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# *Arithmetic Geometry*

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# Arithmetic Geometry

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*Source code: GitHub Repository*

*Version: 0.1.0*

*Last updated: January 16, 2026*

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# Chapter 1

## Valuation on fields

### 1.1 Valuation fields

#### 1.1.1 Absolute values and completion

**Definition 1.1.1.** Let  $\mathbf{k}$  be a field. An *absolute value* on  $\mathbf{k}$  is a function  $\|\cdot\| : \mathbf{k} \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following properties for all  $x, y \in \mathbf{k}$ :

- (a)  $\|x\| = 0$  if and only if  $x = 0$ ;
- (b)  $\|xy\| = \|x\| \cdot \|y\|$ ;
- (c)  $\|x + y\| \leq \|x\| + \|y\|$ .

A field  $\mathbf{k}$  equipped with an absolute value  $\|\cdot\|$  is called a *valuation field*.

**Remark 1.1.2.** Let  $\mathbf{k}$  be a field. Recall that a (additive) valuation on  $\mathbf{k}$  is a function  $v : \mathbf{k}^\times \rightarrow \mathbb{R}$  such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$ ;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$ .

We can extend  $v$  to the whole field  $\mathbf{k}$  by defining  $v(0) = +\infty$ . Fix a real number  $\varepsilon \in (0, 1)$ . Then  $v$  induces an absolute value  $|\cdot|_v : \mathbf{k} \rightarrow \mathbb{R}_+$  defined by  $|x|_v = \varepsilon^{v(x)}$  for each  $x \in \mathbf{k}$ .

The valuation  $v$  defined above is called an *additive valuation*. And an absolute value  $|\cdot|$  on  $\mathbf{k}$  is called a *multiplicative valuation*. In this note, the term *valuation* may refer to either an additive valuation or a multiplicative valuation, depending on the context.

**Example 1.1.3.** Let  $\mathbf{k}$  be a field. The *trivial absolute value* on  $\mathbf{k}$  is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

**Definition 1.1.4.** The (*multiplicative*) *valuation group* of a valuation field  $(\mathbf{k}, \|\cdot\|)$  is defined as the subgroup of  $\mathbb{R}_{>0}$  given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation  $\sqrt{|\mathbf{k}^\times|}$  to denote the set  $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$ .

Note that an absolute value  $\|\cdot\|$  is non-trivial if and only if its valuation group  $|\mathbf{k}^\times|$  is not equal to  $\{1\}$ .

**Definition 1.1.5.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field. We say that  $\mathbf{k}$  is *complete* if the metric  $d(x, y) := \|x - y\|$  makes  $\mathbf{k}$  a complete metric space.

**Lemma 1.1.6.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field and  $(\widehat{\mathbf{k}}, \|\cdot\|)$  its completion as a metric space. Then the operations of addition and multiplication on  $\mathbf{k}$  can be extended to  $\widehat{\mathbf{k}}$  uniquely, making  $(\widehat{\mathbf{k}}, \|\cdot\|)$  a complete valuation field containing  $\mathbf{k}$  as a dense subfield.

*Proof.* Simple analysis. □

**Proposition 1.1.7.** Let  $(\mathbf{k}, \|\cdot\|)$  be a complete valuation field with non-trivial absolute value. Then  $\mathbf{k}$  is uncountable.

*Proof.* Since the absolute value  $\|\cdot\|$  is non-trivial, we can construct a sequence  $\{x_n\}_{n=1}^\infty \subseteq \mathbf{k}$  inductively such that  $\|x_n\| < \|x_{n-1}\|/2$  for any  $n \geq 1$  and  $\|x_0\| < 1$ . Then there is an injective map from  $\mathbb{N}^{\{0,1\}}$  to  $\mathbf{k}$  defined by

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n x_n, \quad a_n \in \{0, 1\}.$$

Since  $\|x_n\| < 2^{-n}$ , the series  $\sum_{n=1}^\infty a_n x_n$  converges in  $\mathbf{k}$ . Note  $\|x_n\| > \|\sum_{m \geq n} x_m\|$  for each  $n$ , we have that the map is injective. Thus  $\mathbf{k}$  is uncountable. □

Unlike the real number field  $\mathbb{R}$ , even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

**Definition 1.1.8.** A valuation field  $(\mathbf{k}, \|\cdot\|)$  is called *spherically complete* if every decreasing sequence of closed balls in  $\mathbf{k}$  has a non-empty intersection.

**Example 1.1.9.** The field  $\mathbb{C}_p$  of  $p$ -adic complex numbers is not spherically complete, see [Yang: to be added](#).

**Example 1.1.10.** Let  $|\cdot|_\infty$  be the usual absolute value on the field  $\mathbb{Q}$  of rational numbers. Then  $(\mathbb{Q}, |\cdot|_\infty)$  is a valuation field. Its completion is the field  $\mathbb{R}$  of real numbers equipped with the usual absolute value.

**Example 1.1.11.** Let  $p$  be a prime number. For any non-zero rational number  $x \in \mathbb{Q}$ , we can write it as  $x = p^n \frac{a}{b}$ , where  $n \in \mathbb{Z}$  and  $a, b \in \mathbb{Z}$  are integers not divisible by  $p$ . The  *$p$ -adic absolute value* on  $\mathbb{Q}$  is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then  $(\mathbb{Q}, |\cdot|_p)$  is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of  $p$ -adic numbers equipped with the  $p$ -adic absolute value; see [Yang: to be added.](#)

**Definition 1.1.12.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field. We say that  $\mathbf{k}$  is *non-archimedean* if its absolute value  $\|\cdot\|$  satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that  $\mathbf{k}$  is *archimedean*.

**Proposition 1.1.13.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field. Then  $\mathbf{k}$  is archimedean if and only if the set  $\{\|n \cdot 1\| : n \in \mathbb{Z}\}$  is unbounded.

*Proof.* [Yang: To be added.](#) □

### 1.1.2 Places on a field

**Definition 1.1.14.** Let  $\mathbf{k}$  be a field. Two absolute values  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $\mathbf{k}$  are said to be *equivalent* if there exists a real number  $c \in (0, \infty)$  such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field  $\mathbf{k}$ . Moreover, the following lemma shows that the converse is also true.

**Lemma 1.1.15.** Let  $\mathbf{k}$  be a field and  $\|\cdot\|_1, \|\cdot\|_2$  be two absolute values on  $\mathbf{k}$ . Then the following statements are equivalent:

- (a)  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent;
- (b)  $\|\cdot\|_1$  and  $\|\cdot\|_2$  induce the same topology on  $\mathbf{k}$ ;
- (c) The unit disks  $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$  and  $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$  are the same.

*Proof.* The implications (a)  $\Rightarrow$  (b) is obvious. Now we prove (b)  $\Rightarrow$  (c). For any  $x \in D_1$ , we have  $x^n \rightarrow 0$  as  $n \rightarrow \infty$  under the absolute value  $\|\cdot\|_1$  and thus under  $\|\cdot\|_2$ . Therefore,  $\|x\|_2^n \rightarrow 0$  as  $n \rightarrow \infty$ , which implies that  $\|x\|_2 < 1$ , i.e.,  $x \in D_2$ . Similarly, we can prove that  $D_2 \subseteq D_1$ .

Finally, we prove (c)  $\Rightarrow$  (a). If  $\|\cdot\|_1$  is trivial, then  $D_1 = \{0\}$  and thus  $\|\cdot\|_2$  is also trivial. In this case, they are equivalent. Suppose that both  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are non-trivial. Pick any  $x, y \notin D_1 = D_2$ . Then there exist real numbers  $\alpha, \beta > 0$  such that  $\|x\|_1 = \|x\|_2^\alpha$  and  $\|y\|_1 = \|y\|_2^\beta$ . Suppose the contrary that  $\alpha \neq \beta$ . Consider the domain  $\Lambda \subseteq \mathbb{Z}^2$  defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since  $\alpha \neq \beta$ , the two lines defined by the equalities are not parallel. Thus  $\Lambda$  is non-empty. Pick

$(n, m) \in \Lambda$  and set  $z := x^n y^{-m}$ . Then we have  $\|z\|_2 < 1$  and  $\|z\|_1 > 1$ , a contradiction.  $\square$

**Definition 1.1.16.** Let  $\mathbf{k}$  be a field. A *place* on  $\mathbf{k}$  is an equivalence class of absolute values on  $\mathbf{k}$ . We denote the set of all places on  $\mathbf{k}$  by  $\text{Pl}_{\mathbf{k}}$ .

**Theorem 1.1.17.** Let  $(\mathbf{k}, \|\cdot\|)$  be an archimedean complete valuation field. Then  $\mathbf{k}$  is isomorphic to either the real number field  $\mathbb{R}$  or the complex number field  $\mathbb{C}$  equipped with the usual absolute value. Yang: To be revised.

*Proof.* Yang: To be added.  $\square$

**Theorem 1.1.18** (Ostrowski's theorem). Every nontrivial absolute value on  $\mathbb{Q}$  is equivalent to either the usual absolute value  $|\cdot|_{\infty}$  or a  $p$ -adic absolute value  $|\cdot|_p$  for some prime number  $p$ . Yang: To be revised.

*Proof.* Yang: To be added.  $\square$

## 1.2 Non-archimedean valuations

### 1.2.1 Topology: Ultra-metric space

We will use  $B(x, r)$  (resp.  $E(x, r)$ ) to denote the open ball (resp. closed ball) with center  $x$  and radius  $r$ .

**Definition 1.2.1.** A metric space  $(X, d)$  is called an *ultra-metric space* if its metric  $d$  satisfies the *strong triangle inequality*:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.$$

**Remark 1.2.2.** The term *ultra-metric space* should be translated into Chinese as “奥特度量空间”. There is no special reason for this translation, except that I insist on using “奥特” to translate “ultra”.

If  $(\mathbf{k}, \|\cdot\|)$  is a non-archimedean field, then the metric  $d(x, y) := \|x - y\|$  on  $\mathbf{k}$  makes  $(\mathbf{k}, d)$  an ultra-metric space.

**Proposition 1.2.3.** Let  $(X, d)$  be an ultra-metric space. Then for any  $x, y, z \in X$ , at least two of the three distances  $d(x, y), d(y, z), d(z, x)$  are equal. And the third distance is less than or equal to the common value of the other two.

*Proof.* Suppose that  $d(x, y) \geq d(y, z)$ . By the strong triangle inequality, we have

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(x, y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} = \max\{d(x, z), d(y, z)\} \leq d(x, y).$$

This shows that  $d(x, y) = \max\{d(x, z), d(y, z)\}$ . Thus either  $d(x, z) = d(x, y) \geq d(y, z)$  or  $d(y, z) = d(x, y) \geq d(x, z)$ .  $\square$



**Proposition 1.2.4.** Let  $(X, d)$  be an ultra-metric space. Let  $D_i$  be (open or closed) ball in  $X$  for  $i = 1, 2$ . If  $D_1 \cap D_2 \neq \emptyset$ , then either  $D_1 \subseteq D_2$  or  $D_2 \subseteq D_1$ .

*Proof.* Suppose that  $D_i$  has center  $x_i$  and radius  $r_i$  for  $i = 1, 2$ . Let  $y \in D_1 \cap D_2$ . We have

$$d(x_1, x_2) \leq \max\{d(x_1, y), d(y, x_2)\}.$$

Without loss of generality, we may assume that  $d(x_1, x_2) \leq d(x_1, y)$ . It follows that  $x_2 \in D_1$  since  $d(x_1, y) < r_1$  (or  $\leq r_1$ ).

If there exists  $z \in D_2 \setminus D_1$ , we claim that  $D_1 \subseteq D_2$ . We have  $d(x_1, z) > d(x_1, x_2)$ . Then by [Proposition 1.2.3](#),

$$r_1 \leq d(x_1, z) = d(x_2, z) \leq r_2.$$

In particular, if  $D_2$  is an open ball, then we have strict inequality  $r_1 < r_2$ . For any  $w \in D_1$ , we have

$$d(x_2, w) \leq \max\{d(x_2, x_1), d(x_1, w)\} \leq r_1 \leq r_2.$$

Thus  $w \in D_2$  whatever  $D_2$  is open or closed, and it shows that  $D_1 \subseteq D_2$ .  $\square$

**Proposition 1.2.5.** Let  $(X, d)$  be an ultra-metric space. Then both  $B(x, r)$  and  $E(x, r)$  are closed and open subsets of  $X$  for any  $x \in X$  and  $r > 0$ .

*Proof.* We show that the sphere  $S(x, r) := \{y \in X \mid d(x, y) = r\}$  is open in  $X$ . Note that if  $y \in S(x, r)$ , then for any  $r' < r$ , we have  $B(y, r') \cap E(x, r) \neq \emptyset$  and  $x \in E(x, r) \setminus B(y, r')$ . Thus by [Proposition 1.2.4](#), we have  $B(y, r') \subseteq E(x, r)$ . If  $B(y, r') \cap B(x, r) \neq \emptyset$ , then by [Proposition 1.2.4](#) again, we have  $B(y, r') \subseteq B(x, r)$ . However,  $y \in B(y, r') \setminus B(x, r)$ , a contradiction. Thus  $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$ . It yields that  $S(x, r) = \bigcup_{y \in S(x, r)} B(y, r/2)$  is open in  $X$ .

Since  $E(x, r) = B(x, r) \cup S(x, r)$  and  $B(x, r) = E(x, r) \setminus S(x, r)$ , both  $B(x, r)$  and  $E(x, r)$  are open and closed in  $X$ .  $\square$

**Corollary 1.2.6.** Let  $(X, d)$  be an ultra-metric space. Then  $X$  is totally disconnected, i.e., the only connected subsets of  $X$  are the set with at most one point.

