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# *Berkovich Space*

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# Berkovich Space

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*Source code:* [github.com/MonkeyUnderMountain/Berkovich\\_Space](https://github.com/MonkeyUnderMountain/Berkovich_Space)

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# Contents

<b>Preface</b>	<b>5</b>
<b>1 Commutative banach algebras</b>	<b>7</b>
1.1 Semi-normed Rings and Modules . . . . .	7
1.1.1 Semi-normed algebraic structures . . . . .	7
1.1.2 banach rings . . . . .	8
1.1.3 Complete field . . . . .	9
1.1.4 Examples . . . . .	10
1.2 Affinoid algebras . . . . .	10
1.2.1 The first properties . . . . .	10
1.2.2 Finite banach module . . . . .	11
<b>2 Affinoid spaces</b>	<b>13</b>
2.1 Spectrum . . . . .	13
2.1.1 Definition . . . . .	13
2.1.2 Examples . . . . .	14
2.2 Affinoid domains . . . . .	16
<b>3 Analytic spaces</b>	<b>19</b>
<b>References</b>	<b>21</b>



# Preface

This document provides an introduction to Berkovich spaces, a fundamental concept in non-archimedean analytic geometry. The theory of Berkovich spaces offers a powerful framework for studying analytic varieties over non-archimedean valued fields, providing a geometric approach that bridges algebraic and analytic methods.

The main references are [[Ber90](#); [BGR84](#)].



# Chapter 1

## Commutative banach algebras

### 1.1 Semi-normed Rings and Modules

#### 1.1.1 Semi-normed algebraic structures

**Definition 1.1.1.** Let  $M$  be an abelian group. A *semi-norm* on  $M$  is a function  $\|\cdot\| : M \rightarrow \mathbb{R}_+$  such that

- $\|0\| = 0$ ;
- $\forall x, y \in M, \|x + y\| \leq \|x\| + \|y\|$ .

If we further have  $\|x\| = 0 \iff x = 0$ , then we say  $\|\cdot\|$  is a *norm*. A *semi-normed abelian group* (resp. *normed abelian group*) is an abelian group equipped with a semi-norm (resp. norm).

**Definition 1.1.2.** Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two semi-norms on an abelian group  $M$ . We say  $\|\cdot\|_1$  is *bounded* by  $\|\cdot\|_2$  if there exists a constant  $C > 0$  such that  $\forall x \in M, \|x\|_1 \leq C\|x\|_2$ .

**Remark 1.1.3.** If two semi-norms (resp. norms) on an abelian group  $M$  are bounded by each other, then they induce the same topology on  $M$ .

**Definition 1.1.4.** Let  $M$  be a semi-normed abelian group and  $N \subseteq M$  be a subgroup. The *residue semi-norm* on the quotient group  $M/N$  is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

**Remark 1.1.5.** The residue semi-norm is a norm if and only if  $N$  is closed in  $M$ .

**Definition 1.1.6.** Let  $M$  and  $N$  be two semi-normed abelian groups. A group homomorphism  $f : M \rightarrow N$  is called *bounded* if there exists a constant  $C > 0$  such that  $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$ . A bounded homomorphism  $f : M \rightarrow N$  is called *admissible* if the induced isomorphism  $M/\ker f \rightarrow \text{Im } f$  is an isometry, i.e.,  $\forall x \in M, \|f(x)\|_N = \inf_{y \in \ker f} \|x + y\|_M$ .

**Definition 1.1.7.** Let  $R$  be a ring (commutative with unity). A *semi-norm* on  $R$  is a semi-norm  $\|\cdot\|$  on the underlying abelian group of  $R$  such that  $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$  and  $\|1\| = 1$ . A *semi-normed ring* is a ring equipped with a semi-norm.

**Definition 1.1.8.** A semi-norm  $\|\cdot\|$  on a ring  $R$  is called *multiplicative* if  $\forall x, y \in R, \|xy\| = \|x\|\|y\|$ . It is called *power-multiplicative* if  $\forall x \in R, \|x^n\| = \|x\|^n$  for all integers  $n \geq 1$ .

**Definition 1.1.9.** Let  $(R, \|\cdot\|_R)$  be a normed ring. A *semi-normed  $R$ -module* is a pair  $(M, \|\cdot\|_M)$  where  $M$  is an  $R$ -module and  $\|\cdot\|_M$  is a semi-norm on the underlying abelian group of  $M$  such that there exists  $C > 0$  with  $\forall a \in R, x \in M, \|ax\|_M \leq C\|a\|_R\|x\|_M$ .

