

# Valuation fields

## 1 Absolute values and completion

**Definition 1.** Let  $\mathbf{k}$  be a field. An *absolute value* on  $\mathbf{k}$  is a function  $\|\cdot\| : \mathbf{k} \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following properties for all  $x, y \in \mathbf{k}$ :

- (a)  $\|x\| = 0$  if and only if  $x = 0$ ;
- (b)  $\|xy\| = \|x\| \cdot \|y\|$ ;
- (c)  $\|x + y\| \leq \|x\| + \|y\|$ .

A field  $\mathbf{k}$  equipped with an absolute value  $\|\cdot\|$  is called a *valuation field*.

**Remark 2.** Let  $\mathbf{k}$  be a field. Recall that a (additive) valuation on  $\mathbf{k}$  is a function  $v : \mathbf{k}^\times \rightarrow \mathbb{R}$  such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$ ;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$ .

We can extend  $v$  to the whole field  $\mathbf{k}$  by defining  $v(0) = +\infty$ . Fix a real number  $\varepsilon \in (0, 1)$ . Then  $v$  induces an absolute value  $|\cdot|_v : \mathbf{k} \rightarrow \mathbb{R}_+$  defined by  $|x|_v = \varepsilon^{v(x)}$  for each  $x \in \mathbf{k}$ .

The valuation  $v$  defined above is called an *additive valuation*. And an absolute value  $|\cdot|$  on  $\mathbf{k}$  is called a *multiplicative valuation*. In this note, the term *valuation* may refer to either an additive valuation or a multiplicative valuation, depending on the context.

**Example 3.** Let  $\mathbf{k}$  be a field. The *trivial absolute value* on  $\mathbf{k}$  is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

**Definition 4.** The (*multiplicative*) *valuation group* of a valuation field  $(\mathbf{k}, \|\cdot\|)$  is defined as the subgroup of  $\mathbb{R}_{>0}$  given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation  $\sqrt[n]{|\mathbf{k}^\times|}$  to denote the set  $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$ .

Note that an absolute value  $\|\cdot\|$  is non-trivial if and only if its valuation group  $|\mathbf{k}^\times|$  is not equal to  $\{1\}$ .

**Definition 5.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field. We say that  $\mathbf{k}$  is *complete* if the metric  $d(x, y) := \|x - y\|$  makes  $\mathbf{k}$  a complete metric space.

**Lemma 6.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field and  $(\widehat{\mathbf{k}}, \|\cdot\|)$  its completion as a metric space. Then the operations of addition and multiplication on  $\mathbf{k}$  can be extended to  $\widehat{\mathbf{k}}$  uniquely, making  $(\widehat{\mathbf{k}}, \|\cdot\|)$  a complete valuation field containing  $\mathbf{k}$  as a dense subfield.

*Proof.* Note that the operations of addition and multiplication on  $\mathbf{k}$  are uniformly continuous with respect to the metric  $d(x, y) = \|x - y\|$ . Thus they can be extended to  $\widehat{\mathbf{k}}$  uniquely.  $\square$

**Proposition 7.** Let  $(\mathbf{k}, \|\cdot\|)$  be a complete valuation field with non-trivial absolute value. Then  $\mathbf{k}$  is uncountable.

*Proof.* Since the absolute value  $\|\cdot\|$  is non-trivial, we can construct a sequence  $\{x_n\}_{n=1}^{\infty} \subseteq \mathbf{k}$  inductively such that  $\|x_n\| < \|x_{n-1}\|/2$  for any  $n \geq 1$  and  $\|x_0\| < 1$ . Then there is an injective map from  $\mathbb{N}^{\{0,1\}}$  to  $\mathbf{k}$  defined by

$$(a_n)_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} a_n x_n, \quad a_n \in \{0, 1\}.$$

Since  $\|x_n\| < 2^{-n}$ , the series  $\sum_{n=1}^{\infty} a_n x_n$  converges in  $\mathbf{k}$ . Note  $\|x_n\| > \|\sum_{m \geq n} x_m\|$  for each  $n$ , we have that the map is injective. Thus  $\mathbf{k}$  is uncountable.  $\square$

Unlike the real number field  $\mathbb{R}$ , even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

**Definition 8.** A valuation field  $(\mathbf{k}, \|\cdot\|)$  is called *spherically complete* if every decreasing sequence of closed balls in  $\mathbf{k}$  has a non-empty intersection.

**Example 9.** The field  $\mathbb{C}_p$  of  $p$ -adic complex numbers is not spherically complete, see [Yang: to be added](#).

**Example 10.** Let  $|\cdot|_{\infty}$  be the usual absolute value on the field  $\mathbb{Q}$  of rational numbers. Then  $(\mathbb{Q}, |\cdot|_{\infty})$  is a valuation field. Its completion is the field  $\mathbb{R}$  of real numbers equipped with the usual absolute value.

