Arithmetic Geometry

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Arithmetic Geometry

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Chapter 1

Valuation on fields

1.1 Valuation fields

1.1.1 Absolute values and completion

Definition 1.1.1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot|:\mathbf{k}\to\mathbb{R}_{\geq 0}$ satisfying the following properties for all $x,y\in\mathbf{k}$:

(a) $\|x\|=0$ if and only if $x=0$;

(b) $\|xy\|=\|x\|\cdot\|y\|$;

(c) $\|x+y\|\leq\|x\|+\|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 1.1.2. Let $\mathbf{k}$ be a field. Recall that a (additive) valuation on $\mathbf{k}$ is a function $v:\mathbf{k}^\times\to\mathbb{R}$ such that

- $\forall x,y\in\mathbf{k}^\times, v(xy)=v(x)+v(y)$;
- $\forall x,y\in\mathbf{k}^\times, v(x+y)\geq\min\{v(x),v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0)=+\infty$. Fix a real number $\varepsilon\in(0,1)$. Then v induces an absolute value $|\cdot|_v:\mathbf{k}\to\mathbb{R}_+$ defined by $|x|_v=\varepsilon^{v(x)}$ for each $x\in\mathbf{k}$.

The valuation v defined above is called an *additive valuation*. And an absolute value $|\cdot|$ on $\mathbf{k}$ is called a *multiplicative valuation*. In this note, the term *valuation* may refer to either an additive valuation or a multiplicative valuation, depending on the context.

Example 1.1.3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\|:=\begin{cases}0, & x=0;\\ 1, & x\neq 0.\end{cases}$$

Definition 1.1.4. The *(multiplicative) valuation group* of a valuation field $(\mathbf{k}, \|\cdot\|)$ is defined as the subgroup of $\mathbb{R}_{>0}$ given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation $\sqrt{|\mathbf{k}^\times|}$ to denote the set $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$.

Note that an absolute value $\|\cdot\|$ is non-trivial if and only if its valuation group $|\mathbf{k}^\times|$ is not equal to $\{1\}$.

Definition 1.1.5. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 1.1.6. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Note that the operations of addition and multiplication on $\mathbf{k}$ are uniformly continuous with respect to the metric $d(x, y) = \|x - y\|$. Thus they can be extended to $\widehat{\mathbf{k}}$ uniquely. □

Proposition 1.1.7. Let $(\mathbf{k}, \|\cdot\|)$ be a complete valuation field with non-trivial absolute value. Then $\mathbf{k}$ is uncountable.

Proof. Since the absolute value $\|\cdot\|$ is non-trivial, we can construct a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathbf{k}$ inductively such that $\|x_n\| < \|x_{n-1}\|/2$ for any $n \geq 1$ and $\|x_0\| < 1$. Then there is an injective map from $\mathbb{N}^{\{0,1\}}$ to $\mathbf{k}$ defined by

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n x_n, \quad a_n \in \{0, 1\}.$$

Since $\|x_n\| < 2^{-n}$, the series $\sum_{n=1}^\infty a_n x_n$ converges in $\mathbf{k}$. Note $\|x_n\| > \|\sum_{m \geq n} x_m\|$ for each n, we have that the map is injective. Thus $\mathbf{k}$ is uncountable. □

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 1.1.8. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 1.1.9. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

Example 1.1.10. Let $|\cdot|_\infty$ be the usual absolute value on the field $\mathbb{Q}$ of rational numbers. Then $(\mathbb{Q}, |\cdot|_\infty)$ is a valuation field. Its completion is the field $\mathbb{R}$ of real numbers equipped with the usual absolute value.

Example 1.1.11. Let p be a prime number. For any non-zero rational number $x \in \mathbb{Q}$, we can write it as $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then $(\mathbb{Q}, |\cdot|_p)$ is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of p-adic numbers equipped with the p-adic absolute value; see Yang: to be added..

Definition 1.1.12. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Example 1.1.13. Let v be an additive valuation on a field $\mathbf{k}$. Then the induced absolute value $|\cdot|_v$ as in Remark 1.1.2 is non-archimedean.

The converse is also true: if $(\mathbf{k}, |\cdot|)$ is a non-archimedean valuation field, then the function $v : \mathbf{k}^\times \to \mathbb{R}$ defined by $v(x) = -\log |x|$ is an additive valuation on $\mathbf{k}$.

The following proposition explains the terminology "archimedean".

Proposition 1.1.14. Let $(\mathbf{k}, |\cdot|)$ be a valuation field. Then $\mathbf{k}$ is archimedean if and only if the set $\{|n \cdot 1| : n \in \mathbb{Z}\}$ is unbounded.

Proof. Sufficiency is obvious. Yang: To be added. □

Theorem 1.1.15 (Ostrowski)**.** Let $(\mathbf{k}, \|\cdot\|)$ be an archimedean complete valuation field. Then there exists an embedding $\iota : \mathbf{k} \hookrightarrow \mathbb{C}$ and $s \in (0, \infty)$ such that

$$\|x\| = |\iota(x)|^s, \quad \forall x \in \mathbf{k},$$

where $|\cdot|$ is the usual absolute value on $\mathbb{C}$. In other words, complete archimedean valuation fields are equivalent to either $\mathbb{R}$ or $\mathbb{C}$ equipped with usual absolute value (see Definition 1.1.16).

Proof. Yang: To be added. □

1.1.2 Places on a field

Definition 1.1.16. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c \in (0, \infty)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$. Moreover, the following lemma shows that the converse is also true.

Lemma 1.1.17. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) $\Rightarrow$ (b) is obvious. Now we prove (b) $\Rightarrow$ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) $\Rightarrow$ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. Suppose the contrary that $\alpha \neq \beta$. Consider the domain $\Lambda \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Λ is non-empty. Pick $(n, m) \in \Lambda$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. $\square$

Definition 1.1.18. Let $\mathbf{k}$ be a field. A *place* on $\mathbf{k}$ is an equivalence class of non-trivial absolute values on $\mathbf{k}$. An archimedean (resp. non-archimedean) place is also called an *finite* (resp. *infinite*) place. We denote the set of all places (resp. finite, infinite places) on $\mathbf{k}$ by $M_{\mathbf{k}}$ (resp. $M_{\mathbf{k}}^f$, $M_{\mathbf{k}}^\infty$). If $\mathbf{l} \subset \mathbf{k}$ is a subfield, we denote by $M_{\mathbf{k}/\mathbf{l}}$ (resp. $M_{\mathbf{k}/\mathbf{l}}^f$, $M_{\mathbf{k}/\mathbf{l}}^\infty$) the set of all places (resp. finite, infinite places) on $\mathbf{k}$ which are trivial on $\mathbf{l}$.

Example 1.1.19. Let $\mathbf{k} = \mathbb{C}(t)$ and $\mathbf{l} = \mathbb{C}$. Then I claim that

$$M_{\mathbb{C}(t)/\mathbb{C}} \cong \{ \text{ prime divisors on } \mathbb{P}_{\mathbb{C}}^1 \}.$$

For each prime divisor P on $\mathbb{P}_{\mathbb{C}}^1$, we can define an additive valuation $\mathrm{Mult}_P : \mathbb{C}(t)^\times \to \mathbb{Z}$ by assigning to each non-zero rational function $f \in \mathbb{C}(t)^\times$ its multiplicity at P. Fix a real number $\varepsilon \in (0, 1)$. Then we obtain a multiplicative valuation (absolute value) $|\cdot|_P$ on $\mathbb{C}(t)$ as in Remark 1.1.2. It is easy to check that the absolute value $|\cdot|_P$ is trivial on $\mathbb{C}$ and that different prime divisors give rise to inequivalent absolute values.

Conversely, given any non-trivial absolute value $|\cdot|$ on $\mathbb{C}(t)$ which is trivial on $\mathbb{C}$, by Proposition 1.1.14 and Example 1.1.13, the absolute value $|\cdot|$ is given by an additive valuation $v : \mathbb{C}(t)^\times \to \mathbb{R}$. Let $\mathcal{O}_v$ be the valuation ring of v and $\mathfrak{m}_v$ its maximal ideal. Then $t \in \mathcal{O}_v$ or $t^{-1} \in \mathcal{O}_v$. Without loss of generality, we assume that $t \in \mathcal{O}_v$. Since $|\cdot|$ is trivial on $\mathbb{C}$, we have $\mathbb{C}[t] \subseteq \mathcal{O}_v$. And we have $\mathfrak{m}_v \cap \mathbb{C}[t] \neq 0$ since otherwise v is trivial on $\mathbb{C}[t]^\times$ and thus on $\mathbb{C}(t)^\times$. It follows that the image of $\mathbb{C}[t]$ under the quotient map $\mathcal{O}_v \to \mathcal{O}_v/\mathfrak{m}_v$ is $\mathbb{C}$. This gives a point $P \in \mathbb{A}_{\mathbb{C}}^1 \subseteq \mathbb{P}_{\mathbb{C}}^1$. Then v is different from the additive valuation Mult_P by a positive scalar multiple via looking at the values on $\mathbb{C}[t]$. Thus we have established the claimed bijection.

Theorem 1.1.20 (Ostrowski)**.** Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to either the usual absolute value $|\cdot|_\infty$ or a p-adic absolute value $|\cdot|_p$ for some prime number p. In other words,

$$M_{\mathbb{Q}} = \{|\cdot|_\infty\} \cup \{|\cdot|_p : p \text{ is a prime number}\}.$$

Proof. Yang: To be added. □

Remark 1.1.21. For every non-archimedean place v on $\mathbb{Q}$ corresponding to a prime number p, we choose the unique normalized absolute value $|\cdot|_v$ in the class v such that $|p|_v = 1/p$. For the archimedean place v on $\mathbb{Q}$, we choose the usual absolute value $|\cdot|_v = |\cdot|_\infty$. Unless otherwise specified, we always use the normalized absolute values on $\mathbb{Q}$.

Remark 1.1.22. For any non-zero rational number $x \in \mathbb{Q}^\times$, one can easily check the *product formula*:

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1.$$

This can be viewed as an arithmetic analogue of the fact on $\mathbb{P}^1_{\mathbb{C}}$ that

$$\sum_{P \in \mathbb{P}^1(\mathbb{C})} \mathrm{Mult}_P(f) = 0$$

for any non-zero rational function $f \in \mathbb{C}(t)^\times$. Indeed, fix a real number $\varepsilon \in (0,1)$. Then by Example 1.1.19, above fact can be rewritten as

$$\prod_{P \in \mathbb{P}^1_{\mathbb{C}}} |f|_P = 1.$$

Theorem 1.1.23 (Artin-Whaples approximations)**.** Let $\mathbf{k}$ be a field and $v_1, v_2, \dots, v_n \in M_{\mathbf{k}}$ be pairwise distinct places on $\mathbf{k}$. For each $i \in \{1,2,\dots,n\}$, let $x_i \in \mathbf{k}$ and $\varepsilon_i > 0$. Then there exists an element $x \in \mathbf{k}$ such that

$$|x - x_i|_{v_i} < \varepsilon_i, \quad \forall i \in \{1,2,\dots,n\}.$$

In particular, the image of the diagonal embedding

$$\mathbf{k} \to \prod_{i=1}^{n} \mathbf{k}_{v_i}$$

is dense, where $\mathbf{k}_{v_i}$ is the completion of $\mathbf{k}$ with respect to the place v_i, and the product is equipped with the product topology.

Proof. Yang: To be added. □

1.2 Non-archimedean valuations

1.2.1 Topology: Ultra-metric space

We will use $B(x,r)$ (resp. $E(x,r)$) to denote the open ball (resp. closed ball) with center x and radius r.

Definition 1.2.1. A metric space (X,d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x,z) \le \max\{d(x,y), d(y,z)\}, \quad \forall x,y,z \in X.$$

Remark 1.2.2. The term *ultra-metric space* should be translated into Chinese as "奥特度量空间". There is no special reason for this translation, except that I insist on using "奥特" to translate "ultra".

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x, y) := \|x - y\|$ on $\mathbf{k}$ makes $(\mathbf{k}, d)$ an ultra-metric space.

Proposition 1.2.3. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x, y) \geq d(y, z)$. By the strong triangle inequality, we have

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(x, y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} = \max\{d(x, z), d(y, z)\} \leq d(x, y).$$

This shows that $d(x, y) = \max\{d(x, z), d(y, z)\}$. Thus either $d(x, z) = d(x, y) \geq d(y, z)$ or $d(y, z) = d(x, y) \geq d(x, z)$. □

Proposition 1.2.4. Let (X, d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1, 2$. If $D_1 \cap D_2 \neq \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1, 2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1, x_2) \leq \max\{d(x_1, y), d(y, x_2)\}.$$

Without loss of generality, we may assume that $d(x_1, x_2) \leq d(x_1, y)$. It follows that $x_2 \in D_1$ since $d(x_1, y) < r_1$ (or $\leq r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1, z) > d(x_1, x_2)$. Then by Proposition 1.2.3,

$$r_1 \leq d(x_1, z) = d(x_2, z) \leq r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2, w) \leq \max\{d(x_2, x_1), d(x_1, w)\} \leq r_1 \leq r_2.$$

Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. □

Proposition 1.2.5. Let (X, d) be an ultra-metric space. Then both $B(x, r)$ and $E(x, r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x, r) := \{y \in X \mid d(x, y) = r\}$ is open in X. Note that if $y \in S(x, r)$, then for any $r' < r$, we have $B(y, r') \cap E(x, r) \neq \varnothing$ and $x \in E(x, r) \setminus B(y, r')$. Thus by Proposition 1.2.4, we have $B(y, r') \subseteq E(x, r)$. If $B(y, r') \cap B(x, r) \neq \varnothing$, then by Proposition 1.2.4 again, we have $B(y, r') \subseteq B(x, r)$. However, $y \in B(y, r') \setminus B(x, r)$, a contradiction. Thus $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$. It yields that $S(x, r) = \bigcup_{y \in S(x,r)} B(y, r/2)$ is open in X.

