Non-archimedean analysis

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The main references for this chapter are [Gou97; Rob00; 李文威 18].

1   Tate algebras

Notation 1.1. Let $T = (T_1, \ldots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \ldots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;

- $\underline{T/r} := (T_1/r_1, T_2/r_2, \ldots, T_n/r_n)$;

- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;

- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \ldots, n$, we have $\alpha_i \leq \beta_i$;

- $E(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \ldots, n\}$ and $B(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \ldots, n\}$ for $x = (x_1, \ldots, x_n) \in \mathbf{k}^n$;

- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

Definition 1.2. Let $\mathbf{k}$ be a complete non-archimedean field. Let $T = (T_1, \ldots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \ldots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$\mathbf{k}\langle \underline{r^{-1}T} \rangle := \mathbf{k}\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 1.3. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ is a non-archimedean multiplicative banach $\mathbf{k}$-algebra with respect to the *gauss norm*

$$\left\|\sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha\right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Yang: For the definition of banach ring, see

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ are two elements in $\mathbf{k}\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{\frac{\varepsilon}{\|g\|}\|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|}\|a_\alpha\| r^\alpha\right\} \leq \varepsilon.$$

Hence $f \cdot g \in \mathbf{k}\{\underline{T/r}\}$ and it shows that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $\mathbf{k}\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f + g\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \leq \sup_{\alpha \in \mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \leq \max\{\|f\|, \|g\|\}$$

and

$$\|f \cdot g\| = \left\|\sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) T^\gamma\right\| = \sup_{\gamma \in \mathbb{N}^n} \left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma$$

$$\leq \sup_{\gamma \in \mathbb{N}^n} \max_{\alpha+\beta=\gamma} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \leq \|f\| \cdot \|g\|.$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\| \cdot \|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \left\|\sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta\right\| r^{\alpha_1+\beta_1} \leq \|f \cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by **??**. Thus the gauss norm is multiplicative.

Step 4. Finally show that $\mathbf{k}\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $\mathbf{k}\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in $\mathbf{k}$. Since $\mathbf{k}$ is complete, set $a_\alpha := \lim_{m \to +\infty} a_{\alpha,m}$ and $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all

$m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \leq 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \leq \varepsilon$$

for all $m > N$. $\qquad\square$

Proposition 1.4. Let $\mathbf{k}$ be a complete non-archimedean field. An element $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ is invertible if and only if $\|a_0\| > \|a_\alpha\| r^\alpha$ for all $\alpha \neq 0$.

Proof. Multiplying by a_0^{-1}, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $\mathbf{k}[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = - \sum_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\| r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for m. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\| r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\{A^{m+1}, A \cdot A^m\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in \mathbf{k}\{\underline{T/r}\}$. $\qquad\square$

Let $\mathbf{k}$ be a complete non-archimedean field. Recall that a derivative operator $\partial : \mathbf{k}\{\underline{T/r}\} \to \mathbf{k}\{\underline{T/r}\}$ is defined as the $\mathbf{k}$-linear map such that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we have Yang: To be revised.

Proposition 1.5. Let $\mathbf{k}$ be a complete non-archimedean field, and ∂ be a derivative operator on $\mathbf{k}\{\underline{T/r}\}$. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we have $\partial(f) \in \mathbf{k}\{\underline{T/r}\}$.

Proof. Yang: We only need to check the case $\partial = \partial/\partial T_1$. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha \in \mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1 - 1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}.$$

Noting that $\mathbf{k}$ is non-archimedean, we have $\|\alpha_1 a_\alpha\| \leq \|a_\alpha\|$. Then

$$\lim_{|\alpha| \to +\infty} \|\alpha_1 a_\alpha\| r_1^{\alpha_1 - 1} r_2^{\alpha_2} \cdots r_n^{\alpha_n} \leq \frac{1}{r_1} \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0.$$

The conclusion follows.   □

2   Analytic functions on closed polydisks

Proposition 2.1. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we can associate a function $F_f : E(0, \underline{r}) \to \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \quad \text{for } x = (x_1, \ldots, x_n) \in E(0, \underline{r}).$$

This defines a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T/r}\}$ to the ring of all functions from $E(0, \underline{r})$ to $\mathbf{k}$.