**Example 11.** Let  $p$  be a prime number. For any non-zero rational number  $x \in \mathbb{Q}$ , we can write it as  $x = p^n \frac{a}{b}$ , where  $n \in \mathbb{Z}$  and  $a, b \in \mathbb{Z}$  are integers not divisible by  $p$ . The  $p$ -adic absolute value on  $\mathbb{Q}$  is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then  $(\mathbb{Q}, |\cdot|_p)$  is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of  $p$ -adic numbers equipped with the  $p$ -adic absolute value; see [Yang: to be added](#).

**Definition 12.** Let  $(\mathbf{k}, \|\cdot\|)$  be a valuation field. We say that  $\mathbf{k}$  is *non-archimedean* if its absolute value  $\|\cdot\|$  satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that  $\mathbf{k}$  is *archimedean*.

**Example 13.** Let  $v$  be an additive valuation on a field  $\mathbf{k}$ . Then the induced absolute value  $|\cdot|_v$  as in Remark 2 is non-archimedean.

The converse is also true: if  $(\mathbf{k}, |\cdot|)$  is a non-archimedean valuation field, then the function  $v : \mathbf{k}^\times \rightarrow \mathbb{R}$  defined by  $v(x) = -\log |x|$  is an additive valuation on  $\mathbf{k}$ .

**Proposition 14.** Let  $(\mathbf{k}, |\cdot|)$  be a valuation field. Then  $\mathbf{k}$  is archimedean if and only if the set  $\{|n \cdot 1| : n \in \mathbb{Z}\}$  is unbounded.

*Proof.* Sufficiency is obvious. Yang: To be added. □

## 2 Places on a field

**Definition 15.** Let  $\mathbf{k}$  be a field. Two absolute values  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $\mathbf{k}$  are said to be *equivalent* if there exists a real number  $c \in (0, \infty)$  such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field  $\mathbf{k}$ . Moreover, the following lemma shows that the converse is also true.

**Lemma 16.** Let  $\mathbf{k}$  be a field and  $\|\cdot\|_1, \|\cdot\|_2$  be two absolute values on  $\mathbf{k}$ . Then the following statements are equivalent:

- (a)  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent;
- (b)  $\|\cdot\|_1$  and  $\|\cdot\|_2$  induce the same topology on  $\mathbf{k}$ ;
- (c) The unit disks  $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$  and  $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$  are the same.

*Proof.* The implications (a)  $\Rightarrow$  (b) is obvious. Now we prove (b)  $\Rightarrow$  (c). For any  $x \in D_1$ , we have  $x^n \rightarrow 0$  as  $n \rightarrow \infty$  under the absolute value  $\|\cdot\|_1$  and thus under  $\|\cdot\|_2$ . Therefore,  $\|x\|_2^n \rightarrow 0$  as  $n \rightarrow \infty$ , which implies that  $\|x\|_2 < 1$ , i.e.,  $x \in D_2$ . Similarly, we can prove that  $D_2 \subseteq D_1$ .

Finally, we prove (c)  $\Rightarrow$  (a). If  $\|\cdot\|_1$  is trivial, then  $D_1 = \{0\}$  and thus  $\|\cdot\|_2$  is also trivial. In this case, they are equivalent. Suppose that both  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are non-trivial. Pick any  $x, y \notin D_1 = D_2$ . Then there exist real numbers  $\alpha, \beta > 0$  such that  $\|x\|_1 = \|x\|_2^\alpha$  and  $\|y\|_1 = \|y\|_2^\beta$ . Suppose the contrary that  $\alpha \neq \beta$ . Consider the domain  $\Lambda \subseteq \mathbb{Z}^2$  defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since  $\alpha \neq \beta$ , the two lines defined by the equalities are not parallel. Thus  $\Lambda$  is non-empty. Pick  $(n, m) \in \Lambda$  and set  $z := x^n y^{-m}$ . Then we have  $\|z\|_2 < 1$  and  $\|z\|_1 > 1$ , a contradiction. □

**Theorem 17** (Ostrowski). Let  $(\mathbf{k}, \|\cdot\|)$  be an archimedean complete valuation field. Then  $\mathbf{k}$  is isomorphic to either the real number field  $\mathbb{R}$  or the complex number field  $\mathbb{C}$  equipped with the usual absolute value.

*Proof.* Yang: To be added. □

**Definition 18.** Let  $\mathbf{k}$  be a field. A *place* on  $\mathbf{k}$  is an equivalence class of non-trivial absolute values on  $\mathbf{k}$ . An archimedean (resp. non-archimedean) place is also called an *finite* (resp. *infinite*) place. We denote the set of all places (resp. finite, infinite places) on  $\mathbf{k}$  by  $M_{\mathbf{k}}$  (resp.  $M_{\mathbf{k}}^f, M_{\mathbf{k}}^\infty$ ). If  $\mathbf{l} \subset \mathbf{k}$  is a subfield, we denote by  $M_{\mathbf{k}/\mathbf{l}}$  (resp.  $M_{\mathbf{k}/\mathbf{l}}^f, M_{\mathbf{k}/\mathbf{l}}^\infty$ ) the set of all places (resp. finite, infinite places) on  $\mathbf{k}$  which are trivial on  $\mathbf{l}$ .

