

# Normed rings and modules

## 1 Semi-normed algebraic structures

**Definition 1.** Let  $G$  be an abelian group. A *semi-norm* on  $G$  is a function  $\|\cdot\| : G \rightarrow \mathbb{R}_{\geq 0}$  such that

- $\|0\| = 0$ ;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$ .

Suppose that  $R$  is a ring (commutative with unity) and  $\|\cdot\|$  is a semi-norm on the underlying abelian group of  $R$ . We further require that

- $\|1\| = 1$ ;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$ .

Suppose that  $(M, \|\cdot\|_M)$  is an  $R$ -module and  $\|\cdot\|_M$  is a semi-norm on the underlying abelian group of  $M$ . We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$ .

Suppose that  $(A, \|\cdot\|_A)$  is an  $R$ -algebra and  $\|\cdot\|_A$  is a semi-norm on the underlying  $R$ -module of  $A$ . We further require that this semi-norm is a semi-norm on the underlying ring of  $A$ .

If we further have  $\forall x, \|x\| = 0 \implies x = 0$ , then we say  $\|\cdot\|$  is a *norm* on the corresponding algebraic structure.

If we replace the triangle inequality  $\|x + y\| \leq \|x\| + \|y\|$  by the stronger inequality  $\|x + y\| \leq \max(\|x\|, \|y\|)$ , then we say  $\|\cdot\|$  is a *non-archimedean* semi-norm.

**Definition 2.** Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two semi-norms on an abelian group (or ring,  $R$ -module,  $R$ -algebra)  $A$ . We say  $\|\cdot\|_1$  is *bounded* by  $\|\cdot\|_2$  if there exists a constant  $C > 0$  such that  $\forall x \in A, \|x\|_1 \leq C\|x\|_2$ . If  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are bounded by each other, we say they are *equivalent*.

**Remark 3.** Equivalent semi-norms induce the same topology on  $A$ . However, the converse is not true in general. Compare with [Lemma 29](#).

Yang: what about on a module?

**Definition 4.** Let  $M$  be a semi-normed abelian group (or ring,  $R$ -module,  $R$ -algebra) and  $N \subseteq M$  be a subgroup (or ideal,  $R$ -submodule, ideal). The *residue semi-norm* on the quotient group  $M/N$  is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

For case of rings or  $R$ -algebras, we multiply a constant to make sure that  $\|1 + N\|_{M/N} = 1$  if necessary.

Unless otherwise specified, we always equip the quotient  $M/N$  with the residue semi-norm.

**Remark 5.** The residue semi-norm is a norm if and only if  $N$  is closed in  $M$ .

**Definition 6.** Let  $M$  and  $N$  be two semi-normed abelian groups (or rings,  $R$ -modules,  $R$ -algebras). A homomorphism  $f : M \rightarrow N$  is called *bounded* if there exists a constant  $C > 0$  such that  $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$ .

A bounded homomorphism  $f : M \rightarrow N$  is called *admissible* if the induced isomorphism  $M/\ker f \rightarrow \text{Im } f$  is an isometry, i.e.,  $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$ .

**Definition 7.** A semi-norm  $\|\cdot\|$  on a ring  $R$  is called *multiplicative* if  $\forall x, y \in R, \|xy\| = \|x\|\|y\|$ . It is called *power-multiplicative* if  $\forall x \in R, \|x^n\| = \|x\|^n$  for all integers  $n \geq 1$ . A multiplicative norm sometimes is called a *(multiplicative) valuation* or an *absolute value*.

**Example 8.** Let  $R$  be arbitrary ring. The *trivial norm* on  $R$  is defined as  $\|x\| = 0$  if  $x = 0$  and  $\|x\| = 1$  if  $x \neq 0$ . The ring  $R$  equipped with the trivial norm is a valuation ring.

**Example 9.** A valuation field  $(\mathbf{k}, |\cdot|)$  can be viewed as a valuation ring.

**Example 10.** Let  $|\cdot| = |\cdot|_\infty$  be the usual absolute value on  $\mathbb{Z}$ . Then  $(\mathbb{Z}, |\cdot|)$  is a valuation ring.

