Rings of Lower Dimension

1 Artinian Rings and Length of Modules

Definition 1. Let A be a ring and M an A module. *A simple module filtration of* M is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i-1} is a simple module, i.e. it has no submodule except 0 and itself. If M has a simple module filtration as above, we define the *length of* M as n and say that M has *finite length*.

The following proposition guarantees the length is well-defined.

Proposition 2. Suppose M has a simple module filtration $M = M_{0,0} \supsetneq M_{1,0} \supsetneq \cdots \supsetneq M_{n,0} = 0$. Then for any other filtration $M = M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,m} = 0$ with $m > n$, there exist $k < m$ such that $M_{0,k} = M_{0,k+1}$.

Proof. We claim that there are at least $0 \leq k_1 < \cdots < k_{m-n} < m$ satisfies that $M_{0,k_i} = M_{0,k_i+1}$. Let $M_{i,j} := M_{i,0} \cap M_{0,j}$. Inductively on n, we can assume that there exist $k_1, \cdots, k_{n-m+1}$ such that $M_{1,k} = M_{1,k+1}$. Consider the sequence

$$M_{0,0}/M_{1,0} \supset (M_{0,1} + M_{1,0})/M_{1,0} \supset \cdots \supset (M_{0,m} + M_{1,0})/M_{1,0} = 0$$

in $M_{0,0}/M_{1,0}$. Since $M_{0,0}/M_{1,0}$ is simple, there is at most one k_i with $M_{0,k_i} + M_{1,0} \neq M_{0,k_i+1} + M_{1,0}$. And note that if $M_{0,k_i} + M_{1,0} = M_{0,k_i+1} + M_{1,0}$ and $M_{0,k_i} \cap M_{1,0} = M_{0,k_i} \cap M_{1,0}$, then $M_{0,k_i} = M_{0,k_i+1}$ by the Five Lemma. □

Example 3. Let A be a ring and $\mathfrak{m} \in \operatorname{mSpec} A$. Then $A/\mathfrak{m}$ is a simple module. Yang: To be completed.

Proposition 4. Let A be a ring and M an A-module. Then M is of finite length iff it satisfies both a.c.c and d.c.c.

Proof. Note that if M has either a strictly ascending chain or a strictly descending chain, M is of infinite length. Conversely, d.c.c guarantee M has a simple submodule and a.c.c guarantee the sequence terminates. □

Proposition 5. The length $l(-)$ is an additive function for modules of finite length. That is, if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ with M_i of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

Proof. The simple module filtrations of M_1 and M_3 will give a simple module filtration of M_2. □

Proposition 6. Let $(A, \mathfrak{m})$ be a local ring. Then A is artinian iff $\mathfrak{m}^n = 0$ for some $n \geq 0$.

Proof. Suppose A is artinian. Then the sequence $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ is stable. It follows that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n. By the Nakayama's Lemma ??, $\mathfrak{m}^n = 0$.

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Conversely, we have

$$\mathfrak{m} \subset \mathfrak{N} \subset \bigcap_{\text{minimal prime ideal}} \mathfrak{p},$$

whence $\mathfrak{m}$ is minimal. □

Proposition 7. Let A be a ring. Then A is artinian iff A is of finite length.

Proof. First we show that A has only finite maximal ideal. Otherwise, consider the set $\{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k\}$. It has a minimal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ and for any maximal ideal $\mathfrak{m}$, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}$. It follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i. Let $\mathfrak{M} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ be the Jacobi radical of A. Consider the sequence $\mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots$ and by Nakayama's Lemma, we have $\mathfrak{M}^k = 0$ for some k. Consider the filtration

$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1^k \supset \mathfrak{m}_1^k \mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0).$$

We have $\mathfrak{m}_1^k \cdots \mathfrak{m}_i^j / \mathfrak{m}_1^k \cdots \mathfrak{m}_i^{j+1}$ is an $A/\mathfrak{m}_i$-vector space. It is artinian and then of finite length. Hence A is of finite length. □

Theorem 8. Let A be a ring. Then A is artinian iff A is noetherian and of dimension 0.

Proof. Suppose A is artinian. Then A is noetherian by Proposition 7. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A/\mathfrak{p}$ is an artinian integral domain. If there is $a \in A/\mathfrak{p}$ is not invertible, consider $(a) \supset (a^2) \supset \cdots$, we see $a = 0$. Hence $\mathfrak{p}$ is maximal and $\dim A = 0$.

Suppose that A is noetherian and of dimension 0. Then every maximal ideal is minimal. In particular, A has only finite maximal ideal $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$. Let $\mathfrak{q}_i$ be the $\mathfrak{p}_i$-component of (0). Then we have $A \hookrightarrow \bigoplus_i A/\mathfrak{q}_i$. We just need to show that $A/\mathfrak{q}_i$ is of finite length as A-module. If $\mathfrak{q}_i \subset \mathfrak{p}_j$, take radical we get $\mathfrak{p}_i \subset \mathfrak{q}_j$ and hence $i = j$. So $A/\mathfrak{q}_i$ is a local ring with maximal ideal $\mathfrak{p}_i A/\mathfrak{q}_i$. Then every element in $\mathfrak{p}_i A/\mathfrak{q}_i$ is nilpotent. Since $\mathfrak{p}_i$ is finitely generated, $(\mathfrak{p}_i A/\mathfrak{q}_i)^k = 0$ for some k. Then $A/\mathfrak{q}_i$ is artinian and then of finite length as $A/\mathfrak{q}_i$-module. Then the conclusion follows. □

2 DVR and Dedekind Domain

Definition 9. A *valuation* on a field K is a function $v : K^\times \to \Gamma$, where Γ is a totally ordered abelian group, such that for all $x, y \in K^\times$:

(i) $v(xy) = v(x) + v(y)$.

(ii) $v(x + y) \geq \min(v(x), v(y))$ if $x + y \neq 0$.

We extend v to K by setting $v(0) = +\infty$, where $+\infty$ is an element greater than all elements of Γ. If Γ is discrete with respect to the order topology, we say that v is a *discrete valuation*.

Example 10. (a) Let $\mathbf{K} = \mathbb{Q}$ and p be a prime number. Let $v : \mathbb{Q}^\times \to \mathbb{Z}$ be defined by $v(a/b) = n$ if $a/b = p^n(c/d)$ with c, d coprime to p. Then v is a discrete valuation on $\mathbb{Q}$.

(b) Let $\mathbf{K} = \mathbf{k}(T)$ be the field of rational functions over a field $\mathbf{k}$. For $f = x^n g \in \mathbf{k}(t)$ with $g(0) \in \mathbf{k}^\times$, let $v(f) = n$.

(c) Let $\mathbf{K} = \mathbb{C}_p$ the p-adic complex numbers. For $x \in \mathbb{C}_p^\times$, let $v(x) = -\log_p |x|_p$. Then v is a valuation on $\mathbb{C}_p$ which is not discrete.

Definition 11 (Discrete Valuation Ring)**.** Let V be a domain with field of fractions $\mathbf{K}$. We say that V is a *discrete valuation ring (DVR)* if there exists a discrete valuation v on $\mathbf{K}$ such that $V = \{x \in \mathbf{K} \mid v(x) \geq 0\}$.

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