# Non-archimedean analysis



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The main references for this chapter are [Gou97; Rob00; 李文威 18].

# 1 Tate algebras

## 1.1 Tate algebras

**Notation 1.1.** Let $T = (T_1, \ldots, T_n)$ be a tuple of $n$ indeterminates, $r = (r_1, \ldots, r_n)$ be a tuple of $n$ positive real numbers, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \ldots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \ldots, n$, we have $\alpha_i \leq \beta_i$;
- $E(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \ldots, n\}$ and $B(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \ldots, n\}$ for $x = (x_1, \ldots, x_n) \in \mathbf{k}^n$;
- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space $X$ indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

**Date**: November 23, 2025, **Author**: Tianle Yang, My Homepage

**Definition 1.2.** Let $\mathbf{k}$ be a complete non-archimedean field. Let $T = (T_1, \ldots, T_n)$ be a tuple of $n$ indeterminates and $r = (r_1, \ldots, r_n)$ be a tuple of $n$ positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$\mathbf{k}\langle \underline{r^{-1}T}\rangle := \mathbf{k}\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{|\alpha|\to+\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

**Proposition 1.3.** Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ is a non-archimedean multiplicative banach $\mathbf{k}$-algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha\in\mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Yang: For the definition of banach ring, see

*Proof.* The proof splits into several parts. Every parts is straightforward and standard.

**Step 1.** We first show that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha$ are two elements in $\mathbf{k}\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Hence $f \cdot g \in \mathbf{k}\{\underline{T/r}\}$ and it shows that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

**Step 2.** Show that the gauss norm is a non-archimedean norm on $\mathbf{k}\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f+g\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \leq \sup_{\alpha\in\mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \leq \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned} \|f\cdot g\| = \left\| \sum_{\gamma\in\mathbb{N}^n} \left( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma \right\| &= \sup_{\gamma\in\mathbb{N}^n} \left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \\ &\leq \sup_{\gamma\in\mathbb{N}^n} \max_{\alpha+\beta=\gamma} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \leq \|f\| \cdot \|g\|. \end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

**Step 3.** Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\| \cdot \|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \left\| \sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta \right\| r^{\alpha_1+\beta_1} \leq \|f\cdot g\|,$$

where the third equality holds since $(\alpha_1, \beta_1)$ is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by Proposition 4.2. Thus the gauss norm is multiplicative.

**Step 4.** Finally show that $\mathbf{k}\{\underline{T}/r\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m}T^\alpha\}$ be a cauchy sequence in $\mathbf{k}\{\underline{T}/r\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\|r^\alpha \leq \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in $\mathbf{k}$. Since $\mathbf{k}$ is complete, set $a_\alpha := \lim_{m\to+\infty} a_{\alpha,m}$ and $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\|r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\|r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\|r^\alpha + \|a_{\alpha,m}\|r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\|r^\alpha \leq 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T}/r\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\|r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\|r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\|r^\alpha \leq \varepsilon$$

for all $m > N$. □

**Proposition 1.4.** Let $\mathbf{k}$ be a complete non-archimedean field. An element $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/r\}$ is invertible if and only if $\|a_0\| > \|a_\alpha\|r^\alpha$ for all $\alpha \neq 0$.

*Proof.* Multiplying by $a_0^{-1}$, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of $f$ in $\mathbf{k}[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta\in\mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma\in\mathbb{N}^n} \left( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = -\sum_{\substack{\alpha+\beta=\gamma \\ \alpha\neq 0}} a_\alpha b_\beta.$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\|r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on $\gamma$ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\|r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha\neq 0}} \|a_\alpha\|r^\alpha \cdot \|b_\beta\|r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\|r^\beta \leq 1.$$

Suppose that the claim holds for $m$. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\|r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\|r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha\neq 0}} \|a_\alpha\|r^\alpha \cdot \|b_\beta\|r^\beta \leq \max\left\{A^{m+1}, A \cdot A^m\right\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in \mathbf{k}\{\underline{T/r}\}$. □

Let $\mathbf{k}$ be a complete non-archimedean field. Recall that the formal derivative operator $\partial_i : \mathbf{k}[[\underline{T}]] \to \mathbf{k}[[\underline{T}]]$ is defined by

$$\frac{\partial}{\partial T_i}\left(\sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha\right) := \sum_{\alpha\in\mathbb{N}^n} \alpha_i a_\alpha T_1^{\alpha_1} \cdots T_i^{\alpha_i - 1} \cdots T_n^{\alpha_n}.$$

**Lemma 1.5.** Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we have $\partial_i(f) \in \mathbf{k}\{\underline{T/r}\}$.

