Convergent and restricted power series

Notation 1. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \dots, n$, we have $\alpha_i \leq \beta_i$;
- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

1 Absolutely convergent power series

Definition 2. Let R be a banach ring and $r > 0$ be a real number. We define the *ring of absolutely convergent power series* over $\mathbf{k}$ with radius r as

$$R\langle T/r \rangle \coloneqq \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm $\| \sum_{n=0}^{\infty} a_n T^n \| := \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r \rangle$ is a banach ring.
For a tuple of n indeterminates $T = (T_1, \dots, T_n)$ and a tuple of n positive real numbers $r = (r_1, \dots, r_n)$, we define

$$R\left\langle \underline{T/r} \right\rangle \coloneqq R\langle T_1/r_1, \dots, T_n/r_n \rangle := R\langle T_1/r, \dots, T_{n-1}/r_{n-1} \rangle \langle T_n/r_n \rangle.$$

Note that if R has trivial norm, then

$$R\langle T/r \rangle = \begin{cases} R[[T]], & \text{if } r < 1; \\ R[T], & \text{if } r \geq 1. \end{cases}$$

Yang: To add the spectral of absolutely convergent power series.

2 Tate algebras

Definition 3. Let R be a non-archimedean banach ring. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$R\langle \underline{r^{-1}T} \rangle := R\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in R, \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

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Proposition 4. Let R be a non-archimedean banach ring. Then the Tate algebra $R\{\underline{T/r}\}$ is a non-archimedean multiplicative banach R-algebra with respect to the *gauss norm*

$$\left\|\sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha\right\| := \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha\in\mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $R\{\underline{T/r}\}$ is a R-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha$ are two nonzero elements in $R\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \le \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{\frac{\varepsilon}{\|g\|}\|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|}\|a_\alpha\| r^\alpha\right\} \le \varepsilon.$$

Hence $f \cdot g \in R\{\underline{T/r}\}$ and it shows that $R\{\underline{T/r}\}$ is a R-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $R\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f+g\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \le \sup_{\alpha\in\mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \le \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned}\|f\cdot g\| = \left\|\sum_{\gamma\in\mathbb{N}^n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) T^\gamma\right\| &= \sup_{\gamma\in\mathbb{N}^n}\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \\ &\le \sup_{\gamma\in\mathbb{N}^n}\max_{\alpha+\beta=\gamma} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0}\cdot\|b_{\beta_0}\| r^{\beta_0} \le \|f\|\cdot\|g\|.\end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\|\cdot\|g\| = \|a_{\alpha_1}\| r^{\alpha_1}\cdot\|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \left\|\sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta\right\| r^{\alpha_1+\beta_1} \le \|f\cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1}\cdot\|b_{\beta_1}\| r^{\beta_1}$ is maximized and by **??**. Thus the gauss norm is multiplicative.

Step 4. Finally show that $R\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $R\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \le \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in R. Since R is complete, set $a_\alpha := \lim_{m\to+\infty} a_{\alpha,m}$ and $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all

$m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \leq 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \leq \varepsilon$$

for all $m > N$. Yang: To be revised, the original version is for a field. □

Example 5. Let R be a non-archimedean banach ring and $A = R\{T\}$ be the Tate algebra in one variable over R. Then we have

$$A^\circ = \left\{ \sum_{n \geq 0} a_n T^n \, : \, |a_n| \leq 1 \text{ for all } n \in \mathbb{N} \right\} = R^\circ\{T\},$$

and

$$A^{\circ\circ} = \left\{ \sum_{n \geq 0} a_n T^n \, : \, |a_n| < 1 \text{ for all } n \in \mathbb{N} \right\} = R^{\circ\circ}\{T\}.$$

Since the norm of items in a restricted power series will tend to 0, we have

$$\widetilde{A} = \widetilde{R}[\underline{T}].$$

Example 6. Let R is a multiplicative non-archimedean banach ring. Set

$$\sqrt{|R|^{-1}} = \{r \in \mathbb{R}_{>0} \, : \, r^{-n} \in |R| \text{ for some } n \in \mathbb{N}_{>0}\}.$$

Fix $r \in \mathbb{R}^n_{>0}$, consider the Tate algebra $A = R\{T/r\}$.

