The p-adic Method

No Cover Image

Use \coverimage{filename} to add an image

阿巴阿巴!

Contents

1 Adelic topology

The main reference of this section is [Xie25]. Fix a finitely generated field $\Bbbk$ over $\overline{\mathbb{Q}}$. Let X be variety over $\Bbbk$.

Let L/K be a field extension. We denote by $M_{L/K}$ (resp. $M_{L,K}$) the set of places of L which restrict to K is trivial (resp. non-trivial).

1.1 Adelic subset

Notation 1.1. Let $\mathbf{k}$ be a subfield of $\Bbbk$ which is finitely generated over $\mathbb{Q}$ and over which X is defined (such $\mathbf{k}$ is called a *defined field* of X). Denote by $I_{\mathbf{k}}$ the set of all embeddings $\sigma : \mathbf{k} \to \mathbb{C}_\sigma$ over $\mathbf{k}$, where $\mathbb{C}_\sigma = \mathbb{C}_p$ or $\mathbb{C}$. Every such σ corresponds to a place $v \in M_{\mathbf{k},\mathbb{Q}}$ by pulling back the standard absolute value on $\mathbb{C}_p$ or $\mathbb{C}$. For each $\sigma \in I_{\mathbf{k}}$, set $E_\sigma = \{\tau : \Bbbk \to \mathbb{C}_v \mid \tau|_{\mathbf{k}} = \sigma\}$. For every $\tau \in I_{\mathbf{k}}$, we have an inclusion map $\phi_\tau : X(\Bbbk) \to X(\mathbb{C}_\tau)$.

On $X(\mathbb{C}_\tau)$, we have the analytic topology induced from $\mathbb{C}_\tau$.

Definition 1.2. Let $\mathbf{k}$ be a defined field of X. Let $\sigma, \sigma_i \in I_{\mathbf{k}}$ and $U \subset X(\mathbb{C}_\sigma), U_i \subseteq X(\mathbb{C}_{\sigma_i})$ be an open subsets in the analytic topology. We define

$$X_{\mathbf{k}}(\sigma, U) := \bigcup_{\tau \in E_\sigma} \phi_\tau^{-1}(U) \subseteq X(\Bbbk)$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) := \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\Bbbk).$$

The subset of form $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is called a *basic adelic open subset* of $X(\Bbbk)$.

Date: February 21, 2026, **Author**: Tianle Yang, My Homepage

Definition 1.3. A *general adelic subset* of $X(\Bbbk)$ is defined to a subset of the form $\pi(B)$ with $\pi : Y \to X$ a flat morphism of varieties and B a basic adelic open subset of $Y(\Bbbk)$.

Remark 1.4. To define a general adelic subset of $X(\Bbbk)$, there are two fields involved: the field $\mathbf{k}$ over which the basic adelic subset is defined, and the field $\mathbf{l}$ over which the morphism $\pi : Y \to X$ is defined.

If we fix a defined field $\mathbf{k}_0$ of X, by [Xie25, Proposition 3.15 (ii) and (v)], we can always choose $\mathbf{l} = \mathbf{k}_0$ and $\mathbf{k}$ is a finite extension of $\mathbf{k}_0$.

Proposition 1.5. The finite union and intersection of general adelic subsets are still general adelic subsets.

By Proposition 1.5, the general adelic subsets form a basis of topology on $X(\Bbbk)$. Hence we have the following definition.

Definition 1.6. The *adelic topology* on $X(\Bbbk)$ is defined to be the topology generated by all general adelic subsets of $X(\Bbbk)$, i.e., an adelic open subset is an arbitrary union of general adelic subsets.

Proposition 1.7. We have the following properties of adelic topology:

(a) adelic topology is finer than Zariski topology;

(b) morphisms of varieties are continuous with respect to adelic topology;

(c) flat morphisms of varieties are open with respect to adelic topology.

Proposition 1.8. The action of $\mathrm{Gal}(\Bbbk/\mathbf{k})$ on $X(\Bbbk)$, namely $(\sigma, x) \mapsto \sigma(x)$, is continuous with respect to the adelic topology on $X(\Bbbk)$ and the profinite topology on $\mathrm{Gal}(\Bbbk/\mathbf{k})$.

Recall that on $\mathbb{P}^1_{\mathbb{Q}}$, the Artin-Whaples Approximation Theorem says that for any finite collection of places $v_1, \ldots, v_n$ of $\mathbb{Q}$ and any open subsets $U_i \subseteq \mathbb{P}^1(\mathbb{Q}_{v_i})$, the intersection $\bigcap_{i=1}^n (U_i \cap \mathbb{P}^1(\mathbb{Q}))$ is non-empty. The following lemma is a generalization and it is the motivation of the definition of adelic subsets.

Theorem 1.9. Adelic topology preserves the irreducibility of varieties. Explicitly, on a variety, the intersection of any finite collection of non-empty adelic open subsets is non-empty.