*Proof.* Suppose that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha\in\mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1-1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}.$$

Noting that $\mathbf{k}$ is non-archimedean, we have $\|\alpha_1 a_\alpha\| \leq \|a_\alpha\|$. Then

$$\lim_{|\alpha|\to+\infty} \|\alpha_1 a_\alpha\| r_1^{\alpha_1-1} r_2^{\alpha_2} \cdots r_n^{\alpha_n} \leq \frac{1}{r_1} \lim_{|\alpha|\to+\infty} \|a_\alpha\| r^\alpha = 0.$$

The conclusion follows. □

## 1.2 Analytic functions on closed polydisks

**Proposition 1.6.** Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we can associate a function $F_f : E(0,\underline{r}) \to \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha \qquad \text{for } x = (x_1, \ldots, x_n) \in E(0,\underline{r}).$$

This defines a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T/r}\}$ to the ring of all functions from $E(0,\underline{r})$ to $\mathbf{k}$.

*Proof.* Given $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \ldots, x_n) \in E(0,\underline{r})$, we have

$$\left\|\sum_{|\alpha|=n} a_\alpha x^\alpha\right\| \leq \max_{|\alpha|=n} \|a_\alpha\| r^\alpha \to 0 \qquad \text{as } n \to +\infty.$$

Hence by Proposition 4.3, the series $F_f(x) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha$ converges in $\mathbf{k}$. This defines a function $F_f : E(0,\underline{r}) \to \mathbf{k}$.

Let $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. Set

$$A_n = \sum_{|\alpha|<n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta|<n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma|<n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) x^\gamma.$$

We need to show that $F_f(x)F_g(x) = \lim A_n B_n = \lim C_n = F_{fg}(x)$. Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta|\geq n}} a_\alpha b_\beta x^{\alpha+\beta}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $n > 2N$, we have

$$\|A_n B_n - C_n\| \le \max_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta| \ge n}} \|a_\alpha\| \|b_\beta\| \|x^{\alpha+\beta}\| < \max\left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \le \varepsilon.$$

Thus $F_f(x) F_g(x) = (F_{fg})(x)$. The addition and scalar multiplication can be verified directly. We thus finish the proof. ☐

**Proposition 1.7.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x, y \in E(0, \underline{r})$, we have

$$\|f(y) - f(x)\|_{\mathbf{k}} \le L \cdot \|y - x\|_\infty,$$

where $L = \max_{1 \le i \le n} \|f\|_g / r_i$.

*Proof.* Set $y - x = (h_1, \dots, h_n)$ and $x^{(0)} = x$, $x^{(i)} = (x_1 + h_1, \dots, x_i + h_i, x_{i+1}, \dots, x_n)$ for $i = 1, \dots, n$. We have

$$\|f(y) - f(x)\|_{\mathbf{k}} \le \max_{1 \le i \le n} \|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}}.$$

We only need to show that for every $i = 1, \dots, n$, we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}} \le \frac{\|f\|_g}{r_i} \|h_i\|.$$

Without loss of generality and for simplicity, we assume that $y = (x_1 + h, x_2, \dots, x_n)$ and $x = (x_1, x_2, \dots, x_n)$. Note that by the strong triangle inequality, we have $\|h\| \le r_1$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\begin{aligned} f(y) - f(x) &= \sum_{\alpha \in \mathbb{N}^n} a_\alpha \left( (x_1 + h)^{\alpha_1} - x_1^{\alpha_1} \right) x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ &= \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k. \end{aligned}$$

Note that

$$\left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| r_1^k \le \|a_\alpha\| r^\alpha \le \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_{\mathbf{k}} \le \max_{\alpha \in \mathbb{N}^n} \max_{1 \le k \le \alpha_1} \left\{ \left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| \|h\|^k \right\} \le \max_k \left\{ \|f\|_g \left( \frac{\|h\|}{r_1} \right)^k \right\} \le \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. ☐

**Lemma 1.8.** Let $\mathbf{k}$ be a complete non-archimedean field. Then we have $\|f(x)\| \le \|f\|$ for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$. In particular, if $f_n \to f$ as $n \to +\infty$ in $\mathbf{k}\{\underline{T/r}\}$, then we have $\|f_n(x) - f(x)\| \to 0$ for every $x \in E(0, \underline{r})$.