*Proof.* Suppose that  $S \subset X$  has at least two distinct points  $x, y \in S$ . Let  $r := d(x, y) > 0$ . Consider the open ball  $B(x, r/2)$ . By [Proposition 1.2.5](#),  $B(x, r/2)$  is both open and closed in  $X$ . Thus  $B(x, r/2) \cap S$  is both open and closed in  $S$ , however, it is non-empty and not equal to  $S$  since it contains  $x$  but not  $y$ . This shows that  $S$  is disconnected.  $\square$

**Proposition 1.2.7.** Let  $(X, d)$  be an ultra-metric space. A sequence  $\{x_n\}$  in  $X$  is cauchy if and only if  $d(x_n, x_{n+1}) \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* The necessity is true for all metric spaces. Suppose that  $d(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ . For any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $d(x_n, x_{n+1}) < \varepsilon$  for all  $n \geq N$ . For any  $m, n \geq N$  with  $m < n$ , by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \dots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that  $\{x_n\}$  is a cauchy sequence.  $\square$

### 1.2.2 Algebra: ring of integers and residue field

Let  $\mathbf{k}$  be a non-archimedean field. Then easily see that  $\{x \in \mathbf{k} : \|x\| \leq 1\}$  is a subring of  $\mathbf{k}$ . Moreover, it is a local ring whose maximal ideal is  $\{x \in \mathbf{k} : \|x\| < 1\}$ .

**Definition 1.2.8.** Let  $\mathbf{k}$  be a non-archimedean field. The *ring of integers* of  $\mathbf{k}$  is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of  $\mathbf{k}$  is defined as

$$\kappa_{\mathbf{k}} := \tilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Set  $I_{r,<} := B(0, r)$  and  $I_{r,\leq} := E(0, r)$  for each  $r \in [0, 1]$ .

**Proposition 1.2.9.** The sets  $I_{r,<}$  and  $I_{r,\leq}$  are ideals of the ring of integers  $\mathbf{k}^\circ$ . Conversely, any ideal of  $\mathbf{k}^\circ$  is of the form  $I_{r,<}$  or  $I_{r,\leq}$  for some  $r \in (0, 1)$ .

*Proof.* Let  $I$  be an ideal of  $\mathbf{k}^\circ$ . Set  $r = \sup\{|a| : a \in I\}$  (resp.  $r = \max\{|a| : a \in I\}$  when the maximum exists). Then, by definition, we have  $I \subset I_{r,<}$  (resp.  $I \subset I_{r,\leq}$ ). For every  $x \in \mathbf{k}^\circ$  with  $|x| < r$  (resp.  $|x| \leq r$ ), there exists  $a \in I$  such that  $|x| \leq |a|$ . Thus,  $|x/a| \leq 1$  and so  $x/a \in \mathbf{k}^\circ$ . Since  $I$  is an ideal, we have  $x = (x/a)a \in I$ . Therefore,  $I_{r,<} \subset I$  (resp.  $I_{r,\leq} \subset I$ ).  $\square$

**Proposition 1.2.10.** Let  $I_r$  be either  $I_{r,<}$  or  $I_{r,\leq}$  for each  $r \in (0, 1)$ . Suppose  $\{r_n \in (0, 1)\}_{n \in \mathbb{N}}$  is a decreasing sequence converging to 0. Then the completion  $\hat{\mathbf{k}}$  of  $\mathbf{k}$  is isomorphic to the projective limit

$$\hat{\mathbf{k}}^\circ \cong \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}.$$

*Proof.* For every  $x \in \hat{\mathbf{k}}^\circ$ , there exists a cauchy sequence  $\{x_m\}_{m \in \mathbb{N}}$  in  $\mathbf{k}^\circ$  converging to  $x$ . Since  $\{r_n\}_{n \in \mathbb{N}}$  converges to 0, for each  $n \in \mathbb{N}$ , there exists  $M_n \in \mathbb{N}$  such that for all  $m, m' \geq M_n$ , we have  $|x_m - x_{m'}| < r_n$ . Thus, the sequence  $\{x_m + I_{r_n}\}_{m \in \mathbb{N}}$  is eventually constant in  $\mathbf{k}^\circ / I_{r_n}$ . Define a map

$$\Phi : \hat{\mathbf{k}}^\circ \rightarrow \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}, \quad x \mapsto \left( \lim_{m \rightarrow \infty} x_m + I_{r_n} \right)_{n \in \mathbb{N}}.$$

It is straightforward to verify that  $\Phi$  is a well-defined ring homomorphism.

Conversely, for every  $(a_n + I_{r_n})_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}$ , we can choose a representative  $a_n \in \mathbf{k}^\circ$  for each  $n$ . We claim that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  is a cauchy sequence in  $\mathbf{k}^\circ$ . Indeed, for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $r_N < \varepsilon$ . For all  $m, n \geq N$ , since  $a_n + I_{r_n}$  maps to  $a_m + I_{r_m}$  under the natural projection, we have  $|a_n - a_m| < r_N < \varepsilon$ . Thus,  $\{a_n\}_{n \in \mathbb{N}}$  converges to some  $x \in \hat{\mathbf{k}}^\circ$ . Easily see that the limit  $x$  is independent of the choice of representatives  $\{a_n\}_{n \in \mathbb{N}}$ . This gives a map

$$\Psi : \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n} \rightarrow \hat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow \infty} a_n.$$

Direct verification shows that  $\Psi = \Phi^{-1}$ .  $\square$

**Corollary 1.2.11.** Let  $\mathbf{k}$  be a non-archimedean field and  $\widehat{\mathbf{k}}$  its completion. Then the residue field  $\mathcal{K}_{\widehat{\mathbf{k}}} \cong \mathcal{K}_{\mathbf{k}}$  under the natural embedding  $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$ .

**Corollary 1.2.12.** Let  $\mathbf{k}$  be a non-archimedean field and  $\widehat{\mathbf{k}}$  its completion. Then the valuation group  $|\widehat{\mathbf{k}}^\times|$  of  $\widehat{\mathbf{k}}$  is equal to the valuation group  $|\mathbf{k}^\times|$  of  $\mathbf{k}$ .

*Proof.* Note that

$$\begin{aligned} r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\ &\iff \widehat{\mathbf{k}}^\circ/I_{r,<} \rightarrow \widehat{\mathbf{k}}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff \mathbf{k}^\circ/I_{r,<} \rightarrow \mathbf{k}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\ &\iff r \in |\mathbf{k}^\times|. \end{aligned}$$

□

**Proposition 1.2.13.** Let  $\mathbf{k}$  be a non-archimedean field with non-trivial valuation. Then  $\mathbf{k}^\circ$  is totally bounded iff  $\mathbf{k}^\circ/I_{r,<}$  and  $\mathbf{k}^\circ/I_{r,\leq}$  are finite for each  $r \in [0, 1]$ . Moreover, if  $\mathbf{k}$  is complete, then it is locally compact iff  $\mathbf{k}^\circ/I_r$  is finite for each  $r \in (0, 1)$ .

**Slogan** “*Locally compact  $\iff$  pro-finite.*”

*Proof.* We just prove the case for  $I_r = I_{r,<}$ . The case for  $I_r = I_{r,\leq}$  is similar.

Suppose that  $\mathbf{k}^\circ/I_r$  is finite for each  $r \in [0, 1]$ . Then for every  $\varepsilon > 0$ , there exists  $r \in (0, 1)$  such that  $r < \varepsilon$  and  $\mathbf{k}^\circ/I_r$  is finite. Let  $\{a_1 + I_r, \dots, a_n + I_r\}$  be the complete set of representatives of  $\mathbf{k}^\circ/I_r$ . Then the balls  $B(a_i, r)$  for  $i = 1, \dots, n$  cover  $\mathbf{k}^\circ$ .

Conversely, suppose that  $\mathbf{k}^\circ/I_r$  is infinite for some  $r \in [0, 1]$ . Then there exists an infinite set  $\{a_n\}$  with  $|a_n| \in [r, 1]$  such that their images in  $\mathbf{k}^\circ/I_r$  are distinct. In particular, for every  $m \neq n$ , we have  $|a_n - a_m| \geq r$ . Any subsequence of  $\{a_n\}$  is not cauchy. Thus,  $\mathbf{k}^\circ$  is not totally bounded. □

**Proposition 1.2.14.** The ring  $\mathbf{k}^\circ$  is noetherian iff  $\mathbf{k}$  is a discrete valuation field.

*Proof.* Note that  $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$  is a multiplicative subgroup. If  $\mathbf{k}$  is not a discrete valuation field, then  $|\mathbf{k}^\times|$  is dense in  $\mathbb{R}_{>0}$ . In particular, there exists a strictly ascending sequence  $r_n \in |\mathbf{k}^\times| \cap (0, 1)$ . Then the ideals  $I_{r_n,\leq}$  form a strictly ascending chain of ideals in  $\mathbf{k}^\circ$ .

The converse is standard since now  $\mathbf{k}^\circ$  is a discrete valuation ring. □

**Proposition 1.2.15.** Let  $\mathbf{k}$  be a complete non-archimedean field. Then  $\mathbf{k}$  is locally compact iff  $\mathbf{k}$  is a discrete valuation field and its residue field  $\mathcal{K}_{\mathbf{k}}$  is finite.

*Proof.* The necessity follows from [Proposition 1.2.13](#). For the sufficiency, suppose that  $\mathbf{k}$  is a discrete valuation field whose residue field  $\mathcal{K}_{\mathbf{k}}$  is finite. Let  $\pi \in \mathbf{k}^\circ$  be a uniformizer. We only need to show that  $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$  is finite for each  $n \in \mathbb{N}$ . Note that there is an isomorphism

$$\pi^{n-1} \mathbf{k}^\circ / \pi^n \mathbf{k}^\circ \cong \mathcal{K}_{\mathbf{k}}, \quad x + \pi^n \mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on  $n$ , we conclude that  $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$  is finite. □

### 1.2.3 Hensel's Lemma

**Theorem 1.2.16** (Hensel's lemma). Let  $\mathbf{k}$  be a complete non-archimedean field and  $F(T) \in \mathbf{k}^\circ[T]$  a monic polynomial. Suppose that the reduction  $f(T) \in \mathcal{K}_{\mathbf{k}}[T]$  of  $F(T)$  factors as

$$f(T) = g(T)h(T),$$

where  $g(T), h(T) \in \mathcal{K}_{\mathbf{k}}[T]$  are monic polynomials that are coprime in  $\mathcal{K}_{\mathbf{k}}[T]$ . Then there exist monic polynomials  $G(T), H(T) \in \mathbf{k}^\circ[T]$  such that

$$F(T) = G(T)H(T),$$

and the reductions of  $G(T), H(T)$  in  $\mathcal{K}_{\mathbf{k}}[T]$  are  $g(T), h(T)$  respectively.

*Proof.* Since  $\gcd(g, h) = 1$  in  $\mathcal{K}_{\mathbf{k}}[T]$ , there exist polynomials  $u(T), v(T) \in \mathcal{K}_{\mathbf{k}}[T]$  such that  $ug + vh = 1$  and  $\deg u < \deg h, \deg v < \deg g$ . Choose lifts  $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$  of  $g(T), h(T), u(T), v(T)$  respectively preserving their degrees such that  $G_0$  and  $H_0$  are monic. Then there exist  $r < 1$  such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \pmod{I_r}, \quad F(T) - G_0(T)H_0(T) \equiv 0 \pmod{I_r},$$

where  $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$ .

We will construct a sequence of monic polynomials  $\{G_n(T)\}_{n \in \mathbb{N}}$  and  $\{H_n(T)\}_{n \in \mathbb{N}}$  in  $\mathbf{k}^\circ[T]$  such that for each  $n \in \mathbb{N}$ ,

$$G_n(T) \equiv G_{n-1}(T) \pmod{I_{r^n}}, \quad H_n(T) \equiv H_{n-1}(T) \pmod{I_{r^n}},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \pmod{I_{r^{n+1}}}.$$

If we have such sequences, then their coefficients converge in the complete ring  $\mathbf{k}^\circ$ . Let  $G(T)$  and  $H(T)$  be the limits of  $\{G_n(T)\}$  and  $\{H_n(T)\}$  respectively. Then we have  $F(T) = G(T)H(T)$  and the reductions of  $G(T), H(T)$  in  $\mathcal{K}_{\mathbf{k}}[T]$  are  $g(T), h(T)$  respectively.

The case  $n = 0$  is done by the above construction. Now suppose that we have constructed  $G_n(T)$  and  $H_n(T)$  for some  $n \geq 0$ . Since  $G_n - G_0 \equiv 0 \pmod{I_r}$  and  $H_n - H_0 \equiv 0 \pmod{I_r}$ , we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \pmod{I_r}.$$

Set  $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$  and  $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$ . Then we have

$$\begin{aligned} (G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T]. \end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

This finishes the induction. □

**Corollary 1.2.17.** Let  $\mathbf{k}$  be a complete non-archimedean field and  $F(T) \in \mathbf{k}^\circ[T]$  a monic polynomial. Suppose that the reduction  $f(T) \in \mathcal{K}_{\mathbf{k}}[T]$  of  $F(T)$  has a simple root  $a \in \mathcal{K}_{\mathbf{k}}$ . Then there exists a root  $\alpha \in \mathbf{k}^\circ$  of  $F(T)$  whose reduction is  $a$ .

