Arithmetic Geometry

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Arithmetic Geometry

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Source code: GitHub Repository

Version: 0.1.0

Last updated: January 10, 2026

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Chapter 1

Valuation on fields

1.1 Valuation fields

1.1.1 Absolute values and completion

Definition 1.1.1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\| \cdot | : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\| \cdot \|$ is called a *valuation field*.

Remark 1.1.2. Let $\mathbf{k}$ be a field. Recall that a (additive) valuation on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $| \cdot |_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

The valuation v defined above is called an *additive valuation*. And an absolute value $| \cdot |$ on $\mathbf{k}$ is called a *multiplicative valuation*. In this note, the term *valuation* may refer to either an additive valuation or a multiplicative valuation, depending on the context.

Example 1.1.3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 1.1.4. The *(multiplicative) valuation group* of a valuation field $(\mathbf{k}, \|\cdot\|)$ is defined as the subgroup of $\mathbb{R}_{>0}$ given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation $\sqrt{|\mathbf{k}^\times|}$ to denote the set $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$.

Note that an absolute value $\|\cdot\|$ is non-trivial if and only if its valuation group $|\mathbf{k}^\times|$ is not equal to $\{1\}$.

Definition 1.1.5. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 1.1.6. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Simple analysis. □

Proposition 1.1.7. Let $(\mathbf{k}, \|\cdot\|)$ be a complete valuation field with non-trivial absolute value. Then $\mathbf{k}$ is uncountable.

Proof. Since the absolute value $\|\cdot\|$ is non-trivial, we can construct a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathbf{k}$ inductively such that $\|x_n\| < \|x_{n-1}\|/2$ for any $n \geq 1$ and $\|x_0\| < 1$. Then there is an injective map from $\mathbb{N}^{\{0,1\}}$ to $\mathbf{k}$ defined by

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n x_n, \quad a_n \in \{0, 1\}.$$

Since $\|x_n\| < 2^{-n}$, the series $\sum_{n=1}^\infty a_n x_n$ converges in $\mathbf{k}$. Note $\|x_n\| > \|\sum_{m\geq n} x_m\|$ for each n, we have that the map is injective. Thus $\mathbf{k}$ is uncountable. □

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 1.1.8. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 1.1.9. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

Example 1.1.10. Let $|\cdot|_\infty$ be the usual absolute value on the field $\mathbb{Q}$ of rational numbers. Then $(\mathbb{Q}, |\cdot|_\infty)$ is a valuation field. Its completion is the field $\mathbb{R}$ of real numbers equipped with the usual absolute value.

Example 1.1.11. Let p be a prime number. For any non-zero rational number $x \in \mathbb{Q}$, we can write it as $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then $(\mathbb{Q}, |\cdot|_p)$ is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of p-adic numbers equipped with the p-adic absolute value; see Yang: to be added..

Definition 1.1.12. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Proposition 1.1.13. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. Then $\mathbf{k}$ is archimedean if and only if the set $\{\|n \cdot 1\| : n \in \mathbb{Z}\}$ is unbounded.

Proof. Yang: To be added. □

1.1.2 Places on a field

Definition 1.1.14. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c \in (0, \infty)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$. Moreover, the following lemma shows that the converse is also true.

Lemma 1.1.15. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) $\Rightarrow$ (b) is obvious. Now we prove (b) $\Rightarrow$ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) $\Rightarrow$ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. Suppose the contrary that $\alpha \neq \beta$. Consider the domain $\Lambda \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Λ is non-empty. Pick

$(n, m) \in \Lambda$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. □

Definition 1.1.16. Let $\mathbf{k}$ be a field. A *place* on $\mathbf{k}$ is an equivalence class of absolute values on $\mathbf{k}$. We denote the set of all places on $\mathbf{k}$ by $\mathrm{Pl}_{\mathbf{k}}$.

Theorem 1.1.17. Let $(\mathbf{k}, \|\cdot\|)$ be an archimedean complete valuation field. Then $\mathbf{k}$ is isomorphic to either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$ equipped with the usual absolute value. Yang: To be revised.

Proof. Yang: To be added. □

Theorem 1.1.18 (Ostrowski's theorem)**.** Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to either the usual absolute value $|\cdot|_\infty$ or a p-adic absolute value $|\cdot|_p$ for some prime number p. Yang: To be revised.

Proof. Yang: To be added. □

1.2 Non-archimedean valuations

1.2.1 Topology: Ultra-metric space

We will use $B(x, r)$ (resp. $E(x, r)$) to denote the open ball (resp. closed ball) with center x and radius r.

Definition 1.2.1. A metric space (X, d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x, z) \le \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.$$

Remark 1.2.2. The term *ultra-metric space* should be translated into Chinese as "奥特度量空间". There is no special reason for this translation, except that I insist on using "奥特" to translate "ultra".

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x, y) := \|x - y\|$ on $\mathbf{k}$ makes $(\mathbf{k}, d)$ an ultra-metric space.

Proposition 1.2.3. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x, y) \ge d(y, z)$. By the strong triangle inequality, we have

$$d(x, z) \le \max\{d(x, y), d(y, z)\} = d(x, y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x, y) \le \max\{d(x, z), d(z, y)\} = \max\{d(x, z), d(y, z)\} \le d(x, y).$$

This shows that $d(x, y) = \max\{d(x, z), d(y, z)\}$. Thus either $d(x, z) = d(x, y) \ge d(y, z)$ or $d(y, z) = d(x, y) \ge d(x, z)$. □

Proposition 1.2.4. Let (X, d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1, 2$. If $D_1 \cap D_2 \neq \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1, 2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1, x_2) \leq \max\{d(x_1, y), d(y, x_2)\}.$$

Without loss of generality, we may assume that $d(x_1, x_2) \leq d(x_1, y)$. It follows that $x_2 \in D_1$ since $d(x_1, y) < r_1$ (or $\leq r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1, z) > d(x_1, x_2)$. Then by Proposition 1.2.3,

$$r_1 \leq d(x_1, z) = d(x_2, z) \leq r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2, w) \leq \max\{d(x_2, x_1), d(x_1, w)\} \leq r_1 \leq r_2.$$

Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. □

Proposition 1.2.5. Let (X, d) be an ultra-metric space. Then both $B(x, r)$ and $E(x, r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x, r) := \{y \in X \mid d(x, y) = r\}$ is open in X. Note that if $y \in S(x, r)$, then for any $r' < r$, we have $B(y, r') \cap E(x, r) \neq \varnothing$ and $x \in E(x, r) \setminus B(y, r')$. Thus by Proposition 1.2.4, we have $B(y, r') \subseteq E(x, r)$. If $B(y, r') \cap B(x, r) \neq \varnothing$, then by Proposition 1.2.4 again, we have $B(y, r') \subseteq B(x, r)$. However, $y \in B(y, r') \setminus B(x, r)$, a contradiction. Thus $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$. It yields that $S(x, r) = \bigcup_{y \in S(x,r)} B(y, r/2)$ is open in X.

Since $E(x, r) = B(x, r) \cup S(x, r)$ and $B(x, r) = E(x, r) \setminus S(x, r)$, both $B(x, r)$ and $E(x, r)$ are open and closed in X. □

Corollary 1.2.6. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x, y) > 0$. Consider the open ball $B(x, r/2)$. By Proposition 1.2.5, $B(x, r/2)$ is both open and closed in X. Thus $B(x, r/2) \cap S$ is both open and closed in S, however, it is non-empty and not equal to S since it contains x but not y. This shows that S is disconnected. □

Proposition 1.2.7. Let (X, d) be an ultra-metric space. A sequence $\{x_n\}$ in X is cauchy if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Proof. The necessity is true for all metric spaces. Suppose that $d(x_n, x) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \varepsilon$ for all $n \geq N$. For any $m, n \geq N$ with $m < n$, by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \dots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that $\{x_n\}$ is a cauchy sequence. □

1.2.2 Algebra: ring of integers and residue field

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 1.2.8. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathscr{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^\circ/\mathbf{k}^{\circ\circ}.$$

Set $I_{r,<} := B(0,r)$ and $I_{r,\leq} := E(0,r)$ for each $r \in [0,1]$.

Proposition 1.2.9. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers $\mathbf{k}^\circ$. Conversely, any ideal of $\mathbf{k}^\circ$ is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0,1)$.

Proof. Let I be an ideal of $\mathbf{k}^\circ$. Set $r = \sup\{|a| : a \in I\}$ (resp. $r = \max\{|a| : a \in I\}$ when the maximum exists). Then, by definition, we have $I \subset I_{r,<}$ (resp. $I \subset I_{r,\leq}$). For every $x \in \mathbf{k}^\circ$ with $|x| < r$ (resp. $|x| \leq r$), there exists $a \in I$ such that $|x| \leq |a|$. Thus, $|x/a| \leq 1$ and so $x/a \in \mathbf{k}^\circ$. Since I is an ideal, we have $x = (x/a)a \in I$. Therefore, $I_{r,<} \subset I$ (resp. $I_{r,\leq} \subset I$). □

Proposition 1.2.10. Let I_r be either $I_{r,<}$ or $I_{r,\leq}$ for each $r \in (0,1)$. Suppose $\{r_n \in (0,1)\}_{n\in\mathbb{N}}$ is a decreasing sequence converging to 0. Then the completion $\widehat{\mathbf{k}}$ of $\mathbf{k}$ is isomorphic to the projective limit

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}.$$

Proof. For every $x \in \widehat{\mathbf{k}}^\circ$, there exists a cauchy sequence $\{x_m\}_{m\in\mathbb{N}}$ in $\mathbf{k}^\circ$ converging to x. Since $\{r_n\}_{n\in\mathbb{N}}$ converges to 0, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that for all $m, m' \geq M_n$, we have $|x_m - x_{m'}| < r_n$. Thus, the sequence $\{x_m + I_{r_n}\}_{m\in\mathbb{N}}$ is eventually constant in $\mathbf{k}^\circ/I_{r_n}$. Define a map

$$\Phi : \widehat{\mathbf{k}}^\circ \to \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}, \quad x \mapsto \left(\lim_{m\to\infty} x_m + I_{r_n}\right)_{n\in\mathbb{N}}.$$

It is straightforward to verify that Φ is a well-defined ring homomorphism.

