

Smoothness

1 Modules of differentials and derivations

In this subsection, let R be a ring and A an R -algebra.

Definition 1 (Derivation). A *derivation* of A over R is an R -linear map $\partial : A \rightarrow M$ with an A -module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module M , the set of all derivations of A over R into M forms an A -module, denoted by $\text{Der}_R(A, M)$.

Given a module homomorphism $f : M \rightarrow N$ of A -modules and a derivation $\partial \in \text{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of A over R into N . This gives a functor $\text{Der}_R(A, -) : \mathbf{Mod}_A \rightarrow \mathbf{Mod}_A$.

Proposition 2. The functor $\text{Der}_R(A, -)$ is representable. The representing object is denoted by $(\Omega_{A/R}, \Psi)$, where $\Psi : \text{Hom}_A(\Omega_{A/R}, -) \rightarrow \text{Der}_R(A, -)$ is given by pullback along a universal R -derivation $d_A : A \rightarrow \Omega_{A/R}$. The module $\Omega_{A/R}$ is called the *module of differentials* of A over R .

Proof. First suppose A is a free R -algebra with a set of generators $a_\lambda, \lambda \in \Lambda$. Then an R -derivation $\partial \in \text{Der}_R(A, M)$ is uniquely determined by its values on the generators a_λ . Let

$$\Omega_{A/R} := \bigoplus_{\lambda \in \Lambda} A \cdot da_\lambda$$

and $d : A \rightarrow \Omega_{A/R}$ be the R -derivation defined by $a_\lambda \mapsto da_\lambda$. For any R -derivation $\partial \in \text{Der}_R(A, M)$, we can define a unique A -module homomorphism $\Phi_\partial : \Omega_{A/R} \rightarrow M$ by sending da_λ to $\partial(a_\lambda)$ such that $\partial = \Phi_\partial \circ d$. This gives a bijection

$$\text{Der}_R(A, M) \cong \text{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_\partial.$$

Now suppose $A = F/I$ is an arbitrary R -algebra, where F is a free R -algebra and I is an ideal of F . Then we can define the module of differentials

$$\Omega_{A/R} := (\Omega_{F/R} \otimes_F A) / \sum_{f \in I} A \cdot df.$$

The R -linear map $d_A : F \otimes_F A \xrightarrow{d_F} \Omega_{F/R} \otimes_F A \rightarrow \Omega_{A/R}$ is a derivation of A over R .

For any R -derivation $\partial \in \text{Der}_R(A, M)$, note that $F \rightarrow A \xrightarrow{\partial} M$ is an R -derivation of F over R into M . Then we get an F -module homomorphism $\Omega_F \rightarrow M$. It gives an A -module homomorphism $\Omega_F \otimes_F A \rightarrow M, df \otimes 1 \mapsto \partial f$. This map factors into $\Omega_F \otimes_F A \rightarrow \Omega_{A/R}$ and $\Phi_\partial : \Omega_{A/R} \rightarrow M$. Since Φ_∂ is A -linear and $\Omega_{A/R}$ is generated by da_λ as A -module, such Φ_∂ is unique. \square

Corollary 3. Suppose A is of finite type over R . Then the module of differentials $\Omega_{A/R}$ is a finitely generated A -module.

Remark 4. Let B be an A -algebra, M an A -module and N a B -module. If there is a homomorphism of A -modules $M \rightarrow N$, then we can extend it to a homomorphism of B -modules $M \otimes_A B \rightarrow N$ by sending $m \otimes b$ to $m \cdot b$. And such extension is unique in the sense of following commutative diagram:

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & \nearrow \exists! & \\ M \otimes_A B & & \end{array}$$

Hence we get a natural bijection

$$\text{Hom}_A(M, N) \cong \text{Hom}_B(M \otimes_A B, N).$$

Proposition 5. Let A, R' be R -algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