**Example 11.** Let  $X$  be a compact Hausdorff topological space. The ring  $\mathcal{C}(X, \mathbb{R})$  of continuous real-valued functions on  $X$  equipped with the norm  $\|f\| = \sup_{x \in X} |f(x)|$  is a normed ring. Its norm is power-multiplicative but not multiplicative in general. It is worth mentioning that the Gelfand-Kolmogorov Theorem saying that we can recover  $X$  from the normed ring  $\mathcal{C}(X, \mathbb{R})$ .

**Definition 12.** A (semi-)norm on an abelian group  $M$  induces a (pseudo-)metric  $d(x, y) = \|x - y\|$  on  $M$ . A (semi-)normed abelian group  $M$  is called *complete* if it is complete as a (pseudo-)metric space.

**Definition 13.** A *banach ring* is a complete normed ring.

**Definition 14.** Let  $(A, \|\cdot\|_A)$  be a normed algebraic structure, e.g., a normed abelian group, a normed ring, or a normed module. The *completion* of  $A$ , denoted by  $\hat{A}$ , is the completion of  $A$  as a metric space. Since  $A$  is dense in its completion and the algebraic operations are uniformly continuous, the algebraic operations on  $A$  can be uniquely extended to the completion.

Let  $R$  be a normed ring and  $M, N$  be semi-normed  $R$ -modules. There is a natural semi-norm on the tensor product  $M \otimes_R N$  defined as

$$\|z\|_{M \otimes_R N} = \inf \left\{ \sum_i \|x_i\|_M \|y_i\|_N : z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N \right\}.$$

**Definition 15.** Let  $R$  be a banach ring and  $M, N$  complete semi-normed  $R$ -modules. The *complete tensor product*  $M \hat{\otimes}_R N$  is defined as the completion of the semi-normed  $R$ -module  $M \otimes_R N$ .

**Example 16.** Let  $R$  be a banach ring and  $r > 0$  be a real number. We define the ring of absolutely convergent power series over  $\mathbf{k}$  with radius  $r$  as

$$R \langle T/r \rangle := \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm  $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$ , the ring  $R \langle T/r \rangle$  is a banach ring.

When  $R = \mathbf{k}$  is a **Yang**: To be checked.

**Example 17.** Yang: Example of complete tensor product.

## 2 Spectral radius

**Definition 18.** Let  $R$  be a banach ring. For each  $f \in R$ , the *spectral radius* of  $f$  is defined as

$$\rho(f) = \lim_{n \rightarrow \infty} \|f^n\|^{1/n}.$$

Since  $\|\cdot\|$  is submultiplicative, the limit defining  $\rho(f)$  exists and equals to  $\inf_{n \geq 1} \|f^n\|^{1/n}$  by Fekete's Subadditive Lemma

**Proposition 19.** Let  $(R, \|\cdot\|)$  be a banach ring. The spectral radius  $\rho(\cdot)$  defines a power-multiplicative semi-norm on  $R$  that is bounded by  $\|\cdot\|$ .

*Proof.* Yang: To be continued. □

**Definition 20.** A banach ring  $R$  is called *uniform* if its norm is power-multiplicative.

**Definition 21.** Let  $R$  be a banach ring. The *uniformization* of  $R$ , denoted by  $R \rightarrow R^u$ , is the banach ring with the universal property among all bounded homomorphisms from  $R$  to uniform banach rings. Yang: To be continued.

**Definition 22.** Let  $R$  be a banach ring. An element  $f \in R$  is called *quasi-nilpotent* if  $\rho(f) = 0$ . All quasi-nilpotent elements of  $R$  form an ideal, denoted by  $\text{Qnil}(R)$ .

**Proposition 23.** Let  $R$  be a banach ring. The completion of  $R/\text{Qnil}(R)$  with respect to the spectral radius  $\rho(\cdot)$  is the uniformization of  $R$ .

