# Smoothness

## 1 Modules of differentials and derivations

In this subsection, let $R$ be a ring and $A$ an $R$-algebra.

**Definition 1** (Derivation)**.** A *derivation* of $A$ over $R$ is an $R$-linear map $\partial : A \to M$ with an $A$-module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module $M$, the set of all derivations of $A$ over $R$ into $M$ forms an $A$-module, denoted by $\mathrm{Der}_R(A, M)$.

Given a module homomorphism $f : M \to N$ of $A$-modules and a derivation $\partial \in \mathrm{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of $A$ over $R$ into $N$.

**Proposition 2.** The functor $\mathrm{Der}_R(A, -)$ is representable. The representing object is denoted by $\Omega_{A/R}$, which is called the *module of differentials* of $A$ over $R$.

*Proof.* First suppose $A$ is a free $R$-algebra with a set of generators $a_\lambda, \lambda \in \Lambda$. Then an $R$-derivation $\partial \in \mathrm{Der}_R(A, M)$ is uniquely determined by its values on the generators $a_\lambda$. Let

$$\Omega_{A/R} := \bigoplus_{\lambda \in \Lambda} A \cdot \mathrm{d}a_\lambda$$

and $\mathrm{d} : A \to \Omega_{A/R}$ be the $R$-derivation defined by $a_\lambda \mapsto \mathrm{d}a_\lambda$. For any $R$-derivation $\partial \in \mathrm{Der}_R(A, M)$, we can define a unique $A$-module homomorphism $\Phi_\partial : \Omega_{A/R} \to M$ by sending $\mathrm{d}a_\lambda$ to $\partial(a_\lambda)$ such that $\partial = \Phi_\partial \circ \mathrm{d}$. This gives a bijection

$$\mathrm{Der}_R(A, M) \cong \mathrm{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_\partial.$$

Now suppose $A = F/I$ is an arbitrary $R$-algebra, where $F$ is a free $R$-algebra and $I$ is an ideal of $F$. Then we can define the module of differentials

$$\Omega_{A/R} := \left(\Omega_{F/R} \otimes_F A\right) / \sum_{f \in I} A \cdot \mathrm{d}f.$$

The $R$-linear map $\mathrm{d}_A : F \otimes_F A \xrightarrow{\mathrm{d}_F} \Omega_{F/R} \otimes_F A \to \Omega_{A/R}$ is a derivation of $A$ over $R$.

For any $R$-derivation $\partial \in \mathrm{Der}_R(A, M)$, note that $F \to A \xrightarrow{\partial} M$ is an $R$-derivation of $F$ over $R$ into $M$. Then we get an $F$-module homomorphism $\Omega_F \to M$. It gives an $A$-module homomorphism $\Omega_F \otimes_F A \to M, \mathrm{d}f \otimes 1 \mapsto \partial f$. This map factors into $\Omega_F \otimes_F A \to \Omega_{A/R}$ and $\Phi_\partial : \Omega_{A/R} \to M$. Since $\Phi_\partial$ is $A$-linear and $\Omega_{A/R}$ is generated by $\mathrm{d}a_\lambda$ as $A$-module, such $\Phi_\partial$ is unique. $\square$

**Date**: August 12, 2025, **Author**: Tianle Yang, My Website

**Corollary 3.** Suppose $A$ is of finite type over $R$. Then the module of differentials $\Omega_{A/R}$ is a finitely generated $A$-module.

**Remark 4.** Let $B$ be an $A$-algebra, $M$ an $A$-module and $N$ a $B$-module. If there is a homomorphism of $A$-modules $M \to N$, then we can extend it to a homomorphism of $B$-modules $M \otimes_A B \to N$ by sending $m \otimes b$ to $m \cdot b$. And such extension is unique in the sense of following commutative diagram:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N . \\ M \otimes_A B \arrow[ur, "\exists!"'] & \end{tikzcd}$$

