

# *Valuations on fields*

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# 1 Valuation fields

## 1.1 Absolute values and completion

**Definition 1.1.** Let  $\mathbf{k}$  be a field. An *absolute value* on  $\mathbf{k}$  is a function  $\|\cdot\| : \mathbf{k} \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following properties for all  $x, y \in \mathbf{k}$ :

- (a)  $\|x\| = 0$  if and only if  $x = 0$ ;
- (b)  $\|xy\| = \|x\| \cdot \|y\|$ ;
- (c)  $\|x + y\| \leq \|x\| + \|y\|$ .

A field  $\mathbf{k}$  equipped with an absolute value  $\|\cdot\|$  is called a *valuation field*.

**Remark 1.2.** Let  $\mathbf{k}$  be a field. Recall that a (additive) valuation on  $\mathbf{k}$  is a function  $v : \mathbf{k}^\times \rightarrow \mathbb{R}$  such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$ ;

- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$ .

We can extend  $v$  to the whole field  $\mathbf{k}$  by defining  $v(0) = +\infty$ . Fix a real number  $\varepsilon \in (0, 1)$ . Then  $v$  induces an absolute value  $|\cdot|_v : \mathbf{k} \rightarrow \mathbb{R}_+$  defined by  $|x|_v = \varepsilon^{v(x)}$  for each  $x \in \mathbf{k}$ .

The valuation  $v$  defined above is called an *additive valuation*. And an absolute value  $|\cdot|$  on  $\mathbf{k}$  is called a *multiplicative valuation*. In this note, the term *valuation* may refer to either an additive valuation or a multiplicative valuation, depending on the context.

**Example 1.3.** Let  $\mathbf{k}$  be a field. The *trivial absolute value* on  $\mathbf{k}$  is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

**Definition 1.4.** The (*multiplicative*) *valuation group* of a valuation field  $(\mathbf{k}, \|\cdot\|)$  is defined as the subgroup of  $\mathbb{R}_{>0}$  given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation  $\sqrt{|\mathbf{k}^\times|}$  to denote the set  $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$ .

Note that an absolute value  $\|\cdot\|$  is non-trivial if and only if its valuation group  $|\mathbf{k}^\times|$  is not equal to  $\{1\}$ .

**Definition 1.5.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field. We say that  $\mathbf{k}$  is *complete* if the metric  $d(x, y) := \|x - y\|$  makes  $\mathbf{k}$  a complete metric space.

**Lemma 1.6.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field and  $(\widehat{\mathbf{k}}, \|\cdot\|)$  its completion as a metric space. Then the operations of addition and multiplication on  $\mathbf{k}$  can be extended to  $\widehat{\mathbf{k}}$  uniquely, making  $(\widehat{\mathbf{k}}, \|\cdot\|)$  a complete valuation field containing  $\mathbf{k}$  as a dense subfield.

*Proof.* Note that the operations of addition and multiplication on  $\mathbf{k}$  are uniformly continuous with respect to the metric  $d(x, y) = \|x - y\|$ . Thus they can be extended to  $\widehat{\mathbf{k}}$  uniquely.  $\square$

**Proposition 1.7.** Let  $(\mathbf{k}, \|\cdot\|)$  be a complete valuation field with non-trivial absolute value. Then  $\mathbf{k}$  is uncountable.

*Proof.* Since the absolute value  $\|\cdot\|$  is non-trivial, we can construct a sequence  $\{x_n\}_{n=1}^\infty \subseteq \mathbf{k}$  inductively such that  $\|x_n\| < \|x_{n-1}\|/2$  for any  $n \geq 1$  and  $\|x_0\| < 1$ . Then there is an injective map from  $\mathbb{N}^{\{0,1\}}$  to  $\mathbf{k}$  defined by

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n x_n, \quad a_n \in \{0, 1\}.$$

Since  $\|x_n\| < 2^{-n}$ , the series  $\sum_{n=1}^\infty a_n x_n$  converges in  $\mathbf{k}$ . Note  $\|x_n\| > \|\sum_{m \geq n} x_m\|$  for each  $n$ , we have that the map is injective. Thus  $\mathbf{k}$  is uncountable.  $\square$

Unlike the real number field  $\mathbb{R}$ , even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

**Definition 1.8.** A valuation field  $(\mathbf{k}, \|\cdot\|)$  is called *spherically complete* if every decreasing sequence of closed balls in  $\mathbf{k}$  has a non-empty intersection.

**Example 1.9.** The field  $\mathbb{C}_p$  of  $p$ -adic complex numbers is not spherically complete, see Yang: to be added.

**Example 1.10.** Let  $|\cdot|_\infty$  be the usual absolute value on the field  $\mathbb{Q}$  of rational numbers. Then  $(\mathbb{Q}, |\cdot|_\infty)$  is a valuation field. Its completion is the field  $\mathbb{R}$  of real numbers equipped with the usual absolute value.

**Example 1.11.** Let  $p$  be a prime number. For any non-zero rational number  $x \in \mathbb{Q}$ , we can write it as  $x = p^n \frac{a}{b}$ , where  $n \in \mathbb{Z}$  and  $a, b \in \mathbb{Z}$  are integers not divisible by  $p$ . The  $p$ -adic absolute value on  $\mathbb{Q}$  is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then  $(\mathbb{Q}, |\cdot|_p)$  is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of  $p$ -adic numbers equipped with the  $p$ -adic absolute value; see Yang: to be added..

**Definition 1.12.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field. We say that  $\mathbf{k}$  is *non-archimedean* if its absolute value  $\|\cdot\|$  satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that  $\mathbf{k}$  is *archimedean*.

**Example 1.13.** Let  $v$  be an additive valuation on a field  $\mathbf{k}$ . Then the induced absolute value  $|\cdot|_v$  as in Remark 1.2 is non-archimedean.

The converse is also true: if  $(\mathbf{k}, |\cdot|)$  is a non-archimedean valuation field, then the function  $v : \mathbf{k}^\times \rightarrow \mathbb{R}$  defined by  $v(x) = -\log|x|$  is an additive valuation on  $\mathbf{k}$ .

**Proposition 1.14.** Let  $(\mathbf{k}, |\cdot|)$  be a valuation field. Then  $\mathbf{k}$  is archimedean if and only if the set  $\{|n \cdot 1| : n \in \mathbb{Z}\}$  is unbounded.

*Proof.* Sufficiency is obvious. Yang: To be added. □

## 1.2 Places on a field

**Definition 1.15.** Let  $\mathbf{k}$  be a field. Two absolute values  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $\mathbf{k}$  are said to be *equivalent* if there exists a real number  $c \in (0, \infty)$  such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field  $\mathbf{k}$ . Moreover, the following lemma shows that the converse is also true.