*Proof.* Yang: To be continued. □

**Example 24.** Let  $R$  be a banach ring and  $r > 0$  be a real number. Consider the ring of absolutely convergent power series  $R\langle T/r \rangle$  defined in Example 16. For each  $f = \sum_{n=0}^{\infty} a_n T^n \in R\langle T/r \rangle$ , we have

$$\rho(f) = \max_{n \geq 0} \|a_n\| r^n.$$

Thus the uniformization of  $R\langle T/r \rangle$  is given by the ring

$$R\{T/r\} = \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \lim_{n \rightarrow \infty} \|a_n\| r^n = 0 \right\},$$

equipped with the norm  $\|\sum_{n=0}^{\infty} a_n T^n\| = \max_{n \geq 0} \|a_n\| r^n$ . Yang: To be revised.

Yang: To be continued...

### 3 Tate algebras

**Notation 25.** Let  $T = (T_1, \dots, T_n)$  be a tuple of  $n$  indeterminates,  $r = (r_1, \dots, r_n)$  be a tuple of  $n$  positive real numbers, and  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \dots T_n^{\alpha_n}$  and  $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \dots r_n^{\alpha_n}$ ;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$ ;
- $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n$ ;
- $\alpha \leq_{\text{total}} \beta$  if and only if for all  $i = 1, \dots, n$ , we have  $\alpha_i \leq \beta_i$ ;
- Let  $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$  be a set of elements in a metric space  $X$  indexed by multi-indices  $\alpha \in \mathbb{N}^n$ . We say that  $\lim_{|\alpha| \rightarrow +\infty} x_\alpha = x \in X$  if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $\alpha \in \mathbb{N}^n$  with  $|\alpha| > N$ , we have  $d(x_\alpha, x) < \varepsilon$ .

**Definition 26.** Let  $R$  be a non-archimedean banach ring. Let  $T = (T_1, \dots, T_n)$  be a tuple of  $n$  indeterminates and  $r = (r_1, \dots, r_n)$  be a tuple of  $n$  positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$R\{\underline{T/r}\} := R\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in R, \lim_{|\alpha| \rightarrow +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

**Proposition 27.** Let  $R$  be a non-archimedean banach ring. Then the Tate algebra  $R\{\underline{T/r}\}$  is a non-archimedean multiplicative banach  $R$ -algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

*Proof.* The proof splits into several parts. Every parts is straightforward and standard.

**Step 1.** We first show that  $R\{\underline{T/r}\}$  is a  $R$ -algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$  and  $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$  are two nonzero elements in  $R\{\underline{T/r}\}$ . Given  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $|\alpha| > N$ , we have  $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$  and  $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$ . For any  $|\gamma| > 2N$ , we have

$$\left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Hence  $f \cdot g \in R\{\underline{T/r}\}$  and it shows that  $R\{\underline{T/r}\}$  is a  $R$ -algebra.

**Step 2.** Show that the gauss norm is a non-archimedean norm on  $R\{\underline{T/r}\}$ .

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f + g\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \leq \sup_{\alpha \in \mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \leq \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned} \|f \cdot g\| &= \left\| \sum_{\gamma \in \mathbb{N}^n} \left( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma \right\| = \sup_{\gamma \in \mathbb{N}^n} \left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \\ &\leq \sup_{\gamma \in \mathbb{N}^n} \max_{\alpha+\beta=\gamma} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \leq \|f\| \cdot \|g\|. \end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed  $\mathbf{k}$ -algebra.

**Step 3.** Show that the gauss norm is multiplicative.

Suppose that  $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$  and  $\|a_\alpha\| r^\alpha < \|f\|$  for all  $\alpha <_{\text{total}} \alpha_1$ . Similar to  $\|b_{\beta_1}\| r^{\beta_1}$ . Then we have

$$\|f\| \cdot \|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \left\| \sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta \right\| r^{\alpha_1+\beta_1} \leq \|f \cdot g\|,$$

where the third equality holds since  $(\alpha_1, \beta_1)$  is the unique pair such that  $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$  is maximized and by Proposition 30. Thus the gauss norm is multiplicative.

