Affinoid algebras

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1 Normed rings and modules

1.1 Semi-normed algebraic structures

Definition 1.1. Let G be an abelian group. A *semi-norm* on G is a function $\|\cdot\| : G \to \mathbb{R}_{\geq 0}$ such that

- $\|0\| = 0$;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$.

Suppose that R is a ring (commutative with unity) and $\|\cdot\|$ is a semi-norm on the underlying abelian group of R. We further require that

- $\|1\| = 1$;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$.

Suppose that $(M, \|\cdot\|_M)$ is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M. We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$.

Suppose that $(A, \|\cdot\|_A)$ is an R-algebra and $\|\cdot\|_A$ is a semi-norm on the underlying R-module of A. We further require that this semi-norm is a semi-norm on the underlying ring of A.

Definition 1.2. Let A be an abelian group (or ring, R-module, R-algebra) equipped with a semi-norm $\|\cdot\|$. We say that $\|\cdot\|$ is a *norm* if $\forall x \in A, \|x\| = 0 \iff x = 0$.

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Definition 1.3. Let A be an abelian group (or ring, R-module, R-algebra) equipped with a semi-norm $\|\cdot\|$. We say that $\|\cdot\|$ is *non-archimedean* if we have the strong triangle inequality $\forall x, y \in A, \|x+y\| \leq \max(\|x\|, \|y\|)$.

Definition 1.4. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group (or ring, R-module, R-algebra) A. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in A, \|x\|_1 \leq C\|x\|_2$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are bounded by each other, we say they are *equivalent*.

Remark 1.5. Equivalent semi-norms induce the same topology on A. However, the converse is not true in general. Compare with **??**.

Yang: what about on a module?

Definition 1.6. Let M be a semi-normed abelian group (or R-module) and $N \subseteq M$ be a subgroup (or R-submodule). The *residue semi-norm* on the quotient group M/N is defined as

$$\|x+N\|_{M/N} = \inf_{y \in N} \|x+y\|_M.$$

Unless otherwise specified, we always equip the quotient M/N with the residue semi-norm.

Remark 1.7. The residue semi-norm is a norm if and only if N is closed in M.

Definition 1.8. Let M and N be two semi-normed abelian groups (or rings, R-modules, R-algebras). A homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$.
A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$.

Definition 1.9. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$. A multiplicative norm sometimes is called a *(multiplicative) valuation* or an *absolute value*.

Example 1.10. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a valuation ring. This norm is non-archimedean and multiplicative.

Example 1.11. A valuation field $(\mathbf{k}, |\cdot|)$ can be viewed as a valuation ring.

Example 1.12. Let $|\cdot| = |\cdot|_\infty$ be the usual absolute value on $\mathbb{Z}$. Then $(\mathbb{Z}, |\cdot|)$ is a valuation ring.

Example 1.13. Let X be a compact Hausdorff topological space. The ring $C(X, \mathbb{R})$ of continuous real-valued functions on X equipped with the norm $\|f\| = \sup_{x \in X} |f(x)|$ is a normed ring. Its norm is power-multiplicative but not multiplicative in general. It is worth mentioning that the Gelfand-Kolmogorov Theorem saying that we can recover X from the normed ring $C(X, \mathbb{R})$.

Example 1.14. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. The action of K on itself by multiplication induces a embedding $K \hookrightarrow \operatorname{End}_{\mathbb{Q}}(K) \cong M_n(\mathbb{Q})$. Consider a norm $\|\cdot\|$ on the matrix ring $M_n(\mathbb{Q})$ and restrict it to K. Then $(K, \|\cdot\|)$ is a normed ring. This norm is even not power-multiplicative in general.

Definition 1.15. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 1.16. A *banach ring* is a complete normed ring.

Proposition 1.17. Let R be a banach ring and $I \subseteq R$ be a closed ideal. Then the residue norm on the quotient ring R/I is a norm for rings.

Proof. We only need to show that $\|1\|_{R/I} = 1$. Yang: To be added. □

Proposition 1.18. Let R be a banach ring. Then the group of invertible elements $R^\times$ is an open subset of R.

Proof. Yang: To be added. □

Corollary 1.19. Let R be a banach ring. Then every maximal ideal of R is closed.

Proof. Yang: To be added. □

Definition 1.20. Let $(A, \|\cdot\|_A)$ be a normed algebraic structure, e.g., a normed abelian group, a normed ring, or a normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a metric space. Since A is dense in its completion and the algebraic operations are uniformly continuous, the algebraic operations on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M\otimes_R N} = \inf\left\{\sum_i \|x_i\|_M \|y_i\|_N : z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N\right\}.$$

Definition 1.21. Let R be a banach ring and M, N complete semi-normed R-modules. The *complete tensor product* $M \widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Example 1.22. Yang: Example of complete tensor product.