**Lemma 1.16.** Let  $\mathbf{k}$  be a field and  $\|\cdot\|_1, \|\cdot\|_2$  be two absolute values on  $\mathbf{k}$ . Then the following statements are equivalent:

- (a)  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent;
- (b)  $\|\cdot\|_1$  and  $\|\cdot\|_2$  induce the same topology on  $\mathbf{k}$ ;
- (c) The unit disks  $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$  and  $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$  are the same.

*Proof.* The implications (a)  $\Rightarrow$  (b) is obvious. Now we prove (b)  $\Rightarrow$  (c). For any  $x \in D_1$ , we have  $x^n \rightarrow 0$  as  $n \rightarrow \infty$  under the absolute value  $\|\cdot\|_1$  and thus under  $\|\cdot\|_2$ . Therefore,  $\|x\|_2^n \rightarrow 0$  as  $n \rightarrow \infty$ , which implies that  $\|x\|_2 < 1$ , i.e.,  $x \in D_2$ . Similarly, we can prove that  $D_2 \subseteq D_1$ .

Finally, we prove (c)  $\Rightarrow$  (a). If  $\|\cdot\|_1$  is trivial, then  $D_1 = \{0\}$  and thus  $\|\cdot\|_2$  is also trivial. In this case, they are equivalent. Suppose that both  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are non-trivial. Pick any  $x, y \notin D_1 = D_2$ . Then there exist real numbers  $\alpha, \beta > 0$  such that  $\|x\|_1 = \|x\|_2^\alpha$  and  $\|y\|_1 = \|y\|_2^\beta$ . Suppose the contrary that  $\alpha \neq \beta$ . Consider the domain  $\Lambda \subseteq \mathbb{Z}^2$  defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since  $\alpha \neq \beta$ , the two lines defined by the equalities are not parallel. Thus  $\Lambda$  is non-empty. Pick  $(n, m) \in \Lambda$  and set  $z := x^n y^{-m}$ . Then we have  $\|z\|_2 < 1$  and  $\|z\|_1 > 1$ , a contradiction.  $\square$

**Theorem 1.17** (Ostrowski). Let  $(\mathbf{k}, \|\cdot\|)$  be an archimedean complete valuation field. Then  $\mathbf{k}$  is isomorphic to either the real number field  $\mathbb{R}$  or the complex number field  $\mathbb{C}$  equipped with the usual absolute value.

*Proof.* Yang: To be added.  $\square$

**Definition 1.18.** Let  $\mathbf{k}$  be a field. A *place* on  $\mathbf{k}$  is an equivalence class of non-trivial absolute values on  $\mathbf{k}$ . An archimedean (resp. non-archimedean) place is also called an *finite* (resp. *infinite*) place. We denote the set of all places (resp. finite, infinite places) on  $\mathbf{k}$  by  $M_{\mathbf{k}}$  (resp.  $M_{\mathbf{k}}^f, M_{\mathbf{k}}^\infty$ ). If  $\mathbf{l} \subset \mathbf{k}$  is a subfield, we denote by  $M_{\mathbf{k}/\mathbf{l}}$  (resp.  $M_{\mathbf{k}/\mathbf{l}}^f, M_{\mathbf{k}/\mathbf{l}}^\infty$ ) the set of all places (resp. finite, infinite places) on  $\mathbf{k}$  which are trivial on  $\mathbf{l}$ .

**Example 1.19.** Let  $\mathbf{k} = \mathbb{C}(t)$  and  $\mathbf{l} = \mathbb{C}$ . Then I claim that

$$M_{\mathbb{C}(t)/\mathbb{C}} \cong \{ \text{prime divisors on } \mathbb{P}_{\mathbb{C}}^1 \}.$$

For each prime divisor  $P$  on  $\mathbb{P}_{\mathbb{C}}^1$ , we can define an additive valuation  $\text{Mult}_P : \mathbb{C}(t)^\times \rightarrow \mathbb{Z}$  by assigning to each non-zero rational function  $f \in \mathbb{C}(t)^\times$  its multiplicity at  $P$ . Fix a real number  $\varepsilon \in (0, 1)$ . Then we obtain a multiplicative valuation (absolute value)  $|\cdot|_P$  on  $\mathbb{C}(t)$  as in Remark 1.2. It is easy to check that the absolute value  $|\cdot|_P$  is trivial on  $\mathbb{C}$  and that different prime divisors give rise to inequivalent absolute values.

Conversely, given any non-trivial absolute value  $|\cdot|$  on  $\mathbb{C}(t)$  which is trivial on  $\mathbb{C}$ , by Proposition 1.14 and Example 1.13, the absolute value  $|\cdot|$  is given by an additive valuation  $v : \mathbb{C}(t)^\times \rightarrow \mathbb{R}$ . Let  $\mathcal{O}_v$  be the valuation ring of  $v$  and  $\mathfrak{m}_v$  its maximal ideal. Then  $t \in \mathcal{O}_v$  or  $t^{-1} \in \mathcal{O}_v$ . Without loss of generality, we assume that  $t \in \mathcal{O}_v$ . Since  $|\cdot|$  is trivial on  $\mathbb{C}$ , we have  $\mathbb{C}[t] \subseteq \mathcal{O}_v$ . And we have  $\mathfrak{m}_v \cap \mathbb{C}[t] \neq 0$  since otherwise  $v$  is trivial on  $\mathbb{C}[t]^\times$  and thus on  $\mathbb{C}(t)^\times$ . It follows that the image of  $\mathbb{C}[t]$  under the quotient map  $\mathcal{O}_v \rightarrow \mathcal{O}_v/\mathfrak{m}_v$  is  $\mathbb{C}$ . This gives a point  $P \in \mathbb{A}_{\mathbb{C}}^1 \subseteq \mathbb{P}_{\mathbb{C}}^1$ . Then  $v$  is different

from the additive valuation  $\text{Mult}_P$  by a positive scalar multiple via looking at the values on  $\mathbb{C}[t]$ . Thus we have established the claimed bijection.

**Theorem 1.20** (Ostrowski). Every nontrivial absolute value on  $\mathbb{Q}$  is equivalent to either the usual absolute value  $|\cdot|_\infty$  or a  $p$ -adic absolute value  $|\cdot|_p$  for some prime number  $p$ . In other words,

$$M_{\mathbb{Q}} = \{|\cdot|_\infty\} \cup \{|\cdot|_p : p \text{ is a prime number}\}.$$

| *Proof.* Yang: To be added. □

**Remark 1.21.** For every non-archimedean place  $v$  on  $\mathbb{Q}$  corresponding to a prime number  $p$ , we choose the unique normalized absolute value  $|\cdot|_v$  in the class  $v$  such that  $|p|_v = 1/p$ . For the archimedean place  $v$  on  $\mathbb{Q}$ , we choose the usual absolute value  $|\cdot|_v = |\cdot|_\infty$ . Unless otherwise specified, we always use the normalized absolute values on  $\mathbb{Q}$ .

