

Affinoid algebras

Definition 1. Let R be a non-archimedean banach ring. A banach R -algebra A is called a *R -affinoid algebra* if there exists an admissible surjective homomorphism

$$\varphi : R\{\underline{T}/\underline{r}\} \twoheadrightarrow A$$

for some $\underline{r} = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$.

If one can choose $r_1 = \dots = r_n = 1$, then we say that A is a *strictly R -affinoid algebra*.

Example 2. Suppose that the norm on R is trivial. Then a strictly R -affinoid algebra is just an R -algebra of finite type equipped with the trivial norm.

Example 3. Let A be an R -affinoid algebra, $f_i, g_j \in A$ and $r_i, s_j \in \mathbb{R}_{>0}$. Define

$$\begin{aligned} A\{\underline{f}/\underline{r}\} &:= A\{\underline{T}/\underline{r}\}/(T_i - f_i), \\ A\{\underline{f}/\underline{r}, \underline{s}/\underline{g}\} &:= A\{\underline{T}/\underline{r}, \underline{S}/\underline{s}\}/(T_i - f_i, g_j S_j - 1). \end{aligned}$$

Suppose that $\underline{g} \in R$ with $(f_i, \underline{g}) = A$. Then we can define

$$A\{(\underline{f}/\underline{g})/\underline{r}\} := A\{\underline{T}/\underline{r}\}/(gT_i - f_i).$$

All of the above three algebras are again R -affinoid algebras. They can be viewed as normed analogues of localizations. **Yang: To be replaced.**

In the rest of this section, we fix a complete non-archimedean field \mathbf{k} and consider \mathbf{k} -affinoid algebras.

1 Strict case

First we consider strictly \mathbf{k} -affinoid algebras, which are well-studied in the era of rigid analytic geometry.

Proposition 4. Let \mathbf{k} be a complete non-archimedean field and $f \in \mathbf{k}\{\underline{T}\}$. Then there exists an automorphism φ of $\mathbf{k}\{\underline{T}\}$ over \mathbf{k} such that $\varphi(f)$ is T_n -distinguished, i.e., $\varphi(f) \in A\{T_n\}$ is distinguished in the variable T_n with $A = \mathbf{k}\{T_1, \dots, T_{n-1}\}$.

Proof. **Yang: To be added.** □

Proposition 5. Let \mathbf{k} be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T}\}$ is noetherian, factorial, and Jacobson.

Proof. **Yang: To be completed.** □

Corollary 6. Strictly \mathbf{k} -affinoid algebras are noetherian.

Proof. **Yang: To be completed.** □

Theorem 7. Let A be a strictly \mathbf{k} -affinoid algebra. Then there exists a finite injective admissible homomorphism

$$\varphi : \mathbf{k}\{T_1, \dots, T_d\} \hookrightarrow A.$$

Proof. Yang: To be completed. □

Proposition 8. Let A be an \mathbf{k} -affinoid algebra. Then there exists a constant $C > 0$ and $N > 0$ such that for all $f \in A$ and $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

In particular, $\text{Qnil}(A) = \text{nil}(A)$.

Furthermore, if A is reduced, we have

$$\|f\| \leq C\rho(f)$$

for all $f \in A$.

Proof. Yang: To be completed. □

2 General case

Proposition 9. Let A be an affinoid \mathbf{k} -algebra. If and only if $\rho(f) \in \sqrt{|\mathbf{k}|} \cup \{0\}$ for all $f \in A$, then A is strict. Yang: To be complete.

Proof. Yang: To be completed. □

Definition 10. Let \mathbf{k} be a non-archimedean field. We define the *ring of restricted Laurent series* over \mathbf{k} as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} := \mathbf{k}\{T/r, r/T\}.$$

Yang: Is \mathbf{K}_r always a field? Yang: Do we have $\mathbf{L}_{\mathbf{k},r} = \text{Frac}(\mathbf{k}\{T/r\})$?

Proposition 11. Let \mathbf{k} be a non-archimedean field. If $r \notin \sqrt{|\mathbf{k}^\times|}$, then \mathbf{K}_r is a complete non-archimedean field with non-trivial absolute value extending that of \mathbf{k} .

Yang: Tensor with \mathbf{K}_r .

Proposition 12. Let A be a \mathbf{k} -affinoid algebra. Then there exists $r_i \in \mathbb{R}_{>0}$ such that

$$\mathbf{K}_{\underline{r}} \hat{\otimes}_{\mathbf{k}} A$$

is a strictly $\mathbf{K}_{\underline{r}}$ -affinoid algebra.

There are three different categories of finite modules over an affinoid algebra A :

- The category \mathbf{Banmod}_A of finite banach A -modules with A -linear maps as morphisms.
- The category \mathbf{Banmod}_A^b of finite banach A -modules with bounded A -linear maps as morphisms.
- The category \mathbf{mod}_A of finite A -modules with all A -linear maps as morphisms.

Theorem 13. Let A be an affinoid \mathbf{k} -algebra. Then the category of finite banach A -modules with bounded A -linear maps as morphisms is equivalent to the category of finite A -modules with A -linear maps as morphisms. *Yang: To be revised.*

For simplicity, we will just write $\text{mod } A$ to denote the category of finite banach A -modules with bounded A -linear maps as morphisms.

Appendix

DRAFT