Proof. Given $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \ldots, x_n) \in E(0, \underline{r})$, we have

$$\left\| \sum_{|\alpha| = n} a_\alpha x^\alpha \right\| \leq \max_{|\alpha| = n} \|a_\alpha\| r^\alpha \to 0 \quad \text{as } n \to +\infty.$$

Hence by **??**, the series $F_f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$ converges in $\mathbf{k}$. This defines a function $F_f : E(0, \underline{r}) \to \mathbf{k}$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. Set

$$A_n = \sum_{|\alpha| < n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta| < n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma| < n} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) x^\gamma.$$

We need to show that $F_f(x) F_g(x) = \lim A_n B_n = \lim C_n = F_{fg}(x)$. Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha| < n, |\beta| < n \\ |\alpha + \beta| \geq n}} a_\alpha b_\beta x^{\alpha + \beta}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon / \|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon / \|f\|$. For any $n > 2N$, we have

$$\|A_n B_n - C_n\| \leq \max_{\substack{|\alpha| < n, |\beta| < n \\ |\alpha + \beta| \geq n}} \|a_\alpha\| \|b_\beta\| \|x^{\alpha + \beta}\| < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Thus $F_f(x) F_g(x) = (F_{fg})(x)$. The addition and scalar multiplication can be verified directly. We thus finish the proof.   □

Proposition 2.2. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x, y \in E(0, \underline{r})$, we have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq L \cdot \|y - x\|_{\infty},$$

where $L = \max_{1 \leq i \leq n} \|f\|_g / r_i$.

Proof. Set $y - x = (h_1, \ldots, h_n)$ and $x^{(0)} = x$, $x^{(i)} = (x_1 + h_1, \ldots, x_i + h_i, x_{i+1}, \ldots, x_n)$ for $i = 1, \ldots, n$. We have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{1 \leq i \leq n} \|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}}.$$

We only need to show that for every $i = 1, \ldots, n$, we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}} \leq \frac{\|f\|_g}{r_i} \|h_i\|.$$

Without loss of generality and for simplicity, we assume that $y = (x_1 + h, x_2, \ldots, x_n)$ and $x = (x_1, x_2, \ldots, x_n)$. Note that by the strong triangle inequality, we have $\|h\| \leq r_1$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$f(y) - f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \left((x_1 + h)^{\alpha_1} - x_1^{\alpha_1}\right) x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$$= \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k.$$

Note that

$$\left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| r_1^k \leq \|a_\alpha\| r^\alpha \leq \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{\alpha \in \mathbb{N}^n} \max_{1 \leq k \leq \alpha_1} \left\{ \left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| \|h\|^k \right\} \leq \max_k \left\{ \|f\|_g \left(\frac{\|h\|}{r_1}\right)^k \right\} \leq \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. $\qquad\square$

Lemma 2.3. Let $\mathbf{k}$ be a complete non-archimedean field. Then we have $\|f(x)\| \leq \|f\|$ for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$. In particular, if $f_n \to f$ as $n \to +\infty$ in $\mathbf{k}\{\underline{T/r}\}$, then we have $\|f_n(x) - f(x)\| \to 0$ for every $x \in E(0, \underline{r})$.

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \ldots, x_n) \in E(0, \underline{r})$. We have

$$\left\| \sum_{|\alpha| < N} a_\alpha x^\alpha \right\| \leq \max_{|\alpha| < N} \|a_\alpha\| r^\alpha \leq \|f\|$$

for every $N \in \mathbb{N}$. Taking $N \to +\infty$, we have $\|f(x)\| \leq \|f\|$. $\qquad\square$

Proposition 2.4. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation, and $\partial_i = \partial/\partial T_i$ be the derivative operator on $\mathbf{k}\{\underline{T/r}\}$ with respect to the indeterminate T_i for $i = 1, \ldots, n$.

Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$, we have

$$F_{\partial_i(f)}(x) = \lim_{h \to 0} \frac{F_f(x_1, \ldots, x_i + h, \ldots, x_n) - F_f(x)}{h}.$$

Proof. Without loss of generality, we can assume that $i = 1$. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $f_n = \sum_{|\alpha| < n} a_\alpha T^\alpha$ for $n \in \mathbb{N}$. Set $x_h = (x_1 + h, x_2 \ldots, x_n)$ and $L_f(h) = (F_f(x_h) - F_f(x))/h$ for $h \in \mathbf{k}^\times$. Note that for fixed h, we have $\lim_{n \to \infty} L_{f_n}(h) = L_f(h)$.