**Remark 1.22.** For any non-zero rational number  $x \in \mathbb{Q}^\times$ , one can easily check the *product formula*:

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1.$$

This can be viewed as an arithmetic analogue of the fact on  $\mathbb{P}_{\mathbb{C}}^1$  that

$$\sum_{P \in \mathbb{P}_{\mathbb{C}}^1} \text{Mult}_P(f) = 0$$

for any non-zero rational function  $f \in \mathbb{C}(t)^\times$ . Indeed, fix a real number  $\varepsilon \in (0, 1)$ . Then by Example 1.19, above fact can be rewritten as

$$\prod_{P \in \mathbb{P}_{\mathbb{C}}^1} |f|_P = 1.$$

**Theorem 1.23** (Artin-Whaples approximations). Let  $\mathbf{k}$  be a field and  $v_1, v_2, \dots, v_n \in M_{\mathbf{k}}$  be pairwise distinct places on  $\mathbf{k}$ . For each  $i \in \{1, 2, \dots, n\}$ , let  $x_i \in \mathbf{k}$  and  $\varepsilon_i > 0$ . Then there exists an element  $x \in \mathbf{k}$  such that

$$|x - x_i|_{v_i} < \varepsilon_i, \quad \forall i \in \{1, 2, \dots, n\}.$$

In particular, the image of the diagonal embedding

$$\mathbf{k} \rightarrow \prod_{i=1}^n \mathbf{k}_{v_i}$$

is dense, where  $\mathbf{k}_{v_i}$  is the completion of  $\mathbf{k}$  with respect to the place  $v_i$ .

| *Proof.* Yang: To be added. □

## 2 Non-archimedean valuations

### 2.1 Topology: Ultra-metric space

We will use  $B(x, r)$  (resp.  $E(x, r)$ ) to denote the open ball (resp. closed ball) with center  $x$  and radius  $r$ .

**Definition 2.1.** A metric space  $(X, d)$  is called an *ultra-metric space* if its metric  $d$  satisfies the *strong triangle inequality*:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.$$

**Remark 2.2.** The term *ultra-metric space* should be translated into Chinese as “奥特度量空间”. There is no special reason for this translation, except that I insist on using “奥特” to translate “ultra”.

If  $(\mathbf{k}, \|\cdot\|)$  is a non-archimedean field, then the metric  $d(x, y) := \|x - y\|$  on  $\mathbf{k}$  makes  $(\mathbf{k}, d)$  an ultra-metric space.

**Proposition 2.3.** Let  $(X, d)$  be an ultra-metric space. Then for any  $x, y, z \in X$ , at least two of the three distances  $d(x, y), d(y, z), d(z, x)$  are equal. And the third distance is less than or equal to the common value of the other two.

*Proof.* Suppose that  $d(x, y) \geq d(y, z)$ . By the strong triangle inequality, we have

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(x, y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} = \max\{d(x, z), d(y, z)\} \leq d(x, z).$$

This shows that  $d(x, y) = \max\{d(x, z), d(y, z)\}$ . Thus either  $d(x, z) = d(x, y) \geq d(y, z)$  or  $d(y, z) = d(x, y) \geq d(x, z)$ .  $\square$

**Proposition 2.4.** Let  $(X, d)$  be an ultra-metric space. Let  $D_i$  be (open or closed) ball in  $X$  for  $i = 1, 2$ . If  $D_1 \cap D_2 \neq \emptyset$ , then either  $D_1 \subseteq D_2$  or  $D_2 \subseteq D_1$ .

*Proof.* Suppose that  $D_i$  has center  $x_i$  and radius  $r_i$  for  $i = 1, 2$ . Let  $y \in D_1 \cap D_2$ . We have

$$d(x_1, x_2) \leq \max\{d(x_1, y), d(y, x_2)\}.$$

Without loss of generality, we may assume that  $d(x_1, x_2) \leq d(x_1, y)$ . It follows that  $x_2 \in D_1$  since  $d(x_1, y) < r_1$  (or  $\leq r_1$ ).

If there exists  $z \in D_2 \setminus D_1$ , we claim that  $D_1 \subseteq D_2$ . We have  $d(x_1, z) > d(x_1, x_2)$ . Then by Proposition 2.3,

$$r_1 \leq d(x_1, z) = d(x_2, z) \leq r_2.$$

In particular, if  $D_2$  is an open ball, then we have strict inequality  $r_1 < r_2$ . For any  $w \in D_1$ , we have

$$d(x_2, w) \leq \max\{d(x_2, x_1), d(x_1, w)\} \leq r_1 < r_2.$$

| Thus  $w \in D_2$  whatever  $D_2$  is open or closed, and it shows that  $D_1 \subseteq D_2$ .  $\square$

**Proposition 2.5.** Let  $(X, d)$  be an ultra-metric space. Then both  $B(x, r)$  and  $E(x, r)$  are closed and open subsets of  $X$  for any  $x \in X$  and  $r > 0$ .

*Proof.* We show that the sphere  $S(x, r) := \{y \in X \mid d(x, y) = r\}$  is open in  $X$ . Note that if  $y \in S(x, r)$ , then for any  $r' < r$ , we have  $B(y, r') \cap E(x, r) \neq \emptyset$  and  $x \in E(x, r) \setminus B(y, r')$ . Thus by Proposition 2.4, we have  $B(y, r') \subseteq E(x, r)$ . If  $B(y, r') \cap B(x, r) \neq \emptyset$ , then by Proposition 2.4 again, we have  $B(y, r') \subseteq B(x, r)$ . However,  $y \in B(y, r') \setminus B(x, r)$ , a contradiction. Thus  $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$ . It yields that  $S(x, r) = \bigcup_{y \in S(x, r)} B(y, r/2)$  is open in  $X$ .

Since  $E(x, r) = B(x, r) \cup S(x, r)$  and  $B(x, r) = E(x, r) \setminus S(x, r)$ , both  $B(x, r)$  and  $E(x, r)$  are open and closed in  $X$ .  $\square$

**Corollary 2.6.** Let  $(X, d)$  be an ultra-metric space. Then  $X$  is totally disconnected, i.e., the only connected subsets of  $X$  are the set with at most one point.