Since $E(x, r) = B(x, r) \cup S(x, r)$ and $B(x, r) = E(x, r) \setminus S(x, r)$, both $B(x, r)$ and $E(x, r)$ are open and closed in X. $\square$

Corollary 1.2.6. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x, y) > 0$. Consider the open ball $B(x, r/2)$. By Proposition 1.2.5, $B(x, r/2)$ is both open and closed in X. Thus $B(x, r/2) \cap S$ is both open and closed in S, however, it is non-empty and not equal to S since it contains x but not y. This shows that S is disconnected. $\square$

Proposition 1.2.7. Let (X, d) be an ultra-metric space. A sequence $\{x_n\}$ in X is cauchy if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Proof. The necessity is true for all metric spaces. Suppose that $d(x_n, x) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \varepsilon$ for all $n \geq N$. For any $m, n \geq N$ with $m < n$, by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \ldots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that $\{x_n\}$ is a cauchy sequence. $\square$

1.2.2 Algebra: ring of integers and residue field

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 1.2.8. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathscr{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Set $I_{r,<} := B(0, r)$ and $I_{r,\leq} := E(0, r)$ for each $r \in [0, 1]$.

Proposition 1.2.9. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers $\mathbf{k}^\circ$. Conversely, any ideal of $\mathbf{k}^\circ$ is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0, 1)$.

Proof. Let I be an ideal of $\mathbf{k}^\circ$. Set $r = \sup\{|a| : a \in I\}$ (resp. $r = \max\{|a| : a \in I\}$ when the maximum exists). Then, by definition, we have $I \subset I_{r,<}$ (resp. $I \subset I_{r,\leq}$). For every $x \in \mathbf{k}^\circ$ with $|x| < r$ (resp. $|x| \leq r$), there exists $a \in I$ such that $|x| \leq |a|$. Thus, $|x/a| \leq 1$ and so $x/a \in \mathbf{k}^\circ$. Since I is an ideal, we have $x = (x/a)a \in I$. Therefore, $I_{r,<} \subset I$ (resp. $I_{r,\leq} \subset I$). $\square$

Proposition 1.2.10. Let I_r be either $I_{r,<}$ or $I_{r,\leq}$ for each $r \in (0, 1)$. Suppose $\{r_n \in (0, 1)\}_{n \in \mathbb{N}}$ is a decreasing sequence converging to 0. Then the completion $\widehat{\mathbf{k}}$ of $\mathbf{k}$ is isomorphic to the projective

limit
$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}.$$

Proof. For every $x \in \widehat{\mathbf{k}}^\circ$, there exists a cauchy sequence $\{x_m\}_{m\in\mathbb{N}}$ in $\mathbf{k}^\circ$ converging to x. Since $\{r_n\}_{n\in\mathbb{N}}$ converges to 0, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that for all $m, m' \geq M_n$, we have $|x_m - x_{m'}| < r_n$. Thus, the sequence $\{x_m + I_{r_n}\}_{m\in\mathbb{N}}$ is eventually constant in $\mathbf{k}^\circ/I_{r_n}$. Define a map
$$\Phi : \widehat{\mathbf{k}}^\circ \to \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}, \quad x \mapsto \left(\lim_{m\to\infty} x_m + I_{r_n}\right)_{n\in\mathbb{N}}.$$
It is straightforward to verify that Φ is a well-defined ring homomorphism.

Conversely, for every $(a_n + I_{r_n})_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}$, we can choose a representative $a_n \in \mathbf{k}^\circ$ for each n. We claim that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathbf{k}^\circ$. Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_N < \varepsilon$. For all $m, n \geq N$, since $a_n + I_{r_n}$ maps to $a_m + I_{r_m}$ under the natural projection, we have $|a_n - a_m| < r_N < \varepsilon$. Thus, $\{a_n\}_{n\in\mathbb{N}}$ converges to some $x \in \widehat{\mathbf{k}}^\circ$. Easily see that the limit x is independent of the choice of representatives $\{a_n\}_{n\in\mathbb{N}}$. This gives a map
$$\Psi : \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n} \to \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} a_n.$$
Direct verification shows that $\Psi = \Phi^{-1}$. □

Corollary 1.2.11. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the residue field $\mathscr{k}_{\widehat{\mathbf{k}}} \cong \mathscr{k}_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$.

Corollary 1.2.12. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the valuation group $|\widehat{\mathbf{k}}^\times|$ of $\widehat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^\times|$ of $\mathbf{k}$.

Proof. Note that
$$\begin{aligned} r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneqq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\ &\iff \widehat{\mathbf{k}}^\circ/I_{r,<} \to \widehat{\mathbf{k}}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff \mathbf{k}^\circ/I_{r,<} \to \mathbf{k}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff I_{r,<} \subsetneqq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\ &\iff r \in |\mathbf{k}^\times|. \end{aligned}$$
□

Proposition 1.2.13. Let $\mathbf{k}$ be a non-archimedean field with non-trivial valuation. Then $\mathbf{k}^\circ$ is totally bounded iff $\mathbf{k}^\circ/I_{r,<}$ and $\mathbf{k}^\circ/I_{r,\leq}$ are finite for each $r \in [0,1]$. Moreover, if $\mathbf{k}$ is complete, then it is locally compact iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$.

Slogan *"Locally compact $\iff$ pro-finite."*

Proof. We just prove the case for $I_r = I_{r,<}$. The case for $I_r = I_{r,\leq}$ is similar.

Suppose that $\mathbf{k}^\circ/I_r$ is finite for each $r \in [0,1]$. Then for every $\varepsilon > 0$, there exists $r \in (0,1)$

such that $r < \varepsilon$ and $\mathbf{k}^\circ/I_r$ is finite. Let $\{a_1 + I_r, \ldots, a_n + I_r\}$ be the complete set of representatives of $\mathbf{k}^\circ/I_r$. Then the balls $B(a_i, r)$ for $i = 1, \ldots, n$ cover $\mathbf{k}^\circ$.

Conversely, suppose that $\mathbf{k}^\circ/I_r$ is infinite for some $r \in [0,1]$. Then there exists an infinite set $\{a_n\}$ with $|a_n| \in [r, 1]$ such that their images in $\mathbf{k}^\circ/I_r$ are distinct. In particular, for every $m \neq n$, we have $|a_n - a_m| \geq r$. Any subsequence of $\{a_n\}$ is not cauchy. Thus, $\mathbf{k}^\circ$ is not totally bounded. □

Proposition 1.2.14. The ring $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field.

Proof. Note that $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$ is a multiplicative subgroup. If $\mathbf{k}$ is not a discrete valuation field, then $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. In particular, there exists a strictly ascending sequence $r_n \in |\mathbf{k}^\times| \cap (0,1)$. Then the ideals $I_{r_n,\leq}$ form a strictly ascending chain of ideals in $\mathbf{k}^\circ$.

The converse is standard since now $\mathbf{k}^\circ$ is a discrete valuation ring. □

Proposition 1.2.15. Let $\mathbf{k}$ be a complete non-archimedean field. Then $\mathbf{k}$ is locally compact iff $\mathbf{k}$ is a discrete valuation field and its residue field $\mathscr{k}_{\mathbf{k}}$ is finite.

Proof. The necessity follows from Proposition 1.2.13. For the sufficiency, suppose that $\mathbf{k}$ is a discrete valuation field whose residue field $\mathscr{k}_{\mathbf{k}}$ is finite. Let $\pi \in \mathbf{k}^\circ$ be a uniformizer. We only need to show that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite for each $n \in \mathbb{N}$. Note that there is an isomorphism

$$\pi^{n-1}\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ \cong \mathscr{k}_{\mathbf{k}}, \quad x + \pi^n\mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on n, we conclude that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite. □

1.2.3 Hensel's Lemma

Theorem 1.2.16 (Hensel's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \mathscr{k}_{\mathbf{k}}[T]$ are monic polynomials that are coprime in $\mathscr{k}_{\mathbf{k}}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\mathscr{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

Proof. Since $\gcd(g, h) = 1$ in $\mathscr{k}_{\mathbf{k}}[T]$, there exist polynomials $u(T), v(T) \in \mathscr{k}_{\mathbf{k}}[T]$ such that $ug + vh = 1$ and $\deg u < \deg h, \deg v < \deg g$. Choose lifts $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$ of $g(T), h(T), u(T), v(T)$ respectively preserving their degrees such that G_0 and H_0 are monic. Then there exist $r < 1$ such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \mod I_r, \quad F(T) - G_0(T)H_0(T) \equiv 0 \mod I_r,$$

where $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$.

We will construct a sequence of monic polynomials $\{G_n(T)\}_{n\in\mathbb{N}}$ and $\{H_n(T)\}_{n\in\mathbb{N}}$ in $\mathbf{k}^\circ[T]$ such that for each $n \in \mathbb{N}$,

$$G_n(T) \equiv G_{n-1}(T) \mod I_{r^n}, \quad H_n(T) \equiv H_{n-1}(T) \mod I_{r^n},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \mod I_{r^{n+1}}.$$

If we have such sequences, then their coefficients converge in the complete ring $\mathbf{k}^\circ$. Let $G(T)$ and $H(T)$ be the limits of $\{G_n(T)\}$ and $\{H_n(T)\}$ respectively. Then we have $F(T) = G(T)H(T)$ and the reductions of $G(T), H(T)$ in $\mathcal{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

The case $n = 0$ is done by the above construction. Now suppose that we have constructed $G_n(T)$ and $H_n(T)$ for some $n \geq 0$. Since $G_n - G_0 \equiv 0 \mod I_r$ and $H_n - H_0 \equiv 0 \mod I_r$, we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \mod I_r.$$

Set $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$ and $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$. Then we have

$$\begin{aligned}(G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T].\end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

This finishes the induction. □

Corollary 1.2.17. Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathcal{k}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $a \in \mathcal{k}_{\mathbf{k}}$. Then there exists a root $\alpha \in \mathbf{k}^\circ$ of $F(T)$ whose reduction is a.

Proof. Since a is a simple root of $f(T)$, we have the factorization $f(T) = (T - a)h(T)$ for some monic polynomial $h(T) \in \mathcal{k}_{\mathbf{k}}[T]$ with $h(a) \neq 0$. Then the result follows from Theorem 1.2.16. □

1.2.4 Newton polygons

Yang: To be filled.

1.3 Finite field extensions

1.3.1 Finite-dimensional vector space

Definition 1.3.1. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|ax\| = |a| \cdot \|x\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Example 1.3.2. Let $\mathbf{k}$ be a valuation field and V a finite-dimensional vector space over $\mathbf{k}$ with basis

$\{e_1, e_2, \dots, e_n\}$. For any $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$, define

$$\|x\|_{\max} := \max_{1\le i\le n} |a_i|.$$

Then $\|\cdot\|_{\max}$ is a norm on V, called the *maximal norm* with respect to the basis $\{e_1, e_2, \dots, e_n\}$.

Example 1.3.3. Setting as in Example 1.3.2, for any $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$, define

$$\|x\|_1 := |a_1| + |a_2| + \cdots + |a_n|.$$

Then $\|\cdot\|_1$ is also a norm on V.

Definition 1.3.4. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \le \|x\|_2 \le C_2\|x\|_1.$$

Lemma 1.3.5. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V.

Proof. The sufficiency is clear. Now suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on V. Hence the unit open ball with respect to $\|\cdot\|_1$ contains a unit open ball with respect to $\|\cdot\|_2$. That is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every $x \in V$ with $\|x\|_1 = 1$, we have $\|x\|_2 \ge C = C\|x\|_1$. By scaling, we get that for every $x \in V$,

$$\|x\|_2 \ge C\|x\|_1.$$

Similar for the other direction, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Proposition 1.3.6. Let V be a normed finite-dimensional vector space over a complete valuation field $\mathbf{k}$. Then V is complete.

Proof. Yang: To be added. □

Theorem 1.3.7. Let V be a finite-dimensional vector space over a complete field $\mathbf{k}$. Then all norms on V are equivalent.

Proof. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis as in Example 1.3.2. Let $\|\cdot\|$ be any norm on V. It suffices to show that $\|\cdot\|$ and $\|\cdot\|_{\max}$ are equivalent. First we have

$$\|y\| \le \sum_{i=1}^{n} |a_i|\|e_i\| \le \left(\sum_{i=1}^{n} \|e_i\|\right) \|y\|_{\max}$$

for any $y = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$. It remains to show that there exists a constant $C > 0$ such that for any $y \in V$,

$$\|y\|_{\max} \le C\|y\|.$$

Yang: To be added. □

Remark 1.3.8. If the base field $\mathbf{k}$ is not complete, then Theorem 1.3.7 may fail. For example, let $\mathbf{k} = \mathbb{Q}$ with the usual absolute value, and let $V = \mathbb{Q}[\alpha]$ with $\alpha^2 - \alpha - 1 = 0$. There are two embeddings of V into $\mathbb{R}$:

$$\iota_1 : a + b\alpha \mapsto a + b\frac{1+\sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b\frac{1-\sqrt{5}}{2}.$$

Define two norms on V by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where $|\cdot|$ is the usual absolute value on $\mathbb{R}$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent since $\iota_2(\alpha^n) \to 0$ as $n \to \infty$ while $\iota_1(\alpha^n) \to \infty$.

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

Lemma 1.3.9. Let $\mathbf{k}$ be a complete field and V a normed finite-dimensional vector space over $\mathbf{k}$. Then

$$\|\cdot\| : \mathrm{End}_{\mathbf{k}}(V) \to \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \mathrm{End}_{\mathbf{k}}(V).$$

Proof. First we show the existence of the supremum, i.e., there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\| \leq C\|x\|$. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis. Since all norms on V are bounded by each other by Theorem 1.3.7, we only need to show that there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\|_1 \leq C\|x\|_{\max}$. Write $T(e_i) = \sum_{j=1}^n a_{ij} e_j$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^n x_i e_i \in V$, we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \left(\sum_{1 \leq i,j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of $\|\cdot\|$ are clear. It remains to show the triangle inequality and sub-multiplicativity. For any $A, B \in \mathrm{End}_{\mathbf{k}}(V)$, we have

$$\frac{\|(A+B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all $x \in V \setminus \{0\}$ gives $\|A + B\| \leq \|A\| + \|B\|$. We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$. Taking supremum we get $\|AB\| \leq \|A\| \cdot \|B\|$. $\square$

1.3.2 Finite field extensions

Lemma 1.3.10. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then there exists an absolute value on $\mathbf{l}$ extending the absolute value on $\mathbf{k}$.