Conversely, for every $(a_n + I_{r_n})_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}$, we can choose a representative $a_n \in \mathbf{k}^\circ$ for each n. We claim that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathbf{k}^\circ$. Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_N < \varepsilon$. For all $m, n \geq N$, since $a_n + I_{r_n}$ maps to $a_m + I_{r_m}$ under the natural projection, we have $|a_n - a_m| < r_N < \varepsilon$. Thus, $\{a_n\}_{n\in\mathbb{N}}$ converges to some $x \in \widehat{\mathbf{k}}^\circ$. Easily see that the limit x is independent of the choice of representatives $\{a_n\}_{n\in\mathbb{N}}$. This gives a map

$$\Psi : \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n} \to \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} a_n.$$

Direct verification shows that $\Psi = \Phi^{-1}$. □

Corollary 1.2.11. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the residue field $\mathscr{k}_{\widehat{\mathbf{k}}} \cong \mathscr{k}_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$.

Corollary 1.2.12. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the valuation group $|\widehat{\mathbf{k}}^\times|$ of $\widehat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^\times|$ of $\mathbf{k}$.

Proof. Note that

$$\begin{aligned} r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneqq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\ &\iff \widehat{\mathbf{k}}^\circ/I_{r,<} \to \widehat{\mathbf{k}}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff \mathbf{k}^\circ/I_{r,<} \to \mathbf{k}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff I_{r,<} \subsetneqq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\ &\iff r \in |\mathbf{k}^\times|. \end{aligned}$$

□

Proposition 1.2.13. Let $\mathbf{k}$ be a non-archimedean field with non-trivial valuation. Then $\mathbf{k}^\circ$ is totally bounded iff $\mathbf{k}^\circ/I_{r,<}$ and $\mathbf{k}^\circ/I_{r,\leq}$ are finite for each $r \in [0,1]$. Moreover, if $\mathbf{k}$ is complete, then it is locally compact iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$.

Slogan *"Locally compact $\iff$ pro-finite."*

Proof. We just prove the case for $I_r = I_{r,<}$. The case for $I_r = I_{r,\leq}$ is similar.

Suppose that $\mathbf{k}^\circ/I_r$ is finite for each $r \in [0,1]$. Then for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that $r < \varepsilon$ and $\mathbf{k}^\circ/I_r$ is finite. Let $\{a_1 + I_r, \dots, a_n + I_r\}$ be the complete set of representatives of $\mathbf{k}^\circ/I_r$. Then the balls $B(a_i, r)$ for $i = 1, \dots, n$ cover $\mathbf{k}^\circ$.

Conversely, suppose that $\mathbf{k}^\circ/I_r$ is infinite for some $r \in [0,1]$. Then there exists an infinite set $\{a_n\}$ with $|a_n| \in [r,1]$ such that their images in $\mathbf{k}^\circ/I_r$ are distinct. In particular, for every $m \neq n$, we have $|a_n - a_m| \geq r$. Any subsequence of $\{a_n\}$ is not cauchy. Thus, $\mathbf{k}^\circ$ is not totally bounded. □

Proposition 1.2.14. The ring $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field.

Proof. Note that $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$ is a multiplicative subgroup. If $\mathbf{k}$ is not a discrete valuation field, then $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. In particular, there exists a strictly ascending sequence $r_n \in |\mathbf{k}^\times| \cap (0,1)$. Then the ideals $I_{r_n,\leq}$ form a strictly ascending chain of ideals in $\mathbf{k}^\circ$.

The converse is standard since now $\mathbf{k}^\circ$ is a discrete valuation ring. □

Proposition 1.2.15. Let $\mathbf{k}$ be a complete non-archimedean field. Then $\mathbf{k}$ is locally compact iff $\mathbf{k}$ is a discrete valuation field and its residue field $\mathscr{k}_{\mathbf{k}}$ is finite.

Proof. The necessity follows from Proposition 1.2.13. For the sufficiency, suppose that $\mathbf{k}$ is a discrete valuation field whose residue field $\mathscr{k}_{\mathbf{k}}$ is finite. Let $\pi \in \mathbf{k}^\circ$ be a uniformizer. We only need to show that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite for each $n \in \mathbb{N}$. Note that there is an isomorphism

$$\pi^{n-1}\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ \cong \mathscr{k}_{\mathbf{k}}, \quad x + \pi^n\mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on n, we conclude that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite. □

1.2.3 Hensel's Lemma

Theorem 1.2.16 (Hensel's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \Bbbk_{\mathbf{k}}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \Bbbk_{\mathbf{k}}[T]$ are monic polynomials that are coprime in $\Bbbk_{\mathbf{k}}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\Bbbk_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

Proof. Since $\gcd(g, h) = 1$ in $\Bbbk_{\mathbf{k}}[T]$, there exist polynomials $u(T), v(T) \in \Bbbk_{\mathbf{k}}[T]$ such that $ug + vh = 1$ and $\deg u < \deg h, \deg v < \deg g$. Choose lifts $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$ of $g(T), h(T), u(T), v(T)$ respectively preserving their degrees such that G_0 and H_0 are monic. Then there exist $r < 1$ such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \mod I_r, \quad F(T) - G_0(T)H_0(T) \equiv 0 \mod I_r,$$

where $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$.

We will construct a sequence of monic polynomials $\{G_n(T)\}_{n\in\mathbb{N}}$ and $\{H_n(T)\}_{n\in\mathbb{N}}$ in $\mathbf{k}^\circ[T]$ such that for each $n \in \mathbb{N}$,

$$G_n(T) \equiv G_{n-1}(T) \mod I_{r^n}, \quad H_n(T) \equiv H_{n-1}(T) \mod I_{r^n},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \mod I_{r^{n+1}}.$$

If we have such sequences, then their coefficients converge in the complete ring $\mathbf{k}^\circ$. Let $G(T)$ and $H(T)$ be the limits of $\{G_n(T)\}$ and $\{H_n(T)\}$ respectively. Then we have $F(T) = G(T)H(T)$ and the reductions of $G(T), H(T)$ in $\Bbbk_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

The case $n = 0$ is done by the above construction. Now suppose that we have constructed $G_n(T)$ and $H_n(T)$ for some $n \geq 0$. Since $G_n - G_0 \equiv 0 \mod I_r$ and $H_n - H_0 \equiv 0 \mod I_r$, we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \mod I_r.$$

Set $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$ and $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$. Then we have

$$\begin{aligned}(G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T].\end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

This finishes the induction. □

Corollary 1.2.17. Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \tilde{k}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $a \in \tilde{k}_{\mathbf{k}}$. Then there exists a root $\alpha \in \mathbf{k}^\circ$ of $F(T)$ whose reduction is a.

Proof. Since a is a simple root of $f(T)$, we have the factorization $f(T) = (T-a)h(T)$ for some monic polynomial $h(T) \in \tilde{k}_{\mathbf{k}}[T]$ with $h(a) \neq 0$. Then the result follows from Theorem 1.2.16. □

1.2.4 Newton polygons

Yang: To be filled.

1.3 Finite field extensions

1.3.1 Finite-dimensional vector space

Definition 1.3.1. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|ax\| = |a| \cdot \|x\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Example 1.3.2. Let $\mathbf{k}$ be a valuation field and V a finite-dimensional vector space over $\mathbf{k}$ with basis $\{e_1, e_2, \dots, e_n\}$. For any $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$, define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then $\|\cdot\|_{\max}$ is a norm on V, called the *maximal norm* with respect to the basis $\{e_1, e_2, \dots, e_n\}$.

Example 1.3.3. Setting as in Example 1.3.2, for any $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$, define

$$\|x\|_1 := |a_1| + |a_2| + \cdots + |a_n|.$$

Then $\|\cdot\|_1$ is also a norm on V.

Definition 1.3.4. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Lemma 1.3.5. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V.

Proof. The sufficiency is clear. Now suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on V. Hence the unit open ball with respect to $\|\cdot\|_1$ contains a unit open ball with respect to $\|\cdot\|_2$. That

is,
$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$
Then for every $x \in V$ with $\|x\|_1 = 1$, we have $\|x\|_2 \geq C = C\|x\|_1$. By scaling, we get that for every $x \in V$,
$$\|x\|_2 \geq C\|x\|_1.$$
Similar for the other direction, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Proposition 1.3.6. Let V be a normed finite-dimensional vector space over a complete valuation field $\mathbf{k}$. Then V is complete.

