Dimension and Depth

There are three numbers measuring the "size" of a local ring $(A, \mathfrak{m})$:

- $\dim A$: the Krull dimension of A.
- $\operatorname{depth} A$: the depth of A.
- $\dim_{\kappa(\mathfrak{m})} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$-vector space.

Somehow the Krull dimension is "homological" and the depth is "cohomological".

Definition 1. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A-module. A sequence $t_1, \cdots, t_n \in I$ is called an *M-regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \cdots, t_{i-1})M$ for all i.

Example 2. Let $A = \mathbf{k}[x, y]/(x^2, xy)$ and $I = (x, y)$. Then $\operatorname{depth}_I A = 0$.

Definition 3. Let A be a noetherian ring. For every $\mathfrak{p} \in \operatorname{Spec} A$, $\mathfrak{p}/\mathfrak{p}^2$ is a vector space over $\kappa(\mathfrak{p})$. The *Zariski's tangent space* $T_{A,\mathfrak{p}}$ of A at $\mathfrak{p}$ is defined as $(\mathfrak{p}/\mathfrak{p}^2)^\vee$, the dual $\kappa(\mathfrak{p})$-vector space of $\mathfrak{p}/\mathfrak{p}^2$.

1 Krull's Principal Ideal Theorem

Theorem 4 (Krull's Principal Ideal Theorem)**.** Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let $\mathfrak{p}$ be a minimal prime ideal among those containing f. Then $\operatorname{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_\mathfrak{p}$, we may assume A is local with maximal ideal $\mathfrak{p}$. Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.

Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is $\mathfrak{q}$-primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_\mathfrak{q} = \mathfrak{q}^{n+1} A_\mathfrak{q}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_\mathfrak{q} = 0$. It follows that $\mathfrak{q}A_\mathfrak{q}$ is minimal, whence $A_\mathfrak{q}$ is artinian. Therefore, $\mathfrak{q}$ is minimal in A. □

Corollary 5. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n. Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.

Choose $\mathfrak{q}$ be a minimal prime ideal among those containing $(\mathfrak{p}_{k-1}, f)$ and contained in $\mathfrak{p}_{k+1}$. Then by Krull's Principal Ideal Theorem 4, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace $\mathfrak{p}_k$ by $\mathfrak{q}_k$, we have $f \in \mathfrak{q}_k \setminus \mathfrak{p}_{k-1}$

Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.

Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$

has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A. It follows that $\dim A/(f) + 1 \leq \dim A$. □

Proposition 6. Let $(A, \mathfrak{m})$ be a local noetherian ring with residue field $\mathbf{k}$. Then the following inequalities hold:

$$\operatorname{depth} A \leq \dim A \leq \dim_{\mathbf{k}} T_{A,\mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary 5.

Let $t_1, \cdots, t_n$ be a $\kappa(\mathfrak{m})$-basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \cdots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \cdots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \cdots, t_n))$. It follows that $\mathfrak{m} = (t_1, \cdots, t_n)$ by Nakayama's Lemma. By Corollary 5,

$$n + \dim A/(t_1, \cdots, t_n) \geq n - 1 + \dim A/(t_1, \cdots, t_{n-1}) \geq \cdots \geq 1 + \dim A/(t_1) \geq \dim A.$$

We conclude the result. □

Definition 7. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property* (R_k) or *is regular in codimension* k if $\forall \xi \in X$ with $\operatorname{codim} Z_\xi \leq k$,

$$\dim_{\kappa(\xi)} T_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

We say that X *verifies property* (S_k) if $\forall \xi \in X$ with $\operatorname{depth} \mathcal{O}_{X,\xi} < k$,

$$\operatorname{depth} \mathcal{O}_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

Example 8. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.

Suppose A verifies (S_1). If $\mathfrak{p} \in \operatorname{Ass} A$, every element in $\mathfrak{p}$ is a zero divisor. Then $\operatorname{depth} A_\mathfrak{p} = 0$. It follows that $\dim A_\mathfrak{p} = 0$ and then $\mathfrak{p}$ is minimal.

Suppose A has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_\mathfrak{p} = 0$. This means every element in $\mathfrak{p}A_\mathfrak{p}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Proposition ??, $\mathfrak{p} = \mathfrak{q}$ for some minimal $\mathfrak{q}$, whence $\dim A_\mathfrak{p} = 0$.

