Non-archimedean analysis

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Contents

The main references for this chapter are [Gou97; Rob00; 李文威 18].

1   Tate algebras

Notation 1.1. Let $T = (T_1, \ldots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \ldots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;

- $\underline{T/r} := (T_1/r_1, T_2/r_2, \ldots, T_n/r_n)$;

- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;

- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \ldots, n$, we have $\alpha_i \leq \beta_i$;

- $E(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \ldots, n\}$ and $B(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \ldots, n\}$ for $x = (x_1, \ldots, x_n) \in \mathbf{k}^n$;

- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

Definition 1.2. Let $\mathbf{k}$ be a complete non-archimedean field. Let $T = (T_1, \ldots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \ldots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring*

of restricted power series) is defined as

$$\mathbf{k}\langle \underline{r^{-1}T} \rangle := \mathbf{k}\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 1.3. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ is a non-archimedean multiplicative banach $\mathbf{k}$-algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Yang: For the definition of banach ring, see

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ are two elements in $\mathbf{k}\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Hence $f \cdot g \in \mathbf{k}\{\underline{T/r}\}$ and it shows that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $\mathbf{k}\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f + g\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \leq \sup_{\alpha \in \mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \leq \max\{\|f\|, \|g\|\}$$

and

$$\|f \cdot g\| = \left\| \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma \right\| = \sup_{\gamma \in \mathbb{N}^n} \left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma$$

$$\leq \sup_{\gamma \in \mathbb{N}^n} \max_{\alpha+\beta=\gamma} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \leq \|f\| \cdot \|g\|.$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\| \cdot \|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \left\| \sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta \right\| r^{\alpha_1+\beta_1} \leq \|f \cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by **??**. Thus the gauss norm is multiplicative.

Step 4. Finally show that $\mathbf{k}\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $\mathbf{k}\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in $\mathbf{k}$. Since $\mathbf{k}$ is complete, set $a_\alpha := \lim_{m \to +\infty} a_{\alpha,m}$ and $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \leq 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \leq \varepsilon$$

for all $m > N$. $\qquad\square$

Proposition 1.4. Let $\mathbf{k}$ be a complete non-archimedean field. An element $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ is invertible if and only if $\|a_0\| > \|a_\alpha\| r^\alpha$ for all $\alpha \neq 0$.

Proof. Multiplying by a_0^{-1}, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $\mathbf{k}[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = - \sum_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\| r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for m. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\| r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\{A^{m+1}, A \cdot A^m\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in \mathbf{k}\{\underline{T/r}\}$. $\qquad\square$

Let $\mathbf{k}$ be a complete non-archimedean field. Recall that the formal derivative operator $\partial_i : \mathbf{k}[[\underline{T}]] \to \mathbf{k}[[\underline{T}]]$ is defined by

$$\frac{\partial}{\partial T_i}\left(\sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha\right) := \sum_{\alpha \in \mathbb{N}^n} \alpha_i a_\alpha T_1^{\alpha_1} \cdots T_i^{\alpha_i - 1} \cdots T_n^{\alpha_n}.$$

Lemma 1.5. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T}/\underline{r}\}$, we have $\partial_i(f) \in \mathbf{k}\{\underline{T}/\underline{r}\}$.

Proof. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/\underline{r}\}$. We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha \in \mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1 - 1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}.$$

Noting that $\mathbf{k}$ is non-archimedean, we have $\|\alpha_1 a_\alpha\| \le \|a_\alpha\|$. Then

$$\lim_{|\alpha| \to +\infty} \|\alpha_1 a_\alpha\| r_1^{\alpha_1 - 1} r_2^{\alpha_2} \cdots r_n^{\alpha_n} \le \frac{1}{r_1} \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0.$$

The conclusion follows. $\square$

2 Analytic functions on closed polydisks

Proposition 2.1. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T}/\underline{r}\}$, we can associate a function $F_f : E(0, \underline{r}) \to \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \quad \text{for } x = (x_1, \dots, x_n) \in E(0, \underline{r}).$$

This defines a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T}/\underline{r}\}$ to the ring of all functions from $E(0, \underline{r})$ to $\mathbf{k}$.