*Proof.* Suppose that  $S \subset X$  has at least two distinct points  $x, y \in S$ . Let  $r := d(x, y) > 0$ . Consider the open ball  $B(x, r/2)$ . By Proposition 2.5,  $B(x, r/2)$  is both open and closed in  $X$ . Thus  $B(x, r/2) \cap S$  is both open and closed in  $S$ , however, it is non-empty and not equal to  $S$  since it contains  $x$  but not  $y$ . This shows that  $S$  is disconnected.  $\square$

**Proposition 2.7.** Let  $(X, d)$  be an ultra-metric space. A sequence  $\{x_n\}$  in  $X$  is cauchy if and only if  $d(x_n, x_{n+1}) \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* The necessity is true for all metric spaces. Suppose that  $d(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ . For any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $d(x_n, x_{n+1}) < \varepsilon$  for all  $n \geq N$ . For any  $m, n \geq N$  with  $m < n$ , by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \dots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that  $\{x_n\}$  is a cauchy sequence.  $\square$

## 2.2 Algebra: ring of integers and residue field

Let  $\mathbf{k}$  be a non-archimedean field. Then easily see that  $\{x \in \mathbf{k} : \|x\| \leq 1\}$  is a subring of  $\mathbf{k}$ . Moreover, it is a local ring whose maximal ideal is  $\{x \in \mathbf{k} : \|x\| < 1\}$ .

**Definition 2.8.** Let  $\mathbf{k}$  be a non-archimedean field. The *ring of integers* of  $\mathbf{k}$  is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of  $\mathbf{k}$  is defined as

$$\mathcal{K}_\mathbf{k} := \tilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Set  $I_{r,<} := B(0, r)$  and  $I_{r,\leq} := E(0, r)$  for each  $r \in [0, 1]$ .

**Proposition 2.9.** The sets  $I_{r,<}$  and  $I_{r,\leq}$  are ideals of the ring of integers  $\mathbf{k}^\circ$ . Conversely, any ideal of  $\mathbf{k}^\circ$  is of the form  $I_{r,<}$  or  $I_{r,\leq}$  for some  $r \in (0, 1)$ .

*Proof.* Let  $I$  be an ideal of  $\mathbf{k}^\circ$ . Set  $r = \sup\{|a| : a \in I\}$  (resp.  $r = \max\{|a| : a \in I\}$  when the maximum exists). Then, by definition, we have  $I \subset I_{r,<}$  (resp.  $I \subset I_{r,\leq}$ ). For every  $x \in \mathbf{k}^\circ$  with  $|x| < r$  (resp.  $|x| \leq r$ ), there exists  $a \in I$  such that  $|x| \leq |a|$ . Thus,  $|x/a| \leq 1$  and so  $x/a \in \mathbf{k}^\circ$ . Since  $I$  is an ideal, we have  $x = (x/a)a \in I$ . Therefore,  $I_{r,<} \subset I$  (resp.  $I_{r,\leq} \subset I$ ).  $\square$

**Proposition 2.10.** Let  $I_r$  be either  $I_{r,<}$  or  $I_{r,\leq}$  for each  $r \in (0, 1)$ . Suppose  $\{r_n \in (0, 1)\}_{n \in \mathbb{N}}$  is a decreasing sequence converging to 0. Then the completion  $\widehat{\mathbf{k}}$  of  $\mathbf{k}$  is isomorphic to the projective limit

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}.$$

*Proof.* For every  $x \in \widehat{\mathbf{k}}^\circ$ , there exists a cauchy sequence  $\{x_m\}_{m \in \mathbb{N}}$  in  $\mathbf{k}^\circ$  converging to  $x$ . Since  $\{r_n\}_{n \in \mathbb{N}}$  converges to 0, for each  $n \in \mathbb{N}$ , there exists  $M_n \in \mathbb{N}$  such that for all  $m, m' \geq M_n$ , we have  $|x_m - x_{m'}| < r_n$ . Thus, the sequence  $\{x_m + I_{r_n}\}_{m \in \mathbb{N}}$  is eventually constant in  $\mathbf{k}^\circ / I_{r_n}$ . Define a map

$$\Phi : \widehat{\mathbf{k}}^\circ \rightarrow \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}, \quad x \mapsto \left( \lim_{m \rightarrow \infty} x_m + I_{r_n} \right)_{n \in \mathbb{N}}.$$

It is straightforward to verify that  $\Phi$  is a well-defined ring homomorphism.

Conversely, for every  $(a_n + I_{r_n})_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}$ , we can choose a representative  $a_n \in \mathbf{k}^\circ$  for each  $n$ . We claim that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  is a cauchy sequence in  $\mathbf{k}^\circ$ . Indeed, for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $r_N < \varepsilon$ . For all  $m, n \geq N$ , since  $a_n + I_{r_n}$  maps to  $a_m + I_{r_m}$  under the natural projection, we have  $|a_n - a_m| < r_N < \varepsilon$ . Thus,  $\{a_n\}_{n \in \mathbb{N}}$  converges to some  $x \in \widehat{\mathbf{k}}^\circ$ . Easily see that the limit  $x$  is independent of the choice of representatives  $\{a_n\}_{n \in \mathbb{N}}$ . This gives a map

$$\Psi : \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n} \rightarrow \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow \infty} a_n.$$

Direct verification shows that  $\Psi = \Phi^{-1}$ .  $\square$

**Corollary 2.11.** Let  $\mathbf{k}$  be a non-archimedean field and  $\widehat{\mathbf{k}}$  its completion. Then the residue field  $\kappa_{\widehat{\mathbf{k}}} \cong \kappa_{\mathbf{k}}$  under the natural embedding  $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$ .

**Corollary 2.12.** Let  $\mathbf{k}$  be a non-archimedean field and  $\widehat{\mathbf{k}}$  its completion. Then the valuation group  $|\widehat{\mathbf{k}}^\times|$  of  $\widehat{\mathbf{k}}$  is equal to the valuation group  $|\mathbf{k}^\times|$  of  $\mathbf{k}$ .

*Proof.* Note that

$$\begin{aligned} r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \not\subseteq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\ &\iff \widehat{\mathbf{k}}^\circ / I_{r,<} \rightarrow \widehat{\mathbf{k}}^\circ / I_{r,\leq} \text{ is not an isomorphism} \\ &\iff \mathbf{k}^\circ / I_{r,<} \rightarrow \mathbf{k}^\circ / I_{r,\leq} \text{ is not an isomorphism} \\ &\iff I_{r,<} \not\subseteq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\ &\iff r \in |\mathbf{k}^\times|. \end{aligned}$$



**Proposition 2.13.** Let  $\mathbf{k}$  be a non-archimedean field with non-trivial valuation. Then  $\mathbf{k}^\circ$  is totally bounded iff  $\mathbf{k}^\circ/I_{r,<}$  and  $\mathbf{k}^\circ/I_{r,\leq}$  are finite for each  $r \in [0, 1]$ . Moreover, if  $\mathbf{k}$  is complete, then it is locally compact iff  $\mathbf{k}^\circ/I_r$  is finite for each  $r \in (0, 1)$ .

**Slogan** “Locally compact  $\iff$  pro-finite.”

*Proof.* We just prove the case for  $I_r = I_{r,<}$ . The case for  $I_r = I_{r,\leq}$  is similar.