One can talk about boundedness, admissibility and residue semi-norms in the contexts of semi-normed rings and semi-normed modules similar to those in semi-normed abelian groups.

### 1.1.2 banach rings

**Definition 1.1.10.** A (semi-)norm on an abelian group  $M$  induces a (pseudo-)metric  $d(x, y) = \|x - y\|$  on  $M$ . A (semi-)normed abelian group  $M$  is called *complete* if it is complete as a (pseudo-)metric space.

**Definition 1.1.11.** A *banach ring* is a complete normed ring.

**Definition 1.1.12.** Let  $(A, \|\cdot\|_A)$  be a (semi-)normed algebraic structure, e.g., a (semi-)normed abelian group, a (semi-)normed ring, or a (semi-)normed module. The *completion* of  $A$ , denoted by  $\hat{A}$ , is the completion of  $A$  as a (pseudo-)metric space. Since  $A$  is dense in its completion, the algebraic operations and (semi-)norms on  $A$  can be uniquely extended to the completion.

Let  $R$  be a normed ring and  $M, N$  be semi-normed  $R$ -modules. There is a natural semi-norm on the tensor product  $M \otimes_R N$  defined as

$$\|z\|_{M \otimes_R N} = \inf \left\{ \sum_i \|x_i\|_M \|y_i\|_N : z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N \right\}.$$

**Definition 1.1.13.** Let  $R$  be a complete normed ring and  $M, N$  complete semi-normed  $R$ -modules. The *complete tensor product*  $M \hat{\otimes}_R N$  is defined as the completion of the semi-normed  $R$ -module  $M \otimes_R N$ .

**Definition 1.1.14.** Let  $R$  be a banach ring. For each  $f \in R$ , the *spectral radius* of  $f$  is defined as

$$\rho(f) = \lim_{n \rightarrow \infty} \|f^n\|^{1/n}.$$

Yang: Since ,  $\rho(f)$  exists.

**Definition 1.1.15.** A banach ring  $R$  is called *uniform* if its norm is power-multiplicative.



**Proposition 1.1.16.** Let  $(R, \|\cdot\|)$  be a banach ring. The spectral radius  $\rho(\cdot)$  defines a power-multiplicative semi-norm on  $R$  that is bounded by  $\|\cdot\|$ .

*Proof.* Yang: To be continued. □

**Definition 1.1.17.** Let  $R$  be a banach ring. An element  $f \in R$  is called *quasi-nilpotent* if  $\rho(f) = 0$ . All quasi-nilpotent elements of  $R$  form an ideal, denoted by  $\text{Qnil}(R)$ .

**Definition 1.1.18.** Let  $R$  be a banach ring. The *uniformization* of  $R$ , denoted by  $R \rightarrow R^u$ , is the banach ring with the universal property among all bounded homomorphisms from  $R$  to uniform banach rings. Yang: To be continued.

**Proposition 1.1.19.** Let  $R$  be a banach ring. The completion of  $R/\text{Qnil}(R)$  with respect to the spectral radius  $\rho(\cdot)$  is the uniformization of  $R$ .

*Proof.* Yang: To be continued. □

### 1.1.3 Complete field

**Definition 1.1.20.** A multiplicative norm on a field is also called an *absolute value*. A *valuation field* is a field equipped with an absolute value.

**Remark 1.1.21.** Let  $\mathbf{k}$  be a field. Recall that a *valuation* on  $\mathbf{k}$  is a function  $v : \mathbf{k}^\times \rightarrow \mathbb{R}$  such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$ ;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$ .

We can extend  $v$  to the whole field  $\mathbf{k}$  by defining  $v(0) = +\infty$ . Fix a real number  $\varepsilon \in (0, 1)$ . Then  $v$  induces an absolute value  $|\cdot|_v : \mathbf{k} \rightarrow \mathbb{R}_+$  defined by  $|x|_v = \varepsilon^{v(x)}$  for each  $x \in \mathbf{k}$ .

In some literature, the valuation  $v$  is called an *additive valuation* and the induced absolute value  $|\cdot|_v$  is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

**Definition 1.1.22.** A valuation field  $(\mathbf{k}, |\cdot|)$  is called *non-Archimedean* if  $\forall x, y \in \mathbf{k}, |x + y| \leq \max\{|x|, |y|\}$ , i.e., the norm satisfies the ultrametric inequality. Otherwise, it is called *Archimedean*.

