Adelic topology

The main reference of this section is [Xie25]. Fix an algebraically closed field $\Bbbk$ which is the algebraic closure of a finitely generated field over $\mathbb{Q}$. Let X be variety over $\Bbbk$.

In this section, we allow that the variety X is not irreducible, i.e., a variety over $\Bbbk$ is a reduced separated scheme of finite type over $\Bbbk$.

Let L/K be a field extension. We denote by $M_{L/K}$ (resp. $M_{L,K}$) the set of places of L which restrict to K is trivial (resp. non-trivial).

1 Adelic topology

Notation 1. Set

$$F(\Bbbk) \coloneqq \{\mathbf{k} \subseteq \Bbbk \mid \mathbf{k} \text{ is finitely generated over } \mathbb{Q} \text{ and } \Bbbk/\mathbf{k} \text{ is algebraic}\}.$$

Denote by $I_{\mathbf{k}}$ the set of all embeddings $\sigma : \mathbf{k} \to \mathbb{C}_\sigma$ over $\mathbf{k}$, where $\mathbb{C}_\sigma = \mathbb{C}_p$ or $\mathbb{C}$. Every such σ corresponds to a place $v \in M_{\mathbf{k},\mathbb{Q}}$ by pulling back the standard absolute value on $\mathbb{C}_p$ or $\mathbb{C}$. For each $\sigma \in I_{\mathbf{k}}$, set $E_\sigma = \{\bar{\sigma} : \Bbbk \to \mathbb{C}_\sigma \mid \bar{\sigma}|_{\mathbf{k}} = \sigma\}$ be the set of embeddings of $\Bbbk$ to $\mathbb{C}_v$ extending σ.

We say that $\mathbf{k}$ is a *defined field of* X if there exists a $\mathbf{k}$-variety $X_{\mathbf{k}}$ such that $X_{\mathbf{k}} \times_{\mathbf{k}} \operatorname{Spec} \Bbbk \cong X$. Such a variety $X_{\mathbf{k}}$ is called a *model of* X *over* $\mathbf{k}$ or a $\mathbf{k}$-*model of* X. For every $\sigma \in I_{\mathbf{k}}$ and $\bar{\sigma} \in E_\sigma$, we have an inclusion map $\phi_{\bar{\sigma}} : X(\Bbbk) \cong X_{\mathbf{k}}(\Bbbk) \to X_{\mathbf{k}}(\mathbb{C}_\sigma)$. Here the first isomorphism is given by the base change $X_{\mathbf{k}} \times_{\mathbf{k}} \operatorname{Spec} \Bbbk \cong X$ and the second map is given by composition with $\operatorname{Spec} \bar{\sigma} : \operatorname{Spec} \mathbb{C}_\sigma \to \operatorname{Spec} \Bbbk$. On $X_{\mathbf{k}}(\mathbb{C}_\sigma)$, we have the analytic topology induced from the norm on $\mathbb{C}_\sigma$.

Definition 2. Let $\mathbf{k}$ be a defined field of X with a model $X_{\mathbf{k}}$. Let $\sigma, \sigma_i \in I_{\mathbf{k}}$ and $U \subset X_{\mathbf{k}}(\mathbb{C}_\sigma), U_i \subseteq X_{\mathbf{k}}(\mathbb{C}_{\sigma_i})$ be an open subsets in the analytic topology. We define

$$X_{\mathbf{k}}(\sigma, U) \coloneqq \bigcup_{\tau \in E_\sigma} \phi_\tau^{-1}(U) \subseteq X(\Bbbk)$$

and

$$X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n) \coloneqq \bigcap_{i=1}^n X_{\mathbf{k}}(\sigma_i, U_i) \subseteq X(\Bbbk).$$

The subset of form $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is called a *basic adelic open subset* of $X(\Bbbk)$.

Remark 3.

Lemma 4. We have the following basic facts of basic adelic subsets:

(a) let $\mathbf{k}'$ be a finite extension of $\mathbf{k}$ and $X_{\mathbf{k}'}(\{\sigma_i', U_i'\}_{i=1}^m)$ be a basic adelic subset of $X(\Bbbk)$ defined by $\mathbf{k}'$;

(b) let $f : Y \to X$ be a morphism of varieties over $\Bbbk$ and $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ be a basic adelic subset of $X(\Bbbk)$ defined by $\mathbf{k}$, then $f^{-1}(X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)) = Y_{\mathbf{k}}(\{\sigma_i, f^{-1}(U_i)\}_{i=1}^n)$;

(c) let $\tau \in \operatorname{Gal}(\Bbbk/\mathbf{k})$ and $X_{\mathbf{k}'}(\{\sigma_i', U_i'\}_{i=1}^m)$ be a basic adelic subset of $X(\Bbbk)$ defined by $\mathbf{k}'$, then

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$$\tau(X_{\mathbf{k}'}(\{\sigma_i', U_i'\}_{i=1}^m)) = X_{\mathbf{k}'}(\{\tau\sigma_i', \tau(U_i')\}_{i=1}^m);$$

Yang:

Definition 5. A *general adelic subset* of $X(\mathbb{k})$ is defined to a subset of the form $\pi(B)$ with $\pi : Y \to X$ a flat morphism of varieties and B a basic adelic open subset of $Y(\mathbb{k})$.