*Proof.* Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \dots, x_n) \in E(0, \underline{r})$. We have

$$\left\| \sum_{|\alpha|<N} a_\alpha x^\alpha \right\| \le \max_{|\alpha|<N} \|a_\alpha\| r^\alpha \le \|f\|$$

for every $N \in \mathbb{N}$. Taking $N \to +\infty$, we have $\|f(x)\| \le \|f\|$. □

**Proposition 1.9.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation, and $\partial_i = \partial/\partial T_i$ be the derivative operator on $\mathbf{k}\{\underline{T/r}\}$ with respect to the indeterminate $T_i$ for $i = 1, \dots, n$. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$, we have

$$F_{\partial_i(f)}(x) = \lim_{h\to 0} \frac{F_f(x_1, \dots, x_i + h, \dots, x_n) - F_f(x)}{h}.$$

*Proof.* Without loss of generality, we can assume that $i = 1$. Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $f_n = \sum_{|\alpha|<n} a_\alpha T^\alpha$ for $n \in \mathbb{N}$. Set $x_h = (x_1 + h, x_2 \dots, x_n)$ and $L_f(h) = (F_f(x_h) - F_f(x))/h$ for $h \in \mathbf{k}^\times$. Note that for fixed $h$, we have $\lim_{n\to\infty} L_{f_n}(h) = L_f(h)$.

We compute $L_{f_n}(h) - F_{\partial f_n}(x)$ explicitly:

$$\begin{aligned} L_{f_n}(h) - F_{\partial f_n}(x) &= \frac{1}{h}\left( \sum_{|\alpha|<n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} h^k x_2^{\alpha_2} \cdots x_n^{\alpha_n} - \sum_{|\alpha|<n} \alpha_1 a_\alpha x_1^{\alpha_1-1} h x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right) \\ &= \sum_{|\alpha|<n} \sum_{k=2}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^{k-1}. \end{aligned}$$

Note that

$$M = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\| r_1^{k-1} \le \|f\|/r_1 < +\infty.$$

Hence

$$\|L_{f_n}(h) - F_{\partial f_n}(x)\| \le \max_{2\le k\le n} \left\{ M \frac{\|h\|^{k-1}}{r_1^{k-1}} \right\} \le M \frac{\|h\|}{r_1}$$

for $h \in \mathbf{k}^\times$ with $\|h\| < r_1$. Taking $n \to +\infty$, we have

$$\|L_f(h) - F_{\partial f}(x)\| \le M \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

Yang: The following should be a theorem.

**Corollary 1.10.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment $f \mapsto F_f$ in Proposition 1.6 is injective.

*Proof.* Note that if $F_f = 0$, then for every $i = 1, \dots, n$, we have $F_{\partial_i(f)} = 0$ by Proposition 1.9. By taking repeated derivatives, we have $F_{\partial^\alpha f} = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Note that $F_{\partial^\alpha f}(0) = \alpha! a_\alpha$. It follows that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n$ and thus $f = 0$. □

**Remark 1.11.** Corollary 1.10 holds for non-archimedean fields of positive characteristic as well. The proof uses Theorem 2.2 and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element $f \in \mathbf{k}\{\underline{T/r}\}$ with the associated function $F_f : E(0,\underline{r}) \to \mathbf{k}$ as in Proposition 1.6.