Proof. Yang: To be added. □

Theorem 1.3.7. Let V be a finite-dimensional vector space over a complete field $\mathbf{k}$. Then all norms on V are equivalent.

Proof. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis as in Example 1.3.2. Let $\|\cdot\|$ be any norm on V. It suffices to show that $\|\cdot\|$ and $\|\cdot\|_{\max}$ are equivalent. First we have
$$\|y\| \leq \sum_{i=1}^{n} |a_i|\|e_i\| \leq \left(\sum_{i=1}^{n} \|e_i\|\right)\|y\|_{\max}$$
for any $y = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$. It remains to show that there exists a constant $C > 0$ such that for any $y \in V$,
$$\|y\|_{\max} \leq C\|y\|.$$
Yang: To be added. □

Remark 1.3.8. If the base field $\mathbf{k}$ is not complete, then Theorem 1.3.7 may fail. For example, let $\mathbf{k} = \mathbb{Q}$ with the usual absolute value, and let $V = \mathbb{Q}[\alpha]$ with $\alpha^2 - \alpha - 1 = 0$. There are two embeddings of V into $\mathbb{R}$:
$$\iota_1 : a + b\alpha \mapsto a + b\frac{1+\sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b\frac{1-\sqrt{5}}{2}.$$
Define two norms on V by
$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$
where $|\cdot|$ is the usual absolute value on $\mathbb{R}$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent since $\iota_2(\alpha^n) \to 0$ as $n \to \infty$ while $\iota_1(\alpha^n) \to \infty$.

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

Lemma 1.3.9. Let $\mathbf{k}$ be a complete field and V a normed finite-dimensional vector space over $\mathbf{k}$. Then
$$\|\cdot\| : \mathrm{End}_{\mathbf{k}}(V) \to \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V\setminus\{0\}} \frac{\|T(x)\|}{\|x\|}$$
defines a norm on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ satisfying
$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \mathrm{End}_{\mathbf{k}}(V).$$

Proof. First we show the existence of the supremum, i.e., there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\| \leq C\|x\|$. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis. Since all norms on V are bounded by each other by Theorem 1.3.7, we only need to show that there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\|_1 \leq C\|x\|_{\max}$. Write $T(e_i) = \sum_{j=1}^n a_{ij} e_j$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^n x_i e_i \in V$, we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \left(\sum_{1 \leq i,j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of $\|\cdot\|$ are clear. It remains to show the triangle inequality and sub-multiplicativity. For any $A, B \in \mathrm{End}_{\mathbf{k}}(V)$, we have

$$\frac{\|(A+B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all $x \in V \setminus \{0\}$ gives $\|A + B\| \leq \|A\| + \|B\|$. We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$. Taking supremum we get $\|AB\| \leq \|A\| \cdot \|B\|$. □

1.3.2 Finite field extensions

Lemma 1.3.10. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then there exists an absolute value on $\mathbf{l}$ extending the absolute value on $\mathbf{k}$.

Proof. Fix a norm $\|\cdot\|_V$ on the $\mathbf{k}$-vector space $V = \mathbf{l}$. The norm $\|\cdot\|_V$ induces an operator norm $\|\cdot\|_{\mathrm{op}}$ on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ as in Lemma 1.3.9. For any $a \in \mathbf{l}$, let $\mu_a \in \mathrm{End}_{\mathbf{k}}(V)$ be the $\mathbf{k}$-linear map defined by multiplication by a. Note that $a \mapsto \mu_a$ gives a embedding of $\mathbf{k}$-algebras and if $a \in \mathbf{k}$, $\|\mu_a\|_{\mathrm{op}} = \|a\|_{\mathbf{k}}$. Thus the restriction of $\|\cdot\|_{\mathrm{op}}$ to $\mathbf{l}$ gives an norm on $\mathbf{l}$ extending that on $\mathbf{k}$. The normed ring $(\mathbf{l}, \|\cdot\|_{\mathrm{op}})$ is a Banach ring since it is a finite-dimensional vector space over the complete field $\mathbf{k}$. By Theorem 4.1.7, there exists a multiplicative seminorm $\|\cdot\|_{\mathbf{l}}$ on $\mathbf{l}$ bounded by $\|\cdot\|_{\mathrm{op}}$. In particular, $\|\cdot\|_{\mathbf{l}}$ is bounded by $\|\cdot\|_{\mathbf{k}}$ on $\mathbf{k}$. On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus $\|\cdot\|_{\mathbf{l}}$ extends the absolute value on $\mathbf{k}$. □

Theorem 1.3.11. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the absolute value on $\mathbf{l}$ which extends the absolute value on $\mathbf{k}$ is uniquely determined by the absolute value on $\mathbf{k}$. Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where $n = [\mathbf{l} : \mathbf{k}]$ and $N_{\mathbf{l}/\mathbf{k}}$ is the norm map from $\mathbf{l}$ to $\mathbf{k}$.

Proof. Let $\|\cdot\|_{\mathbf{l}}$ be arbitrary absolute value on $\mathbf{l}$ extending that on $\mathbf{k}$. We will show that $\|\cdot\|_{\mathbf{l}}$ must be equal to $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$. For any $a \in \mathbf{l}$, set $b = a^n / N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$. Then $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that $\|b\|_{\mathbf{l}} = 1$ whenever $N_{\mathbf{l}/\mathbf{k}}(b) = 1$.

Note that the norm map $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \to \mathbf{k}$ is the determinant of the $\mathbf{k}$-linear map $\mu_b \in \mathrm{End}_{\mathbf{k}}(V)$ defined by multiplication by b. Hence it is continuous on $\mathbf{l}$ (since it is a polynomial in the entries of the matrix representation). If $\|b\|_{\mathbf{l}} < 1$, then $\|b^m\|_{\mathbf{l}} \to 0$ as $m \to \infty$. Thus $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \to 0$ as $m \to \infty$, contradicting the fact that $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$ for all m. Similarly, if $\|b\|_{\mathbf{l}} > 1$, then just consider b^{-1}. □

Proposition 1.3.12. Let $\mathbf{k}$ be an algebraically closed valuation field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

Proof. Let $f \in \widehat{\mathbf{k}}[X]$ be a non-constant polynomial. We will show that f has a root in $\widehat{\mathbf{k}}$. Take a sequence of polynomials $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbf{k}[X]$ converging to f coefficient-wisely and of the same degree d. Since $\mathbf{k}$ is algebraically closed, each f_n splits completely in $\mathbf{k}$ and hence in $\widehat{\mathbf{k}}$. Write $f_n(X) = \prod_{i=1}^{d}(X - \alpha_{n,i})$ with $\alpha_{n,i} \in \widehat{\mathbf{k}}$.

Let $\mathbf{l}$ be a finite extension of $\widehat{\mathbf{k}}$ such that f has a root α in $\mathbf{l}$. For every $\varepsilon > 0$, if there are infinitely many n such that $\alpha_{n,i} \notin B(\alpha, \varepsilon)$ for all $1 \le i \le d$, then we have $|f_n(\alpha)| \ge \varepsilon^d$ for infinitely many n, contradicting the fact that $f_n(\alpha) \to f(\alpha) = 0$. Thus for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \ge N$, there exists $1 \le i \le d$ with $\alpha_{n,i} \in B(\alpha, \varepsilon)$. That is, we can find a sequence $\alpha_{n,i_n} \in \mathbf{k}$ converging to α. Since $\widehat{\mathbf{k}}$ is complete, we have $\alpha \in \widehat{\mathbf{k}}$. □

1.3.3 Ramification and inertia

In this subsection, we study the extensions of absolute values on finite field extensions. Note that we do not assume the base field to be complete.

Definition 1.3.13. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. We denote by $w|v$ if $w \in M_L$ is an absolute value on L extending v. For each $w|v$, we define the *ramification index* $e(w|v)$ and the *inertia degree* $f(w|v)$ by

$$e(w|v) := [|\hat{L}^\times|_w : |\widehat{K}^\times|_v], \quad f(w|v) := \frac{[\hat{L} : \widehat{K}]}{e(w|v)},$$

where $\widehat{K}$ and $\hat{L}$ are the completions of K and L with respect to v and w, respectively.

Lemma 1.3.14. Suppose that v is non-archimedean and $\mathscr{k}_v$ and ℓ_w are the residue fields of K and L with respect to v and w, respectively. Then we have

$$f(w|v) = [\ell_w : \mathscr{k}_v].$$

Remark 1.3.15. Yang: To be added.

Theorem 1.3.16. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. Then we have

$$\sum_{w|v} e(w|v) f(w|v) = [L : K].$$

Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. We have

$$L \otimes_K K_v \cong \prod_{w|v} L_w,$$

where the product is taken over all absolute values $w \in M_L$ extending v.

Theorem 1.3.17. Let $\mathbf{k}$ be a number field. Then

$$M_{\mathbf{k}}^{\infty} = \{\text{embeddings } \sigma : \mathbf{k} \to \mathbb{C}\}$$

and

$$M_{\mathbf{k}}^{\mathrm{f}} = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_{\mathbf{k}}\}.$$

Yang: To be revised.

