\mathfrak{p}}$;

Moreover, if $X$ is smooth (resp. $f$ is a morphism, resp. $f$ is étale), then we can further require that $\mathcal{X}$ is smooth over $\operatorname{Spec} R$ (resp. $\not f$ is a morphism, resp. $\not f$ is étale over $\operatorname{Spec} R$).

Yang: We can embedd $R$ into $\mathbb{C}_p$ for some $p$.

## 2.3 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

**Theorem 2.6** (ref.[Xie25, Proposition 3.24])**.** Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$, $X$ a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$.
There exists an iteration $g = f^m$ of $f$, an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \geq 3$, an analytic open subset $U \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \to U$ such that

(1) $U \cong (\mathbb{C}_p^\circ)^d$ analytically, where $d = \dim X$;

(2) $g$ is well-defined on $U$, $U$ is invariant under $g$ and $\|g|_U - \mathrm{id}_U\| < 1/p$;

(3) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

**Example 2.7.** Let $X = E \times E$ with $E$ an elliptic curve without complex multiplication defined over a number field $\mathbf{k}$, and let $f : X \to X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. Yang: To be continued.

# 3 Applications

## 3.1 Existence of non-preperiodic points

**Theorem 3.1** (ref.[Ame11, Corollary 9])**.** Let $\mathbf{k}$ be an algebraically closed field of characteristic $0$, $X$ a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$. Then there exists a basic adelic subset $U \subset X(\mathbf{k})$ such that the forward orbit $O_f(x) = \{f^n(x) : n \geq 0\}$ is well-defined and infinite for every $x \in U$.

## 3.2 DML conjecture for étale morphisms

**Theorem 3.2** (ref.[BGT10, Theorem 1.3])**.** Let $\mathbf{k}$ be a field of characteristic $0$, $X$ a variety defined over $\mathbf{k}$, and $f : X \to X$ an étale morphism defined over $\mathbf{k}$. The DML conjecture holds for $(X, f)$.

### 3.3 DML conjecture for adelic general points

## References


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