Proof. We check the universal property of $\Omega_{A/R} \otimes_A A'$. First, the map

$$d_{A'} : A \otimes_R R' \rightarrow \Omega_{A/R} \otimes_R R' \cong \Omega_{A/R} \otimes_A A', \quad a \otimes r \mapsto da \otimes r$$

is an R' -derivation of A' into $\Omega_{A/R} \otimes_A A'$. For any R' -derivation $\partial' : A' \rightarrow M$ into an A' -module M , we can compose it with the homomorphism $A' \rightarrow A$ and get an R -derivation $\partial : A \rightarrow M$. By the universal property of $\Omega_{A/R}$, there is a unique A -module homomorphism $\Phi : \Omega_{A/R} \rightarrow M$ such that $\partial = \Phi \circ d_A$. Then we can extend it to an A' -module homomorphism $\Phi' : \Omega_{A/R} \otimes_A A' \rightarrow M$ by Remark 4. By the construction, we have $\Phi' \circ d_{A'} = \partial'$. \square

Proposition 6. Let A be an R -algebra and S a multiplicative set of A . Then we have an isomorphism

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

Proof. Let

$$d_{S^{-1}A} : S^{-1}A \rightarrow S^{-1}\Omega_{A/R}, \quad \frac{a}{s} \mapsto \frac{sda - ads}{s^2}.$$

By direct computation, $d_{S^{-1}A}$ is an R -derivation of $S^{-1}A$ over R into $S^{-1}\Omega_{A/R}$. For any R -derivation $\partial : S^{-1}A \rightarrow M$ into an $S^{-1}A$ -module M , we can get an $S^{-1}A$ -module homomorphism $\Phi' : S^{-1}\Omega_{A/R} \rightarrow M$ as proof of Proposition 5. We have

$$\partial\left(s \cdot \frac{a}{s}\right) = s\partial\left(\frac{a}{s}\right) + \frac{a}{s}\partial s.$$

It follows that

$$\partial\left(\frac{a}{s}\right) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(da) - a\Phi'(ds)}{s^2} = \Phi'\left(\frac{sda - ads}{s^2}\right).$$

Thus, $\Phi' \circ d_{S^{-1}A} = \partial$. \square

Theorem 7. Let A be an R -algebra and B an A -algebra. Then there is a natural short exact sequence

$$\Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow \Omega_{B/A} \rightarrow 0$$

of B -modules.

Proof. Let $d_{A/R} : A \rightarrow \Omega_{A/R}$ be the R -derivation of A over R . The map $A \rightarrow B \xrightarrow{d_{B/R}} \Omega_{B/R}$ induces a B -linear map

$$u : \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R}, \quad d_{A/R}(a) \otimes b \mapsto bd_{B/R}(a).$$

The map $d_{B/A}$ is an A -derivation and hence R -derivation. Then it induces a B -linear map

$$v : \Omega_{B/R} \rightarrow \Omega_{B/A}, \quad d_{B/R}(b) \mapsto d_{B/A}(b).$$

Since $\Omega_{B/A}$ is generated by elements of the form $d_{B/A}(b)$ for $b \in B$, the map v is surjective. And clearly $d_{B/A}(a) = ad_{B/A}(1) = 0$ for $a \in A$.

Consider the composition $B \xrightarrow{d_{B/R}} \Omega_{B/R} \rightarrow \Omega_{B/R}/\mathfrak{J}u$. For every $a \in A, b \in B$, we have

$$[d_{B/R}(ab)] = [bd_{B/R}(a) + ad_{B/R}(b)] = [bd_{B/R}(a)] + [ad_{B/A}(b)] = [ad_{B/A}(b)].$$

Hence it is indeed an A -derivation of B . Then it induces a B -linear map

$$\varphi : \Omega_{B/A} \rightarrow \Omega_{B/R}/\mathfrak{J}u, \quad d_{B/A}(b) \mapsto [d_{B/R}(b)].$$

The map φ is surjective since $\Omega_{B/R}$ is generated by elements of the form $d_{B/R}(b)$ for $b \in B$. Note that the composition

$$\Omega_{B/A} \xrightarrow{\varphi} \Omega_{B/R}/\mathfrak{J}u \rightarrow \Omega_{B/A}/\text{Ker } v$$

is the identity map. Thus, φ is injective and hence an isomorphism. In particular, we have $\text{Ker } v = \mathfrak{J}u$. \square

Remark 8. The exact sequence in Theorem 7 is left exact if and only if every R -derivation of A into B -module extends to an R -derivation of B into B -module.