Proposition 1.3.18 (Product formula)**.** Let $\mathbf{k}$ be a number field. For each $x \in \mathbf{k}^{\times}$, we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^{\infty}; \\ 1, & v \in M_{\mathbf{k}}^{0}. \end{cases}$$

Yang: To be revised.

Remark 1.3.19. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. Suppose that v is non-archimedean. Yang: To be rewritten.

1.4 Artin-Whaples approximations

Theorem 1.4.1 (Artin-Whaples approximations)**.** Let K be a field, and let $v_1, v_2, \dots, v_n$ be pairwise inequivalent nontrivial absolute values on K. For any $a_1, a_2, \dots, a_n \in K$ and any $\varepsilon > 0$, there exists an element $x \in K$ such that

$$|x - a_i|_{v_i} < \varepsilon$$

for all $1 \le i \le n$. Yang: To be checked.

1.4.1 Geometric version

Theorem 1.4.2. Let $\mathbf{k}$ be a field with algebraic closure $\bar{\mathbf{k}}$. Let X be a normal, projective, geometrically integral variety over $\mathbf{k}$. Let $x_1, x_2, \dots, x_n \in X(\mathbf{k})$ be closed points lying over pairwise distinct points of X. Let $v_1, v_2, \dots, v_n \in M_{\mathbf{k}}$ be pairwise inequivalent absolute values on $\mathbf{k}$. For every $i = 1, 2, \dots, n$, let U_i be an open neighborhood of x_i in $X(\bar{\mathbf{k}})$ with respect to the topology induced by v_i. Then there exists a rational point $x \in X(\mathbf{k})$ such that $x \in U_i$ for all $1 \le i \le n$. Yang: To be revsied.

Yang: This gives [Xie25, Proposition 3.9]

Chapter 2

Non-archimedean analysis

2.1 Local theory I: functions

2.1.1 Tate algebras

Notation 2.1.1. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \dots, n$, we have $\alpha_i \leq \beta_i$;
- $E(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \dots, n\}$ and $B(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \dots, n\}$ for $x = (x_1, \dots, x_n) \in \mathbf{k}^n$;
- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

Definition 2.1.2. Let $\mathbf{k}$ be a complete non-archimedean field. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$\mathbf{k}\langle \underline{r^{-1}T} \rangle := \mathbf{k}\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 2.1.3. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ is a non-archimedean multiplicative banach $\mathbf{k}$-algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Yang: For the definition of banach ring, see

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha$ are two elements in $\mathbf{k}\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \le \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{\frac{\varepsilon}{\|g\|}\|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|}\|a_\alpha\| r^\alpha\right\} \le \varepsilon.$$

Hence $f \cdot g \in \mathbf{k}\{\underline{T/r}\}$ and it shows that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $\mathbf{k}\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f+g\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \le \sup_{\alpha\in\mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \le \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned}\|f\cdot g\| &= \left\|\sum_{\gamma\in\mathbb{N}^n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) T^\gamma\right\| = \sup_{\gamma\in\mathbb{N}^n}\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \\ &\le \sup_{\gamma\in\mathbb{N}^n}\max_{\alpha+\beta=\gamma} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \le \|f\| \cdot \|g\|.\end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\| \cdot \|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \left\|\sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta\right\| r^{\alpha_1+\beta_1} \le \|f\cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by Proposition 1.2.3. Thus the gauss norm is multiplicative.

Step 4. Finally show that $\mathbf{k}\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $\mathbf{k}\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \le \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in $\mathbf{k}$. Since $\mathbf{k}$ is complete, set $a_\alpha := \lim_{m\to+\infty} a_{\alpha,m}$ and $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \le \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \le 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \leq \varepsilon$$

for all $m > N$. □

Proposition 2.1.4. Let $\mathbf{k}$ be a complete non-archimedean field. An element $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ is invertible if and only if $\|a_0\| > \|a_\alpha\| r^\alpha$ for all $\alpha \neq 0$.

Proof. Multiplying by a_0^{-1}, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $\mathbf{k}[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = -\sum_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\| r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for m. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\| r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\left\{A^{m+1}, A \cdot A^m\right\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in \mathbf{k}\{\underline{T/r}\}$. □

Let $\mathbf{k}$ be a complete non-archimedean field. Recall that the formal derivative operator $\partial_i : \mathbf{k}[[\underline{T}]] \to \mathbf{k}[[\underline{T}]]$ is defined by

$$\frac{\partial}{\partial T_i} \left(\sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right) := \sum_{\alpha \in \mathbb{N}^n} \alpha_i a_\alpha T_1^{\alpha_1} \cdots T_i^{\alpha_i - 1} \cdots T_n^{\alpha_n}.$$

Lemma 2.1.5. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we have $\partial_i(f) \in \mathbf{k}\{\underline{T/r}\}$.

Proof. Suppose that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha\in\mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1-1}T_2^{\alpha_2}\cdots T_n^{\alpha_n}.$$

Noting that $\mathbf{k}$ is non-archimedean, we have $\|\alpha_1 a_\alpha\| \le \|a_\alpha\|$. Then

$$\lim_{|\alpha|\to+\infty} \|\alpha_1 a_\alpha\| r_1^{\alpha_1-1} r_2^{\alpha_2}\cdots r_n^{\alpha_n} \le \frac{1}{r_1}\lim_{|\alpha|\to+\infty}\|a_\alpha\| r^\alpha = 0.$$

The conclusion follows. □

2.1.2 Analytic functions on closed polydisks

Proposition 2.1.6. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we can associate a function $F_f : E(0,\underline{r}) \to \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha \qquad \text{for } x=(x_1,\dots,x_n)\in E(0,\underline{r}).$$

This defines a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T/r}\}$ to the ring of all functions from $E(0,\underline{r})$ to $\mathbf{k}$.

Proof. Given $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x=(x_1,\dots,x_n)\in E(0,\underline{r})$, we have

$$\left\|\sum_{|\alpha|=n} a_\alpha x^\alpha\right\| \le \max_{|\alpha|=n}\|a_\alpha\| r^\alpha \to 0 \qquad \text{as } n\to+\infty.$$

Hence by Proposition 1.2.7, the series $F_f(x) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha$ converges in $\mathbf{k}$. This defines a function $F_f : E(0,\underline{r}) \to \mathbf{k}$.

Let $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. Set

$$A_n = \sum_{|\alpha|<n} a_\alpha x^\alpha,\quad B_n = \sum_{|\beta|<n} b_\beta x^\beta,\quad C_n = \sum_{|\gamma|<n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right)x^\gamma.$$

We need to show that $F_f(x)F_g(x) = \lim A_nB_n = \lim C_n = F_{fg}(x)$. Note that

$$A_nB_n - C_n = \sum_{\substack{|\alpha|<n,|\beta|<n\\ |\alpha+\beta|\ge n}} a_\alpha b_\beta x^{\alpha+\beta}.$$

Given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for all $|\alpha|>N$, we have $\|a_\alpha\|r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\|r^\alpha < \varepsilon/\|f\|$. For any $n>2N$, we have

$$\|A_nB_n - C_n\| \le \max_{\substack{|\alpha|<n,|\beta|<n\\ |\alpha+\beta|\ge n}} \|a_\alpha\|\|b_\beta\|\|x^{\alpha+\beta}\| < \max\left\{\frac{\varepsilon}{\|g\|}\|b_\beta\|r^\beta, \frac{\varepsilon}{\|f\|}\|a_\alpha\|r^\alpha\right\} \le \varepsilon.$$

Thus $F_f(x)F_g(x) = (F_{fg})(x)$. The addition and scalar multiplication can be verified directly. We thus finish the proof. □

Proposition 2.1.7. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x, y \in E(0, \underline{r})$, we have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq L \cdot \|y - x\|_\infty,$$

where $L = \max_{1 \leq i \leq n} \|f\|_g / r_i$.

Proof. Set $y - x = (h_1, \ldots, h_n)$ and $x^{(0)} = x$, $x^{(i)} = (x_1 + h_1, \ldots, x_i + h_i, x_{i+1}, \ldots, x_n)$ for $i = 1, \ldots, n$. We have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{1 \leq i \leq n} \|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}}.$$

We only need to show that for every $i = 1, \ldots, n$, we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}} \leq \frac{\|f\|_g}{r_i} \|h_i\|.$$

Without loss of generality and for simplicity, we assume that $y = (x_1 + h, x_2, \ldots, x_n)$ and $x = (x_1, x_2, \ldots, x_n)$. Note that by the strong triangle inequality, we have $\|h\| \leq r_1$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\begin{aligned} f(y) - f(x) &= \sum_{\alpha \in \mathbb{N}^n} a_\alpha \left((x_1 + h)^{\alpha_1} - x_1^{\alpha_1} \right) x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ &= \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k. \end{aligned}$$

Note that

$$\left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| r_1^k \leq \|a_\alpha\| r^\alpha \leq \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{\alpha \in \mathbb{N}^n} \max_{1 \leq k \leq \alpha_1} \left\{ \left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| \|h\|^k \right\} \leq \max_k \left\{ \|f\|_g \left(\frac{\|h\|}{r_1} \right)^k \right\} \leq \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

Lemma 2.1.8. Let $\mathbf{k}$ be a complete non-archimedean field. Then we have $\|f(x)\| \leq \|f\|$ for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$. In particular, if $f_n \to f$ as $n \to +\infty$ in $\mathbf{k}\{\underline{T/r}\}$, then we have $\|f_n(x) - f(x)\| \to 0$ for every $x \in E(0, \underline{r})$.