Proof. Given $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/\underline{r}\}$ and $x = (x_1, \dots, x_n) \in E(0, \underline{r})$, we have

$$\left\|\sum_{|\alpha|=n} a_\alpha x^\alpha\right\| \le \max_{|\alpha|=n} \|a_\alpha\| r^\alpha \to 0 \quad \text{as } n \to +\infty.$$

Hence by **??**, the series $F_f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$ converges in $\mathbf{k}$. This defines a function $F_f : E(0, \underline{r}) \to \mathbf{k}$.

Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/\underline{r}\}$. Set

$$A_n = \sum_{|\alpha|<n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta|<n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma|<n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) x^\gamma.$$

We need to show that $F_f(x) F_g(x) = \lim A_n B_n = \lim C_n = F_{fg}(x)$. Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta|\ge n}} a_\alpha b_\beta x^{\alpha+\beta}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha <$

$\varepsilon/\|f\|$. For any $n > 2N$, we have

$$\|A_n B_n - C_n\| \leq \max_{\substack{|\alpha|<n,|\beta|<n \\ |\alpha+\beta|\geq n}} \|a_\alpha\|\|b_\beta\|\|x^{\alpha+\beta}\| < \max\left\{\frac{\varepsilon}{\|g\|}\|b_\beta\|r^\beta, \frac{\varepsilon}{\|f\|}\|a_\alpha\|r^\alpha\right\} \leq \varepsilon.$$

Thus $F_f(x)F_g(x) = (F_{fg})(x)$. The addition and scalar multiplication can be verified directly. We thus finish the proof. □

Proposition 2.2. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x, y \in E(0,\underline{r})$, we have

$$\|f(y) - f(x)\|_\mathbf{k} \leq L \cdot \|y - x\|_\infty,$$

where $L = \max_{1 \leq i \leq n} \|f\|_g/r_i$.

Proof. Set $y - x = (h_1, \ldots, h_n)$ and $x^{(0)} = x$, $x^{(i)} = (x_1 + h_1, \ldots, x_i + h_i, x_{i+1}, \ldots, x_n)$ for $i = 1, \ldots, n$. We have

$$\|f(y) - f(x)\|_\mathbf{k} \leq \max_{1 \leq i \leq n} \|f(x^{(i)}) - f(x^{(i-1)})\|_\mathbf{k}.$$

We only need to show that for every $i = 1, \ldots, n$, we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_\mathbf{k} \leq \frac{\|f\|_g}{r_i}\|h_i\|.$$

Without loss of generality and for simplicity, we assume that $y = (x_1 + h, x_2, \ldots, x_n)$ and $x = (x_1, x_2, \ldots, x_n)$. Note that by the strong triangle inequality, we have $\|h\| \leq r_1$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$f(y) - f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \left((x_1 + h)^{\alpha_1} - x_1^{\alpha_1}\right) x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$$= \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k.$$

Note that

$$\left\|\binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\right\| r_1^k \leq \|a_\alpha\| r^\alpha \leq \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_\mathbf{k} \leq \max_{\alpha \in \mathbb{N}^n} \max_{1 \leq k \leq \alpha_1} \left\{\left\|\binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\right\| \|h\|^k\right\} \leq \max_k \left\{\|f\|_g \left(\frac{\|h\|}{r_1}\right)^k\right\} \leq \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

Lemma 2.3. Let $\mathbf{k}$ be a complete non-archimedean field. Then we have $\|f(x)\| \leq \|f\|$ for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0,\underline{r})$. In particular, if $f_n \to f$ as $n \to +\infty$ in $\mathbf{k}\{\underline{T/r}\}$, then we have $\|f_n(x) - f(x)\| \to 0$ for every $x \in E(0,\underline{r})$.

