

# Smoothness

## 1 Modules of differentials and derivations

In this subsection, let  $R$  be a ring and  $A$  an  $R$ -algebra.

**Definition 1** (Derivation). A *derivation* of  $A$  over  $R$  is an  $R$ -linear map  $\partial : A \rightarrow M$  with an  $A$ -module such that for all  $a, b \in A$ , we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module  $M$ , the set of all derivations of  $A$  over  $R$  into  $M$  forms an  $A$ -module, denoted by  $\text{Der}_R(A, M)$ .

Given a module homomorphism  $f : M \rightarrow N$  of  $A$ -modules and a derivation  $\partial \in \text{Der}_R(A, M)$ , the map  $f \circ \partial$  is a derivation of  $A$  over  $R$  into  $N$ .

**Proposition 2.** The functor  $\text{Der}_R(A, -)$  is representable. The representing object is denoted by  $\Omega_{A/R}$ , which is called the *module of differentials* of  $A$  over  $R$ .

*Proof.* First suppose  $A$  is a free  $R$ -algebra with a set of generators  $a_\lambda, \lambda \in \Lambda$ . Then an  $R$ -derivation  $\partial \in \text{Der}_R(A, M)$  is uniquely determined by its values on the generators  $a_\lambda$ . Let

$$\Omega_{A/R} := \bigoplus_{\lambda \in \Lambda} A \cdot da_\lambda$$

and  $d : A \rightarrow \Omega_{A/R}$  be the  $R$ -derivation defined by  $a_\lambda \mapsto da_\lambda$ . For any  $R$ -derivation  $\partial \in \text{Der}_R(A, M)$ , we can define a unique  $A$ -module homomorphism  $\Phi_\partial : \Omega_{A/R} \rightarrow M$  by sending  $da_\lambda$  to  $\partial(a_\lambda)$  such that  $\partial = \Phi_\partial \circ d$ . This gives a bijection

$$\text{Der}_R(A, M) \cong \text{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_\partial.$$

Now suppose  $A = F/I$  is an arbitrary  $R$ -algebra, where  $F$  is a free  $R$ -algebra and  $I$  is an ideal of  $F$ . Then we can define the module of differentials

$$\Omega_{A/R} := (\Omega_{F/R} \otimes_F A) / \sum_{f \in I} A \cdot df.$$

The  $R$ -linear map  $d_A : F \otimes_F A \xrightarrow{d_F} \Omega_{F/R} \otimes_F A \rightarrow \Omega_{A/R}$  is a derivation of  $A$  over  $R$ .

For any  $R$ -derivation  $\partial \in \text{Der}_R(A, M)$ , note that  $F \rightarrow A \xrightarrow{\partial} M$  is an  $R$ -derivation of  $F$  over  $R$  into  $M$ . Then we get an  $F$ -module homomorphism  $\Omega_F \rightarrow M$ . It gives an  $A$ -module homomorphism  $\Omega_F \otimes_F A \rightarrow M, df \otimes 1 \mapsto \partial f$ . This map factors into  $\Omega_{F/R} \otimes_F A \rightarrow \Omega_{A/R}$  and  $\Phi_\partial : \Omega_{A/R} \rightarrow M$ . Since  $\Phi_\partial$  is  $A$ -linear and  $\Omega_{A/R}$  is generated by  $da_\lambda$  as  $A$ -module, such  $\Phi_\partial$  is unique.  $\square$

**Corollary 3.** Suppose  $A$  is of finite type over  $R$ . Then the module of differentials  $\Omega_{A/R}$  is a finitely generated  $A$ -module.