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \ldots, x_n) \in E(0, \underline{r})$. We have

$$\left\| \sum_{|\alpha| < N} a_\alpha x^\alpha \right\| \leq \max_{|\alpha| < N} \|a_\alpha\| r^\alpha \leq \|f\|$$

for every $N \in \mathbb{N}$. Taking $N \to +\infty$, we have $\|f(x)\| \leq \|f\|$. □

Proposition 2.1.9. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation, and $\partial_i = \partial / \partial T_i$ be the derivative operator on $\mathbf{k}\{\underline{T/r}\}$ with respect to the indeterminate T_i for $i = 1, \ldots, n$.

Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$, we have

$$F_{\partial_i(f)}(x) = \lim_{h \to 0} \frac{F_f(x_1, \ldots, x_i + h, \ldots, x_n) - F_f(x)}{h}.$$

Proof. Without loss of generality, we can assume that $i = 1$. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $f_n = \sum_{|\alpha|<n} a_\alpha T^\alpha$ for $n \in \mathbb{N}$. Set $x_h = (x_1 + h, x_2 \ldots, x_n)$ and $L_f(h) = (F_f(x_h) - F_f(x))/h$ for $h \in \mathbf{k}^\times$. Note that for fixed h, we have $\lim_{n\to\infty} L_{f_n}(h) = L_f(h)$.

We compute $L_{f_n}(h) - F_{\partial f_n}(x)$ explicitly:

$$\begin{aligned} L_{f_n}(h) - F_{\partial f_n}(x) &= \frac{1}{h}\left(\sum_{|\alpha|<n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} h^k x_2^{\alpha_2} \cdots x_n^{\alpha_n} - \sum_{|\alpha|<n} \alpha_1 a_\alpha x_1^{\alpha_1 - 1} h x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right) \\ &= \sum_{|\alpha|<n} \sum_{k=2}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^{k-1}. \end{aligned}$$

Note that

$$M = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\| r_1^{k-1} \leq \|f\|/r_1 < +\infty.$$

Hence

$$\|L_{f_n}(h) - F_{\partial f_n}(x)\| \leq \max_{2 \leq k \leq n} \left\{ M \frac{\|h\|^{k-1}}{r_1^{k-1}} \right\} \leq M \frac{\|h\|}{r_1}$$

for $h \in \mathbf{k}^\times$ with $\|h\| < r_1$. Taking $n \to +\infty$, we have

$$\|L_f(h) - F_{\partial f}(x)\| \leq M \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. ☐

Yang: The following should be a theorem.

Corollary 2.1.10. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment $f \mapsto F_f$ in Proposition 2.1.6 is injective.

Proof. Note that if $F_f = 0$, then for every $i = 1, \ldots, n$, we have $F_{\partial_i(f)} = 0$ by Proposition 2.1.9. By taking repeated derivatives, we have $F_{\partial^\alpha f} = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Note that $F_{\partial^\alpha f}(0) = \alpha! a_\alpha$. It follows that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n$ and thus $f = 0$. ☐

Remark 2.1.11. Corollary 2.1.10 holds for non-archimedean fields of positive characteristic as well. The proof uses Theorem 2.3.2 and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element $f \in \mathbf{k}\{\underline{T/r}\}$ with the associated function $F_f : E(0, \underline{r}) \to \mathbf{k}$ as in Proposition 2.1.6.

Proposition 2.1.12. Let $\mathbb{k}$ be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra $\mathbb{k}\{\underline{T/r}\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in E(0, \underline{r})} \|f(x)\|_{\mathbb{k}}.$$

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbb{k}\{\underline{T/r}\}$. We write $f = g + h$ with $g = \sum_{\alpha \in S} a_\alpha T^\alpha$ and $h = \sum_{\alpha \notin S} a_\alpha T^\alpha$, where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that S is a non-empty finite set and $\|h\| < \|f\|$. By Lemma 2.1.8, we have $\|h(x)\| < \|f\|$ for every $x \in E(0, \underline{r})$. It suffices to show that $\|g\|_{\sup} = \|g\|$.

Since $\mathbb{k}$ is algebraically closed, $|\mathbb{k}^\times|$ is dense in $\mathbb{R}_{>0}$. For every pair $\alpha, \beta \in S$ with $\alpha \neq \beta$, the set $\{t \in \mathbb{R}^n_{>0} : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$ is a proper closed subset of $\mathbb{R}^n_{>0}$. Thus we can find $t_m \in |\mathbb{k}^\times|^n$ such that $t_m < r$, $t_m \to r$ as $m \to +\infty$ and for every $\alpha, \beta \in S$ with $\alpha \neq \beta$, we have $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$ for all m. For each m, we can find $x_m \in E(0, \underline{r})$ such that $\|x_m^\alpha\| = t_m^\alpha$ for every $\alpha \in S$ since $t_m \in |\mathbb{k}^\times|^n$. It follows that

$$\|g(x_m)\| = \max_{\alpha \in S} \|a_\alpha\| \|x_m^\alpha\| = \max_{\alpha \in S} \|a_\alpha\| t_m^\alpha \to \|g\| \quad \text{as } m \to +\infty.$$

Thus $\|g\|_{\sup} = \|g\|$. □

Remark 2.1.13. If $\mathbf{k}$ is locally compact (hence not algebraically closed), the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ do not coincide with the supremum norm. For example, consider the Tate algebra $\mathbb{Q}_p\{T\}$. The element $f = T^p - T$ has gauss norm $\|f\| = 1$. However, for every $x \in E(0,1) = \mathbb{Z}_p$, we have $f(x) = x^p - x \equiv 0 \mod p$. Thus $\|f(x)\|_p \leq 1/p$ and $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$.

Remark 2.1.14. Recall that in classical complex analysis, the closure of the polynomial ring $\mathbb{C}[T_1, \dots, T_n]$ with respect to the supremum norm on a closed polydisc $E(0, \underline{r}) \subset \mathbb{C}^n$ is the ring of all complex-valued continuous functions which are analytic on its interior $B(0, \underline{r})$.

2.2 Local theory II: maps

2.2.1

Yang: Recall the Runge theorem in complex analysis.

Definition 2.2.1. Let $\mathbf{k}$ be a complete non-archimedean field. A function $f : E(0, \underline{r}) \to \mathbf{k}$ is called *analytic* if there exists $F \in \mathbf{k}\{\underline{T/r}\}$ such that $f = F$ as functions from $E(0, \underline{r})$ to $\mathbf{k}$. Yang: To be revised.

Yang: Composition of analytic functions.

Definition 2.2.2.

Proposition 2.2.3.

Theorem 2.2.4 (Implicit Function Theorem over Non-Archimedean Fields)**.** Let

2.3 Analytic functions in one variable

Proposition 2.3.1. Let $\mathbf{k}$ be a complete non-archimedean field and $f = \sum_{n=0}^{+\infty} a_n T^n \in \mathbf{k}[[T]]$. Set

$$R := \frac{1}{\limsup_{n\to+\infty} \|a_n\|^{1/n}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Then we have

(a) the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$ and diverges for all $x \in \mathbf{k}$ with $\|x\| > R$;

(b) if $R < +\infty$, the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| = R$ if and only if $\lim_{n\to+\infty} \|a_n\| R^n = 0$.

Proof. By Proposition 1.2.7, we only need to check when the terms $a_n x^n$ tend to zero as $n \to +\infty$. If $\|x\| < R$, there exists $r \in (0,1)$ such that $\|x\| < r^2 R$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\|a_n\|^{1/n} < 1/(rR)$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n < \|a_n\| (r^2 R)^n < (r^2 R)^n \cdot \frac{1}{(rR)^n} = r^n \to 0.$$

Thus the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$.

Suppose that $\|x\| > R$. There exists $s > 1$ such that $\|x\| > R/s$. By the definition of R, there exist infinitely many $n \in \mathbb{N}$ such that $\|a_n\|^{1/n} > s/R$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n > \|a_n\| \frac{R^n}{s^n} > \left(\frac{s}{R}\right)^n \cdot \frac{R^n}{s^n} = 1.$$

Thus the series $f(x)$ diverges for all $x \in \mathbf{k}$ with $\|x\| > R$.

Finally, the case $\|x\| = R$ is direct from Proposition 1.2.7. Yang: To be revised. □

Theorem 2.3.2 (Strassman)**.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$ be an analytic function. Suppose that $\|a_N\| r^N > \|a_n\| r^n$ for all $n > N$. Then f has at most N zeros in the closed ball $E(0, r)$.

Proof. We induct on N. The case $N = 0$ is direct from Proposition 2.1.4. Suppose that the conclusion holds for $N - 1$. Let x be a zero of f in $E(0, r)$. Set

$$g(T) = \frac{f(T) - f(x)}{T - x} = \sum_{k=0}^{+\infty} \left(\sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n \geq k+1} \|a_n\| r^{n-1} \to 0 \quad \text{as } k \to \infty.$$

It follows that $g(T) \in \mathbf{k}\{T/r\}$.