Suppose that $r \in \sqrt{|R|^{-1}}$. Let n be the minimal positive integer such that $r^n \in |R|^{-1}$ and

$$\widetilde{M}_k := \{a \in R \, : \, |a| = r^{-nk}\} / \{a \in R \, : \, |a| < r^{-nk}\}.$$

For $a_m T^m$ with $n \nmid m$, we have$\|a_m T^m\| = |a_m| r^m \leq 1 \implies |a_m| r^m < 1$. Hence

$$\widetilde{R\{T/r\}} = \widetilde{R} \oplus \widetilde{M}_1 T^n \oplus \widetilde{M}_2 T^{2n} \oplus \widetilde{M}_3 T^{3n} \oplus \cdots.$$

In case $R = \mathbf{k}$ is a non-archimedean field, we have $\widetilde{M}_k \cong \widetilde{\mathbf{k}}$ by choosing an element $c \in \mathbf{k}$ with $|c| = r^{-n}$. Hence

$$\widetilde{\mathbf{k}\{T/r\}} \cong \mathscr{k}_{\mathbf{k}}[T^n].$$

Suppose that $r \notin \sqrt{|R|^{-1}}$. Then for every $\|a_n T^n\| = a_n r^n \leq 1$, we have $|a_n| < 1$. It follows that

$$\widetilde{R\{T/r\}} = \widetilde{R}.$$

3 Weierstrass preparation

Definition 7. Let R be a non-archimedean banach ring and $A = R\{T/r\}$. For $f = \sum_{a_n \in \mathbb{N}} a_n T^n \in A$, we define the *degree* of f as

$$\deg f := \max\{n \in \mathbb{N} : \|a_n\| r^n = \|f\|\}.$$

It is interesting to note that if R has trivial norm, then $\deg f$ coincides with the usual degree of polynomials when $r \geq 1$ and the order of formal power series when $r < 1$.

Definition 8. Let R be a non-archimedean banach ring and $A = R\{T/r\}$. A restricted power series $f = \sum_{n \in \mathbb{N}} a_n T^n \in A$ of degree d is said to be *distinguished* if a_d is invertible in R.

Proposition 9. Let R be a non-archimedean banach ring. An element f is invertible if and only if $\deg f = 0$ and the constant item of f is invertible in R.

Proof. Multiplying by a_0^{-1}, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $R[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = - \sum_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} a_\alpha b_\beta .$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\| r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for m. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\| r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\left\{A^{m+1}, A \cdot A^m\right\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in R\{\underline{T/r}\}$. Yang: To be revised. $\square$

Proposition 10. Let $\mathbf{k}$ be a complete non-archimedean field and $r > 0$ be a positive real number. Then the Tate algebra $\mathbf{k}\{T/r\}$ is an euclidean domain with respect to the degree defined in Definition 7. Yang: To be added.

Proof. Let $f, g \in \mathbf{k}\{T/r\}$ be two elements with $g \neq 0$. Denote $n = \deg f$ and $m = \deg g$. We need to find $q, r \in \mathbf{k}\{T/r\}$ such that

$$f = q \cdot g + r, \quad \deg r < \deg g.$$

Yang: To be added. □

Definition 11. Let R be a non-archimedean banach ring and $A = R\{T/r\}$. A Weierstrass polynomial is a monic polynomial $P \in A[T] \subset R\{T/r\}$ whose two degrees as a polynomial and as a restricted power series coincide.

Theorem 12 (Weierstrass preparation theorem)**.** Let R be a non-archimedean banach ring. Let $f \in R\{T/r\}$ be a distinguished restricted power series of degree d. Then there exists a unique Weierstrass polynomial $p \in R[T]$ of degree d and a unique unit $u \in R\{T/r\}$ such that

$$f = p \cdot u.$$

Yang: To be checked.

Proof. Yang: To be added. □

Remark 13. In my knowledge, there are at least three different versions of Weierstrass preparation theorem under different settings:

- The classical Weierstrass preparation in complex analysis;
- The Weierstrass preparation for formal power series over complete noetherian local rings;
- The Weierstrass preparation for Tate algebras over non-archimedean banach rings.

Let $(R, \mathfrak{m})$ be a complete noetherian local ring. Note that there is also a Weierstrass preparation theorem for formal power series over R stating that for every formal power series $f \in R[[T]]$ whose reduction $\overline{f} \in (R/\mathfrak{m})[[T]]$ is of order d, there exists a unique monic polynomial $p \in R[T]$ of degree d and a unique unit $u \in R[[T]]$ such that

$$p \equiv T^d \mod \mathfrak{m}, \quad f = p \cdot u.$$

Yang: To be continued.

Appendix