Associated prime ideals

1 Associated prime ideals

Definition 1 (Associated prime ideals)**.** Let A be a noetherian ring and M an A-module. The *associated prime ideals* of M are the prime ideals $\mathfrak{p}$ of form $\operatorname{Ann}(x)$ for some $x \in M$. The set of associated prime ideals of M is denoted by $\operatorname{Ass}(M)$.

Example 2. Let $A = \Bbbk[x,y]/(xy)$ and $M = A$. First we see that $(x) = \operatorname{Ann} y, (y) = \operatorname{Ann} x \in \operatorname{Ass} M$. Then we check other prime ideals. For (x,y), if $xf = yf = 0$, then $f \in (x) \cap (y) = (0)$. If $(x-a) = \operatorname{Ann} f$ for some f, note that $y \in (x-a)$ for $a \in \Bbbk^*$, then $f \in (x)$. Hence $f = 0$. Therefore $\operatorname{Ass} M = \{(x),(y)\}$.

Example 3. Let $A = \Bbbk[x,y]/(x^2,xy)$ and $M = A$. The underlying space of $\operatorname{Spec} A$ is the y-axis since $\sqrt{(x^2,xy)} = (x)$. First note that $(x) = \operatorname{Ann} y, (x,y) = \operatorname{Ann} x \in \operatorname{Ass} M$. For $(x, y-a)$ with $a \in \Bbbk^*$, easily see that $xf = (y-a)f = 0$ implies $f = 0$ since $A = \Bbbk \cdot x \oplus \Bbbk[y]$ as $\Bbbk$-vector space. Hence $\operatorname{Ass} M = \{(x),(x,y)\}$.

Lemma 4. Let A be a noetherian ring and M an A-module. Then the maximal element of the set

$$\{\operatorname{Ann} x \colon x \in M_{\mathfrak{p}}, x \neq 0\}$$

belongs to $\operatorname{Ass} M$.

Proof. We just need to show that such $\operatorname{Ann} x$ is prime. Otherwise, there exist $a, b \in A$ such that $ab \in \operatorname{Ann} x$ but $a, b \notin \operatorname{Ann} x$. It follows that $\operatorname{Ann} x \subsetneq \operatorname{Ann} ax$ since $b \in \operatorname{Ann} ax \setminus \operatorname{Ann} x$. This contradicts the maximality of $\operatorname{Ann} x$. □

An element $a \in A$ is called a zero divisor for M if $M \to aM, m \mapsto am$ is not injective.

Corollary 5. Let A be a noetherian ring and M an A-module. Then

$$\{\text{zero divisors for } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}.$$

Lemma 6. Let A be a noetherian ring and M an A-module. Then $\mathfrak{p} \in \operatorname{Ass}_A M$ iff $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Proof. Suppose $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Ann} y_0/c$ with $y_0 \in M$ and $c \in A \setminus \mathfrak{p}$. For $a \in \operatorname{Ann} y_0$, $ay_0 = 0$. Then $a/1 \in \mathfrak{p}A_{\mathfrak{p}}$. It follows that $a \in \mathfrak{p}$. Hence $\operatorname{Ann} y_0 \subset \mathfrak{p}$.

Inductively, if $\operatorname{Ann} y_n \subsetneq \mathfrak{p}$, then there exists $b_n \in A \setminus \mathfrak{p}$ such that $y_{n+1} := b_n y_n$, $\operatorname{Ann} y_{n+1} \subset \mathfrak{p}$ and $\operatorname{Ann} y_n \subsetneq \operatorname{Ann} y_{n+1}$. To see this, choose $a_n \in \mathfrak{p} \setminus \operatorname{Ann} y_n$. Then $(a_n/1)y_n = 0$ since $a_n/1 \in \mathfrak{p}A_{\mathfrak{p}}$. By definition, there exist $b_n \in A \setminus \mathfrak{p}$ such that $a_n b_n y_n = 0$. This process must terminate since A is noetherian. Thus $\operatorname{Ann} y_n = \mathfrak{p}$ for some n. Hence $\mathfrak{p} \in \operatorname{Ass}_A M$.