**Remark 4.** Let  $B$  be an  $A$ -algebra,  $M$  an  $A$ -module and  $N$  a  $B$ -module. If there is a homomorphism of  $A$ -modules  $M \rightarrow N$ , then we can extend it to a homomorphism of  $B$ -modules  $M \otimes_A B \rightarrow N$  by sending  $m \otimes b$  to  $m \cdot b$ . And such extension is unique in the sense of following commutative diagram:

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & \nearrow \exists! & \\ M \otimes_A B & & \end{array}$$

Hence we get a natural bijection

$$\text{Hom}_A(M, N) \cong \text{Hom}_B(M \otimes_A B, N).$$

**Proposition 5.** Let  $A, R'$  be  $R$ -algebras and  $A' := A \otimes_R R'$ . Then the module of differentials  $\Omega_{A'/R'}$  is isomorphic to  $\Omega_{A/R} \otimes_A A'$ .

*Proof.* We check the universal property of  $\Omega_{A/R} \otimes_A A'$ . First, the map

$$d_{A'} : A \otimes_R R' \rightarrow \Omega_{A/R} \otimes_R R' \cong \Omega_{A/R} \otimes_A A', \quad a \otimes r \mapsto da \otimes r$$

is an  $R'$ -derivation of  $A'$  into  $\Omega_{A/R} \otimes_A A'$ . For any  $R'$ -derivation  $\partial' : A' \rightarrow M$  into an  $A'$ -module  $M$ , we can compose it with the homomorphism  $A' \rightarrow A$  and get an  $R$ -derivation  $\partial : A \rightarrow M$ . By the universal property of  $\Omega_{A/R}$ , there is a unique  $A$ -module homomorphism  $\Phi : \Omega_{A/R} \rightarrow M$  such that  $\partial = \Phi \circ d_A$ . Then we can extend it to an  $A'$ -module homomorphism  $\Phi' : \Omega_{A/R} \otimes_A A' \rightarrow M$  by Remark 4. By the construction, we have  $\Phi' \circ d_{A'} = \partial'$ .  $\square$

**Proposition 6.** Let  $A$  be an  $R$ -algebra and  $S$  a multiplicative set of  $A$ . Then we have an isomorphism

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

*Proof.* Let

$$d_{S^{-1}A} : S^{-1}A \rightarrow S^{-1}\Omega_{A/R}, \quad \frac{a}{s} \mapsto \frac{sda - ads}{s^2}.$$

By direct computation,  $d_{S^{-1}A}$  is an  $R$ -derivation of  $S^{-1}A$  over  $R$  into  $S^{-1}\Omega_{A/R}$ . For any  $R$ -derivation  $\partial : S^{-1}A \rightarrow M$  into an  $S^{-1}A$ -module  $M$ , we can get an  $S^{-1}A$ -module homomorphism  $\Phi' : S^{-1}\Omega_{A/R} \rightarrow M$  as proof of Proposition 5. We have

$$\partial\left(s \cdot \frac{a}{s}\right) = s\partial\left(\frac{a}{s}\right) + \frac{a}{s}\partial s.$$

It follows that

$$\partial\left(\frac{a}{s}\right) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(da) - a\Phi'(ds)}{s^2} = \Phi'\left(\frac{sda - ads}{s^2}\right).$$

Thus,  $\Phi' \circ d_{S^{-1}A} = \partial$ .  $\square$

**Theorem 7.** Let  $A$  be an  $R$ -algebra and  $B$  an  $A$ -algebra. Then there is a natural short exact sequence

$$\Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow \Omega_{B/A} \rightarrow 0$$

of  $B$ -modules.

*Proof.* Let  $d_{A/R} : A \rightarrow \Omega_{A/R}$  be the  $R$ -derivation of  $A$  over  $R$ . The map  $A \rightarrow B \xrightarrow{d_{B/R}} \Omega_{B/R}$  induces a  $B$ -linear map

$$u : \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R}, \quad d_{A/R}(a) \otimes b \mapsto bd_{B/R}(a).$$

The map  $d_{B/A}$  is an  $A$ -derivation and hence  $R$ -derivation. Then it induces a  $B$ -linear map

$$\nu : \Omega_{B/R} \rightarrow \Omega_{B/A}, \quad d_{B/R}(b) \mapsto d_{B/A}(b).$$

Since  $\Omega_{B/A}$  is generated by elements of the form  $d_{B/A}(b)$  for  $b \in B$ , the map  $\nu$  is surjective. And clearly  $d_{B/A}(a) = ad_{B/A}(1) = 0$  for  $a \in A$ .