Suppose that  $\mathbf{k}^\circ/I_r$  is finite for each  $r \in [0, 1]$ . Then for every  $\varepsilon > 0$ , there exists  $r \in (0, 1)$  such that  $r < \varepsilon$  and  $\mathbf{k}^\circ/I_r$  is finite. Let  $\{a_1 + I_r, \dots, a_n + I_r\}$  be the complete set of representatives of  $\mathbf{k}^\circ/I_r$ . Then the balls  $B(a_i, r)$  for  $i = 1, \dots, n$  cover  $\mathbf{k}^\circ$ .

Conversely, suppose that  $\mathbf{k}^\circ/I_r$  is infinite for some  $r \in [0, 1]$ . Then there exists an infinite set  $\{a_n\}$  with  $|a_n| \in [r, 1]$  such that their images in  $\mathbf{k}^\circ/I_r$  are distinct. In particular, for every  $m \neq n$ , we have  $|a_n - a_m| \geq r$ . Any subsequence of  $\{a_n\}$  is not cauchy. Thus,  $\mathbf{k}^\circ$  is not totally bounded.  $\square$

**Proposition 2.14.** The ring  $\mathbf{k}^\circ$  is noetherian iff  $\mathbf{k}$  is a discrete valuation field.

*Proof.* Note that  $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$  is a multiplicative subgroup. If  $\mathbf{k}$  is not a discrete valuation field, then  $|\mathbf{k}^\times|$  is dense in  $\mathbb{R}_{>0}$ . In particular, there exists a strictly ascending sequence  $r_n \in |\mathbf{k}^\times| \cap (0, 1)$ . Then the ideals  $I_{r_n,\leq}$  form a strictly ascending chain of ideals in  $\mathbf{k}^\circ$ .

The converse is standard since now  $\mathbf{k}^\circ$  is a discrete valuation ring.  $\square$

**Proposition 2.15.** Let  $\mathbf{k}$  be a complete non-archimedean field. Then  $\mathbf{k}$  is locally compact iff  $\mathbf{k}$  is a discrete valuation field and its residue field  $\mathcal{k}_\mathbf{k}$  is finite.

*Proof.* The necessity follows from Proposition 2.13. For the sufficiency, suppose that  $\mathbf{k}$  is a discrete valuation field whose residue field  $\mathcal{k}_\mathbf{k}$  is finite. Let  $\pi \in \mathbf{k}^\circ$  be a uniformizer. We only need to show that  $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$  is finite for each  $n \in \mathbb{N}$ . Note that there is an isomorphism

$$\pi^{n-1} \mathbf{k}^\circ / \pi^n \mathbf{k}^\circ \cong \mathcal{k}_\mathbf{k}, \quad x + \pi^n \mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on  $n$ , we conclude that  $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$  is finite.  $\square$

## 2.3 Hensel's Lemma

**Theorem 2.16** (Hensel's lemma). Let  $\mathbf{k}$  be a complete non-archimedean field and  $F(T) \in \mathbf{k}^\circ[T]$  a monic polynomial. Suppose that the reduction  $f(T) \in \mathcal{k}_\mathbf{k}[T]$  of  $F(T)$  factors as

$$f(T) = g(T)h(T),$$

where  $g(T), h(T) \in \mathcal{k}_\mathbf{k}[T]$  are monic polynomials that are coprime in  $\mathcal{k}_\mathbf{k}[T]$ . Then there exist monic polynomials  $G(T), H(T) \in \mathbf{k}^\circ[T]$  such that

$$F(T) = G(T)H(T),$$

and the reductions of  $G(T), H(T)$  in  $\mathcal{k}_\mathbf{k}[T]$  are  $g(T), h(T)$  respectively.

*Proof.* Since  $\gcd(g, h) = 1$  in  $\mathcal{K}_\mathbf{k}[T]$ , there exist polynomials  $u(T), v(T) \in \mathcal{K}_\mathbf{k}[T]$  such that  $ug + vh = 1$  and  $\deg u < \deg h, \deg v < \deg g$ . Choose lifts  $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$  of  $g(T), h(T), u(T), v(T)$  respectively preserving their degrees such that  $G_0$  and  $H_0$  are monic. Then there exist  $r < 1$  such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \pmod{I_r}, \quad F(T) - G_0(T)H_0(T) \equiv 0 \pmod{I_r},$$

where  $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$ .

We will construct a sequence of monic polynomials  $\{G_n(T)\}_{n \in \mathbb{N}}$  and  $\{H_n(T)\}_{n \in \mathbb{N}}$  in  $\mathbf{k}^\circ[T]$  such that for each  $n \in \mathbb{N}$ ,

$$G_n(T) \equiv G_{n-1}(T) \pmod{I_{r^n}}, \quad H_n(T) \equiv H_{n-1}(T) \pmod{I_{r^n}},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \pmod{I_{r^{n+1}}}.$$

If we have such sequences, then their coefficients converge in the complete ring  $\mathbf{k}^\circ$ . Let  $G(T)$  and  $H(T)$  be the limits of  $\{G_n(T)\}$  and  $\{H_n(T)\}$  respectively. Then we have  $F(T) = G(T)H(T)$  and the reductions of  $G(T), H(T)$  in  $\mathcal{K}_\mathbf{k}[T]$  are  $g(T), h(T)$  respectively.

The case  $n = 0$  is done by the above construction. Now suppose that we have constructed  $G_n(T)$  and  $H_n(T)$  for some  $n \geq 0$ . Since  $G_n - G_0 \equiv 0 \pmod{I_r}$  and  $H_n - H_0 \equiv 0 \pmod{I_r}$ , we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \pmod{I_r}.$$

Set  $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$  and  $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$ . Then we have

$$\begin{aligned} (G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T]. \end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

This finishes the induction. □

**Corollary 2.17.** Let  $\mathbf{k}$  be a complete non-archimedean field and  $F(T) \in \mathbf{k}^\circ[T]$  a monic polynomial. Suppose that the reduction  $f(T) \in \mathcal{K}_\mathbf{k}[T]$  of  $F(T)$  has a simple root  $a \in \mathcal{K}_\mathbf{k}$ . Then there exists a root  $\alpha \in \mathbf{k}^\circ$  of  $F(T)$  whose reduction is  $a$ .

*Proof.* Since  $a$  is a simple root of  $f(T)$ , we have the factorization  $f(T) = (T - a)h(T)$  for some monic polynomial  $h(T) \in \mathcal{K}_\mathbf{k}[T]$  with  $h(a) \neq 0$ . Then the result follows from [Theorem 2.16](#). □

## 2.4 Newton polygons

Yang: To be filled.

### 3 Finite field extensions

#### 3.1 Finite-dimensional vector space

**Definition 3.1.** Let  $\mathbf{k}$  be a valuation field and  $V$  a vector space over  $\mathbf{k}$ . A *norm* on  $V$  is a function  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following properties for all  $x, y \in V$  and  $a \in \mathbf{k}$ :

- (a)  $\|x\| = 0$  if and only if  $x = 0$ ;
- (b)  $\|ax\| = |a| \cdot \|x\|$ ;
- (c)  $\|x + y\| \leq \|x\| + \|y\|$ .