Hence we get a natural bijection

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(M \otimes_A B, N).$$

**Proposition 5.** Let $A, R'$ be $R$-algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

*Proof.* We check the universal property of $\Omega_{A/R} \otimes_A A'$. First, the map

$$\mathrm{d}_{A'} : A \otimes_R R' \to \Omega_{A/R} \otimes_R R' \cong \Omega_{A/R} \otimes_A A', \quad a \otimes r \mapsto \mathrm{d}a \otimes r$$

is an $R'$-derivation of $A'$ into $\Omega_{A/R} \otimes_A A'$. For any $R'$-derivation $\partial' : A' \to M$ into an $A'$-module $M$, we can compose it with the homomorphism $A' \to A$ and get an $R$-derivation $\partial : A \to M$. By the universal property of $\Omega_{A/R}$, there is a unique $A$-module homomorphism $\Phi : \Omega_{A/R} \to M$ such that $\partial = \Phi \circ \mathrm{d}_A$. Then we can extend it to an $A'$-module homomorphism $\Phi' : \Omega_{A/R} \otimes_A A' \to M$ by Remark 4. By the construction, we have $\Phi' \circ \mathrm{d}_{A'} = \partial'$. □

**Proposition 6.** Let $A$ be an $R$-algebra and $S$ a multiplicative set of $A$. Then we have an isomorphism

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

*Proof.* Let

$$\mathrm{d}_{S^{-1}A} : S^{-1}A \to S^{-1}\Omega_{A/R}, \quad \frac{a}{s} \mapsto \frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}.$$

By direct computation, $\mathrm{d}_{S^{-1}A}$ is an $R$-derivation of $S^{-1}A$ over $R$ into $S^{-1}\Omega_{A/R}$. For any $R$-derivation $\partial : S^{-1}A \to M$ into an $S^{-1}A$-module $M$, we can get an $S^{-1}A$-module homomorphism $\Phi' : S^{-1}\Omega_{A/R} \to M$ as proof of Proposition 5. We have

$$\partial(s \cdot \frac{a}{s}) = s\partial(\frac{a}{s}) + \frac{a}{s}\partial s.$$

It follows that

$$\partial(\frac{a}{s}) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(\mathrm{d}a) - a\Phi'(\mathrm{d}s)}{s^2} = \Phi'(\frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}).$$

Thus, $\Phi' \circ \mathrm{d}_{S^{-1}A} = \partial$. □

**Theorem 7.** Let $A$ be an $R$-algebra and $B$ an $A$-algebra. Then there is a natural short exact sequence

$$\Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to \Omega_{B/A} \to 0$$

of $B$-modules.

*Proof.* Let $\mathrm{d}_{A/R} : A \to \Omega_{A/R}$ be the $R$-derivation of $A$ over $R$. The map $A \to B \xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R}$ induces a $B$-linear map

$$u : \Omega_{A/R} \otimes_A B \to \Omega_{B/R}, \quad \mathrm{d}_{A/R}(a) \otimes b \mapsto b\mathrm{d}_{B/R}(a).$$

The map $\mathrm{d}_{B/A}$ is an $A$-derivation and hence $R$-derivation. Then it induces a $B$-linear map

$$v : \Omega_{B/R} \to \Omega_{B/A}, \quad \mathrm{d}_{B/R}(b) \mapsto \mathrm{d}_{B/A}(b).$$

Since $\Omega_{B/A}$ is generated by elements of the form $\mathrm{d}_{B/A}(b)$ for $b \in B$, the map $v$ is surjective. And clearly $\mathrm{d}_{B/A}(a) = a\mathrm{d}_{B/A}(1) = 0$ for $a \in A$.

Consider the composition $B \xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R} \to \Omega_{B/R}/\Im u$. For every $a \in A, b \in B$, we have

$$[\mathrm{d}_{B/R}(ab)] = [b\mathrm{d}_{B/R}(a) + a\mathrm{d}_{B/R}(b)] = [b\mathrm{d}_{B/R}(a)] + [a\mathrm{d}_{B/A}(b)] = [a\mathrm{d}_{B/A}(b)].$$

Hence it is indeed an $A$-derivation of $B$. Then it induces a $B$-linear map

$$\varphi : \Omega_{B/A} \to \Omega_{B/R}/\Im u, \quad \mathrm{d}_{B/A}(b) \mapsto [\mathrm{d}_{B/R}(b)].$$

The map $\varphi$ is surjective since $\Omega_{B/R}$ is generated by elements of the form $\mathrm{d}_{B/R}(b)$ for $b \in B$. Note that the composition

$$\Omega_{B/A} \xrightarrow{\varphi} \Omega_{B/R}/\Im u \to \Omega_{B/A}/\operatorname{Ker} v$$

is the identity map. Thus, $\varphi$ is injective and hence an isomorphism. In particular, we have $\operatorname{Ker} v = \Im u$. □

**Remark 8.** The exact sequence in Theorem 7 is left exact if and only if every $R$-derivation of $A$ into $B$-module extends to an $R$-derivation of $B$ into $B$-module.