To be completed.

Theorem 9. Let A be an R -algebra and I an ideal of A . Set $B := A/I$. Then there is a natural short exact sequence

$$I/I^2 \rightarrow \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow 0$$

of B -modules.

Proof. Suppose $A = F/\mathfrak{b}$ for some free R -algebra F and an ideal \mathfrak{b} of F . Let \mathfrak{a} be the preimage of I in F . Let $d\mathfrak{b}$ (resp. $d\mathfrak{a}$) denote the image of \mathfrak{b} (resp. \mathfrak{a}) in $\Omega_{F/R}$. Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B / (d\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B / (d\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \rightarrow (d\mathfrak{a} \otimes_F B) / (d\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows. \square

Definition 10. Let \mathbf{k} be a field and A an integral \mathbf{k} -algebra of finite type of dimension n . We say A is *smooth at* $\mathfrak{p} \in \text{Spec } A$ if the module of differentials $\Omega_{A,\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module of rank n .

Example 11. Let \mathbf{K}/\mathbf{k} be a finite generated field extension and \mathbf{k}' be the algebraic closure of \mathbf{k} in \mathbf{K} . Then

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \text{trdeg}(\mathbf{K}/\mathbf{k}) + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}},$$

and $\dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}} = 0$ if and only if \mathbf{k}' is separable over \mathbf{k} .

First suppose $\mathbf{K} = \mathbf{k}'$ is algebraic over \mathbf{k} . Suppose \mathbf{k}'/\mathbf{k} is separable. For every $\alpha \in \mathbf{k}'$, suppose $f(\alpha) = 0$ for $f \in \mathbf{k}[T]$. Then $df(\alpha) = f'(\alpha)d\alpha = 0$. By the separability of \mathbf{k}'/\mathbf{k} , we have $f'(\alpha) \neq 0$. It follows that $d\alpha = 0$. Conversely, let $\alpha \in \mathbf{k}'$ be an inseparable element over \mathbf{k} . Since $\mathbf{k}[\alpha] \rightarrow \mathbf{k}[\alpha], \alpha^n \mapsto n\alpha^{n-1}$ is a non-zero R -derivation, we have $\Omega_{\mathbf{k}[\alpha]/\mathbf{k}} \neq 0$. By induction on number of generated elements, choosing a middle field $\mathbf{k} \subset \mathbf{k}'' \subset \mathbf{k}'$, at least one of $\Omega_{\mathbf{k}''/\mathbf{k}}$ and $\Omega_{\mathbf{k}'/\mathbf{k}''}$ is non-zero. Then $\Omega_{\mathbf{K}/\mathbf{k}} \neq 0$ by Theorem 7.

Then suppose $\mathbf{k}' = \mathbf{k}$. By the Noether's Normalization Lemma, we can find a finite set of elements $T_1, \dots, T_n \in \mathbf{K}$ such that \mathbf{K} is algebraic over $\mathbf{k}'(T_1, \dots, T_n)$. Note that we can choose T_i such that $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$ is separable. To see this, if $\alpha \in \mathbf{K}$ is an inseparable element over $\mathbf{k}'(T_1, \dots, T_n)$, then by replacing a suitable T_i with α , we reduce the inseparable degree of $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$.