Lemma 1.10 (ref. [Xie25, Proposition 3.9], cf. [MZ14, Theorem 1.2])**.** For any finite collection of basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$, the intersection

$$\bigcap_{i=1}^{n} X_{\mathbf{k}}(\sigma_i, U_i)$$

is non-empty.

Slogan *On a variety, the small open ball in one place will be dense with respect to other places.*

Yang: Is $\mathbb{A}^1(\mathbb{Q})$ adelic closed in $\mathbb{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?

Yang: Describe all adelic closed subset in $\mathbb{A}^1(\overline{\mathbb{Q}})$.

2 Interpolation

The main reference of this section is [Ame11; BGT10; Poo14; Xie25].

2.1 Interpolation of analytic maps

In this subsection, we find the interpolation of analytic maps on an analytic disk. Fix a complete non-archimedean field $\mathbf{k}$ of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$ for some prime p. Set $r_p = p^{-1/(p-1)}$. We use the method of difference operators given in [Poo14].

Let $E = E(0,1) = \{x \in \mathbf{k}^d \mid \|x\| \le 1\}$ be the closed unit ball in $\mathbf{k}^d$ and $\Phi : E \to E$ be an analytic map, i.e., $\Phi \in \mathbf{k}^\circ\{\underline{X}\}^d$. Here the norm on $\mathbf{k}^d$ or $\mathbf{k}^\circ\{\underline{X}\}^d$ is the supremum norm, i.e., $|x| = \max_{1\le i\le d} |x_i|$. For every analytic map h from E to E, we define

$$\Delta(h) := h \circ \Phi - h, \quad \Delta^n(h) := \Delta(\Delta^{n-1}(h)) \text{ for } n \ge 1,$$

and $\Delta^0(h) = h$. Note that $\Delta^n(h)$ is still an analytic map from E to E by the strong triangle inequality.

Lemma 2.1. We have the following binomial theorem:

$$\sum_{m=0}^{n} \binom{n}{m} \Delta^m(\mathrm{id}_E) = \Phi^n.$$

Proof. By induction, we have

$$\begin{aligned}
\Delta^m(\mathrm{id}_E) &= \Delta\left(\sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k\right) \\
&= \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^{k+1} - \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^{m-1-k} \Phi^k \\
&= \sum_{k=0}^{m} \left(\binom{m-1}{k-1} (-1)^{m-k} - \binom{m-1}{k} (-1)^{m-1-k}\right) \Phi^k \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Phi^k.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{m=0}^{n} \binom{n}{m} \Delta^m(\mathrm{id}_E) &= \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} \binom{m}{k} (-1)^{m-k} \Phi^k \\
&= \sum_{k=0}^{n} \sum_{m=k}^{n} \binom{n}{k} \binom{n-k}{m-k} (-1)^{m-k} \Phi^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \Phi^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \\
&= \binom{n}{n} \Phi^n + \sum_{k=0}^{n-1} \binom{n}{k} \Phi^k \cdot (1-1)^{n-k} \\
&= \Phi^n.
\end{aligned}$$

We finish the proof. $\square$

Lemma 2.2. Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{X}\}^d$ satisfies $\|\Phi - \mathrm{id}_E\| \leq r$. Then

$$\|\Delta^n(\mathrm{id}_E)\| \leq r^n$$

Proof. By Yang: ref, we have

$$\|\Delta(h)\| = \|h \circ \Phi - h\| \leq \|h\| \cdot \|\Phi - \mathrm{id}_E\| \leq r\|h\|.$$

Hence by induction, the result follows. □

Theorem 2.3 (ref.[Poo14, Theorem 1], cf.[BGT10, Theorem 3.3]). Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{X}\}^d$ satisfies $r := \|\Phi - \mathrm{id}_E\| < r_p$. Then there exists a unique function $F \in \mathbf{k}\{\underline{X}, T/s\}^d$, such that for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in E$,

$$F(x, n) = \Phi^n(x).$$

Here s is any real number with $1 < s < r_p/r$.

Proof. Consider the formal series

$$F(\underline{X}, T) := \sum_{n=0}^{\infty} \binom{T}{n} \Delta^n(\mathrm{id}_E)(\underline{X}).$$

Recall the Newton's binomial function

$$\binom{T}{n} := \frac{T(T-1)\cdots(T-n+1)}{n!} \in \mathbf{k}[T].$$

Since $\binom{T}{n}$ is a polynomial in T and $\Delta^n(\mathrm{id}_E)(\underline{X}) \in \mathbf{k}^\circ\{\underline{X}\}^d$, we have $f_n = \binom{T}{n}\Delta^n(\mathrm{id}_E)(\underline{X}) \in \mathbf{k}\{\underline{X}, T\}^d$. Note that for each $n \in \mathbb{Z}_{\geq 0}$, then $|n!|_{\mathbf{k}} \geq r_p^n$. Hence we have

$$\left\| \binom{T}{n} \right\| \leq s^n r_p^{-n}$$

since $s > 1$. By Lemma 2.2, we have

$$\|f_n\| \leq \left\| \binom{T}{n} \right\| \cdot \|\Delta^n(\mathrm{id}_E)\| \leq s^n r_p^{-n} r^n = (sr/r_p)^n.$$

Since $sr/r_p < 1$, the series $F(\underline{X}, T) = \sum_{n=0}^{\infty} f_n$ converges in $\mathbf{k}\{\underline{X}, T/s\}^d$. By Lemma 2.1, we have $F(x, n) = \Phi^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in E$. The uniqueness of F follows from Yang: ref. □

Yang: If f is invertible, can we see that g is unique? Yang: It seems right.