**Definition 1.1.23.** A *complete field* is a valuation field which is complete as a metric space.

**Lemma 1.1.24.** Let  $\mathbf{k}$  be a non-Archimedean complete field. Then the set  $\mathbf{k}^\circ = \{x \in \mathbf{k} : |x| \leq 1\}$  is a subring of  $\mathbf{k}$ , which is a local ring. Moreover, the set  $\mathbf{k}^{\circ\circ} = \{x \in \mathbf{k} : |x| < 1\}$  is the maximal ideal of  $\mathbf{k}^\circ$ .

**Definition 1.1.25.** Let  $\mathbf{k}$  be a non-Archimedean complete field. The subring  $\mathbf{k}^\circ$  is called the *ring of integers* of  $\mathbf{k}$ . The set  $\mathbf{k}^{\circ\circ} = \{x \in \mathbf{k} : |x| < 1\}$  is the maximal ideal of  $\mathbf{k}^\circ$ . The residue field  $\hat{\mathbf{k}} = \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}$  is called the *residue field* of  $\mathbf{k}$ . Yang: To be revised.

### 1.1.4 Examples

**Example 1.1.26.** Let  $R$  be arbitrary ring. The *trivial norm* on  $R$  is defined as  $\|x\| = 0$  if  $x = 0$  and  $\|x\| = 1$  if  $x \neq 0$ . The ring  $R$  equipped with the trivial norm is a normed ring.

**Example 1.1.27.** The fields  $\mathbb{C}$  and  $\mathbb{R}$  equipped with the usual absolute value are complete fields.

**Example 1.1.28.** The field  $\mathbb{Q}_p$  of  $p$ -adic numbers equipped with the  $p$ -adic norm is a complete non-Archimedean field.

**Example 1.1.29.** Let  $R$  be a banach ring and  $r > 0$  be a real number. We define the ring of absolutely convergent power series over  $\mathbf{k}$  with radius  $r$  as

$$R\langle T/r \rangle := \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm  $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$ , the ring  $R\langle T/r \rangle$  is a banach ring.

When  $R = \mathbf{k}$  is a **Yang: To be checked.**

**Example 1.1.30.** Let  $\mathbf{k}$  be a non-Archimedean complete field. We define

$$\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\} := \left\{ \sum_{I \in \mathbb{N}^n} a_I T^I \in \mathbf{k}[[T_1, \dots, T_n]] : \lim_{|I| \rightarrow \infty} |a_I| r^I = 0 \right\},$$

where  $r = (r_1, \dots, r_n)$  is an  $n$ -tuple of positive real numbers,  $T^I = T_1^{i_1} \dots T_n^{i_n}$  for  $I = (i_1, \dots, i_n)$ , and  $|I| = i_1 + \dots + i_n$ . Equipped with the norm  $\|\sum_{I \in \mathbb{N}^n} a_I T^I\| = \sup_{I \in \mathbb{N}^n} |a_I| r^I$ , the affinoid  $\mathbf{k}$ -algebra  $\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$  is a banach  $\mathbf{k}$ -algebra. This is called the *Tate algebra* over  $\mathbf{k}$  with polyradius  $r$  equipped with the *Gauss norm*. We will denote  $\mathbf{k}\{\underline{T}/r\} = \mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$  for simplicity.

**Yang: To be continued...**

## 1.2 Affinoid algebras

### 1.2.1 The first properties

**Definition 1.2.1.** Let  $\mathbf{k}$  be a non-archimedean field. A banach  $\mathbf{k}$ -algebra  $A$  is called a *affinoid  $\mathbf{k}$ -algebra* if there exists an admissible surjective homomorphism

$$\varphi : \mathbf{k}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \twoheadrightarrow A$$

for some  $n \in \mathbb{N}$  and  $r_1, \dots, r_n \in \mathbb{R}_{>0}$ .

If one can choose  $r_1 = \dots = r_n = 1$ , then we say that  $A$  is a *strict affinoid  $\mathbf{k}$ -algebra*.

