Derived Functors

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Derived Functors

In this section, fix an abelian category $\mathcal{A}$.

1 Resolution

Definition 1 (Resolution)**.** Let $A \in \mathcal{A}$. A *projective resolution* (resp. *flat resolution, free resolution*) of A is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0,$$

where each P_i is a projective (resp. flat, free) object in $\mathcal{A}$.
An *injective resolution* of A is an exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots \to I^n \to \cdots,$$

where each I^i is an injective object in $\mathcal{A}$.

Proposition 2. Let $P_\bullet : \cdots \to P_1 \to P_0 \to A \to 0$ and $Q_\bullet : \cdots \to Q_1 \to Q_0 \to B \to 0$ be complexes in $\mathcal{A}$ such that P_i is projective and $Q_\bullet$ is exact. Given a morphism $f : A \to B$, there exists a morphism of complexes $f_\bullet : P_\bullet \to Q_\bullet$ such that $f_0 = f$. In particular, any two such morphism of complexes are homotopic.
Dually, let $I^\bullet : 0 \to A \to I^0 \to I^1 \to \cdots$ and $J^\bullet : 0 \to B \to J^0 \to J^1 \to \cdots$ be complexes in $\mathcal{A}$ such that J^i is injective and $I^\bullet$ is exact. Given a morphism $f : A \to B$, there exists a morphism of complexes $f^\bullet : I^\bullet \to J^\bullet$ such that $f^0 = f$. In particular, any two such morphism of complexes are homotopic.

Proof. Yang: To be completed. □

Definition 3. For an object $A \in \mathcal{A}$, the *projective dimension* of A, denoted $\operatorname{proj.dim} A$, is the smallest integer n such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

of A of length n. If no such n exists, we set $\operatorname{proj.dim} A = \infty$.
Dually, the *injective dimension* of A, denoted $\operatorname{inj.dim} A$, is the smallest integer n such that there exists an injective resolution

$$0 \to A \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to 0$$

of A of length n. If no such n exists, we set $\operatorname{inj.dim} A = \infty$.