For every $n > N$, we have

$$\|a_N\| > \|a_n\| r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\| \sum_{n=N}^{N+m} a_n x^{n-N} \right\| = \|a_N\|$$

for every $m \in \mathbb{N}$ by Proposition 1.2.3. Take $m \to +\infty$, we have $\|b_{N-1}\| = \|a_N\|$. For every $k > N-1$, we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n\| r^{n-1} \leq \max_{n > N} \|a_n\| r^{n-1} < \|a_N\| r^{N-1} = \|b_{N-1}\| r^{N-1}.$$

By the induction hypothesis, g has at most $N-1$ zeros in $E(0,r)$. It follows that f has at most N zeros in $E(0,r)$ since $f(T) = (T-x) \cdot g(T)$. □

2.3.1 Entire functions

2.3.2 Maximum principle

2.4 Elementary functions

2.4.1 Exponential and logarithmic functions

Fix a prime number p in the following and consider $\mathbf{k}$ being a complete non-archimedean field with $|p| = p^{-1}$. Let $r_p := p^{-1/(p-1)}$.

Construction 2.4.1. The *exponential function* exp is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The *logarithmic function* log is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Proposition 2.4.2. We have the following properties:

(a) the exponential function exp converges on the open disk $B(0, r_p)$;

(b) the logarithmic function log converges on the open disk $B(1,1)$;

(c) $|\exp(x) - 1| = |x|$ and $|\log(1+x)| = |x|$ for all $x \in B(0, r_p)$ or $x \in B(1, r_p)$ respectively;

(d) endow $B(0, r_p)$ with the group structure induced by addition in $\mathbf{k}$ and $B(1, r_p)$ with the group structure induced by multiplication in $\mathbf{k}$, then $\exp : B(0, r_p) \to B(1, r_p)$ is an isometric group isomorphism with inverse $\log : B(1, r_p) \to B(0, r_p)$.

Proof. For the convergent radius of exponential function, by Lemma 2.4.3, noting that

$$\liminf_{n\to+\infty} \frac{s_n}{n} = 0,$$

we have

$$\limsup_{n\to+\infty} |n!|_p^{-1/n} = \limsup_{n\to+\infty} p^{v_p(n!)/n} = p^{\limsup_{n\to+\infty}(1-(s_n/n))/(p-1)} = p^{1/(p-1)}.$$

That is, the convergent radius of the exponential function is $r_p = p^{-1/(p-1)}$. Considering $n = p^m$, we have

$$|p^m!|_p r_p^n = p^{(p^m-1)/(p-1)} \cdot p^{-p^m/(p-1)} = p^{-1/(p-1)} \neq 0.$$

Hence the convergent domain of the exponential function is $B(0, r_p)$.

For the logarithmic function, we have

$$\limsup_{n\to+\infty} |n|_p^{-1/n} = \limsup_{n\to+\infty} p^{v_p(n)/n} = p^0 = 1.$$

And $|1/(np+1)|_p = 1$ for all $n \in \mathbb{N}$. Thus, the convergent domain of the logarithmic function is $B(1,1)$.

For $x \in B(0, r_p)$, we have

$$\left|\frac{x^{n-1}}{n!}\right|_p < r_p^{n-1} \cdot p^{v_p(n!)} = p^{v_p(n!)-(n-1)/(p-1)} \le 1.$$

Hence $|x^n/n!|_p < |x|_p$ for all $n \ge 2$ and thus

$$|\exp(x) - 1|_p = \left|\sum_{n=1}^{+\infty} \frac{x^n}{n!}\right|_p = |x|_p.$$

For $x + 1 \in B(1, r_p)$, setting $|x|_p = p^{-t}$ with $t \ge 1/(p-1)$, we have

$$\left|\frac{x^{n-1}}{n}\right|_p = p^{v_p(n)-t(n-1)} \le p^{v_p(n!)-t(n-1)} \le p^{(1/(p-1)-t)(n-1)} \le 1, \quad \forall n \ge 2.$$

Similarly, we have $|x^n/n|_p < |x|_p$ and hence $|\log(1+x)|_p = |x|_p$.

The identities

$$\begin{aligned} \exp(X+Y) &= \exp(X) \cdot \exp(Y), \\ \log((1+X)(1+Y)) &= \log(1+X) + \log(1+Y), \\ \exp(\log(1+X)) &= 1 + X, \\ \log(\exp(X)) &= X \end{aligned}$$

are purely formal and holds for indeterminates X and Y. Easy to check that $\exp(X+Y), \log(1+X) + \log(1+Y) \in \mathbf{k}\{X/r_p, Y/r_p\}$. Thus, the assertion (d) follows from (c) and Proposition 2.1.6. □

Recall the following useful lemma regarding the p-adic valuation of factorials.

Lemma 2.4.3. Let p be a prime number and $n \in \mathbb{N}$, write $n = \sum_{k=0}^{m} a_k p^k$ in the p-adic expansion and set $s_n := \sum_{k=0}^{m} a_k$. Then

$$v_p(n!) = \frac{n - s_n}{p - 1}.$$

Proof. Yang: To be added. ☐

Corollary 2.4.4. Let $\mathbf{k}$ be a complete non-archimedean field with $|p| = p^{-1}$. The multiplication group

$$\mathbf{k}^\times \cong |\mathbf{k}^\times| \times \mathscr{k}_{\mathbf{k}}^\times \times \mathbf{k}^{\circ\circ}$$

where $\mathscr{k}_{\mathbf{k}}$ is the residue field of $\mathbf{k}$. Yang: To be revised.

Proof. Yang: To be added. ☐

Proposition 2.4.5. Suppose that $\mathbf{k} = \mathbb{k}$ is algebraically closed. The logarithmic function log defines a surjective group homomorphism $1 + \mathbb{k}^{\circ\circ} \to \mathbb{k}$ with kernel the group μ_{p^∞} of all p-power roots of unity. Yang: To be checked.

Proof. ☐

Yang: continuation of exponential and logarithmic

2.4.2 Mahler series

Notation 2.4.6. We use $\binom{x}{n}$ to denote the *binomial polynomial* defined by

$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}.$$

Definition 2.4.7. Fix a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbf{k}$. The *Mahler series* associated to $\{a_n\}$ is defined to be the formal series

$$f(x) := \sum_{n=0}^{+\infty} a_n \binom{x}{n}.$$

Yang: To be checked.

Proposition 2.4.8.

Theorem 2.4.9. The series converges.

Chapter 3

Affinoid algebras

3.1 Normed rings and modules

3.1.1 Semi-normed algebraic structures

Definition 3.1.1. Let G be an abelian group. A *semi-norm* on G is a function $\|\cdot\| : G \to \mathbb{R}_{\geq 0}$ such that

- $\|0\| = 0$;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$.

Suppose that R is a ring (commutative with unity) and $\|\cdot\|$ is a semi-norm on the underlying abelian group of R. We further require that

- $\|1\| = 1$;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$.

Suppose that $(M, \|\cdot\|_M)$ is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M. We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$.

Suppose that $(A, \|\cdot\|_A)$ is an R-algebra and $\|\cdot\|_A$ is a semi-norm on the underlying R-module of A. We further require that this semi-norm is a semi-norm on the underlying ring of A.

Definition 3.1.2. Let A be an abelian group (or ring, R-module, R-algebra) equipped with a semi-norm $\|\cdot\|$. If $\forall x \in A, \|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*.

Yang: Note that this definition of semi-normed module is a little different of [**Ber90**]

Definition 3.1.3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group (or ring, R-module, R-algebra) A. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in A, \|x\|_1 \leq C\|x\|_2$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are bounded by each other, we say they are *equivalent*.

Remark 3.1.4. Equivalent semi-norms induce the same topology on A.

Definition 3.1.5. Let M be a semi-normed abelian group (or ring, R-module, R-algebra) and $N \subseteq M$ be a subgroup (or ideal, R-submodule, ideal). The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Yang: Is this always a semi-norm? In particular, $\|1\| = 1$?

Unless otherwise specified, we always equip the quotient M/N with the residue semi-norm.

Remark 3.1.6. The residue semi-norm is a norm if and only if N is closed in M.

Definition 3.1.7. Let M and N be two semi-normed abelian groups (or rings, R-modules, R-algebras). A homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$.

A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$.

Definition 3.1.8. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$.

3.1.2 Banach rings

Definition 3.1.9. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 3.1.10. A *banach ring* is a complete normed ring.

Yang: The counterpart of prime ideal is multiplicative seminorm.

Definition 3.1.11. Let $(A, \|\cdot\|_A)$ be a (semi-)normed algebraic structure, e.g., a (semi-)normed abelian group, a (semi-)normed ring, or a (semi-)normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a (pseudo-)metric space. Since A is dense in its completion, the algebraic operations and (semi-)norms on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M \otimes_R N} = \inf \left\{ \sum_i \|x_i\|_M \|y_i\|_N \, : \, z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N \right\}.$$

Definition 3.1.12. Let R be a complete normed ring and M, N complete semi-normed R-modules. The *complete tensor product* $M \widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Definition 3.1.13. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Yang: Since , $\rho(f)$ exists.

Definition 3.1.14. A banach ring R is called *uniform* if its norm is power-multiplicative.

Proposition 3.1.15. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 3.1.16. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\operatorname{Qnil}(R)$.