Proof. Fix a norm $\|\cdot\|_V$ on the $\mathbf{k}$-vector space $V = \mathbf{l}$. The norm $\|\cdot\|_V$ induces an operator norm $\|\cdot\|_{\mathrm{op}}$ on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ as in Lemma 1.3.9. For any $a \in \mathbf{l}$, let $\mu_a \in \mathrm{End}_{\mathbf{k}}(V)$ be the $\mathbf{k}$-linear map defined by multiplication by a. Note that $a \mapsto \mu_a$ gives a embedding of $\mathbf{k}$-algebras and if $a \in \mathbf{k}$, $\|\mu_a\|_{\mathrm{op}} = \|a\|_{\mathbf{k}}$. Thus the restriction of $\|\cdot\|_{\mathrm{op}}$ to $\mathbf{l}$ gives an norm on $\mathbf{l}$ extending that on $\mathbf{k}$. The normed ring $(\mathbf{l}, \|\cdot\|_{\mathrm{op}})$ is a Banach ring since it is a finite-dimensional vector space over the complete field $\mathbf{k}$. By Theorem 4.1.4, there exists a multiplicative seminorm $\|\cdot\|_{\mathbf{l}}$ on $\mathbf{l}$ bounded by $\|\cdot\|_{\mathrm{op}}$. In particular, $\|\cdot\|_{\mathbf{l}}$ is bounded by $\|\cdot\|_{\mathbf{k}}$ on $\mathbf{k}$. On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus $\|\cdot\|_{\mathbf{l}}$ extends the absolute value on $\mathbf{k}$. □

Theorem 1.3.11. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the absolute value on $\mathbf{l}$ which extends the absolute value on $\mathbf{k}$ is uniquely determined by the absolute value on $\mathbf{k}$. Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where $n = [\mathbf{l} : \mathbf{k}]$ and $N_{\mathbf{l}/\mathbf{k}}$ is the norm map from $\mathbf{l}$ to $\mathbf{k}$.

Proof. Let $\|\cdot\|_{\mathbf{l}}$ be arbitrary absolute value on $\mathbf{l}$ extending that on $\mathbf{k}$. We will show that $\|\cdot\|_{\mathbf{l}}$ must be equal to $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$. For any $a \in \mathbf{l}$, set $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$. Then $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that $\|b\|_{\mathbf{l}} = 1$ whenever $N_{\mathbf{l}/\mathbf{k}}(b) = 1$.

Note that the norm map $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \to \mathbf{k}$ is the determinant of the $\mathbf{k}$-linear map $\mu_b \in \mathrm{End}_{\mathbf{k}}(V)$ defined by multiplication by b. Hence it is continuous on $\mathbf{l}$ (since it is a polynomial in the entries of the matrix representation). If $\|b\|_{\mathbf{l}} < 1$, then $\|b^m\|_{\mathbf{l}} \to 0$ as $m \to \infty$. Thus $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \to 0$ as $m \to \infty$, contradicting the fact that $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$ for all m. Similarly, if $\|b\|_{\mathbf{l}} > 1$, then just consider b^{-1}. □

Proposition 1.3.12. Let $\mathbf{k}$ be an algebraically closed valuation field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

Proof. Let $f \in \widehat{\mathbf{k}}[X]$ be a non-constant polynomial. We will show that f has a root in $\widehat{\mathbf{k}}$. Take a sequence of polynomials $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbf{k}[X]$ converging to f coefficient-wisely and of the same degree d. Since $\mathbf{k}$ is algebraically closed, each f_n splits completely in $\mathbf{k}$ and hence in $\widehat{\mathbf{k}}$. Write $f_n(X) = \prod_{i=1}^{d}(X - \alpha_{n,i})$ with $\alpha_{n,i} \in \widehat{\mathbf{k}}$.

Let $\mathbf{l}$ be a finite extension of $\widehat{\mathbf{k}}$ such that f has a root α in $\mathbf{l}$. For every $\varepsilon > 0$, if there are infinitely many n such that $\alpha_{n,i} \notin B(\alpha, \varepsilon)$ for all $1 \le i \le d$, then we have $|f_n(\alpha)| \ge \varepsilon^d$ for infinitely many n, contradicting the fact that $f_n(\alpha) \to f(\alpha) = 0$. Thus for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \ge N$, there exists $1 \le i \le d$ with $\alpha_{n,i} \in B(\alpha, \varepsilon)$. That is, we can find a sequence $\alpha_{n,i_n} \in \mathbf{k}$ converging to α. Since $\widehat{\mathbf{k}}$ is complete, we have $\alpha \in \widehat{\mathbf{k}}$. □

Theorem 1.3.13 (Krasner's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field, and $\alpha, \beta \in \overline{\mathbf{k}}$. Denote by $\alpha_1, \alpha_2, \dots, \alpha_n$ the conjugates of α over $\mathbf{k}$ with $\alpha_1 = \alpha$. If

$$|\beta - \alpha| < |\alpha - \alpha_i|, \quad \forall i = 2, \dots, n,$$

then $\alpha \in \mathbf{k}(\beta)$.

Proof. Yang: To be added. □

1.3.3 Ramification and inertia

In this subsection, we study the extensions of absolute values on finite field extensions. Note that we do not assume the base field to be complete.

Definition 1.3.14. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. We denote by $w|v$ if $w \in M_L$ is an absolute value on L extending v. For each $w|v$, we define the *ramification index* $e(w|v)$ and the *inertia degree* $f(w|v)$ by

$$e(w|v) := [|\hat{L}^\times|_w : |\widehat{K}^\times|_v], \quad f(w|v) := \frac{[\hat{L} : \widehat{K}]}{e(w|v)},$$

where $\widehat{K}$ and $\hat{L}$ are the completions of K and L with respect to v and w, respectively.

Lemma 1.3.15. Suppose that v is non-archimedean and $\mathscr{k}_v$ and ℓ_w are the residue fields of K and L with respect to v and w, respectively. Then we have

$$f(w|v) = [\ell_w : \mathscr{k}_v].$$

Remark 1.3.16. Yang: To be added.

Theorem 1.3.17. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. Then we have

$$\sum_{w|v} e(w|v) f(w|v) = [L : K].$$

Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. We have

$$L \otimes_K K_v \cong \prod_{w|v} L_w,$$

where the product is taken over all absolute values $w \in M_L$ extending v.

Theorem 1.3.18. Let $\mathbf{k}$ be a number field. Then

$$M_{\mathbf{k}}^{\infty} = \{\text{embeddings } \sigma : \mathbf{k} \to \mathbb{C}\}$$

and

$$M_{\mathbf{k}}^{\mathrm{f}} = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_{\mathbf{k}}\}.$$

Yang: To be revised.

Proposition 1.3.19 (Product formula)**.** Let $\mathbf{k}$ be a number field. For each $x \in \mathbf{k}^\times$, we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^\infty; \\ 1, & v \in M_{\mathbf{k}}^0. \end{cases}$$

Yang: To be revised.

Remark 1.3.20. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. Suppose that v is non-archimedean. Yang: To be rewritten.

1.4 Example: p-adic fields

1.4.1 p-adic fields

Construction 1.4.1. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_\mathfrak{p}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_\mathfrak{p} : K^\times \to \mathbb{Z}$ the corresponding discrete valuation. The $\mathfrak{p}$*-adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_\mathfrak{p} := N(\mathfrak{p})^{-v_\mathfrak{p}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the $\mathfrak{p}$-adic absolute value $|\cdot|_\mathfrak{p}$ is denoted by $K_\mathfrak{p}$, called the $\mathfrak{p}$*-adic field*.

We just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p in the following.

Example 1.4.2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The p*-adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Easily see that $|\mathbb{Q}_p^\times|_p = p^\mathbb{Z}$ and $\kappa_{\mathbb{Q}_p} \cong \mathbb{F}_p$.

Unlike the field of real numbers $\mathbb{R}$, the p-adic field $\mathbb{Q}_p$ has many finite extensions.

Proposition 1.4.3. The field $\mathbb{Q}_p$ has algebraic extensions of arbitrarily large degree.

Proof. Since there are infinitely many irreducible monic polynomials over the finite field $\mathbb{F}_p$, consider any lift of such an irreducible monic polynomial to a monic polynomial with coefficients in $\mathbb{Z}_p$. If the lift is not irreducible over $\mathbb{Q}_p$, then the factorization of the lift gives a nontrivial factorization of its reduction modulo p since the factors can be chosen to be monic and have coefficients in $\mathbb{Z}_p$, which contradicts the irreducibility of the original polynomial over $\mathbb{F}_p$. Thus, the lift is irreducible over $\mathbb{Q}_p$.

On the other hand, note that $|\mathbb{Q}_p^\times|_p = p^{\mathbb{Z}}$. It follows that $f(T) = T^n - p$ is irreducible over $\mathbb{Q}_p$ for every integer $n \geq 1$. Otherwise, suppose we have a monic factorization $f(T) = g(T)h(T)$ with $g(T), h(T) \in \mathbb{Z}_p[T]$ and $\deg g, \deg h < n$. Then by considering the reduction modulo p, we have $g(0), h(0) \equiv 0 \mod p$. It follows that $|f(0)|_p = |g(0)h(0)|_p \leq p^{-2}$, which contradicts $|f(0)|_p = |p|_p = p^{-1}$. □

Let

$$R := \bigcup_{n=1}^{+\infty} \mathbb{Q}_p(\mu_{p^n})$$

and

$$K := \bigcup_{n=1}^{+\infty} \mathbb{Q}_p(\bigcup_{p \nmid m} \mu_m).$$

Yang: R is a totally ramified extension and K is the maximal unramified extension.

1.4.2 Completion

Proposition 1.4.4. The algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is not complete with respect to the extension of the p-adic absolute value $|\cdot|_p$.

Proof. Let $(a_n)_{n\geq 1}$ be a sequence in $\overline{\mathbb{Q}_p}$ such that

$$[\mathbb{Q}_p(\{a_n\}) : \mathbb{Q}_p] = +\infty.$$

Set $s_1 = 0$ and

$$s_n = s_{n-1} + p^{k_n} a_n$$

for a suitable $k_n \in \mathbb{N}$ such that

$$|p^{k_n} a_n|_p \leq \frac{|p^{k_{n-1}} a_{n-1}|_p}{2}, \quad |p^{k_n} a_n|_p < |s_{n-1} - \sigma(s_{n-1})|_p \quad \forall \sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}_p}/\mathbb{Q}_p\right) \text{ with } \sigma(s_{n-1}) \neq s_{n-1}.$$

Then the sequence $(s_n)_{n\geq 1}$ is a Cauchy sequence. Let $s \in \widehat{\overline{\mathbb{Q}_p}}$ be the limit of $(s_n)_{n\geq 1}$. We have $|s - s_{n-1}|_p = |p^{k_n} a_n|_p < |s_{n-1} - \sigma(s_{n-1})|_p$ for all $n \geq 1$ and all $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}_p}/\mathbb{Q}_p\right)$ with $\sigma(s_{n-1}) \neq s_{n-1}$. By Krasner's lemma (Theorem 1.3.13), it follows that $s_n \in \mathbb{Q}_p(s)$ for all $n \geq 1$. Thus, we have

$$[\mathbb{Q}_p(s) : \mathbb{Q}_p] \geq [\mathbb{Q}_p(\{a_n\}) : \mathbb{Q}_p] = +\infty,$$

which implies that $s \notin \overline{\mathbb{Q}_p}$. □

Construction 1.4.5. Let p be a prime number. The *field* $\mathbb{C}_p$ *of* p*-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$.

The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$ by Proposition 1.3.12. By Corollaries 1.2.11 and 1.2.12, we have

$$|\mathbb{C}_p^\times|_p = |\overline{\mathbb{Q}_p}^\times|_p = p^{\mathbb{Q}}, \quad \Bbbk_{\mathbb{C}_p} \cong \Bbbk_{\overline{\mathbb{Q}_p}} \cong \overline{\mathbb{F}_p}.$$

Proposition 1.4.6. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Proof. Yang: To be completed. □

Yang: For example, see p-adic fields for beginners.

Construction 1.4.7. Let p be a prime number. Yang: We construct the *spherically complete p-adic field Ω_p*. Yang: To be completed.

Yang: does Ω_p has the same card as $\mathbb{R}$?

Chapter 2

Normed algebras

2.1 Normed rings and modules

2.1.1 Semi-normed algebraic structures

Definition 2.1.1. Let G be an abelian group. A *semi-norm* on G is a function $\|\cdot\| : G \to \mathbb{R}_{\geq 0}$ such that

- $\|0\| = 0$;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$.

Suppose that R is a ring (commutative with unity) and $\|\cdot\|$ is a semi-norm on the underlying abelian group of R. We further require that

- $\|1\| = 1$;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$.

Suppose that $(M, \|\cdot\|_M)$ is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M. We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$.

Suppose that $(A, \|\cdot\|_A)$ is an R-algebra and $\|\cdot\|_A$ is a semi-norm on the underlying R-module of A. We further require that this semi-norm is a semi-norm on the underlying ring of A.

Definition 2.1.2. Let A be an abelian group (or ring, R-module, R-algebra) equipped with a semi-norm $\|\cdot\|$. We say that $\|\cdot\|$ is a *norm* if $\forall x \in A, \|x\| = 0 \iff x = 0$.

Definition 2.1.3. Let A be an abelian group (or ring, R-module, R-algebra) equipped with a semi-norm $\|\cdot\|$. We say that $\|\cdot\|$ is *non-archimedean* if we have the strong triangle inequality $\forall x, y \in A, \|x + y\| \leq \max(\|x\|, \|y\|)$.

