Some fundamental Results

Yang: To be completed

1 Rings and modules

In the appendix and all the note, the "ring" is always commutative and with identity. We denote by $\operatorname{Spec} A$ the set of prime ideals of a ring A. We denote by $\operatorname{mSpec} A$ the set of maximal ideals of A. Let $I \subset A$ be an ideal of A. We define

$$V(I) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} A : I \subset \mathfrak{p}\}.$$

Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A. We define

$$(\mathfrak{a} : \mathfrak{b}) \coloneqq \{a \in A : a\mathfrak{b} \subset \mathfrak{a}\}.$$

This is an ideal of A.

Let $\operatorname{rad}(A)$ be the Jacobian radical of A, i.e., the intersection of all maximal ideals of A. Let $\operatorname{nil}(A)$ be the nilradical of A, i.e., the ideal of A consisting of all nilpotent elements.

Proposition 1. Let A be a ring. Then we have

$$\operatorname{nil}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}.$$

Proof. Yang: To be completed. □

Proposition 2. Let A be a ring, $\mathfrak{p}, \mathfrak{p}_i$ prime ideals of A and $\mathfrak{a}, \mathfrak{a}_i$ ideals of A.

(a) Suppose $\mathfrak{a} \subset \bigcup_{i=1}^n \mathfrak{p}_i$. Then there exists i such that $\mathfrak{a} \subset \mathfrak{p}_i$.

(b) Suppose $\bigcap_{i=1}^n \mathfrak{a}_i \subset \mathfrak{p}$. Then there exists i such that $\mathfrak{a}_i \subset \mathfrak{p}$.

Proof. Yang: To be completed. □

Let M be an A-module. We say that M is *finite* if there exists an exact sequence

$$A^n \to M \to 0.$$

We say that M is *finite presented* if there exists an exact sequence

$$A^m \to A^n \to M \to 0.$$

If A is a noetherian ring, then every finite A-module is finite presented.

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Definition 3. Let A be a ring and M an A-module. The *support* of M is defined as

$$\operatorname{Supp} M \coloneqq \{\mathfrak{p} \in \operatorname{Spec} A \colon M_{\mathfrak{p}} \neq 0\}.$$

The *annihilator* of M is defined as

$$\operatorname{Ann} M \coloneqq \{a \in A \colon aM = 0\}.$$

This is an ideal of A.

Proposition 4. Let A be a ring and M a finite A-module. Then $\operatorname{Supp} M = V(\operatorname{Ann} M)$. In particular, $\operatorname{Supp} M$ is a closed subset of $\operatorname{Spec} A$.

Proof. Yang: To be completed. □

2 Localization

Definition 5. Let A be a ring and $S \subset A$ a multiplicative subset, i.e., $1 \in S$ and $s_1, s_2 \in S$ implies $s_1 s_2 \in S$. Let M be an A-module. The *localization* of M at S is defined as

$$S^{-1}M \coloneqq M \times S/\sim,$$

where $(m, s) \sim (n, t)$ if there exists $u \in S$ such that $u(tm - sn) = 0$. We denote the equivalence class of (m, s) by $\frac{m}{s}$ or m/s.
The localization $S^{-1}A$ is still a ring and hence an A-algebra. The localization $S^{-1}M$ is an $S^{-1}A$-module. If $M = B$ is an A-algebra, then $S^{-1}B$ is an $S^{-1}A$-algebra.

Example 6. Let A be a ring, $\mathfrak{p}$ a prime ideal of A and M an A-module. Then $S = A \setminus \mathfrak{p}$ is a multiplicative subset. The localization $S^{-1}M$ is denoted by $M_{\mathfrak{p}}$ and called the localization of M at $\mathfrak{p}$.

Let $f \in A$ be an element. Then $S = \{f^n \colon n \geq 0\}$ is a multiplicative subset. The localization $S^{-1}M$ is denoted by $M[1/f]$.

Proposition 7. Let A be a ring, $S \subset A$ a multiplicative subset and M an A-module. Then the localization $S^{-1}M$ satisfies the following universal property: for any A-module N such that every element of S acts on N as an automorphism, there is a unique morphism of A-modules $S^{-1}M \to N$ making the following diagram commute:

$$\begin{tikzcd}
M \arrow[r] \arrow[dr] & S^{-1}M \arrow[d, dashed] \\
& N
\end{tikzcd}$$

Yang: To be completed.

