Residue fields and reductions

1 Recover non-archimedean complete fields algebraically

In this subsection, let $\mathbf{k}$ be a non-archimedean field. Set $I_{r,<} := \{x \in \mathbf{k} : \|x\| < r\}$ and $I_{r,\leq} := \{x \in \mathbf{k} : \|x\| \leq r\}$ for each $r \in (0,1)$.

Proposition 1. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers $\mathbf{k}^\circ$. Conversely, any ideal of $\mathbf{k}^\circ$ is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0,1)$. Yang: To be checked.

Proof. Yang: To be checked. □

Proposition 2. We have

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{r\in(0,1)} \mathbf{k}^\circ/I_r.$$

Yang: To be checked.

Proposition 3. Let $\mathbf{k}$ be a non-archimedean field. Then $\mathbf{k}$ is totally bounded iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$. Moreover, if $\mathbf{k}$ is complete, then it is locally compact iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$. Yang: To be checked.

Slogan *"Locally compact $\iff$ pro-finite."*

Proof. □

Proposition 4. The ring $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field. Yang: To be revised.

Proposition 5. Let $\mathbf{k}$ be a complete non-archimedean field. Then $\mathbf{k}$ is locally compact iff $\mathbf{k}$ is a discrete valuation field and its residue field $\mathscr{k}_{\mathbf{k}}$ is finite. Yang: To be checked.

Proof. Yang: To be added. □

2 Hensel's Lemma

Theorem 6 (Hensel's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \mathscr{k}_{\mathbf{k}}[T]$ are monic polynomials that are coprime in $\mathscr{k}_{\mathbf{k}}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\mathscr{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively. Yang: To be checked.

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Proof. Yang: To be added. □

Corollary 7. Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^{\circ}[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $\alpha \in \mathscr{k}_{\mathbf{k}}$. Then there exists a root $a \in \mathbf{k}^{\circ}$ of $F(T)$ whose reduction is α. Yang: To be revised.

Proof. Yang: To be added. □

3 Newton polygons

Yang: To be filled.

Appendix