Definition 2.1.4. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group (or ring, R-module, R-algebra) A. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in A, \|x\|_1 \leq C\|x\|_2$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are bounded by each other, we say they are *equivalent*.

Remark 2.1.5. Equivalent semi-norms induce the same topology on A. However, the converse is not true in general. Compare with Lemma 1.1.17.

Yang: what about on a module?

Definition 2.1.6. Let M be a semi-normed abelian group (or R-module) and $N \subseteq M$ be a subgroup (or R-submodule). The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Unless otherwise specified, we always equip the quotient M/N with the residue semi-norm.

Remark 2.1.7. The residue semi-norm is a norm if and only if N is closed in M.

Definition 2.1.8. Let M and N be two semi-normed abelian groups (or rings, R-modules, R-algebras). A homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$.
A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$.

Definition 2.1.9. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$. A multiplicative norm sometimes is called a *(multiplicative) valuation* or an *absolute value*.

Example 2.1.10. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a valuation ring. This norm is non-archimedean and multiplicative.

Example 2.1.11. A valuation field $(\mathbf{k}, |\cdot|)$ can be viewed as a valuation ring.

Example 2.1.12. Let $|\cdot| = |\cdot|_\infty$ be the usual absolute value on $\mathbb{Z}$. Then $(\mathbb{Z}, |\cdot|)$ is a valuation ring.

Example 2.1.13. Let X be a compact Hausdorff topological space. The ring $C(X, \mathbb{R})$ of continuous real-valued functions on X equipped with the norm $\|f\| = \sup_{x \in X} |f(x)|$ is a normed ring. Its norm is power-multiplicative but not multiplicative in general. It is worth mentioning that the Gelfand-Kolmogorov Theorem saying that we can recover X from the normed ring $C(X, \mathbb{R})$.

Example 2.1.14. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. The action of K on itself by multiplication induces a embedding $K \hookrightarrow \operatorname{End}_{\mathbb{Q}}(K) \cong M_n(\mathbb{Q})$. Consider a norm $\|\cdot\|$ on the matrix ring $M_n(\mathbb{Q})$ and restrict it to K. Then $(K, \|\cdot\|)$ is a normed ring. This norm is even not power-multiplicative in general.

Definition 2.1.15. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 2.1.16. A *banach ring* is a complete normed ring.

Proposition 2.1.17. Let R be a banach ring and $I \subseteq R$ be a closed ideal. Then the residue norm on the quotient ring R/I is a norm for rings.

Proof. We only need to show that $\|1\|_{R/I} = 1$. Yang: To be added. □

Proposition 2.1.18. Let R be a banach ring. Then the group of invertible elements $R^\times$ is an open subset of R.

Proof. Yang: To be added. □

Corollary 2.1.19. Let R be a banach ring. Then every maximal ideal of R is closed.

Proof. Yang: To be added. □

Definition 2.1.20. Let $(A, \|\cdot\|_A)$ be a normed algebraic structure, e.g., a normed abelian group, a normed ring, or a normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a metric space. Since A is dense in its completion and the algebraic operations are uniformly continuous, the algebraic operations on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M\otimes_R N} = \inf\left\{\sum_i \|x_i\|_M \|y_i\|_N : z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N\right\}.$$

Definition 2.1.21. Let R be a banach ring and M, N complete semi-normed R-modules. The *complete tensor product* $M \widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Example 2.1.22. Yang: Example of complete tensor product.

Definition 2.1.23. Let R be a non-archimedean banach ring. We define

$$R^\circ = \{f \in R : \rho(f) \leq 1\}, \quad R^{\circ\circ} = \{f \in R : \rho(f) < 1\}.$$

The *reduction ring* of R is defined as the quotient ring

$$\widetilde{R} = R^\circ / R^{\circ\circ}.$$

For a non-archimedean field $\mathbf{k}$, its reduction ring $\widetilde{\mathbf{k}} = \mathscr{k}_{\mathbf{k}}$ is just the residue field of its valuation ring.

Example 2.1.24. Let R be a ring equipped with the trivial norm. Then we have $R^\circ = R$ and $R^{\circ\circ} = \mathrm{nil}(R)$. Hence the reduction ring $\widetilde{R}$ is isomorphic to the reduced ring $R_{\mathrm{red}} = R/\mathrm{nil}(R)$.

2.1.2 Spectral radius

Definition 2.1.25. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Since $\|\cdot\|$ is submultiplicative, the limit defining $\rho(f)$ exists and equals to $\inf_{n\geq 1} \|f^n\|^{1/n}$ by Fekete's Subadditive Lemma.

Example 2.1.26. Consider the normed ring $(K, \|\cdot\|)$ in Example 2.1.14. Suppose that $\|\cdot\|$ on $M_n(\mathbb{Q})$ is given by the 2-norm induced by the euclidean norm on $\mathbb{Q}^n$. Then for a matrix $A \in M_n(\mathbb{Q})$, its spectral radius $\rho(A)$ is given by the largest absolute value of eigenvalues of A. Hence for each $a \in K$, its spectral radius $\rho(a)$ is given by the largest absolute value of embeddings of a into $\mathbb{C}$, i.e.,

$$\rho(a) = \max_{\sigma : K \hookrightarrow \mathbb{C}} |\sigma(a)|.$$

Proposition 2.1.27. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 2.1.28. A banach ring R is called *uniform* if its norm is power-multiplicative.

Definition 2.1.29. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Definition 2.1.30. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\mathrm{Qnil}(R)$.

Proposition 2.1.31. Let R be a banach ring. The completion of $R/\mathrm{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

Yang: To be continued...

2.2 Convergent and restricted power series

Notation 2.2.1. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;

- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \dots, n$, we have $\alpha_i \leq \beta_i$;
- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

2.2.1 Absolutely convergent power series

Definition 2.2.2. Let R be a banach ring and $r > 0$ be a real number. We define the *ring of absolutely convergent power series* over $\mathbf{k}$ with radius r as

$$R\langle T/r\rangle := \left\{\sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty\right\}.$$

Equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| := \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r\rangle$ is a banach ring.
For a tuple of n indeterminates $T = (T_1, \dots, T_n)$ and a tuple of n positive real numbers $r = (r_1, \dots, r_n)$, we define

$$R\left\langle \underline{T/r}\right\rangle := R\langle T_1/r_1, \dots, T_n/r_n\rangle := R\langle T_1/r, \dots, T_{n-1}/r_{n-1}\rangle\langle T_n/r_n\rangle.$$

Note that if R has trivial norm, then

$$R\langle T/r\rangle = \begin{cases} R[[T]], & \text{if } r < 1; \\ R[T], & \text{if } r \geq 1. \end{cases}$$

Yang: To add the spectral of absolutely convergent power series.

2.2.2 Tate algebras

Definition 2.2.3. Let R be a non-archimedean banach ring. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$R\langle \underline{r^{-1}T}\rangle := R\{\underline{r^{-1}T}\} := \left\{\sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in R, \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0\right\}.$$

Proposition 2.2.4. Let R be a non-archimedean banach ring. Then the Tate algebra $R\{\underline{T/r}\}$ is a non-archimedean multiplicative banach R-algebra with respect to the *gauss norm*

$$\left\|\sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha\right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $R\{\underline{T/r}\}$ is a R-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ are two nonzero elements in $R\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{\frac{\varepsilon}{\|g\|}\|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|}\|a_\alpha\| r^\alpha\right\} \leq \varepsilon.$$

Hence $f \cdot g \in R\{\underline{T/r}\}$ and it shows that $R\{\underline{T/r}\}$ is a R-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $R\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f+g\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \leq \sup_{\alpha\in\mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \leq \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned}\|f\cdot g\| = \left\|\sum_{\gamma\in\mathbb{N}^n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) T^\gamma\right\| &= \sup_{\gamma\in\mathbb{N}^n}\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \\ &\leq \sup_{\gamma\in\mathbb{N}^n}\max_{\alpha+\beta=\gamma} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \leq \|f\|\cdot\|g\|.\end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\|\cdot\|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \left\|\sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta\right\| r^{\alpha_1+\beta_1} \leq \|f\cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by Proposition 1.2.3. Thus the gauss norm is multiplicative.

Step 4. Finally show that $R\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $R\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in R. Since R is complete, set $a_\alpha := \lim_{m\to+\infty} a_{\alpha,m}$ and $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \leq 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \leq \varepsilon$$

for all $m > N$. Yang: To be revised, the original version is for a field. □

Example 2.2.5. Let R be a non-archimedean banach ring and $A = R\{T\}$ be the Tate algebra in one variable over R. Then we have

$$A^\circ = \left\{ \sum_{n\geq 0} a_n T^n \,:\, |a_n| \leq 1 \text{ for all } n \in \mathbb{N} \right\} = R^\circ\{T\},$$

and

$$A^{\circ\circ} = \left\{ \sum_{n\geq 0} a_n T^n \,:\, |a_n| < 1 \text{ for all } n \in \mathbb{N} \right\} = R^{\circ\circ}\{T\}.$$

Since the norm of items in a restricted power series will tend to 0, we have

$$\widetilde{A} = \widetilde{R}[\underline{T}].$$

Example 2.2.6. Let R is a multiplicative non-archimedean banach ring. Set

$$\sqrt{|R|^{-1}} = \{r \in \mathbb{R}_{>0} \,:\, r^{-n} \in |R| \text{ for some } n \in \mathbb{N}_{>0}\}.$$

Fix $r \in \mathbb{R}^n_{>0}$, consider the Tate algebra $A = R\{T/r\}$.

Suppose that $r \in \sqrt{|R|^{-1}}$. Let n be the minimal positive integer such that $r^n \in |R|^{-1}$ and

$$\widetilde{M}_k := \{a \in R \,:\, |a| = r^{-nk}\}/\{a \in R \,:\, |a| < r^{-nk}\}.$$

For $a_m T^m$ with $n \nmid m$, we have$\|a_m T^m\| = |a_m| r^m \leq 1 \implies |a_m| r^m < 1$. Hence

$$\widetilde{R\{T/r\}} = \widetilde{R} \oplus \widetilde{M}_1 T^n \oplus \widetilde{M}_2 T^{2n} \oplus \widetilde{M}_3 T^{3n} \oplus \cdots.$$

In case $R = \mathbf{k}$ is a non-archimedean field, we have $\widetilde{M}_k \cong \widetilde{\mathbf{k}}$ by choosing an element $c \in \mathbf{k}$ with $|c| = r^{-n}$. Hence

$$\widetilde{\mathbf{k}\{T/r\}} \cong \Bbbk_{\mathbf{k}}[T^n].$$

Suppose that $r \notin \sqrt{|R|^{-1}}$. Then for every $\|a_n T^n\| = a_n r^n \leq 1$, we have $|a_n| < 1$. It follows that

$$\widetilde{R\{T/r\}} = \widetilde{R}.$$

2.2.3 Weierstrass preparation

Definition 2.2.7. Let R be a non-archimedean banach ring and $A = R\{T/r\}$. For $f = \sum_{a_n \in \mathbb{N}} a_n T^n \in A$, we define the *degree* of f as

$$\deg f := \max\{n \in \mathbb{N} \,:\, \|a_n\| r^n = \|f\|\}.$$

It is interesting to note that if R has trivial norm, then $\deg f$ coincides with the usual degree of polynomials when $r \geq 1$ and the order of formal power series when $r < 1$.

Definition 2.2.8. Let R be a non-archimedean banach ring and $A = R\{T/r\}$. A restricted power series $f = \sum_{n\in\mathbb{N}} a_n T^n \in A$ of degree d is said to be *distinguished* if a_d is invertible in R.

Proposition 2.2.9. Let R be a non-archimedean banach ring. An element f is invertible if and only if $\deg f = 0$ and the constant item of f is invertible in R.

Proof. Multiplying by a_0^{-1}, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $R[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta\in\mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma\in\mathbb{N}^n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right)T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = -\sum_{\substack{\alpha+\beta=\gamma\\ \alpha\neq 0}} a_\alpha b_\beta.$$

Let $A = \|f-1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\| r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma\\ \alpha\neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for m. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\| r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma\\ \alpha\neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\left\{A^{m+1}, A \cdot A^m\right\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in R\{\underline{T/r}\}$. Yang: To be revised. □

Proposition 2.2.10. Let $\mathbf{k}$ be a complete non-archimedean field and $r > 0$ be a positive real number. Then the Tate algebra $\mathbf{k}\{T/r\}$ is an euclidean domain with respect to the degree defined in Definition 2.2.7. Yang: To be added.

Proof. Let $f, g \in \mathbf{k}\{T/r\}$ be two elements with $g \neq 0$. Denote $n = \deg f$ and $m = \deg g$. We need to find $q, r \in \mathbf{k}\{T/r\}$ such that

$$f = q \cdot g + r, \quad \deg r < \deg g.$$

Yang: To be added. □

Definition 2.2.11. Let R be a non-archimedean banach ring and $A = R\{T/r\}$. A Weierstrass polynomial is a monic polynomial $P \in A[T] \subset R\{T/r\}$ whose two degrees as a polynomial and as a restricted power series coincide.

Theorem 2.2.12 (Weierstrass preparation theorem)**.** Let R be a non-archimedean banach ring. Let $f \in R\{T/r\}$ be a distinguished restricted power series of degree d. Then there exists a unique Weierstrass polynomial $p \in R[T]$ of degree d and a unique unit $u \in R\{T/r\}$ such that

$$f = p \cdot u.$$

Yang: To be checked.

Proof. Yang: To be added. □

Remark 2.2.13. In my knowledge, there are at least three different versions of Weierstrass preparation theorem under different settings:

- The classical Weierstrass preparation in complex analysis;
- The Weierstrass preparation for formal power series over complete noetherian local rings;
- The Weierstrass preparation for Tate algebras over non-archimedean banach rings.

Let $(R, \mathfrak{m})$ be a complete noetherian local ring. Note that there is also a Weierstrass preparation theorem for formal power series over R stating that for every formal power series $f \in R[[T]]$ whose reduction $\overline{f} \in (R/\mathfrak{m})[[T]]$ is of order d, there exists a unique monic polynomial $p \in R[T]$ of degree d and a unique unit $u \in R[[T]]$ such that

$$p \equiv T^d \mod \mathfrak{m}, \quad f = p \cdot u.$$

Yang: To be continued.

