The **p** -adic Method

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Contents

1   Adelic topology

The main reference of this section is [Xie25].

Let $\mathbf{k}$ be a number field, and let $M_{\mathbf{k}}$ be the set of places of $\mathbf{k}$.

1.1   Basic adelic subset

1.2   General adelic subset

1.3   Adelic topology

2   Interpolation

2.1   Interpolation of analytic maps

The main reference of this section is [Ame11; BGT10; Poo14]. We first state the main theorem of this section.

> **Theorem 2.1** (ref.[Poo14, Theorem 1] cf.[BGT10, Theorem 3.3])**.** Let $\mathbf{k}$ be a complete non-archimedean field of characteristic 0 with $|p|_{\mathbf{k}} = 1/p$. Set $r_p = p^{-1/(p-1)}$. Let $E = E(0,1) = \{x \in \mathbf{k}^d \mid \|x\| \leq 1\}$ be the closed unit ball in $\mathbf{k}^d$. Here the norm on $\mathbf{k}^d$ or $\mathbf{k}^\circ\{\underline{T}\}^d$ is the supremum norm, i.e., $|x| = \max_{1 \leq i \leq d} |x_i|$.
>
> Suppose that $\Phi = (\Phi_1, \cdots, \Phi_d) \in \mathbf{k}^\circ\{\underline{T}\}^d$ satisfies $\|\Phi - \mathrm{id}_E\| \leq r_p$. Then there exists a function

$F \in \mathbf{k}\{\underline{T}, S\}^d$ such that for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in E$,

$$F(x, n) = \Phi^n(x).$$

Proof. Yang: To be added. □

Yang: If f is invertible, can we see that g is unique?

2.2 Interpolation on an analytic open subset of morphisms

The main reference of this section is [Xie25, Section 3.2]. We first state the main theorem of this section.

Theorem 2.2 (ref.[Xie25, Proposition 3.24]). Let $\mathbf{k}$ be a finitely generated field over $\mathbb{Q}$, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$.

There exists an iteration $g = f^m$ of f, an embedding $\mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime $p \geq 3$, an analytic open subset $U \subseteq X(\mathbb{C}_p)$ and an analytic map $\Phi : \mathbb{C}_p^\circ \times U \to U$ such that

(1) $U \cong (\mathbb{C}_p^\circ)^d$ analytically, where $d = \dim X$;

(2) g is well-defined on U, U is invariant under g and $\|g|_U - \mathrm{id}_U\| < 1/p$;

(3) $\Phi(n, x) = g^n(x)$ for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in U$;

Lemma 2.3. Yang: There is a good model for (X, f).

Example 2.4. Let $X = E \times E$ with E an elliptic curve without complex multiplication defined over a number field $\mathbf{k}$, and let $f : X \to X$ be the endomorphism defined by $(a, b) \mapsto (a + b, b)$. Yang: To be continued.

3 Applications

3.1 Existence of non-preperiodic points

Theorem 3.1 (ref.[Ame11, Corollary 9]). Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, X a projective variety defined over $\mathbf{k}$, and $f : X \dashrightarrow X$ a dominant rational self-map defined over $\mathbf{k}$. Then there exists a point $x \in X(\mathbf{k})$ such that the forward orbit $O_f(x) = \{f^n(x) : n \geq 0\}$ is well-defined and infinite.

3.2 DML conjecture for étale morphisms

3.3 DML conjecture for adelic general points

References

[Ame11]   E Amerik. "Existence of non-preperiodic algebraic points for a rational self-map of infinite order". In: *Mathematical Research Letters* 18.2 (2011), pp. 251–256 (cit. on pp. 1, 2).

[BGT10]   Jason P Bell, Dragos Ghioca, and Thomas J Tucker. "The dynamical Mordell-Lang problem for étale maps". In: *American journal of mathematics* 132.6 (2010), pp. 1655–1675 (cit. on p. 1).

[Poo14]   Bjorn Poonen. "p-adic interpolation of iterates". In: *Bulletin of the London Mathematical Society* 46.3 (2014), pp. 525–527 (cit. on p. 1).

[Xie25]   Junyi Xie. "The existence of Zariski dense orbits for endomorphisms of projective surfaces". In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on pp. 1, 2).