To be completed.

**Theorem 9.** Let $A$ be an $R$-algebra and $I$ an ideal of $A$. Set $B \coloneqq A/I$. Then there is a natural short exact sequence

$$I/I^2 \to \Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to 0$$

of $B$-modules.

*Proof.* Suppose $A = F/\mathfrak{b}$ for some free $R$-algebra $F$ and an ideal $\mathfrak{b}$ of $F$. Let $\mathfrak{a}$ be the preimage of $I$ in $F$. Let $\mathrm{d}\mathfrak{b}$ (resp. $\mathrm{d}\mathfrak{a}$) denote the image of $\mathfrak{b}$ (resp. $\mathfrak{a}$) in $\Omega_{F/R}$. Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \to (\mathrm{d}\mathfrak{a} \otimes_F B)/(\mathrm{d}\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows. □

**Definition 10.** Let $\mathbf{k}$ be a field and $A$ an integral $\mathbf{k}$-algebra of finite type of dimension $n$. We say $A$ is *smooth at* $\mathfrak{p} \in \operatorname{Spec} A$ if the module of differentials $\Omega_{A,\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank $n$.

**Example 11.** Let $\mathbf{K}/\mathbf{k}$ be a finite generated field extension and $\mathbf{k}'$ be the algebraic closure of $\mathbf{k}$ in $\mathbf{K}$. Then

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \operatorname{trdeg}(\mathbf{K}/\mathbf{k}) + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}},$$

and $\dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}} = 0$ if and only if $\mathbf{k}'$ is separable over $\mathbf{k}$.

First suppose $\mathbf{K} = \mathbf{k}'$ is algebraic over $\mathbf{k}$. Suppose $\mathbf{k}'/\mathbf{k}$ is separable. For every $\alpha \in \mathbf{k}'$, suppose $f(\alpha) = 0$ for $f \in \mathbf{k}[T]$. Then $\mathrm{d}f(\alpha) = f'(\alpha)\mathrm{d}\alpha = 0$. By the separability of $\mathbf{k}'/\mathbf{k}$, we have $f'(\alpha) \neq 0$. It follows that $\mathrm{d}\alpha = 0$. Conversely, let $\alpha \in \mathbf{k}'$ be a inseparable element over $\mathbf{k}$. Since $\mathbf{k}[\alpha] \to \mathbf{k}[\alpha], \alpha^n \mapsto n\alpha^{n-1}$ is a non-zero $R$-derivation, we have $\Omega_{\mathbf{k}[\alpha]/\mathbf{k}} \neq 0$. By induction on number of generated elements, choosing a middle field $\mathbf{k} \subset \mathbf{k}'' \subset \mathbf{k}'$, at least one of $\Omega_{\mathbf{k}''/\mathbf{k}}$ and $\Omega_{\mathbf{k}'/\mathbf{k}''}$ is non-zero. Then $\Omega_{\mathbf{K}/\mathbf{k}} \neq 0$ by Theorem 7.

Then suppose $\mathbf{k}' = \mathbf{k}$. By the Noether's Normalization Lemma, we can find a finite set of elements $T_1, \cdots, T_n \in \mathbf{K}$ such that $\mathbf{K}$ is algebraic over $\mathbf{k}'(T_1, \dots, T_n)$. Note that we can choose $T_i$ such that $\mathbf{K}/\mathbf{k}'(T_1, \cdots, T_n)$ is separable. To see this, if $\alpha \in \mathbf{K}$ is an inseparable element over $\mathbf{k}'(T_1, \cdots, T_n)$, then by replacing a suitable $T_i$ with $\alpha$, we reduce the inseparable degree of $\mathbf{K}/\mathbf{k}'(T_1, \cdots, T_n)$.

