Vanishing in positive characteristic

1 Preliminaries

Let $\mathbf{A}$ be an abelian category. The category $\mathsf{C}(\mathbf{A})$ of complexes in $\mathbf{A}$ is defined as follows: the objects are complexes $X^\bullet$ in $\mathbf{A}$, and the morphisms are morphisms of complexes. For every $X^\bullet \in \mathrm{Obj}(\mathsf{C}(\mathbf{A}))$, the object X^n is the n-th component of the complex, and the morphism $\mathrm{d}^n : X^n \to X^{n+1}$ is the differential.

We denote $X[k]$ by the complex obtained by shifting $X^\bullet$ by k, that is,

$$X[k]^n = X^{n+k}, \quad \mathrm{d}^n_{X[k]} = (-1)^k \mathrm{d}^{n+k}_{X^\bullet}.$$

Given a morphism $f : X^\bullet \to Y^\bullet$ in $\mathsf{C}(\mathbf{A})$, we define the map cone $\mathrm{Cone}(f)^\bullet \in \mathsf{C}(\mathbf{A})$ by

$$\mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n, \quad \mathrm{d}^n_{\mathrm{Cone}(f)} = \begin{bmatrix} \mathrm{d}^{n+1}_X & \\ f^{n+1} & \mathrm{d}^n_Y \end{bmatrix},$$

Using the notation of shifting, we can also write

$$\mathrm{Cone}(f)^\bullet = \left(X[1]^\bullet \oplus Y^\bullet, \begin{bmatrix} \mathrm{d}_{X[1]} & \\ f[1] & \mathrm{d}_Y \end{bmatrix} \right).$$

Yang: Check that the cone is a complex.

The category $\mathsf{K}(\mathbf{A})$ is defined by

$$\mathrm{Obj}(\mathsf{K}(\mathbf{A})) = \mathrm{Obj}(\mathsf{C}(\mathbf{A})), \quad \mathrm{Hom}_{\mathsf{K}(\mathbf{A})}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{\mathsf{C}(\mathbf{A})}(X^\bullet, Y^\bullet)/\{\text{homotopy}\}.$$

A homomorphism $f^\bullet : X^\bullet \to Y^\bullet$ is said to be a *quasi-isomorphism* if the induced map $H^n(f^\bullet) : H^n(X^\bullet) \to H^n(Y^\bullet)$ is an isomorphism for all n.

Example 1. Let $\mathbf{A}$ be an abelian category and A an object in $\mathbf{A}$. Let $A \xrightarrow{i} I^\bullet$ be an injective resolution of A. Then the complex $I^\bullet$ is a complex in $\mathsf{C}(\mathbf{A})$, and the morphism

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] \arrow[d, "i"] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r, "d^0"] & I^1 \arrow[r, "d^1"] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

is a quasi-isomorphism in $\mathsf{K}(\mathbf{A})$.

Definition 2. A *triangle* in $\mathsf{K}(\mathbf{A})$ (or $\mathsf{C}(\mathbf{A})$) is a diagram of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X[1]^\bullet$$

such that f, g, and h are morphisms of complexes.

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For every $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathsf{C}(\mathbf{A})$, we can construct a triangle

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \to \mathrm{Cone}(f^\bullet)^\bullet \to X[1]^\bullet,$$

where the morphism $Y^\bullet \to \mathrm{Cone}(f^\bullet)^\bullet$ is the natural inclusion, and the morphism $\mathrm{Cone}(f^\bullet)^\bullet \to X[1]^\bullet$ is the natural projection. The triangle which is isomorphic to the above triangle in $\mathsf{K}(\mathbf{A})$ is called *distinguished triangle*.

Definition 3 (Truncation functor)**.** The *truncated functor* $\tau^{>0} : \mathsf{K}(\mathbf{A}) \to \mathsf{K}(\mathbf{A})$ is defined by

$$\tau^{>0}(X^\bullet)^n = \left(\cdots \to 0 \to \operatorname{coker} d^0 \to X^1 \to X^2 \to \cdots\right).$$

Yang: On cohomological level, we have

$$H^n(\tau^{>0}(X^\bullet)) = \begin{cases} 0, & n \leq 0, \\ H^n(X^\bullet), & n > 0. \end{cases}$$

Definition 4 (Derived category)**.** Let $\mathbf{A}$ be an abelian category. The *derived category* $\mathsf{D}(\mathbf{A})$ is defined by the following universal property: for any

Proposition 5. Let $\mathbf{A}$ be an abelian category with enough injectives. Then for every object $X \in \mathsf{D}^+(\mathbf{A})$, there exists an isomorphism $X \to I$ in $\mathsf{D}^+(\mathbf{A})$ such I^n is an injective object in $\mathbf{A}$ for all n.

Definition 6. Such an isomorphism $X \to I$ is called an *injective resolution* of X.

Definition 7 (Right Derived functor)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories and $F : \mathbf{A} \to \mathbf{B}$ a left exact functor. The *right derived functor* of F is a datum (T, α) fitting into the following diagram

$$\begin{tikzcd} \mathsf{K}^+(\mathbf{A}) \arrow[r, "K^+(F)"] \arrow[d] & \mathsf{K}^+(\mathbf{B}) \arrow[d] \\ \mathsf{D}^+(\mathbf{A}) \arrow[r, "T"] \arrow[ur, Rightarrow, "\alpha"] & \mathsf{D}^+(\mathbf{B}) \end{tikzcd}$$

that satisfies for every additive functor $G : \mathsf{D}^+(\mathbf{A}) \to \mathsf{D}^+(\mathbf{B})$ preserving distinguish triangles and the shifting $X \mapsto X[1]$, the map