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Conversely, suppose $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$. If $(a/s)(x/1) = 0 \in M_{\mathfrak{p}}$, there exist $t \in A \setminus \mathfrak{p}$ such that $tax = 0$. It follows that $ta \in \mathfrak{p}$ and then $(a/s) \in \mathfrak{p}A_{\mathfrak{p}}$. Hence $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. □

Proposition 7. We have $\operatorname{Ass} M \subset \operatorname{Supp} M$. Moreover, if $\mathfrak{p} \in \operatorname{Supp} M$ satisfies $V(\mathfrak{p})$ is an irreducible component of $\operatorname{Supp} M$, then $\mathfrak{p} \in \operatorname{Ass} M$.

Proof. For any $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$, we have $A/\mathfrak{p} \cong A \cdot x \subset M$. Tensoring with $A_{\mathfrak{p}}$ gives $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$ since $A_{\mathfrak{p}}$ is flat. Hence $M_{\mathfrak{p}} \neq 0$ and $\mathfrak{p} \in \operatorname{Supp} M$.

Now suppose $\mathfrak{p} \in \operatorname{Supp} M$ and $V(\mathfrak{p})$ is an irreducible component of $\operatorname{Supp} M$. First we show that $\mathfrak{p} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $x \in M_{\mathfrak{p}}$ such that $\operatorname{Ann} x$ is maximal in the set

$$\{\operatorname{Ann} x \colon x \in M_{\mathfrak{p}}, x \neq 0\}.$$

Then we claim that $\operatorname{Ann} x = \mathfrak{p}A_{\mathfrak{p}}$. First, $\operatorname{Ann} x$ is prime by Lemma 4. If $\operatorname{Ann} x \neq \mathfrak{p}$, then $V(\operatorname{Ann} x) \supset V(\mathfrak{p})$. This implies that $\operatorname{Ann} x \notin \operatorname{Supp} M_{\mathfrak{p}}$ since $\operatorname{Supp} M_{\mathfrak{p}} = \operatorname{Supp} M \cap \operatorname{Spec} A_{\mathfrak{p}}$. This is a contradiction. Thus $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. By Lemma 6, we have $\mathfrak{p} \in \operatorname{Ass} M$. □

Remark 8. The existence of irreducible component is guaranteed by Zorn's Lemma.

Definition 9. A prime ideal $\mathfrak{p} \in \operatorname{Ass} M$ is called *embedded* if $V(\mathfrak{p})$ is not an irreducible component of $\operatorname{Supp} M$.

Example 10. For $M = A = \Bbbk[x, y]/(x^2, xy)$, the origin (x, y) is an embedded point.

Proposition 11. If we have exact sequence $0 \to M_1 \to M_2 \to M_3$, then $\operatorname{Ass} M_2 \subset \operatorname{Ass} M_1 \cup \operatorname{Ass} M_3$.

Proof. Let $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M_2 \setminus \operatorname{Ass} M_1$. Then the image $[x]$ of x in M_3 is not equal to 0. We have that $\operatorname{Ann} x \subset \operatorname{Ann}[x]$. If $a \in \operatorname{Ann}[x] \setminus \operatorname{Ann} x$, then $ax \in M_1$. Since $\operatorname{Ann} x \subsetneq \operatorname{Ann} ax$, there is $b \in \operatorname{Ann} ax \setminus \operatorname{Ann} x$. However, it implies $ba \in \operatorname{Ann} x$, and then $a \in \operatorname{Ann} x$ since $\operatorname{Ann} x$ is prime, which is a contradiction. □

Corollary 12. If M is finitely generated, then the set $\operatorname{Ass} M$ is finite.