*Proof.* Since  $a$  is a simple root of  $f(T)$ , we have the factorization  $f(T) = (T - a)h(T)$  for some monic polynomial  $h(T) \in \mathcal{K}_{\mathbf{k}}[T]$  with  $h(a) \neq 0$ . Then the result follows from [Theorem 1.2.16](#).  $\square$

## 1.2.4 Newton polygons

Yang: To be filled.

# 1.3 Finite field extensions

## 1.3.1 Finite-dimensional vector space

**Definition 1.3.1.** Let  $\mathbf{k}$  be a valuation field and  $V$  a vector space over  $\mathbf{k}$ . A *norm* on  $V$  is a function  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following properties for all  $x, y \in V$  and  $a \in \mathbf{k}$ :

- (a)  $\|x\| = 0$  if and only if  $x = 0$ ;
- (b)  $\|ax\| = |a| \cdot \|x\|$ ;
- (c)  $\|x + y\| \leq \|x\| + \|y\|$ .

**Example 1.3.2.** Let  $\mathbf{k}$  be a valuation field and  $V$  a finite-dimensional vector space over  $\mathbf{k}$  with basis  $\{e_1, e_2, \dots, e_n\}$ . For any  $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$ , define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then  $\|\cdot\|_{\max}$  is a norm on  $V$ , called the *maximal norm* with respect to the basis  $\{e_1, e_2, \dots, e_n\}$ .

**Example 1.3.3.** Setting as in [Example 1.3.2](#), for any  $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$ , define

$$\|x\|_1 := |a_1| + |a_2| + \dots + |a_n|.$$

Then  $\|\cdot\|_1$  is also a norm on  $V$ .

**Definition 1.3.4.** Let  $\mathbf{k}$  be a valuation field and  $V$  a vector space over  $\mathbf{k}$ . Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $V$  are said to be *equivalent* if there exist positive constants  $C_1, C_2 > 0$  such that for all  $x \in V$ ,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

**Lemma 1.3.5.** Let  $\mathbf{k}$  be a valuation field and  $V$  a vector space over  $\mathbf{k}$ . Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $V$  are equivalent if and only if they induce the same topology on  $V$ .

*Proof.* The sufficiency is clear. Now suppose that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  induce the same topology on  $V$ . Hence the unit open ball with respect to  $\|\cdot\|_1$  contains a unit open ball with respect to  $\|\cdot\|_2$ . That

is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every  $x \in V$  with  $\|x\|_1 = 1$ , we have  $\|x\|_2 \geq C = C\|x\|_1$ . By scaling, we get that for every  $x \in V$ ,

$$\|x\|_2 \geq C\|x\|_1.$$

Similar for the other direction, we conclude that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent.  $\square$

**Proposition 1.3.6.** Let  $V$  be a normed finite-dimensional vector space over a complete valuation field  $\mathbf{k}$ . Then  $V$  is complete.

*Proof.* Yang: To be added.  $\square$

**Theorem 1.3.7.** Let  $V$  be a finite-dimensional vector space over a complete field  $\mathbf{k}$ . Then all norms on  $V$  are equivalent.

*Proof.* Fix a basis  $\{e_1, e_2, \dots, e_n\}$  of  $V$  and let  $\|\cdot\|_{\max}$  be the maximal norm with respect to this basis as in Example 1.3.2. Let  $\|\cdot\|$  be any norm on  $V$ . It suffices to show that  $\|\cdot\|$  and  $\|\cdot\|_{\max}$  are equivalent. First we have

$$\|y\| \leq \sum_{i=1}^n |a_i| \|e_i\| \leq \left( \sum_{i=1}^n \|e_i\| \right) \|y\|_{\max}$$

for any  $y = a_1 e_1 + a_2 e_2 + \dots + a_n e_n \in V$ . It remains to show that there exists a constant  $C > 0$  such that for any  $y \in V$ ,

$$\|y\|_{\max} \leq C\|y\|.$$

Yang: To be added.  $\square$

**Remark 1.3.8.** If the base field  $\mathbf{k}$  is not complete, then Theorem 1.3.7 may fail. For example, let  $\mathbf{k} = \mathbb{Q}$  with the usual absolute value, and let  $V = \mathbb{Q}[\alpha]$  with  $\alpha^2 - \alpha - 1 = 0$ . There are two embeddings of  $V$  into  $\mathbb{R}$ :

$$\iota_1 : a + b\alpha \mapsto a + b\frac{1+\sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b\frac{1-\sqrt{5}}{2}.$$

Define two norms on  $V$  by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where  $|\cdot|$  is the usual absolute value on  $\mathbb{R}$ . Then  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are not equivalent since  $\iota_2(\alpha^n) \rightarrow 0$  as  $n \rightarrow \infty$  while  $\iota_1(\alpha^n) \rightarrow \infty$ .

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

**Lemma 1.3.9.** Let  $\mathbf{k}$  be a complete field and  $V$  a normed finite-dimensional vector space over  $\mathbf{k}$ . Then

$$\|\cdot\| : \text{End}_{\mathbf{k}}(V) \rightarrow \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the  $\mathbf{k}$ -vector space  $\text{End}_{\mathbf{k}}(V)$  satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \text{End}_{\mathbf{k}}(V).$$

*Proof.* First we show the existence of the supremum, i.e., there exists  $C > 0$  such that for all  $x \in V \setminus \{0\}$ ,  $\|T(x)\| \leq C\|x\|$ . Fix a basis  $\{e_1, e_2, \dots, e_n\}$  of  $V$  and let  $\|\cdot\|_{\max}$  be the maximal norm with respect to this basis. Since all norms on  $V$  are bounded by each other by [Theorem 1.3.7](#), we only need to show that there exists  $C > 0$  such that for all  $x \in V \setminus \{0\}$ ,  $\|T(x)\|_1 \leq C\|x\|_{\max}$ . Write  $T(e_i) = \sum_{j=1}^n a_{ij}e_j$  for  $1 \leq i \leq n$ . For any  $x = \sum_{i=1}^n x_i e_i \in V$ , we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^n \left( \sum_{i=1}^n a_{ij}x_i \right) e_j \right\|_1 = \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij}x_i \right| \leq \left( \sum_{1 \leq i, j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of  $\|\cdot\|$  are clear. It remains to show the triangle inequality and sub-multiplicativity. For any  $A, B \in \text{End}_{\mathbf{k}}(V)$ , we have

$$\frac{\|(A+B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all  $x \in V \setminus \{0\}$  gives  $\|A+B\| \leq \|A\| + \|B\|$ . We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence  $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$ . Taking supremum we get  $\|AB\| \leq \|A\| \cdot \|B\|$ .  $\square$

### 1.3.2 Finite field extensions

**Lemma 1.3.10.** Let  $\mathbf{k}$  be a complete field and  $\mathbf{l}$  a finite extension of  $\mathbf{k}$ . Then there exists an absolute value on  $\mathbf{l}$  extending the absolute value on  $\mathbf{k}$ .

*Proof.* Fix a norm  $\|\cdot\|_V$  on the  $\mathbf{k}$ -vector space  $V = \mathbf{l}$ . The norm  $\|\cdot\|_V$  induces an operator norm  $\|\cdot\|_{\text{op}}$  on the  $\mathbf{k}$ -vector space  $\text{End}_{\mathbf{k}}(V)$  as in [Lemma 1.3.9](#). For any  $a \in \mathbf{l}$ , let  $\mu_a \in \text{End}_{\mathbf{k}}(V)$  be the  $\mathbf{k}$ -linear map defined by multiplication by  $a$ . Note that  $a \mapsto \mu_a$  gives an embedding of  $\mathbf{k}$ -algebras and if  $a \in \mathbf{k}$ ,  $\|\mu_a\|_{\text{op}} = \|a\|_{\mathbf{k}}$ . Thus the restriction of  $\|\cdot\|_{\text{op}}$  to  $\mathbf{l}$  gives a norm on  $\mathbf{l}$  extending that on  $\mathbf{k}$ . The normed ring  $(\mathbf{l}, \|\cdot\|_{\text{op}})$  is a Banach ring since it is a finite-dimensional vector space over the complete field  $\mathbf{k}$ . By [Theorem 4.1.9](#), there exists a multiplicative seminorm  $\|\cdot\|_{\mathbf{l}}$  on  $\mathbf{l}$  bounded by  $\|\cdot\|_{\text{op}}$ . In particular,  $\|\cdot\|_{\mathbf{l}}$  is bounded by  $\|\cdot\|_{\mathbf{k}}$  on  $\mathbf{k}$ . On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus  $\|\cdot\|_{\mathbf{l}}$  extends the absolute value on  $\mathbf{k}$ .  $\square$

**Theorem 1.3.11.** Let  $\mathbf{k}$  be a complete field and  $\mathbf{l}$  a finite extension of  $\mathbf{k}$ . Then the absolute value on  $\mathbf{l}$  which extends the absolute value on  $\mathbf{k}$  is uniquely determined by the absolute value on  $\mathbf{k}$ . Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where  $n = [\mathbf{l} : \mathbf{k}]$  and  $N_{\mathbf{l}/\mathbf{k}}$  is the norm map from  $\mathbf{l}$  to  $\mathbf{k}$ .

*Proof.* Let  $\|\cdot\|_{\mathbf{l}}$  be arbitrary absolute value on  $\mathbf{l}$  extending that on  $\mathbf{k}$ . We will show that  $\|\cdot\|_{\mathbf{l}}$  must be equal to  $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$ . For any  $a \in \mathbf{l}$ , set  $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$ . Then  $N_{\mathbf{l}/\mathbf{k}}(b) = 1$  and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that  $\|b\|_{\mathbf{l}} = 1$  whenever  $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ .

Note that the norm map  $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \rightarrow \mathbf{k}$  is the determinant of the  $\mathbf{k}$ -linear map  $\mu_b \in \text{End}_{\mathbf{k}}(V)$  defined by multiplication by  $b$ . Hence it is continuous on  $\mathbf{l}$  (since it is a polynomial in the entries of the matrix representation). If  $\|b\|_{\mathbf{l}} < 1$ , then  $\|b^m\|_{\mathbf{l}} \rightarrow 0$  as  $m \rightarrow \infty$ . Thus  $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \rightarrow 0$  as  $m \rightarrow \infty$ , contradicting the fact that  $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$  for all  $m$ . Similarly, if  $\|b\|_{\mathbf{l}} > 1$ , then just consider  $b^{-1}$ .  $\square$

**Proposition 1.3.12.** Let  $\mathbf{k}$  be an algebraically closed valuation field. Then its completion  $\hat{\mathbf{k}}$  is also algebraically closed.

*Proof.* Let  $f \in \hat{\mathbf{k}}[X]$  be a non-constant polynomial. We will show that  $f$  has a root in  $\hat{\mathbf{k}}$ . Take a sequence of polynomials  $\{f_n\}_{n \in \mathbb{N}}$  in  $\mathbf{k}[X]$  converging to  $f$  coefficient-wisely and of the same degree  $d$ . Since  $\mathbf{k}$  is algebraically closed, each  $f_n$  splits completely in  $\mathbf{k}$  and hence in  $\hat{\mathbf{k}}$ . Write  $f_n(X) = \prod_{i=1}^d (X - \alpha_{n,i})$  with  $\alpha_{n,i} \in \hat{\mathbf{k}}$ .

Let  $\mathbf{l}$  be a finite extension of  $\hat{\mathbf{k}}$  such that  $f$  has a root  $\alpha$  in  $\mathbf{l}$ . For every  $\varepsilon > 0$ , if there are infinitely many  $n$  such that  $\alpha_{n,i} \notin B(\alpha, \varepsilon)$  for all  $1 \leq i \leq d$ , then we have  $|f_n(\alpha)| \geq \varepsilon^d$  for infinitely many  $n$ , contradicting the fact that  $f_n(\alpha) \rightarrow f(\alpha) = 0$ . Thus for every  $\varepsilon > 0$ , there exists  $N > 0$  such that for all  $n \geq N$ , there exists  $1 \leq i \leq d$  with  $\alpha_{n,i} \in B(\alpha, \varepsilon)$ . That is, we can find a sequence  $\alpha_{n,i_n} \in \mathbf{k}$  converging to  $\alpha$ . Since  $\hat{\mathbf{k}}$  is complete, we have  $\alpha \in \hat{\mathbf{k}}$ .  $\square$

### 1.3.3 Ramification and inertia

In this subsection, we study the extensions of absolute values on finite field extensions. Note that we do not assume the base field to be complete.

**Definition 1.3.13.** Let  $L/K$  be a finite field extension, and  $v \in M_K$  an absolute value on  $K$ . We denote by  $w|v$  if  $w \in M_L$  is an absolute value on  $L$  extending  $v$ . For each  $w|v$ , we define the *ramification index*  $e(w|v)$  and the *inertia degree*  $f(w|v)$  by

$$e(w|v) := [|\hat{L}^\times|_w : |\hat{K}^\times|_v], \quad f(w|v) := \frac{[\hat{L} : \hat{K}]}{e(w|v)},$$

where  $\hat{K}$  and  $\hat{L}$  are the completions of  $K$  and  $L$  with respect to  $v$  and  $w$ , respectively.