**Proposition 1.12.** Let $\Bbbk$ be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra $\Bbbk\{\underline{T/r}\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in E(0,\underline{r})} \|f(x)\|_{\Bbbk}.$$

*Proof.* Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \Bbbk\{\underline{T/r}\}$. We write $f = g + h$ with $g = \sum_{\alpha \in S} a_\alpha T^\alpha$ and $h = \sum_{\alpha \notin S} a_\alpha T^\alpha$, where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that $S$ is a non-empty finite set and $\|h\| < \|f\|$. By Lemma 1.8, we have $\|h(x)\| < \|f\|$ for every $x \in E(0,\underline{r})$. It suffices to show that $\|g\|_{\sup} = \|g\|$.

Since $\Bbbk$ is algebraically closed, $|\Bbbk^\times|$ is dense in $\mathbb{R}_{>0}$. For every pair $\alpha, \beta \in S$ with $\alpha \neq \beta$, the set $\{t \in \mathbb{R}^n_{>0} : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$ is a proper closed subset of $\mathbb{R}^n_{>0}$. Thus we can find $t_m \in |\Bbbk^\times|^n$ such that $t_m < r$, $t_m \to r$ as $m \to +\infty$ and for every $\alpha, \beta \in S$ with $\alpha \neq \beta$, we have $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$ for all $m$. For each $m$, we can find $x_m \in E(0,\underline{r})$ such that $\|x_m^\alpha\| = t_m^\alpha$ for every $\alpha \in S$ since $t_m \in |\Bbbk^\times|^n$. It follows that

$$\|g(x_m)\| = \max_{\alpha \in S} \|a_\alpha\| \|x_m^\alpha\| = \max_{\alpha \in S} \|a_\alpha\| t_m^\alpha \to \|g\| \quad \text{as } m \to +\infty.$$

Thus $\|g\|_{\sup} = \|g\|$. □

**Remark 1.13.** If $\mathbf{k}$ is not algebraically closed, the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ may not coincide with the supremum norm. For example, consider the Tate algebra $\mathbb{Q}_p\{T\}$. The element $f = T^p - T$ has gauss norm $\|f\| = 1$. However, for every $x \in E(0,1) = \mathbb{Z}_p$, we have $f(x) = x^p - x \equiv 0 \mod p$. Thus $\|f(x)\|_p \leq 1/p$ and $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$.

**Remark 1.14.** Recall the Weierstrass-Stone theorem in classical analysis which states that the closure of the polynomial ring $\mathbb{C}[T_1, \ldots, T_n]$ with respect to the supremum norm on a closed polydisc $E \subset \mathbb{C}^n$ is the ring of all complex-valued continuous functions on $E$. Yang: This is wrong. In the context of non-archimedean analysis, Proposition 1.12 can be viewed as an analogue of this theorem. It states that the closure of the polynomial ring $\Bbbk[T_1, \ldots, T_n]$ with respect to the supremum norm on a closed polydisc $E(0,\underline{r}) \subset \Bbbk^n$ is the Tate algebra $\Bbbk\{\underline{T/r}\}$.

From this perspective, the Tate algebra can be viewed as the "correct" analogue of the ring of continuous functions on a closed polydisc in non-archimedean analysis.

Yang: Recall the Runge theorem in complex analysis.

**Definition 1.15.** Let $\mathbf{k}$ be a complete non-archimedean field. A function $f : E(0,\underline{r}) \to \mathbf{k}$ is called *analytic* if there exists $F \in \mathbf{k}\{\underline{T/r}\}$ such that $f = F$ as functions from $E(0,\underline{r})$ to $\mathbf{k}$. Yang: To be revised.

Yang: Composition of analytic functions.

# 2 Analytic functions in one variable

**Proposition 2.1.** Let $\mathbf{k}$ be a complete non-archimedean field and $f = \sum_{n=0}^{+\infty} a_n T^n \in \mathbf{k}[[T]]$. Set

$$R := \frac{1}{\limsup_{n\to+\infty} \|a_n\|^{1/n}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Then we have

(a) the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$ and diverges for all $x \in \mathbf{k}$ with $\|x\| > R$;

(b) if $R < +\infty$, the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| = R$ if and only if $\lim_{n\to+\infty} \|a_n\| R^n = 0$.

*Proof.* By Proposition 4.3, we only need to check when the terms $a_n x^n$ tend to zero as $n \to +\infty$. If $\|x\| < R$, there exists $r \in (0,1)$ such that $\|x\| < r^2 R$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\|a_n\|^{1/n} < 1/(rR)$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n < \|a_n\| (r^2 R)^n < (r^2 R)^n \cdot \frac{1}{(rR)^n} = r^n \to 0.$$

Thus the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$.