Proof. For $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$, we know that the submodule M_1 generated by x is isomorphic to $A/\mathfrak{p}$. Inductively, we can choose M_n be the preimage of a submodule of M/M_{n-1} which is isomorphic to $A/\mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass} M/M_{n-1}$. We can take an ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$. Since M is finitely generated, this is a finite sequence. Then the conclusion follows by Proposition 11. □

2 Primary decomposition

Definition 13. An A-module is called *co-primary* if $\operatorname{Ass} M$ has a single element. Let M be an A-module and $N \subset M$ a submodule. Then N is called *primary* if M/N is co-primary. If $\operatorname{Ass} M/N = \{\mathfrak{p}\}$, then N is called $\mathfrak{p}$-primary.

Remark 14. This definition coincide with primary ideals in the case $M = A$. Recall an ideal $\mathfrak{q} \subset A$ is called *primary* if $\forall ab \in \mathfrak{p}$, $a \notin \mathfrak{q}$ implies $b^n \in \mathfrak{q}$ for some n.

Let $\mathfrak{q}$ be a $\mathfrak{q}$-primary ideal. Since $\operatorname{Supp} A/\mathfrak{q} = \{\mathfrak{p}\}$, $\mathfrak{p} \in \operatorname{Ass} A/\mathfrak{q}$. Suppose $\operatorname{Ann}[a] \in \operatorname{Ass} A/\mathfrak{q}$. Then $\mathfrak{p} \subset \operatorname{Ann}[a]$ since $V(\mathfrak{p}) = \operatorname{Supp} A/\mathfrak{q}$. If $b \in \operatorname{Ann}[a]$, then $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Hence $b^n \in \mathfrak{q}$, and then $b \in \mathfrak{p}$. This shows that $\operatorname{Ass} A/\mathfrak{q} = \{\mathfrak{p}\}$ and $\mathfrak{q}$ is $\mathfrak{p}$-primary as an A-submodule.

Let $\mathfrak{q} \subset A$ be a $\mathfrak{p}$-primary A-submodule. First we have $\mathfrak{p} = \sqrt{\mathfrak{q}}$ since $V(\mathfrak{p})$ is the unique irreducible component of $\operatorname{Supp} A/\mathfrak{q}$. Suppose $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Then $b \in \operatorname{Ann}[a] \subset \mathfrak{p}$ since $\mathfrak{p}$ is the unique maximal element in $\{\operatorname{Ann}[c] : c \in A \setminus \mathfrak{q}\}$. This implies that $b^n \in \mathfrak{q}$.

Definition 15. Let A be a noetherian ring, M an A-module and $N \subset M$ a submodule. A *minimal primary decomposition* of N in M is a finite set of primary submodules $\{Q_i\}_{i=1}^n$ such that

$$N = \bigcap_{i=1}^n Q_i,$$

no Q_i can be omitted and $\operatorname{Ass} M/Q_i$ are pairwise distinct. For $\operatorname{Ass} M/Q_i = \{\mathfrak{p}\}$, Q_i is called belonging to $\mathfrak{p}$.

Indeed, if $N \subset M$ admits a minimal primary decomposition $N = \bigcap Q_i$ with Q_i belonging to $\mathfrak{p}$, then $\operatorname{Ass}(M/N) = \{\mathfrak{p}_i\}$. For given i, consider $N_i := \bigcap_{j \neq i} Q_j$, then $N_i/N \cong (N_i + Q_i)/Q_i$. Since $N_i \neq N$, $\operatorname{Ass} N_i/N \neq \emptyset$. On the other hand, $\operatorname{Ass} N_i/N \subset \operatorname{Ass} M/Q_i = \{\mathfrak{p}\}$. It follows that $\operatorname{Ass} N_i/N = \{\mathfrak{p}_i\}$, whence $\mathfrak{p}_i \in \operatorname{Ass} M/N$. Conversely, we have an injection $M/N \hookrightarrow \bigoplus M/Q_i$, so $\operatorname{Ass} M/N \subset \bigcup \operatorname{Ass} M/Q_i$. Due to this, if Q_i belongs to $\mathfrak{p}$, we also say that Q_i is the $\mathfrak{p}$-component of N.

