Finite field extensions

1 Finite-dimensional vector space

Definition 1. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|ax\| = |a| \cdot \|x\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Lemma 3. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V.

Proof. Yang: To be added. □

Proposition 4. Let V be a finite-dimensional vector space over a complete non-archimedean field $\mathbf{k}$. Then all norms on V are equivalent. Yang: To be checked.

Proof. Yang: To be added. □

Proposition 5. Let V be a finite-dimensional vector space over a complete valuation field $\mathbf{k}$. Then V is complete with respect to any norm on V.

2 Finite field extensions

Construction 6. Let $\mathbf{k}$ be a valuation field and $\mathbf{l}$ a finite extension of $\mathbf{k}$ with degree $n = [\mathbf{l} : \mathbf{k}]$. For any $a \in \mathbf{l}$, define

$$|a|_{\mathbf{l}} := |N_{\mathbf{l}/\mathbf{k}}(a)|_{\mathbf{k}}^{1/n},$$

where $N_{\mathbf{l}/\mathbf{k}}(a)$ is the norm of a from $\mathbf{l}$ to $\mathbf{k}$. By Lemma 7, $|\cdot|_{\mathbf{l}}$ is an absolute value on $\mathbf{l}$ extending the absolute value $|\cdot|_{\mathbf{k}}$ on $\mathbf{k}$. Yang: To be checked.

Lemma 7. Let $\mathbf{k}$ be a valuation field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the function $|\cdot|_{\mathbf{l}}$ defined in Construction 6 is an absolute value on $\mathbf{l}$ extending the absolute value on $\mathbf{k}$.
Moreover, if $\mathbf{k}$ is non-archimedean, then so is $\mathbf{l}$.

Proof. Yang: To be added. □

Date: November 3, 2025, **Author**: Tianle Yang, My Homepage

Proposition 8. Let $\mathbf{k}$ be a complete non-archimedean field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the absolute value on $\mathbf{l}$ is uniquely determined by the absolute value on $\mathbf{k}$. Yang: To be checked.

Proof. Yang: To be added. □

Remark 9. Yang: I want to discuss some compatiblity of extension and completion.

Proposition 10. Let $\mathbf{k}$ be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed. Yang: To be checked.

Proof. Yang: To be added. □

Appendix