Suppose that $\|x\| > R$. There exists $s > 1$ such that $\|x\| > R/s$. By the definition of $R$, there exist infinitely many $n \in \mathbb{N}$ such that $\|a_n\|^{1/n} > s/R$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n > \|a_n\| \frac{R^n}{s^n} > \left(\frac{s}{R}\right)^n \cdot \frac{R^n}{s^n} = 1.$$

Thus the series $f(x)$ diverges for all $x \in \mathbf{k}$ with $\|x\| > R$.

Finally, the case $\|x\| = R$ is direct from Proposition 4.3. Yang: To be revised. □

**Theorem 2.2** (Strassman)**.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$ be an analytic function. Suppose that $\|a_N\| r^N > \|a_n\| r^n$ for all $n > N$. Then $f$ has at most $N$ zeros in the closed ball $E(0, r)$.

*Proof.* We induct on $N$. The case $N = 0$ is direct from Proposition 1.4. Suppose that the conclusion holds for $N - 1$. Let $x$ be a zero of $f$ in $E(0, r)$. Set

$$g(T) = \frac{f(T) - f(x)}{T - x} = \sum_{k=0}^{+\infty} \left( \sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n \geq k+1} \|a_n\| r^{n-1} \to 0 \quad \text{as } k \to \infty.$$

It follows that $g(T) \in \mathbf{k}\{T/r\}$.

For every $n > N$, we have

$$\|a_N\| > \|a_n\| r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\| \sum_{n=N}^{N+m} a_n x^{n-N} \right\| = \|a_N\|$$

for every $m \in \mathbb{N}$ by Proposition 4.2. Take $m \to +\infty$, we have $\|b_{N-1}\| = \|a_N\|$. For every $k > N-1$, we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n\| r^{n-1} \leq \max_{n > N} \|a_n\| r^{n-1} < \|a_N\| r^{N-1} = \|b_{N-1}\| r^{N-1}.$$

By the induction hypothesis, $g$ has at most $N-1$ zeros in $E(0,r)$. It follows that $f$ has at most $N$ zeros in $E(0,r)$ since $f(T) = (T-x)\cdot g(T)$. □

## 2.1 Entire functions

## 2.2 Maximum principle

# 3 Elementary functions

## 3.1 Exponential and logarithmic functions

Fix a prime number $p$ in the following and consider $\mathbf{k}$ being a complete non-archimedean field with $|p| = p^{-1}$. Let $r_p := p^{-1/(p-1)}$.

**Construction 3.1.** The *exponential function* exp is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The *logarithmic function* log is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

**Proposition 3.2.** We have the following properties:

(a) the exponential function exp converges on the open disk $B(0, r_p)$;

(b) the logarithmic function log converges on the open disk $B(1,1)$;

(c) $|\exp(x) - 1| = |x|$ and $|\log(1+x)| = |x|$ for all $x \in B(0, r_p)$ or $x \in B(1, r_p)$ respectively;

(d) endow $B(0, r_p)$ with the group structure induced by addition in $\mathbf{k}$ and $B(1, r_p)$ with the group structure induced by multiplication in $\mathbf{k}$, then $\exp : B(0, r_p) \to B(1, r_p)$ is an isometric group isomorphism with inverse $\log : B(1, r_p) \to B(0, r_p)$.

*Proof.* For the convergent radius of exponential function, by Lemma 3.3, noting that

$$\liminf_{n\to+\infty}\frac{s_n}{n}=0,$$

we have

$$\limsup_{n\to+\infty}|n!|_p^{-1/n}=\limsup_{n\to+\infty}p^{v_p(n!)/n}=p^{\limsup_{n\to+\infty}(1-(s_n/n))/(p-1)}=p^{1/(p-1)}.$$

That is, the convergent radius of the exponential function is $r_p=p^{-1/(p-1)}$. Considering $n=p^m$, we have

$$|p^m!|_p r_p^n=p^{(p^m-1)/(p-1)}\cdot p^{-p^m/(p-1)}=p^{-1/(p-1)}\neq 0.$$

Hence the convergent domain of the exponential function is $B(0,r_p)$.

