

Finite field extensions

1 Finite-dimensional vector space

Definition 1. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . A *norm* on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

- (a) $\|x\| = 0$ if and only if $x = 0$;
- (b) $\|ax\| = |a| \cdot \|x\|$;
- (c) $\|x + y\| \leq \|x\| + \|y\|$.

Example 2. Let \mathbf{k} be a valuation field and V a finite-dimensional vector space over \mathbf{k} with basis $\{e_1, e_2, \dots, e_n\}$. For any $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$, define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then $\|\cdot\|_{\max}$ is a norm on V , called the *maximal norm* with respect to the basis $\{e_1, e_2, \dots, e_n\}$.

Example 3. Setting as in Example 2, for any $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$, define

$$\|x\|_1 := |a_1| + |a_2| + \dots + |a_n|.$$

Then $\|\cdot\|_1$ is also a norm on V .

Definition 4. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Lemma 5. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V .

Proof. The sufficiency is clear. Now suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on V . Hence the unit open ball with respect to $\|\cdot\|_1$ contains a unit open ball with respect to $\|\cdot\|_2$. That is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every $x \in V$ with $\|x\|_1 = 1$, we have $\|x\|_2 \geq C = C\|x\|_1$. By scaling, we get that for every $x \in V$,

$$\|x\|_2 \geq C\|x\|_1.$$

Similar for the other direction, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. \square

Proposition 6. Let V be a normed finite-dimensional vector space over a complete valuation field \mathbf{k} . Then V is complete.

| *Proof.* Yang: To be added. □

Theorem 7. Let V be a finite-dimensional vector space over a complete field \mathbf{k} . Then all norms on V are equivalent.

Proof. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis as in [Example 2](#). Let $\|\cdot\|$ be any norm on V . It suffices to show that $\|\cdot\|$ and $\|\cdot\|_{\max}$ are equivalent. First we have

$$\|y\| \leq \sum_{i=1}^n |a_i| \|e_i\| \leq \left(\sum_{i=1}^n \|e_i\| \right) \|y\|_{\max}$$

for any $y = a_1 e_1 + a_2 e_2 + \dots + a_n e_n \in V$. It remains to show that there exists a constant $C > 0$ such that for any $y \in V$,

$$\|y\|_{\max} \leq C \|y\|.$$

Yang: To be added. □

Remark 8. If the base field \mathbf{k} is not complete, then [Theorem 7](#) may fail. For example, let $\mathbf{k} = \mathbb{Q}$ with the usual absolute value, and let $V = \mathbb{Q}[\alpha]$ with $\alpha^2 - \alpha - 1 = 0$. There are two embeddings of V into \mathbb{R} :

$$\iota_1 : a + b\alpha \mapsto a + b \frac{1 + \sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b \frac{1 - \sqrt{5}}{2}.$$

Define two norms on V by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where $|\cdot|$ is the usual absolute value on \mathbb{R} . Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent since $\iota_2(\alpha^n) \rightarrow 0$ as $n \rightarrow \infty$ while $\iota_1(\alpha^n) \rightarrow \infty$.

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

Lemma 9. Let \mathbf{k} be a complete field and V a normed finite-dimensional vector space over \mathbf{k} . Then

$$\|\cdot\| : \text{End}_{\mathbf{k}}(V) \rightarrow \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the \mathbf{k} -vector space $\text{End}_{\mathbf{k}}(V)$ satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \text{End}_{\mathbf{k}}(V).$$

Proof. First we show the existence of the supremum, i.e., there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\| \leq C\|x\|$. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis. Since all norms on V are bounded by each other by [Theorem 7](#), we only need to show that there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\|_1 \leq C\|x\|_{\max}$. Write $T(e_i) = \sum_{j=1}^n a_{ij} e_j$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^n x_i e_i \in V$, we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \left(\sum_{1 \leq i, j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of $\|\cdot\|$ are clear. It remains to show the triangle inequality

and sub-multiplicativity. For any $A, B \in \text{End}_{\mathbf{k}}(V)$, we have

$$\frac{\|(A + B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all $x \in V \setminus \{0\}$ gives $\|A + B\| \leq \|A\| + \|B\|$. We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$. Taking supremum we get $\|AB\| \leq \|A\| \cdot \|B\|$. \square