**Definition 1.2.2.** Let  $\mathbf{k}$  be a non-archimedean field. We define the *ring of restricted Laurent series* over  $\mathbf{k}$  as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n : a_n \in \mathbf{k}, \lim_{|n| \rightarrow \infty} |a_n| r^n = 0 \right\}$$

equipped with the norm

$$\|f\| = \sup_{n \in \mathbb{Z}} |a_n| r^n.$$

Yang: Is  $\mathbf{K}_r$  always a field? Yang: Do we have  $\mathbf{L}_{\mathbf{k},r} = \text{Frac}(\mathbf{k}\{T/r\})$ ?

**Proposition 1.2.3.** Let  $\mathbf{k}$  be a non-archimedean field. If  $r \notin \sqrt{|\mathbf{k}^\times|}$ , then  $\mathbf{K}_r$  is a complete non-archimedean field with non-trivial absolute value extending that of  $\mathbf{k}$ .

**Proposition 1.2.4.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. Then  $A$  is noetherian, and every ideal of  $A$  is closed.

**Proposition 1.2.5.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. Then there exists a constant  $C > 0$  and  $N > 0$  such that for all  $f \in A$  and  $n \geq N$ , we have

$$\|f^n\| \leq C \rho(f)^n.$$

**Proposition 1.2.6.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. If and only if  $\rho(f) \in \sqrt{|\mathbf{k}|}$  for all  $f \in A$ , then  $A$  is strict. Yang: To be complete.

## 1.2.2 Finite banach module

There are three different categories of finite modules over an affinoid algebra  $A$ :

- The category  $\mathbf{Banmod}_A$  of finite banach  $A$ -modules with  $A$ -linear maps as morphisms.
- The category  $\mathbf{Banmod}_A^b$  of finite banach  $A$ -modules with bounded  $A$ -linear maps as morphisms.
- The category  $\mathbf{mod}_A$  of finite  $A$ -modules with all  $A$ -linear maps as morphisms.

**Theorem 1.2.7.** Let  $A$  be an affinoid  $\mathbf{k}$ -algebra. Then the category of finite banach  $A$ -modules with bounded  $A$ -linear maps as morphisms is equivalent to the category of finite  $A$ -modules with  $A$ -linear maps as morphisms. Yang: To be revised.

For simplicity, we will just write  $\mathbf{mod}_A$  to denote the category of finite banach  $A$ -modules with bounded  $A$ -linear maps as morphisms.



# Chapter 2

## Affinoid spaces

### 2.1 Spectrum

#### 2.1.1 Definition

**Definition 2.1.1.** Let  $R$  be a Banach ring. The *spectrum*  $\mathcal{M}(R)$  of  $R$  is defined as the set of all multiplicative semi-norms on  $R$  that are bounded with respect to the given norm on  $R$ . For every point  $x \in \mathcal{M}(R)$ , we denote the corresponding multiplicative semi-norm by  $|\cdot|_x$ . We equip  $\mathcal{M}(R)$  with the weakest topology such that for each  $f \in R$ , the evaluation map  $\mathcal{M}(R) \rightarrow \mathbb{R}_{\geq 0}$ , defined by  $x \mapsto |f|_x =: f(x)$ , is continuous.

**Definition 2.1.2.** Let  $\varphi : R \rightarrow S$  be a bounded ring homomorphism of Banach rings. The *pullback* map  $\mathcal{M}(\varphi) : \mathcal{M}(S) \rightarrow \mathcal{M}(R)$  is defined by  $\mathcal{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$  for each  $x \in \mathcal{M}(S)$ .

**Definition 2.1.3.** Let  $R$  be a Banach ring. A *character* of  $R$  is a bounded ring homomorphism  $\chi : R \rightarrow K$ , where  $K$  is a completed field. Two characters  $\chi_1 : R \rightarrow K_1$  and  $\chi_2 : R \rightarrow K_2$  are said to be *equivalent* if there exists a commutative diagram of bounded ring homomorphisms

$$\begin{array}{ccccc} & & R & & \\ \chi_1 \swarrow & & \downarrow & \searrow \chi_2 & \\ K_1 & \longleftarrow & K & \longrightarrow & K_2 \end{array}$$

for some completed field  $K$ .

**Proposition 2.1.4.** Let  $R$  be a Banach ring. The spectrum  $\mathcal{M}(R)$  is in bijection with the equivalence classes of characters of  $R$ .