Definition 1.23. Let R be a non-archimedean banach ring. We define

$$R^\circ = \{f \in R : \rho(f) \leq 1\}, \quad R^{\circ\circ} = \{f \in R : \rho(f) < 1\}.$$

The *reduction ring* of R is defined as the quotient ring

$$\widetilde{R} = R^\circ / R^{\circ\circ}.$$

For a non-archimedean field $\mathbf{k}$, its reduction ring $\widetilde{\mathbf{k}} = \Bbbk_{\mathbf{k}}$ is just the residue field of its valuation ring.

Example 1.24. Let R be a ring equipped with the trivial norm. Then we have $R^\circ = R$ and $R^{\circ\circ} = \mathrm{nil}(R)$. Hence the reduction ring $\widetilde{R}$ is isomorphic to the reduced ring $R_{\mathrm{red}} = R/\mathrm{nil}(R)$.

1.2 Spectral radius

Definition 1.25. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Since $\|\cdot\|$ is submultiplicative, the limit defining $\rho(f)$ exists and equals to $\inf_{n\geq 1} \|f^n\|^{1/n}$ by Fekete's Subadditive Lemma.

Example 1.26. Consider the normed ring $(K, \|\cdot\|)$ in Example 1.14. Suppose that $\|\cdot\|$ on $M_n(\mathbb{Q})$ is given by the 2-norm induced by the euclidean norm on $\mathbb{Q}^n$. Then for a matrix $A \in M_n(\mathbb{Q})$, its spectral radius $\rho(A)$ is given by the largest absolute value of eigenvalues of A. Hence for each $a \in K$, its spectral radius $\rho(a)$ is given by the largest absolute value of embeddings of a into $\mathbb{C}$, i.e.,

$$\rho(a) = \max_{\sigma : K \hookrightarrow \mathbb{C}} |\sigma(a)|.$$

Proposition 1.27. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 1.28. A banach ring R is called *uniform* if its norm is power-multiplicative.

Definition 1.29. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Definition 1.30. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\mathrm{Qnil}(R)$.

Proposition 1.31. Let R be a banach ring. The completion of $R/\mathrm{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

Yang: To be continued...

2 Convergent and restricted power series

Notation 2.1. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;

- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \dots, n$, we have $\alpha_i \leq \beta_i$;
- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

2.1 Absolutely convergent power series

Definition 2.2. Let R be a banach ring and $r > 0$ be a real number. We define the *ring of absolutely convergent power series* over $\mathbf{k}$ with radius r as

$$R\langle T/r \rangle \coloneqq \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] \,:\, \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm $\| \sum_{n=0}^{\infty} a_n T^n \| := \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r \rangle$ is a banach ring.
For a tuple of n indeterminates $T = (T_1, \dots, T_n)$ and a tuple of n positive real numbers $r = (r_1, \dots, r_n)$, we define

$$R\left\langle \underline{T/r} \right\rangle \coloneqq R\langle T_1/r_1, \dots, T_n/r_n \rangle := R\langle T_1/r, \dots, T_{n-1}/r_{n-1} \rangle \langle T_n/r_n \rangle.$$

2.2 Tate algebras

Definition 2.3. Let R be a non-archimedean banach ring. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$R\langle \underline{r^{-1}T} \rangle := R\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in R, \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 2.4. Let R be a non-archimedean banach ring. Then the Tate algebra $R\{\underline{T/r}\}$ is a non-archimedean multiplicative banach R-algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $R\{\underline{T/r}\}$ is a R-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ are two nonzero elements in $R\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon / \|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon / \|f\|$. For any $|\gamma| > 2N$, we have

$$\left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Hence $f \cdot g \in R\{\underline{T/r}\}$ and it shows that $R\{\underline{T/r}\}$ is a R-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $R\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f+g\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \le \sup_{\alpha\in\mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \le \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned} \|f\cdot g\| &= \left\| \sum_{\gamma\in\mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma \right\| = \sup_{\gamma\in\mathbb{N}^n} \left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \\ &\le \sup_{\gamma\in\mathbb{N}^n} \max_{\alpha+\beta=\gamma} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \le \|f\|\cdot\|g\|. \end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed **k**-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\|\cdot\|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \left\| \sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta \right\| r^{\alpha_1+\beta_1} \le \|f\cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by **??**. Thus the gauss norm is multiplicative.