Since $\mathbf{K}/\mathbf{k}'(T_1, \cdots, T_n)$ is finite, every $\mathbf{k}$-derivation of $\mathbf{k}'(T_1, \cdots, T_n)$ into $\mathbf{K}$-module extends to a $\mathbf{k}$-derivation of $\mathbf{K}$ into $\mathbf{K}$-module. Then by Remark 8, we have

$$0 \to \Omega_{\mathbf{k}'(T_1, \cdots, T_n)/\mathbf{k}} \otimes_{\mathbf{k}'(T_1, \cdots, T_n)} \mathbf{K} \to \Omega_{\mathbf{K}/\mathbf{k}} \to \Omega_{\mathbf{K}/\mathbf{k}'(T_1, \cdots, T_n)} \to 0.$$

Finally, note that every $\mathbf{k}$-derivation $\partial$ of $\mathbf{k}'$ into $\mathbf{K}$-module can be extended to $\mathbf{k}'[T_1, \cdots, T_n]$ by setting $\partial T_i = 0$. Thus, we have

$$0 \to \Omega_{\mathbf{k}'/\mathbf{k}} \otimes_{\mathbf{k}'} \mathbf{k}'[T_1, \cdots, T_n] \to \Omega_{\mathbf{k}'[T_1, \cdots, T_n]/\mathbf{k}} \to \Omega_{\mathbf{k}'[T_1, \cdots, T_n]/\mathbf{k}'} \to 0.$$

This follows that

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}'} + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}}.$$

## 2 Applications to affine varieties

Let $\mathbf{k}$ be arbitrary field, $A = \mathbf{k}[T_1, \dots, T_n]$ and $\mathfrak{m}$ a maximal ideal of $A$ such that $\kappa(\mathfrak{m})$ is separable over $\mathbf{k}$. We try to give an explanation of Zariski's tangent space at $\mathfrak{m}$ using the language of derivation. We know that $\Omega_{A/\mathbf{k}} = \bigoplus_{i=1}^n A\mathrm{d}T_i$, thus $\Omega_{A_\mathfrak{m}/\mathbf{k}} \cong \bigoplus_{i=1}^n A_\mathfrak{m}\mathrm{d}T_i$. Then

$$\operatorname{Der}_{\mathbf{k}}(A_\mathfrak{m}, A_\mathfrak{m}) \cong \operatorname{Hom}_{\mathbf{k}}(\Omega_{A_\mathfrak{m}/\mathbf{k}}, A_\mathfrak{m}) \cong \bigoplus_{i=1}^n A_\mathfrak{m}\partial_i,$$

where $\partial_i \in \operatorname{Der}_{\mathbf{k}}(A_\mathfrak{m}, A_\mathfrak{m})$ is the derivation defined by $\mathrm{d}T_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. It coincides with the usual derivation $f \mapsto \partial f/\partial T_i$. Consider the restriction of $\partial_i$ to $\mathfrak{m}$ and take values in the residue field $\kappa(\mathfrak{m})$, we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1,\cdots,\partial_n)^T} A_{\mathfrak{m}}^n \to \kappa(\mathfrak{m})^n.$$

Since $\kappa(\mathfrak{m})$ is separable over $\mathbf{k}$, we claim that $\operatorname{Ker}\Phi = \mathfrak{m}^2$. Indeed, by Remark 12, we can write every $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ as $\sum_i a_i g_i$. Then

$$\frac{\partial f}{\partial T_i} = a_i \frac{\partial g_i}{\partial T_i} + g_i \frac{\partial a_i}{\partial T_i}.$$

Since $g_i$ is separable, the image of $\partial g_i / \partial T_i$ in $\kappa(\mathfrak{m})$ is not zero. Hence $\Phi(f) \neq 0$. By the claim, $\Phi$ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$ of $\kappa(\mathfrak{m})$-vector spaces. Then we get

$$T_{A,\mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where $x \in \mathbb{A}_{\mathbf{k}}^n$ is the point corresponding to $\mathfrak{m}$. This coincides with the usual tangent space at $x$ in language of differential geometry.

**Remark 12.** Let $\mathbf{k}$ be arbitrary field, $A = \mathbf{k}[T_1, \cdots, T_n]$ and $g_i$ irreducible polynomials in one variable $T_i$ over $\mathbf{k}$. Then for every $f \in A$, we can write

$$f = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{Z}_{\geq 0}^n} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \leq \deg g_i.$$

This is called the *Taylor expansion of $f$ with respect to* $g_1, \cdots, g_n$.