**Lemma 1.3.14.** Suppose that  $v$  is non-archimedean and  $\kappa_v$  and  $\ell_w$  are the residue fields of  $K$  and  $L$  with respect to  $v$  and  $w$ , respectively. Then we have

$$f(w|v) = [\ell_w : \kappa_v].$$

**Remark 1.3.15.** Yang: To be added.

**Theorem 1.3.16.** Let  $L/K$  be a finite field extension, and  $v \in M_K$  an absolute value on  $K$ . Then we have

$$\sum_{w|v} e(w|v)f(w|v) = [L : K].$$



Let  $L/K$  be a finite field extension, and  $v \in M_K$  an absolute value on  $K$ . We have

$$L \otimes_K K_v \cong \prod_{w|v} L_w,$$

where the product is taken over all absolute values  $w \in M_L$  extending  $v$ .

**Theorem 1.3.17.** Let  $\mathbf{k}$  be a number field. Then

$$M_{\mathbf{k}}^{\infty} = \{\text{embeddings } \sigma : \mathbf{k} \rightarrow \mathbb{C}\}$$

and

$$M_{\mathbf{k}}^f = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_{\mathbf{k}}\}.$$

Yang: To be revised.

**Proposition 1.3.18** (Product formula). Let  $\mathbf{k}$  be a number field. For each  $x \in \mathbf{k}^{\times}$ , we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^{\infty}; \\ 1, & v \in M_{\mathbf{k}}^0. \end{cases}$$

Yang: To be revised.

**Remark 1.3.19.** Let  $L/K$  be a finite field extension, and  $v \in M_K$  an absolute value on  $K$ . Suppose that  $v$  is non-archimedean. Yang: To be rewritten.

## 1.4 Artin-Whaples approximations

**Theorem 1.4.1** (Artin-Whaples approximations). Let  $K$  be a field, and let  $v_1, v_2, \dots, v_n$  be pairwise inequivalent nontrivial absolute values on  $K$ . For any  $a_1, a_2, \dots, a_n \in K$  and any  $\varepsilon > 0$ , there exists an element  $x \in K$  such that

$$|x - a_i|_{v_i} < \varepsilon$$

for all  $1 \leq i \leq n$ . Yang: To be checked.

### 1.4.1 Geometric version

**Theorem 1.4.2.** Let  $\mathbf{k}$  be a field with algebraic closure  $\mathbb{k}$ . Let  $X$  be a normal, projective, geometrically integral variety over  $\mathbf{k}$ . Let  $x_1, x_2, \dots, x_n \in X(\mathbf{k})$  be closed points lying over pairwise distinct points of  $X$ . Let  $v_1, v_2, \dots, v_n \in M_{\mathbf{k}}$  be pairwise inequivalent absolute values on  $\mathbf{k}$ . For every  $i = 1, 2, \dots, n$ , let  $U_i$  be an open neighborhood of  $x_i$  in  $X(\mathbb{k})$  with respect to the topology induced by  $v_i$ . Then there exists a rational point  $x \in X(\mathbf{k})$  such that  $x \in U_i$  for all  $1 \leq i \leq n$ . Yang: To be revised.

Yang: This gives [Xie25, Proposition 3.9]



## Chapter 2

# Non-archimedean analysis

## 2.1 Local theory I: functions

### 2.1.1 Tate algebras

**Notation 2.1.1.** Let  $T = (T_1, \dots, T_n)$  be a tuple of  $n$  indeterminates,  $r = (r_1, \dots, r_n)$  be a tuple of  $n$  positive real numbers, and  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \dots T_n^{\alpha_n}$  and  $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \dots r_n^{\alpha_n}$ ;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$ ;
- $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n$ ;
- $\alpha \leq_{\text{total}} \beta$  if and only if for all  $i = 1, \dots, n$ , we have  $\alpha_i \leq \beta_i$ ;
- $E(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \dots, n\}$  and  $B(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \dots, n\}$  for  $x = (x_1, \dots, x_n) \in \mathbf{k}^n$ ;
- Let  $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$  be a set of elements in a metric space  $X$  indexed by multi-indices  $\alpha \in \mathbb{N}^n$ . We say that  $\lim_{|\alpha| \rightarrow +\infty} x_\alpha = x \in X$  if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $\alpha \in \mathbb{N}^n$  with  $|\alpha| > N$ , we have  $d(x_\alpha, x) < \varepsilon$ .

**Definition 2.1.2.** Let  $\mathbf{k}$  be a complete non-archimedean field. Let  $T = (T_1, \dots, T_n)$  be a tuple of  $n$  indeterminates and  $r = (r_1, \dots, r_n)$  be a tuple of  $n$  positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$\mathbf{k}\langle \underline{T/r} \rangle := \mathbf{k}\{\underline{T/r}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{|\alpha| \rightarrow +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

**Proposition 2.1.3.** Let  $\mathbf{k}$  be a complete non-archimedean field. Then the Tate algebra  $\mathbf{k}\{\underline{T/r}\}$  is a non-archimedean multiplicative banach  $\mathbf{k}$ -algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

**Yang:** For the definition of banach ring, see

*Proof.* The proof splits into several parts. Every parts is straightforward and standard.

**Step 1.** We first show that  $\mathbf{k}\{\underline{T/r}\}$  is a  $\mathbf{k}$ -algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$  and  $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$  are two elements in  $\mathbf{k}\{\underline{T/r}\}$ . Given  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $|\alpha| > N$ , we have  $\|a_\alpha\|r^\alpha < \varepsilon/\|g\|$  and  $\|b_\alpha\|r^\alpha < \varepsilon/\|f\|$ . For any  $|\gamma| > 2N$ , we have

$$\left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\|r^\alpha \cdot \|b_\beta\|r^\beta < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\|r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\|r^\alpha \right\} \leq \varepsilon.$$

Hence  $f \cdot g \in \mathbf{k}\{\underline{T/r}\}$  and it shows that  $\mathbf{k}\{\underline{T/r}\}$  is a  $\mathbf{k}$ -algebra.

**Step 2.** Show that the gauss norm is a non-archimedean norm on  $\mathbf{k}\{\underline{T/r}\}$ .

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f + g\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha + b_\alpha\|r^\alpha \leq \sup_{\alpha \in \mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\}r^\alpha \leq \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned} \|f \cdot g\| &= \left\| \sum_{\gamma \in \mathbb{N}^n} \left( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma \right\| = \sup_{\gamma \in \mathbb{N}^n} \left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \\ &\leq \sup_{\gamma \in \mathbb{N}^n} \max_{\alpha+\beta=\gamma} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \leq \|f\| \cdot \|g\|. \end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed  $\mathbf{k}$ -algebra.

**Step 3.** Show that the gauss norm is multiplicative.

Suppose that  $\|f\| = \|a_{\alpha_1}\|r^{\alpha_1}$  and  $\|a_\alpha\|r^\alpha < \|f\|$  for all  $\alpha <_{\text{total}} \alpha_1$ . Similar to  $\|b_{\beta_1}\|r^{\beta_1}$ . Then we have

$$\|f\| \cdot \|g\| = \|a_{\alpha_1}\|r^{\alpha_1} \cdot \|b_{\beta_1}\|r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \left\| \sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta \right\| r^{\alpha_1+\beta_1} \leq \|f \cdot g\|,$$

where the third equality holds since  $(\alpha_1, \beta_1)$  is the unique pair such that  $\|a_{\alpha_1}\|r^{\alpha_1} \cdot \|b_{\beta_1}\|r^{\beta_1}$  is maximized and by [Proposition 1.2.3](#). Thus the gauss norm is multiplicative.

**Step 4.** Finally show that  $\mathbf{k}\{\underline{T/r}\}$  is complete with respect to the gauss norm.

Let  $\{f_m = \sum a_{\alpha,m} T^\alpha\}$  be a cauchy sequence in  $\mathbf{k}\{\underline{T/r}\}$ . We have

$$\|a_{\alpha,m} - a_{\alpha,l}\|r^\alpha \leq \|f_m - f_l\|.$$

Thus for each  $\alpha \in \mathbb{N}^n$ , the sequence  $\{a_{\alpha,m}\}$  is a cauchy sequence in  $\mathbf{k}$ . Since  $\mathbf{k}$  is complete, set  $a_\alpha := \lim_{m \rightarrow +\infty} a_{\alpha,m}$  and  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$ . Given  $\varepsilon > 0$ , there exists  $M \in \mathbb{N}$  such that for all  $m, l > M$ , we have  $\|f_m - f_l\| < \varepsilon$ . Fixing  $m > M$ , there exists  $N \in \mathbb{N}$  such that for all  $|\alpha| > N$ , we have  $\|a_{\alpha,m}\|r^\alpha < \varepsilon$ . Hence for all  $|\alpha| > N$  and  $l > M$ , we have

$$\|a_{\alpha,l}\|r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\|r^\alpha + \|a_{\alpha,m}\|r^\alpha < 2\varepsilon.$$

Taking  $l \rightarrow +\infty$ , we have  $\|a_\alpha\|r^\alpha \leq 2\varepsilon$  for all  $|\alpha| > N$ . It follows that  $f \in \mathbf{k}\{\underline{T/r}\}$ .

For every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $m, l > N$ , we have  $\|f_m - f_l\| < \varepsilon$ . Thus for all  $\alpha \in \mathbb{N}^n$  and  $m, l > N$ , we have

$$\|a_{\alpha, m} - a_{\alpha, l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking  $l \rightarrow +\infty$ , we have  $\|a_{\alpha, m} - a_\alpha\| r^\alpha \leq \varepsilon$  for all  $m > N$ . It follows that

$$\|f - f_m\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha - a_{\alpha, m}\| r^\alpha \leq \varepsilon$$

for all  $m > N$ . □

**Proposition 2.1.4.** Let  $\mathbf{k}$  be a complete non-archimedean field. An element  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/r\}$  is invertible if and only if  $\|a_0\| > \|a_\alpha\| r^\alpha$  for all  $\alpha \neq 0$ .

*Proof.* Multiplying by  $a_0^{-1}$ , we can reduce to the case  $a_0 = 1$ . Let  $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$  be the inverse of  $f$  in  $\mathbf{k}[[\underline{T}]]$ . Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every  $\gamma \neq 0 \in \mathbb{N}^n$ ,

$$b_\gamma = - \sum_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let  $A = \|f - 1\| < 1$ . We show that for every  $m \in \mathbb{N}$ , there exists  $C_m > 0$  such that for all  $\alpha \in \mathbb{N}^n$  with  $|\alpha| \geq C_m$ , we have  $\|b_\alpha\| r^\alpha \leq A^m$ . For  $m = 0$ , note that  $b_0 = 1$ . By induction on  $\gamma$  with respect to the total order  $\leq_{\text{total}}$ , we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for  $m$ . There exists  $D_{m+1} \in \mathbb{N}$  such that for all  $\alpha \in \mathbb{N}^n$  with  $|\alpha| \geq D_{m+1}$ , we have  $\|a_\alpha\| r^\alpha \leq A^{m+1}$ . Set  $C_{m+1} = C_m + D_{m+1} + 1$ . For any  $\gamma \in \mathbb{N}^n$  with  $|\gamma| \geq C_{m+1}$ , we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\{A^{m+1}, A \cdot A^m\} = A^{m+1}$$

since either  $|\alpha| \geq D_{m+1}$  or  $|\beta| \geq C_m$ . Thus by induction, we have  $\|b_\alpha\| r^\alpha \rightarrow 0$  as  $|\alpha| \rightarrow +\infty$ . It follows that  $g \in \mathbf{k}\{\underline{T}/r\}$ . □

Let  $\mathbf{k}$  be a complete non-archimedean field. Recall that the formal derivative operator  $\partial_i : \mathbf{k}[[\underline{T}]] \rightarrow \mathbf{k}[[\underline{T}]]$  is defined by

$$\frac{\partial}{\partial T_i} \left( \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right) := \sum_{\alpha \in \mathbb{N}^n} \alpha_i a_\alpha T_1^{\alpha_1} \cdots T_i^{\alpha_i-1} \cdots T_n^{\alpha_n}.$$

**Lemma 2.1.5.** Let  $\mathbf{k}$  be a complete non-archimedean field. Then for every  $f \in \mathbf{k}\{\underline{T}/r\}$ , we have  $\partial_i(f) \in \mathbf{k}\{\underline{T}/r\}$ .

*Proof.* Suppose that  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/\underline{r}\}$ . We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha \in \mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1-1} T_2^{\alpha_2} \dots T_n^{\alpha_n}.$$

Noting that  $\mathbf{k}$  is non-archimedean, we have  $\|\alpha_1 a_\alpha\| \leq \|a_\alpha\|$ . Then

$$\lim_{|\alpha| \rightarrow +\infty} \|\alpha_1 a_\alpha\| r_1^{\alpha_1-1} r_2^{\alpha_2} \dots r_n^{\alpha_n} \leq \frac{1}{r_1} \lim_{|\alpha| \rightarrow +\infty} \|a_\alpha\| r^\alpha = 0.$$

The conclusion follows.  $\square$

## 2.1.2 Analytic functions on closed polydisks

**Proposition 2.1.6.** Let  $\mathbf{k}$  be a complete non-archimedean field. Then for every  $f \in \mathbf{k}\{\underline{T}/\underline{r}\}$ , we can associate a function  $F_f : E(0, \underline{r}) \rightarrow \mathbf{k}$  defined by

$$F_f(x) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \quad \text{for } x = (x_1, \dots, x_n) \in E(0, \underline{r}).$$

This defines a homomorphism of  $\mathbf{k}$ -algebras from  $\mathbf{k}\{\underline{T}/\underline{r}\}$  to the ring of all functions from  $E(0, \underline{r})$  to  $\mathbf{k}$ .