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \ldots, x_n) \in E(0, \underline{r})$. We have

$$\left\| \sum_{|\alpha| < N} a_\alpha x^\alpha \right\| \leq \max_{|\alpha| < N} \|a_\alpha\| r^\alpha \leq \|f\|$$

for every $N \in \mathbb{N}$. Taking $N \to +\infty$, we have $\|f(x)\| \leq \|f\|$. $\qquad\square$

Proposition 2.4. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation, and $\partial_i = \partial/\partial T_i$ be the derivative operator on $\mathbf{k}\{\underline{T/r}\}$ with respect to the indeterminate T_i for $i = 1, \ldots, n$. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$, we have

$$F_{\partial_i(f)}(x) = \lim_{h \to 0} \frac{F_f(x_1, \ldots, x_i + h, \ldots, x_n) - F_f(x)}{h}.$$

Proof. Without loss of generality, we can assume that $i = 1$. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $f_n = \sum_{|\alpha| < n} a_\alpha T^\alpha$ for $n \in \mathbb{N}$. Set $x_h = (x_1 + h, x_2 \ldots, x_n)$ and $L_f(h) = (F_f(x_h) - F_f(x))/h$ for $h \in \mathbf{k}^\times$. Note that for fixed h, we have $\lim_{n \to \infty} L_{f_n}(h) = L_f(h)$.

We compute $L_{f_n}(h) - F_{\partial f_n}(x)$ explicitly:

$$L_{f_n}(h) - F_{\partial f_n}(x) = \frac{1}{h}\left(\sum_{|\alpha| < n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} h^k x_2^{\alpha_2} \cdots x_n^{\alpha_n} - \sum_{|\alpha| < n} \alpha_1 a_\alpha x_1^{\alpha_1 - 1} h x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right)$$

$$= \sum_{|\alpha| < n} \sum_{k=2}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^{k-1}.$$

Note that

$$M = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\| r_1^{k-1} \leq \|f\|/r_1 < +\infty.$$

Hence

$$\|L_{f_n}(h) - F_{\partial f_n}(x)\| \leq \max_{2 \leq k \leq n} \left\{ M \frac{\|h\|^{k-1}}{r_1^{k-1}} \right\} \leq M \frac{\|h\|}{r_1}$$

for $h \in \mathbf{k}^\times$ with $\|h\| < r_1$. Taking $n \to +\infty$, we have

$$\|L_f(h) - F_{\partial f}(x)\| \leq M \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. $\qquad\square$

Corollary 2.5. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment $f \mapsto F_f$ in Proposition 2.1 is injective.

Proof. Note that if $F_f = 0$, then for every $i = 1, \ldots, n$, we have $F_{\partial_i(f)} = 0$ by Proposition 2.4. By taking repeated derivatives, we have $F_{\partial^\alpha f} = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Note that $F_{\partial^\alpha f}(0) = \alpha! a_\alpha$. It follows that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n$ and thus $f = 0$. $\qquad\square$

Remark 2.6. Corollary 2.5 holds for non-archimedean fields of positive characteristic as well. The proof uses **??** and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element $f \in \mathbf{k}\{\underline{T/r}\}$ with the associated function $F_f : E(0, \underline{r}) \to \mathbf{k}$ as in Proposition 2.1.

Proposition 2.7. Let $\Bbbk$ be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra $\Bbbk\{\underline{T}/\underline{r}\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in E(0,\underline{r})} \|f(x)\|_{\Bbbk}.$$

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \Bbbk\{\underline{T}/\underline{r}\}$. We write $f = g + h$ with $g = \sum_{\alpha \in S} a_\alpha T^\alpha$ and $h = \sum_{\alpha \notin S} a_\alpha T^\alpha$, where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that S is a non-empty finite set and $\|h\| < \|f\|$. By Lemma 2.3, we have $\|h(x)\| < \|f\|$ for every $x \in E(0,\underline{r})$. It suffices to show that $\|g\|_{\sup} = \|g\|$.