Example 2.4. Let $\mathbf{k} = \mathbb{Q}_p$ with $p \geq 3$, and let $\Phi : E \to E$ be the analytic map defined by $\Phi(x) = px^2 + x$. Then we have $\|\Phi - \mathrm{id}_E\| = \|pT^2\| = 1/p < r_p$. Yang: To be checked.

2.2 Interpolation on an analytic open subset of morphisms

Let $f : X \dashrightarrow X$ be a dominant rational self-map of a projective variety of dimension d defined over a finitely generated field $\mathbf{k}$ over $\mathbb{Q}$. We try to find some analytic local interpolation of the iterates of f on $X(\mathbb{C}_p)$ for some prime p.

Lemma 2.5. There exists a subring $R \subseteq \mathbf{k}$ of finite type over $\mathbb{Z}$, a projective scheme $\mathcal{X}$ over $\operatorname{Spec} R$ with generic fiber X, and a rational self-map $\mathscr{f} : \mathcal{X} \dashrightarrow \mathcal{X}$ over $\operatorname{Spec} R$ with generic fiber f such that

(a) for every prime ideal $\mathfrak{p}$ of R, the special fiber $\mathcal{X}_{\mathfrak{p}}$ is geometrically integral and of the same dimension as X;

(b) the union of non-smooth locus of $\mathcal{X}$ and indeterminacy locus, non-étale locus of $\mathscr{f}$ does not contain any entire special fiber $\mathcal{X}_{\mathfrak{p}}$;

Moreover, if X is smooth (resp. f is a morphism, resp. f is étale), then we can further require that $\mathcal{X}$ is smooth over $\operatorname{Spec} R$ (resp. $\mathscr{f}$ is a morphism, resp. $\mathscr{f}$ is étale over $\operatorname{Spec} R$).

Yang: We can embedd R into $\mathbb{C}_p$ for some p.

Theorem 2.6 (ref.[Xie25, Proposition 3.24])**.** There exists an iteration $g = f^m$ of f, an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \geq 3$, an analytic open subset $U \cong (\mathbb{C}_p^\circ)^d \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \to U$ such that

(a) g is well-defined on U, U is invariant under g and $\|g|_U - \mathrm{id}_U\| < 1/p$;

(b) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

Example 2.7. Let $X = E \times E$ with E an elliptic curve without complex multiplication defined over a number field $\mathbf{k}$, and let $f : X \to X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. Yang: To be continued.

3 Applications

3.1 Existence of non-preperiodic points

Theorem 3.1 (ref.[Ame11, Corollary 9])**.** Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$. Then there exists a basic adelic subset $U \subset X(\mathbf{k})$ such that the forward orbit $O_f(x) = \{f^n(x) : n \geq 0\}$ is well-defined and infinite for every $x \in U$.

3.2 DML conjecture for étale morphisms

Theorem 3.2 (ref.[BGT10, Theorem 1.3])**.** Let $\mathbf{k}$ be a field of characteristic 0, X a variety defined over $\mathbf{k}$, and $f : X \to X$ an étale morphism defined over $\mathbf{k}$. The DML conjecture holds for (X, f).

3.3 DML conjecture for adelic general points

References

[Ame11] E Amerik. “Existence of non-preperiodic algebraic points for a rational self-map of infinite order”. In: *Mathematical Research Letters* 18.2 (2011), pp. 251–256 (cit. on pp. 3, 5).

[BGT10] Jason P Bell, Dragos Ghioca, and Thomas J Tucker. “The dynamical Mordell-Lang problem for étale maps”. In: *American journal of mathematics* 132.6 (2010), pp. 1655–1675 (cit. on pp. 3–5).

[MZ14] Vincenzo Mantova and Umberto Zannier. “Artin-Whaples approximations of bounded degree in algebraic varieties”. In: *Proceedings of the American Mathematical Society* 142.9 (2014), pp. 2953–2964 (cit. on p. 2).

[Poo14] Bjorn Poonen. “p-adic interpolation of iterates”. In: *Bulletin of the London Mathematical Society* 46.3 (2014), pp. 525–527 (cit. on pp. 3, 4).

[Xie25] Junyi Xie. “The existence of Zariski dense orbits for endomorphisms of projective surfaces”. In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on pp. 1–3, 5).