Finite Algebra and Normality

Let R be a ring and A be an R-algebra. We say that A is *of finite type* over R if there exists a surjective R-algebra homomorphism $R[T_1, \cdots, T_n] \to A$ for some $n \geq 0$. We say that A is finite over R if it is finite as an R-module.

1　Finite algebra

Let A be a ring and B a finite A-algebra.

Example 1. Let K be a number field. Then O_K is a finite $\mathbb{z}$-algebra. To be completed.

Lemma 2. Let $A \subset B$ be noetherian rings such that B is finite over A. Then the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Proof. For $\mathfrak{p} \in \operatorname{Spec} A$, let $S := A - \mathfrak{p}$ and denote $S^{-1}B$ by $B_\mathfrak{p}$. Then we have $A_\mathfrak{p} \hookrightarrow B_\mathfrak{p}$ and $B_\mathfrak{p}$ is finite over $A_\mathfrak{p}$. Let $\mathfrak{P}B_\mathfrak{p}$ be a maximal ideal of $B_\mathfrak{p}$. We claim that $\mathfrak{P}B_\mathfrak{p} \cap A_\mathfrak{p}$ is maximal. Indeed, consider $A_\mathfrak{p}/(\mathfrak{P} \cap A_\mathfrak{p}) \hookrightarrow B_\mathfrak{p}/\mathfrak{P}B_\mathfrak{p}$, the latter is finite over the former. This enforces $A_\mathfrak{p}/(\mathfrak{P}B_\mathfrak{p} \cap A_\mathfrak{p})$ be a field. Hence $\mathfrak{P}B_\mathfrak{p} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, and then $\mathfrak{P} \cap A = \mathfrak{p}$. $\qquad\square$

Proposition 3. Let $A \subset B$ be noetherian rings such that B is finite over A. Then $\dim A = \dim B$.

Proof. If we have a sequence $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ of prime ideals in B, then there exists $f \in \mathfrak{P}_2 \setminus \mathfrak{P}_1$. Since B is finite over A, there exist $a_1, \cdots, a_n \in A$ such that

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0.$$

Then $a_n \in \mathfrak{P}_2 \cap A$. If $a_n \in \mathfrak{P}_1$, $f^{n-1} + \cdots + a_{n_1} \in \mathfrak{P}_1$ since $f \notin \mathfrak{P}_1$. Then $a_{n-1} \in \mathfrak{P}_2$. Repeat the process, it will terminate, whence $\mathfrak{P}_1 \cap A \subsetneq \mathfrak{P}_2 \cap A$. Otherwise, we have $f^n \in a_1 B + \cdots + a_n B \subset \mathfrak{P}_1$.

Conversely, suppose we have $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} A$ with $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Choose $\mathfrak{P}_1 \in \operatorname{Spec} B$ such that $\mathfrak{P}_1 \cap A = \mathfrak{p}_1$, then we have $A/\mathfrak{p}_1 \subset B/\mathfrak{P}_1$. Let $\mathfrak{P}_2$ be the preimage of the prime ideal in $B/\mathfrak{P}_1$ which is over image of $\mathfrak{p}_2$ in $A/\mathfrak{p}_1$. Proposition 2 guarantees that such $\mathfrak{P}_2$ exists. Then we get $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$. Repeat this progress, we get $\dim B \geq \dim A$. $\qquad\square$

To be completed

Definition 4. An integral domain A is called *normal* if it is integrally closed in its field of fractions $\operatorname{Frac}(A)$.

Lemma 5. Let $A \subset C$ be rings and B the integral closure of A in C, S a multiplicatively closed subset of A. Then the integral closure of $S^{-1}A$ in $S^{-1}C$ is $S^{-1}B$.

Proof. For every $b \in B$ and $\forall s \in S$, there exists $a_i \in A$ s.t.

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then
$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s^1}\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0.$$

Hence b/s is integral over $S^{-1}A$, $S^{-1}B$ is integral over $S^{-1}A$.