2.3 Affinoid algebras

Definition 2.3.1. Let R be a non-archimedean banach ring. A banach R-algebra A is called a *R-affinoid algebra* if there exists an admissible surjective homomorphism

$$\varphi : R\{\underline{T/r}\} \twoheadrightarrow A$$

for some $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$.
If one can choose $r_1 = \dots = r_n = 1$, then we say that A is a *strictly R-affinoid algebra*.

Example 2.3.2. Suppose that the norm on R is trivial. Then a strictly R-affinoid algebra is just an R-algebra of finite type equipped with the trivial norm.

Example 2.3.3. Let A be an R-affinoid algebra, $f_i, g_j \in A$ and $r_i, s_j \in \mathbb{R}_{>0}$. Define

$$\begin{aligned} A\left\{\underline{f/r}\right\} &:= A\left\{\underline{T/r}\right\}/(T_i - f_i), \\ A\left\{\underline{f/r}, \underline{s/g}\right\} &:= A\left\{\underline{T/r}, \underline{S/s}\right\}/(T_i - f_i, g_j S_j - 1). \end{aligned}$$

Suppose that $g \in R$ with $(f_i, g) = A$. Then we can define

$$A\left\{\underline{(f/g)/r}\right\} := A\left\{\underline{T/r}\right\}/(gT_i - f_i).$$

All of the above three algebras are again R-affinoid algebras. They can be viewed as normed analogues of localizations. Yang: To be replaced.

In the rest of this section, we fix a complete non-archimedean field $\mathbf{k}$ and consider $\mathbf{k}$-affinoid algebras.

2.3.1 Strict case

First we consider strictly $\mathbf{k}$-affinoid algebras, which are well-studied in the era of rigid analytic geometry.

Proposition 2.3.4. Let $\mathbf{k}$ be a complete non-archimedean field and $f \in \mathbf{k}\{\underline{T}\}$. Then there exists an automorphism φ of $\mathbf{k}\{\underline{T}\}$ over $\mathbf{k}$ such that $\varphi(f)$ is T_n-distinguished, i.e., $\varphi(f) \in A\{T_n\}$ is distinguished in the variable T_n with $A = \mathbf{k}\{T_1, \dots, T_{n-1}\}$.

Proof. Yang: To be added. ☐

Proposition 2.3.5. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T}\}$ is noetherian, factorial, and Jacobson.

Proof. Yang: To be completed. ☐

Corollary 2.3.6. Strictly $\mathbf{k}$-affinoid algebras are noetherian.

Proof. Yang: To be completed. ☐

Theorem 2.3.7. Let A be a strictly $\mathbf{k}$-affinoid algebra. Then there exists a finite injective admissible homomorphism

$$\varphi : \mathbf{k}\{T_1, \dots, T_d\} \hookrightarrow A.$$

Proof. Yang: To be completed. ☐

Proposition 2.3.8. Let A be an $\mathbf{k}$-affinoid algebra. Then there exists a constant $C > 0$ and $N > 0$ such that for all $f \in A$ and $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

In particular, $\mathrm{Qnil}(A) = \mathrm{nil}(A)$.
Furthermore, if A is reduced, we have

$$\|f\| \leq C\rho(f)$$

for all $f \in A$.

Proof. Yang: To be completed. ☐

2.3.2 General case

Proposition 2.3.9. Let A be an affinoid $\mathbf{k}$-algebra. If and only if $\rho(f) \in \sqrt{|\mathbf{k}|} \cup \{0\}$ for all $f \in A$, then A is strict. Yang: To be complete.

Proof. Yang: To be completed. ☐

Definition 2.3.10. Let $\mathbf{k}$ be a non-archimedean field. We define the *ring of restricted Laurent series* over $\mathbf{k}$ as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} := \mathbf{k}\{T/r, r/T\}.$$

Yang: Is $\mathbf{K}_r$ always a field? Yang: Do we have $\mathbf{L}_{\mathbf{k},r} = \mathrm{Frac}(\mathbf{k}\{T/r\})$?

Proposition 2.3.11. Let $\mathbf{k}$ be a non-archimedean field. If $r \notin \sqrt{|\mathbf{k}^\times|}$, then $\mathbf{K}_r$ is a complete non-archimedean field with non-trivial absolute value extending that of $\mathbf{k}$.

Yang: Tensor with $\mathbf{K}_r$.

Proposition 2.3.12. Let A be a $\mathbf{k}$-affinoid algebra. Then there exists $r_i \in \mathbb{R}_{>0}$ such that

$$\mathbf{K}_{\underline{r}} \widehat{\otimes}_{\mathbf{k}} A$$

is a strictly $\mathbf{K}_{\underline{r}}$-affinoid algebra.

There are three different categories of finite modules over an affinoid algebra A:

- The category $\mathbf{Banmod}_A$ of finite banach A-modules with A-linear maps as morphisms.
- The category $\mathbf{Banmod}_A^b$ of finite banach A-modules with bounded A-linear maps as morphisms.
- The category $\mathbf{mod}_A$ of finite A-modules with all A-linear maps as morphisms.

Theorem 2.3.13. Let A be an affinoid $\mathbf{k}$-algebra. Then the category of finite banach A-modules with bounded A-linear maps as morphisms is equivalent to the category of finite A-modules with A-linear maps as morphisms. Yang: To be revised.

For simplicity, we will just write mod_A to denote the category of finite banach A-modules with bounded A-linear maps as morphisms.

Chapter 3

Non-archimedean analysis

3.1 Local theory I: functions

3.1.1 Analytic functions on closed polydisks

Proposition 3.1.1. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we can associate a function $F_f : E(0,\underline{r}) \to \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \qquad \text{for } x = (x_1, \dots, x_n) \in E(0, \underline{r}).$$

This defines a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T/r}\}$ to the ring of all functions from $E(0,\underline{r})$ to $\mathbf{k}$.

Proof. Given $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \dots, x_n) \in E(0,\underline{r})$, we have

$$\left\| \sum_{|\alpha|=n} a_\alpha x^\alpha \right\| \le \max_{|\alpha|=n} \|a_\alpha\| r^\alpha \to 0 \qquad \text{as } n \to +\infty.$$

Hence by Proposition 1.2.7, the series $F_f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$ converges in $\mathbf{k}$. This defines a function $F_f : E(0,\underline{r}) \to \mathbf{k}$.

Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. Set

$$A_n = \sum_{|\alpha|<n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta|<n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma|<n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) x^\gamma.$$

We need to show that $F_f(x) F_g(x) = \lim A_n B_n = \lim C_n = F_{fg}(x)$. Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta| \ge n}} a_\alpha b_\beta x^{\alpha+\beta}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon / \|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon / \|f\|$. For any $n > 2N$, we have

$$\|A_n B_n - C_n\| \le \max_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta| \ge n}} \|a_\alpha\| \|b_\beta\| \|x^{\alpha+\beta}\| < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \le \varepsilon.$$

Thus $F_f(x)F_g(x) = (F_{fg})(x)$. The addition and scalar multiplication can be verified directly. We thus finish the proof. ☐

Proposition 3.1.2. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x, y \in E(0, \underline{r})$, we have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq L \cdot \|y - x\|_\infty,$$

where $L = \max_{1\leq i\leq n} \|f\|_g / r_i$.

Proof. Set $y - x = (h_1, \dots, h_n)$ and $x^{(0)} = x$, $x^{(i)} = (x_1 + h_1, \dots, x_i + h_i, x_{i+1}, \dots, x_n)$ for $i = 1, \dots, n$. We have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{1\leq i\leq n} \|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}}.$$

We only need to show that for every $i = 1, \dots, n$, we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}} \leq \frac{\|f\|_g}{r_i}\|h_i\|.$$

Without loss of generality and for simplicity, we assume that $y = (x_1 + h, x_2, \dots, x_n)$ and $x = (x_1, x_2, \dots, x_n)$. Note that by the strong triangle inequality, we have $\|h\| \leq r_1$.

Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\begin{aligned} f(y) - f(x) &= \sum_{\alpha\in\mathbb{N}^n} a_\alpha \left((x_1 + h)^{\alpha_1} - x_1^{\alpha_1}\right) x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ &= \sum_{\alpha\in\mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k. \end{aligned}$$

Note that

$$\left\|\binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\right\| r_1^k \leq \|a_\alpha\| r^\alpha \leq \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{\alpha\in\mathbb{N}^n} \max_{1\leq k\leq\alpha_1} \left\{\left\|\binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\right\| \|h\|^k\right\} \leq \max_k \left\{\|f\|_g \left(\frac{\|h\|}{r_1}\right)^k\right\} \leq \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. ☐

Lemma 3.1.3. Let $\mathbf{k}$ be a complete non-archimedean field. Then we have $\|f(x)\| \leq \|f\|$ for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$. In particular, if $f_n \to f$ as $n \to +\infty$ in $\mathbf{k}\{\underline{T/r}\}$, then we have $\|f_n(x) - f(x)\| \to 0$ for every $x \in E(0, \underline{r})$.

Proof. Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \dots, x_n) \in E(0, \underline{r})$. We have

$$\left\|\sum_{|\alpha|<N} a_\alpha x^\alpha\right\| \leq \max_{|\alpha|<N} \|a_\alpha\| r^\alpha \leq \|f\|$$

for every $N \in \mathbb{N}$. Taking $N \to +\infty$, we have $\|f(x)\| \leq \|f\|$. ☐

Let $\mathbf{k}$ be a complete non-archimedean field. Recall that the formal derivative operator $\partial_i : \mathbf{k}[[\underline{T}]] \to$

$\mathbf{k}[[\underline{T}]]$ is defined by

$$\frac{\partial}{\partial T_i}\left(\sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha\right) := \sum_{\alpha\in\mathbb{N}^n} \alpha_i a_\alpha T_1^{\alpha_1}\cdots T_i^{\alpha_i-1}\cdots T_n^{\alpha_n}.$$

Lemma 3.1.4. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f\in\mathbf{k}\{\underline{T/r}\}$, we have $\partial_i(f)\in\mathbf{k}\{\underline{T/r}\}$.

Proof. Suppose that $f=\sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha\in\mathbf{k}\{\underline{T/r}\}$. We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha\in\mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1-1}T_2^{\alpha_2}\cdots T_n^{\alpha_n}.$$

Noting that $\mathbf{k}$ is non-archimedean, we have $\|\alpha_1 a_\alpha\|\leq\|a_\alpha\|$. Then

$$\lim_{|\alpha|\to+\infty}\|\alpha_1 a_\alpha\| r_1^{\alpha_1-1}r_2^{\alpha_2}\cdots r_n^{\alpha_n}\leq\frac{1}{r_1}\lim_{|\alpha|\to+\infty}\|a_\alpha\|r^\alpha=0.$$

The conclusion follows. □

Proposition 3.1.5. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation, and $\partial_i=\partial/\partial T_i$ be the derivative operator on $\mathbf{k}\{\underline{T/r}\}$ with respect to the indeterminate T_i for $i=1,\ldots,n$. Then for every $f\in\mathbf{k}\{\underline{T/r}\}$ and $x\in E(0,\underline{r})$, we have

$$F_{\partial_i(f)}(x)=\lim_{h\to 0}\frac{F_f(x_1,\ldots,x_i+h,\ldots,x_n)-F_f(x)}{h}.$$

Proof. Without loss of generality, we can assume that $i=1$. Let $f=\sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha\in\mathbf{k}\{\underline{T/r}\}$ and $f_n=\sum_{|\alpha|<n} a_\alpha T^\alpha$ for $n\in\mathbb{N}$. Set $x_h=(x_1+h,x_2\ldots,x_n)$ and $L_f(h)=(F_f(x_h)-F_f(x))/h$ for $h\in\mathbf{k}^\times$. Note that for fixed h, we have $\lim_{n\to\infty}L_{f_n}(h)=L_f(h)$.

We compute $L_{f_n}(h)-F_{\partial f_n}(x)$ explicitly:

$$\begin{aligned} L_{f_n}(h)-F_{\partial f_n}(x) &= \frac{1}{h}\left(\sum_{|\alpha|<n}\sum_{k=1}^{\alpha_1}\binom{\alpha_1}{k}a_\alpha x_1^{\alpha_1-k}h^k x_2^{\alpha_2}\cdots x_n^{\alpha_n}-\sum_{|\alpha|<n}\alpha_1 a_\alpha x_1^{\alpha_1-1}hx_2^{\alpha_2}\cdots x_n^{\alpha_n}\right)\\ &= \sum_{|\alpha|<n}\sum_{k=2}^{\alpha_1}\binom{\alpha_1}{k}a_\alpha x_1^{\alpha_1-k}x_2^{\alpha_2}\cdots x_n^{\alpha_n}h^{k-1}. \end{aligned}$$

Note that

$$M=\sup_{\alpha\in\mathbb{N}^n}\|a_\alpha x_1^{\alpha_1-k}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\|r_1^{k-1}\leq\|f\|/r_1<+\infty.$$

Hence

$$\|L_{f_n}(h)-F_{\partial f_n}(x)\|\leq\max_{2\leq k\leq n}\left\{M\frac{\|h\|^{k-1}}{r_1^{k-1}}\right\}\leq M\frac{\|h\|}{r_1}$$

for $h\in\mathbf{k}^\times$ with $\|h\|<r_1$. Taking $n\to+\infty$, we have

$$\|L_f(h)-F_{\partial f}(x)\|\leq M\frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

Yang: The following should be a theorem.

Corollary 3.1.6. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment $f \mapsto F_f$ in Proposition 3.1.1 is injective.