Step 4. Finally show that $R\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $R\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \le \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in R. Since R is complete, set $a_\alpha := \lim_{m\to+\infty} a_{\alpha,m}$ and $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \le \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \le 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \le \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \le \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \le \varepsilon$$

for all $m > N$. □

Example 2.5. Let R be a non-archimedean banach ring and $A = R\{T\}$ be the Tate algebra in one variable over R. Then we have

$$A^\circ = \left\{ \sum_{n\geq 0} a_n T^n \,:\, |a_n| \leq 1 \text{ for all } n \in \mathbb{N} \right\} = R^\circ\{T\},$$

and

$$A^{\circ\circ} = \left\{ \sum_{n\geq 0} a_n T^n \,:\, |a_n| < 1 \text{ for all } n \in \mathbb{N} \right\} = R^{\circ\circ}\{T\}.$$

Since the norm of items in a restricted power series will tend to 0, we have

$$\widetilde{A} = \widetilde{R}[\underline{T}].$$

Example 2.6. Let R is a multiplicative non-archimedean banach ring. Set

$$\sqrt{|R|^{-1}} = \{r \in \mathbb{R}_{>0} \,:\, r^{-n} \in |R| \text{ for some } n \in \mathbb{N}_{>0}\}.$$

Fix $r \in \mathbb{R}^n_{>0}$, consider the Tate algebra $A = R\{T/r\}$.

Suppose that $r \in \sqrt{|R|^{-1}}$. Let n be the minimal positive integer such that $r^n \in |R|^{-1}$ and

$$\widetilde{M}_k := \{a \in R \,:\, |a| = r^{-nk}\}/\{a \in R \,:\, |a| < r^{-nk}\}.$$

For a_mT^m with $n \nmid m$, we have$\|a_mT^m\| = |a_m|r^m \leq 1 \implies |a_m|r^m < 1$. Hence

$$\widetilde{R\{T/r\}} = \widetilde{R} \oplus \widetilde{M}_1 T^n \oplus \widetilde{M}_2 T^{2n} \oplus \widetilde{M}_3 T^{3n} \oplus \cdots.$$

In case $R = \mathbf{k}$ is a non-archimedean field, we have $\widetilde{M}_k \cong \widetilde{\mathbf{k}}$ by choosing an element $c \in \mathbf{k}$ with $|c| = r^{-n}$. Hence

$$\widetilde{\mathbf{k}\{T/r\}} \cong \mathscr{k}_{\mathbf{k}}[T^n].$$

Suppose that $r \notin \sqrt{|R|^{-1}}$. Then for every $\|a_nT^n\| = a_nr^n \leq 1$, we have $|a_n| < 1$. It follows that

$$\widetilde{R\{T/r\}} = \widetilde{R}.$$

2.3 Weierstrass division

Definition 2.7. Let R be a non-archimedean banach ring and $r \in \mathbb{R}_{>0}$. A restricted power series $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in R\{\underline{T/r}\}$ is said to be *distinguished in the variable T_n of degree d* if

- $a_\alpha \in R$ is a unit for $\alpha = (0, \ldots, 0, d)$;
- $\|a_\alpha\| r^\alpha < \|a_{(0,\ldots,0,d)}\| r_n^d$ for all $\alpha_n < d$.

Yang: To be revised.

Proposition 2.8. Let R be a non-archimedean banach ring. An element $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in R\{\underline{T/r}\}$ is invertible if and only if a_0 is invertible in R and $\|a_0\| > \|a_\alpha\| r^\alpha$ for all $\alpha \neq 0$.

Proof. Multiplying by a_0^{-1}, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $R[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = - \sum_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\| r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for m. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\| r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\left\{A^{m+1}, A \cdot A^m\right\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in R\{\underline{T/r}\}$. $\square$

Theorem 2.9 (Weierstrass preparation theorem)**.** Let $\mathbf{k}$ be a complete non-archimedean field. Let $f \in \mathbf{k}\{\underline{T/r}\}$ be a restricted power series that is distinguished in the variable T_n of degree d, i.e.,

$$f = \sum_{\alpha \in \mathbb{N}^{n-1}} a_\alpha T^\alpha + \sum_{\alpha_n \geq 1} a_\alpha T^\alpha$$

with $a_{(0,\ldots,0,d)}$ being a unit in $\mathbf{k}\{\underline{T/r}\}$ and $\|a_\alpha\| r^\alpha < \|a_{(0,\ldots,0,d)}\| r_n^d$ for all $\alpha_n < d$. Then there exists a unique monic polynomial $P \in \mathbf{k}\{\underline{T/r}\}[T_n]$ of degree d in T_n and a unique unit $U \in \mathbf{k}\{\underline{T/r}\}$ such that

$$f = P \cdot U.$$

Yang: To be checked.