Since $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$ is finite, every \mathbf{k} -derivation of $\mathbf{k}'(T_1, \dots, T_n)$ into \mathbf{K} -module extends to a \mathbf{k} -derivation of \mathbf{K} into \mathbf{K} -module. Then by Remark 8, we have

$$0 \rightarrow \Omega_{\mathbf{k}'(T_1, \dots, T_n)/\mathbf{k}} \otimes_{\mathbf{k}'(T_1, \dots, T_n)} \mathbf{K} \rightarrow \Omega_{\mathbf{K}/\mathbf{k}} \rightarrow \Omega_{\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)} \rightarrow 0.$$

Finally, note that every \mathbf{k} -derivation ∂ of \mathbf{k}' into \mathbf{K} -module can be extended to $\mathbf{k}'[T_1, \dots, T_n]$ by setting $\partial T_i = 0$. Thus, we have

$$0 \rightarrow \Omega_{\mathbf{k}'/\mathbf{k}} \otimes_{\mathbf{k}'} \mathbf{k}'[T_1, \dots, T_n] \rightarrow \Omega_{\mathbf{k}'[T_1, \dots, T_n]/\mathbf{k}} \rightarrow \Omega_{\mathbf{k}'[T_1, \dots, T_n]/\mathbf{k}'} \rightarrow 0.$$

This follows that

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}'} + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}}.$$

2 Applications to affine varieties

Let \mathbf{k} be arbitrary field, $A = \mathbf{k}[T_1, \dots, T_n]$ and \mathfrak{m} a maximal ideal of A such that $\kappa(\mathfrak{m})$ is separable over \mathbf{k} . We try to give an explanation of Zariski's tangent space at \mathfrak{m} using the language of derivation. We know that $\Omega_{A/\mathbf{k}} = \bigoplus_{i=1}^n A dT_i$, thus $\Omega_{A_{\mathfrak{m}}/\mathbf{k}} \cong \bigoplus_{i=1}^n A_{\mathfrak{m}} dT_i$. Then

$$\text{Der}_{\mathbf{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}}) \cong \text{Hom}_{\mathbf{k}}(\Omega_{A_{\mathfrak{m}}/\mathbf{k}}, A_{\mathfrak{m}}) \cong \bigoplus_{i=1}^n A_{\mathfrak{m}} \partial_i,$$

where $\partial_i \in \text{Der}_{\mathbf{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ is the derivation defined by $dT_i \mapsto 1$ and $dT_j \mapsto 0$ for $j \neq i$. It coincides with the usual derivation $f \mapsto \partial f / \partial T_i$. Consider the restriction of ∂_i to \mathfrak{m} and take values in the residue field $\kappa(\mathfrak{m})$, we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1, \dots, \partial_n)^T} A_{\mathfrak{m}}^n \rightarrow \kappa(\mathfrak{m})^n.$$

Since $\kappa(\mathfrak{m})$ is separable over \mathbf{k} , we claim that $\text{Ker } \Phi = \mathfrak{m}^2$. Indeed, by Remark 12, we can write every $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ as $\sum_i a_i g_i$. Then

$$\frac{\partial f}{\partial T_i} = a_i \frac{\partial g_i}{\partial T_i} + g_i \frac{\partial a_i}{\partial T_i}.$$

Since g_i is separable, the image of $\partial g_i / \partial T_i$ in $\kappa(\mathfrak{m})$ is not zero. Hence $\Phi(f) \neq 0$. By the claim, Φ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$ of $\kappa(\mathfrak{m})$ -vector spaces. Then we get

$$T_{A,\mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where $x \in \mathbf{a}_{\mathbf{k}}^n$ is the point corresponding to \mathfrak{m} . This coincides with the usual tangent space at x in language of differential geometry.

Remark 12. Let \mathbf{k} be arbitrary field, $A = \mathbf{k}[T_1, \dots, T_n]$ and g_i irreducible polynomials in one variable T_i over \mathbf{k} . Then for every $f \in A$, we can write

$$f = \sum_{I=(i_1, \dots, i_n) \in \mathbf{z}_{\geq 0}^n} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \leq \deg g_i.$$

This is called the *Taylor expansion of f with respect to g_1, \dots, g_n* .