When $n = 1$, it follows from division algorithm. For $n > 1$, we can use induction on $n$. Let $\mathbf{K} = \mathbf{k}(T_1, \cdots, T_{n-1})$. Then we can write $f$ as

$$f = \sum_{i=0}^r a_i g_n^i, \quad a_i \in \mathbf{K}[T_n], \quad \deg a_i < \deg g_n.$$

Comparing the coefficients of two sides from the highest degree of $T_n$ to the lowest degree, we see that

$$a_i \in \mathbf{k}[T_1, \cdots, T_{n-1}].$$

By induction hypothesis, the conclusion follows.

Let $B = A/I$ be a $\mathbf{k}$ of finite type, $I = (F_1, \ldots, F_m) \subset \mathfrak{m}$ and $\mathfrak{n}$ the image of $\mathfrak{m}$ in $B$. We have an exact sequence of $\kappa(\mathfrak{m})$-vector spaces

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} : \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of $F_1, \ldots, F_m$ is the $m \times n$ matrix

$$J(F_1, \ldots, F_m) := \left(\frac{\partial F_i}{\partial T_j}\right)_{1\leq i\leq m, 1\leq j\leq n}$$

with entries in $B$.

**Theorem 13.** Setting as above. Then $B$ is regular at $\mathfrak{n}$ if and only if the Jacobian matrix $J$ has maximal rank $n - \dim B_{\mathfrak{n}}$ after taking values in the residue field $\kappa(\mathfrak{m})$.

*Proof.* We have an exact sequence

$$0 \to T_{B,\mathfrak{n}} \to T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa^m \to 0,$$

where $\Psi$ sends $\partial \in T_{A,\mathfrak{m}}$ to $(\partial(F_1), \dots, \partial(F_m))^T$. Note that the matrix of $\Psi$ is just $J^T$, the transpose of the Jacobian matrix. Hence

$$\operatorname{rank} J = n - \dim_\kappa T_{B,\mathfrak{n}} \le n - \dim B_{\mathfrak{n}}$$

and the equality holds if and only if $B$ is regular at $\mathfrak{n}$. □

**Remark 14.** If $\kappa(\mathfrak{m})$ is not separable over $\mathbf{k}$, then we still have the inequality

$$\operatorname{rank} J \le n - \dim B_{\mathfrak{n}}.$$

Indeed, in any case, we have an exact sequence

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Hence $\dim_\kappa I/(I \cap \mathfrak{m}^2) = n - \dim B_{\mathfrak{n}}$. There is a $\kappa(\mathfrak{m})$-linear map

$$I/(I \cap \mathfrak{m}^2) \to \kappa(\mathfrak{m})^n, \quad [f] \mapsto (\partial_1(f), \dots, \partial_n(f))^T,$$

and every row of the Jacobian matrix $J$ is in the image of this map. Thus, the rank of $J$ is at most $n - \dim B_{\mathfrak{n}}$.

Hence if $\operatorname{rank} J = n - \dim B_{\mathfrak{n}}$, we can still see that $B$ is regular at $\mathfrak{n}$. However, the converse does not hold in general.

**Proposition 15.** Let $\mathbf{k}$ be a field, $\Bbbk$ the algebraic closure of $\mathbf{k}$, $A$ a $\mathbf{k}$-algebra of finite type and $A_{\Bbbk} \coloneqq A \otimes_{\mathbf{k}} \mathbf{k}$. Suppose $A_{\Bbbk}$ is integral. Let $\mathfrak{m} \in \operatorname{mSpec} A$ and $\mathfrak{m}'$ be a maximal ideal of $A_{\Bbbk}$ lying over $\mathfrak{m}$. Then

(a) If $A_{\Bbbk}$ is regular at $\mathfrak{m}'$, then $A$ is regular at $\mathfrak{m}$;

(b) suppose $\kappa(\mathfrak{m})$ is separable over $\mathbf{k}$, the converse holds.

*Proof.* Regarding $J_{\mathfrak{m}}$ and $J_{\mathfrak{m}'}$ as matrices with entries in $\Bbbk$, they are the same and hence have the same rank. If $A_{\Bbbk}$ is regular at $\mathfrak{m}'$, since $\kappa(\mathfrak{m}) = \Bbbk$, then $\operatorname{rank} J_{\mathfrak{m}'} = n - \dim A_{\Bbbk,\mathfrak{m}'}$. Note that $\dim A_{\Bbbk,\mathfrak{m}'} = \operatorname{trdeg}(\mathscr{K}(A_{\Bbbk})/\Bbbk) = \operatorname{trdeg}(\mathscr{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}}$, we have $\operatorname{rank} J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence $A$ is regular at $\mathfrak{m}$.