**Example 3.2.** Let  $\mathbf{k}$  be a valuation field and  $V$  a finite-dimensional vector space over  $\mathbf{k}$  with basis  $\{e_1, e_2, \dots, e_n\}$ . For any  $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$ , define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then  $\|\cdot\|_{\max}$  is a norm on  $V$ , called the *maximal norm* with respect to the basis  $\{e_1, e_2, \dots, e_n\}$ .

**Example 3.3.** Setting as in Example 3.2, for any  $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$ , define

$$\|x\|_1 := |a_1| + |a_2| + \dots + |a_n|.$$

Then  $\|\cdot\|_1$  is also a norm on  $V$ .

**Definition 3.4.** Let  $\mathbf{k}$  be a valuation field and  $V$  a vector space over  $\mathbf{k}$ . Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $V$  are said to be *equivalent* if there exist positive constants  $C_1, C_2 > 0$  such that for all  $x \in V$ ,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

**Lemma 3.5.** Let  $\mathbf{k}$  be a valuation field and  $V$  a vector space over  $\mathbf{k}$ . Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $V$  are equivalent if and only if they induce the same topology on  $V$ .

*Proof.* The sufficiency is clear. Now suppose that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  induce the same topology on  $V$ . Hence the unit open ball with respect to  $\|\cdot\|_1$  contains a unit open ball with respect to  $\|\cdot\|_2$ . That is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every  $x \in V$  with  $\|x\|_1 = 1$ , we have  $\|x\|_2 \geq C = C\|x\|_1$ . By scaling, we get that for every  $x \in V$ ,

$$\|x\|_2 \geq C\|x\|_1.$$

Similar for the other direction, we conclude that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent.  $\square$

**Proposition 3.6.** Let  $V$  be a normed finite-dimensional vector space over a complete valuation field  $\mathbf{k}$ . Then  $V$  is complete.

*Proof.* Yang: To be added.  $\square$

**Theorem 3.7.** Let  $V$  be a finite-dimensional vector space over a complete field  $\mathbf{k}$ . Then all norms on  $V$  are equivalent.

*Proof.* Fix a basis  $\{e_1, e_2, \dots, e_n\}$  of  $V$  and let  $\|\cdot\|_{\max}$  be the maximal norm with respect to this basis as in [Example 3.2](#). Let  $\|\cdot\|$  be any norm on  $V$ . It suffices to show that  $\|\cdot\|$  and  $\|\cdot\|_{\max}$  are equivalent. First we have

$$\|y\| \leq \sum_{i=1}^n |a_i| \|e_i\| \leq \left( \sum_{i=1}^n \|e_i\| \right) \|y\|_{\max}$$

for any  $y = a_1 e_1 + a_2 e_2 + \dots + a_n e_n \in V$ . It remains to show that there exists a constant  $C > 0$  such that for any  $y \in V$ ,

$$\|y\|_{\max} \leq C \|y\|.$$

Yang: To be added. □

**Remark 3.8.** If the base field  $\mathbf{k}$  is not complete, then [Theorem 3.7](#) may fail. For example, let  $\mathbf{k} = \mathbb{Q}$  with the usual absolute value, and let  $V = \mathbb{Q}[\alpha]$  with  $\alpha^2 - \alpha - 1 = 0$ . There are two embeddings of  $V$  into  $\mathbb{R}$ :

$$\iota_1 : a + b\alpha \mapsto a + b \frac{1 + \sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b \frac{1 - \sqrt{5}}{2}.$$

Define two norms on  $V$  by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where  $|\cdot|$  is the usual absolute value on  $\mathbb{R}$ . Then  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are not equivalent since  $\iota_2(\alpha^n) \rightarrow 0$  as  $n \rightarrow \infty$  while  $\iota_1(\alpha^n) \rightarrow \infty$ .

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

**Lemma 3.9.** Let  $\mathbf{k}$  be a complete field and  $V$  a normed finite-dimensional vector space over  $\mathbf{k}$ . Then

$$\|\cdot\| : \text{End}_{\mathbf{k}}(V) \rightarrow \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the  $\mathbf{k}$ -vector space  $\text{End}_{\mathbf{k}}(V)$  satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \text{End}_{\mathbf{k}}(V).$$

*Proof.* First we show the existence of the supremum, i.e., there exists  $C > 0$  such that for all  $x \in V \setminus \{0\}$ ,  $\|T(x)\| \leq C \|x\|$ . Fix a basis  $\{e_1, e_2, \dots, e_n\}$  of  $V$  and let  $\|\cdot\|_{\max}$  be the maximal norm with respect to this basis. Since all norms on  $V$  are bounded by each other by [Theorem 3.7](#), we only need to show that there exists  $C > 0$  such that for all  $x \in V \setminus \{0\}$ ,  $\|T(x)\|_1 \leq C \|x\|_{\max}$ . Write  $T(e_i) = \sum_{j=1}^n a_{ij} e_j$  for  $1 \leq i \leq n$ . For any  $x = \sum_{i=1}^n x_i e_i \in V$ , we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^n \left( \sum_{i=1}^n a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \left( \sum_{1 \leq i, j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of  $\|\cdot\|$  are clear. It remains to show the triangle inequality

and sub-multiplicativity. For any  $A, B \in \text{End}_{\mathbf{k}}(V)$ , we have

$$\frac{\|(A + B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all  $x \in V \setminus \{0\}$  gives  $\|A + B\| \leq \|A\| + \|B\|$ . We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence  $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$ . Taking supremum we get  $\|AB\| \leq \|A\| \cdot \|B\|$ .  $\square$

## 3.2 Finite field extensions

**Lemma 3.10.** Let  $\mathbf{k}$  be a complete field and  $\mathbf{l}$  a finite extension of  $\mathbf{k}$ . Then there exists an absolute value on  $\mathbf{l}$  extending the absolute value on  $\mathbf{k}$ .

