Valued Fields

1 Setup

Let $\mathbf{k}$ be a field. A *norm* on $\mathbf{k}$ is a function $\|\cdot\| : \mathbf{k} \to \mathbb{R}_+$ such that

- (non-degeneracy) $\|x\| = 0 \iff x = 0$;
- (normalization) $\|1\| = 1$;
- (multiplicativity) $\forall x, y \in \mathbf{k}, \|xy\| = \|x\|\|y\|$;
- (triangle inequality) $\forall x, y \in \mathbf{k}, \|x + y\| \leq \|x\| + \|y\|$.

A norm is called *non-Archimedean* if it satisfies the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}.$$

A field $\mathbf{k}$ equipped with a non-Archimedean norm is called a *non-Archimedean field*. A norm induces a metric $d(x, y) = \|x - y\|$ on $\mathbf{k}$. With this metric, $\mathbf{k}$ is a metric space. We say $\mathbf{k}$ is *complete* if it is complete as a metric space.

2 Fundamental fact about ultra-metric space

Appendix

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