We compute $L_{f_n}(h) - F_{\partial f_n}(x)$ explicitly:

$$L_{f_n}(h) - F_{\partial f_n}(x) = \frac{1}{h}\left(\sum_{|\alpha| < n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} h^k x_2^{\alpha_2} \cdots x_n^{\alpha_n} - \sum_{|\alpha| < n} \alpha_1 a_\alpha x_1^{\alpha_1 - 1} h x_2^{\alpha_2} \cdots x_n^{\alpha_n}\right)$$

$$= \sum_{|\alpha| < n} \sum_{k=2}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^{k-1}.$$

Note that

$$M = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\| r_1^{k-1} \leq \|f\|/r_1 < +\infty.$$

Hence

$$\|L_{f_n}(h) - F_{\partial f_n}(x)\| \leq \max_{2 \leq k \leq n}\left\{M \frac{\|h\|^{k-1}}{r_1^{k-1}}\right\} \leq M \frac{\|h\|}{r_1}$$

for $h \in \mathbf{k}^\times$ with $\|h\| < r_1$. Taking $n \to +\infty$, we have

$$\|L_f(h) - F_{\partial f}(x)\| \leq M \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. $\qquad\square$

Corollary 2.5. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment $f \mapsto F_f$ in Proposition 2.1 is injective.

Proof. Note that if $F_f = 0$, then for every $i = 1, \ldots, n$, we have $F_{\partial_i(f)} = 0$ by Proposition 2.4. By taking repeated derivatives, we have $F_{\partial^\alpha f} = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Note that $F_{\partial^\alpha f}(0) = \alpha! a_\alpha$. It follows that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n$ and thus $f = 0$. $\qquad\square$

Remark 2.6. Corollary 2.5 holds for non-archimedean fields of positive characteristic as well. The proof uses Theorem 2.10 and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element $f \in \mathbf{k}\{\underline{T/r}\}$ with the associated function $F_f : E(0, \underline{r}) \to \mathbf{k}$ as in Proposition 2.1.

Proposition 2.7. Let $\Bbbk$ be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra $\Bbbk\{\underline{T/r}\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in E(0, \underline{r})} \|f(x)\|_{\Bbbk}.$$

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \Bbbk\{\underline{T}/r\}$. We write $f = g+h$ with $g = \sum_{\alpha \in S} a_\alpha T^\alpha$ and $h = \sum_{\alpha \notin S} a_\alpha T^\alpha$, where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that S is a non-empty finite set and $\|h\| < \|f\|$. By Lemma 2.3, we have $\|h(x)\| < \|f\|$ for every $x \in E(0,\underline{r})$. It suffices to show that $\|g\|_{\sup} = \|g\|$.

Since $\Bbbk$ is algebraically closed, $|\Bbbk^\times|$ is dense in $\mathbb{R}_{>0}$. For every pair $\alpha, \beta \in S$ with $\alpha \neq \beta$, the set $\{t \in \mathbb{R}_{>0}^n : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$ is a proper closed subset of $\mathbb{R}_{>0}^n$. Thus we can find $t_m \in |\Bbbk^\times|^n$ such that $t_m < r$, $t_m \to r$ as $m \to +\infty$ and for every $\alpha, \beta \in S$ with $\alpha \neq \beta$, we have $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$ for all m. For each m, we can find $x_m \in E(0,\underline{r})$ such that $\|x_m^\alpha\| = t_m^\alpha$ for every $\alpha \in S$ since $t_m \in |\Bbbk^\times|^n$. It follows that

$$\|g(x_m)\| = \max_{\alpha \in S} \|a_\alpha\| \|x_m^\alpha\| = \max_{\alpha \in S} \|a_\alpha\| t_m^\alpha \to \|g\| \quad \text{as } m \to +\infty.$$

Thus $\|g\|_{\sup} = \|g\|$. $\qquad\square$

Remark 2.8. If $\mathbf{k}$ is not algebraically closed, the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T}/r\}$ may not coincide with the supremum norm. For example, consider the Tate algebra $\mathbb{Q}_p\{T\}$. The element $f = T^p - T$ has gauss norm $\|f\| = 1$. However, for every $x \in E(0,1) = \mathbb{Z}_p$, we have $f(x) = x^p - x \equiv 0$ mod p. Thus $\|f(x)\|_p \leq 1/p$ and $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$.

