

# The $p$ -adic Method



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# Contents

<b>1 Adelic topology</b>	<b>1</b>
1.1 Basic adelic subset . . . . .	1
1.2 General adelic subset . . . . .	2
1.3 Adelic topology . . . . .	2
<b>2 Interpolation</b>	<b>2</b>
2.1 Pick integral model . . . . .	2
2.2 Interpolation of analytic maps . . . . .	2
2.3 Interpolation on an analytic open subset of morphisms . . . . .	2
<b>3 Applications</b>	<b>3</b>
3.1 Existence of non-preperiodic points . . . . .	3
3.2 DML conjecture for étale morphisms . . . . .	3
3.3 DML conjecture for adelic general points . . . . .	3
<b>References</b>	<b>3</b>

## 1 Adelic topology

The main reference of this section is [Xie25].

### 1.1 Basic adelic subset

Let  $\mathbf{k}$  be a finitely generated field over  $\mathbb{Q}$  and  $\mathbb{k}$  be its algebraic closure.

Let  $\mathcal{I}_{\mathbf{k}}$  be the set of all embeddings  $\sigma : \mathbf{k} \rightarrow \mathbb{C}_\sigma$  over  $\mathbf{k}$ , where  $\mathbb{C}_\sigma = \mathbb{C}_p$  or  $\mathbb{C}$  according to whether  $v$  is non-archimedean or archimedean. For each  $\sigma \in \mathcal{I}_{\mathbf{k}}$ , set  $\mathcal{I}_\sigma = \{\tau : \mathbb{k} \rightarrow \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$ . For every  $\tau \in \mathcal{I}_{\mathbf{k}}$ , we have an induced map  $\phi_\tau : X(\mathbb{k}) \rightarrow X(\mathbb{C}_\tau)$ .

On  $X(\mathbb{C}_\tau)$ , we have the analytic topology induced from  $\mathbb{C}_\tau$ .

**Definition 1.1.** Let  $\sigma, \sigma_i \in \mathcal{I}_{\mathbf{k}}$  and let  $U, U_i \subseteq X(\mathbb{C}_\sigma)$  be an open subset in the analytic topology.

We define

$$X_{\mathbf{k}}(\sigma, U) := \bigcup_{\tau \in \mathcal{I}_\sigma} \phi_\tau^{-1}(U) \subseteq X(\mathbb{k})$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) := \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\mathbb{k}).$$

The subset of form  $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$  is called a *basic adelic open subset* of  $X(\mathbb{k})$ , where  $U \subseteq X(\mathbb{C}_\sigma)$  is an open subset in the analytic topology.

**Theorem 1.2** (ref. [Xie25, Proposition 3.9] cf. []). Suppose that  $X$  is irreducible. Then for any finite collection of places  $\sigma_1, \dots, \sigma_n \in M_{\mathbf{k}}$  and any basic adelic open subsets  $X_{\mathbf{k}}(\sigma_i, U_i)$  for  $i = 1, \dots, n$ , the intersection

$$\bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

## 1.2 General adelic subset

## 1.3 Adelic topology

Yang: Is  $A^1(\mathbb{Q})$  adelic closed in  $A^1(\overline{\mathbb{Q}})$  with adelic topology? If so, why?

Yang: Describe all adelic closed subset in  $A^1(\overline{\mathbb{Q}})$ .

## 2 Interpolation

### 2.1 Pick integral model

### 2.2 Interpolation of analytic maps

The main reference of this section is [Ame11; BGT10; Poo14]. We first state the main theorem of this section.

**Theorem 2.1** (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3]). Let  $\mathbf{k}$  be a complete non-archimedean field of characteristic 0 with  $|p|_{\mathbf{k}} = 1/p$ . Set  $r_p = p^{-1/(p-1)}$ . Let  $E = E(0, 1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$  be the closed unit ball in  $\mathbf{k}^d$ . Here the norm on  $\mathbf{k}^d$  or  $\mathbf{k}^{\circ}\{\underline{T}\}^d$  is the supremum norm, i.e.,  $|x| = \max_{1 \leq i \leq d} |x_i|$ .