Proof. Note that if $F_f = 0$, then for every $i = 1, \dots, n$, we have $F_{\partial_i(f)} = 0$ by Proposition 3.1.5. By taking repeated derivatives, we have $F_{\partial^\alpha f} = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Note that $F_{\partial^\alpha f}(0) = \alpha! a_\alpha$. It follows that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n$ and thus $f = 0$. □

Remark 3.1.7. Corollary 3.1.6 holds for non-archimedean fields of positive characteristic as well. The proof uses Theorem 3.3.2 and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element $f \in \mathbf{k}\{\underline{T/r}\}$ with the associated function $F_f : E(0, \underline{r}) \to \mathbf{k}$ as in Proposition 3.1.1.

Proposition 3.1.8. Let $\Bbbk$ be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra $\Bbbk\{\underline{T/r}\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in E(0,\underline{r})} \|f(x)\|_{\Bbbk}.$$

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \Bbbk\{\underline{T/r}\}$. We write $f = g + h$ with $g = \sum_{\alpha \in S} a_\alpha T^\alpha$ and $h = \sum_{\alpha \notin S} a_\alpha T^\alpha$, where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that S is a non-empty finite set and $\|h\| < \|f\|$. By Lemma 3.1.3, we have $\|h(x)\| < \|f\|$ for every $x \in E(0, \underline{r})$. It suffices to show that $\|g\|_{\sup} = \|g\|$.

Since $\Bbbk$ is algebraically closed, $|\Bbbk^\times|$ is dense in $\mathbb{R}_{>0}$. For every pair $\alpha, \beta \in S$ with $\alpha \neq \beta$, the set $\{t \in \mathbb{R}^n_{>0} : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$ is a proper closed subset of $\mathbb{R}^n_{>0}$. Thus we can find $t_m \in |\Bbbk^\times|^n$ such that $t_m < r$, $t_m \to r$ as $m \to +\infty$ and for every $\alpha, \beta \in S$ with $\alpha \neq \beta$, we have $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$ for all m. For each m, we can find $x_m \in E(0, \underline{r})$ such that $\|x_m^\alpha\| = t_m^\alpha$ for every $\alpha \in S$ since $t_m \in |\Bbbk^\times|^n$. It follows that

$$\|g(x_m)\| = \max_{\alpha \in S} \|a_\alpha\| \|x_m^\alpha\| = \max_{\alpha \in S} \|a_\alpha\| t_m^\alpha \to \|g\| \quad \text{as } m \to +\infty.$$

Thus $\|g\|_{\sup} = \|g\|$. □

Remark 3.1.9. If $\mathbf{k}$ is locally compact (hence not algebraically closed), the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ do not coincide with the supremum norm. For example, consider the Tate algebra $\mathbb{Q}_p\{T\}$. The element $f = T^p - T$ has gauss norm $\|f\| = 1$. However, for every $x \in E(0,1) = \mathbb{Z}_p$, we have $f(x) = x^p - x \equiv 0 \mod p$. Thus $\|f(x)\|_p \leq 1/p$ and $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$.

Remark 3.1.10. Recall that in classical complex analysis, the closure of the polynomial ring $\mathbb{C}[T_1, \dots, T_n]$ with respect to the supremum norm on a closed polydisc $E(0, \underline{r}) \subset \mathbb{C}^n$ is the ring of all complex-valued continuous functions which are analytic on its interior $B(0, \underline{r})$.

Yang: Invertiblity of a function

3.2 Local theory II: maps

Let $\mathbf{k}$ be a complete non-archimedean field.

3.2.1 The first properties

Yang: Recall the Runge theorem in complex analysis.

Definition 3.2.1. A map $f : (E(0,r) \subset \mathbf{k}^n) \to \mathbf{k}^m$ is called *analytic* if there exists power series $f_1, \dots, f_m \in \mathbf{k}\{\underline{T/r}\}$ such that for any $x \in E(0,r)$, we have

$$f(x) = (f_1(x), \dots, f_m(x)).$$

Yang: To be revised.

Yang: Composition of analytic functions.

Definition 3.2.2. A map $f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^m$ is called *analytic* if there exists power series $f_1, \dots, f_m \in \mathbf{k}\{\underline{T/r}\}$ such that for any $x \in E(0,\underline{r})$, we have

$$f(x) = (f_1(x), \dots, f_m(x)).$$

Proposition 3.2.3. Let $f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^m$ and $g : (E(0,\underline{s}) \subset \mathbf{k}^m) \to \mathbf{k}^l$ be two analytic maps such that $f(E(0,\underline{r})) \subset E(0,\underline{s})$. Then the composition $g \circ f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^l$ is also analytic.
Furthermore, if $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_l)$ with $f_i = \sum_\alpha a_{i,\alpha} \underline{T}^\alpha$ and $g_j = \sum_\beta b_{j,\beta} \underline{T}^\beta$, then the composition $g \circ f = (h_1, \dots, h_l)$ with

$$h_j = \sum_\beta b_{j,\beta} f^\beta = \sum_\beta b_{j,\beta} f_1^{\beta_1} \cdots f_m^{\beta_m}.$$

Yang: To be checked. Yang: To be revised.

Proof. Yang: To be completed. □

3.2.2 Inverse and implicit function

Definition 3.2.4. Let $f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^m$ be an analytic map. The *tangent map* $\mathrm{d}f_0 : \mathbf{k}^n \to \mathbf{k}^m$ of f at 0 is defined to be the linear map given by the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial T_j}(0) \right)_{1 \le i \le m, 1 \le j \le n}.$$

Yang: To be checked.

Theorem 3.2.5 (Inverse Function Theorem over Non-Archimedean Fields)**.** Let $f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^n$ be an analytic map. Suppose that $f(0) = 0$ and the tangent map $\mathrm{d}f_0 : \mathbf{k}^n \to \mathbf{k}^n$ is an isomorphism.

Then there exist $E(0,\underline{r'}) \subset E(0,\underline{r})$, $E(0,\underline{s'}) \subset f(E(0,\underline{r}))$ and an analytic map $g : (E(0,\underline{s'}) \subset \mathbf{k}^n) \to \mathbf{k}^n$ such that

$$f \circ g = \mathrm{id}_{E(0,\underline{s'})}, \quad g \circ f = \mathrm{id}_{E(0,\underline{r'})}.$$

Proof. Yang: To be completed. ☐

Theorem 3.2.6 (Implicit Function Theorem over Non-Archimedean Fields). Let $f : (E(0,\underline{r}) \subset \mathbf{k}^{n+m}) \to \mathbf{k}^m, (x_1, \cdots, x_n, y_1, \ldots, y_m) \mapsto f(x,y)$ be an analytic map. Suppose that $f(0) = 0$ and the Jacobian matrix $\left(\partial_j f_i(0)\right)_{1\le i,j\le m}$ is invertible.
Then there exist $\underline{r'} = (r'_1, \ldots, r'_n)$ with each $r'_i > 0$ and an analytic map $g : (E(0,\underline{r'}) \subset \mathbf{k}^n) \to \mathbf{k}^m$ such that for any $x \in E(0,\underline{r'})$,

$$f(x,y) = 0 \iff y = g(x).$$

Proof. Yang: To be completed. ☐

3.3 Analytic functions in one variable

Proposition 3.3.1. Let $\mathbf{k}$ be a complete non-archimedean field and $f = \sum_{n=0}^{+\infty} a_n T^n \in \mathbf{k}[[T]]$. Set

$$R := \frac{1}{\limsup_{n\to+\infty} \|a_n\|^{1/n}} \in \mathbb{R}_{\ge 0} \cup \{+\infty\}.$$

Then we have

(a) the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$ and diverges for all $x \in \mathbf{k}$ with $\|x\| > R$;

(b) if $R < +\infty$, the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| = R$ if and only if $\lim_{n\to+\infty} \|a_n\| R^n = 0$.

Proof. By Proposition 1.2.7, we only need to check when the terms $a_n x^n$ tend to zero as $n \to +\infty$. If $\|x\| < R$, there exists $r \in (0,1)$ such that $\|x\| < r^2 R$. Then there exists $N \in \mathbb{N}$ such that for all $n \ge N$, we have $\|a_n\|^{1/n} < 1/(rR)$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n < \|a_n\| (r^2 R)^n < (r^2 R)^n \cdot \frac{1}{(rR)^n} = r^n \to 0.$$

Thus the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$.

Suppose that $\|x\| > R$. There exists $s > 1$ such that $\|x\| > R/s$. By the definition of R, there exist infinitely many $n \in \mathbb{N}$ such that $\|a_n\|^{1/n} > s/R$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n > \|a_n\| \frac{R^n}{s^n} > \left(\frac{s}{R}\right)^n \cdot \frac{R^n}{s^n} = 1.$$

Thus the series $f(x)$ diverges for all $x \in \mathbf{k}$ with $\|x\| > R$.

Finally, the case $\|x\| = R$ is direct from Proposition 1.2.7. Yang: To be revised. ☐

Theorem 3.3.2 (Strassman). Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$ be an analytic function. Suppose that $\|a_N\| r^N > \|a_n\| r^n$ for all $n > N$. Then f has at most N zeros in the closed ball $E(0,r)$.

Proof. We induct on N. The case $N = 0$ is direct from ??. Suppose that the conclusion holds for $N-1$. Let x be a zero of f in $E(0,r)$. Set

$$g(T) = \frac{f(T)-f(x)}{T-x} = \sum_{k=0}^{+\infty} \left(\sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n \geq k+1} \|a_n\| r^{n-1} \to 0 \quad \text{as } k \to \infty.$$

It follows that $g(T) \in \mathbf{k}\{T/r\}$.

For every $n > N$, we have

$$\|a_N\| > \|a_n\| r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\| \sum_{n=N}^{N+m} a_n x^{n-N} \right\| = \|a_N\|$$

for every $m \in \mathbb{N}$ by Proposition 1.2.3. Take $m \to +\infty$, we have $\|b_{N-1}\| = \|a_N\|$. For every $k > N-1$, we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n\| r^{n-1} \leq \max_{n > N} \|a_n\| r^{n-1} < \|a_N\| r^{N-1} = \|b_{N-1}\| r^{N-1}.$$

By the induction hypothesis, g has at most $N-1$ zeros in $E(0,r)$. It follows that f has at most N zeros in $E(0,r)$ since $f(T) = (T-x) \cdot g(T)$. □

Yang: Does the proof mean that $\mathbf{k}\{T\}$ with $v(f) := n$ such that $a_n = \max a_i$ and $a_n > a_m$ for all $m > n$ is an Euclidean ring?

Yang: There exist $f \in \mathbf{k}\{T\}$ with $f(a) \neq 0$ for all $|a| \leq 1$ but $1/f \notin \mathbf{k}\{T\}$. Yang: Is this right?

3.3.1 Entire functions

3.3.2 Maximum principle

3.4 Elementary functions

3.4.1 Exponential and logarithmic functions

Fix a prime number p in the following and consider $\mathbf{k}$ being a complete non-archimedean field with $|p| = p^{-1}$. Let $r_p := p^{-1/(p-1)}$.

Construction 3.4.1. The *exponential function* exp is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The *logarithmic function* log is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Proposition 3.4.2. We have the following properties:

(a) the exponential function exp converges on the open disk $B(0, r_p)$;

(b) the logarithmic function log converges on the open disk $B(1, 1)$;

(c) $|\exp(x) - 1| = |x|$ and $|\log(1+x)| = |x|$ for all $x \in B(0, r_p)$ or $x \in B(1, r_p)$ respectively;

(d) endow $B(0, r_p)$ with the group structure induced by addition in $\mathbf{k}$ and $B(1, r_p)$ with the group structure induced by multiplication in $\mathbf{k}$, then $\exp : B(0, r_p) \to B(1, r_p)$ is an isometric group isomorphism with inverse $\log : B(1, r_p) \to B(0, r_p)$.

Proof. For the convergent radius of exponential function, by Lemma 3.4.3, noting that

$$\liminf_{n \to +\infty} \frac{s_n}{n} = 0,$$

we have

$$\limsup_{n \to +\infty} |n!|_p^{-1/n} = \limsup_{n \to +\infty} p^{v_p(n!)/n} = p^{\limsup_{n \to +\infty}(1-(s_n/n))/(p-1)} = p^{1/(p-1)}.$$

That is, the convergent radius of the exponential function is $r_p = p^{-1/(p-1)}$. Considering $n = p^m$, we have

$$|p^m!|_p r_p^n = p^{(p^m-1)/(p-1)} \cdot p^{-p^m/(p-1)} = p^{-1/(p-1)} \neq 0.$$

Hence the convergent domain of the exponential function is $B(0, r_p)$.

For the logarithmic function, we have

$$\limsup_{n \to +\infty} |n|_p^{-1/n} = \limsup_{n \to +\infty} p^{v_p(n)/n} = p^0 = 1.$$

And $|1/(np+1)|_p = 1$ for all $n \in \mathbb{N}$. Thus, the convergent domain of the logarithmic function is $B(1, 1)$.

For $x \in B(0, r_p)$, we have

$$\left| \frac{x^{n-1}}{n!} \right|_p < r_p^{n-1} \cdot p^{v_p(n!)} = p^{v_p(n!)-(n-1)/(p-1)} \leq 1.$$

Hence $|x^n/n!|_p < |x|_p$ for all $n \geq 2$ and thus

$$|\exp(x) - 1|_p = \left| \sum_{n=1}^{+\infty} \frac{x^n}{n!} \right|_p = |x|_p.$$

For $x+1 \in B(1, r_p)$, setting $|x|_p = p^{-t}$ with $t \geq 1/(p-1)$, we have

$$\left|\frac{x^{n-1}}{n}\right|_p = p^{v_p(n)-t(n-1)} \leq p^{v_p(n!)-t(n-1)} \leq p^{(1/(p-1)-t)(n-1)} \leq 1, \quad \forall n \geq 2.$$

Similarly, we have $|x^n/n|_p < |x|_p$ and hence $|\log(1+x)|_p = |x|_p$.

The identities

$$\begin{aligned} \exp(X+Y) &= \exp(X) \cdot \exp(Y), \\ \log((1+X)(1+Y)) &= \log(1+X) + \log(1+Y), \\ \exp(\log(1+X)) &= 1+X, \\ \log(\exp(X)) &= X \end{aligned}$$

are purely formal and holds for indeterminates X and Y. Easy to check that $\exp(X+Y), \log(1+X)+\log(1+Y) \in \mathbf{k}\{X/r_p, Y/r_p\}$. Thus, the assertion (d) follows from (c) and Proposition 3.1.1. □

Recall the following useful lemma regarding the p-adic valuation of factorials.