When $n = 1$, it follows from division algorithm. For $n > 1$, we can use induction on n . Let $\mathbf{K} = \mathbf{k}(T_1, \dots, T_{n-1})$. Then we can write f as

$$f = \sum_{i=0}^r a_i g_n^i, \quad a_i \in \mathbf{K}[T_n], \quad \deg a_i < \deg g_n.$$

Comparing the coefficients of two sides from the highest degree of T_n to the lowest degree, we see that

$$a_i \in \mathbf{k}[T_1, \dots, T_{n-1}].$$

By induction hypothesis, the conclusion follows.

Let $B = A/I$ be a \mathbf{k} of finite type, $I = (F_1, \dots, F_m) \subset \mathfrak{m}$ and \mathfrak{n} the image of \mathfrak{m} in B . We have an exact sequence of $\kappa(\mathfrak{m})$ -vector spaces

$$0 \rightarrow I/(I \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} : \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of F_1, \dots, F_m is the $m \times n$ matrix

$$J(F_1, \dots, F_m) := \left(\frac{\partial F_i}{\partial T_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

with entries in B .

Theorem 13. Setting as above. Then B is regular at \mathfrak{n} if and only if the Jacobian matrix J has maximal rank $n - \dim B_{\mathfrak{n}}$ after taking values in the residue field $\kappa(\mathfrak{m})$.

Proof. We have an exact sequence

$$0 \rightarrow T_{B,\mathfrak{n}} \rightarrow T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa^m \rightarrow 0,$$

where Ψ sends $\partial \in T_{A,\mathfrak{m}}$ to $(\partial(F_1), \dots, \partial(F_m))^T$. Note that the matrix of Ψ is just J^T , the transpose of the Jacobian matrix. Hence

$$\text{rank } J = n - \dim_{\kappa} T_{B,\mathfrak{n}} \leq n - \dim B_{\mathfrak{n}}$$

| and the equality holds if and only if B is regular at \mathfrak{n} . □

Remark 14. If $\kappa(\mathfrak{m})$ is not separable over \mathbf{k} , then we still have the inequality

$$\text{rank } J \leq n - \dim B_{\mathfrak{n}}.$$

Indeed, in any case, we have an exact sequence

$$0 \rightarrow I/(I \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0.$$

Hence $\dim_{\kappa} I/(I \cap \mathfrak{m}^2) = n - \dim B_{\mathfrak{n}}$. There is a $\kappa(\mathfrak{m})$ -linear map

$$I/(I \cap \mathfrak{m}^2) \rightarrow \kappa(\mathfrak{m})^n, \quad [f] \mapsto (\partial_1(f), \dots, \partial_n(f))^T,$$

and every row of the Jacobian matrix J is in the image of this map. Thus, the rank of J is at most $n - \dim B_{\mathfrak{n}}$.

Hence if $\text{rank } J = n - \dim B_{\mathfrak{n}}$, we can still see that B is regular at \mathfrak{n} . However, the converse does not hold in general.

Proposition 15. Let \mathbf{k} be a field, \mathbb{k} the algebraic closure of \mathbf{k} , A a \mathbf{k} -algebra of finite type and $A_{\mathbb{k}} := A \otimes_{\mathbf{k}} \mathbb{k}$. Suppose $A_{\mathbb{k}}$ is integral. Let $\mathfrak{m} \in \text{mSpec } A$ and \mathfrak{m}' be a maximal ideal of $A_{\mathbb{k}}$ lying over \mathfrak{m} . Then

- (a) If $A_{\mathbb{k}}$ is regular at \mathfrak{m}' , then A is regular at \mathfrak{m} ;
- (b) suppose $\kappa(\mathfrak{m})$ is separable over \mathbf{k} , the converse holds.

Proof. Regarding $J_{\mathfrak{m}}$ and $J_{\mathfrak{m}'}$ as matrices with entries in \mathbb{k} , they are the same and hence have the same rank. If $A_{\mathbb{k}}$ is regular at \mathfrak{m}' , since $\kappa(\mathfrak{m}) = \mathbb{k}$, then $\text{rank } J_{\mathfrak{m}'} = n - \dim A_{\mathbb{k}, \mathfrak{m}'}$. Note that $\dim A_{\mathbb{k}, \mathfrak{m}'} = \text{trdeg}(\mathcal{K}(A_{\mathbb{k}})/\mathbb{k}) = \text{trdeg}(\mathcal{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}}$, we have $\text{rank } J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence A is regular at \mathfrak{m} .

