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# Contents



# 1 Valuations

Let $\mathbf{k}$ be a field. Usually, we consider $\mathbf{k}$ to be a number field or a function field.

## 1.1 Definition

**Definition 1.1.** An *absolute value* or a *valuation* on $\mathbf{k}$ is a function $|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

- $|x| = 0$ if and only if $x = 0$;
- $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbf{k}$;
- $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbf{k}$.

**Remark 1.2.** Recall that a *additive valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend $v$ to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then $v$ induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation $v$ is called an *valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the multiplicative valuation (i.e., absolute value).

**Example 1.3.** Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

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**Definition 1.4.** An absolute value $|\cdot|$ on a field $\mathbf{k}$ is called *non-archimedean* if it satisfies the strong triangle inequality

$$|x+y| \leq \max\{|x|,|y|\} \quad \text{for all } x,y \in \mathbf{k}.$$

Otherwise, it is called *archimedean*.

**Proposition 1.5.**

**Notation 1.6.** Let $\mathbf{k}$ be a field. We denote by $M_{\mathbf{k}}$ the set of all absolute values (i.e., valuations) on $\mathbf{k}$. For each $v \in M_{\mathbf{k}}$, we also call $v$ a *place* of $\mathbf{k}$.

## 1.2 Non-archimedean place

## 1.3 Number field case

In this section, let $\mathbf{k}$ be a number field.

**Theorem 1.7.** Let $\mathbf{k}$ be a number field. Then

$$M_{\mathbf{k}}^{\infty} = \{\text{embeddings } \sigma : \mathbf{k} \to \mathbb{C}\}$$

and

$$M_{\mathbf{k}}^{\mathrm{f}} = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_{\mathbf{k}}\}.$$

Yang: To be revised.

**Proposition 1.8** (Product formula)**.** Let $\mathbf{k}$ be a number field. For each $x \in \mathbf{k}^{\times}$, we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^{\infty}; \\ 1, & v \in M_{\mathbf{k}}^{0}. \end{cases}$$

Yang: To be revised.

# 2 Finite field extensions

**Theorem 2.1.** Let $L/K$ be a finite field extension, and $v \in M_K$ an absolute value on $K$. Then we have

$$\sum_{w|v} e(w/v) f(w/v) = [L : K].$$

**Definition 2.2.** Yang: To be added.

# 3 Hilbert's irreducibility theorem

The main reference for this section is [Ser08], which is an excellent book focusing on the inverse Galois problem.

**Theorem 3.1** (Hilbert's irreducibility theorem)**.** Let $f(x,t) \in \mathbb{Q}[x,t]$ be an irreducible polynomial in two variables with rational coefficients. Then there exist infinitely many rational numbers $t_0 \in \mathbb{Q}$ such that the specialized polynomial $f(x,t_0) \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$. Yang: To be checked.

## 3.1 Application to inverse Galois problem

# References

[Ser08] Jean-Pierre Serre. *Topics in Galois Theory*. Second. Vol. 1. Research Notes in Mathematics. With notes by Henri Darmon. Wellesley, MA: A K Peters, Ltd., 2008. ISBN: 978-1568814124 (cit. on p. 3).