*Proof.* Yang: To be completed □

**Proposition 2.1.5.** Let  $R$  be a Banach ring. For each  $x \in \mathcal{M}(R)$ , let  $\wp_x$  be the kernel of the multiplicative semi-norm  $|\cdot|_x$ . Then  $\wp_x$  is a closed prime ideal of  $R$ , and  $x \mapsto \wp_x$  defines a continuous map from  $\mathcal{M}(R)$  to  $\text{Spec}(R)$  equipped with the Zariski topology.

*Proof.* Yang: To be completed □

**Definition 2.1.6.** Let  $R$  be a Banach ring. For each  $x \in \mathcal{M}(R)$ , the *completed residue field* at the point  $x$  is defined as the completion of the residue field  $\kappa(x) = \text{Frac}(R/\mathfrak{o}_x)$  with respect to the multiplicative norm induced by the semi-norm  $|\cdot|_x$ , denoted by  $\mathcal{H}(x)$ .

**Definition 2.1.7.** Let  $R$  be a Banach ring. The *Gel'fand transform* of  $R$  is the bounded ring homomorphism

$$\Gamma : R \rightarrow \prod_{x \in \mathcal{M}(R)} \mathcal{H}(x), \quad f \mapsto (f(x))_{x \in \mathcal{M}(R)},$$

where the norm on the product  $\prod_{x \in \mathcal{M}(R)} \mathcal{H}(x)$  is given by the supremum norm.

**Proposition 2.1.8.** The Gel'fand transform  $\Gamma : R \rightarrow \prod_{x \in \mathcal{M}(R)} \mathcal{H}(x)$  of a Banach ring  $R$  factors through the uniformization  $R^u$  of  $R$ , and the induced map  $R^u \rightarrow \prod_{x \in \mathcal{M}(R)} \mathcal{H}(x)$  is an isometric embedding. Yang: To be checked.

**Theorem 2.1.9.** Let  $R$  be a Banach ring. The spectrum  $\mathcal{M}(R)$  is a nonempty compact Hausdorff space.

*Proof.* Yang: To be continued. □

**Lemma 2.1.10.** Let  $\{K_i\}_{i \in I}$  be a family of completed fields. Consider the Banach ring  $R = \prod_{i \in I} K_i$  equipped with the product norm. The spectrum  $\mathcal{M}(R)$  is homeomorphic to the Stone-Čech compactification of the discrete space  $I$ .

**Remark 2.1.11.** The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

**Proposition 2.1.12.** Let  $K/k$  be a Galois extension of complete fields, and let  $R$  be a Banach  $k$ -algebra. The Galois group  $\text{Gal}(K/k)$  acts on the spectrum  $\mathcal{M}(R \hat{\otimes}_k K)$  via

$$g \cdot x : f \mapsto |(1 \otimes g^{-1})(f)|_x$$

for each  $g \in \text{Gal}(K/k)$ ,  $x \in \mathcal{M}(R \hat{\otimes}_k K)$  and  $f \in R \hat{\otimes}_k K$ . Moreover, the natural map  $\mathcal{M}(R \hat{\otimes}_k K) \rightarrow \mathcal{M}(R)$  induces a homeomorphism

$$\mathcal{M}(R \hat{\otimes}_k K) / \text{Gal}(K/k) \xrightarrow{\sim} \mathcal{M}(R).$$

Yang: To be checked.

## 2.1.2 Examples

**Example 2.1.13.** Let  $(\mathbf{k}, |\cdot|)$  be a complete valuation field. The spectrum  $\mathcal{M}(\mathbf{k})$  consists of a single point corresponding to the given absolute value  $|\cdot|$  on  $\mathbf{k}$ . Yang: To be checked.

**Example 2.1.14.** Consider the Banach ring  $(\mathbb{Z}, \|\cdot\|)$  with  $\|\cdot\| = |\cdot|_\infty$  is the usual absolute value norm on  $\mathbb{Z}$ . Let  $|\cdot|_p$  denote the  $p$ -adic norm for each prime number  $p$ , i.e.,  $|n|_p = p^{-v_p(n)}$  for each

$n \in \mathbb{Z}$ , where  $v_p(n)$  is the  $p$ -adic valuation of  $n$ . The spectrum

$$\mathcal{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0, 1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0, \infty)\} \cup \{|\cdot|_0\},$$

where  $|a|_p^\infty := \lim_{\alpha \rightarrow \infty} |a|_p^\alpha$  for each  $a \in \mathbb{Z}$  and  $|\cdot|_0$  is the trivial norm on  $\mathbb{Z}$ . **Yang: To be checked.**

**Spectrum of Tate algebra in one variable** Let  $\mathbf{k}$  be a complete non-archimedean field, and let  $A = \mathbf{k}\{T/r\}$ . We list some types of points in the spectrum  $\mathcal{M}(A)$ .