Consider the composition  $B \xrightarrow{d_{B/R}} \Omega_{B/R} \rightarrow \Omega_{B/R}/\mathfrak{J}u$ . For every  $a \in A, b \in B$ , we have

$$[d_{B/R}(ab)] = [bd_{B/R}(a) + ad_{B/R}(b)] = [bd_{B/R}(a)] + [ad_{B/R}(b)] = [ad_{B/A}(b)].$$

Hence it is indeed an  $A$ -derivation of  $B$ . Then it induces a  $B$ -linear map

$$\varphi : \Omega_{B/A} \rightarrow \Omega_{B/R}/\mathfrak{J}u, \quad d_{B/A}(b) \mapsto [d_{B/R}(b)].$$

The map  $\varphi$  is surjective since  $\Omega_{B/R}$  is generated by elements of the form  $d_{B/R}(b)$  for  $b \in B$ . Note that the composition

$$\Omega_{B/A} \xrightarrow{\varphi} \Omega_{B/R}/\mathfrak{J}u \rightarrow \Omega_{B/A}/\text{Ker } \nu$$

is the identity map. Thus,  $\varphi$  is injective and hence an isomorphism. In particular, we have  $\text{Ker } \nu = \mathfrak{J}u$ .  $\square$

**Remark 8.** The exact sequence in Theorem 7 is left exact if and only if every  $R$ -derivation of  $A$  into  $B$ -module extends to an  $R$ -derivation of  $B$  into  $B$ -module.

To be completed.

**Theorem 9.** Let  $A$  be an  $R$ -algebra and  $I$  an ideal of  $A$ . Set  $B := A/I$ . Then there is a natural short exact sequence

$$I/I^2 \rightarrow \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow 0$$

of  $B$ -modules.

*Proof.* Suppose  $A = F/\mathfrak{b}$  for some free  $R$ -algebra  $F$  and an ideal  $\mathfrak{b}$  of  $F$ . Let  $\mathfrak{a}$  be the preimage of  $I$  in  $F$ . Let  $d\mathfrak{b}$  (resp.  $d\mathfrak{a}$ ) denote the image of  $\mathfrak{b}$  (resp.  $\mathfrak{a}$ ) in  $\Omega_{F/R}$ . Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B / (d\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B / (d\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \rightarrow (d\mathfrak{a} \otimes_F B) / (d\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows.  $\square$

**Definition 10.** Let  $\mathbf{k}$  be a field and  $A$  an integral  $\mathbf{k}$ -algebra of finite type of dimension  $n$ . We say  $A$  is *smooth at*  $\mathfrak{p} \in \text{Spec } A$  if the module of differentials  $\Omega_{A,\mathfrak{p}}$  is a free  $A_{\mathfrak{p}}$ -module of rank  $n$ .

**Example 11.** Let  $\mathbf{K}/\mathbf{k}$  be a finite generated field extension and  $\mathbf{k}'$  be the algebraic closure of  $\mathbf{k}$  in  $\mathbf{K}$ . Then

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \text{trdeg}(\mathbf{K}/\mathbf{k}) + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}},$$

and  $\dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}} = 0$  if and only if  $\mathbf{k}'$  is separable over  $\mathbf{k}$ .

