Valuations

Let $\mathbf{k}$ be a field. Usually, we consider $\mathbf{k}$ to be a number field or a function field.

1 Definition

Definition 1. An *absolute value* or a *valuation* on $\mathbf{k}$ is a function $|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

- $|x| = 0$ if and only if $x = 0$;
- $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbf{k}$;
- $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbf{k}$.

Remark 2. Recall that a *additive valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the multiplicative valuation (i.e., absolute value).

Example 3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 4. An absolute value $|\cdot|$ on a field $\mathbf{k}$ is called *non-archimedean* if it satisfies the strong triangle inequality

$$|x + y| \leq \max\{|x|, |y|\} \quad \text{for all } x, y \in \mathbf{k}.$$

Otherwise, it is called *archimedean*.

Proposition 5.

Notation 6. Let $\mathbf{k}$ be a field. We denote by $M_{\mathbf{k}}$ the set of all absolute values (i.e., valuations) on $\mathbf{k}$. For each $v \in M_{\mathbf{k}}$, we also call v a *place* of $\mathbf{k}$.

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2 Non-archimedean place

3 Number field case

In this section, let $\mathbf{k}$ be a number field.

Theorem 7. Let $\mathbf{k}$ be a number field. Then

$$M_{\mathbf{k}}^{\infty} = \{\text{embeddings } \sigma : \mathbf{k} \to \mathbb{C}\}$$

and

$$M_{\mathbf{k}}^{\mathrm{f}} = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_{\mathbf{k}}\}.$$

Yang: To be revised.

Proposition 8 (Product formula)**.** Let $\mathbf{k}$ be a number field. For each $x \in \mathbf{k}^{\times}$, we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^{\infty}; \\ 1, & v \in M_{\mathbf{k}}^{0}. \end{cases}$$

Yang: To be revised.

Appendix