Valuation fields

1 Absolute values and completion

Definition 1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field.*

Remark 2. Let $\mathbf{k}$ be a field. Recall that a (additive) valuation on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;

- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

The valuation v defined above is called an *additive valuation.* And an absolute value $|\cdot|$ on $\mathbf{k}$ is called a *multiplicative valuation.* In this note, the term *valuation* may refer to either an additive valuation or a multiplicative valuation, depending on the context.

Example 3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 4. The *(multiplicative) valuation group* of a valuation field $(\mathbf{k}, \|\cdot\|)$ is defined as the subgroup of $\mathbb{R}_{>0}$ given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation $\sqrt{|\mathbf{k}^\times|}$ to denote the set $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$.

Note that an absolute value $\|\cdot\|$ is non-trivial if and only if its valuation group $|\mathbf{k}^\times|$ is not equal to $\{1\}$.

Definition 5. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 6. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Note that the operations of addition and multiplication on $\mathbf{k}$ are uniformly continuous with respect to the metric $d(x, y) = \|x - y\|$. Thus they can be extended to $\widehat{\mathbf{k}}$ uniquely. $\square$

Proposition 7. Let $(\mathbf{k}, \|\cdot\|)$ be a complete valuation field with non-trivial absolute value. Then $\mathbf{k}$ is uncountable.

Proof. Since the absolute value $\|\cdot\|$ is non-trivial, we can construct a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbf{k}$ inductively such that $\|x_n\| < \|x_{n-1}\|/2$ for any $n \geq 1$ and $\|x_0\| < 1$. Then there is an injective map from $\mathbb{N}^{\{0,1\}}$ to $\mathbf{k}$ defined by

$$(a_n)_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} a_n x_n, \quad a_n \in \{0, 1\}.$$

Since $\|x_n\| < 2^{-n}$, the series $\sum_{n=1}^{\infty} a_n x_n$ converges in $\mathbf{k}$. Note $\|x_n\| > \|\sum_{m \geq n} x_m\|$ for each n, we have that the map is injective. Thus $\mathbf{k}$ is uncountable. $\square$

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 8. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 9. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

Example 10. Let $|\cdot|_{\infty}$ be the usual absolute value on the field $\mathbb{Q}$ of rational numbers. Then $(\mathbb{Q}, |\cdot|_{\infty})$ is a valuation field. Its completion is the field $\mathbb{R}$ of real numbers equipped with the usual absolute value.

Example 11. Let p be a prime number. For any non-zero rational number $x \in \mathbb{Q}$, we can write it as $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then $(\mathbb{Q}, |\cdot|_p)$ is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of p-adic numbers equipped with the p-adic absolute value; see Yang: to be added..

Definition 12. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Example 13. Let v be an additive valuation on a field $\mathbf{k}$. Then the induced absolute value $|\cdot|_v$ as in Remark 2 is non-archimedean.

The converse is also true: if $(\mathbf{k}, |\cdot|)$ is a non-archimedean valuation field, then the function $v : \mathbf{k}^\times \to \mathbb{R}$ defined by $v(x) = -\log|x|$ is an additive valuation on $\mathbf{k}$.

The following proposition explains the terminology "archimedean".

Proposition 14. Let $(\mathbf{k}, |\cdot|)$ be a valuation field. Then $\mathbf{k}$ is archimedean if and only if the set $\{|n \cdot 1| : n \in \mathbb{Z}\}$ is unbounded.

Proof. Sufficiency is obvious. Yang: To be added. □

Theorem 15 (Ostrowski)**.** Let $(\mathbf{k}, \|\cdot\|)$ be an archimedean complete valuation field. Then there exists an embedding $\iota : \mathbf{k} \hookrightarrow \mathbb{C}$ and $s \in (0, \infty)$ such that

$$\|x\| = |\iota(x)|^s, \quad \forall x \in \mathbf{k},$$

where $|\cdot|$ is the usual absolute value on $\mathbb{C}$. In other words, complete archimedean valuation fields are equivalent to either $\mathbb{R}$ or $\mathbb{C}$ equipped with usual absolute value (see Definition 16).

Proof. Yang: To be added. □

2 Places on a field

Definition 16. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c \in (0, \infty)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$. Moreover, the following lemma shows that the converse is also true.

Lemma 17. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) $\Rightarrow$ (b) is obvious. Now we prove (b) $\Rightarrow$ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) $\Rightarrow$ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. Suppose the

contrary that $\alpha \neq \beta$. Consider the domain $\Lambda \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Λ is non-empty. Pick $(n, m) \in \Lambda$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. $\square$

Definition 18. Let $\mathbf{k}$ be a field. A *place* on $\mathbf{k}$ is an equivalence class of non-trivial absolute values on $\mathbf{k}$. An archimedean (resp. non-archimedean) place is also called an *finite* (resp. *infinite*) place. We denote the set of all places (resp. finite, infinite places) on $\mathbf{k}$ by $M_{\mathbf{k}}$ (resp. $M_{\mathbf{k}}^f$, $M_{\mathbf{k}}^\infty$).
If $\mathbf{l} \subset \mathbf{k}$ is a subfield, we denote by $M_{\mathbf{k}/\mathbf{l}}$ (resp. $M_{\mathbf{k}/\mathbf{l}}^f$, $M_{\mathbf{k}/\mathbf{l}}^\infty$) the set of all places (resp. finite, infinite places) on $\mathbf{k}$ which are trivial on $\mathbf{l}$.