Remark 2.9. Recall the Weierstrass-Stone theorem in classical analysis which states that the closure of the polynomial ring $\mathbb{C}[T_1, \ldots, T_n]$ with respect to the supremum norm on a closed polydisc $E \subset \mathbb{C}^n$ is the ring of all complex-valued continuous functions on E.

In the context of non-archimedean analysis, Proposition 2.7 can be viewed as an analogue of this theorem. It states that the closure of the polynomial ring $\Bbbk[T_1, \ldots, T_n]$ with respect to the supremum norm on a closed polydisc $E(0,\underline{r}) \subset \Bbbk^n$ is the Tate algebra $\Bbbk\{\underline{T}/r\}$.

From this perspective, the Tate algebra can be viewed as the "correct" analogue of the ring of continuous functions on a closed polydisc in non-archimedean analysis.

Theorem 2.10 (Strassman)**.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$ be an analytic function. Suppose that $\|a_N\| r^N > \|a_n\| r^n$ for all $n > N$. Then f has at most N zeros in the closed ball $E(0,r)$.

Proof. We induct on N. The case $N = 0$ is direct from Proposition 1.4. Suppose that the conclusion holds for $N - 1$. Let x be a zero of f in $E(0,r)$. Set

$$g(T) = \frac{f(T) - f(x)}{T - x} = \sum_{k=0}^{+\infty} \left(\sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n \geq k+1} \|a_n\| r^{n-1} \to 0 \quad \text{as } k \to \infty.$$

It follows that $g(T) \in \mathbf{k}\{T/r\}$.

For every $n > N$, we have

$$\|a_N\| > \|a_n\|r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\|\sum_{n=N}^{N+m} a_n x^{n-N}\right\| = \|a_N\|$$

for every $m \in \mathbb{N}$ by **??**. Take $m \to +\infty$, we have $\|b_{N-1}\| = \|a_N\|$. For every $k > N - 1$, we have

$$\|b_k\|r^k = \max_{n \geq k+1}\|a_n\|r^{n-1} \leq \max_{n > N}\|a_n\|r^{n-1} < \|a_N\|r^{N-1} = \|b_{N-1}\|r^{N-1}.$$

By the induction hypothesis, g has at most $N - 1$ zeros in $E(0, r)$. It follows that f has at most N zeros in $E(0, r)$ since $f(T) = (T - x) \cdot g(T)$. $\qquad\square$

3 Elementary functions

3.1 Elementary functions

Lemma 3.1. Let p be a prime number and $n \in \mathbb{N}$. We have $v_p(n!) =$.

Yang: Exponential, logarithmic, and the interpolation functions.

Fix a prime number p in the following and consider $\mathbf{k} = \mathbb{Q}_p, \mathbb{C}_p$, or Ω_p. Let $r_p := p^{-1/(p-1)}$.

Construction 3.2. The *exponential function* $\exp : \mathbf{k} \to \mathbf{k}$ is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The radius of convergence of $\exp(x)$ is $+\infty$ if $p = 2$ and $p^{-1/(p-1)}$ if $p > 2$.

The *logarithmic function* $\log : 1 + \mathbf{k}^{\circ\circ} \to \mathbf{k}$ is defined by the power series

$$\log(1 + x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

The radius of convergence of $\log(1 + x)$ is 1.

Moreover, for every x in the domain of convergence of exp and every y in the domain of convergence of log, we have

$$\log(\exp(x)) = x, \quad \exp(\log(y)) = y.$$

Yang: To be checked.

Definition 3.3. Let

Theorem 3.4. The series converges.

4   Appendix

Theorem 4.1. Let R be a Banach ring. The spectrum $\mathcal{M}(R)$ is nonempty.

References

[Gou97]   Fernando Q. Gouvêa. *p-adic Numbers: An Introduction.* 2nd ed. Universitext. Berlin,
           Heidelberg: Springer, 1997. ISBN: 978-3-642-59058-0. DOI: 10.1007/978-3-642-59058-0.
           URL: https://link.springer.com/book/10.1007/978-3-642-59058-0 (cit. on p. 1).

[Rob00]   Alain M Robert. *A course in p-adic analysis.* Vol. 198. Graduate Texts in Mathematics.
           Springer Science & Business Media, 2000 (cit. on p. 1).

[李文威 18] 李文威. 代数学方法（第一卷）基础架构. 现代数学基础. 版面字数: 612 千字, 全书页数:
           448 页. 北京: 高等教育出版社, 2018. ISBN: 978-7-04-050725-6 (cit. on p. 1).