First suppose  $\mathbf{K} = \mathbf{k}'$  is algebraic over  $\mathbf{k}$ . Suppose  $\mathbf{k}'/\mathbf{k}$  is separable. For every  $\alpha \in \mathbf{k}'$ , suppose  $f(\alpha) = 0$  for  $f \in \mathbf{k}[T]$ . Then  $df(\alpha) = f'(\alpha)d\alpha = 0$ . By the separability of  $\mathbf{k}'/\mathbf{k}$ , we have  $f'(\alpha) \neq 0$ . It follows that  $d\alpha = 0$ . Conversely, let  $\alpha \in \mathbf{k}'$  be an inseparable element over  $\mathbf{k}$ . Since  $\mathbf{k}[\alpha] \rightarrow \mathbf{k}[\alpha], \alpha^n \mapsto n\alpha^{n-1}$  is a non-zero  $R$ -derivation, we have  $\Omega_{\mathbf{k}[\alpha]/\mathbf{k}} \neq 0$ . By induction on number of generated elements, choosing a middle field  $\mathbf{k} \subset \mathbf{k}'' \subset \mathbf{k}'$ , at least one of  $\Omega_{\mathbf{k}''/\mathbf{k}}$  and  $\Omega_{\mathbf{k}'/\mathbf{k}''}$  is non-zero. Then  $\Omega_{\mathbf{K}/\mathbf{k}} \neq 0$  by Theorem 7.

Then suppose  $\mathbf{k}' = \mathbf{k}$ . By the Noether's Normalization Lemma, we can find a finite set of elements  $T_1, \dots, T_n \in \mathbf{K}$  such that  $\mathbf{K}$  is algebraic over  $\mathbf{k}'(T_1, \dots, T_n)$ . Note that we can choose  $T_i$  such that  $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$  is separable. To see this, if  $\alpha \in \mathbf{K}$  is an inseparable element over  $\mathbf{k}'(T_1, \dots, T_n)$ , then by replacing a suitable  $T_i$  with  $\alpha$ , we reduce the inseparable degree of  $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$ .

Since  $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$  is finite, every  $\mathbf{k}$ -derivation of  $\mathbf{k}'(T_1, \dots, T_n)$  into  $\mathbf{K}$ -module extends to a  $\mathbf{k}$ -derivation of  $\mathbf{K}$  into  $\mathbf{K}$ -module. Then by Remark 8, we have

$$0 \rightarrow \Omega_{\mathbf{k}'(T_1, \dots, T_n)/\mathbf{k}} \otimes_{\mathbf{k}'(T_1, \dots, T_n)} \mathbf{K} \rightarrow \Omega_{\mathbf{K}/\mathbf{k}} \rightarrow \Omega_{\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)} \rightarrow 0.$$

Finally, note that every  $\mathbf{k}$ -derivation  $\partial$  of  $\mathbf{k}'$  into  $\mathbf{K}$ -module can be extended to  $\mathbf{k}'[T_1, \dots, T_n]$  by setting  $\partial T_i = 0$ . Thus, we have

$$0 \rightarrow \Omega_{\mathbf{k}'/\mathbf{k}} \otimes_{\mathbf{k}'} \mathbf{k}'[T_1, \dots, T_n] \rightarrow \Omega_{\mathbf{k}'[T_1, \dots, T_n]/\mathbf{k}} \rightarrow \Omega_{\mathbf{k}'[T_1, \dots, T_n]/\mathbf{k}'} \rightarrow 0.$$

This follows that

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}'} + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}}.$$

## 2 Applications to affine varieties

Let  $\mathbf{k}$  be arbitrary field,  $A = \mathbf{k}[T_1, \dots, T_n]$  and  $\mathfrak{m}$  a maximal ideal of  $A$  such that  $\kappa(\mathfrak{m})$  is separable over  $\mathbf{k}$ . We try to give an explanation of Zariski's tangent space at  $\mathfrak{m}$  using the language of derivation. We know that  $\Omega_{A/\mathbf{k}} = \bigoplus_{i=1}^n A dT_i$ , thus  $\Omega_{A_{\mathfrak{m}}/\mathbf{k}} \cong \bigoplus_{i=1}^n A_{\mathfrak{m}} dT_i$ . Then

