Elementary functions

1 Exponential and logarithmic functions

Fix a prime number p in the following and consider $\mathbf{k}$ being a complete non-archimedean field with $|p| = p^{-1}$. Let $r_p := p^{-1/(p-1)}$.

Construction 1. The *exponential function* $\exp : \mathbf{k} \to \mathbf{k}$ is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The *logarithmic function* $\log : 1 + \mathbf{k}^{\circ\circ} \to \mathbf{k}$ is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Yang: To be checked.

Recall the following useful lemma regarding the p-adic valuation of factorials.

Lemma 2. Let p be a prime number and $n \in \mathbb{N}$. We have

$$v_p(n!) = \sum_{k=1}^{+\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Proof. Yang: To be added. □

Yang: Exponential, logarithmic, and the interpolation functions.

Proposition 3. We have the following properties:

(a) the exponential function exp converges on the closed disk $E(0, r_p)$;

(b) the logarithmic function log converges on the open disk $E(1, 1)$;

(c) endow $E(0, r_p)$ with the group structure induced by addition in $\mathbf{k}$ and $E(1, 1)$ with the group structure induced by multiplication in $\mathbf{k}$, then $\exp : E(0, r_p) \to E(1, 1)$ is a group isomorphism with inverse $\log : E(1, 1) \to E(0, r_p)$.

Yang: To be checked.

Proof. Yang: To be added. □

2 Mahler sequence

Notation 4. We use $\binom{x}{n}$ to denote the *binomial polynomial* defined by

$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}.$$

Definition 5.

Theorem 6. The series converges.