2 Finite field extensions

Theorem 10. Let \mathbf{k} be a complete field and \mathbf{l} a finite extension of \mathbf{k} . Then the absolute value on \mathbf{l} is uniquely determined by the absolute value on \mathbf{k} . Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where $n = [\mathbf{l} : \mathbf{k}]$ and $N_{\mathbf{l}/\mathbf{k}}$ is the norm map from \mathbf{l} to \mathbf{k} .

Proof. Fix a norm $\|\cdot\|_V$ on the \mathbf{k} -vector space $V = \mathbf{l}$. The norm $\|\cdot\|_V$ induces an operator norm $\|\cdot\|_{\text{op}}$ on the \mathbf{k} -vector space $\text{End}_{\mathbf{k}}(V)$ as in Lemma 9. For any $a \in \mathbf{l}$, let $\mu_a \in \text{End}_{\mathbf{k}}(V)$ be the \mathbf{k} -linear map defined by multiplication by a . Note that $a \mapsto \mu_a$ is a \mathbf{k} -algebra homomorphism and if $a \in \mathbf{k}$, $\|\mu_a\|_{\text{op}} = \|a\|_{\mathbf{k}}$.

Step 1. Show that

$$\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}} = \lim_{m \rightarrow \infty} \sqrt[m]{\|\mu_{a^m}\|_{\text{op}}} =: \rho(a^n).$$

The existence of the limit $\rho(a) = \lim_{m \rightarrow \infty} \sqrt[m]{\|\mu_{a^m}\|_{\text{op}}}$ follows from the sub-multiplicativity of the operator norm. And note that $\rho(a^n) = \rho(a)^n$ for all $a \in \mathbf{l}$ and $n \in \mathbb{N}$. We can assume that $a \neq 0$. Let $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$. Then $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ and

$$\rho(b) = \lim_{m \rightarrow \infty} \sqrt[m]{\|\mu_{b^m}\|_{\text{op}}} = \lim_{m \rightarrow \infty} \sqrt[m]{\frac{\|\mu_{a^{mn}}\|_{\text{op}}}{\|N_{\mathbf{l}/\mathbf{k}}(a)^m\|_{\mathbf{k}}}} = \frac{\rho(a^n)}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}$$

since $N_{\mathbf{l}/\mathbf{k}}$ is multiplicative and $N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{k}$. Thus it suffices to show that $\rho(b) = 1$.

Suppose that $\rho(b) < 1$. Then there exists $\delta \in (0, 1)$ and $M > 0$ such that for all $m \geq M$, $\|\mu_{b^m}\|_{\text{op}} < \delta^m$. In particular, we have $\mu_{b^m} \rightarrow 0$ as $m \rightarrow \infty$ in the normed vector space $\text{End}_{\mathbf{k}}(V)$. However, note that \det is continuous on $\text{End}_{\mathbf{k}}(V)$ (since it is a polynomial in the entries of the matrix representation). Thus we have $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \rightarrow 0$ as $m \rightarrow \infty$, contradicting the fact that $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$ for all m .

if $\rho(b) > 1$, Yang: We can not get $\rho(b^{-1}) < 1$ directly.

Step 2. Show that

$$\rho(a + b) \leq \rho(a) + \rho(b)$$

and hence $\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$ defines an absolute value on \mathbf{l} extending that on \mathbf{k} .

Step 3. Show the uniqueness of the absolute value on \mathbf{l} extending that on \mathbf{k} .

Yang: To be added. \square

| **Remark 11.** Yang: I want to discuss some compatibility of extension and completion.

| **Proposition 12.** Let \mathbf{k} be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

| *Proof.* Yang: To be added. □

Appendix

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