*Proof.* Fix a norm  $\|\cdot\|_V$  on the  $\mathbf{k}$ -vector space  $V = \mathbf{l}$ . The norm  $\|\cdot\|_V$  induces an operator norm  $\|\cdot\|_{\text{op}}$  on the  $\mathbf{k}$ -vector space  $\text{End}_{\mathbf{k}}(V)$  as in Lemma 3.9. For any  $a \in \mathbf{l}$ , let  $\mu_a \in \text{End}_{\mathbf{k}}(V)$  be the  $\mathbf{k}$ -linear map defined by multiplication by  $a$ . Note that  $a \mapsto \mu_a$  gives a embedding of  $\mathbf{k}$ -algebras and if  $a \in \mathbf{k}$ ,  $\|\mu_a\|_{\text{op}} = \|a\|_{\mathbf{k}}$ . Thus the restriction of  $\|\cdot\|_{\text{op}}$  to  $\mathbf{l}$  gives an norm on  $\mathbf{l}$  extending that on  $\mathbf{k}$ . The normed ring  $(\mathbf{l}, \|\cdot\|_{\text{op}})$  is a Banach ring since it is a finite-dimensional vector space over the complete field  $\mathbf{k}$ . By ??, there exists a multiplicative seminorm  $\|\cdot\|_{\mathbf{l}}$  on  $\mathbf{l}$  bounded by  $\|\cdot\|_{\text{op}}$ . In particular,  $\|\cdot\|_{\mathbf{l}}$  is bounded by  $\|\cdot\|_{\mathbf{k}}$  on  $\mathbf{k}$ . On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus  $\|\cdot\|_{\mathbf{l}}$  extends the absolute value on  $\mathbf{k}$ .  $\square$

**Theorem 3.11.** Let  $\mathbf{k}$  be a complete field and  $\mathbf{l}$  a finite extension of  $\mathbf{k}$ . Then the absolute value on  $\mathbf{l}$  which extends the absolute value on  $\mathbf{k}$  is uniquely determined by the absolute value on  $\mathbf{k}$ . Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where  $n = [\mathbf{l} : \mathbf{k}]$  and  $N_{\mathbf{l}/\mathbf{k}}$  is the norm map from  $\mathbf{l}$  to  $\mathbf{k}$ .

*Proof.* Let  $\|\cdot\|_{\mathbf{l}}$  be arbitrary absolute value on  $\mathbf{l}$  extending that on  $\mathbf{k}$ . We will show that  $\|\cdot\|_{\mathbf{l}}$  must be equal to  $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$ . For any  $a \in \mathbf{l}$ , set  $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$ . Then  $N_{\mathbf{l}/\mathbf{k}}(b) = 1$  and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that  $\|b\|_{\mathbf{l}} = 1$  whenever  $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ .

Note that the norm map  $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \rightarrow \mathbf{k}$  is the determinant of the  $\mathbf{k}$ -linear map  $\mu_b \in \text{End}_{\mathbf{k}}(V)$  defined by multiplication by  $b$ . Hence it is continuous on  $\mathbf{l}$  (since it is a polynomial in the entries of the matrix representation). If  $\|b\|_{\mathbf{l}} < 1$ , then  $\|b^m\|_{\mathbf{l}} \rightarrow 0$  as  $m \rightarrow \infty$ . Thus  $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \rightarrow 0$  as  $m \rightarrow \infty$ , contradicting the fact that  $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$  for all  $m$ . Similarly, if  $\|b\|_{\mathbf{l}} > 1$ , then just consider  $b^{-1}$ .  $\square$

**Proposition 3.12.** Let  $\mathbf{k}$  be an algebraically closed valuation field. Then its completion  $\widehat{\mathbf{k}}$  is also algebraically closed.

*Proof.* Let  $f \in \widehat{\mathbf{k}}[X]$  be a non-constant polynomial. We will show that  $f$  has a root in  $\widehat{\mathbf{k}}$ . Take a sequence of polynomials  $\{f_n\}_{n \in \mathbb{N}}$  in  $\mathbf{k}[X]$  converging to  $f$  coefficient-wisely and of the same degree  $d$ . Since  $\mathbf{k}$  is algebraically closed, each  $f_n$  splits completely in  $\mathbf{k}$  and hence in  $\widehat{\mathbf{k}}$ . Write  $f_n(X) = \prod_{i=1}^d (X - \alpha_{n,i})$  with  $\alpha_{n,i} \in \widehat{\mathbf{k}}$ .

Let  $\mathbf{l}$  be a finite extension of  $\widehat{\mathbf{k}}$  such that  $f$  has a root  $\alpha$  in  $\mathbf{l}$ . For every  $\varepsilon > 0$ , if there are infinitely many  $n$  such that  $\alpha_{n,i} \notin B(\alpha, \varepsilon)$  for all  $1 \leq i \leq d$ , then we have  $|f_n(\alpha)| \geq \varepsilon^d$  for infinitely many  $n$ , contradicting the fact that  $f_n(\alpha) \rightarrow f(\alpha) = 0$ . Thus for every  $\varepsilon > 0$ , there exists  $N > 0$  such that for all  $n \geq N$ , there exists  $1 \leq i \leq d$  with  $\alpha_{n,i} \in B(\alpha, \varepsilon)$ . That is, we can find a sequence  $\alpha_{n,i_n} \in \mathbf{k}$  converging to  $\alpha$ . Since  $\widehat{\mathbf{k}}$  is complete, we have  $\alpha \in \widehat{\mathbf{k}}$ .  $\square$

### 3.3 Ramification and inertia

In this subsection, we study the extensions of absolute values on finite field extensions. Note that we do not assume the base field to be complete.

**Definition 3.13.** Let  $L/K$  be a finite field extension, and  $v \in M_K$  an absolute value on  $K$ . We denote by  $w|v$  if  $w \in M_L$  is an absolute value on  $L$  extending  $v$ . For each  $w|v$ , we define the *ramification index*  $e(w|v)$  and the *inertia degree*  $f(w|v)$  by

$$e(w|v) := [|\hat{L}^\times|_w : |\hat{K}^\times|_v], \quad f(w|v) := \frac{[\hat{L} : \hat{K}]}{e(w|v)},$$

where  $\hat{K}$  and  $\hat{L}$  are the completions of  $K$  and  $L$  with respect to  $v$  and  $w$ , respectively.

**Lemma 3.14.** Suppose that  $v$  is non-archimedean and  $\mathcal{K}_v$  and  $\mathcal{L}_w$  are the residue fields of  $K$  and  $L$  with respect to  $v$  and  $w$ , respectively. Then we have

$$f(w|v) = [\mathcal{L}_w : \mathcal{K}_v].$$

**Remark 3.15.** Yang: To be added.

**Theorem 3.16.** Let  $L/K$  be a finite field extension, and  $v \in M_K$  an absolute value on  $K$ . Then we have

$$\sum_{w|v} e(w|v)f(w|v) = [L : K].$$

Let  $L/K$  be a finite field extension, and  $v \in M_K$  an absolute value on  $K$ . We have

$$L \otimes_K K_v \cong \prod_{w|v} L_w,$$

where the product is taken over all absolute values  $w \in M_L$  extending  $v$ .

**Theorem 3.17.** Let  $\mathbf{k}$  be a number field. Then

$$M_\mathbf{k}^\infty = \{\text{embeddings } \sigma : \mathbf{k} \rightarrow \mathbb{C}\}$$

and

$$M_\mathbf{k}^f = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_\mathbf{k}\}.$$

Yang: To be revised.