Since $\Bbbk$ is algebraically closed, $|\Bbbk^\times|$ is dense in $\mathbb{R}_{>0}$. For every pair $\alpha, \beta \in S$ with $\alpha \neq \beta$, the set $\{t \in \mathbb{R}^n_{>0} : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$ is a proper closed subset of $\mathbb{R}^n_{>0}$. Thus we can find $t_m \in |\Bbbk^\times|^n$ such that $t_m < r$, $t_m \to r$ as $m \to +\infty$ and for every $\alpha, \beta \in S$ with $\alpha \neq \beta$, we have $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$ for all m. For each m, we can find $x_m \in E(0,\underline{r})$ such that $\|x_m^\alpha\| = t_m^\alpha$ for every $\alpha \in S$ since $t_m \in |\Bbbk^\times|^n$. It follows that

$$\|g(x_m)\| = \max_{\alpha \in S} \|a_\alpha\| \|x_m^\alpha\| = \max_{\alpha \in S} \|a_\alpha\| t_m^\alpha \to \|g\| \quad \text{as } m \to +\infty.$$

Thus $\|g\|_{\sup} = \|g\|$. $\qquad\square$

Remark 2.8. If $\mathbf{k}$ is not algebraically closed, the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T}/\underline{r}\}$ may not coincide with the supremum norm. For example, consider the Tate algebra $\mathbb{Q}_p\{T\}$. The element $f = T^p - T$ has gauss norm $\|f\| = 1$. However, for every $x \in E(0,1) = \mathbb{Z}_p$, we have $f(x) = x^p - x \equiv 0$ mod p. Thus $\|f(x)\|_p \leq 1/p$ and $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$.

Remark 2.9. Recall the Weierstrass-Stone theorem in classical analysis which states that the closure of the polynomial ring $\mathbb{C}[T_1, \ldots, T_n]$ with respect to the supremum norm on a closed polydisc $E \subset \mathbb{C}^n$ is the ring of all complex-valued continuous functions on E. Yang: This is wrong. In the context of non-archimedean analysis, Proposition 2.7 can be viewed as an analogue of this theorem. It states that the closure of the polynomial ring $\Bbbk[T_1, \ldots, T_n]$ with respect to the supremum norm on a closed polydisc $E(0,\underline{r}) \subset \Bbbk^n$ is the Tate algebra $\Bbbk\{\underline{T}/\underline{r}\}$.

From this perspective, the Tate algebra can be viewed as the "correct" analogue of the ring of continuous functions on a closed polydisc in non-archimedean analysis.

Yang: Recall the Runge theorem in complex analysis.

Definition 2.10. Let $\mathbf{k}$ be a complete non-archimedean field. A function $f : E(0,\underline{r}) \to \mathbf{k}$ is called *analytic* if there exists $F \in \mathbf{k}\{\underline{T}/\underline{r}\}$ such that $f = F$ as functions from $E(0,\underline{r})$ to $\mathbf{k}$. Yang: To be revised.

3   Analytic functions in one variable

Proposition 3.1. Let $\mathbf{k}$ be a complete non-archimedean field and $f = \sum_{n=0}^{+\infty} a_n T^n \in \mathbf{k}[[T]]$. Set

$$R := \frac{1}{\limsup_{n \to +\infty} \|a_n\|^{1/n}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Then we have

(a) the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$ and diverges for all $x \in \mathbf{k}$ with $\|x\| > R$;

(b) if $R < +\infty$, the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| = R$ if and only if $\lim_{n \to +\infty} \|a_n\| R^n = 0$.

Proof. By **??**, we only need to check when the terms $a_n x^n$ tend to zero as $n \to +\infty$. If $\|x\| < R$, there exists $r \in (0, 1)$ such that $\|x\| < r^2 R$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\|a_n\|^{1/n} < 1/(rR)$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n < \|a_n\| (r^2 R)^n < (r^2 R)^n \cdot \frac{1}{(rR)^n} = r^n \to 0.$$

Thus the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$.