*Proof.* Given  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/\underline{r}\}$  and  $x = (x_1, \dots, x_n) \in E(0, \underline{r})$ , we have

$$\left\| \sum_{|\alpha|=n} a_\alpha x^\alpha \right\| \leq \max_{|\alpha|=n} \|a_\alpha\| r^\alpha \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Hence by Proposition 1.2.7, the series  $F_f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$  converges in  $\mathbf{k}$ . This defines a function  $F_f : E(0, \underline{r}) \rightarrow \mathbf{k}$ .

Let  $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/\underline{r}\}$ . Set

$$A_n = \sum_{|\alpha| < n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta| < n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma| < n} \left( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) x^\gamma.$$

We need to show that  $F_f(x)F_g(x) = \lim A_n B_n = \lim C_n = F_{fg}(x)$ . Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha| < n, |\beta| < n \\ |\alpha+\beta| \geq n}} a_\alpha b_\beta x^{\alpha+\beta}.$$

Given  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $|\alpha| > N$ , we have  $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$  and  $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$ . For any  $n > 2N$ , we have

$$\|A_n B_n - C_n\| \leq \max_{\substack{|\alpha| < n, |\beta| < n \\ |\alpha+\beta| \geq n}} \|a_\alpha\| \|b_\beta\| \|x^{\alpha+\beta}\| < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Thus  $F_f(x)F_g(x) = (F_{fg})(x)$ . The addition and scalar multiplication can be verified directly. We thus finish the proof.  $\square$

**Proposition 2.1.7.** Let  $\mathbf{k}$  be a complete non-archimedean field with non-trivial valuation. Then for every  $f \in \mathbf{k}\{\underline{T}/\underline{r}\}$  and  $x, y \in E(0, \underline{r})$ , we have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq L \cdot \|y - x\|_{\infty},$$

where  $L = \max_{1 \leq i \leq n} \|f\|_g / r_i$ .

*Proof.* Set  $y - x = (h_1, \dots, h_n)$  and  $x^{(0)} = x$ ,  $x^{(i)} = (x_1 + h_1, \dots, x_i + h_i, x_{i+1}, \dots, x_n)$  for  $i = 1, \dots, n$ . We have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{1 \leq i \leq n} \|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}}.$$

We only need to show that for every  $i = 1, \dots, n$ , we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}} \leq \frac{\|f\|_g}{r_i} \|h_i\|.$$

Without loss of generality and for simplicity, we assume that  $y = (x_1 + h, x_2, \dots, x_n)$  and  $x = (x_1, x_2, \dots, x_n)$ . Note that by the strong triangle inequality, we have  $\|h\| \leq r_1$ .

Let  $f = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} T^{\alpha} \in \mathbf{k}\{\underline{T}/\underline{r}\}$ . We have

$$\begin{aligned} f(y) - f(x) &= \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} ((x_1 + h)^{\alpha_1} - x_1^{\alpha_1}) x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ &= \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_{\alpha} x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k. \end{aligned}$$

Note that

$$\left\| \binom{\alpha_1}{k} a_{\alpha} x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| r_1^k \leq \|a_{\alpha}\| r^{\alpha} \leq \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{\alpha \in \mathbb{N}^n} \max_{1 \leq k \leq \alpha_1} \left\{ \left\| \binom{\alpha_1}{k} a_{\alpha} x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| \|h\|^k \right\} \leq \max_k \left\{ \|f\|_g \left( \frac{\|h\|}{r_1} \right)^k \right\} \leq \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows.  $\square$

**Lemma 2.1.8.** Let  $\mathbf{k}$  be a complete non-archimedean field. Then we have  $\|f(x)\| \leq \|f\|$  for every  $f \in \mathbf{k}\{\underline{T}/\underline{r}\}$  and  $x \in E(0, \underline{r})$ . In particular, if  $f_n \rightarrow f$  as  $n \rightarrow +\infty$  in  $\mathbf{k}\{\underline{T}/\underline{r}\}$ , then we have  $\|f_n(x) - f(x)\| \rightarrow 0$  for every  $x \in E(0, \underline{r})$ .

*Proof.* Let  $f = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} T^{\alpha} \in \mathbf{k}\{\underline{T}/\underline{r}\}$  and  $x = (x_1, \dots, x_n) \in E(0, \underline{r})$ . We have

$$\left\| \sum_{|\alpha| < N} a_{\alpha} x^{\alpha} \right\| \leq \max_{|\alpha| < N} \|a_{\alpha}\| r^{\alpha} \leq \|f\|$$

for every  $N \in \mathbb{N}$ . Taking  $N \rightarrow +\infty$ , we have  $\|f(x)\| \leq \|f\|$ .  $\square$

**Proposition 2.1.9.** Let  $\mathbf{k}$  be a complete non-archimedean field with non-trivial valuation, and  $\partial_i = \partial/\partial T_i$  be the derivative operator on  $\mathbf{k}\{\underline{T}/\underline{r}\}$  with respect to the indeterminate  $T_i$  for  $i = 1, \dots, n$ .

Then for every  $f \in \mathbf{k}\{\underline{T}/\underline{r}\}$  and  $x \in E(0, \underline{r})$ , we have

$$F_{\partial_i(f)}(x) = \lim_{h \rightarrow 0} \frac{F_f(x_1, \dots, x_i + h, \dots, x_n) - F_f(x)}{h}.$$

*Proof.* Without loss of generality, we can assume that  $i = 1$ . Let  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/\underline{r}\}$  and  $f_n = \sum_{|\alpha| < n} a_\alpha T^\alpha$  for  $n \in \mathbb{N}$ . Set  $x_h = (x_1 + h, x_2, \dots, x_n)$  and  $L_f(h) = (F_f(x_h) - F_f(x))/h$  for  $h \in \mathbf{k}^\times$ . Note that for fixed  $h$ , we have  $\lim_{n \rightarrow \infty} L_{f_n}(h) = L_f(h)$ .

We compute  $L_{f_n}(h) - F_{\partial f_n}(x)$  explicitly:

$$\begin{aligned} L_{f_n}(h) - F_{\partial f_n}(x) &= \frac{1}{h} \left( \sum_{|\alpha| < n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} h^k x_2^{\alpha_2} \dots x_n^{\alpha_n} - \sum_{|\alpha| < n} \alpha_1 a_\alpha x_1^{\alpha_1-1} h x_2^{\alpha_2} \dots x_n^{\alpha_n} \right) \\ &= \sum_{|\alpha| < n} \sum_{k=2}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \dots x_n^{\alpha_n} h^{k-1}. \end{aligned}$$

Note that

$$M = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha x_1^{\alpha_1-1} x_2^{\alpha_2} \dots x_n^{\alpha_n}\| r_1^{k-1} \leq \|f\|/r_1 < +\infty.$$

Hence

$$\|L_{f_n}(h) - F_{\partial f_n}(x)\| \leq \max_{2 \leq k \leq n} \left\{ M \frac{\|h\|^{k-1}}{r_1^{k-1}} \right\} \leq M \frac{\|h\|}{r_1}$$

for  $h \in \mathbf{k}^\times$  with  $\|h\| < r_1$ . Taking  $n \rightarrow +\infty$ , we have

$$\|L_f(h) - F_{\partial f}(x)\| \leq M \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

**Yang:** The following should be a theorem.

**Corollary 2.1.10.** Let  $\mathbf{k}$  be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment  $f \mapsto F_f$  in [Proposition 2.1.6](#) is injective.

*Proof.* Note that if  $F_f = 0$ , then for every  $i = 1, \dots, n$ , we have  $F_{\partial_i(f)} = 0$  by [Proposition 2.1.9](#). By taking repeated derivatives, we have  $F_{\partial_\alpha f} = 0$  for every multi-index  $\alpha \in \mathbb{N}^n$ . Note that  $F_{\partial_\alpha f}(0) = \alpha! a_\alpha$ . It follows that  $a_\alpha = 0$  for every  $\alpha \in \mathbb{N}^n$  and thus  $f = 0$ . □

**Remark 2.1.11.** [Corollary 2.1.10](#) holds for non-archimedean fields of positive characteristic as well. The proof uses [Theorem 2.3.2](#) and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element  $f \in \mathbf{k}\{\underline{T}/\underline{r}\}$  with the associated function  $F_f : E(0, \underline{r}) \rightarrow \mathbf{k}$  as in [Proposition 2.1.6](#).

**Proposition 2.1.12.** Let  $\mathbf{k}$  be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra  $\mathbf{k}\{\underline{T}/\underline{r}\}$  coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in E(0, \underline{r})} \|f(x)\|_{\mathbf{k}}.$$



*Proof.* Let  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/r\}$ . We write  $f = g + h$  with  $g = \sum_{\alpha \in S} a_\alpha T^\alpha$  and  $h = \sum_{\alpha \notin S} a_\alpha T^\alpha$ , where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that  $S$  is a non-empty finite set and  $\|h\| < \|f\|$ . By Lemma 2.1.8, we have  $\|h(x)\| < \|f\|$  for every  $x \in E(0, \underline{r})$ . It suffices to show that  $\|g\|_{\sup} = \|g\|$ .

Since  $\mathbf{k}$  is algebraically closed,  $|\mathbf{k}^\times|$  is dense in  $\mathbb{R}_{>0}$ . For every pair  $\alpha, \beta \in S$  with  $\alpha \neq \beta$ , the set  $\{t \in \mathbb{R}_{>0}^n : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$  is a proper closed subset of  $\mathbb{R}_{>0}^n$ . Thus we can find  $t_m \in |\mathbf{k}^\times|^n$  such that  $t_m < r$ ,  $t_m \rightarrow r$  as  $m \rightarrow +\infty$  and for every  $\alpha, \beta \in S$  with  $\alpha \neq \beta$ , we have  $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$  for all  $m$ . For each  $m$ , we can find  $x_m \in E(0, \underline{r})$  such that  $\|x_m^\alpha\| = t_m^\alpha$  for every  $\alpha \in S$  since  $t_m \in |\mathbf{k}^\times|^n$ . It follows that

$$\|g(x_m)\| = \max_{\alpha \in S} \|a_\alpha\| \|x_m^\alpha\| = \max_{\alpha \in S} \|a_\alpha\| t_m^\alpha \rightarrow \|g\| \quad \text{as } m \rightarrow +\infty.$$

Thus  $\|g\|_{\sup} = \|g\|$ . □

**Remark 2.1.13.** If  $\mathbf{k}$  is locally compact (hence not algebraically closed), the gauss norm on the Tate algebra  $\mathbf{k}\{\underline{T}/r\}$  do not coincide with the supremum norm. For example, consider the Tate algebra  $\mathbb{Q}_p\{T\}$ . The element  $f = T^p - T$  has gauss norm  $\|f\| = 1$ . However, for every  $x \in E(0, 1) = \mathbb{Z}_p$ , we have  $f(x) = x^p - x \equiv 0 \pmod{p}$ . Thus  $\|f(x)\|_p \leq 1/p$  and  $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$ .

**Remark 2.1.14.** Recall that in classical complex analysis, the closure of the polynomial ring  $\mathbb{C}[T_1, \dots, T_n]$  with respect to the supremum norm on a closed polydisc  $E(0, \underline{r}) \subset \mathbb{C}^n$  is the ring of all complex-valued continuous functions which are analytic on its interior  $B(0, \underline{r})$ .

## 2.2 Local theory II: maps

Let  $\mathbf{k}$  be a complete non-archimedean field.

### 2.2.1 The first properties

Yang: Recall the Runge theorem in complex analysis.

**Definition 2.2.1.** A map  $f : (E(0, r) \subset \mathbf{k}^n) \rightarrow \mathbf{k}^m$  is called *analytic* if there exists power series  $f_1, \dots, f_m \in \mathbf{k}\{\underline{T}/r\}$  such that for any  $x \in E(0, r)$ , we have

$$f(x) = (f_1(x), \dots, f_m(x)).$$

Yang: To be revised.

Yang: Composition of analytic functions.

**Definition 2.2.2.** A map  $f : (E(0, \underline{r}) \subset \mathbf{k}^n) \rightarrow \mathbf{k}^m$  is called *analytic* if there exists power series  $f_1, \dots, f_m \in \mathbf{k}\{\underline{T}/r\}$  such that for any  $x \in E(0, \underline{r})$ , we have

$$f(x) = (f_1(x), \dots, f_m(x)).$$

**Proposition 2.2.3.** Let  $f : (E(0, \underline{r}) \subset \mathbf{k}^n) \rightarrow \mathbf{k}^m$  and  $g : (E(0, \underline{s}) \subset \mathbf{k}^m) \rightarrow \mathbf{k}^l$  be two analytic maps such that  $f(E(0, \underline{r})) \subset E(0, \underline{s})$ . Then the composition  $g \circ f : (E(0, \underline{r}) \subset \mathbf{k}^n) \rightarrow \mathbf{k}^l$  is also analytic.

Furthermore, if  $f = (f_1, \dots, f_m)$  and  $g = (g_1, \dots, g_l)$  with  $f_i = \sum_{\alpha} a_{i,\alpha} T^{\alpha}$  and  $g_j = \sum_{\beta} b_{j,\beta} T^{\beta}$ , then the composition  $g \circ f = (h_1, \dots, h_l)$  with

$$h_j = \sum_{\beta} b_{j,\beta} f^{\beta} = \sum_{\beta} b_{j,\beta} f_1^{\beta_1} \cdots f_m^{\beta_m}.$$

Yang: To be checked. Yang: To be revised.