Proposition 8. The natural map $A \to S^{-1}A$ is injective if and only if S contains no zero divisors.

Proposition 9. Let A be a ring, $S \subset A$ a multiplicative subset and M an A-module. Then we have a natural isomorphism of $S^{-1}A$-modules

$$S^{-1}M \cong M \otimes_A S^{-1}A.$$

Proposition 10. The localization $S^{-1}A$ is a flat A-algebra.

3 Chain conditions

Definition 11. Let A be a ring. We say that A is *noetherian* (resp. *artinian*) if every ascending (resp. descending) chain of ideals of A stabilizes.

Proposition 12. Let A be a ring. The following are equivalent:

(a) A is noetherian.

(b) Every ideal of A is finitely generated.

(c) Every non-empty set of ideals of A has a maximal element (with respect to inclusion).

Proof. Yang: To be completed. □

Theorem 13 (Hilbert's Basis Theorem)**.** If A is a noetherian ring, then $A[x]$ is noetherian.

Proof. Yang: To be completed. □

Remark 14. By a similar argument replacing $\deg f$ by $\operatorname{ord} f$, we can show that if A is noetherian, then the formal power series ring $A[[x]]$ is also noetherian.

4 Nakayama's Lemma

Theorem 15 (Nakayama's Lemma)**.** Let A be a ring and $\mathfrak{M}$ be its Jacobi radical. Suppose M is a finitely generated A-module. If $\mathfrak{a}M = M$ for $\mathfrak{a} \subset \mathfrak{M}$, then $M = 0$.

Proof. Suppose M is generated by $x_1, \cdots, x_n$. Since $M = \mathfrak{a}M$, formally we have $(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(\mathfrak{a})$. Then $(\Phi - \mathrm{id})(x_1, \cdots, x_n)^T = 0$. Note that $\det(\Phi - \mathrm{id}) = 1 + a$ for $a \in \mathfrak{a} \subset \mathfrak{M}$. Then $\Phi - \mathrm{id}$ is invertible and then $M = 0$. □

Remark 16. The finiteness of M is crucial in Nakayama's Lemma. For example, let $\overline{\mathbb{Z}}$ be the ring of algebraic integers in $\overline{\mathbb{Q}}$. Choose a non-zero prime ideal $\mathfrak{p}$ of $\overline{\mathbb{Z}}$. Then we have that $\mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}} = \mathfrak{p}^2\overline{\mathbb{Z}}_{\mathfrak{p}}$. Indeed, if $a \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$, let $b = \sqrt{a} \in \overline{\mathbb{Z}}_{\mathfrak{p}}$. Then $b^2 = a \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$ and whence $b \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$ since $\mathfrak{p}$ is prime. It follows that $a = b^2 \in \mathfrak{p}^2\overline{\mathbb{Z}}_{\mathfrak{p}}$.

Proposition 17 (Geometric form of Nakayama's Lemma)**.** Let $X = \operatorname{Spec} A$ be an affine scheme, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proof. Yang: To be completed. □

Corollary 18. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. Then the function $x \mapsto \dim_{\kappa(x)} \mathcal{F}|_x$ is upper semicontinuous.

Proof. Yang: To be completed. □

5 Nullstellensatz

Let $\mathbf{k}$ be a field and $\Bbbk$ be its algebraic closure.

Theorem 19 (Noether's Normalization Lemma)**.** Let A be a $\mathbf{k}$-algebra of finite type. Then there is an injection $\mathbf{k}[T_1, \cdots, T_d] \hookrightarrow A$ such that A is finite over $\mathbf{k}[T_1, \cdots, T_d]$.

Remark 20. Here A does not need to be integral. For example,

Theorem 21 (Hilbert's Nullstellensatz)**.** Let A be a $\mathbf{k}$-algebra of finite type.

(a) If $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a finite extension of $\mathbf{k}$.

(b) Suppose that $\mathbf{k}$ is algebraically closed and $A = \mathbf{k}[x_1, \cdots, x_n]/\mathfrak{a}$. Then there is a bijection between the set of maximal ideals of A and the set $\{(a_1, \cdots, a_n) \in \mathbf{k}^n : f(a_1, \cdots, a_n) = 0, \forall f \in \mathfrak{a}\}$.