Spectrum

Let $\mathbf{k}$ be a spherically complete non-archimedean field which is algebraically closed and $A = \mathbf{k}[T]$. We want to consider the "analytic structure" on $\mathrm{mSpec}\,A$. However, unlike the complex case, the set $\mathrm{mSpec}\,A$ is totally disconnected with respect to the topology induced by the absolute value on $\mathbf{k}$ (??). To overcome this difficulty, Berkovich uses multiplicative semi-norms to "fill in the gaps" between the points in $\mathrm{mSpec}\,A$, leading to the notion of the spectrum of a Banach ring.

We first consider the local model. Hence we should consider the Tate algebra $\mathbf{k}\{T\}$ instead of the polynomial ring $\mathbf{k}[T]$. Yang: The maximal ideal of $\mathbf{k}\{T\}$ corresponding to the point in the disk $\{a \in \mathbf{k} : a \leq 1\}$. Yang: Closed or open disk?

1 Definition

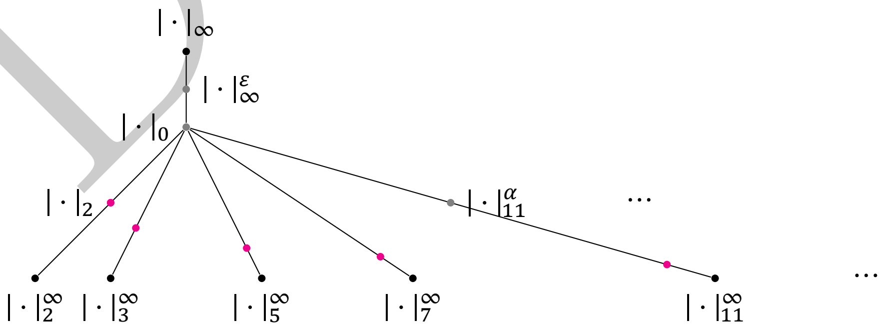
Definition 1. Let R be a Banach ring. The *spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$.
We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_{\geq 0}$, defined by $x \mapsto |f|_x =: f(x)$, is continuous.

Example 2. Let $(\mathbf{k}, |\cdot|)$ be a complete valuation field. The spectrum $\mathscr{M}(\mathbf{k})$ consists of a single point corresponding to the given absolute value $|\cdot|$ on $\mathbf{k}$.

Example 3. Consider the Banach ring $(\mathbb{Z}, \|\cdot\|)$ with $\|\cdot\| = |\cdot|_\infty$ is the usual absolute value norm on $\mathbb{Z}$. Let $|\cdot|_p$ denote the p-adic norm for each prime number p, i.e., $|n|_p = p^{-v_p(n)}$ for each $n \in \mathbb{Z}$, where $v_p(n)$ is the p-adic valuation of n. The spectrum

$$\mathscr{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0,1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0,\infty]\} \cup \{|\cdot|_0\},$$

where $|a|_p^\infty := \lim_{\alpha\to\infty} |a|_p^\alpha$ for each $a \in \mathbb{Z}$ and $|\cdot|_0$ is the trivial norm on $\mathbb{Z}$.



Yang: To be checked.

Definition 4. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $\mathscr{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$ for each $x \in \mathscr{M}(S)$.

Date: January 16, 2026, **Author**: Tianle Yang, My Homepage

Proposition 5. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The pullback map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is continuous.

Proof. Yang: To be completed. □

Definition 6. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, the *completed residue field* at the point x is defined as the completion of the residue field $\kappa(x) = \mathrm{Frac}(R/\wp_x)$ with respect to the multiplicative norm induced by the semi-norm $|\cdot|_x$, denoted by $\mathscr{H}(x)$.

Definition 7. Let R be a Banach ring. The *Gel'fand transform* of R is the bounded ring homomorphism

$$\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x), \quad f \mapsto (f(x))_{x \in \mathscr{M}(R)},$$

where the norm on the product $\prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is given by the supremum norm.

Proposition 8. The Gel'fand transform $\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ of a Banach ring R factors through the uniformization R^u of R, and the induced map $R^u \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is an isometric embedding. Yang: To be checked.

Theorem 9. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

Proof. Yang: To be continued. □

Lemma 10. Let $\{K_i\}_{i \in I}$ be a family of completed fields. Consider the Banach ring $R = \prod_{i \in I} K_i$ equipped with the product norm. The spectrum $\mathscr{M}(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I.

Remark 11. The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

Theorem 12. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a compact Hausdorff space.

Proof. Yang: To be added. □

Proposition 13. Let K/k be a Galois extension of complete fields, and let R be a Banach k-algebra. The Galois group $\mathrm{Gal}(K/k)$ acts on the spectrum $\mathscr{M}(R \widehat{\otimes}_k K)$ via

$$g \cdot x : f \mapsto |(1 \otimes g^{-1})(f)|_x$$

for each $g \in \mathrm{Gal}(K/k)$, $x \in \mathscr{M}(R \widehat{\otimes}_k K)$ and $f \in R \widehat{\otimes}_k K$. Moreover, the natural map $\mathscr{M}(R \widehat{\otimes}_k K) \to \mathscr{M}(R)$ induces a homeomorphism

$$\mathscr{M}(R \widehat{\otimes}_k K)/\,\mathrm{Gal}(K/k) \xrightarrow{\sim} \mathscr{M}(R).$$

Yang: To be checked.