**Proposition 3.18** (Product formula). Let  $\mathbf{k}$  be a number field. For each  $x \in \mathbf{k}^\times$ , we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^\infty; \\ 1, & v \in M_{\mathbf{k}}^0. \end{cases}$$

Yang: To be revised.

**Remark 3.19.** Let  $L/K$  be a finite field extension, and  $v \in M_K$  an absolute value on  $K$ . Suppose that  $v$  is non-archimedean. Yang: To be rewritten.

## 4 Example: $p$ -adic fields

### 4.1 $p$ -adic fields

**Construction 4.1.** Let  $K$  be a number field and  $\mathfrak{p}$  be a prime ideal of the ring of integers  $\mathcal{O}_K$  of  $K$ . Considering the localization  $(\mathcal{O}_K)_{\mathfrak{p}}$  of  $\mathcal{O}_K$  at  $\mathfrak{p}$ , which is a discrete valuation ring, denote by  $v_{\mathfrak{p}} : K^\times \rightarrow \mathbb{Z}$  the corresponding discrete valuation. The  $p$ -adic absolute value on  $K$  associated to  $\mathfrak{p}$  is defined as

$$|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}, \quad \forall x \in K,$$

where  $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$  is the norm of  $\mathfrak{p}$ .

The completion of  $K$  with respect to the  $p$ -adic absolute value  $|\cdot|_{\mathfrak{p}}$  is denoted by  $K_{\mathfrak{p}}$ , called the  $\mathfrak{p}$ -adic field.

We just focus on the case  $K = \mathbb{Q}$  and  $\mathfrak{p} = (p)$  for a prime number  $p$  in the following.

**Example 4.2.** Let  $p$  be a prime number. For every  $r \in \mathbb{Q}$ , we can write  $r$  as  $r = p^n \frac{a}{b}$ , where  $n \in \mathbb{Z}$  and  $a, b \in \mathbb{Z}$  are integers not divisible by  $p$ . The  $p$ -adic absolute value on  $\mathbb{Q}$  is defined as

$$|r|_p := p^{-n}.$$

The  $p$ -adic field  $\mathbb{Q}_p$  can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \mid n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For  $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$  with  $a_n \neq 0$ , its  $p$ -adic absolute value is given by  $|x|_p = p^{-n}$ . The operations of addition and multiplication on  $\mathbb{Q}_p$  are defined similarly as those on decimal expansions.

Unlike the field of real numbers  $\mathbb{R}$ , the  $p$ -adic field  $\mathbb{Q}_p$  has many finite extensions.

**Proposition 4.3.** There are infinitely many irreducible polynomials over the  $p$ -adic field  $\mathbb{Q}_p$ .

*Proof.* Since there are infinitely many irreducible monic polynomials over the finite field  $\mathbb{F}_p$ , consider any lift of such an irreducible monic polynomial to a monic polynomial with coefficients in  $\mathbb{Z}_p$ . If

the lift is not irreducible over  $\mathbb{Q}_p$ , then the factorization of the lift gives a nontrivial factorization of its reduction modulo  $p$  since the factors can be chosen to be monic and have coefficients in  $\mathbb{Z}_p$ , which contradicts the irreducibility of the original polynomial over  $\mathbb{F}_p$ . Thus, the lift is irreducible over  $\mathbb{Q}_p$ .

On the other hand, note that  $|\mathbb{Q}_p^\times|_p = p^{\mathbb{Z}}$ . It follows that  $f(T) = T^n - p$  is irreducible over  $\mathbb{Q}_p$  for every integer  $n \geq 1$ . Otherwise, suppose we have a monic factorization  $f(T) = g(T)h(T)$  with  $g(T), h(T) \in \mathbb{Z}_p[T]$  and  $\deg g, \deg h < n$ . Then by considering the reduction modulo  $p$ , we have  $g(0), h(0) \equiv 0 \pmod{p}$ . It follows that  $|f(0)|_p = |g(0)h(0)|_p \leq p^{-2}$ , which contradicts  $|f(0)|_p = |p|_p = p^{-1}$ .  $\square$

## 4.2 Completion

**Proposition 4.4.** The algebraic closure  $\overline{\mathbb{Q}_p}$  of  $\mathbb{Q}_p$  is not complete with respect to the extension of the  $p$ -adic absolute value  $|\cdot|_p$ .

*Proof.* Yang: To be completed.  $\square$

**Construction 4.5.** Let  $p$  be a prime number. The field  $\mathbb{C}_p$  of  $p$ -adic complex numbers is defined as the completion of the algebraic closure of  $\mathbb{Q}_p$  with respect to the unique extension of the  $p$ -adic absolute value  $|\cdot|_p$  on  $\mathbb{Q}_p$ .

The field  $\mathbb{C}_p$  is algebraically closed and complete with respect to  $|\cdot|_p$  by Proposition 3.12. By Corollaries 2.11 and 2.12, we have

$$|\mathbb{C}_p^\times|_p = |\overline{\mathbb{Q}_p}^\times|_p = p^{\mathbb{Q}}, \quad \kappa_{\mathbb{C}_p} \cong \kappa_{\overline{\mathbb{Q}_p}} \cong \overline{\mathbb{F}_p}.$$

**Proposition 4.6.** The field  $\mathbb{C}_p$  of  $p$ -adic complex numbers is not spherically complete.

*Proof.* Yang: To be completed.  $\square$

**Construction 4.7.** Let  $p$  be a prime number. Yang: We construct the *spherically complete  $p$ -adic field*  $\Omega_p$ . Yang: To be completed.

## 4.3 Elementary functions

**Lemma 4.8.** Let  $p$  be a prime number and  $n \in \mathbb{N}$ . We have  $v_p(n!) =$ .

Yang: Exponential, logarithmic, and the interpolation functions.

Fix a prime number  $p$  in the following and consider  $\mathbf{k} = \mathbb{Q}_p, \mathbb{C}_p$ , or  $\Omega_p$ . Let  $r_p := p^{-1/(p-1)}$ .

**Construction 4.9.** The exponential function  $\exp : \mathbf{k} \rightarrow \mathbf{k}$  is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The radius of convergence of  $\exp(x)$  is  $+\infty$  if  $p = 2$  and  $p^{-1/(p-1)}$  if  $p > 2$ .

The logarithmic function  $\log : 1 + \mathbf{k}^\circ \rightarrow \mathbf{k}$  is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

The radius of convergence of  $\log(1 + x)$  is 1.

Moreover, for every  $x$  in the domain of convergence of  $\exp$  and every  $y$  in the domain of convergence of  $\log$ , we have

$$\log(\exp(x)) = x, \quad \exp(\log(y)) = y.$$

Yang: To be checked.

**Definition 4.10.** Let

**Theorem 4.11.** The series converges.

## References

- [Xie25] Junyi Xie. “The existence of Zariski dense orbits for endomorphisms of projective surfaces”. In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62.