Group objects

Let $\mathbf{C}$ be a category with terminal object $*$ and finite products.

Definition 1. A *group object* in $\mathbf{C}$ is an object G together with morphisms

- $m : G \times G \to G$ (multiplication),
- $e : * \to G$ (identity),
- $i : G \to G$ (inverse),

such that the following diagrams commute:

- Associativity:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- Identity:

$$\begin{tikzcd} * \times G \arrow[r, "e \times \mathrm{id}_G"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "\mathrm{id}_G \times e"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd}$$

- Inverse:

$$\begin{tikzcd} G \arrow[r, "\Delta"] & G \times G \arrow[r, "\mathrm{id}_G \times i"] & G \times G \arrow[r, "m"] & G \\ & * \arrow[ul, "e"'] & & \end{tikzcd}$$

Yang: To be checked.

Example 2. In the category of sets, a group object is just a group in the usual sense. In the category of topological spaces, a group object is a topological group. In the category of smooth manifolds, a group object is a Lie group. Yang:

Definition 3. Let G be a group object in a category $\mathbf{C}$ and X an object in $\mathbf{C}$. A *group action* of G on X is a morphism

$$\sigma : G \times X \to X$$

such that the following diagrams commute:

- Identity:

$$\begin{tikzcd} * \times X \arrow[r, "e \times \mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

Date: January 26, 2026, **Author**: Tianle Yang, My Homepage

- Compatibility:

$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \sigma"'] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

Definition 4. Let G be a group object in a category $\mathbf{C}$ acting on objects X, Y via actions σ_X, σ_Y respectively. A morphism $f : X \to Y$ is said to be G-*invariant* if the following diagram commutes:

$$\begin{tikzcd} G \times X \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{id}_G \times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\sigma_Y"'] & Y \end{tikzcd}$$

Yang:

Definition 5. Let G be a group object in a category $\mathbf{C}$ acting on an object X via action σ. A *categorical quotient* of X by G is an object Q with trivial G-action, together with a G-invariant morphism $q : X \to Q$, such that for any object Y with trivial G-action and any G-invariant morphism $f : X \to Y$, there exists a unique morphism $\overline{f} : Q \to Y$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "f"'] & Q \arrow[d, dashed, "\overline{f}"] \\ & Y \end{tikzcd}$$

Yang: Everything above need to be checked.

Appendix