Conversely, suppose $A$ is regular at $\mathfrak{m}$ and $\kappa(\mathfrak{m})$ is separable over $\mathbf{k}$. Then $\operatorname{rank} J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence $A_{\Bbbk}$ is regular at $\mathfrak{m}'$. To be modified. □

**Proposition 16.** Let $\mathbf{k}$ be a field and $A$ an integral $\mathbf{k}$-algebra of finite type and of dimension $n$. Let $\mathbb{k}$ be the algebraic closure of $\mathbf{k}$ and $A_{\mathbb{k}} \coloneqq A \otimes_{\mathbf{k}} \mathbb{k}$. Then $A$ is smooth at $\mathfrak{p} \in \operatorname{Spec} A$ if and only if $A_{\mathbb{k}}$ is regular at every $\mathfrak{m}'$ over $\mathfrak{m}$.

*Proof.* Since $\Omega_{A_{\mathbb{k}}/\mathbb{k}} \cong \Omega_{A/\mathbf{k}} \otimes_A A_{\mathbb{k}}$ is free of rank $n$ if and only if $\Omega_{A/\mathbf{k}}$ is free of rank $n$, we can assume that $\mathbf{k} = \mathbb{k}$. If $A$ is smooth at $\mathfrak{p}$, then $\Omega_{A_{\mathfrak{p}}/\mathbf{k}} \cong \bigoplus A_{\mathfrak{p}} \mathrm{d}f_i$ is free of rank $n$. Let $\mathfrak{P}_i \in \operatorname{Der}_{\mathbb{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ be the derivation defined by $\mathrm{d}f_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. Then we have $\partial_i f_j = \delta_{ij}$ for $1 \leq i, j \leq n$. Then similar to above argument, we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{(\partial_1, \dots, \partial_n)^T} \mathbb{k}^n.$$

This shows that $A_{\mathbb{k}}$ is regular at $\mathfrak{m}$.

Conversely, suppose $A_{\mathbb{k}}$ is regular at $\mathfrak{m}$. Note that $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A,\mathbb{k}} \otimes_A \mathbb{k}$ is surjective since $\Omega_{A_{\mathbb{k}}/\mathbb{k}} = 0$. Then by Nakayama's lemma, $\Omega_{A_{\mathfrak{m}}/\mathbb{k}}$ is generated by $n$ elements as an $A_{\mathfrak{m}}$-module.

Note that $\dim_{\mathscr{K}(A)} \Omega_{\mathscr{K}(A)/\mathbf{k}} = \operatorname{trdeg}(\mathscr{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}} = n$. By induction on transcendental degree.

By Nakayama's Lemma, $\Omega_{A_{\mathfrak{m}}/\mathbf{k}}$ is free of rank $n$ as an $A_{\mathfrak{m}}$-module.

To be completed. □

**Example 17.** Let $\mathbf{k}$ be an imperfect field of characteristic $p > 2$. Suppose $\alpha = \beta^p \in \mathbf{k}$ and $\beta$ is not in $\mathbf{k}$. Let $A = \mathbf{k}[x, y]/(x^2 - y^p - \alpha)$ and $\mathfrak{m} = (x, y^p - \alpha) = (x)$. Note that $\mathfrak{m}$ is principal, so $A$ is regular at $\mathfrak{m}$. However,

$$J_{\mathfrak{m}} = \left( \frac{\partial}{\partial x}(x^2 - y^p - \alpha), \frac{\partial}{\partial y}(y^p - \alpha) \right) = (2x, 0) = (0, 0) \in M_{1\times 2}(\kappa(\mathfrak{m})).$$

Thus, $A$ is not smooth at $\mathfrak{m}$. From the view of differentials, we have

$$\Omega_{A_{\mathfrak{m}}/\mathbf{k}} = A_{\mathfrak{m}}\mathrm{d}x \oplus A_{\mathfrak{m}}\mathrm{d}y / A_{\mathfrak{m}} \cdot x\mathrm{d}x = \kappa(\mathfrak{m})\mathrm{d}x \oplus A_{\mathfrak{m}}\mathrm{d}y,$$

which is not free as an $A_{\mathfrak{m}}$-module.