$$\text{Der}_{\mathbf{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}}) \cong \text{Hom}_{\mathbf{k}}(\Omega_{A_{\mathfrak{m}}/\mathbf{k}}, A_{\mathfrak{m}}) \cong \bigoplus_{i=1}^n A_{\mathfrak{m}} \partial_i,$$

where  $\partial_i \in \text{Der}_{\mathbf{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$  is the derivation defined by  $dT_i \mapsto 1$  and  $dT_j \mapsto 0$  for  $j \neq i$ . It coincides with the usual derivation  $f \mapsto \partial f / \partial T_i$ . Consider the restriction of  $\partial_i$  to  $\mathfrak{m}$  and take values in the residue field  $\kappa(\mathfrak{m})$ , we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1, \dots, \partial_n)^T} A_{\mathfrak{m}}^n \rightarrow \kappa(\mathfrak{m})^n.$$

Since  $\kappa(\mathfrak{m})$  is separable over  $\mathbf{k}$ , we claim that  $\text{Ker } \Phi = \mathfrak{m}^2$ . Indeed, by Remark 12, we can write every  $f \in \mathfrak{m} \setminus \mathfrak{m}^2$  as  $\sum_i a_i g_i$ . Then

$$\frac{\partial f}{\partial T_i} = a_i \frac{\partial g_i}{\partial T_i} + g_i \frac{\partial a_i}{\partial T_i}.$$

Since  $g_i$  is separable, the image of  $\partial g_i / \partial T_i$  in  $\kappa(\mathfrak{m})$  is not zero. Hence  $\Phi(f) \neq 0$ . By the claim,  $\Phi$  induces an isomorphism  $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$  of  $\kappa(\mathfrak{m})$ -vector spaces. Then we get

$$T_{A,\mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where  $x \in \mathbf{a}_{\mathbf{k}}^n$  is the point corresponding to  $\mathfrak{m}$ . This coincides with the usual tangent space at  $x$  in language of differential geometry.

**Remark 12.** Let  $\mathbf{k}$  be arbitrary field,  $A = \mathbf{k}[T_1, \dots, T_n]$  and  $g_i$  irreducible polynomials in one variable  $T_i$  over  $\mathbf{k}$ . Then for every  $f \in A$ , we can write

$$f = \sum_{I=(i_1, \dots, i_n) \in \mathbf{z}_{\geq 0}^n} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \leq \deg g_i.$$

This is called the *Taylor expansion of  $f$  with respect to  $g_1, \dots, g_n$* .

When  $n = 1$ , it follows from division algorithm. For  $n > 1$ , we can use induction on  $n$ . Let  $\mathbf{K} = \mathbf{k}(T_1, \dots, T_{n-1})$ . Then we can write  $f$  as

$$f = \sum_{i=0}^r a_i g_n^i, \quad a_i \in \mathbf{K}[T_n], \quad \deg a_i < \deg g_n.$$

Comparing the coefficients of two sides from the highest degree of  $T_n$  to the lowest degree, we see that

$$a_i \in \mathbf{k}[T_1, \dots, T_{n-1}].$$

By induction hypothesis, the conclusion follows.

Let  $B = A/I$  be a  $\mathbf{k}$  of finite type,  $I = (F_1, \dots, F_m) \subset \mathfrak{m}$  and  $\mathfrak{n}$  the image of  $\mathfrak{m}$  in  $B$ . We have an exact sequence of  $\kappa(\mathfrak{m})$ -vector spaces

$$0 \rightarrow I/(I \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} : \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of  $F_1, \dots, F_m$  is the  $m \times n$  matrix

$$J(F_1, \dots, F_m) := \left( \frac{\partial F_i}{\partial T_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

with entries in  $B$ .

**Theorem 13.** Setting as above. Then  $B$  is regular at  $\mathfrak{n}$  if and only if the Jacobian matrix  $J$  has maximal rank  $n - \dim B_{\mathfrak{n}}$  after taking values in the residue field  $\kappa(\mathfrak{m})$ .