2 Reduction map and kernel map

Lemma 14. Let R be a ring. For every $\mathfrak{p} \in \operatorname{Spec} R$, we endow the residue field $\kappa(\mathfrak{p}) = \operatorname{Frac}(R/\mathfrak{p})$ with the trivial norm. Then the Zariski topology on $\operatorname{Spec} R$ is the weakest topology such that for each $f \in R$, the evaluation map $\operatorname{Spec} R \to \mathbb{R}$, defined by $\mathfrak{p} \mapsto |f|_{\kappa(\mathfrak{p})}$, is continuous.

Proof. Yang: To be completed. □

Proposition 15. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, let $\wp_x$ be the kernel of the multiplicative semi-norm $|\cdot|_x$. Then $\wp_x$ is a closed prime ideal of R, and $x \mapsto \wp_x$ defines a continuous map from $\mathscr{M}(R)$ to $\operatorname{Spec}(R)$ equipped with the Zariski topology.

Proof. Yang: To be completed □

Example 16. Let us consider the spectrum $\mathscr{M}(\mathbb{Z})$ in Example 3. Under the kernel map $\mathscr{M}(\mathbb{Z}) \to \operatorname{Spec}(\mathbb{Z})$, the points $|\cdot|_p^\infty$ for each prime number p are mapped to the prime ideal (p), the other above points are all mapped to the zero ideal (0).

Construction 17. Suppose that R is a non-archimedean Banach ring with valuation subring R° and maximal ideal $R^{\circ\circ}$. For each $x \in \mathscr{M}(R)$, there is an induced homomorphism $R^\circ \to \mathscr{H}(x)^\circ$ between the valuation subrings. Furthermore, we have an induced homomorphism between the residue rings $\widetilde{R} = R^\circ/R^{\circ\circ} \to \mathscr{k}_{\mathscr{H}(x)}$. This gives rise to the reduction map from $\mathscr{M}(R)$ to $\operatorname{Spec}(\widetilde{R})$.

There is a natural reduction map from the spectrum $\mathscr{M}(R)$ to the spectrum $\operatorname{Spec}(\widetilde{R} = R^\circ/R^{\circ\circ})$ of the residue ring $\widetilde{R}$.

Yang: To be continued.

3 Spectrum of Tate algebras

Spectrum of Tate algebra in one variable Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{T/r\}$. We list some types of points in the spectrum $\mathscr{M}(A)$.

Construction 18. For each $a \in \mathbf{k}$ with $|a| \le r$, we have the *type I* point x_a corresponding to the evaluation at a, i.e., $|f|_{x_a} := |f(a)|$ for each $f \in A$.

For each closed disk $E = E(a,s) := \{b \in \mathbf{k} : |b-a| \le s\}$ with center $a \in \mathbf{k}$ and radius $s \le r$, we have the point $x_{a,s}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} := \sup_{b \in E(a,s)} |f(b)|$$

for each $f \in A$. If $s \in |\mathbf{k}^\times|$, then the point x_E is called a *type II* point; otherwise, it is called a *type III* point.

Let $\{E^{(s)}\}_s$ be a family of closed disks in $\mathbf{k}$ such that $E^{(s)}$ is of radius s, $E^{(s_1)} \subsetneq E^{(s_2)}$ for any $s_1 < s_2$ and $\bigcap_s E^{(s)} = \varnothing$. Then we have the point $x_{\{E^{(s)}\}}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E^{(s)}\}}} := \inf_s |f|_{x_{E^{(s)}}}$$

for each $f \in A$. Such a point is called a *type IV* point.

Yang: To be completed.

Proposition 19. The points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ can be classified into four types as described above. Yang: To be checked

Proof. Yang: To be completed. □

Proposition 20. The completed residue fields of the four types of points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ are described as follows:

- type I point x_a: $\mathscr{H}(x_a)$ is isomorphic to $\mathbf{k}$;
- type II point $x_{a,s}$: $\mathscr{H}(x_{a,s}) \cong \mathscr{k}_{\mathbf{k}}((t))$;
- type III point $x_{a,s}$: $\mathscr{k}_{\mathscr{H}(x_{a,s})} \cong \mathscr{k}_{\mathbf{k}}$ and the value group $|\mathscr{H}(x_{a,s})^{\times}|$ is generated by $|\mathbf{k}^{\times}|$ and s;
- type IV point $x_{\{E(s)\}}$: $\mathscr{H}(x_{\{E(s)\}})$ is an immediate extension of $\mathbf{k}$.

Yang: To be checked.

Example 21. The completed residue field $\mathscr{H}(x_a)$ for a type I point x_a with $a \in \mathbf{k}$ and $|a| \leq r$ is isomorphic to $\mathbf{k}$. Yang: To be complete.

Spectrum of Tate algebra in servel variables Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$. We can consider the spectrum $\mathscr{M}(A)$ similarly.

Appendix