Proposition 16. Suppose $N \subset M$ has a minimal primary decomposition. If $\mathfrak{p} \in \operatorname{Ass} M/N$ is not embedded, then the $\mathfrak{p}$ component of N is unique. Explicitly, we have $Q = \nu^{-1}(N_{\mathfrak{p}})$, where $\nu : M \to M_{\mathfrak{p}}$.

Proof. First we show that $Q = \nu^{-1}(Q_{\mathfrak{p}})$. Clearly $Q \subset \nu^{-1}(Q_{\mathfrak{p}})$. Suppose $x \in \nu^{-1}(Q_{\mathfrak{p}})$. Then there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in Q$. That is, $[sx] = 0 \in M/Q$. If $[x] \neq 0$, we have $s \in \operatorname{Ann}[x] \subset \mathfrak{p}$. This contradiction enforces $Q = \nu^{-1}(Q_{\mathfrak{p}})$.

Then we show that $N_{\mathfrak{p}} = Q_{\mathfrak{p}}$. Just need to show that for $\mathfrak{p}' \neq \mathfrak{p}$ and the $\mathfrak{p}'$ component Q' of N, $Q'_{\mathfrak{p}} = M_{\mathfrak{p}}$. Since $\mathfrak{p}$ is not embedded, $\mathfrak{p}' \not\subset \mathfrak{p}$. Then $\mathfrak{p} \notin V(\mathfrak{p}) = \operatorname{Supp} M/Q'$. So $M_{\mathfrak{p}}/Q'_{\mathfrak{p}} = 0$. ☐

Example 17. If $\mathfrak{p}$ is embedded, then its components may not be unique. For example, let $M = A = \Bbbk[x,y]/(x^2, xy)$. Then for every $n \in \mathbb{Z}_{\geq 1}$, $(x) \cap (x^2, xy, y^n)$ is a minimal primary decomposition of $(0) \subset M$.

Let A be a noetherian ring and $\mathfrak{p} \subset A$ a prime ideal. We consider the $\mathfrak{p}$ component of $\mathfrak{p}^n$, which is called n-th symbolic power of $\mathfrak{p}$, denoted by $\mathfrak{p}^{(n)}$. We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A$. In general, $\mathfrak{p}^{(n)}$ is not equal to $\mathfrak{p}^n$; see below example.

Example 18. Let $A = \mathbf{k}[x,y,z,w]/(y^2 - zx^2, yz - xw)$ and $\mathfrak{p} = (y,z,w)$. We have $z = y^2/x^2, w = yz/x \in \mathfrak{p}^2 A_{\mathfrak{p}}$, whence $\mathfrak{p}^2 A_{\mathfrak{p}} = (z,w) \neq \mathfrak{p}^2$.

Theorem 19. Let A be a noetherian ring and M an A-module. Then for every $\mathfrak{p} \in \operatorname{Ass} M$, there is a $\mathfrak{p}$-primary submodule $Q(\mathfrak{p})$ such that

$$(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M} Q(\mathfrak{p}).$$

Proof. Consider the set

$$\mathcal{N} := \{N \subset M : \mathfrak{p} \notin \operatorname{Ass} N\}.$$

Note that $\operatorname{Ass} \bigcup N_i = \bigcup \operatorname{Ass} N_i$ by definition of associated prime ideals. Then it is easy to check that $\mathcal{N}$ satisfies the conditions of Zorn's Lemma. Hence $\mathcal{N}$ has a maximal element $Q(\mathfrak{p})$. We claim that $Q(\mathfrak{p})$ is $\mathfrak{p}$-primary. If there is $\mathfrak{p}' \neq \mathfrak{p} \in \operatorname{Ass} M/Q(\mathfrak{p})$, then there is a submodule $N' \cong A/\mathfrak{p}$. Let N'' be the preimage of N' in M. We have $Q(\mathfrak{p}) \subsetneq N''$ and $N'' \in \mathcal{N}$. This is a contradiction. By the fact $\operatorname{Ass} \bigcap N_i = \bigcap \operatorname{Ass} N_i$, we get the conclusion. □

Corollary 20. Let A be a noetherian ring and M a finite A-module. Then every submodule of M has a minimal primary decomposition.