Lemma 3.4.3. Let p be a prime number and $n \in \mathbb{N}$, write $n = \sum_{k=0}^{m} a_k p^k$ in the p-adic expansion and set $s_n := \sum_{k=0}^{m} a_k$. Then

$$v_p(n!) = \frac{n - s_n}{p-1}.$$

Proof. Yang: To be added. □

Corollary 3.4.4. Let $\mathbf{k}$ be a complete non-archimedean field with $|p| = p^{-1}$. The multiplication group

$$\mathbf{k}^\times \cong |\mathbf{k}^\times| \times \mathcal{k}_{\mathbf{k}}^\times \times \mathbf{k}^{\circ\circ}$$

where $\mathcal{k}_{\mathbf{k}}$ is the residue field of $\mathbf{k}$. Yang: To be revised.

Proof. Yang: To be added. □

Proposition 3.4.5. Suppose that $\mathbf{k} = \Bbbk$ is algebraically closed. The logarithmic function log defines a surjective group homomorphism $1 + \Bbbk^{\circ\circ} \to \Bbbk$ with kernel the group μ_{p^∞} of all p-power roots of unity. Yang: To be checked.

Proof. □

Yang: continuation of exponential and logarithmic

3.4.2 Mahler series

Notation 3.4.6. We use $\binom{x}{n}$ to denote the *binomial polynomial* defined by

$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}.$$

Definition 3.4.7. Fix a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbf{k}$. The *Mahler series* associated to $\{a_n\}$ is defined to be the formal series

$$f(x) := \sum_{n=0}^{+\infty} a_n \binom{x}{n}.$$

Yang: To be checked.

Proposition 3.4.8.

Theorem 3.4.9. The series converges.

Chapter 4

Berkovich spaces

4.1 Spectrum

Let $\mathbf{k}$ be a spherically complete non-archimedean field which is algebraically closed and $A = \mathbf{k}[T]$. We want to consider the "analytic structure" on $\mathsf{mSpec}\,A$. However, unlike the complex case, the set $\mathsf{mSpec}\,A$ is totally disconnected with respect to the topology induced by the absolute value on $\mathbf{k}$ (Corollary 1.2.6). To overcome this difficulty, Berkovich uses multiplicative semi-norms to "fill in the gaps" between the points in $\mathsf{mSpec}\,A$, leading to the notion of the spectrum of a Banach ring.

4.1.1 Definition

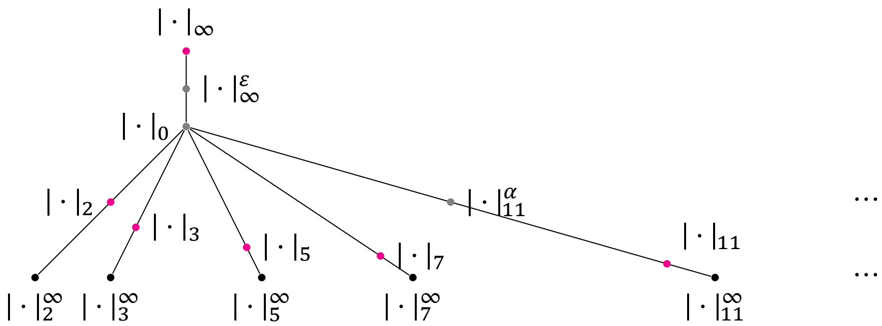
Definition 4.1.1. Let R be a Banach ring. The *Berkovich spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$.
We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_{\geq 0}$, defined by $x \mapsto |f|_x =: f(x)$, is continuous.

Example 4.1.2. Let $(\mathbf{k}, |\cdot|)$ be a complete valuation field. The Berkovich spectrum $\mathscr{M}(\mathbf{k})$ consists of a single point corresponding to the given absolute value $|\cdot|$ on $\mathbf{k}$.

Example 4.1.3. Consider the Banach ring $(\mathbb{Z}, \|\cdot\|)$ with $\|\cdot\| = |\cdot|_\infty$ is the usual absolute value norm on $\mathbb{Z}$. Let $|\cdot|_p$ denote the p-adic norm for each prime number p, i.e., $|n|_p = p^{-v_p(n)}$ for each $n \in \mathbb{Z}$, where $v_p(n)$ is the p-adic valuation of n. The Berkovich spectrum

$$\mathscr{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0,1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0,\infty]\} \cup \{|\cdot|_0\},$$

where $|a|_p^\infty := \lim_{\alpha\to\infty} |a|_p^\alpha$ for each $a \in \mathbb{Z}$ and $|\cdot|_0$ is the trivial norm on $\mathbb{Z}$.



Yang: To be continued.

Theorem 4.1.4. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Note that the pullback of residue norm on the residue field $R/\mathfrak{m}$ is bounded with respect to the given norm on R. Replacing R by the completion of $R/\mathfrak{m}$, we may assume that R is a complete field. Consider the set

$$\Sigma = \{\text{norm on } R \text{ bounded by the given norm } \|\cdot\|\}$$

with the partial order defined by boundedness. Since for a descending chain in Σ, the infimum is a norm, by Zorn's lemma, there exists a minimal element $|\cdot| \in \Sigma$.

We claim that $|\cdot|$ is multiplicative. Since the spectral radius $\rho(f) = \lim_{n\to\infty} |f^n|^{1/n}$ associated to $|\cdot|$ is power-multiplicative and bounded by $|\cdot|$, by minimality of $|\cdot|$, we have $\rho(f) = |f|$ for each $f \in R$. Thus $|\cdot|$ is power-multiplicative. If $|\cdot|$ is not multiplicative, then there exist $a, b \in R \setminus \{0\}$ such that $|ab| < |a||b|$. Then $|b| \leq |a^{-1}||ab| < |a^{-1}||a||b|$, which implies that $|a||a^{-1}| > 1$. Set $r = |a|^{-1} < |a^{-1}|$ and consider $R\langle T/r\rangle$. Since $r \cdot |a| = 1$, we have that

$$\left|\sum_{n=0}^{\infty} a^n T^n\right| = \sum_{n=0}^{\infty} |a^n| r^n = \sum_{n=0}^{\infty} |a|^n r^n = \sum_{n=0}^{\infty} 1 = \infty.$$

The power series is not convergent in $R\langle T/r\rangle$ and hence $1 - aT$ is not invertible in $R\langle T/r\rangle$. Let $\mathfrak{n}$ be a maximal ideal of $R\langle T/r\rangle$ containing $1 - aT$. Consider $R \to R\langle T/r\rangle \to R\langle T/r\rangle/\mathfrak{n}$. Since R is a field, the composition is injective. The residue norm on $R\langle T/r\rangle/\mathfrak{n}$ induces a norm $|\cdot|'$ on R bounded by $|\cdot|$. Note that $|a^{-1}|' \leq |T| = r = |a|^{-1} < |a^{-1}|$, contradicting the minimality of $|\cdot|$. □

Definition 4.1.5. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $\mathscr{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$ for each $x \in \mathscr{M}(S)$.

Note that $\mathscr{M}(\varphi)(f^{-1}(V)) = \varphi(f)^{-1}(V)$ for each $f \in R$ and open subset $V \subset \mathbb{R}_{\geq 0}$. Hence the pullback map is continuous.

Notation 4.1.6. Let R be a Banach ring and $x \in \mathscr{M}(R)$. We denote by $|\cdot|_x$ the multiplicative semi-norm on R corresponding to the point x. Its kernel $\{f \in R : |f|_x = 0\}$ is a closed prime ideal of R, denoted by $\wp_x$.

Definition 4.1.7. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, the *completed residue field* at the point x is defined as the completion of the residue field $\kappa(x) = \mathrm{Frac}(R/\wp_x)$ with respect to the multiplicative norm induced by the semi-norm $|\cdot|_x$, denoted by $\mathscr{H}(x)$.

Example 4.1.8. Consider the Banach ring $(\mathbb{Z}, |\cdot|_\infty)$ as in Example 4.1.3. We have

- $x = |\cdot|_\infty^\varepsilon$ for some $\varepsilon \in (0,1]$: $\wp_x = (0)$ and $\mathscr{H}(x) \cong \mathbb{R}$ with the absolute value norm raised to the power ε;
- $x = |\cdot|_0$: $\wp_x = (0)$ and $\mathscr{H}(x) \cong \mathbb{Q}$ with the trivial norm;
- $x = |\cdot|_p^\alpha$ for some prime number p and $\alpha \in (0,\infty)$: $\wp_x = (0)$ and $\mathscr{H}(x) \cong \mathbb{Q}_p$ with the p-adic norm raised to the power α;
- $x = |\cdot|_p^\infty$ for some prime number p: $\wp_x = (p)$ and $\mathscr{H}(x) \cong \mathbb{F}_p$ with the trivial norm.

Definition 4.1.9. Let R be a Banach ring. The *Gel'fand transform* of R is the bounded ring homomorphism

$$\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x), \quad f \mapsto (f(x))_{x \in \mathscr{M}(R)},$$

where the norm on the product $\prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is given by the supremum norm.

Proposition 4.1.10. The Gel'fand transform $\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ of a Banach ring R factors through the uniformization R^u of R, and the induced map $R^u \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is an isometric embedding. Yang: To be checked.

Proof. Yang: To be added. □

Lemma 4.1.11. Let $\{K_i\}_{i \in I}$ be a family of completed fields. Consider the Banach ring $R = \prod_{i \in I} K_i$ equipped with the product norm. The spectrum $\mathscr{M}(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I.

Proof. Yang: To be added. □

Remark 4.1.12. The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

Theorem 4.1.13. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a compact Hausdorff space.

Proof. Yang: To be added. □

Proposition 4.1.14. Let K/k be a Galois extension of complete fields, and let R be a Banach k-algebra. The Galois group $\mathsf{Gal}(K/k)$ acts on the spectrum $\mathscr{M}(R \widehat{\otimes}_k K)$ via

$$g \cdot x : f \mapsto |(1 \otimes g^{-1})(f)|_x$$

for each $g \in \mathsf{Gal}(K/k)$, $x \in \mathscr{M}(R \widehat{\otimes}_k K)$ and $f \in R \widehat{\otimes}_k K$. Moreover, the natural map $\mathscr{M}(R \widehat{\otimes}_k K) \to \mathscr{M}(R)$ induces a homeomorphism

$$\mathscr{M}(R \widehat{\otimes}_k K) / \mathsf{Gal}(K/k) \xrightarrow{\sim} \mathscr{M}(R).$$

Yang: To be checked.

Proof. Yang: To be added. □

4.1.2 Reduction map and kernel map

Proposition 4.1.15. Let R be a Banach ring. The kernel map $\mathscr{M}(R) \to \mathsf{Spec}(R), x \mapsto \wp_x$ is continuous with respect to the Zariski topology on $\mathsf{Spec}(R)$.

Proof. Let $D(f) = \{f \neq 0\} \subset \mathsf{Spec}(R)$ be a principal open subset for some $f \in R$. The preimage of $D(f)$ under the kernel map is just the set $\{x \in \mathscr{M}(R) : |f|_x > 0\} = f^{-1}(\mathbb{R}_{>0})$, which is open in $\mathscr{M}(R)$ by definition of the topology on $\mathscr{M}(R)$. $\square$

Example 4.1.16. Let us consider the spectrum $\mathscr{M}(\mathbb{Z})$ in Example 4.1.3. Under the kernel map $\mathscr{M}(\mathbb{Z}) \to \mathsf{Spec}(\mathbb{Z})$, the points $|\cdot|_p^\infty$ for each prime number p are mapped to the prime ideal (p), the other above points are all mapped to the zero ideal (0).

Yang: Is this surjective? what is its fiber?

Proposition 4.1.17. Yang: To be added.

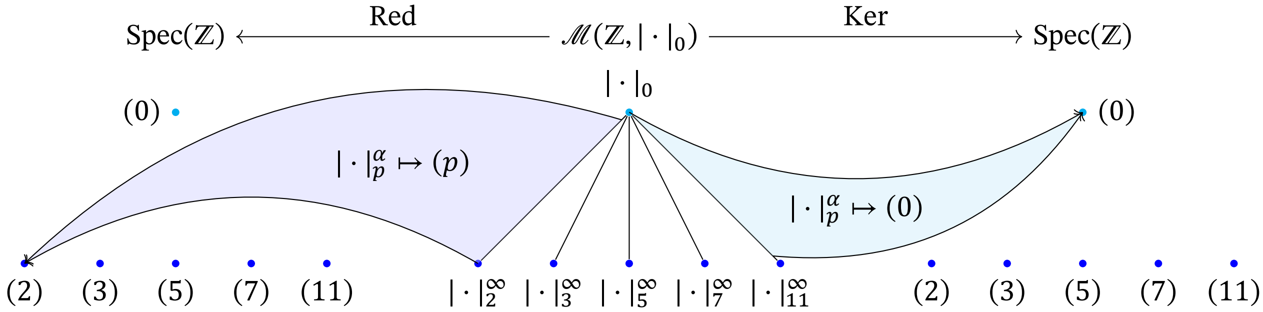
Construction 4.1.18. Suppose that R is a non-archimedean Banach ring with valuation subring R° and maximal ideal $R^{\circ\circ}$. For each $x \in \mathscr{M}(R)$, there is an induced homomorphism $R^\circ \to \mathscr{H}(x)^\circ$ between the valuation subrings. Furthermore, we have an induced homomorphism between the residue rings $\widetilde{R} = R^\circ/R^{\circ\circ} \to \kappa_{\mathscr{H}(x)}$. This gives rise to the *reduction map*

$$\mathsf{Red} : \mathscr{M}(R) \to \mathsf{Spec}(\widetilde{R}), \quad x \mapsto \ker(\widetilde{R} \to \kappa_{\mathscr{H}(x)}).$$

Example 4.1.19. Let $(\mathbb{Z}, |\cdot|_0)$ be the Banach ring with the trivial norm. The reduction ring is $\widetilde{\mathbb{Z}} = \mathbb{Z}$.

