Affinoid domains

Consider $X = \mathscr{M}(A)$ with $A = \mathbf{k}\{T_1, \dots, T_n\}$. Yang: Not every open subset of X gives an affinoid space, that is, the completion of the ring of analytic functions on that open subset is not necessarily an affinoid algebra. Yang: Right? example?

1 Definition

Definition 1. Let A be a $\mathbf{k}$-affinoid algebra, and let $X = \mathscr{M}(A)$ be the associated affinoid space. A closed subset $V \subseteq X$ is called an *affinoid domain* if there exists a $\mathbf{k}$-affinoid algebra A_V and a morphism of $\mathbf{k}$-affinoid algebras $\varphi : A \to A_V$ satisfying the following universal property: for every bounded homomorphism of $\mathbf{k}$-affinoid algebras $\psi : A \to B$ such that the induced map on spectra $\mathscr{M}(\psi) : \mathscr{M}(B) \to X$ has its image contained in V, there exists a unique bounded homomorphism $\theta : A_V \to B$ such that the following diagram commutes:

$$\begin{tikzcd} & A_V \arrow[dr, dashed, "\theta"] & \\ A \arrow[ur, "\varphi"] \arrow[rr, "\psi"'] & & B \end{tikzcd}$$

In this case, we say that V is represented by the affinoid algebra A_V.

Slogan *A closed subset $V \subset X$ is an affinoid domain if the functor "$\mathrm{Mor}(-, V)$" is representable.*

Yang: Why we consider closed subset rather that open subset?

Construction 2. Let $f = (f_1, \dots, f_n)$ be a tuple of elements in A and $r = (r_1, \dots, r_n)$ be a tuple of positive real numbers. Consider the closed subset of X:

$$X\left(\underline{f/r}\right) := \{x \in X : |f_i(x)| \le r_i, 1 \le i \le n\}.$$

Such a closed subset is called a *Weierstrass domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\{\underline{f/r}\right\} := A\{f_1/r_1, \dots, f_n/r_n\}.$$

Yang: The domain $X\left(\underline{f/r}\right)$ is represented by $A\left\{\underline{f/r}\right\}$.

Construction 3. Let $f = (f_1, \dots, f_n), g = (g_1, \dots, g_m)$ be two tuples of elements in A and $r = (r_1, \dots, r_n), s = (s_1, \dots, s_m)$ be two tuples of positive real numbers. Consider the following closed subset of X:

$$X\left(\underline{f/r}; \underline{g/s}^{-1}\right) := \left\{x \in X : |f_i(x)| \le r_i, |g_j(x)| \ge s_j, 1 \le i \le n, 1 \le j \le m\right\}.$$

Such a closed subset is called a *Laurent domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\{\underline{f/r}; \underline{g/s}^{-1}\right\} := A\{f_1/r_1, \dots, f_n/r_n, g_1^{-1}/s_1, \dots, g_m^{-1}/s_m\}.$$

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Yang: The domain $X\left(\underline{f/r};\underline{g/s}^{-1}\right)$ is represented by $A\left\{\underline{f/r};\underline{g/s}^{-1}\right\}$.

Construction 4. Let $f=(f_1,\dots,f_n), g$ be elements in A such that the ideal generated by them is the whole algebra A. Set $p=(p_1,\dots,p_n)$ be a tuple of positive real numbers. We define the following closed subset of X:

$$X\left(\underline{f/p},g\right):=\{x\in X:\ |f_i(x)|\le p_i|g(x)|, 1\le i\le n\}.$$

Such a closed subset is called a *rational domain* of X. Moreover, we can define a **k**-affinoid algebra

$$A\left\langle \underline{f/p},g^{-1}\right\rangle := A\left\langle \frac{f_1}{p_1 g},\dots,\frac{f_n}{p_n g}\right\rangle,$$

which is the quotient of the Tate algebra

$$A\langle T_1,\dots,T_n\rangle$$

by the ideal generated by the elements $p_i g T_i - f_i$ for $1\le i\le n$. There is a natural bounded homomorphism $\varphi: A\to A\langle \underline{f/p},g^{-1}\rangle$ induced by the inclusion. It can be shown that the closed subset $X(\underline{f/p},g)$ is an affinoid domain represented by the affinoid algebra $A\langle\underline{f/p},g^{-1}\rangle$. Yang: To be checked

Yang: We have a sequence of inclusion:

$$\{\text{Weierstrass domains}\}\subseteq\{\text{Laurent domains}\}\subseteq\{\text{Rational domains}\}\subseteq\{\text{Affinoid domains}\}.$$

Proposition 5. Let A be a **k**-affinoid algebra, and let $X=\mathscr{M}(A)$ be the associated affinoid space. Let $V\subseteq X$ be an affinoid domain represented by the **k**-affinoid algebra A_V. Then the natural bounded homomorphism $\varphi: A\to A_V$ is flat.
We have $\mathscr{M}(A_V)\cong V$.

2 The Grothendieck topology of affinoid domains

Appendix