Hermitian line bundles

Let K be a number field with ring of integers $\mathcal{O}_K$. Let $\mathcal{X}$ be an arithmetic variety over $\operatorname{Spec}\mathcal{O}_K$ of relative dimension n. Yang: flat,separated,of finite type,

1 Definitions and examples

Definition 1. A *hermitian line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, \{\|\cdot\|_\sigma\}_{\sigma\in M_K^\infty})$ on $\mathcal{X}$ consists of

- a line bundle $\mathcal{L}$ on $\mathcal{X}$;
- for each embedding $\sigma : K \hookrightarrow \mathbb{C}$ (equivalently, each infinite place $\sigma \in M_K^\infty$), a smooth hermitian metric $\|\cdot\|_\sigma$ on the complex line bundle $\mathcal{L}_\sigma(\mathbb{C})$ over $\mathcal{X}_\sigma(\mathbb{C})$, which is invariant under the complex conjugation map.

Yang: To be checked.

Definition 2. Let $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ be hermitian line bundles on $\mathcal{X}$.

- The *tensor product* $\overline{\mathcal{L}}_1 \otimes \overline{\mathcal{L}}_2$ is defined as the line bundle $\mathcal{L}_1 \otimes \mathcal{L}_2$ on $\mathcal{X}$ equipped with the metrics $\|s_1 \otimes s_2\|_\sigma = \|s_1\|_{\sigma,1} \cdot \|s_2\|_{\sigma,2}$ on $(\mathcal{L}_1 \otimes \mathcal{L}_2)_\sigma(\mathbb{C})$
- The *dual* $\overline{\mathcal{L}}_1^\vee$ is defined as the dual line bundle $\mathcal{L}_1^\vee$ on $\mathcal{X}$ equipped with the dual metrics $\|\cdot\|_{\sigma,1}^\vee$ on $(\mathcal{L}_1^\vee)_\sigma(\mathbb{C})$ for each $\sigma \in M_K^\infty$.

Yang: To be checked.

Definition 3. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of arithmetic varieties over $\operatorname{Spec}\mathcal{O}_K$. Let $\overline{\mathcal{L}} = (\mathcal{L}, \{\|\cdot\|_\sigma\}_{\sigma\in M_K^\infty})$ be a hermitian line bundle on $\mathcal{X}$. The *pullback* $f^*\overline{\mathcal{L}}$ is defined as the line bundle $f^*\mathcal{L}$ on $\mathcal{Y}$ equipped with the metrics $\|s\|'_\sigma = \|f_\sigma^* s\|_\sigma$ on $(f^*\mathcal{L})_\sigma(\mathbb{C})$ for each $\sigma \in M_K^\infty$, where $f_\sigma : \mathcal{Y}_\sigma(\mathbb{C}) \to \mathcal{X}_\sigma(\mathbb{C})$ is the induced morphism of complex manifolds. Yang: To be checked.

Definition 4. Let $\overline{\mathcal{L}} = (\mathcal{L}, \{\|\cdot\|_\sigma\}_{\sigma\in M_K^\infty})$ be a hermitian line bundle on $\mathcal{X}$. The set

$$H^0(\mathcal{X}, \overline{\mathcal{L}}) := \{s \in H^0(\mathcal{X}, \mathcal{L}) : \|s\|_\sigma \leq 1 \text{ for all } \sigma \in M_K^\infty\}.$$

Yang: To be checked.

Proposition 5. The set $H^0(\mathcal{X}, \overline{\mathcal{L}})$ is finite.

Appendix