If $c/s \in S^{-1}C$ is integral over $S^{-1}A$, then $\exists a_i \in S^{-1}A$ s.t.
$$\left(\frac{c}{s}\right)^n + a_1\left(\frac{c}{s}\right)^{n-1} + \cdots + a_n = 0.$$

Then
$$c^n + a_1 s c^{n-1} + \cdots + a_n s^n = 0 \in S^{-1}C$$

Then $\exists t \in S$ s.t.
$$t(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0 \in C.$$

Then
$$(ct)^n + a_1 st(ct)^{n-1} + \cdots + a_n s^n t^n = t^n(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0.$$

Hence ct is integral over A, then $ct \in B$. Then $c/s = (ct)/(st) \in S^{-1}B$. This completes the proof. □

Proposition 6. Normality is a local property. That is, for an integral domain A, TFAE:

  (i) A is normal.

  (ii) For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_\mathfrak{p}$ is normal.

  (iii) For any maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$, the localization $A_\mathfrak{m}$ is normal.

Proof. When A is normal, $A_\mathfrak{p}$ is normal by Lemma 5.

Assume that $A_\mathfrak{m}$ is normal for every $\mathfrak{m} \in \operatorname{mSpec} A$. If A is not normal, let $\tilde{A}$ be the integral closure of A in $\operatorname{Frac} A$, $\tilde{A}/A$ is a nonzero A-module. Suppose $\mathfrak{p} \in \operatorname{Supp} \tilde{A}/A$ and $\mathfrak{p} \subset \mathfrak{m}$. We have $\tilde{A}_\mathfrak{m}/A_\mathfrak{m} = 0$ and $\tilde{A}_\mathfrak{p}/A_\mathfrak{p} = (\tilde{A}_\mathfrak{m}/A_\mathfrak{m})_\mathfrak{p} \neq 0$. This is a contradiction. □

Proposition 7. Let A be a normal ring. Then $A[X]$ is also normal.

Definition 8. A scheme X is called *normal* if the local ring $\mathscr{O}_{X,\xi}$ is normal for any point $\xi \in X$. A ring A is called *normal* if $\operatorname{Spec} A$ is normal.

Remark 9. For a general ring A, let $S := A \setminus (\bigcup_{\mathfrak{p} \in \operatorname{Ass} A} \mathfrak{p}) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} A} A \setminus \mathfrak{p}$. Then S is a multiplicative set. The localization $S^{-1}A$ is called *the total ring of fractions* of A.

Suppose A is reduced and $\operatorname{Ass} A = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Denote its total ring of fractions by Q. Note that elements in Q are either unit or zero divisor. Hence any maximal ideal $\mathfrak{m}$ is contained in $\bigcup \mathfrak{p}_i Q$, whence contained in some $\mathfrak{p}_i Q$. Thus $\mathfrak{p}_i Q$ are maximal ideals. And we have $\bigcap \mathfrak{p}_i Q = 0$. By the Chinese Remainder Theorem, we have $Q = \prod Q/\mathfrak{p}_i Q = \prod A_{\mathfrak{p}_i}$.

Let A be a reduced ring with total ring of fractions Q. Then A is normal iff A is integral closed in Q. If A is normal, then for every $\mathfrak{p} \in \operatorname{Spec} A$, $A_\mathfrak{p}$ is integral. Then there is unique minimal prime ideal

$\mathfrak{p}_i \subset \mathfrak{p}$. In particular, any two minimal prime ideal are relatively prime. By the Chinese Remainder Theorem, $A = \prod A/\mathfrak{p}_i$. Just need to check $A/\mathfrak{p}_i$ is integral closed in $A_{\mathfrak{p}_i}$. This is clear by check pointwise.

Conversely, suppose A is integral closed in Q. Let e_i be the unit element of $A_{\mathfrak{p}_i}$. It belongs to A since $e_i^2 - e_i = 0$. Since $1 = e_1 + \cdots + e_n$ and $e_i e_j = \delta_{ij}$, we have $A = \prod A e_i$. Since $A e_i$ is integral closed in $A_{\mathfrak{p}_i}$, it is normal. Hence A is normal.

Lemma 10. Let A be a normal ring. Then A verifies (R_1) and (S_2).

