Normed rings and modules

1 Semi-normed algebraic structures

Definition 1. Let G be an abelian group. A *semi-norm* on G is a function $\|\cdot\| : G \to \mathbb{R}_{\geq 0}$ such that

- $\|0\| = 0$;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$.

Suppose that R is a ring (commutative with unity) and $\|\cdot\|$ is a semi-norm on the underlying abelian group of R. We further require that

- $\|1\| = 1$;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$.

Suppose that $(M, \|\cdot\|_M)$ is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M. We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$.

Suppose that $(A, \|\cdot\|_A)$ is an R-algebra and $\|\cdot\|_A$ is a semi-norm on the underlying R-module of A. We further require that this semi-norm is a semi-norm on the underlying ring of A.

Definition 2. Let A be an abelian group (or ring, R-module, R-algebra) equipped with a semi-norm $\|\cdot\|$. We say that $\|\cdot\|$ is a *norm* if $\forall x \in A, \|x\| = 0 \iff x = 0$.

Definition 3. Let A be an abelian group (or ring, R-module, R-algebra) equipped with a semi-norm $\|\cdot\|$. We say that $\|\cdot\|$ is *non-archimedean* if we have the strong triangle inequality $\forall x, y \in A, \|x+y\| \leq \max(\|x\|, \|y\|)$.

Definition 4. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group (or ring, R-module, R-algebra) A. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in A, \|x\|_1 \leq C\|x\|_2$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are bounded by each other, we say they are *equivalent*.

Remark 5. Equivalent semi-norms induce the same topology on A. However, the converse is not true in general. Compare with Lemma 32.

Yang: what about on a module?

Definition 6. Let M be a semi-normed abelian group (or R-module) and $N \subseteq M$ be a subgroup (or R-submodule). The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Unless otherwise specified, we always equip the quotient M/N with the residue semi-norm.

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Remark 7. The residue semi-norm is a norm if and only if N is closed in M.

Definition 8. Let M and N be two semi-normed abelian groups (or rings, R-modules, R-algebras). A homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$.
A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$.

Definition 9. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$. A multiplicative norm sometimes is called a *(multiplicative) valuation* or an *absolute value.*

Example 10. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a valuation ring. This norm is non-archimedean and multiplicative.

Example 11. A valuation field $(\mathbf{k}, |\cdot|)$ can be viewed as a valuation ring.

Example 12. Let $|\cdot| = |\cdot|_\infty$ be the usual absolute value on $\mathbb{Z}$. Then $(\mathbb{Z}, |\cdot|)$ is a valuation ring.

Example 13. Let X be a compact Hausdorff topological space. The ring $C(X, \mathbb{R})$ of continuous real-valued functions on X equipped with the norm $\|f\| = \sup_{x \in X} |f(x)|$ is a normed ring. Its norm is power-multiplicative but not multiplicative in general. It is worth mentioning that the Gelfand-Kolmogorov Theorem saying that we can recover X from the normed ring $C(X, \mathbb{R})$.

Example 14. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. The action of K on itself by multiplication induces a embedding $K \hookrightarrow \operatorname{End}_{\mathbb{Q}}(K) \cong M_n(\mathbb{Q})$. Consider a norm $\|\cdot\|$ on the matrix ring $M_n(\mathbb{Q})$ and restrict it to K. Then $(K, \|\cdot\|)$ is a normed ring. This norm is even not power-multiplicative in general.

Definition 15. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 16. A *banach ring* is a complete normed ring.

Proposition 17. Let R be a banach ring and $I \subseteq R$ be a closed ideal. Then the residue norm on the quotient ring R/I is a norm for rings.

Proof. We only need to show that $\|1\|_{R/I} = 1$. Yang: To be added. □

Proposition 18. Let R be a banach ring. Then the group of invertible elements $R^\times$ is an open subset of R.

Proof. Yang: To be added. □

Corollary 19. Let R be a banach ring. Then every maximal ideal of R is closed.

Proof. Yang: To be added. □

Definition 20. Let $(A, \|\cdot\|_A)$ be a normed algebraic structure, e.g., a normed abelian group, a normed ring, or a normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a metric space. Since A is dense in its completion and the algebraic operations are uniformly continuous, the algebraic operations on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M\otimes_R N} = \inf\left\{\sum_i \|x_i\|_M \|y_i\|_N \,:\, z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N\right\}.$$

Definition 21. Let R be a banach ring and M, N complete semi-normed R-modules. The *complete tensor product* $M \widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Example 22. Yang: Example of complete tensor product.

Definition 23. Let R be a non-archimedean banach ring. We define

$$R^\circ = \{f \in R \,:\, \rho(f) \leq 1\}, \quad R^{\circ\circ} = \{f \in R \,:\, \rho(f) < 1\}.$$

The *reduction ring* of R is defined as the quotient ring

$$\widetilde{R} = R^\circ / R^{\circ\circ}.$$

Example 24. Let R be a ring equipped with the trivial norm. Then we have $R^\circ = R$ and $R^{\circ\circ} = \operatorname{nil}(R)$. Hence the reduction ring $\widetilde{R}$ is isomorphic to the reduced ring $R_{\mathrm{red}} = R/\operatorname{nil}(R)$.

2 Spectral radius

Definition 25. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Since $\|\cdot\|$ is submultiplicative, the limit defining $\rho(f)$ exists and equals to $\inf_{n\geq 1} \|f^n\|^{1/n}$ by Fekete's Subadditive Lemma.

Example 26. Consider the normed ring $(K, \|\cdot\|)$ in Example 14. Suppose that $\|\cdot\|$ on $M_n(\mathbb{Q})$ is given by the 2-norm induced by the euclidean norm on $\mathbb{Q}^n$. Then for a matrix $A \in M_n(\mathbb{Q})$, its spectral radius $\rho(A)$ is given by the largest absolute value of eigenvalues of A. Hence for each $a \in K$, its spectral radius $\rho(a)$ is given by the largest absolute value of embeddings of a into $\mathbb{C}$, i.e.,

$$\rho(a) = \max_{\sigma : K \hookrightarrow \mathbb{C}} |\sigma(a)|.$$

Proposition 27. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 28. A banach ring R is called *uniform* if its norm is power-multiplicative.

Definition 29. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Definition 30. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\mathrm{Qnil}(R)$.

Proposition 31. Let R be a banach ring. The completion of $R/\mathrm{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

Yang: To be continued...

Appendix

Lemma 32. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proposition 33. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.