The $\mathfrak{p}$-adic Method

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Contents

1 Adelic topology

The main reference of this section is [Xie25].

1.1 Basic adelic subset

Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$ and $\Bbbk$ be its algebraic closure.

Let X be a variety over $\mathbf{k}$ with $\mathbf{k}$ a finitely generated field over $\mathbb{Q}$. Set $M_{\mathbf{k}}$ be the set of places of $\mathbf{k}$. For each $v \in M_{\mathbf{k}}$.

Let $\mathcal{J}_{\mathbf{k}}$ be the set of all embeddings $\sigma : K \to \mathbb{C}_\sigma$ over $\mathbf{k}$, where $\mathbb{C}_\sigma = \mathbb{C}_p$ or $\mathbb{C}$ according to whether v is non-archimedean or archimedean. For each $\sigma \in \mathcal{J}_{\mathbf{k}}$, set $\mathcal{J}_\sigma = \{\tau : \Bbbk \to \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$. For every $\tau \in \mathcal{J}_{\Bbbk}$, we have an induced map $\phi_\tau : X(\Bbbk) \to X(\mathbb{C}_\tau)$.

On $X(\mathbb{C}_\tau)$, we have the analytic topology induced from $\mathbb{C}_\tau$.

Definition 1.1.
$$X_{\mathbf{k}}(\sigma, U) := \bigcup_{\tau \in \mathcal{J}_\sigma} \phi_\tau^{-1}(U) \subseteq X(\Bbbk)$$
is called a *basic adelic open subset* of $X(\Bbbk)$, where $U \subseteq X(\mathbb{C}_\sigma)$ is an open subset in the analytic topology.

Theorem 1.2. Suppose that X is irreducible. Then for any finite collection of places $\sigma_1, \ldots, \sigma_n \in M_{\mathbf{k}}$

and any basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$ for $i = 1, \dots, n$, the intersection

$$\bigcap_{i=1}^{n} X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

1.2 General adelic subset

1.3 Adelic topology

Yang: Is $\mathbb{A}^1(\mathbb{Q})$ adelic closed in $\mathbb{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?

Yang: Describe all adelic closed subset in $\mathbb{A}^1(\overline{\mathbb{Q}})$.

2 Interpolation

2.1 Interpolation of analytic maps

The main reference of this section is [Ame11; BGT10; Poo14]. We first state the main theorem of this section.

Theorem 2.1 (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3])**.** Let $\mathbf{k}$ be a complete non-archimedean field of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$. Set $r_p = p^{-1/(p-1)}$. Let $E = E(0,1) = \{x \in \mathbf{k}^d \mid \|x\| \le 1\}$ be the closed unit ball in $\mathbf{k}^d$. Here the norm on $\mathbf{k}^d$ or $\mathbf{k}^\circ\{\underline{T}\}^d$ is the supremum norm, i.e., $|x| = \max_{1\le i\le d} |x_i|$.
Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$ satisfies $\|\Phi - \mathrm{id}_E\| \le r_p$. Then there exists a function $F \in \mathbf{k}\{\underline{T}, S\}^d$ such that for each $n \in \mathbb{Z}_{\ge 0}$ and each $x \in E$,

$$F(x, n) = \Phi^n(x).$$

Proof. Yang: To be added. ☐

Yang: If f is invertible, can we see that g is unique?

Example 2.2. Let $\mathbf{k} = \mathbb{Q}_p$ with $p \ge 3$, and let $\Phi : E \to E$ be the analytic map defined by $\Phi(x) = x^p$. Then we have $\|\Phi - \mathrm{id}_E\| = \|x^p - x\|_{\sup} = 1/p < r_p$. Yang: To be checked.

2.2 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

Theorem 2.3 (ref.[Xie25, Proposition 3.24])**.** Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$.
There exists an iteration $g = f^m$ of f, an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \ge 3$, an analytic open subset $U \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \to U$ such that

(1) $U \cong (\mathbb{C}_p^\circ)^d$ analytically, where $d = \dim X$;

(2) g is well-defined on U, U is invariant under g and $\|g|_U - \mathrm{id}_U\| < 1/p$;

(3) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

Lemma 2.4. Yang: There is a good model for (X, f).

Example 2.5. Let $X = E \times E$ with E an elliptic curve without complex multiplication defined over a number field $\mathbf{k}$, and let $f : X \to X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. Yang: To be continued.

3 Applications

3.1 Existence of non-preperiodic points

Theorem 3.1 (ref.[Ame11, Corollary 9])**.** Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$. Then there exists a point $x \in X(\mathbf{k})$ such that the forward orbit $O_f(x) = \{f^n(x) : n \geq 0\}$ is well-defined and infinite.

3.2 DML conjecture for étale morphisms

3.3 DML conjecture for adelic general points

References

[Ame11] E Amerik. "Existence of non-preperiodic algebraic points for a rational self-map of infinite order". In: *Mathematical Research Letters* 18.2 (2011), pp. 251–256 (cit. on pp. 2, 3).

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[Xie25] Junyi Xie. "The existence of Zariski dense orbits for endomorphisms of projective surfaces". In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on pp. 1, 2).