Example: p-adic fields

1 p-adic fields

Construction 1. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_{\mathfrak{p}}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_{\mathfrak{p}} : K^{\times} \to \mathbb{Z}$ the corresponding discrete valuation. The $\mathfrak{p}$*-adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the $\mathfrak{p}$-adic absolute value $|\cdot|_{\mathfrak{p}}$ is denoted by $K_{\mathfrak{p}}$, called the $\mathfrak{p}$*-adic field*.

We just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p in the following.

Example 2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The p*-adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Easily see that $|\mathbb{Q}_p^{\times}|_p = p^{\mathbb{Z}}$ and $\mathscr{k}_{\mathbb{Q}_p} \cong \mathbb{F}_p$.

Unlike the field of real numbers $\mathbb{R}$, the p-adic field $\mathbb{Q}_p$ has many finite extensions.

Proposition 3. The field $\mathbb{Q}_p$ has algebraic extensions of arbitrarily large degree.

Proof. Since there are infinitely many irreducible monic polynomials over the finite field $\mathbb{F}_p$, consider any lift of such an irreducible monic polynomial to a monic polynomial with coefficients in $\mathbb{Z}_p$. If the lift is not irreducible over $\mathbb{Q}_p$, then the factorization of the lift gives a nontrivial factorization of its reduction modulo p since the factors can be chosen to be monic and have coefficients in $\mathbb{Z}_p$, which contradicts the irreducibility of the original polynomial over $\mathbb{F}_p$. Thus, the lift is irreducible over $\mathbb{Q}_p$.

On the other hand, note that $|\mathbb{Q}_p^{\times}|_p = p^{\mathbb{Z}}$. It follows that $f(T) = T^n - p$ is irreducible over $\mathbb{Q}_p$ for every integer $n \geq 1$. Otherwise, suppose we have a monic factorization $f(T) = g(T)h(T)$ with $g(T), h(T) \in \mathbb{Z}_p[T]$ and $\deg g, \deg h < n$. Then by considering the reduction modulo p, we have $g(0), h(0) \equiv 0 \mod p$. It follows that $|f(0)|_p = |g(0)h(0)|_p \leq p^{-2}$, which contradicts $|f(0)|_p = |p|_p = p^{-1}$. □

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Let
$$R := \bigcup_{n=1}^{+\infty} \mathbb{Q}_p(\mu_{p^n})$$
and
$$K := \bigcup_{n=1}^{+\infty} \mathbb{Q}_p(\bigcup_{p\nmid m} \mu_m).$$
Yang: R is a totally ramified extension and K is the maximal unramified extension.

2 Completion

Proposition 4. The algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is not complete with respect to the extension of the p-adic absolute value $|\cdot|_p$.

Proof. Let $(a_n)_{n\geq 1}$ be a sequence in $\overline{\mathbb{Q}_p}$ such that
$$[\mathbb{Q}_p(\{a_n\}) : \mathbb{Q}_p] = +\infty.$$
Set $s_1 = 0$ and
$$s_n = s_{n-1} + p^{k_n} a_n$$
for a suitable $k_n \in \mathbb{N}$ such that
$$|p^{k_n} a_n|_p \leq \frac{|p^{k_{n-1}} a_{n-1}|_p}{2}, \quad |p^{k_n} a_n|_p < |s_{n-1} - \sigma(s_{n-1})|_p \quad \forall \sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}_p}/\mathbb{Q}_p\right) \text{ with } \sigma(s_{n-1}) \neq s_{n-1}.$$
Then the sequence $(s_n)_{n\geq 1}$ is a Cauchy sequence. Let $s \in \widehat{\overline{\mathbb{Q}_p}}$ be the limit of $(s_n)_{n\geq 1}$. We have $|s - s_{n-1}|_p = |p^{k_n} a_n|_p < |s_{n-1} - \sigma(s_{n-1})|_p$ for all $n \geq 1$ and all $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}_p}/\mathbb{Q}_p\right)$ with $\sigma(s_{n-1}) \neq s_{n-1}$. By Krasner's lemma (Theorem 11), it follows that $s_n \in \mathbb{Q}_p(s)$ for all $n \geq 1$. Thus, we have
$$[\mathbb{Q}_p(s) : \mathbb{Q}_p] \geq [\mathbb{Q}_p(\{a_n\}) : \mathbb{Q}_p] = +\infty,$$
which implies that $s \notin \overline{\mathbb{Q}_p}$. □

Construction 5. Let p be a prime number. The *field* $\mathbb{C}_p$ *of p-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$.

The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$ by Proposition 8. By Corollaries 9 and 10, we have
$$|\mathbb{C}_p^\times|_p = |\overline{\mathbb{Q}_p}^\times|_p = p^{\mathbb{Q}}, \quad \Bbbk_{\mathbb{C}_p} \cong \Bbbk_{\overline{\mathbb{Q}_p}} \cong \overline{\mathbb{F}_p}.$$

Proposition 6. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Proof. Yang: To be completed. □

Yang: For example, see p-adic fields for beginners.

Construction 7. Let p be a prime number. Yang: We construct the *spherically complete p-adic field* Ω_p. Yang: To be completed.

Yang: does Ω_p has the same card as $\mathbb{R}$?

Appendix

Proposition 8. Let $\mathbf{k}$ be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

Corollary 9. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the residue field $\Bbbk_{\widehat{\mathbf{k}}} \cong \Bbbk_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$.

Corollary 10. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the valuation group $|\widehat{\mathbf{k}}^\times|$ of $\widehat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^\times|$ of $\mathbf{k}$.

Theorem 11 (Krasner's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field, and $\alpha, \beta \in \overline{\mathbf{k}}$. Denote by $\alpha_1, \alpha_2, \ldots, \alpha_n$ the conjugates of α over $\mathbf{k}$ with $\alpha_1 = \alpha$. If

$$|\beta - \alpha| < |\alpha - \alpha_i|, \quad \forall i = 2, \ldots, n,$$

then $\alpha \in \mathbf{k}(\beta)$.