Example 9. Let A be a noetherian ring. Then A is reduced iff it verifies (R_0) and (S_1).

Suppose A is reduced. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be all minimal prime ideals of A. We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where $\mathfrak{N}$ is the nilradical of A. Hence A has no embedded point. Since $A_\mathfrak{p}$ is artinian, local and reduced, $A_\mathfrak{p}$ is a field and hence regular.

Conversely, let $\operatorname{Ass} A$ be equal to $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Then every $\mathfrak{p}_i$ is minimal by (S_1). Let f be in $\mathfrak{N}$. Then the image of f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0), $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where $\mathfrak{q}_i$ is the $\mathfrak{p}_i$ component of (0) in A. Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

2 Cohen-Macaulay rings

Definition 10 (Cohen-Macaulay)**.** A noetherian local ring $(A, \mathfrak{m})$ is called *Cohen-Macaulay* if $\dim A = \operatorname{depth} A$. A noetherian ring A is called *Cohen-Macaulay* if for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is Cohen-Macaulay. This is equivalent to that A verifies (S_k) for all $k \geq 0$.

Definition 11. Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension d. A sequence $t_1, \cdots, t_d \in \mathfrak{m}$ is called a *system of parameters* if To be completed.

Example 12 (Non Cohen-Macaulay rings)**.** To be completed.

Corollary 13. Let A be a noetherian ring, M a finite A-module and $a \in A$ an M-regular element. Then $\operatorname{depth} M = \operatorname{depth} M/aM + 1$.

Corollary 14. Let A be a noetherian ring $a \in A$ a nonzero divisor. Then A verifies (S_d) iff A/aA verifies (S_{d-1}).

Definition 15. An ideal I of a noetherian ring A is called *unmixed* if

$$\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \operatorname{Ass}(A/I).$$

Here $\operatorname{ht}(I)$ is defined as

$$\operatorname{ht}(I) := \inf\{\operatorname{ht}(\mathfrak{p}) : I \subset \mathfrak{p}\}.$$

We say that *the unmixedness theorem holds for a noetherian ring* A if any ideal $I \subset A$ generated by $\operatorname{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme* X if $\mathcal{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Proposition 16. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Proof. We can assume that $X = \operatorname{Spec} A$ is affine.

Suppose X is Cohen-Macaulay. Let $I \subset A$ be an ideal generated by $a_1, \cdots, a_r$ with $r = \operatorname{ht}(I)$. We claim that $a_1, \cdots, a_r$ is an A-regular sequence. If so, we get that the unmixedness theorem holds for A by applying Example 8 on A/I. Since $\operatorname{ht}(a_1, \cdots, a_{r-1}) \leq r - 1$ by Krull's Principal Ideal Theorem 4 and $\operatorname{ht}(a_1, \cdots, a_r) = r \leq \operatorname{ht}(a_1, \cdots, a_{r-1}) + 1$, we have $\operatorname{ht}(a_1, \cdots, a_{r-1}) = r - 1$. By induction on r, we can assume that $a_1, \cdots, a_{r-1}$ is an A-regular sequence. Hence any prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1, \cdots, a_{r-1})$ has height $r-1$. Now suppose a_r is a zero divisor in $A/(a_1, \cdots, a_{r-1})$. Then there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1, \cdots, a_{r-1})$ such that $a_r \in \mathfrak{p}$. Then $I \subset \mathfrak{p}$ and $\operatorname{ht}(I) \leq r - 1$. This contradicts that $\operatorname{ht}(I) = r$.

Suppose the unmixedness theorem holds for A. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal with $\operatorname{ht}(\mathfrak{p}) = r$. Then $\mathfrak{p} \in \operatorname{Ass} A$ if and only if $\operatorname{ht}(\mathfrak{p}) = 0$. If $r > 0$, there is a nonzero divisor $a \in \mathfrak{p}$. By Krull's Principal Ideal Theorem 4, $\operatorname{ht}(\mathfrak{p}A/aA) = r - 1$. Inductively, we can find a regular sequence $a_1, \cdots, a_r$ in $\mathfrak{p}$. Then $\operatorname{depth} A_{\mathfrak{p}} = r$. □

Theorem 17. Let A be a noetherian ring of dimension d. The following are equivalent: To be completed.