Conversely, suppose A is regular at \mathfrak{m} and $\kappa(\mathfrak{m})$ is separable over \mathbf{k} . Then $\text{rank } J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence $A_{\mathbb{k}}$ is regular at \mathfrak{m}' . To be modified. □

Proposition 16. Let \mathbf{k} be a field and A an integral \mathbf{k} -algebra of finite type and of dimension n . Let \mathbb{k} be the algebraic closure of \mathbf{k} and $A_{\mathbb{k}} := A \otimes_{\mathbf{k}} \mathbb{k}$. Then A is smooth at $\mathfrak{p} \in \text{Spec } A$ if and only if $A_{\mathbb{k}}$ is regular at every \mathfrak{m}' over \mathfrak{p} .

Proof. Since $\Omega_{A_{\mathbb{k}}/\mathbb{k}} \cong \Omega_{A/\mathbf{k}} \otimes_A A_{\mathbb{k}}$ is free of rank n if and only if $\Omega_{A/\mathbf{k}}$ is free of rank n , we can assume that $\mathbf{k} = \mathbb{k}$. If A is smooth at \mathfrak{p} , then $\Omega_{A_{\mathfrak{p}}/\mathbb{k}} \cong \bigoplus A_{\mathfrak{p}} df_i$ is free of rank n . Let $\mathfrak{P}_i \in \text{Der}_{\mathbb{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ be the derivation defined by $df_i \mapsto 1$ and $dT_j \mapsto 0$ for $j \neq i$. Then we have $\partial_i f_j = \delta_{ij}$ for $1 \leq i, j \leq n$. Then similar to above argument, we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{(\partial_1, \dots, \partial_n)^T} \mathbb{k}^n.$$

This shows that $A_{\mathbb{k}}$ is regular at \mathfrak{m} .

Conversely, suppose $A_{\mathbb{k}}$ is regular at \mathfrak{m} . Note that $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \Omega_{A_{\mathbb{k}}/\mathbb{k}} \otimes_{\mathbb{k}} \mathbb{k}$ is surjective since $\Omega_{A_{\mathbb{k}}/\mathbb{k}} = 0$. Then by Nakayama's lemma, $\Omega_{A_{\mathfrak{m}}/\mathbb{k}}$ is generated by n elements as an $A_{\mathfrak{m}}$ -module.

Note that $\dim_{\mathcal{K}(A)} \Omega_{\mathcal{K}(A)/\mathbb{k}} = \text{trdeg}(\mathcal{K}(A)/\mathbb{k}) = \dim A_{\mathfrak{m}} = n$. By induction on transcendental

degree.

By Nakayama's Lemma, $\Omega_{A_m/\mathbf{k}}$ is free of rank n as an A_m -module.

To be completed. □

Example 17. Let \mathbf{k} be an imperfect field of characteristic $p > 2$. Suppose $\alpha = \beta^p \in \mathbf{k}$ and β is not in \mathbf{k} . Let $A = \mathbf{k}[x, y]/(x^2 - y^p - \alpha)$ and $\mathfrak{m} = (x, y^p - \alpha) = (x)$. Note that \mathfrak{m} is principal, so A is regular at \mathfrak{m} . However,

$$J_{\mathfrak{m}} = \left(\frac{\partial}{\partial x}(x^2 - y^p - \alpha), \frac{\partial}{\partial y}(y^p - \alpha) \right) = (2x, 0) = (0, 0) \in M_{1 \times 2}(\kappa(\mathfrak{m})).$$

Thus, A is not smooth at \mathfrak{m} . From the view of differentials, we have

$$\Omega_{A_m/\mathbf{k}} = A_m dx \oplus A_m dy / A_m \cdot x dx = \kappa(\mathfrak{m}) dx \oplus A_m dy,$$

which is not free as an A_m -module.