Definition 3.1.17. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Proposition 3.1.18. Let R be a banach ring. The completion of $R/\operatorname{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

3.1.3 Complete tensor product

3.1.4 Examples

Example 3.1.19. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a normed ring.

Example 3.1.20. The fields $\mathbb{C}$ and $\mathbb{R}$ equipped with the usual absolute value are complete fields.

Example 3.1.21. The field $\mathbb{Q}_p$ of p-adic numbers equipped with the p-adic norm is a complete non-Archimedean field.

Example 3.1.22. Let R be a banach ring and $r > 0$ be a real number. We define the ring of absolutely convergent power series over $\mathbf{k}$ with radius r as

$$R\langle T/r\rangle := \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r\rangle$ is a banach ring.

When $R = \mathbf{k}$ is a Yang: To be checked.

Example 3.1.23. Let $\mathbf{k}$ be a non-Archimedean complete field. We define

$$\mathbf{k}\{T_1/r_1, \ldots, T_n/r_n\} := \left\{ \sum_{I\in\mathbb{N}^n} a_I T^I \in \mathbf{k}[[T_1, \ldots, T_n]] : \lim_{|I|\to\infty} |a_I| r^I = 0 \right\},$$

where $r = (r_1, \ldots, r_n)$ is an n-tuple of positive real numbers, $T^I = T_1^{i_1} \cdots T_n^{i_n}$ for $I = (i_1, \ldots, i_n)$, and

$|I| = i_1 + \cdots + i_n$. Equipped with the norm $\| \sum_{I \in \mathbb{N}^n} a_I T^I \| = \sup_{I \in \mathbb{N}^n} |a_I| r^I$, the affinoid $\mathbf{k}$-algebra $\mathbf{k}\{T_1/r_1, \ldots, T_n/r_n\}$ is a banach $\mathbf{k}$-algebra. This is called the *Tate algebra* over $\mathbf{k}$ with polyradius r equipped with the *Gauss norm*. We will denote $\mathbf{k}\{\underline{T/r}\} = \mathbf{k}\{T_1/r_1, \ldots, T_n/r_n\}$ for simplicity.

Yang: To be continued...

3.2 Affinoid algebras

3.2.1 The first properties

Definition 3.2.1. Let $\mathbf{k}$ be a non-archimedean field. A banach $\mathbf{k}$-algebra A is called a *affinoid* $\mathbf{k}$-*algebra* if there exists an admissible surjective homomorphism

$$\varphi : \mathbf{k}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \twoheadrightarrow A$$

for some $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in \mathbb{R}_{>0}$.
If one can choose $r_1 = \cdots = r_n = 1$, then we say that A is a *strict affinoid* $\mathbf{k}$-*algebra*.

Definition 3.2.2. Let $\mathbf{k}$ be a non-archimedean field. We define the *ring of restricted Laurent series* over $\mathbf{k}$ as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n : a_n \in \mathbf{k}, \lim_{|n| \to \infty} |a_n| r^n = 0 \right\}$$

equipped with the norm

$$\|f\| = \sup_{n \in \mathbb{Z}} |a_n| r^n.$$

Yang: Is $\mathbf{K}_r$ always a field? Yang: Do we have $\mathbf{L}_{\mathbf{k},r} = \mathrm{Frac}(\mathbf{k}\{T/r\})$?

Proposition 3.2.3. Let $\mathbf{k}$ be a non-archimedean field. If $r \notin \sqrt{|\mathbf{k}^\times|}$, then $\mathbf{K}_r$ is a complete non-archimedean field with non-trivial absolute value extending that of $\mathbf{k}$.

Proposition 3.2.4. Let A be an affinoid $\mathbf{k}$-algebra. Then A is noetherian, and every ideal of A is closed.

Proposition 3.2.5. Let A be an affinoid $\mathbf{k}$-algebra. Then there exists a constant $C > 0$ and $N > 0$ such that for all $f \in A$ and $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

Proposition 3.2.6. Let A be an affinoid $\mathbf{k}$-algebra. If and only if $\rho(f) \in \sqrt{|\mathbf{k}|}$ for all $f \in A$, then A is strict. Yang: To be complete.

3.2.2 Noetherian normalization theorem

3.3 Finite modules

3.3.1 Finite banach module

There are three different categories of finite modules over an affinoid algebra A:

- The category $\mathbf{Banmod}_A$ of finite banach A-modules with A-linear maps as morphisms.
- The category $\mathbf{Banmod}_A^b$ of finite banach A-modules with bounded A-linear maps as morphisms.
- The category $\mathbf{mod}_A$ of finite A-modules with all A-linear maps as morphisms.

Theorem 3.3.1. Let A be an affinoid $\mathbf{k}$-algebra. Then the category of finite banach A-modules with bounded A-linear maps as morphisms is equivalent to the category of finite A-modules with A-linear maps as morphisms. Yang: To be revised.

For simplicity, we will just write mod_A to denote the category of finite banach A-modules with bounded A-linear maps as morphisms.

Chapter 4

Berkovich spaces

4.1 Spectrum

Let $\mathbf{k}$ be a spherically complete non-archimedean field which is algebraically closed and $A = \mathbf{k}[T]$. We want to consider the "analytic structure" on $\operatorname{mSpec} A$. However, unlike the complex case, the set $\operatorname{mSpec} A$ is totally disconnected with respect to the topology induced by the absolute value on $\mathbf{k}$ (Corollary 1.2.6). To overcome this difficulty, Berkovich uses multiplicative semi-norms to "fill in the gaps" between the points in $\operatorname{mSpec} A$, leading to the notion of the spectrum of a Banach ring.

We first consider the local model. Hence we should consider the Tate algebra $\mathbf{k}\{T\}$ instead of the polynomial ring $\mathbf{k}[T]$. Yang: The maximal ideal of $\mathbf{k}\{T\}$ corresponding to the point in the disk $\{a \in \mathbf{k} : a \leq 1\}$. Yang: Closed or open disk?

4.1.1 Definition

Definition 4.1.1. Let R be a Banach ring. The *spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$. We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_{\geq 0}$, defined by $x \mapsto |f|_x =: f(x)$, is continuous.

Definition 4.1.2. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $\mathscr{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$ for each $x \in \mathscr{M}(S)$.

Proposition 4.1.3. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, let $\wp_x$ be the kernel of the multiplicative semi-norm $|\cdot|_x$. Then $\wp_x$ is a closed prime ideal of R, and $x \mapsto \wp_x$ defines a continuous map from $\mathscr{M}(R)$ to $\operatorname{Spec}(R)$ equipped with the Zariski topology.

Proof. Yang: To be completed □

Definition 4.1.4. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, the *completed residue field* at the point x is defined as the completion of the residue field $\kappa(x) = \operatorname{Frac}(R/\wp_x)$ with respect to the multiplicative norm induced by the semi-norm $|\cdot|_x$, denoted by $\mathscr{H}(x)$.

Definition 4.1.5. Let R be a Banach ring. The *Gel'fand transform* of R is the bounded ring homomorphism

$$\Gamma : R \to \prod_{x\in\mathscr{M}(R)} \mathscr{H}(x), \quad f \mapsto (f(x))_{x\in\mathscr{M}(R)},$$

where the norm on the product $\prod_{x\in\mathscr{M}(R)} \mathscr{H}(x)$ is given by the supremum norm.

Proposition 4.1.6. The Gel'fand transform $\Gamma : R \to \prod_{x\in\mathscr{M}(R)} \mathscr{H}(x)$ of a Banach ring R factors through the uniformization R^u of R, and the induced map $R^u \to \prod_{x\in\mathscr{M}(R)} \mathscr{H}(x)$ is an isometric embedding. Yang: To be checked.

Theorem 4.1.7. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

Proof. Yang: To be continued. □

Lemma 4.1.8. Let $\{K_i\}_{i\in I}$ be a family of completed fields. Consider the Banach ring $R = \prod_{i\in I} K_i$ equipped with the product norm. The spectrum $\mathscr{M}(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I.

Remark 4.1.9. The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

Theorem 4.1.10. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a compact Hausdorff space.

Proof. Yang: To be added. □

Proposition 4.1.11. Let K/k be a Galois extension of complete fields, and let R be a Banach k-algebra. The Galois group $\mathrm{Gal}(K/k)$ acts on the spectrum $\mathscr{M}(R\widehat{\otimes}_k K)$ via

$$g\cdot x : f \mapsto |(1\otimes g^{-1})(f)|_x$$

for each $g \in \mathrm{Gal}(K/k)$, $x \in \mathscr{M}(R\widehat{\otimes}_k K)$ and $f \in R\widehat{\otimes}_k K$. Moreover, the natural map $\mathscr{M}(R\widehat{\otimes}_k K) \to \mathscr{M}(R)$ induces a homeomorphism

$$\mathscr{M}(R\widehat{\otimes}_k K)/\,\mathrm{Gal}(K/k) \xrightarrow{\sim} \mathscr{M}(R).$$

Yang: To be checked.

4.1.2 Examples

Example 4.1.12. Let $(\mathbf{k}, |\cdot|)$ be a complete valuation field. The spectrum $\mathscr{M}(\mathbf{k})$ consists of a single point corresponding to the given absolute value $|\cdot|$ on $\mathbf{k}$. Yang: To be checked.