*Proof.* Yang: To be completed. □

## 2.2.2 Inverse and implicit function

**Definition 2.2.4.** Let  $f : (E(0, \underline{r}) \subset \mathbf{k}^n) \rightarrow \mathbf{k}^m$  be an analytic map. The *tangent map*  $df_0 : \mathbf{k}^n \rightarrow \mathbf{k}^m$  of  $f$  at  $0$  is defined to be the linear map given by the Jacobian matrix

$$\left( \frac{\partial f_i}{\partial T_j}(0) \right)_{1 \leq i \leq m, 1 \leq j \leq n}.$$

Yang: To be checked.

**Theorem 2.2.5** (Inverse Function Theorem over Non-Archimedean Fields). Let  $f : (E(0, \underline{r}) \subset \mathbf{k}^n) \rightarrow \mathbf{k}^n$  be an analytic map. Suppose that  $f(0) = 0$  and the tangent map  $df_0 : \mathbf{k}^n \rightarrow \mathbf{k}^n$  is an isomorphism.

Then there exist  $E(0, \underline{r}') \subset E(0, \underline{r})$ ,  $E(0, \underline{s}') \subset f(E(0, \underline{r}))$  and an analytic map  $g : (E(0, \underline{s}') \subset \mathbf{k}^n) \rightarrow \mathbf{k}^n$  such that

$$f \circ g = \text{id}_{E(0, \underline{s}')} \quad g \circ f = \text{id}_{E(0, \underline{r}')}.$$

*Proof.* Yang: To be completed. □

**Theorem 2.2.6** (Implicit Function Theorem over Non-Archimedean Fields). Let  $f : (E(0, \underline{r}) \subset \mathbf{k}^{n+m}) \rightarrow \mathbf{k}^m$ ,  $(x_1, \dots, x_n, y_1, \dots, y_m) \mapsto f(x, y)$  be an analytic map. Suppose that  $f(0) = 0$  and the Jacobian matrix  $(\partial_j f_i(0))_{1 \leq i, j \leq m}$  is invertible.

Then there exist  $\underline{r}' = (r'_1, \dots, r'_n)$  with each  $r'_i > 0$  and an analytic map  $g : (E(0, \underline{r}') \subset \mathbf{k}^n) \rightarrow \mathbf{k}^m$  such that for any  $x \in E(0, \underline{r}')$ ,

$$f(x, y) = 0 \iff y = g(x).$$

*Proof.* Yang: To be completed. □

## 2.3 Analytic functions in one variable

**Proposition 2.3.1.** Let  $\mathbf{k}$  be a complete non-archimedean field and  $f = \sum_{n=0}^{+\infty} a_n T^n \in \mathbf{k}[[T]]$ . Set

$$R := \frac{1}{\limsup_{n \rightarrow +\infty} \|a_n\|^{1/n}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Then we have

- (a) the series  $f(x)$  converges for all  $x \in \mathbf{k}$  with  $\|x\| < R$  and diverges for all  $x \in \mathbf{k}$  with  $\|x\| > R$ ;
- (b) if  $R < +\infty$ , the series  $f(x)$  converges for all  $x \in \mathbf{k}$  with  $\|x\| = R$  if and only if  $\lim_{n \rightarrow +\infty} \|a_n\| R^n = 0$ .

*Proof.* By Proposition 1.2.7, we only need to check when the terms  $a_n x^n$  tend to zero as  $n \rightarrow +\infty$ . If  $\|x\| < R$ , there exists  $r \in (0, 1)$  such that  $\|x\| < r^2 R$ . Then there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $\|a_n\|^{1/n} < 1/(rR)$  and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n < \|a_n\| (r^2 R)^n < (r^2 R)^n \cdot \frac{1}{(rR)^n} = r^n \rightarrow 0.$$

Thus the series  $f(x)$  converges for all  $x \in \mathbf{k}$  with  $\|x\| < R$ .

Suppose that  $\|x\| > R$ . There exists  $s > 1$  such that  $\|x\| > R/s$ . By the definition of  $R$ , there exist infinitely many  $n \in \mathbb{N}$  such that  $\|a_n\|^{1/n} > s/R$  and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n > \|a_n\| \frac{R^n}{s^n} > \left(\frac{s}{R}\right)^n \cdot \frac{R^n}{s^n} = 1.$$

Thus the series  $f(x)$  diverges for all  $x \in \mathbf{k}$  with  $\|x\| > R$ .

Finally, the case  $\|x\| = R$  is direct from Proposition 1.2.7. Yang: To be revised. □

**Theorem 2.3.2** (Strassman). Let  $\mathbf{k}$  be a complete non-archimedean field with non-trivial valuation and  $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$  be an analytic function. Suppose that  $\|a_N\| r^N > \|a_n\| r^n$  for all  $n > N$ . Then  $f$  has at most  $N$  zeros in the closed ball  $E(0, r)$ .

*Proof.* We induct on  $N$ . The case  $N = 0$  is direct from Proposition 2.1.4. Suppose that the conclusion holds for  $N - 1$ . Let  $x$  be a zero of  $f$  in  $E(0, r)$ . Set

$$g(T) = \frac{f(T) - f(x)}{T - x} = \sum_{k=0}^{+\infty} \left( \sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n \geq k+1} \|a_n\| r^{n-1} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

It follows that  $g(T) \in \mathbf{k}\{T/r\}$ .

For every  $n > N$ , we have

$$\|a_N\| > \|a_n\| r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\| \sum_{n=N}^{N+m} a_n x^{n-N} \right\| = \|a_N\|$$

for every  $m \in \mathbb{N}$  by [Proposition 1.2.3](#). Take  $m \rightarrow +\infty$ , we have  $\|b_{N-1}\| = \|a_N\|$ . For every  $k > N-1$ , we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n\| r^{n-1} \leq \max_{n > N} \|a_n\| r^{n-1} < \|a_N\| r^{N-1} = \|b_{N-1}\| r^{N-1}.$$

By the induction hypothesis,  $g$  has at most  $N-1$  zeros in  $E(0, r)$ . It follows that  $f$  has at most  $N$  zeros in  $E(0, r)$  since  $f(T) = (T - x) \cdot g(T)$ .  $\square$

### 2.3.1 Entire functions

### 2.3.2 Maximum principle

## 2.4 Elementary functions

### 2.4.1 Exponential and logarithmic functions

Fix a prime number  $p$  in the following and consider  $\mathbf{k}$  being a complete non-archimedean field with  $|p| = p^{-1}$ . Let  $r_p := p^{-1/(p-1)}$ .

**Construction 2.4.1.** The *exponential function*  $\exp$  is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The *logarithmic function*  $\log$  is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

**Proposition 2.4.2.** We have the following properties:

- (a) the exponential function  $\exp$  converges on the open disk  $B(0, r_p)$ ;
- (b) the logarithmic function  $\log$  converges on the open disk  $B(1, 1)$ ;
- (c)  $|\exp(x) - 1| = |x|$  and  $|\log(1+x)| = |x|$  for all  $x \in B(0, r_p)$  or  $x \in B(1, r_p)$  respectively;
- (d) endow  $B(0, r_p)$  with the group structure induced by addition in  $\mathbf{k}$  and  $B(1, r_p)$  with the group structure induced by multiplication in  $\mathbf{k}$ , then  $\exp : B(0, r_p) \rightarrow B(1, r_p)$  is an isometric group isomorphism with inverse  $\log : B(1, r_p) \rightarrow B(0, r_p)$ .

*Proof.* For the convergent radius of exponential function, by [Lemma 2.4.3](#), noting that

$$\liminf_{n \rightarrow +\infty} \frac{s_n}{n} = 0,$$

we have

$$\limsup_{n \rightarrow +\infty} |n!|_p^{-1/n} = \limsup_{n \rightarrow +\infty} p^{v_p(n!)/n} = p^{\limsup_{n \rightarrow +\infty} (1 - (s_n/n))/(p-1)} = p^{1/(p-1)}.$$

That is, the convergent radius of the exponential function is  $r_p = p^{-1/(p-1)}$ . Considering  $n = p^m$ , we have

$$|p^m!|_p r_p^n = p^{(p^m-1)/(p-1)} \cdot p^{-p^m/(p-1)} = p^{-1/(p-1)} \neq 0.$$

Hence the convergent domain of the exponential function is  $B(0, r_p)$ .

For the logarithmic function, we have

$$\limsup_{n \rightarrow +\infty} |n|_p^{-1/n} = \limsup_{n \rightarrow +\infty} p^{v_p(n)/n} = p^0 = 1.$$

And  $|1/(np+1)|_p = 1$  for all  $n \in \mathbb{N}$ . Thus, the convergent domain of the logarithmic function is  $B(1, 1)$ .

For  $x \in B(0, r_p)$ , we have

$$\left| \frac{x^{n-1}}{n!} \right|_p < r_p^{n-1} \cdot p^{v_p(n!)} = p^{v_p(n!)-(n-1)/(p-1)} \leq 1.$$

Hence  $|x^n/n!|_p < |x|_p$  for all  $n \geq 2$  and thus

$$|\exp(x) - 1|_p = \left| \sum_{n=1}^{+\infty} \frac{x^n}{n!} \right|_p = |x|_p.$$

For  $x+1 \in B(1, r_p)$ , setting  $|x|_p = p^{-t}$  with  $t \geq 1/(p-1)$ , we have

$$\left| \frac{x^{n-1}}{n} \right|_p = p^{v_p(n)-(n-1)t} \leq p^{v_p(n!)-t(n-1)} \leq p^{(1/(p-1)-t)(n-1)} \leq 1, \quad \forall n \geq 2.$$

Similarly, we have  $|x^n/n|_p < |x|_p$  and hence  $|\log(1+x)|_p = |x|_p$ .

The identities

$$\exp(X+Y) = \exp(X) \cdot \exp(Y),$$

$$\log((1+X)(1+Y)) = \log(1+X) + \log(1+Y),$$

$$\exp(\log(1+X)) = 1+X,$$

$$\log(\exp(X)) = X$$

are purely formal and holds for indeterminates  $X$  and  $Y$ . Easy to check that  $\exp(X+Y), \log(1+X) + \log(1+Y) \in \mathbf{k}\{X/r_p, Y/r_p\}$ . Thus, the assertion (d) follows from (c) and [Proposition 2.1.6](#).  $\square$

Recall the following useful lemma regarding the  $p$ -adic valuation of factorials.

**Lemma 2.4.3.** Let  $p$  be a prime number and  $n \in \mathbb{N}$ , write  $n = \sum_{k=0}^m a_k p^k$  in the  $p$ -adic expansion

and set  $s_n := \sum_{k=0}^m a_k$ . Then

$$v_p(n!) = \frac{n - s_n}{p - 1}.$$

*Proof.* Yang: To be added. □

**Corollary 2.4.4.** Let  $\mathbf{k}$  be a complete non-archimedean field with  $|p| = p^{-1}$ . The multiplication group

$$\mathbf{k}^\times \cong |\mathbf{k}^\times| \times \mathcal{K}_{\mathbf{k}}^\times \times \mathbf{k}^{\circ\circ}$$

where  $\mathcal{K}_{\mathbf{k}}$  is the residue field of  $\mathbf{k}$ . Yang: To be revised.

*Proof.* Yang: To be added. □

**Proposition 2.4.5.** Suppose that  $\mathbf{k} = \mathbb{k}$  is algebraically closed. The logarithmic function  $\log$  defines a surjective group homomorphism  $1 + \mathbb{k}^{\circ\circ} \rightarrow \mathbb{k}$  with kernel the group  $\mu_{p^\infty}$  of all  $p$ -power roots of unity. Yang: To be checked.

*Proof.* □

Yang: continuation of exponential and logarithmic

## 2.4.2 Mahler series

**Notation 2.4.6.** We use  $\binom{x}{n}$  to denote the *binomial polynomial* defined by

$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}.$$

**Definition 2.4.7.** Fix a sequence  $\{a_n\}_{n \in \mathbb{N}}$  in  $\mathbf{k}$ . The *Mahler series* associated to  $\{a_n\}$  is defined to be the formal series

$$f(x) := \sum_{n=0}^{+\infty} a_n \binom{x}{n}.$$

Yang: To be checked.

**Proposition 2.4.8.**

**Theorem 2.4.9.** The series converges.

# Chapter 3

## Affinoid algebras

### 3.1 Normed rings and modules

#### 3.1.1 Semi-normed algebraic structures

**Definition 3.1.1.** Let  $G$  be an abelian group. A *semi-norm* on  $G$  is a function  $\|\cdot\| : G \rightarrow \mathbb{R}_{\geq 0}$  such that

- $\|0\| = 0$ ;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$ .

Suppose that  $R$  is a ring (commutative with unity) and  $\|\cdot\|$  is a semi-norm on the underlying abelian group of  $R$ . We further require that

- $\|1\| = 1$ ;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$ .

Suppose that  $(M, \|\cdot\|_M)$  is an  $R$ -module and  $\|\cdot\|_M$  is a semi-norm on the underlying abelian group of  $M$ . We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$ .

Suppose that  $(A, \|\cdot\|_A)$  is an  $R$ -algebra and  $\|\cdot\|_A$  is a semi-norm on the underlying  $R$ -module of  $A$ . We further require that this semi-norm is a semi-norm on the underlying ring of  $A$ .

**Definition 3.1.2.** Let  $A$  be an abelian group (or ring,  $R$ -module,  $R$ -algebra) equipped with a semi-norm  $\|\cdot\|$ . If  $\forall x \in A, \|x\| = 0 \iff x = 0$ , then we say  $\|\cdot\|$  is a *norm*.