Example 19. Let $\mathbf{k} = \mathbb{C}(t)$ and $\mathbf{l} = \mathbb{C}$. Then I claim that

$$M_{\mathbb{C}(t)/\mathbb{C}} \cong \{ \text{ prime divisors on } \mathbb{P}_{\mathbb{C}}^1 \}.$$

For each prime divisor P on $\mathbb{P}_{\mathbb{C}}^1$, we can define an additive valuation $\mathrm{Mult}_P : \mathbb{C}(t)^\times \to \mathbb{Z}$ by assigning to each non-zero rational function $f \in \mathbb{C}(t)^\times$ its multiplicity at P. Fix a real number $\varepsilon \in (0, 1)$. Then we obtain a multiplicative valuation (absolute value) $|\cdot|_P$ on $\mathbb{C}(t)$ as in Remark 2. It is easy to check that the absolute value $|\cdot|_P$ is trivial on $\mathbb{C}$ and that different prime divisors give rise to inequivalent absolute values.

Conversely, given any non-trivial absolute value $|\cdot|$ on $\mathbb{C}(t)$ which is trivial on $\mathbb{C}$, by Proposition 14 and Example 13, the absolute value $|\cdot|$ is given by an additive valuation $v : \mathbb{C}(t)^\times \to \mathbb{R}$. Let $\mathcal{O}_v$ be the valuation ring of v and $\mathfrak{m}_v$ its maximal ideal. Then $t \in \mathcal{O}_v$ or $t^{-1} \in \mathcal{O}_v$. Without loss of generality, we assume that $t \in \mathcal{O}_v$. Since $|\cdot|$ is trivial on $\mathbb{C}$, we have $\mathbb{C}[t] \subseteq \mathcal{O}_v$. And we have $\mathfrak{m}_v \cap \mathbb{C}[t] \neq 0$ since otherwise v is trivial on $\mathbb{C}[t]^\times$ and thus on $\mathbb{C}(t)^\times$. It follows that the image of $\mathbb{C}[t]$ under the quotient map $\mathcal{O}_v \to \mathcal{O}_v/\mathfrak{m}_v$ is $\mathbb{C}$. This gives a point $P \in \mathbb{A}_{\mathbb{C}}^1 \subseteq \mathbb{P}_{\mathbb{C}}^1$. Then v is different from the additive valuation Mult_P by a positive scalar multiple via looking at the values on $\mathbb{C}[t]$. Thus we have established the claimed bijection.

Theorem 20 (Ostrowski). Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to either the usual absolute value $|\cdot|_\infty$ or a p-adic absolute value $|\cdot|_p$ for some prime number p. In other words,

$$M_{\mathbb{Q}} = \{|\cdot|_\infty\} \cup \{|\cdot|_p : p \text{ is a prime number}\}.$$

Proof. Yang: To be added. $\square$

Remark 21. For every non-archimedean place v on $\mathbb{Q}$ corresponding to a prime number p, we choose the unique normalized absolute value $|\cdot|_v$ in the class v such that $|p|_v = 1/p$. For the archimedean place v on $\mathbb{Q}$, we choose the usual absolute value $|\cdot|_v = |\cdot|_\infty$. Unless otherwise specified, we always use the normalized absolute values on $\mathbb{Q}$.

Remark 22. For any non-zero rational number $x \in \mathbb{Q}^\times$, one can easily check the *product formula*:

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1.$$

This can be viewed as an arithmetic analogue of the fact on $\mathbb{P}^1_{\mathbb{C}}$ that

$$\sum_{P \in \mathbb{P}^1(\mathbb{C})} \mathrm{Mult}_P(f) = 0$$

for any non-zero rational function $f \in \mathbb{C}(t)^\times$. Indeed, fix a real number $\varepsilon \in (0,1)$. Then by Example 19, above fact can be rewritten as

$$\prod_{P \in \mathbb{P}^1_{\mathbb{C}}} |f|_P = 1.$$

Theorem 23 (Artin-Whaples approximations). Let $\mathbf{k}$ be a field and $v_1, v_2, \ldots, v_n \in M_{\mathbf{k}}$ be pairwise distinct places on $\mathbf{k}$. For each $i \in \{1, 2, \ldots, n\}$, let $x_i \in \mathbf{k}$ and $\varepsilon_i > 0$. Then there exists an element $x \in \mathbf{k}$ such that

$$|x - x_i|_{v_i} < \varepsilon_i, \quad \forall i \in \{1, 2, \ldots, n\}.$$

In particular, the image of the diagonal embedding

$$\mathbf{k} \to \prod_{i=1}^n \mathbf{k}_{v_i}$$

is dense, where $\mathbf{k}_{v_i}$ is the completion of $\mathbf{k}$ with respect to the place v_i, and the product is equipped with the product topology.

Proof. Yang: To be added. $\qquad\square$