For each  $a \in \mathbf{k}$  with  $|a| \leq r$ , we have the *type I* point  $x_a$  corresponding to the evaluation at  $a$ , i.e.,  $|f|_{x_a} := |f(a)|$  for each  $f \in A$ . For each closed disk  $E = E(a, s) := \{b \in \mathbf{k} : |b - a| \leq s\}$  with center  $a \in \mathbf{k}$  and radius  $s \leq r$ , we have the point  $x_{a,s}$  corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} := \sup_{b \in E(a,s)} |f(b)|$$

for each  $f \in A$ . If  $s \in |\mathbf{k}^\times|$ , then the point  $x_E$  is called a *type II* point; otherwise, it is called a *type III* point.

Let  $\{E^{(s)}\}_s$  be a family of closed disks in  $\mathbf{k}$  such that  $E^{(s)}$  is of radius  $s$ ,  $E^{(s_1)} \subsetneq E^{(s_2)}$  for any  $s_1 < s_2$  and  $\bigcap_s E^{(s)} = \emptyset$ . Then we have the point  $x_{\{E^{(s)}\}}$  corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E^{(s)}\}}} := \inf_s |f|_{x_{E^{(s)}}}$$

for each  $f \in A$ . Such a point is called a *type IV* point.

**Yang: To be completed.**

**Proposition 2.1.15.** Let  $\mathbf{k}$  be a complete non-archimedean field, and let  $r > 0$  be a positive real number. Consider the Tate algebra  $\mathbf{k}\{r^{-1}T\}$  equipped with the Gauss norm. The points in the spectrum  $\mathcal{M}(\mathbf{k}\{r^{-1}T\})$  can be classified into four types as described above. **Yang: To be checked**

*Proof.* **Yang: To be completed.** □

**Proposition 2.1.16.** Let  $\mathbf{k}$  be a complete non-archimedean field, and let  $r > 0$  be a positive real number. Consider the Tate algebra  $\mathbf{k}\{r^{-1}T\}$  equipped with the Gauss norm. The completed residue fields of the four types of points in the spectrum  $\mathcal{M}(\mathbf{k}\{r^{-1}T\})$  are described as follows:

- For a type I point  $x_a$  with  $a \in \mathbf{k}$  and  $|a| \leq r$ , the completed residue field  $\mathcal{H}(x_a)$  is isomorphic to  $\mathbf{k}$ .
- For a type II point  $x_{a,s}$  with  $a \in \mathbf{k}$  and  $s \in |\mathbf{k}^\times|$ , the completed residue field  $\mathcal{H}(x_{a,s})$  is isomorphic to the field of Laurent series over the residue field  $\mathcal{K}_{\mathbf{k}}$ , i.e.,  $\mathcal{K}_{\mathbf{k}}((t))$ .
- For a type III point  $x_{a,s}$  with  $a \in \mathbf{k}$  and  $s \notin |\mathbf{k}^\times|$ , the completed residue field  $\mathcal{H}(x_{a,s})$  is isomorphic to a transcendental extension of  $\mathcal{K}_{\mathbf{k}}$  of degree one.
- For a type IV point  $x_{\{E^{(s)}\}}$ , the completed residue field  $\mathcal{H}(x_{\{E^{(s)}\}})$  is isomorphic to a transcendental extension of  $\mathcal{K}_{\mathbf{k}}$  of infinite degree.

**Yang: To be checked.**

**Example 2.1.17.** The completed residue field  $\mathcal{H}(x_a)$  for a type I point  $x_a$  with  $a \in \mathbf{k}$  and  $|a| \leq r$  is isomorphic to  $\mathbf{k}$ . *Yang: To be complete.*