Yang: Note that this definition of semi-normed module is a little different of [Ber90]

**Definition 3.1.3.** Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two semi-norms on an abelian group (or ring,  $R$ -module,  $R$ -algebra)  $A$ . We say  $\|\cdot\|_1$  is *bounded* by  $\|\cdot\|_2$  if there exists a constant  $C > 0$  such that  $\forall x \in A, \|x\|_1 \leq C\|x\|_2$ . If  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are bounded by each other, we say they are *equivalent*.

**Remark 3.1.4.** Equivalent semi-norms induce the same topology on  $A$ .

**Definition 3.1.5.** Let  $M$  be a semi-normed abelian group (or ring,  $R$ -module,  $R$ -algebra) and  $N \subseteq M$  be a subgroup (or ideal,  $R$ -submodule, ideal). The *residue semi-norm* on the quotient group  $M/N$  is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Yang: Is this always a semi-norm? In particular,  $\|1\| = 1$ ?

Unless otherwise specified, we always equip the quotient  $M/N$  with the residue semi-norm.

**Remark 3.1.6.** The residue semi-norm is a norm if and only if  $N$  is closed in  $M$ .

**Definition 3.1.7.** Let  $M$  and  $N$  be two semi-normed abelian groups (or rings,  $R$ -modules,  $R$ -algebras). A homomorphism  $f : M \rightarrow N$  is called *bounded* if there exists a constant  $C > 0$  such that  $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$ .

A bounded homomorphism  $f : M \rightarrow N$  is called *admissible* if the induced isomorphism  $M/\ker f \rightarrow \operatorname{Im} f$  is an isometry, i.e.,  $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$ .

**Definition 3.1.8.** A semi-norm  $\|\cdot\|$  on a ring  $R$  is called *multiplicative* if  $\forall x, y \in R, \|xy\| = \|x\|\|y\|$ . It is called *power-multiplicative* if  $\forall x \in R, \|x^n\| = \|x\|^n$  for all integers  $n \geq 1$ .

### 3.1.2 Banach rings

**Definition 3.1.9.** A (semi-)norm on an abelian group  $M$  induces a (pseudo-)metric  $d(x, y) = \|x - y\|$  on  $M$ . A (semi-)normed abelian group  $M$  is called *complete* if it is complete as a (pseudo-)metric space.

**Definition 3.1.10.** A *banach ring* is a complete normed ring.

Yang: The counterpart of prime ideal is multiplicative seminorm.

**Definition 3.1.11.** Let  $(A, \|\cdot\|_A)$  be a (semi-)normed algebraic structure, e.g., a (semi-)normed abelian group, a (semi-)normed ring, or a (semi-)normed module. The *completion* of  $A$ , denoted by  $\hat{A}$ , is the completion of  $A$  as a (pseudo-)metric space. Since  $A$  is dense in its completion, the algebraic operations and (semi-)norms on  $A$  can be uniquely extended to the completion.

Let  $R$  be a normed ring and  $M, N$  be semi-normed  $R$ -modules. There is a natural semi-norm on the tensor product  $M \otimes_R N$  defined as

$$\|z\|_{M \otimes_R N} = \inf \left\{ \sum_i \|x_i\|_M \|y_i\|_N : z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N \right\}.$$

**Definition 3.1.12.** Let  $R$  be a complete normed ring and  $M, N$  complete semi-normed  $R$ -modules. The *complete tensor product*  $M \hat{\otimes}_R N$  is defined as the completion of the semi-normed  $R$ -module  $M \otimes_R N$ .



**Definition 3.1.13.** Let  $R$  be a banach ring. For each  $f \in R$ , the *spectral radius* of  $f$  is defined as

$$\rho(f) = \lim_{n \rightarrow \infty} \|f^n\|^{1/n}.$$

Yang: Since ,  $\rho(f)$  exists.

**Definition 3.1.14.** A banach ring  $R$  is called *uniform* if its norm is power-multiplicative.

**Proposition 3.1.15.** Let  $(R, \|\cdot\|)$  be a banach ring. The spectral radius  $\rho(\cdot)$  defines a power-multiplicative semi-norm on  $R$  that is bounded by  $\|\cdot\|$ .

*Proof.* Yang: To be continued. □

**Definition 3.1.16.** Let  $R$  be a banach ring. An element  $f \in R$  is called *quasi-nilpotent* if  $\rho(f) = 0$ . All quasi-nilpotent elements of  $R$  form an ideal, denoted by  $\text{Qnil}(R)$ .

**Definition 3.1.17.** Let  $R$  be a banach ring. The *uniformization* of  $R$ , denoted by  $R \rightarrow R^u$ , is the banach ring with the universal property among all bounded homomorphisms from  $R$  to uniform banach rings. Yang: To be continued.

**Proposition 3.1.18.** Let  $R$  be a banach ring. The completion of  $R/\text{Qnil}(R)$  with respect to the spectral radius  $\rho(\cdot)$  is the uniformization of  $R$ .

*Proof.* Yang: To be continued. □

### 3.1.3 Complete tensor product

#### 3.1.4 Examples

**Example 3.1.19.** Let  $R$  be arbitrary ring. The *trivial norm* on  $R$  is defined as  $\|x\| = 0$  if  $x = 0$  and  $\|x\| = 1$  if  $x \neq 0$ . The ring  $R$  equipped with the trivial norm is a normed ring.

**Example 3.1.20.** The fields  $\mathbb{C}$  and  $\mathbb{R}$  equipped with the usual absolute value are complete fields.

**Example 3.1.21.** The field  $\mathbb{Q}_p$  of  $p$ -adic numbers equipped with the  $p$ -adic norm is a complete non-Archimedean field.

**Example 3.1.22.** Let  $R$  be a banach ring and  $r > 0$  be a real number. We define the ring of absolutely convergent power series over  $\mathbf{k}$  with radius  $r$  as

$$R \langle T/r \rangle := \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm  $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$ , the ring  $R \langle T/r \rangle$  is a banach ring.

When  $R = \mathbf{k}$  is a Yang: To be checked.

**Example 3.1.23.** Let  $\mathbf{k}$  be a non-Archimedean complete field. We define

$$\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\} := \left\{ \sum_{I \in \mathbb{N}^n} a_I T^I \in \mathbf{k}[[T_1, \dots, T_n]] : \lim_{|I| \rightarrow \infty} |a_I| r^I = 0 \right\},$$

where  $r = (r_1, \dots, r_n)$  is an  $n$ -tuple of positive real numbers,  $T^I = T_1^{i_1} \dots T_n^{i_n}$  for  $I = (i_1, \dots, i_n)$ , and

$|I| = i_1 + \cdots + i_n$ . Equipped with the norm  $\|\sum_{I \in \mathbb{N}^n} a_I T^I\| = \sup_{I \in \mathbb{N}^n} |a_I| r^I$ , the affinoid  $\mathbf{k}$ -algebra  $\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$  is a banach  $\mathbf{k}$ -algebra. This is called the *Tate algebra* over  $\mathbf{k}$  with polyradius  $r$  equipped with the *Gauss norm*. We will denote  $\mathbf{k}\{\underline{T}/r\} = \mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$  for simplicity.

Yang: To be continued...

## 3.2 Affinoid algebras

### 3.2.1 The first properties

**Definition 3.2.1.** Let  $\mathbf{k}$  be a non-archimedean field. A banach  $\mathbf{k}$ -algebra  $A$  is called a *affinoid  $\mathbf{k}$ -algebra* if there exists an admissible surjective homomorphism

$$\varphi : \mathbf{k}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \twoheadrightarrow A$$

for some  $n \in \mathbb{N}$  and  $r_1, \dots, r_n \in \mathbb{R}_{>0}$ .

If one can choose  $r_1 = \cdots = r_n = 1$ , then we say that  $A$  is a *strict affinoid  $\mathbf{k}$ -algebra*.

**Definition 3.2.2.** Let  $\mathbf{k}$  be a non-archimedean field. We define the *ring of restricted Laurent series* over  $\mathbf{k}$  as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n : a_n \in \mathbf{k}, \lim_{|n| \rightarrow \infty} |a_n| r^n = 0 \right\}$$

equipped with the norm

$$\|f\| = \sup_{n \in \mathbb{Z}} |a_n| r^n.$$

Yang: Is  $\mathbf{K}_r$  always a field? Yang: Do we have  $\mathbf{L}_{\mathbf{k},r} = \text{Frac}(\mathbf{k}\{T/r\})$ ?

**Proposition 3.2.3.** Let  $\mathbf{k}$  be a non-archimedean field. If  $r \notin \sqrt{|\mathbf{k}^\times|}$ , then  $\mathbf{K}_r$  is a complete non-archimedean field with non-trivial absolute value extending that of  $\mathbf{k}$ .

**Proposition 3.2.4.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. Then  $A$  is noetherian, and every ideal of  $A$  is closed.

*Proof.* Yang: To be completed. □

**Proposition 3.2.5.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. Then there exists a constant  $C > 0$  and  $N > 0$  such that for all  $f \in A$  and  $n \geq N$ , we have

$$\|f^n\| \leq C \rho(f)^n.$$

*Proof.* Yang: To be completed. □

**Proposition 3.2.6.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. If and only if  $\rho(f) \in \sqrt{|\mathbf{k}|}$  for all  $f \in A$ , then  $A$  is strict. Yang: To be complete.

*Proof.* Yang: To be completed. □

### 3.2.2 Noetherian normalization theorem

**Theorem 3.2.7.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. Then there exists a finite injective homomorphism

$$\varphi : \mathbf{k}\{r_1^{-1}T_1, \dots, r_d^{-1}T_d\} \hookrightarrow A$$

for some  $d \in \mathbb{N}$  and  $r_1, \dots, r_d \in \mathbb{R}_{>0}$ . **Yang: To be checked.**

### 3.2.3 Tate algebras and Weierstrass division

### 3.2.4 Reduction

**Definition 3.2.8.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. We define the *reduction* of  $A$  as

$$\tilde{A} = A^\circ / A^{\circ\circ},$$

where

$$A^\circ = \{f \in A : \|f\|_{\sup} \leq 1\}, \quad A^{\circ\circ} = \{f \in A : \|f\|_{\sup} < 1\}.$$

## 3.3 Finite modules

### 3.3.1 Finite banach module

There are three different categories of finite modules over an affinoid algebra  $A$ :

- The category  $\mathbf{Banmod}_A$  of finite banach  $A$ -modules with  $A$ -linear maps as morphisms.
- The category  $\mathbf{Banmod}_A^b$  of finite banach  $A$ -modules with bounded  $A$ -linear maps as morphisms.
- The category  $\mathbf{mod}_A$  of finite  $A$ -modules with all  $A$ -linear maps as morphisms.

**Theorem 3.3.1.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. Then the category of finite banach  $A$ -modules with bounded  $A$ -linear maps as morphisms is equivalent to the category of finite  $A$ -modules with  $A$ -linear maps as morphisms. **Yang: To be revised.**

For simplicity, we will just write  $\mathbf{mod}_A$  to denote the category of finite banach  $A$ -modules with bounded  $A$ -linear maps as morphisms.



# Chapter 4

## Berkovich spaces

### 4.1 Spectrum

Let  $\mathbf{k}$  be a spherically complete non-archimedean field which is algebraically closed and  $A = \mathbf{k}[T]$ . We want to consider the “analytic structure” on  $\mathbf{mSpec} A$ . However, unlike the complex case, the set  $\mathbf{mSpec} A$  is totally disconnected with respect to the topology induced by the absolute value on  $\mathbf{k}$  (Corollary 1.2.6). To overcome this difficulty, Berkovich uses multiplicative semi-norms to “fill in the gaps” between the points in  $\mathbf{mSpec} A$ , leading to the notion of the spectrum of a Banach ring.

We first consider the local model. Hence we should consider the Tate algebra  $\mathbf{k}\{T\}$  instead of the polynomial ring  $\mathbf{k}[T]$ . Yang: The maximal ideal of  $\mathbf{k}\{T\}$  corresponding to the point in the disk  $\{a \in \mathbf{k} : a \leq 1\}$ . Yang: Closed or open disk?

#### 4.1.1 Definition

**Definition 4.1.1.** Let  $R$  be a Banach ring. The *spectrum*  $\mathcal{M}(R)$  of  $R$  is defined as the set of all multiplicative semi-norms on  $R$  that are bounded with respect to the given norm on  $R$ . For every point  $x \in \mathcal{M}(R)$ , we denote the corresponding multiplicative semi-norm by  $|\cdot|_x$ .

We equip  $\mathcal{M}(R)$  with the weakest topology such that for each  $f \in R$ , the evaluation map  $\mathcal{M}(R) \rightarrow \mathbb{R}_{\geq 0}$ , defined by  $x \mapsto |f|_x =: f(x)$ , is continuous.

**Example 4.1.2.** Let  $(\mathbf{k}, |\cdot|)$  be a complete valuation field. The spectrum  $\mathcal{M}(\mathbf{k})$  consists of a single point corresponding to the given absolute value  $|\cdot|$  on  $\mathbf{k}$ . Yang: To be checked.

**Example 4.1.3.** Consider the Banach ring  $(\mathbb{Z}, \|\cdot\|)$  with  $\|\cdot\| = |\cdot|_\infty$  is the usual absolute value norm on  $\mathbb{Z}$ . Let  $|\cdot|_p$  denote the  $p$ -adic norm for each prime number  $p$ , i.e.,  $|n|_p = p^{-v_p(n)}$  for each  $n \in \mathbb{Z}$ , where  $v_p(n)$  is the  $p$ -adic valuation of  $n$ . The spectrum

$$\mathcal{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0, 1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0, \infty]\} \cup \{|\cdot|_0\},$$

where  $|a|_p^\infty := \lim_{\alpha \rightarrow \infty} |a|_p^\alpha$  for each  $a \in \mathbb{Z}$  and  $|\cdot|_0$  is the trivial norm on  $\mathbb{Z}$ . Yang: To be checked.