Theorem 2.10 (Weierstrass division theorem)**.** Let $\mathbf{k}$ be a complete non-archimedean field. Let $f \in \mathbf{k}\{\underline{T/r}\}$ be a restricted power series that is distinguished in the variable T_n of degree d. Then for every $g \in \mathbf{k}\{\underline{T/r}\}$, there exists a unique $Q \in \mathbf{k}\{\underline{T/r}\}$ and a unique polynomial $R \in \mathbf{k}\{\underline{T/r}\}[T_n]$ of degree less than d in T_n such that

$$g = Q \cdot f + R.$$

Yang: To be checked.

Proposition 2.11. Let $\mathbf{k}$ be a complete non-archimedean field and $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$. Then

$$\operatorname{Spec} \mathbf{k}\{\underline{T/r}\} = \{\},$$

where

3 Affinoid algebras

3.1 The first properties

Definition 3.1. Let $\mathbf{k}$ be a non-archimedean field. A banach $\mathbf{k}$-algebra A is called a *affinoid* $\mathbf{k}$-*algebra* if there exists an admissible surjective homomorphism

$$\varphi : \mathbf{k}\{\underline{T/r}\} \twoheadrightarrow A$$

for some $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$.
If one can choose $r_1 = \cdots = r_n = 1$, then we say that A is a *strict affinoid* $\mathbf{k}$-*algebra*.

Example 3.2. Yang: To be added.

Example 3.3. Let A be an affinoid $\mathbf{k}$-algebra and $f_i, g_i \in A$. Then the normed localization

$$A\{(f_i/g_i)/r_i\}_{i=1}^n = A \otimes_{\mathbf{k}\{\underline{T}\}} \mathbf{k}\{\underline{T/r}, \underline{S}\}/(g_i T_i - f_i)_{i=1}^n$$

is again an affinoid $\mathbf{k}$-algebra. Yang: To be added.

Proposition 3.4. Let A be an affinoid $\mathbf{k}$-algebra. Then A is noetherian, and every ideal of A is closed.

Proof. Yang: To be completed. □

Proposition 3.5. Let A be an affinoid $\mathbf{k}$-algebra. Then there exists a constant $C > 0$ and $N > 0$ such that for all $f \in A$ and $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

In particular, $\operatorname{Qnil}(A) = \operatorname{nil}(A)$.
Furthermore, if A is reduced, we have

$$\|f\| \leq C\rho(f)$$

for all $f \in A$.

Proof. Yang: To be completed. □

Proposition 3.6. Let A be an affinoid $\mathbf{k}$-algebra. If and only if $\rho(f) \in \sqrt{|\mathbf{k}|} \cup \{0\}$ for all $f \in A$, then A is strict. Yang: To be complete.

Proof. Yang: To be completed. □

Definition 3.7. Let $\mathbf{k}$ be a non-archimedean field. We define the *ring of restricted Laurent series* over $\mathbf{k}$ as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} = \left\{ \sum_{n\in\mathbb{Z}} a_n T^n \,:\, a_n \in \mathbf{k}, \lim_{|n|\to\infty} |a_n| r^n = 0 \right\}$$

equipped with the norm

$$\|f\| = \sup_{n\in\mathbb{Z}} |a_n| r^n.$$

Yang: Is $\mathbf{K}_r$ always a field? Yang: Do we have $\mathbf{L}_{\mathbf{k},r} = \mathrm{Frac}(\mathbf{k}\{T/r\})$?

Proposition 3.8. Let $\mathbf{k}$ be a non-archimedean field. If $r \notin \sqrt{|\mathbf{k}^\times|}$, then $\mathbf{K}_r$ is a complete non-archimedean field with non-trivial absolute value extending that of $\mathbf{k}$.

Yang: Tensor with $\mathbf{K}_r$.

Theorem 3.9. Let A be a strict affinoid $\mathbf{k}$-algebra. Then there exists a finite injective admissible homomorphism

$$\varphi : \mathbf{k}\{\underline{T}\} \hookrightarrow A$$

for some $d \in \mathbb{N}$ and $r_1, \ldots, r_d \in \mathbb{R}_{>0}$. Yang: To be checked.

3.2 Finite modules over affinoid algebras

There are three different categories of finite modules over an affinoid algebra A:

- The category $\mathbf{Banmod}_A$ of finite banach A-modules with A-linear maps as morphisms.
- The category $\mathbf{Banmod}_A^b$ of finite banach A-modules with bounded A-linear maps as morphisms.
- The category $\mathbf{mod}_A$ of finite A-modules with all A-linear maps as morphisms.

Theorem 3.10. Let A be an affinoid $\mathbf{k}$-algebra. Then the category of finite banach A-modules with bounded A-linear maps as morphisms is equivalent to the category of finite A-modules with A-linear maps as morphisms. Yang: To be revised.

For simplicity, we will just write mod_A to denote the category of finite banach A-modules with bounded A-linear maps as morphisms.