**Example 19.** Let  $\mathbf{k} = \mathbb{C}(t)$  and  $\mathbf{l} = \mathbb{C}$ . Then I claim that

$$M_{\mathbb{C}(t)/\mathbb{C}} \cong \{ \text{prime divisors on } \mathbb{P}_{\mathbb{C}}^1 \}.$$

For each prime divisor  $P$  on  $\mathbb{P}_{\mathbb{C}}^1$ , we can define an additive valuation  $\text{Mult}_P : \mathbb{C}(t)^\times \rightarrow \mathbb{Z}$  by assigning to each non-zero rational function  $f \in \mathbb{C}(t)^\times$  its multiplicity at  $P$ . Fix a real number  $\varepsilon \in (0, 1)$ . Then we obtain a multiplicative valuation (absolute value)  $|\cdot|_P$  on  $\mathbb{C}(t)$  as in Remark 2. It is easy to check that the absolute value  $|\cdot|_P$  is trivial on  $\mathbb{C}$  and that different prime divisors give rise to inequivalent absolute values.

Conversely, given any non-trivial absolute value  $|\cdot|$  on  $\mathbb{C}(t)$  which is trivial on  $\mathbb{C}$ , by Proposition 14 and Example 13, the absolute value  $|\cdot|$  is given by an additive valuation  $v : \mathbb{C}(t)^\times \rightarrow \mathbb{R}$ . Let  $\mathcal{O}_v$  be the valuation ring of  $v$  and  $\mathfrak{m}_v$  its maximal ideal. Then  $t \in \mathcal{O}_v$  or  $t^{-1} \in \mathcal{O}_v$ . Without loss of generality, we assume that  $t \in \mathcal{O}_v$ . Since  $|\cdot|$  is trivial on  $\mathbb{C}$ , we have  $\mathbb{C}[t] \subseteq \mathcal{O}_v$ . And we have  $\mathfrak{m}_v \cap \mathbb{C}[t] \neq 0$  since otherwise  $v$  is trivial on  $\mathbb{C}[t]^\times$  and thus on  $\mathbb{C}(t)^\times$ . It follows that the image of  $\mathbb{C}[t]$  under the quotient map  $\mathcal{O}_v \rightarrow \mathcal{O}_v/\mathfrak{m}_v$  is  $\mathbb{C}$ . This gives a point  $P \in \mathbb{A}_{\mathbb{C}}^1 \subseteq \mathbb{P}_{\mathbb{C}}^1$ . Then  $v$  is different from the additive valuation  $\text{Mult}_P$  by a positive scalar multiple via looking at the values on  $\mathbb{C}[t]$ . Thus we have established the claimed bijection.

**Theorem 20** (Ostrowski). Every nontrivial absolute value on  $\mathbb{Q}$  is equivalent to either the usual absolute value  $|\cdot|_\infty$  or a  $p$ -adic absolute value  $|\cdot|_p$  for some prime number  $p$ . In other words,

$$M_{\mathbb{Q}} = \{ |\cdot|_\infty \} \cup \{ |\cdot|_p : p \text{ is a prime number} \}.$$

*Proof.* Yang: To be added. □

**Remark 21.** For every non-archimedean place  $v$  on  $\mathbb{Q}$  corresponding to a prime number  $p$ , we choose the unique normalized absolute value  $|\cdot|_v$  in the class  $v$  such that  $|p|_v = 1/p$ . For the archimedean place  $v$  on  $\mathbb{Q}$ , we choose the usual absolute value  $|\cdot|_v = |\cdot|_\infty$ . Unless otherwise specified, we always use the normalized absolute values on  $\mathbb{Q}$ .

**Remark 22.** For any non-zero rational number  $x \in \mathbb{Q}^\times$ , one can easily check the *product formula*:

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1.$$

This can be viewed as an arithmetic analogue of the fact on  $\mathbb{P}_{\mathbb{C}}^1$  that

$$\sum_{P \in \mathbb{P}^1(\mathbb{C})} \text{Mult}_P(f) = 0$$

for any non-zero rational function  $f \in \mathbb{C}(t)^\times$ . Indeed, fix a real number  $\varepsilon \in (0, 1)$ . Then by Example 19, above fact can be rewritten as

$$\prod_{P \in \mathbb{P}_{\mathbb{C}}^1} |f|_P = 1.$$

**Theorem 23** (Artin-Whaples approximations). Let  $\mathbf{k}$  be a field and  $v_1, v_2, \dots, v_n \in M_{\mathbf{k}}$  be pairwise distinct places on  $\mathbf{k}$ . For each  $i \in \{1, 2, \dots, n\}$ , let  $x_i \in \mathbf{k}$  and  $\varepsilon_i > 0$ . Then there exists an element  $x \in \mathbf{k}$  such that

$$|x - x_i|_{v_i} < \varepsilon_i, \quad \forall i \in \{1, 2, \dots, n\}.$$

In particular, the image of the diagonal embedding

$$\mathbf{k} \rightarrow \prod_{i=1}^n \mathbf{k}_{v_i}$$

is dense, where  $\mathbf{k}_{v_i}$  is the completion of  $\mathbf{k}$  with respect to the place  $v_i$ .

*Proof.* Yang: To be added.

□

DRAFT