Remark 6. To define a general adelic subset of $X(\mathbb{k})$, there are two fields involved: the field $\mathbf{k}$ over which the basic adelic subset is defined, and the field $\mathbf{l}$ over which the morphism $\pi : Y \to X$ is defined.

If we fix a defined field $\mathbf{k}_0$ of X, by [Xie25, Proposition 3.15 (ii) and (v)], we can always choose $\mathbf{l} = \mathbf{k}_0$ and $\mathbf{k}$ is a finite extension of $\mathbf{k}_0$.

Lemma 7. The finite union and intersection of general adelic subsets are still general adelic subsets.

Proof. Let B_1, B_2 be two general adelic subsets of $X(\mathbb{k})$ defined by $\pi_i : Y_i \to X$ and basic adelic subsets $A_i \subseteq Y_i(\mathbb{k})$ for $i = 1, 2$. Let $Y = Y_1 \times_X Y_2$ and $\pi : Y \to X$ be the natural morphism. Since we work over a field of characteristic zero, Y is reduced. Hence $\pi : Y \to X$ is a flat morphism between varieties. Denote by $C_i = p_i^{-1}(A_i)$ with $p_i : Y \to Y_i$ the natural projection for $i = 1, 2$.

Yang: To be added. □

By Lemma 7, the general adelic subsets form a basis of topology on $X(\mathbb{k})$. Hence we have the following definition.

Definition 8. The *adelic topology* on $X(\mathbb{k})$ is defined to be the topology generated by all general adelic subsets of $X(\mathbb{k})$, i.e., an adelic open subset is an arbitrary union of general adelic subsets.

Proposition 9. We have the following properties of adelic topology:

(a) adelic topology is finer than Zariski topology;

(b) morphisms of varieties are continuous with respect to adelic topology;

(c) flat morphisms of varieties are open with respect to adelic topology.

Proof. Yang: To be added. □

Proposition 10. Let $\mathbf{k}$ be a defined field of X. The action of $\mathsf{Gal}(\mathbb{k}/\mathbf{k})$ on $X(\mathbb{k})$, namely $(\sigma, x) \mapsto \sigma(x)$, is continuous with respect to the adelic topology on $X(\mathbb{k})$ and the profinite topology on $\mathsf{Gal}(\mathbb{k}/\mathbf{k})$.

Proof. Yang: To be added. □

Recall that on $\mathbb{P}^1_{\mathbb{Q}}$, the Artin-Whaples Approximation Theorem says that for any finite collection of places $v_1, \dots, v_n$ of $\mathbb{Q}$ and any open subsets $U_i \subseteq \mathbb{P}^1(\mathbb{Q}_{v_i})$, the intersection $\bigcap_{i=1}^n (U_i \cap \mathbb{P}^1(\mathbb{Q}))$ is non-empty. The following lemma is a generalization and it is the motivation of the definition of adelic subsets.

Theorem 11. Adelic topology preserves the irreducibility of varieties. Explicitly, on a variety, the intersection of any finite collection of non-empty adelic open subsets is non-empty.

Proof. Yang: To be added. □

Lemma 12 (ref. [Xie25, Proposition 3.9], cf. [MZ14, Theorem 1.2])**.** For any finite collection of basic adelic open subsets $X_{\mathbf{k}}(\sigma_i, U_i)$, the intersection $X_{\mathbf{k}}(\{\sigma_i, U_i\}_{i=1}^n)$ is non-empty.

Slogan *On a variety, the small open ball in one place will be dense with respect to other places.*

Proof. Yang: To be added. □

2 Examples and relation to dynamics

Yang: Is $\mathbb{A}^1(\mathbb{Q})$ adelic closed in $\mathbb{A}^1(\overline{\mathbb{Q}})$ with adelic topology? If so, why?

Yang: Describe all adelic closed subset in $\mathbb{A}^1(\overline{\mathbb{Q}})$.

Appendix

References

[MZ14] Vincenzo Mantova and Umberto Zannier. "Artin-Whaples approximations of bounded degree in algebraic varieties". In: *Proceedings of the American Mathematical Society* 142.9 (2014), pp. 2953–2964 (cit. on p. 3).

[Xie25] Junyi Xie. "The existence of Zariski dense orbits for endomorphisms of projective surfaces". In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on pp. 1–3).