# Hermitian line bundles on rings of integers

Let $K$ be a number field, and $\mathcal{O}_K$ be its ring of integers.

## 1 Hermitian vector bundles

**Definition 1.** A *hermitian coherent sheaf* $\overline{\mathcal{F}}$ on $\operatorname{Spec} \mathcal{O}_K$ consists of

- a finitely generated $\mathcal{O}_K$-module $\mathcal{F}$;
- for each embedding $\sigma : K \hookrightarrow \mathbb{C}$, a Hermitian metric $\|\cdot\|_\sigma$ on the complex vector space $\mathcal{F}_\sigma := \mathcal{F} \otimes_\sigma \mathbb{C}$.

Yang: To be revised.

**Definition 2.** Let $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ be hermitian coherent sheaves on $\operatorname{Spec} \mathcal{O}_K$. A *morphism* $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ is an $\mathcal{O}_K$-module homomorphism $f : \mathcal{F} \to \mathcal{G}$ such that for each embedding $\sigma : K \hookrightarrow \mathbb{C}$, the induced map $f_\sigma : \mathcal{F}_\sigma \to \mathcal{G}_\sigma$ is a contraction with respect to the Hermitian metrics, i.e.,

$$\|f_\sigma(v)\|_\sigma \leq \|v\|_\sigma \quad \text{for all } v \in \mathcal{F}_\sigma.$$

Yang: To be revised.

**Definition 3.** A *hermitian vector bundle* $\overline{\mathcal{E}}$ on $\operatorname{Spec} \mathcal{O}_K$ consists of

- a finitely generated projective $\mathcal{O}_K$-module $\mathcal{E}$;
- for each embedding $\sigma : K \hookrightarrow \mathbb{C}$, a Hermitian metric $\|\cdot\|_\sigma$ on the complex vector space $\mathcal{E}_\sigma := \mathcal{E} \otimes_\sigma \mathbb{C}$.

Yang: To be revised.

## 2 Line bundles and divisors

**Definition 4.** A *hermitian line bundle* $\overline{\mathcal{L}}$ on $\operatorname{Spec} \mathcal{O}_K$ consists of

- a line bundle $\mathcal{L}$ on $\operatorname{Spec} \mathcal{O}_K$, i.e., a finitely generated projective $\mathcal{O}_K$-module of rank 1;
- for each embedding $\sigma : K \hookrightarrow \mathbb{C}$, a Hermitian metric $\|\cdot\|_\sigma$ on the complex line $\mathcal{L}_\sigma := \mathcal{L} \otimes_\sigma \mathbb{C}$.

Yang: To be revised.

**Date**: January 28, 2026, **Author**: Tianle Yang, My Homepage

**Definition 5.** Let $\overline{\mathcal{L}}$ be a hermitian line bundle on $\operatorname{Spec} \mathcal{O}_K$. The *degree* of $\overline{\mathcal{L}}$ is defined as

$$\deg \overline{\mathcal{L}} := \log \#(\mathcal{L}/\mathcal{O}_K s) - \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \|s\|_\sigma,$$

where $s$ is any non-zero section of $\mathcal{L}$. Yang: To be revised

**Definition 6.** An *arithmetic divisor* on $\operatorname{Spec} \mathcal{O}_K$ is a pair $(D, g)$, where

- $D$ is a Weil divisor on $\operatorname{Spec} \mathcal{O}_K$, i.e., a formal sum of closed points with integer coefficients;
- $g = (g_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}}$ is a collection of Green's functions $g_\sigma : (\operatorname{Spec} \mathcal{O}_K)_\sigma(\mathbb{C}) \setminus \{D\} \to \mathbb{R}$ for each embedding $\sigma : K \hookrightarrow \mathbb{C}$.

Yang: To be revised

**Definition 7.** Let $(D, g)$ be an arithmetic divisor on $\operatorname{Spec} \mathcal{O}_K$. The *degree* of $(D, g)$ is defined as

$$\deg(D, g) := \sum_{x \in \operatorname{Spec} \mathcal{O}_K} n_x \log N(x) + \sum_{\sigma : K \hookrightarrow \mathbb{C}} g_\sigma(x_\sigma),$$

where $D = \sum_x n_x x$ and $N(x)$ is the norm of the closed point $x$. Yang: To be revised

**Proposition 8.** There is a one-to-one correspondence between isomorphism classes of hermitian line bundles on $\operatorname{Spec} \mathcal{O}_K$ and arithmetic divisors on $\operatorname{Spec} \mathcal{O}_K$. Moreover, this correspondence preserves degrees, i.e., for a hermitian line bundle $\overline{\mathcal{L}}$ corresponding to an arithmetic divisor $(D, g)$, we have

$$\widehat{\deg} \overline{\mathcal{L}} = \widehat{\deg}(D, g).$$

Yang: To be revised

**Proposition 9.** Let $K$ be a number field with ring of integers $\mathcal{O}_K$. There is an exact sequence

$$0 \to \frac{\bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \mathbb{R}}{\log |\mathcal{O}_K^\times|} \to \widehat{\mathrm{Cl}}(\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_K) \to 0,$$

where $\widehat{\mathrm{Cl}}(\mathcal{O}_K)$ is the arithmetic class group of hermitian line bundles on $\operatorname{Spec} \mathcal{O}_K$, and $\mathrm{Cl}(\mathcal{O}_K)$ is the usual class group of $\mathcal{O}_K$. Yang: To be revised

## 3 Application: geometry of numbers

# Appendix