- $x = |\cdot|_p^\alpha$ for some prime number p and $\alpha \in (0,\infty]$: $\kappa_{\mathscr{H}(x)} \cong \mathbb{F}_p$ and the induced homomorphism $\widetilde{\mathbb{Z}} = \mathbb{Z} \to \kappa_{\mathscr{H}(x)} = \mathbb{F}_p$ is the natural projection $\mathbb{Z} \to \mathbb{F}_p$;
- $x = |\cdot|_0$: $\kappa_{\mathscr{H}(x)} \cong \mathbb{Q}$ and the induced homomorphism $\widetilde{\mathbb{Z}} = \mathbb{Z} \to \kappa_{\mathscr{H}(x)} = \mathbb{Q}$ is the natural inclusion $\mathbb{Z} \to \mathbb{Q}$.

The following diagram illustrates the reduction map and the kernel map for the spectrum $\mathscr{M}(\mathbb{Z}, |\cdot|_0)$:



Proposition 4.1.20. Let R be a non-archimedean Banach ring and $\widetilde{U} \subset \mathsf{Spec}(\widetilde{R})$ be a Zariski open subset. Then the preimage $\mathsf{Red}^{-1}(\widetilde{U})$ is a closed subset of $\mathscr{M}(R)$.

Proof. Yang: To be completed. $\square$

4.1.3 Spectrum of Tate algebras

Spectrum of Tate algebra in one variable Let $\Bbbk$ be an algebraically closed complete non-archimedean field, and let $A = \Bbbk\{T/r\}$. We list some types of points in the spectrum $\mathscr{M}(A)$.

Construction 4.1.21. For each $a \in \Bbbk$ with $|a| \leq r$, we have the *type I* point x_a corresponding to the evaluation at a, i.e., $|f|_{x_a} := |f(a)|$ for each $f \in A$.

For each closed disk $E = E(a,s) := \{b \in \Bbbk : |b-a| \leq s\}$ with center $a \in \Bbbk$ and radius $s \leq r$, we have the point $x_E = x_{a,s}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} = |f|_{x_{a,s}} := \sup_{b \in E(a,s)} |f(b)|$$

for each $f \in A$. If $s \in |\Bbbk^\times|$, then the point x_E is called a *type II* point; otherwise, it is called a *type III* point.

Let $E_n = E(a_n, s_n)$ be a sequence of closed disks in $\Bbbk$ such that $E_{n+1} \subsetneq E_n$ and $\bigcap_n E_n = \varnothing$. Then we have the point $x_{\{E_n\}} = x_{\{a_n,s_n\}}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E_n\}}} = |f|_{x_{\{a_n,s_n\}}} := \inf_n |f|_{x_{E_n}}$$

for each $f \in A$. Such a point is called a *type IV* point. Yang: To be completed. Check the definition of type IV points.

Proposition 4.1.22. The points in the spectrum $\mathscr{M}(\Bbbk\{r^{-1}T\})$ can be classified into four types as described above.

Proof. Fix $x \in \mathscr{M}(\Bbbk\{r^{-1}T\})$, set

$$s = \inf_{a \in \Bbbk} |T-a|_x \leq r, \quad E = \{a \in \Bbbk : |T-a|_x = s\} \subset E(0,r).$$

Case 1. $E \neq \varnothing$ and $s = 0$.

By assumption, there exists $a \in E$ such that $|T-a|_x = 0$. Note that if $f(a) = 0$, then $T - a \mid f$ in $\Bbbk\{r^{-1}T\}$ and hence $|f|_x = |T-a|_x |g|_x = 0$. Then we have

$$|f(a)| = ||f(a)| - |f(a) - f|_x| \leq |f|_x \leq |f(a)| + |f(a) - f|_x = |f(a)|$$

for each $f \in \Bbbk\{r^{-1}T\}$, which implies that $|f|_x = |f(a)|$. Thus x is a type I point x_a.

Case 2. $E \neq \varnothing$ and $s > 0$.

Let $a \in E$. Note that for every $b \in \Bbbk$, we have

$$|a-b| \leq \max\{|T-a|_x, |T-b|_x\} = |T-b|_x.$$

First we show that $E = E(a,s)$. For each $b \in E(a,s)$, we have $|T-b|_x \leq \max\{|T-a|_x, |a-b|\} = s$, which implies that $b \in E$. Conversely, for each $b \in E$, we have $|a-b| \leq \max\{|T-a|_x, |T-b|_x\} = s$.

Let $f \in \Bbbk[T]$ be a polynomial. Write $f = \prod_{i=1}^n (T - c_i)$ for some $c_1, \dots, c_n \in \Bbbk$. Then we have

$$|f|_x = \prod_{i=1}^n |T-c_i|_x \geq \prod_{i=1}^n |b-c_i| = |f(b)|, \quad \forall b \in E.$$

I claim that for every $\varepsilon \in (0,1)$, there exists $b \in E$ such that $\varepsilon|T-c_i|_x < |b-c_i|$ for each $i = 1, \dots, n$. Indeed, if $c_i \notin E$, then $|T-c_i|_x = |b-c_i|$ for each $b \in E$. Hence we only need to consider the case when $c_i \in E$. Since $\Bbbk$ is algebraically closed, $E(a,s) \setminus \bigcup_{i=1}^n E(c_i, \varepsilon s) \neq \varnothing$. Choose b in the set. Then we have

$$|f(b)| \geq \prod_{i=1}^n \varepsilon |T - c_i|_x = \varepsilon^n |f|_x.$$

Thus $|f|_x = \sup_{b \in E} |f(b)|$ for each polynomial $f \in \Bbbk[T]$. Since polynomials are dense in $\Bbbk\{r^{-1}T\}$, we have $|f|_x = \sup_{b \in E} |f(b)|$ for each $f \in \Bbbk\{r^{-1}T\}$. Therefore, x is the point $x_E = x_{a,s}$, which is of type II or type III depending on whether $s \in |\Bbbk^\times|$ or not.

Case 3. $E = \varnothing$.

Set $E_n = \{a \in \Bbbk : |T - a|_x \leq s + 1/n\}$ and $a_n \in E_n$ for each $n \in \mathbb{N}$. By the similar argument as in Case 2, we have $E_n = E(a_n, s + 1/n)$. Note that E_n is a decreasing sequence of closed disks with $\bigcap_n E_n = E = \varnothing$.

For $c \in \Bbbk$, there exists N such that $\forall n \geq N$, we have

$$c \notin E_n \implies |T - c|_x > |T - a_n|_x \implies |T - c|_x = |a_n - c|.$$

Thus

$$\inf_n |T - c|_{E_n} = \inf_n |a_n - c| = |T - c|_x.$$

By multiplicativity, we have $\inf_n |f|_{E_n} = |f|_x$ for each polynomial $f \in \Bbbk[T]$. And then by density of polynomials, the equality holds for each $f \in \Bbbk\{r^{-1}T\}$. Therefore, $x = x_{\{E_n\}} = x_{\{a_n, s_n\}}$ is of type IV. $\square$

Proposition 4.1.23. The completed residue fields of the four types of points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ are described as follows:

- type I point x_a: $\mathscr{H}(x_a)$ is isomorphic to $\Bbbk$;
- type II point $x_{a,s}$: $\mathscr{H}(x_{a,s}) \cong \tilde{k}_{\Bbbk}((t))$;
- type III point $x_{a,s}$: $\tilde{k}_{\mathscr{H}(x_{a,s})} \cong \tilde{k}_{\Bbbk}$ and the value group $|\mathscr{H}(x_{a,s})^\times|$ is generated by $|\Bbbk^\times|$ and s;
- type IV point $x_{\{a_n, s_n\}}$: $\mathscr{H}(x_{\{a_n,s_n\}})$ is an immediate extension of $\Bbbk$.

Yang: To be checked.

Proof. Yang: To be completed. $\square$

Example 4.1.24. The completed residue field $\mathscr{H}(x_a)$ for a type I point x_a with $a \in \Bbbk$ and $|a| \leq r$ is isomorphic to $\Bbbk$. Yang: To be complete.

Example 4.1.25. Let $\Bbbk$ be a complete algebraically closed non-archimedean field and $A = \Bbbk\{T/r\}$. We have $\widetilde{A} \cong \tilde{k}_{\Bbbk}[T]$. For a point $x_a \in \mathscr{M}(A)$ of type I corresponding to $a \in \Bbbk$ with $|a| \leq r$ (see Construction 4.1.21), the induced homomorphism $\widetilde{A} = \tilde{k}_{\Bbbk}[T] \to \tilde{k}_{\mathscr{H}(x_a)} = \tilde{k}_{\Bbbk}$ is given by $T \mapsto a$ mod $\Bbbk^{\circ\circ}$.

Yang: To be continued.

Spectrum of Tate algebra in servel variables Let $\Bbbk$ be a complete non-archimedean field, and let $A = \Bbbk\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$. We can consider the spectrum $\mathscr{M}(A)$ similarly.

4.2 Affinoid domains

Consider $X = \mathscr{M}(A)$ with $A = \mathbf{k}\{T_1, \ldots, T_n\}$. Yang: Not every open subset of X gives an affinoid space, that is, the completion of the ring of analytic functions on that open subset is not necessarily an affinoid algebra. Yang: Right? example?

4.2.1 Definition

Definition 4.2.1. Let A be a $\mathbf{k}$-affinoid algebra, and let $X = \mathscr{M}(A)$ be the associated affinoid space. A closed subset $V \subseteq X$ is called an *affinoid domain* if there exists a $\mathbf{k}$-affinoid algebra A_V and a morphism of $\mathbf{k}$-affinoid algebras $\varphi : A \to A_V$ satisfying the following universal property: for every bounded homomorphism of $\mathbf{k}$-affinoid algebras $\psi : A \to B$ such that the induced map on spectra $\mathscr{M}(\psi) : \mathscr{M}(B) \to X$ has its image contained in V, there exists a unique bounded homomorphism $\theta : A_V \to B$ such that the following diagram commutes:

$$\begin{tikzcd} & A_V \arrow[dr, "\theta"] & \\ A \arrow[ur, "\varphi"] \arrow[rr, "\psi"'] & & B \end{tikzcd}$$

In this case, we say that V is represented by the affinoid algebra A_V.

Slogan *A closed subset $V \subset X$ is an affinoid domain if the functor "$\mathrm{Mor}(-, V)$" is representable.*

Yang: Why we consider closed subset rather that open subset?

Construction 4.2.2. Let $f = (f_1, \ldots, f_n)$ be a tuple of elements in A and $r = (r_1, \ldots, r_n)$ be a tuple of positive real numbers. Consider the closed subset of X:

$$X\left(\underline{f/r}\right) := \{x \in X : |f_i(x)| \le r_i, 1 \le i \le n\}.$$

Such a closed subset is called a *Weierstrass domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\{\underline{f/r}\right\} := A\{f_1/r_1, \ldots, f_n/r_n\}.$$

Yang: The domain $X\left(\underline{f/r}\right)$ is represented by $A\left\{\underline{f/r}\right\}$.

Construction 4.2.3. Let $f = (f_1, \ldots, f_n), g = (g_1, \ldots, g_m)$ be two tuples of elements in A and $r = (r_1, \ldots, r_n), s = (s_1, \ldots, s_m)$ be two tuples of positive real numbers. Consider the following closed subset of X:

$$X\left(\underline{f/r}; \underline{g/s}^{-1}\right) := \left\{x \in X : |f_i(x)| \le r_i, |g_j(x)| \ge s_j, 1 \le i \le n, 1 \le j \le m\right\}.$$

Such a closed subset is called a *Laurent domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\{\underline{f/r}; \underline{g/s}^{-1}\right\} := A\left\{f_1/r_1, \ldots, f_n/r_n, g_1^{-1}/s_1, \ldots, g_m^{-1}/s_m\right\}.$$

Yang: The domain $X\left(\underline{f/r}; \underline{g/s}^{-1}\right)$ is represented by $A\left\{\underline{f/r}; \underline{g/s}^{-1}\right\}$.

Construction 4.2.4. Let $f = (f_1, \dots, f_n), g$ be elements in A such that the ideal generated by them is the whole algebra A. Set $p = (p_1, \dots, p_n)$ be a tuple of positive real numbers. We define the following closed subset of X:

$$X\left(\underline{f/p}, g\right) := \{x \in X : |f_i(x)| \le p_i |g(x)|, 1 \le i \le n\}.$$

Such a closed subset is called a *rational domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\langle \underline{f/p}, g^{-1}\right\rangle := A\left\langle \frac{f_1}{p_1 g}, \dots, \frac{f_n}{p_n g}\right\rangle,$$

which is the quotient of the Tate algebra

$$A\langle T_1, \dots, T_n\rangle$$

by the ideal generated by the elements $p_i g T_i - f_i$ for $1 \le i \le n$. There is a natural bounded homomorphism $\varphi : A \to A\langle \underline{f/p}, g^{-1}\rangle$ induced by the inclusion. It can be shown that the closed subset $X(\underline{f/p}, g)$ is an affinoid domain represented by the affinoid algebra $A\langle \underline{f/p}, g^{-1}\rangle$. Yang: To be checked

Yang: We have a sequence of inclusion:

$$\{\text{Weierstrass domains}\} \subseteq \{\text{Laurent domains}\} \subseteq \{\text{Rational domains}\} \subseteq \{\text{Affinoid domains}\}.$$

Proposition 4.2.5. Let A be a $\mathbf{k}$-affinoid algebra, and let $X = \mathscr{M}(A)$ be the associated affinoid space. Let $V \subseteq X$ be an affinoid domain represented by the $\mathbf{k}$-affinoid algebra A_V. Then the natural bounded homomorphism $\varphi : A \to A_V$ is flat.
We have $\mathscr{M}(A_V) \cong V$.

4.2.2 The Grothendieck topology of affinoid domains

Chapter 5

Arakelov geometry

5.1

5.2

5.3

References

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