Valuations on fields

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Contents

1 Valuations

Let $\mathbf{k}$ be a field. Usually, we consider $\mathbf{k}$ to be a number field or a function field.

1.1 Definition

Definition 1.1. An *absolute value* or a *valuation* on $\mathbf{k}$ is a function $|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

- $|x| = 0$ if and only if $x = 0$;
- $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbf{k}$;
- $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbf{k}$.

Remark 1.2. Recall that a *additive valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the multiplicative valuation (i.e., absolute value).

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Example 1.3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 1.4. An absolute value $|\cdot|$ on a field $\mathbf{k}$ is called *non-archimedean* if it satisfies the strong triangle inequality

$$|x+y| \leq \max\{|x|, |y|\} \quad \text{for all } x, y \in \mathbf{k}.$$

Otherwise, it is called *archimedean*.

Proposition 1.5.

Notation 1.6. Let $\mathbf{k}$ be a field. We denote by $M_{\mathbf{k}}$ the set of Yang: equivalence class all absolute values (i.e., valuations) on $\mathbf{k}$. For each $v \in M_{\mathbf{k}}$, we also call v a *place* of $\mathbf{k}$.

1.2 Non-archimedean place

1.3 Places of field of rational numbers

In this section, let $\mathbf{k}$ be a number field.

Theorem 1.7 (Ostrowski's theorem)**.** Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to either the usual absolute value $|\cdot|_\infty$ or a p-adic absolute value $|\cdot|_p$ for some prime number p.

2 Finite field extensions

2.1 Ramification and inertia

Theorem 2.1. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. Then we have

$$\sum_{w|v} e(w/v) f(w/v) = [L : K].$$

Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. We have

$$L \otimes_K K_v \cong \prod_{w|v} L_w,$$

where the product is taken over all absolute values $w \in M_L$ extending v.

Definition 2.2. Yang: To be added.

2.2 Places of number fields

Theorem 2.3. Let $\mathbf{k}$ be a number field. Then

$$M_{\mathbf{k}}^{\infty} = \{\text{embeddings } \sigma : \mathbf{k} \to \mathbb{C}\}$$

and

$$M_{\mathbf{k}}^{\mathrm{f}} = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_{\mathbf{k}}\}.$$

Yang: To be revised.

Proposition 2.4 (Product formula)**.** Let $\mathbf{k}$ be a number field. For each $x \in \mathbf{k}^{\times}$, we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^{\infty}; \\ 1, & v \in M_{\mathbf{k}}^{0}. \end{cases}$$

Yang: To be revised.

3 Artin-Whaples approximations

Theorem 3.1 (Artin-Whaples approximations)**.** Let K be a field, and let $v_1, v_2, \dots, v_n$ be pairwise inequivalent nontrivial absolute values on K. For any $a_1, a_2, \dots, a_n \in K$ and any $\varepsilon > 0$, there exists an element $x \in K$ such that

$$|x - a_i|_{v_i} < \varepsilon$$

for all $1 \leq i \leq n$. Yang: To be checked.

3.1 Geometric version

Theorem 3.2. Let $\mathbf{k}$ be a field with algebraic closure $\bar{\mathbb{k}}$. Let X be a normal, projective, geometrically integral variety over $\mathbf{k}$. Let $x_1, x_2, \dots, x_n \in X(\mathbf{k})$ be closed points lying over pairwise distinct points of X. Let $v_1, v_2, \dots, v_n \in M_{\mathbf{k}}$ be pairwise inequivalent absolute values on $\mathbf{k}$. For every $i = 1, 2, \dots, n$, let U_i be an open neighborhood of x_i in $X(\bar{\mathbb{k}})$ with respect to the topology induced by v_i. Then there exists a rational point $x \in X(\mathbf{k})$ such that $x \in U_i$ for all $1 \leq i \leq n$. Yang: To be revsied.

Yang: This gives [Xie25, Proposition 3.9]

References

[Ser08] Jean-Pierre Serre. *Topics in Galois Theory*. Second. Vol. 1. Research Notes in Mathematics. With notes by Henri Darmon. Wellesley, MA: A K Peters, Ltd., 2008. ISBN: 978-1568814124.

[Xie25] Junyi Xie. "The existence of Zariski dense orbits for endomorphisms of projective surfaces". In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on p. 3).