Proof. To be completed. □

3 Regular rings

Definition 18. A noetherian ring A is said to be *regular at* $\mathfrak{p} \in \operatorname{Spec} A$ if we have

$$\dim_{\kappa(\mathfrak{p})} T_{A,\mathfrak{p}} = \dim A_{\mathfrak{p}},$$

where $\dim A_{\mathfrak{p}}$ is the Krull dimension of the local ring $A_{\mathfrak{p}}$.
A noetherian ring A is said to be *regular* if it is regular at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. This is equivalent to the condition that A verifies (R_k) for all $k \geq 0$.

Remark 19. A noetherian ring A is regular if and only if it is regular at every maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$. The proof uses homological tools; see Theorem ?? and Corollary ??.

Definition 20. Let A be a noetherian ring that is regular at $\mathfrak{p} \in \operatorname{Spec} A$. A sequence $t_1, \cdots, t_n \in \mathfrak{p}$ is called a *regular system of parameters* at $\mathfrak{p}$ if their images form a basis of the $\kappa(\mathfrak{p})$-vector space $\mathfrak{p}/\mathfrak{p}^2$.

Proposition 21. Let $(A, \mathfrak{m})$ be a noetherian local ring that is regular at $\mathfrak{m}$. Let $t_1, \cdots, t_n$ be a regular system of parameters at $\mathfrak{m}$, $\mathfrak{p}_i = (t_1, \cdots, t_i)$ and $\mathfrak{p}_0 = (0)$. Then $\mathfrak{p}_i$ is a prime ideal of height i, and $A/\mathfrak{p}_i$ is a regular local ring for all i. In particular, regular local ring is integral, and the regular system of parameters $t_1, \cdots, t_n$ is a regular sequence in A.

Proof. By the Krull's Principal Ideal Theorem 4, we have

$$n - 1 = \dim A - 1 \leq \dim A/(t_1) \leq \dim_{\kappa(\mathfrak{m}/(t_1))} T_{A/(t_1), \mathfrak{m}/(t_1)} \leq n - 1.$$

Hence $\dim A/(t_1) = n - 1$ and $\operatorname{ht}(t_1) = 1$. Since $t_2, \cdots, t_n$ generate $\mathfrak{m}/(t_1)$, we have that $A/(t_1)$ is regular at $\mathfrak{m}/(t_1)$ and the images of $t_2, \cdots, t_n$ form a regular system of parameters.

For integrality, we induct on the dimension of A. If $\dim A = 0$, then A is a field and hence integral. Suppose $\dim A > 0$, let $\mathfrak{q}$ be a minimal prime ideal of A. Then $t_1 \notin \mathfrak{q}$. We have

$$n - 1 = \dim A - 1 \leq \dim A/(\mathfrak{q} + t_1 A) \leq \dim_{\kappa(\mathfrak{q}/(t_1))} T_{A/(\mathfrak{q}+t_1 A), \mathfrak{q}/(t_1)} \leq n - 1.$$

By similar arguments, we have $A/(\mathfrak{q} + t_1 A)$ is regular at $\mathfrak{m}/(\mathfrak{q} + t_1 A)$. By induction hypothesis, both of $A/t_1 A$ and $A/(\mathfrak{q} + t_1 A)$ are integral and of dimension $n - 1$. Hence $t_1 A = t_1 A + \mathfrak{q}$, i.e. $\mathfrak{q} \subset t_1 A$. For every $a = b t_1 \in \mathfrak{q}$, we have $b \in \mathfrak{q}$ since $t_1 \notin \mathfrak{q}$. Then $\mathfrak{q} \subset t_1 \mathfrak{q} \subset \mathfrak{m} \mathfrak{q}$. By Nakayama's Lemma, $\mathfrak{q} = 0$, whence A is integral. □

Corollary 22. A regular noetherian ring is Cohen-Macaulay.

Corollary 23. A regular noetherian ring is normal.

Remark 24. Indeed we can show a stronger result: a noetherian regular local ring is a UFD; see ref.