## 2.2 Affinoid domains

**Definition 2.2.1.** Let  $A$  be a  $\mathbf{k}$ -affinoid algebra, and let  $X = \mathcal{M}(A)$  be the associated affinoid space. A closed subset  $V \subseteq X$  is called an *affinoid domain* if there exists a  $\mathbf{k}$ -affinoid algebra  $A_V$  and a morphism of  $\mathbf{k}$ -affinoid algebras  $\varphi : A \rightarrow A_V$  satisfying the following universal property: for every bounded homomorphism of  $\mathbf{k}$ -affinoid algebras  $\psi : A \rightarrow B$  such that the induced map on spectra  $\mathcal{M}(\psi) : \mathcal{M}(B) \rightarrow X$  has its image contained in  $V$ , there exists a unique bounded homomorphism  $\theta : A_V \rightarrow B$  such that the following diagram commutes:

$$\begin{array}{ccc} & A_V & \\ \varphi \nearrow & & \searrow \theta \\ A & \xrightarrow{\psi} & B \end{array}$$

In this case, we say that  $V$  is represented by the affinoid algebra  $A_V$ .

**Slogan** *A closed subset  $V \subset X$  is an affinoid domain if the functor “ $\text{Mor}(-, V)$ ” is representable.*

*Yang: Why we consider closed subset rather than open subset?*

**Construction 2.2.2.** Let  $f = (f_1, \dots, f_n), g = (g_1, \dots, g_m)$  be two tuples of elements in  $A$ . Set  $p = (p_1, \dots, p_n)$  and  $q = (q_1, \dots, q_m)$  be two tuples of positive real numbers. We define the following closed subsets of  $X$ :

$$X(\underline{f/p}, \underline{q/g}) := \{x \in X : |f_i(x)| \leq p_i, |g_j(x)| \geq q_j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Such a closed subset is called a *Weierstrass domain* of  $X$ . Moreover, we can define a  $\mathbf{k}$ -affinoid algebra

$$A\langle \underline{f/p}, \underline{q/g} \rangle := A\left\langle \frac{f_1}{p_1}, \dots, \frac{f_n}{p_n}, \frac{q_1}{g_1}, \dots, \frac{q_m}{g_m} \right\rangle,$$

which is the quotient of the Tate algebra

$$A\langle T_1, \dots, T_n, S_1, \dots, S_m \rangle$$

by the ideal generated by the elements  $p_i T_i - f_i$  for  $1 \leq i \leq n$  and  $g_j S_j - q_j$  for  $1 \leq j \leq m$ . There is a natural bounded homomorphism  $\varphi : A \rightarrow A\langle \underline{f/p}, \underline{q/g} \rangle$  induced by the inclusion. It can be shown that the closed subset  $X(\underline{f/p}, \underline{q/g})$  is an affinoid domain represented by the affinoid algebra  $A\langle \underline{f/p}, \underline{q/g} \rangle$ . *Yang: To be checked*

**Construction 2.2.3.** Let  $f = (f_1, \dots, f_n), g$  be elements in  $A$  such that the ideal generated by them is the whole algebra  $A$ . Set  $p = (p_1, \dots, p_n)$  be a tuple of positive real numbers. We define the following closed subset of  $X$ :

$$X(\underline{f/p}, g) := \{x \in X : |f_i(x)| \leq p_i |g(x)|, 1 \leq i \leq n\}.$$



Such a closed subset is called a *rational domain* of  $X$ . Moreover, we can define a  $\mathbf{k}$ -affinoid algebra

$$A\langle \underline{f/p}, g^{-1} \rangle := A\left\langle \frac{f_1}{p_1 g}, \dots, \frac{f_n}{p_n g} \right\rangle,$$

which is the quotient of the Tate algebra

$$A\langle T_1, \dots, T_n \rangle$$

by the ideal generated by the elements  $p_i g T_i - f_i$  for  $1 \leq i \leq n$ . There is a natural bounded homomorphism  $\varphi : A \rightarrow A\langle \underline{f/p}, g^{-1} \rangle$  induced by the inclusion. It can be shown that the closed subset  $X(\underline{f/p}, g)$  is an affinoid domain represented by the affinoid algebra  $A\langle \underline{f/p}, g^{-1} \rangle$ . Yang: To be checked

**Proposition 2.2.4.** Let  $A$  be a  $\mathbf{k}$ -affinoid algebra, and let  $X = \mathcal{M}(A)$  be the associated affinoid space. Let  $V \subseteq X$  be an affinoid domain represented by the  $\mathbf{k}$ -affinoid algebra  $A_V$ . Then the natural bounded homomorphism  $\varphi : A \rightarrow A_V$  is flat.

We have  $\mathcal{M}(A_V) \cong V$ .



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## Chapter 3

# Analytic spaces



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# References

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