**Step 4.** Finally show that  $R\{\underline{T}/r\}$  is complete with respect to the gauss norm.

Let  $\{f_m = \sum a_{\alpha,m} T^\alpha\}$  be a cauchy sequence in  $R\{\underline{T}/r\}$ . We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\|.$$

Thus for each  $\alpha \in \mathbb{N}^n$ , the sequence  $\{a_{\alpha,m}\}$  is a cauchy sequence in  $R$ . Since  $R$  is complete, set  $a_\alpha := \lim_{m \rightarrow +\infty} a_{\alpha,m}$  and  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$ . Given  $\varepsilon > 0$ , there exists  $M \in \mathbb{N}$  such that for all  $m, l > M$ , we have  $\|f_m - f_l\| < \varepsilon$ . Fixing  $m > M$ , there exists  $N \in \mathbb{N}$  such that for all  $|\alpha| > N$ , we have  $\|a_{\alpha,m}\| r^\alpha < \varepsilon$ . Hence for all  $|\alpha| > N$  and  $l > M$ , we have

$$\|a_{\alpha,l}\| r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking  $l \rightarrow +\infty$ , we have  $\|a_\alpha\| r^\alpha \leq 2\varepsilon$  for all  $|\alpha| > N$ . It follows that  $f \in \mathbf{k}\{\underline{T}/r\}$ .

For every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $m, l > N$ , we have  $\|f_m - f_l\| < \varepsilon$ . Thus for all  $\alpha \in \mathbb{N}^n$  and  $m, l > N$ , we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking  $l \rightarrow +\infty$ , we have  $\|a_{\alpha,m} - a_\alpha\| r^\alpha \leq \varepsilon$  for all  $m > N$ . It follows that

$$\|f - f_m\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \leq \varepsilon$$

for all  $m > N$ . □

**Proposition 28.** Let  $R$  be a non-archimedean banach ring. An element  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in R\{\underline{T}/r\}$  is invertible if and only if  $a_0$  is invertible in  $R$  and  $\|a_0\| > \|a_\alpha\| r^\alpha$  for all  $\alpha \neq 0$ .

*Proof.* Multiplying by  $a_0^{-1}$ , we can reduce to the case  $a_0 = 1$ . Let  $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$  be the inverse

of  $f$  in  $R[[T]]$ . Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every  $\gamma \neq 0 \in \mathbb{N}^n$ ,

$$b_\gamma = - \sum_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let  $A = \|f - 1\| < 1$ . We show that for every  $m \in \mathbb{N}$ , there exists  $C_m > 0$  such that for all  $\alpha \in \mathbb{N}^n$  with  $|\alpha| \geq C_m$ , we have  $\|b_\alpha\| r^\alpha \leq A^m$ . For  $m = 0$ , note that  $b_0 = 1$ . By induction on  $\gamma$  with respect to the total order  $\leq_{\text{total}}$ , we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for  $m$ . There exists  $D_{m+1} \in \mathbb{N}$  such that for all  $\alpha \in \mathbb{N}^n$  with  $|\alpha| \geq D_{m+1}$ , we have  $\|a_\alpha\| r^\alpha \leq A^{m+1}$ . Set  $C_{m+1} = C_m + D_{m+1} + 1$ . For any  $\gamma \in \mathbb{N}^n$  with  $|\gamma| \geq C_{m+1}$ , we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\{A^{m+1}, A \cdot A^m\} = A^{m+1}$$

since either  $|\alpha| \geq D_{m+1}$  or  $|\beta| \geq C_m$ . Thus by induction, we have  $\|b_\alpha\| r^\alpha \rightarrow 0$  as  $|\alpha| \rightarrow +\infty$ . It follows that  $g \in R\{\underline{T}/r\}$ .  $\square$

## Appendix

**Lemma 29.** Let  $\mathbf{k}$  be a field and  $\|\cdot\|_1, \|\cdot\|_2$  be two absolute values on  $\mathbf{k}$ . Then the following statements are equivalent:

- (a)  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent;
- (b)  $\|\cdot\|_1$  and  $\|\cdot\|_2$  induce the same topology on  $\mathbf{k}$ ;
- (c) The unit disks  $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$  and  $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$  are the same.

**Proposition 30.** Let  $(X, d)$  be an ultra-metric space. Then for any  $x, y, z \in X$ , at least two of the three distances  $d(x, y), d(y, z), d(z, x)$  are equal. And the third distance is less than or equal to the common value of the other two.