Example 4.1.13. Consider the Banach ring $(\mathbb{Z}, \|\cdot\|)$ with $\|\cdot\| = |\cdot|_\infty$ is the usual absolute value norm on $\mathbb{Z}$. Let $|\cdot|_p$ denote the p-adic norm for each prime number p, i.e., $|n|_p = p^{-v_p(n)}$ for each $n \in \mathbb{Z}$, where $v_p(n)$ is the p-adic valuation of n. The spectrum

$$\mathscr{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0,1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0,\infty]\} \cup \{|\cdot|_0\},$$

where $|a|_p^\infty := \lim_{\alpha\to\infty} |a|_p^\alpha$ for each $a \in \mathbb{Z}$ and $|\cdot|_0$ is the trivial norm on $\mathbb{Z}$. Yang: To be checked.

Spectrum of Tate algebra in one variable Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{T/r\}$. We list some types of points in the spectrum $\mathscr{M}(A)$.

For each $a \in \mathbf{k}$ with $|a| \le r$, we have the *type I* point x_a corresponding to the evaluation at a, i.e., $|f|_{x_a} := |f(a)|$ for each $f \in A$. For each closed disk $E = E(a,s) := \{b \in \mathbf{k} : |b-a| \le s\}$ with center $a \in \mathbf{k}$ and radius $s \le r$, we have the point $x_{a,s}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} := \sup_{b \in E(a,s)} |f(b)|$$

for each $f \in A$. If $s \in |\mathbf{k}^\times|$, then the point x_E is called a *type II* point; otherwise, it is called a *type III* point.

Let $\{E^{(s)}\}_s$ be a family of closed disks in $\mathbf{k}$ such that $E^{(s)}$ is of radius s, $E^{(s_1)} \subsetneq E^{(s_2)}$ for any $s_1 < s_2$ and $\bigcap_s E^{(s)} = \varnothing$. Then we have the point $x_{\{E^{(s)}\}}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E^{(s)}\}}} := \inf_s |f|_{x_{E^{(s)}}}$$

for each $f \in A$. Such a point is called a *type IV* point.

Yang: To be completed.

Proposition 4.1.14. Let $\mathbf{k}$ be a complete non-archimedean field, and let $r > 0$ be a positive real number. Consider the Tate algebra $\mathbf{k}\{r^{-1}T\}$ equipped with the Gauss norm. The points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ can be classified into four types as described above. Yang: To be checked

Proof. Yang: To be completed. □

Proposition 4.1.15. Let $\mathbf{k}$ be a complete non-archimedean field, and let $r > 0$ be a positive real number. Consider the Tate algebra $\mathbf{k}\{r^{-1}T\}$ equipped with the Gauss norm. The completed residue fields of the four types of points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ are described as follows:

- For a type I point x_a with $a \in \mathbf{k}$ and $|a| \le r$, the completed residue field $\mathscr{H}(x_a)$ is isomorphic to $\mathbf{k}$.
- For a type II point $x_{a,s}$ with $a \in \mathbf{k}$ and $s \in |\mathbf{k}^\times|$, the completed residue field $\mathscr{H}(x_{a,s})$ is isomorphic to the field of Laurent series over the residue field $\mathscr{k}_{\mathbf{k}}$, i.e., $\mathscr{k}_{\mathbf{k}}((t))$.
- For a type III point $x_{a,s}$ with $a \in \mathbf{k}$ and $s \notin |\mathbf{k}^\times|$, the completed residue field $\mathscr{H}(x_{a,s})$ is isomorphic to a transcendental extension of $\mathscr{k}_{\mathbf{k}}$ of degree one.
- For a type IV point $x_{\{E^{(s)}\}}$, the completed residue field $\mathscr{H}(x_{\{E^{(s)}\}})$ is isomorphic to a transcendental extension of $\mathscr{k}_{\mathbf{k}}$ of infinite degree.

Yang: To be checked.

Example 4.1.16. The completed residue field $\mathscr{H}(x_a)$ for a type I point x_a with $a \in \mathbf{k}$ and $|a| \le r$ is isomorphic to $\mathbf{k}$. Yang: To be complete.

Spectrum of Tate algebra in servel variables Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$. We can consider the spectrum $\mathscr{M}(A)$ similarly.

4.2 Affinoid domains

Consider $X = \mathscr{M}(A)$ with $A = \mathbf{k}\{T_1, \dots, T_n\}$. Yang: Not every open subset of X gives an affinoid space, that is, the completion of the ring of analytic functions on that open subset is not necessarily an affinoid algebra. Yang: Right? example?

4.2.1 Definition

Definition 4.2.1. Let A be a $\mathbf{k}$-affinoid algebra, and let $X = \mathscr{M}(A)$ be the associated affinoid space. A closed subset $V \subseteq X$ is called an *affinoid domain* if there exists a $\mathbf{k}$-affinoid algebra A_V and a morphism of $\mathbf{k}$-affinoid algebras $\varphi : A \to A_V$ satisfying the following universal property: for every bounded homomorphism of $\mathbf{k}$-affinoid algebras $\psi : A \to B$ such that the induced map on spectra $\mathscr{M}(\psi) : \mathscr{M}(B) \to X$ has its image contained in V, there exists a unique bounded homomorphism $\theta : A_V \to B$ such that the following diagram commutes:

$$\begin{tikzcd} & A_V \arrow[dr, "\theta"] & \\ A \arrow[ur, "\varphi"] \arrow[rr, "\psi"'] & & B \end{tikzcd}$$

In this case, we say that V is represented by the affinoid algebra A_V.

Slogan *A closed subset $V \subset X$ is an affinoid domain if the functor "$\mathrm{Mor}(-, V)$" is representable.*

Yang: Why we consider closed subset rather that open subset?

Construction 4.2.2. Let $f = (f_1, \dots, f_n)$ be a tuple of elements in A and $r = (r_1, \dots, r_n)$ be a tuple of positive real numbers. Consider the closed subset of X:

$$X\left(\underline{f/r}\right) := \{x \in X : |f_i(x)| \le r_i, 1 \le i \le n\}.$$

Such a closed subset is called a *Weierstrass domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\{\underline{f/r}\right\} := A\{f_1/r_1, \dots, f_n/r_n\}.$$

Yang: The domain $X\left(\underline{f/r}\right)$ is represented by $A\left\{\underline{f/r}\right\}$.

Construction 4.2.3. Let $f = (f_1, \dots, f_n), g = (g_1, \dots, g_m)$ be two tuples of elements in A and $r = (r_1, \dots, r_n), s = (s_1, \dots, s_m)$ be two tuples of positive real numbers. Consider the following closed subset of X:

$$X\left(\underline{f/r}; \underline{g/s}^{-1}\right) := \left\{x \in X : |f_i(x)| \le r_i, |g_j(x)| \ge s_j, 1 \le i \le n, 1 \le j \le m\right\}.$$

Such a closed subset is called a *Laurent domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\{\underline{f/r}; \underline{g/s}^{-1}\right\} := A\left\{f_1/r_1, \dots, f_n/r_n, g_1^{-1}/s_1, \dots, g_m^{-1}/s_m\right\}.$$

Yang: The domain $X\left(\underline{f/r}; \underline{g/s}^{-1}\right)$ is represented by $A\left\{\underline{f/r}; \underline{g/s}^{-1}\right\}$.

Construction 4.2.4. Let $f = (f_1, \dots, f_n), g$ be elements in A such that the ideal generated by them is the whole algebra A. Set $p = (p_1, \dots, p_n)$ be a tuple of positive real numbers. We define the following closed subset of X:

$$X\left(\underline{f/p}, g\right) := \{x \in X : |f_i(x)| \leq p_i|g(x)|, 1 \leq i \leq n\}.$$

Such a closed subset is called a *rational domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\langle \underline{f/p}, g^{-1}\right\rangle := A\left\langle \frac{f_1}{p_1 g}, \dots, \frac{f_n}{p_n g}\right\rangle,$$

which is the quotient of the Tate algebra

$$A\langle T_1, \dots, T_n\rangle$$

by the ideal generated by the elements $p_i g T_i - f_i$ for $1 \leq i \leq n$. There is a natural bounded homomorphism $\varphi : A \to A\langle \underline{f/p}, g^{-1}\rangle$ induced by the inclusion. It can be shown that the closed subset $X(\underline{f/p}, g)$ is an affinoid domain represented by the affinoid algebra $A\langle \underline{f/p}, g^{-1}\rangle$. Yang: To be checked

Yang: We have a sequence of inclusion:

$$\{\text{Weierstrass domains}\} \subseteq \{\text{Laurent domains}\} \subseteq \{\text{Rational domains}\} \subseteq \{\text{Affinoid domains}\}.$$

Proposition 4.2.5. Let A be a $\mathbf{k}$-affinoid algebra, and let $X = \mathscr{M}(A)$ be the associated affinoid space. Let $V \subseteq X$ be an affinoid domain represented by the $\mathbf{k}$-affinoid algebra A_V. Then the natural bounded homomorphism $\varphi : A \to A_V$ is flat.
We have $\mathscr{M}(A_V) \cong V$.

4.2.2 The Grothendieck topology of affinoid domains

Chapter 5

Varieties

Chapter 6

Height pairings

References

[Xie25] Junyi Xie. “The existence of Zariski dense orbits for endomorphisms of projective surfaces”. In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on p. 13).