Suppose that $\|x\| > R$. There exists $s > 1$ such that $\|x\| > R/s$. By the definition of R, there exist infinitely many $n \in \mathbb{N}$ such that $\|a_n\|^{1/n} > s/R$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n > \|a_n\| \frac{R^n}{s^n} > \left(\frac{s}{R}\right)^n \cdot \frac{R^n}{s^n} = 1.$$

Thus the series $f(x)$ diverges for all $x \in \mathbf{k}$ with $\|x\| > R$.

Finally, the case $\|x\| = R$ is direct from **??**. Yang: To be revised. $\square$

Theorem 3.2 (Strassman). Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$ be an analytic function. Suppose that $\|a_N\| r^N > \|a_n\| r^n$ for all $n > N$. Then f has at most N zeros in the closed ball $E(0, r)$.

Proof. We induct on N. The case $N = 0$ is direct from Proposition 1.4. Suppose that the conclusion holds for $N - 1$. Let x be a zero of f in $E(0, r)$. Set

$$g(T) = \frac{f(T) - f(x)}{T - x} = \sum_{k=0}^{+\infty} \left(\sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n \geq k+1} \|a_n\| r^{n-1} \to 0 \quad \text{as } k \to \infty.$$

It follows that $g(T) \in \mathbf{k}\{T/r\}$.

For every $n > N$, we have

$$\|a_N\| > \|a_n\| r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\| \sum_{n=N}^{N+m} a_n x^{n-N} \right\| = \|a_N\|$$

for every $m \in \mathbb{N}$ by ??. Take $m \to +\infty$, we have $\|b_{N-1}\| = \|a_N\|$. For every $k > N - 1$, we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n\| r^{n-1} \leq \max_{n > N} \|a_n\| r^{n-1} < \|a_N\| r^{N-1} = \|b_{N-1}\| r^{N-1}.$$

By the induction hypothesis, g has at most $N - 1$ zeros in $E(0, r)$. It follows that f has at most N zeros in $E(0, r)$ since $f(T) = (T - x) \cdot g(T)$. □

3.1 Entire functions

3.2 Maximum principle

4 Elementary functions

4.1 Exponential and logarithmic functions

Fix a prime number p in the following and consider $\mathbf{k}$ being a complete non-archimedean field with $|p| = p^{-1}$. Let $r_p := p^{-1/(p-1)}$.

Construction 4.1. The *exponential function* $\exp : \mathbf{k} \to \mathbf{k}$ is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The *logarithmic function* $\log : 1 + \mathbf{k}^{\circ\circ} \to \mathbf{k}$ is defined by the power series

$$\log(1 + x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Yang: To be checked.

Recall the following useful lemma regarding the p-adic valuation of factorials.

Lemma 4.2. Let p be a prime number and $n \in \mathbb{N}$. We have

$$v_p(n!) = \sum_{k=1}^{+\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Proof. Yang: To be added. □

Yang: Exponential, logarithmic, and the interpolation functions.

Proposition 4.3. We have the following properties:

(a) the exponential function exp converges on the open disk $B(0, r_p)$;

(b) the logarithmic function log converges on the open disk $B(1, 1)$;

(c) endow $B(0, r_p)$ with the group structure induced by addition in $\mathbf{k}$ and $B(1, r_p)$ with the group structure induced by multiplication in $\mathbf{k}$, then $\exp : B(0, r_p) \to B(1, r_p)$ is a group isomorphism with inverse $\log : B(1, r_p) \to B(0, r_p)$.

Yang: To be checked.

Proof. Yang: To be added. □

Proposition 4.4. The logarithmic function log defines a group homomorphism $1 + \mathbf{k}^{\circ\circ} \to \mathbf{k}$ with kernel the group μ_{p^∞} of all p-power roots of unity. Yang: To be checked.

Yang: continuation of exponential and logarithmic

4.2 Mahler series

Notation 4.5. We use $\binom{x}{n}$ to denote the *binomial polynomial* defined by

$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}.$$

Definition 4.6.

Theorem 4.7. The series converges.

5 Appendix

Theorem 5.1. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

References

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