

Adelic topology

The main reference of this section is [Xie25]. Fix a finitely generated field \mathbb{k} over $\overline{\mathbb{Q}}$. Let X be variety over \mathbb{k} .

Let L/K be a field extension. We denote by $M_{L/K}$ (resp. $M_{L,K}$) the set of places of L which restrict to K is trivial (resp. non-trivial).

1 Adelic subset

Notation 1. Let \mathbf{k} be a subfield of \mathbb{k} which is finitely generated over \mathbb{Q} and over which X is defined (such \mathbf{k} is called a *defined field* of X). Denote by $I_{\mathbf{k}}$ the set of all embeddings $\sigma : \mathbf{k} \rightarrow \mathbb{C}_{\sigma}$ over \mathbf{k} , where $\mathbb{C}_{\sigma} = \mathbb{C}_p$ or \mathbb{C} . Every such σ corresponds to a place $v \in M_{\mathbf{k}, \mathbb{Q}}$ by pulling back the standard absolute value on \mathbb{C}_p or \mathbb{C} . For each $\sigma \in I_{\mathbf{k}}$, set $E_{\sigma} = \{\tau : \mathbb{k} \rightarrow \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$. For every $\tau \in I_{\mathbf{k}}$, we have an inclusion map $\phi_{\tau} : X(\mathbb{k}) \rightarrow X(\mathbb{C}_{\tau})$.

On $X(\mathbb{C}_{\tau})$, we have the analytic topology induced from \mathbb{C}_{τ} .

Definition 2. Let \mathbf{k} be a defined field of X . Let $\sigma, \sigma_i \in I_{\mathbf{k}}$ and $U \subset X(\mathbb{C}_{\sigma}), U_i \subseteq X(\mathbb{C}_{\sigma_i})$ be an open subsets in the analytic topology. We define

$$X_{\mathbf{k}}(\sigma, U) := \bigcup_{\tau \in E_{\sigma}} \phi_{\tau}^{-1}(U) \subseteq X(\mathbb{k})$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) := \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\mathbb{k}).$$

The subset of form $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is called a *basic adelic open subset* of $X(\mathbb{k})$.

Definition 3. A *general adelic subset* of $X(\mathbb{k})$ is defined to a subset of the form $\pi(B)$ with $\pi : Y \rightarrow X$ a flat morphism of varieties and B a basic adelic open subset of $Y(\mathbb{k})$.

Remark 4. To define a general adelic subset of $X(\mathbb{k})$, there are two fields involved: the field \mathbf{k} over which the basic adelic subset is defined, and the field \mathbf{l} over which the morphism $\pi : Y \rightarrow X$ is defined.

If we fix a defined field \mathbf{k}_0 of X , by [Xie25, Proposition 3.15 (ii) and (v)], we can always choose $\mathbf{l} = \mathbf{k}_0$ and \mathbf{k} is a finite extension of \mathbf{k}_0 .

Proposition 5. The finite union and intersection of general adelic subsets are still general adelic subsets.

By Proposition 5, the general adelic subsets form a basis of topology on $X(\mathbb{k})$. Hence we have the following definition.

Definition 6. The *adelic topology* on $X(\mathbb{k})$ is defined to be the topology generated by all general adelic subsets of $X(\mathbb{k})$, i.e., an adelic open subset is an arbitrary union of general adelic subsets.

Proposition 7. We have the following properties of adelic topology:

- (a) adelic topology is finer than Zariski topology (in other words, every Zariski open subset is adelic open);
- (b) morphisms of varieties are continuous with respect to adelic topology;
- (c) flat morphisms of varieties are open with respect to adelic topology.

Proposition 8. The action of $\text{Gal}(\mathbb{k}/\mathbf{k})$ on $X(\mathbb{k})$, namely $(\sigma, x) \mapsto \sigma(x)$, is continuous with respect to the adelic topology on $X(\mathbb{k})$ and the profinite topology on $\text{Gal}(\mathbb{k}/\mathbf{k})$.

Recall that on $\mathbb{P}_{\mathbb{Q}}^1$, the Artin-Whaples Approximation Theorem says that for any finite collection of places v_1, \dots, v_n of \mathbb{Q} and any open subsets $U_i \subseteq \mathbb{P}^1(\mathbb{Q}_{v_i})$, the intersection $\bigcap_{i=1}^n (U_i \cap \mathbb{P}^1(\mathbb{Q}))$ is non-empty. The following lemma is a generalization and it is the motivation of the definition of adelic subsets.

Theorem 9. Adelic topology preserves the irreducibility of varieties. Explicitly, on a variety, the intersection of any finite collection of non-empty adelic open subsets is non-empty.

Lemma 10 (ref. [Xie25, Proposition 3.9], cf. [MZ14, Theorem 1.2]). For any finite collection of basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$, the intersection

$$\bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

Slogan *On a variety, the small open ball in one place will be dense with respect to other places.*

Yang: Is $\mathbf{A}^1(\mathbb{Q})$ adelic closed in $\mathbf{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?

Yang: Describe all adelic closed subset in $\mathbf{A}^1(\overline{\mathbb{Q}})$.

Appendix

References

- [MZ14] Vincenzo Mantova and Umberto Zannier. “Artin-Whaples approximations of bounded degree in algebraic varieties”. In: *Proceedings of the American Mathematical Society* 142.9 (2014), pp. 2953–2964 (cit. on p. 2).
- [Xie25] Junyi Xie. “The existence of Zariski dense orbits for endomorphisms of projective surfaces”. In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on pp. 1, 2).