For the logarithmic function, we have

$$\limsup_{n\to+\infty}|n|_p^{-1/n}=\limsup_{n\to+\infty}p^{v_p(n)/n}=p^0=1.$$

And $|1/(np+1)|_p=1$ for all $n\in\mathbb{N}$. Thus, the convergent domain of the logarithmic function is $B(1,1)$.

For $x\in B(0,r_p)$, we have

$$\left|\frac{x^{n-1}}{n!}\right|_p<r_p^{n-1}\cdot p^{v_p(n!)}=p^{v_p(n!)-(n-1)/(p-1)}\leq 1.$$

Hence $|x^n/n!|_p<|x|_p$ for all $n\geq 2$ and thus

$$|\exp(x)-1|_p=\left|\sum_{n=1}^{+\infty}\frac{x^n}{n!}\right|_p=|x|_p.$$

For $x+1\in B(1,r_p)$, setting $|x|_p=p^{-t}$ with $t\geq 1/(p-1)$, we have

$$\left|\frac{x^{n-1}}{n}\right|_p=p^{v_p(n)-t(n-1)}\leq p^{v_p(n!)-t(n-1)}\leq p^{(1/(p-1)-t)(n-1)}\leq 1,\quad\forall n\geq 2.$$

Similarly, we have $|x^n/n|_p<|x|_p$ and hence $|\log(1+x)|_p=|x|_p$.

The identities

$$\begin{aligned}\exp(X+Y)&=\exp(X)\cdot\exp(Y),\\ \log((1+X)(1+Y))&=\log(1+X)+\log(1+Y),\\ \exp(\log(1+X))&=1+X,\\ \log(\exp(X))&=X\end{aligned}$$

are purely formal and holds for indeterminates $X$ and $Y$. Easy to check that $\exp(X+Y),\log(1+X)+\log(1+Y)\in\mathbf{k}\{X/r_p,Y/r_p\}$. Thus, the assertion (d) follows from (c) and Proposition 1.6. □

Recall the following useful lemma regarding the $p$-adic valuation of factorials.

**Lemma 3.3.** Let $p$ be a prime number and $n\in\mathbb{N}$, write $n=\sum_{k=0}^m a_kp^k$ in the $p$-adic expansion

and set $s_n := \sum_{k=0}^{m} a_k$. Then
$$v_p(n!) = \frac{n - s_n}{p - 1}.$$

*Proof.* Yang: To be added. □

**Corollary 3.4.** Let $\mathbf{k}$ be a complete non-archimedean field with $|p| = p^{-1}$. The multiplication group
$$\mathbf{k}^\times \cong |\mathbf{k}^\times| \times k_{\mathbf{k}}^\times \times \mathbf{k}^{\circ\circ}$$
where $k_{\mathbf{k}}$ is the residue field of $\mathbf{k}$. Yang: To be revised.

*Proof.* Yang: To be added. □

**Proposition 3.5.** Suppose that $\mathbf{k} = \Bbbk$ is algebraically closed. The logarithmic function log defines a surjective group homomorphism $1 + \Bbbk^{\circ\circ} \to \Bbbk$ with kernel the group $\mu_{p^\infty}$ of all $p$-power roots of unity. Yang: To be checked.

*Proof.* □

Yang: continuation of exponential and logarithmic

## 3.2 Mahler series

**Notation 3.6.** We use $\binom{x}{n}$ to denote the *binomial polynomial* defined by
$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}.$$

**Definition 3.7.** Fix a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbf{k}$. The *Mahler series* associated to $\{a_n\}$ is defined to be the formal series
$$f(x) := \sum_{n=0}^{+\infty} a_n \binom{x}{n}.$$
Yang: To be checked.

**Proposition 3.8.**

**Theorem 3.9.** The series converges.

# 4 Appendix

**Theorem 4.1.** Let $R$ be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

**Proposition 4.2.** Let $(X, d)$ be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.

**Proposition 4.3.** Let $(X, d)$ be an ultra-metric space. A sequence $\{x_n\}$ in $X$ is cauchy if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

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