*Proof.* We have an exact sequence

$$0 \rightarrow T_{B,\mathfrak{n}} \rightarrow T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa^m \rightarrow 0,$$

where  $\Psi$  sends  $\partial \in T_{A,\mathfrak{m}}$  to  $(\partial(F_1), \dots, \partial(F_m))^T$ . Note that the matrix of  $\Psi$  is just  $J^T$ , the transpose of the Jacobian matrix. Hence

$$\text{rank } J = n - \dim_{\kappa} T_{B,\mathfrak{n}} \leq n - \dim B_{\mathfrak{n}}$$

| and the equality holds if and only if  $B$  is regular at  $\mathfrak{n}$ . □

**Remark 14.** If  $\kappa(\mathfrak{m})$  is not separable over  $\mathbf{k}$ , then we still have the inequality

$$\text{rank } J \leq n - \dim B_{\mathfrak{n}}.$$

Indeed, in any case, we have an exact sequence

$$0 \rightarrow I/(I \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0.$$

Hence  $\dim_{\kappa} I/(I \cap \mathfrak{m}^2) = n - \dim B_{\mathfrak{n}}$ . There is a  $\kappa(\mathfrak{m})$ -linear map

$$I/(I \cap \mathfrak{m}^2) \rightarrow \kappa(\mathfrak{m})^n, \quad [f] \mapsto (\partial_1(f), \dots, \partial_n(f))^T,$$

and every row of the Jacobian matrix  $J$  is in the image of this map. Thus, the rank of  $J$  is at most  $n - \dim B_{\mathfrak{n}}$ .

Hence if  $\text{rank } J = n - \dim B_{\mathfrak{n}}$ , we can still see that  $B$  is regular at  $\mathfrak{n}$ . However, the converse does not hold in general.

**Proposition 15.** Let  $\mathbf{k}$  be a field,  $\mathbb{k}$  the algebraic closure of  $\mathbf{k}$ ,  $A$  a  $\mathbf{k}$ -algebra of finite type and  $A_{\mathbb{k}} := A \otimes_{\mathbf{k}} \mathbb{k}$ . Suppose  $A_{\mathbb{k}}$  is integral. Let  $\mathfrak{m} \in \text{mSpec } A$  and  $\mathfrak{m}'$  be a maximal ideal of  $A_{\mathbb{k}}$  lying over  $\mathfrak{m}$ . Then

- (a) If  $A_{\mathbb{k}}$  is regular at  $\mathfrak{m}'$ , then  $A$  is regular at  $\mathfrak{m}$ ;
- (b) suppose  $\kappa(\mathfrak{m})$  is separable over  $\mathbf{k}$ , the converse holds.

*Proof.* Regarding  $J_{\mathfrak{m}}$  and  $J_{\mathfrak{m}'}$  as matrices with entries in  $\mathbb{k}$ , they are the same and hence have the same rank. If  $A_{\mathbb{k}}$  is regular at  $\mathfrak{m}'$ , since  $\kappa(\mathfrak{m}) = \mathbb{k}$ , then  $\text{rank } J_{\mathfrak{m}'} = n - \dim A_{\mathbb{k}, \mathfrak{m}'}$ . Note that  $\dim A_{\mathbb{k}, \mathfrak{m}'} = \text{trdeg}(\mathcal{K}(A_{\mathbb{k}})/\mathbb{k}) = \text{trdeg}(\mathcal{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}}$ , we have  $\text{rank } J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$ . Hence  $A$  is regular at  $\mathfrak{m}$ .