Suppose that  $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathbf{k}^{\circ}\{\underline{T}\}^d$  satisfies  $\|\Phi - \text{id}_E\| \leq r_p$ . Then there exists a function  $F \in \mathbf{k}\{\underline{T}, S\}^d$  such that for each  $n \in \mathbb{Z}_{\geq 0}$  and each  $x \in E$ ,

$$F(x, n) = \Phi^n(x).$$

| *Proof.* Yang: To be added. □

Yang: If  $f$  is invertible, can we see that  $g$  is unique? Yang: It seems right.

**Example 2.2.** Let  $\mathbf{k} = \mathbb{Q}_p$  with  $p \geq 3$ , and let  $\Phi: E \rightarrow E$  be the analytic map defined by  $\Phi(x) = px^2 + x$ . Then we have  $\|\Phi - \text{id}_E\| = \|pT^2\| = 1/p < r_p$ . Yang: To be checked.

### 2.3 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

**Theorem 2.3** (ref.[Xie25, Proposition 3.24]). Let  $\mathbf{k}$  be a finitely generated field over  $\mathbb{Q}$ ,  $X$  a projective variety defined over  $\mathbf{k}$ , and  $f: X \dashrightarrow X$  a dominant rational self-map defined over  $\mathbf{k}$ .

There exists an iteration  $g = f^m$  of  $f$ , an embedding  $\mathbf{k} \hookrightarrow \mathbb{C}_p$  for some prime  $p \geq 3$ , an analytic open subset  $U \subseteq X(\mathbb{C}_p)$  and an analytic map  $\Phi : \mathbb{C}_p^\circ \times U \rightarrow U$  such that

- (1)  $U \cong (\mathbb{C}_p^\circ)^d$  analytically, where  $d = \dim X$ ;
- (2)  $g$  is well-defined on  $U$ ,  $U$  is invariant under  $g$  and  $\|g|_U - \text{id}_U\| < 1/p$ ;
- (3)  $\Phi(n, x) = g^n(x)$  for each  $n \in \mathbb{Z}_{\geq 0}$  and each  $x \in U$ ;

**Lemma 2.4.** *Yang: There is a good model for  $(X, f)$ .*

**Example 2.5.** Let  $X = E \times E$  with  $E$  an elliptic curve without complex multiplication defined over a number field  $\mathbf{k}$ , and let  $f : X \rightarrow X$  be the endomorphism defined by  $(a, b) \mapsto (a + b, b)$ . *Yang: To be continued.*

## 3 Applications

### 3.1 Existence of non-preperiodic points

**Theorem 3.1** (ref.[Ame11, Corollary 9]). Let  $\mathbf{k}$  be an algebraically closed field of characteristic 0,  $X$  a projective variety defined over  $\mathbf{k}$ , and  $f : X \dashrightarrow X$  a dominant rational self-map defined over  $\mathbf{k}$ . Then there exists a basic adelic subset  $U \subset X(\mathbf{k})$  such that the forward orbit  $O_f(x) = \{f^n(x) : n \geq 0\}$  is well-defined and infinite for every  $x \in U$ .

### 3.2 DML conjecture for étale morphisms

**Theorem 3.2** (ref.[BGT10, Theorem 1.3]). Let  $\mathbf{k}$  be a field of characteristic 0,  $X$  a variety defined over  $\mathbf{k}$ , and  $f : X \rightarrow X$  an étale morphism defined over  $\mathbf{k}$ . The DML conjecture holds for  $(X, f)$ .

### 3.3 DML conjecture for adelic general points

## References

- [Ame11] E Amerik. “Existence of non-preperiodic algebraic points for a rational self-map of infinite order”. In: *Mathematical Research Letters* 18.2 (2011), pp. 251–256 (cit. on pp. 2, 3).
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- [Xie25] Junyi Xie. “The existence of Zariski dense orbits for endomorphisms of projective surfaces”. In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on pp. 1, 2).