

Finite field extensions

1 Finite-dimensional vector space

Definition 1. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . A *norm* on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

- (a) $\|x\| = 0$ if and only if $x = 0$;
- (b) $\|ax\| = |a| \cdot \|x\|$;
- (c) $\|x + y\| \leq \|x\| + \|y\|$.

Example 2. Let \mathbf{k} be a valuation field and V a finite-dimensional vector space over \mathbf{k} with basis $\{e_1, e_2, \dots, e_n\}$. For any $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$, define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then $\|\cdot\|_{\max}$ is a norm on V , called the *maximal norm* with respect to the basis $\{e_1, e_2, \dots, e_n\}$.

Example 3. Setting as in [Example 2](#), for any $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$, define

$$\|x\|_1 := |a_1| + |a_2| + \dots + |a_n|.$$

Then $\|\cdot\|_1$ is also a norm on V .

Definition 4. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Lemma 5. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V .

Proof. The sufficiency is clear. Now suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on V . Hence the unit open ball with respect to $\|\cdot\|_1$ contains a unit open ball with respect to $\|\cdot\|_2$. That is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every $x \in V$ with $\|x\|_1 = 1$, we have $\|x\|_2 \geq C = C\|x\|_1$. By scaling, we get that for every $x \in V$,

$$\|x\|_2 \geq C\|x\|_1.$$

Similar for the other direction, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. \square

Proposition 6. Let V be a normed finite-dimensional vector space over a complete valuation field \mathbf{k} . Then V is complete.

Proof. **Yang:** To be added. □

Theorem 7. Let V be a finite-dimensional vector space over a complete field \mathbf{k} . Then all norms on V are equivalent.

Proof. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis as in [Example 2](#). Let $\|\cdot\|$ be any norm on V . It suffices to show that $\|\cdot\|$ and $\|\cdot\|_{\max}$ are equivalent. First we have

$$\|y\| \leq \sum_{i=1}^n |a_i| \|e_i\| \leq \left(\sum_{i=1}^n \|e_i\| \right) \|y\|_{\max}$$

for any $y = a_1 e_1 + a_2 e_2 + \dots + a_n e_n \in V$. It remains to show that there exists a constant $C > 0$ such that for any $y \in V$,

$$\|y\|_{\max} \leq C \|y\|.$$

Yang: To be added. □

Remark 8. If the base field \mathbf{k} is not complete, then [Theorem 7](#) may fail. For example, let $\mathbf{k} = \mathbb{Q}$ with the usual absolute value, and let $V = \mathbb{Q}[\alpha]$ with $\alpha^2 - \alpha - 1 = 0$. There are two embeddings of V into \mathbb{R} :

$$\iota_1 : a + b\alpha \mapsto a + b \frac{1 + \sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b \frac{1 - \sqrt{5}}{2}.$$

Define two norms on V by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where $|\cdot|$ is the usual absolute value on \mathbb{R} . Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent since $\iota_2(\alpha^n) \rightarrow 0$ as $n \rightarrow \infty$ while $\iota_1(\alpha^n) \rightarrow \infty$.

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

Lemma 9. Let \mathbf{k} be a complete field and V a normed finite-dimensional vector space over \mathbf{k} . Then

$$\|\cdot\| : \text{End}_{\mathbf{k}}(V) \rightarrow \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the \mathbf{k} -vector space $\text{End}_{\mathbf{k}}(V)$ satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \text{End}_{\mathbf{k}}(V).$$

Proof. First we show the existence of the supremum, i.e., there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\| \leq C \|x\|$. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis. Since all norms on V are bounded by each other by [Theorem 7](#), we only need to show that there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\|_1 \leq C \|x\|_{\max}$. Write $T(e_i) = \sum_{j=1}^n a_{ij} e_j$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^n x_i e_i \in V$, we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \left(\sum_{1 \leq i, j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of $\|\cdot\|$ are clear. It remains to show the triangle inequality

and sub-multiplicativity. For any $A, B \in \text{End}_{\mathbf{k}}(V)$, we have

$$\frac{\|(A+B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all $x \in V \setminus \{0\}$ gives $\|A+B\| \leq \|A\| + \|B\|$. We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$. Taking supremum we get $\|AB\| \leq \|A\| \cdot \|B\|$. \square

2 Finite field extensions

Lemma 10. Let \mathbf{k} be a complete field and \mathbf{l} a finite extension of \mathbf{k} . Then there exists an absolute value on \mathbf{l} extending the absolute value on \mathbf{k} .

Proof. Fix a norm $\|\cdot\|_V$ on the \mathbf{k} -vector space $V = \mathbf{l}$. The norm $\|\cdot\|_V$ induces an operator norm $\|\cdot\|_{\text{op}}$ on the \mathbf{k} -vector space $\text{End}_{\mathbf{k}}(V)$ as in Lemma 9. For any $a \in \mathbf{l}$, let $\mu_a \in \text{End}_{\mathbf{k}}(V)$ be the \mathbf{k} -linear map defined by multiplication by a . Note that $a \mapsto \mu_a$ gives an embedding of \mathbf{k} -algebras and if $a \in \mathbf{k}$, $\|\mu_a\|_{\text{op}} = \|a\|_{\mathbf{k}}$. Thus the restriction of $\|\cdot\|_{\text{op}}$ to \mathbf{l} gives an norm on \mathbf{l} extending that on \mathbf{k} . The normed ring $(\mathbf{l}, \|\cdot\|_{\text{op}})$ is a Banach ring since it is a finite-dimensional vector space over the complete field \mathbf{k} . By ??, there exists a multiplicative seminorm $\|\cdot\|_{\mathbf{l}}$ on \mathbf{l} bounded by $\|\cdot\|_{\text{op}}$. In particular, $\|\cdot\|_{\mathbf{l}}$ is bounded by $\|\cdot\|_{\mathbf{k}}$ on \mathbf{k} . On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus $\|\cdot\|_{\mathbf{l}}$ extends the absolute value on \mathbf{k} . \square