Conversely, suppose  $A$  is regular at  $\mathfrak{m}$  and  $\kappa(\mathfrak{m})$  is separable over  $\mathbf{k}$ . Then  $\text{rank } J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$ . Hence  $A_{\mathbb{k}}$  is regular at  $\mathfrak{m}'$ . To be modified. □

**Proposition 16.** Let  $\mathbf{k}$  be a field and  $A$  an integral  $\mathbf{k}$ -algebra of finite type and of dimension  $n$ . Let  $\mathbb{k}$  be the algebraic closure of  $\mathbf{k}$  and  $A_{\mathbb{k}} := A \otimes_{\mathbf{k}} \mathbb{k}$ . Then  $A$  is smooth at  $\mathfrak{p} \in \text{Spec } A$  if and only if  $A_{\mathbb{k}}$  is regular at every  $\mathfrak{m}'$  over  $\mathfrak{p}$ .

*Proof.* Since  $\Omega_{A_{\mathbb{k}}/\mathbb{k}} \cong \Omega_{A/\mathbf{k}} \otimes_A A_{\mathbb{k}}$  is free of rank  $n$  if and only if  $\Omega_{A/\mathbf{k}}$  is free of rank  $n$ , we can assume that  $\mathbf{k} = \mathbb{k}$ . If  $A$  is smooth at  $\mathfrak{p}$ , then  $\Omega_{A_{\mathfrak{p}}/\mathbb{k}} \cong \bigoplus A_{\mathfrak{p}} df_i$  is free of rank  $n$ . Let  $\mathfrak{P}_i \in \text{Der}_{\mathbb{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$  be the derivation defined by  $df_i \mapsto 1$  and  $dT_j \mapsto 0$  for  $j \neq i$ . Then we have  $\partial_i f_j = \delta_{ij}$  for  $1 \leq i, j \leq n$ . Then similar to above argument, we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{(\partial_1, \dots, \partial_n)^T} \mathbb{k}^n.$$

This shows that  $A_{\mathbb{k}}$  is regular at  $\mathfrak{m}$ .

Conversely, suppose  $A_{\mathbb{k}}$  is regular at  $\mathfrak{m}$ . Note that  $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \Omega_{A_{\mathbb{k}}/\mathbb{k}} \otimes_{\mathbb{k}} \mathbb{k}$  is surjective since  $\Omega_{A_{\mathbb{k}}/\mathbb{k}} = 0$ . Then by Nakayama's lemma,  $\Omega_{A_{\mathfrak{m}}/\mathbb{k}}$  is generated by  $n$  elements as an  $A_{\mathfrak{m}}$ -module.

Note that  $\dim_{\mathcal{K}(A)} \Omega_{\mathcal{K}(A)/\mathbf{k}} = \text{trdeg}(\mathcal{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}} = n$ . By induction on transcendental

degree.

By Nakayama's Lemma,  $\Omega_{A_m/\mathbf{k}}$  is free of rank  $n$  as an  $A_m$ -module.

To be completed. □

**Example 17.** Let  $\mathbf{k}$  be an imperfect field of characteristic  $p > 2$ . Suppose  $\alpha = \beta^p \in \mathbf{k}$  and  $\beta$  is not in  $\mathbf{k}$ . Let  $A = \mathbf{k}[x, y]/(x^2 - y^p - \alpha)$  and  $\mathfrak{m} = (x, y^p - \alpha) = (x)$ . Note that  $\mathfrak{m}$  is principal, so  $A$  is regular at  $\mathfrak{m}$ . However,

$$J_{\mathfrak{m}} = \left( \frac{\partial}{\partial x}(x^2 - y^p - \alpha), \frac{\partial}{\partial y}(y^p - \alpha) \right) = (2x, 0) = (0, 0) \in M_{1 \times 2}(\kappa(\mathfrak{m})).$$

Thus,  $A$  is not smooth at  $\mathfrak{m}$ . From the view of differentials, we have

$$\Omega_{A_m/\mathbf{k}} = A_m dx \oplus A_m dy / A_m \cdot x dx = \kappa(\mathfrak{m}) dx \oplus A_m dy,$$

which is not free as an  $A_m$ -module.