**Definition 4.1.4.** Let  $\varphi : R \rightarrow S$  be a bounded ring homomorphism of Banach rings. The *pullback* map  $\mathcal{M}(\varphi) : \mathcal{M}(S) \rightarrow \mathcal{M}(R)$  is defined by  $\mathcal{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$  for each  $x \in \mathcal{M}(S)$ .

**Proposition 4.1.5.** Let  $\varphi : R \rightarrow S$  be a bounded ring homomorphism of Banach rings. The pullback map  $\mathcal{M}(\varphi) : \mathcal{M}(S) \rightarrow \mathcal{M}(R)$  is continuous.

*Proof.* Yang: To be completed. □

**Definition 4.1.6.** Let  $R$  be a Banach ring. For each  $x \in \mathcal{M}(R)$ , the *completed residue field* at the point  $x$  is defined as the completion of the residue field  $\kappa(x) = \text{Frac}(R/\mathfrak{o}_x)$  with respect to the multiplicative norm induced by the semi-norm  $|\cdot|_x$ , denoted by  $\mathcal{H}(x)$ .

**Definition 4.1.7.** Let  $R$  be a Banach ring. The *Gel'fand transform* of  $R$  is the bounded ring homomorphism

$$\Gamma : R \rightarrow \prod_{x \in \mathcal{M}(R)} \mathcal{H}(x), \quad f \mapsto (f(x))_{x \in \mathcal{M}(R)},$$

where the norm on the product  $\prod_{x \in \mathcal{M}(R)} \mathcal{H}(x)$  is given by the supremum norm.

**Proposition 4.1.8.** The Gel'fand transform  $\Gamma : R \rightarrow \prod_{x \in \mathcal{M}(R)} \mathcal{H}(x)$  of a Banach ring  $R$  factors through the uniformization  $R^u$  of  $R$ , and the induced map  $R^u \rightarrow \prod_{x \in \mathcal{M}(R)} \mathcal{H}(x)$  is an isometric embedding. Yang: To be checked.

**Theorem 4.1.9.** Let  $R$  be a Banach ring. The spectrum  $\mathcal{M}(R)$  is nonempty.

*Proof.* Yang: To be continued. □

**Lemma 4.1.10.** Let  $\{K_i\}_{i \in I}$  be a family of completed fields. Consider the Banach ring  $R = \prod_{i \in I} K_i$  equipped with the product norm. The spectrum  $\mathcal{M}(R)$  is homeomorphic to the Stone-Ćech compactification of the discrete space  $I$ .

**Remark 4.1.11.** The Stone-Ćech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

**Theorem 4.1.12.** Let  $R$  be a Banach ring. The spectrum  $\mathcal{M}(R)$  is a compact Hausdorff space.

*Proof.* Yang: To be added. □

**Proposition 4.1.13.** Let  $K/k$  be a Galois extension of complete fields, and let  $R$  be a Banach  $k$ -algebra. The Galois group  $\text{Gal}(K/k)$  acts on the spectrum  $\mathcal{M}(R \hat{\otimes}_k K)$  via

$$g \cdot x : f \mapsto |(1 \otimes g^{-1})(f)|_x$$

for each  $g \in \text{Gal}(K/k)$ ,  $x \in \mathcal{M}(R \hat{\otimes}_k K)$  and  $f \in R \hat{\otimes}_k K$ . Moreover, the natural map  $\mathcal{M}(R \hat{\otimes}_k K) \rightarrow \mathcal{M}(R)$  induces a homeomorphism

$$\mathcal{M}(R \hat{\otimes}_k K) / \text{Gal}(K/k) \xrightarrow{\sim} \mathcal{M}(R).$$

Yang: To be checked.

### 4.1.2 Reduction map and kernel map

**Proposition 4.1.14.** Let  $R$  be a Banach ring. For each  $x \in \mathcal{M}(R)$ , let  $\wp_x$  be the kernel of the multiplicative semi-norm  $|\cdot|_x$ . Then  $\wp_x$  is a closed prime ideal of  $R$ , and  $x \mapsto \wp_x$  defines a continuous map from  $\mathcal{M}(R)$  to  $\text{Spec}(R)$  equipped with the Zariski topology.

*Proof.* Yang: To be completed □

Suppose that  $R$  is a non-archimedean Banach ring with valuation subring  $R^\circ$  and maximal ideal  $R^{\circ\circ}$ . There is a natural reduction map from the spectrum  $\mathcal{M}(R)$  to the spectrum  $\text{Spec}(\tilde{R} = R^\circ/R^{\circ\circ})$  of the residue ring  $\tilde{R}$ .

Yang: To be continued.

### 4.1.3 Spectrum of Tate algebras

**Spectrum of Tate algebra in one variable** Let  $\mathbf{k}$  be a complete non-archimedean field, and let  $A = \mathbf{k}\{T/r\}$ . We list some types of points in the spectrum  $\mathcal{M}(A)$ .

For each  $a \in \mathbf{k}$  with  $|a| \leq r$ , we have the *type I* point  $x_a$  corresponding to the evaluation at  $a$ , i.e.,  $|f|_{x_a} := |f(a)|$  for each  $f \in A$ . For each closed disk  $E = E(a, s) := \{b \in \mathbf{k} : |b - a| \leq s\}$  with center  $a \in \mathbf{k}$  and radius  $s \leq r$ , we have the point  $x_{a,s}$  corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} := \sup_{b \in E(a,s)} |f(b)|$$

for each  $f \in A$ . If  $s \in |\mathbf{k}^\times|$ , then the point  $x_E$  is called a *type II* point; otherwise, it is called a *type III* point.

Let  $\{E^{(s)}\}_s$  be a family of closed disks in  $\mathbf{k}$  such that  $E^{(s)}$  is of radius  $s$ ,  $E^{(s_1)} \subsetneq E^{(s_2)}$  for any  $s_1 < s_2$  and  $\bigcap_s E^{(s)} = \emptyset$ . Then we have the point  $x_{\{E^{(s)}\}}$  corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E^{(s)}\}}} := \inf_s |f|_{x_{E^{(s)}}}$$

for each  $f \in A$ . Such a point is called a *type IV* point.

Yang: To be completed.

**Proposition 4.1.15.** The points in the spectrum  $\mathcal{M}(\mathbf{k}\{r^{-1}T\})$  can be classified into four types as described above. Yang: To be checked

*Proof.* Yang: To be completed. □

**Proposition 4.1.16.** The completed residue fields of the four types of points in the spectrum  $\mathcal{M}(\mathbf{k}\{r^{-1}T\})$  are described as follows:

- type I point  $x_a$ :  $\mathcal{H}(x_a)$  is isomorphic to  $\mathbf{k}$ ;
- type II point  $x_{a,s}$ :  $\mathcal{H}(x_{a,s}) \cong \hat{\mathbf{k}}((t))$ ;
- type III point  $x_{a,s}$ :  $\hat{\mathcal{H}}(x_{a,s}) \cong \hat{\mathbf{k}}$  and the value group  $|\mathcal{H}(x_{a,s})^\times|$  is generated by  $|\mathbf{k}^\times|$  and  $s$ ;
- type IV point  $x_{\{E^{(s)}\}}$ :  $\mathcal{H}(x_{\{E^{(s)}\}})$  is an immediate extension of  $\mathbf{k}$ .

Yang: To be checked.

**Example 4.1.17.** The completed residue field  $\mathcal{H}(x_a)$  for a type I point  $x_a$  with  $a \in \mathbf{k}$  and  $|a| \leq r$  is isomorphic to  $\mathbf{k}$ . Yang: To be complete.

**Spectrum of Tate algebra in several variables** Let  $\mathbf{k}$  be a complete non-archimedean field, and let  $A = \mathbf{k}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ . We can consider the spectrum  $\mathcal{M}(A)$  similarly.

## 4.2 Affinoid domains

Consider  $X = \mathcal{M}(A)$  with  $A = \mathbf{k}\{T_1, \dots, T_n\}$ . Yang: Not every open subset of  $X$  gives an affinoid space, that is, the completion of the ring of analytic functions on that open subset is not necessarily an affinoid algebra. Yang: Right? example?

### 4.2.1 Definition

**Definition 4.2.1.** Let  $A$  be a  $\mathbf{k}$ -affinoid algebra, and let  $X = \mathcal{M}(A)$  be the associated affinoid space. A closed subset  $V \subseteq X$  is called an *affinoid domain* if there exists a  $\mathbf{k}$ -affinoid algebra  $A_V$  and a morphism of  $\mathbf{k}$ -affinoid algebras  $\varphi : A \rightarrow A_V$  satisfying the following universal property: for every bounded homomorphism of  $\mathbf{k}$ -affinoid algebras  $\psi : A \rightarrow B$  such that the induced map on spectra  $\mathcal{M}(\psi) : \mathcal{M}(B) \rightarrow X$  has its image contained in  $V$ , there exists a unique bounded homomorphism  $\theta : A_V \rightarrow B$  such that the following diagram commutes:

$$\begin{array}{ccc} & A_V & \\ \varphi \nearrow & & \searrow \theta \\ A & \xrightarrow{\psi} & B \end{array}$$

In this case, we say that  $V$  is represented by the affinoid algebra  $A_V$ .

**Slogan** A closed subset  $V \subset X$  is an affinoid domain if the functor “ $\text{Mor}(-, V)$ ” is representable.

Yang: Why we consider closed subset rather than open subset?

**Construction 4.2.2.** Let  $f = (f_1, \dots, f_n)$  be a tuple of elements in  $A$  and  $r = (r_1, \dots, r_n)$  be a tuple of positive real numbers. Consider the closed subset of  $X$ :

$$X(\underline{f/r}) := \{x \in X : |f_i(x)| \leq r_i, 1 \leq i \leq n\}.$$

Such a closed subset is called a *Weierstrass domain* of  $X$ . Moreover, we can define a  $\mathbf{k}$ -affinoid algebra

$$A\{\underline{f/r}\} := A\{f_1/r_1, \dots, f_n/r_n\}.$$

Yang: The domain  $X(\underline{f/r})$  is represented by  $A\{\underline{f/r}\}$ .

**Construction 4.2.3.** Let  $f = (f_1, \dots, f_n), g = (g_1, \dots, g_m)$  be two tuples of elements in  $A$  and  $r = (r_1, \dots, r_n), s = (s_1, \dots, s_m)$  be two tuples of positive real numbers. Consider the following closed



subset of  $X$ :

$$X(\underline{f/r}; \underline{g/s}^{-1}) := \{x \in X : |f_i(x)| \leq r_i, |g_j(x)| \geq s_j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Such a closed subset is called a *Laurent domain* of  $X$ . Moreover, we can define a  $\mathbf{k}$ -affinoid algebra

$$A\{\underline{f/r}; \underline{g/s}^{-1}\} := A\{f_1/r_1, \dots, f_n/r_n, g_1^{-1}/s_1, \dots, g_m^{-1}/s_m\}.$$

Yang: The domain  $X(\underline{f/r}; \underline{g/s}^{-1})$  is represented by  $A\{\underline{f/r}; \underline{g/s}^{-1}\}$ .

**Construction 4.2.4.** Let  $f = (f_1, \dots, f_n), g$  be elements in  $A$  such that the ideal generated by them is the whole algebra  $A$ . Set  $p = (p_1, \dots, p_n)$  be a tuple of positive real numbers. We define the following closed subset of  $X$ :

$$X(\underline{f/p}, g) := \{x \in X : |f_i(x)| \leq p_i |g(x)|, 1 \leq i \leq n\}.$$

Such a closed subset is called a *rational domain* of  $X$ . Moreover, we can define a  $\mathbf{k}$ -affinoid algebra

$$A\langle \underline{f/p}, g^{-1} \rangle := A\left\langle \frac{f_1}{p_1 g}, \dots, \frac{f_n}{p_n g} \right\rangle,$$

which is the quotient of the Tate algebra

$$A\langle T_1, \dots, T_n \rangle$$

by the ideal generated by the elements  $p_i g T_i - f_i$  for  $1 \leq i \leq n$ . There is a natural bounded homomorphism  $\varphi : A \rightarrow A\langle \underline{f/p}, g^{-1} \rangle$  induced by the inclusion. It can be shown that the closed subset  $X(\underline{f/p}, g)$  is an affinoid domain represented by the affinoid algebra  $A\langle \underline{f/p}, g^{-1} \rangle$ . Yang: To be checked

Yang: We have a sequence of inclusion:

$$\{\text{Weierstrass domains}\} \subseteq \{\text{Laurent domains}\} \subseteq \{\text{Rational domains}\} \subseteq \{\text{Affinoid domains}\}.$$

**Proposition 4.2.5.** Let  $A$  be a  $\mathbf{k}$ -affinoid algebra, and let  $X = \mathcal{M}(A)$  be the associated affinoid space. Let  $V \subseteq X$  be an affinoid domain represented by the  $\mathbf{k}$ -affinoid algebra  $A_V$ . Then the natural bounded homomorphism  $\varphi : A \rightarrow A_V$  is flat.

We have  $\mathcal{M}(A_V) \cong V$ .

## 4.2.2 The Grothendieck topology of affinoid domains



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# Chapter 5

## Varieties



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# Chapter 6

## Height pairings



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