Theorem 11. Let \mathbf{k} be a complete field and \mathbf{l} a finite extension of \mathbf{k} . Then the absolute value on \mathbf{l} which extends the absolute value on \mathbf{k} is uniquely determined by the absolute value on \mathbf{k} . Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where $n = [\mathbf{l} : \mathbf{k}]$ and $N_{\mathbf{l}/\mathbf{k}}$ is the norm map from \mathbf{l} to \mathbf{k} .

Proof. Let $\|\cdot\|_{\mathbf{l}}$ be arbitrary absolute value on \mathbf{l} extending that on \mathbf{k} . We will show that $\|\cdot\|_{\mathbf{l}}$ must be equal to $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$. For any $a \in \mathbf{l}$, set $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$. Then $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that $\|b\|_{\mathbf{l}} = 1$ whenever $N_{\mathbf{l}/\mathbf{k}}(b) = 1$.

Note that the norm map $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \rightarrow \mathbf{k}$ is the determinant of the \mathbf{k} -linear map $\mu_b \in \text{End}_{\mathbf{k}}(V)$ defined by multiplication by b . Hence it is continuous on \mathbf{l} (since it is a polynomial in the entries of the matrix representation). If $\|b\|_{\mathbf{l}} < 1$, then $\|b^m\|_{\mathbf{l}} \rightarrow 0$ as $m \rightarrow \infty$. Thus $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \rightarrow 0$ as $m \rightarrow \infty$, contradicting the fact that $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$ for all m . Similarly, if $\|b\|_{\mathbf{l}} > 1$, then just consider b^{-1} . \square

Proposition 12. Let \mathbf{k} be an algebraically closed valuation field. Then its completion $\hat{\mathbf{k}}$ is also algebraically closed.

Proof. Let $f \in \hat{\mathbf{k}}[X]$ be a non-constant polynomial. We will show that f has a root in $\hat{\mathbf{k}}$. Take a sequence of polynomials $\{f_n\}_{n \in \mathbb{N}}$ in $\mathbf{k}[X]$ converging to f coefficient-wisely and of the same degree d . Since \mathbf{k} is algebraically closed, each f_n splits completely in \mathbf{k} and hence in $\hat{\mathbf{k}}$. Write $f_n(X) = \prod_{i=1}^d (X - \alpha_{n,i})$ with $\alpha_{n,i} \in \hat{\mathbf{k}}$.

Let \mathbf{l} be a finite extension of $\hat{\mathbf{k}}$ such that f has a root α in \mathbf{l} . For every $\varepsilon > 0$, if there are infinitely many n such that $\alpha_{n,i} \notin B(\alpha, \varepsilon)$ for all $1 \leq i \leq d$, then we have $|f_n(\alpha)| \geq \varepsilon^d$ for infinitely many n , contradicting the fact that $f_n(\alpha) \rightarrow f(\alpha) = 0$. Thus for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$, there exists $1 \leq i \leq d$ with $\alpha_{n,i} \in B(\alpha, \varepsilon)$. That is, we can find a sequence $\alpha_{n,i_n} \in \mathbf{k}$ converging to α . Since $\hat{\mathbf{k}}$ is complete, we have $\alpha \in \hat{\mathbf{k}}$. \square

3 Ramification and inertia

In this subsection, we study the extensions of absolute values on finite field extensions. Note that we do not assume the base field to be complete.

Definition 13. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K . We denote by $w|v$ if $w \in M_L$ is an absolute value on L extending v . For each $w|v$, we define the *ramification index* $e(w|v)$ and the *inertia degree* $f(w|v)$ by

$$e(w|v) := [|\hat{L}^\times|_w : |\hat{K}^\times|_v], \quad f(w|v) := \frac{[\hat{L} : \hat{K}]}{e(w|v)},$$

where \hat{K} and \hat{L} are the completions of K and L with respect to v and w , respectively.

Lemma 14. Suppose that v is non-archimedean and κ_v and ℓ_w are the residue fields of K and L with respect to v and w , respectively. Then we have

$$f(w|v) = [\ell_w : \kappa_v].$$

Remark 15. Yang: To be added.

Theorem 16. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K . Then we have

$$\sum_{w|v} e(w|v)f(w|v) = [L : K].$$

Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K . We have

$$L \otimes_K K_v \cong \prod_{w|v} L_w,$$

where the product is taken over all absolute values $w \in M_L$ extending v .

Theorem 17. Let \mathbf{k} be a number field. Then

$$M_{\mathbf{k}}^\infty = \{\text{embeddings } \sigma : \mathbf{k} \rightarrow \mathbb{C}\}$$

and

$$M_{\mathbf{k}}^f = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_{\mathbf{k}}\}.$$

Yang: To be revised.

Proposition 18 (Product formula). Let \mathbf{k} be a number field. For each $x \in \mathbf{k}^\times$, we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^\infty; \\ 1, & v \in M_{\mathbf{k}}^0. \end{cases}$$

Yang: To be revised.

Remark 19. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K . Suppose that v is non-archimedean. Yang: To be rewritten.

Appendix