Proof. Since all properties are local, we can assume A is integral and local.

For (S_2), by Example **??**, we only need to show that $\mathrm{Ass}_A A/f$ has no embedded point. Let $\mathfrak{p} = (f : g) =\in \mathrm{Ass}_A A/fA$ and $t := f/g \in \mathrm{Frac}\, A$. After Replacing A by $A_{\mathfrak{p}}$, we can assume that $\mathfrak{p}$ is maximal. By definition, $t^{-1}\mathfrak{p} \subset A$. If $t^{-1}\mathfrak{p} \subset \mathfrak{p}$, suppose $\mathfrak{p}$ is generated by $(x_1, \cdots, x_n)$ and $t^{-1}(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(A)$. There is a monic polynomial $\chi(T) \in A[T]$ vanishing Φ. Then $\chi(t^{-1}) = 0$ and $t^{-1} \in A$. This is impossible by definition of t. Then $t^{-1}\mathfrak{p} = A$, and $\mathfrak{p} = (t)$ is principal. By Krull's Principal Ideal Theorem **??**, $\mathrm{ht}(\mathfrak{p}) = 1$.

Now we show that A verifies (R_1). Suppose $(A, \mathfrak{m})$ is local of dimension 1. Choosing $a \in \mathfrak{m}$, A/a is of dimension 0. Then by **??**, $\mathfrak{m}^n \subset aA$ for some $n \geq 1$. Suppose $\mathfrak{m}^{n-1} \not\subset aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$ and let $t = a/b$. By construction, $t^{-1} \notin A$ and $t^{-1}\mathfrak{m} \subset A$. After similar argument, we see that $\mathfrak{m} = tA$, whence A is regular. $\square$

Lemma 11. Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension 1. Then A is normal iff A is regular.

Proof. By lemma 10, we just need to show that regularity implies normality.

Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since A is regular, $\mathfrak{m} = (t)$. Let $I \subset \mathfrak{m}$ be an ideal. If $I \subset \bigcap_n \mathfrak{m}^n$, then for every $a \in I$, there exists a_n such that $a = a_n t^n$. Then we get an ascending chain of ideals $(a_1) \subset (a_2) \subset \cdots$. Hence $a = 0$ by Nakayama's Lemma. Suppose I is not zero. Then there is some n such that $I \subset \mathfrak{m}^n$ and $I \not\subset \mathfrak{m}^{n+1}$. For every $at^n \in I \setminus \mathfrak{m}^{n+1}$, $a \notin \mathfrak{m}$, whence a is a unit in A. Then $I = (t^n)$. Hence A is PID and hence normal. $\square$

Proposition 12. Let A be a noetherian integral domain of dimension ≥ 1 verifying (S_2). Then

$$A = \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A, \mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

Proof. Clearly $A \subset \bigcap A_{\mathfrak{p}}$. Let $t = f/g \in \bigcap A_{\mathfrak{p}}$. Since $f \in gA_{\mathfrak{p}}$ and we have $gA = \bigcap(gA_{\mathfrak{p}} \cap A)$, $f \in gA$. It follows that $t \in A$. $\square$

Theorem 13 (Serre's criterion for normality)**.** Let X be a locally noetherian scheme. Then X is normal if and only if it verifies (R_1) and (S_2).

Proof. One direction has been proved in Lemma 10. Suppose X verifies (R_1) and (S_2). Again we can assume $X = \mathrm{Spec}\, A$ is affine and A is local. By Remark 9, we just need to show that A is integral closed in its total ring of fractions Q. Suppose we have

$$\left(\frac{a}{b}\right)^n + c_1 \left(\frac{a}{b}\right)^{n-1} + \cdots + c_n = 0 \in Q.$$

Since A verifies (S_2), $bA = \bigcap v_\mathfrak{p}^{-1}(b_\mathfrak{p} A_\mathfrak{p})$. So it is sufficient to show that $a_\mathfrak{p} \in b_\mathfrak{p} A_\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) = 1$. Note that $A_\mathfrak{p}$ is regular and hence normal by Lemma 11. Then above equation gives us desired result. $\qquad\square$