$$\begin{array}{c} \mathsf{D}^+(\mathbf{A}) \xrightarrow{T} \mathsf{D}^+(\mathbf{B}) \\ \Downarrow \beta \\ \mathsf{D}^+(\mathbf{A}) \xrightarrow{G} \mathsf{D}^+(\mathbf{B}) \end{array} \quad \mapsto \quad \begin{array}{c} \mathsf{K}^+(\mathbf{A}) \xrightarrow{K^+(F)} \mathsf{K}^+(\mathbf{B}) \to \mathsf{D}^+(\mathbf{B}) \\ \Downarrow \alpha \\ \mathsf{K}^+(\mathbf{A}) \to \mathsf{D}^+(\mathbf{A}) \xrightarrow{T} \mathsf{D}^+(\mathbf{B}) \\ \Downarrow \beta \circ_h \mathrm{id} \\ \mathsf{K}^+(\mathbf{A}) \to \mathsf{D}^+(\mathbf{A}) \xrightarrow{G} \mathsf{D}^+(\mathbf{B}) \end{array}$$

is bijective.

Such functor is unique up to isomorphism, and denoted by $\mathsf{R}F$.

Proposition 8. Let $\mathbf{A}$ be an abelian category with enough injectives, and $F : \mathbf{A} \to \mathbf{B}$ a left exact functor. Then the right derived functor $\mathsf{R}F$ is given by

$$\mathsf{R}F(X^\bullet) = F(I^\bullet),$$

where $I^\bullet$ is an injective resolution of $X^\bullet$.

2 An example

Fix a base ring $T = \mathbb{Z}_p[[u]]$ for some prime $p > 0$ and let $x = (p, T)$ be the maximal ideal of T. Let $Z = \mathbb{P}^1_T$ be the projective line over T. Choose a covering of Z by two affine open subschemes $U_0 = \mathrm{Spec}(T[v])$ and $U_1 = \mathrm{Spec}(T[1/v])$. Let $I = (p, T, v) \subset T[v]$ be the ideal of the closed point $z \in U_0 \subset Z$.

Let $\pi : X = \mathrm{Bl}_p Z \to Z$ be the blow-up of Z at the point z. We try to describe it explicitly. Consider the blow-up $\mathrm{Proj}\, T[v][pW, uW, vW]$ of U_0 at the point z, where W is a formal variable to denote grading. It is covered by

$$\begin{aligned} U_{01} &= \mathrm{Spec}\left(T[v]\left[\frac{uW}{pW}, \frac{vW}{pW}\right]\right) \cong, \\ U_{02} &= \mathrm{Spec}\left(T[v]\left[\frac{pW}{uW}, \frac{vW}{uW}\right]\right) \cong, \\ U_{03} &= \mathrm{Spec}\left(T[v]\left[\frac{pW}{vW}, \frac{uW}{vW}\right]\right) \cong . \end{aligned}$$

Reduce to the special fiber, they become

$$\begin{aligned} U_{01,x} &= \mathrm{Spec}\left(\mathbb{F}_p\left[\frac{uW}{pW}, \frac{vW}{pW}\right]\right), \\ U_{02,x} &= \mathrm{Spec}\left(\mathbb{F}_p\left[\frac{pW}{uW}, \frac{vW}{uW}\right]\right), \\ U_{03,x} &= \mathrm{Spec}\left(\mathbb{F}_p[v]\left[\frac{pW}{vW}, \frac{uW}{vW}\right]/(v\frac{pW}{vW}, v\frac{uW}{vW})\right) \cong \mathrm{Spec}\left(\mathbb{F}_p[v, \alpha, \beta]/(v\alpha, v\beta)\right). \end{aligned}$$

Glue these three affine schemes and $U_{1,x}$ together, we obtain the special fiber X_x, which consists of two components $\mathbb{P}^1_{\mathbb{F}_p}$ and $\mathbb{P}^2_{\mathbb{F}_p}$ meeting at one point. It follows that the exceptional divisor E of the blow-up $\pi : X \to Z$ is isomorphic to $\mathbb{P}^2_{\mathbb{F}_p}$.

Reduce to the fiber $p = 0$, we have

$$\begin{aligned} U_{01,p} &= \mathrm{Spec}\left(\mathbb{F}_p\left[\frac{uW}{pW}, \frac{vW}{pW}\right]\right), \\ U_{02,p} &= \mathrm{Spec}\left(\mathbb{F}_p[[u]]\left[\frac{pW}{uW}, \frac{vW}{uW}\right]/\left(u\frac{pW}{uW}\right)\right), \\ U_{03,p} &= \mathrm{Spec}\left(\mathbb{F}_p[[u]]\left[v, \frac{pW}{vW}, \frac{uW}{vW}\right]/\left(v\frac{pW}{vW}, v\frac{uW}{vW} - u\right)\right). \end{aligned}$$

Let $\mathcal{L} := \pi^*\mathcal{O}_Z(1)$ be the pullback of the line bundle $\mathcal{O}_Z(1)$ on Z. Yang: Then $\mathcal{L}$ is nef and big. We use this example to compute

$$\mathsf{R}\Gamma_x(\mathsf{R}\Gamma(X_{p=0}, \mathcal{L}))$$

Definition 9. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. For $s \in \Gamma(X, \mathcal{F})$, we define the *support* of s to be the closed subset $\{x \in X \mid s_x \neq 0\}$. Let $Y \subset X$ be a closed subset. The section with support in Y is defined to be the set

$$\Gamma_Y(X, \mathcal{F}) = \{s \in \Gamma(X, \mathcal{F}) \mid \operatorname{Supp} s \subset Y\}.$$

Compute $\mathsf{R}\Gamma_x(\mathsf{R}\Gamma(Z_{p=0}, \mathcal{O}_Z(1)))$