

Semi-normed Rings and Modules

1 Semi-normed algebraic structures

Definition 1. Let G be an abelian group. A *semi-norm* on G is a function $\|\cdot\| : G \rightarrow \mathbb{R}_{\geq 0}$ such that

- $\|0\| = 0$;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$.

Suppose that R is a ring (commutative with unity) and $\|\cdot\|$ is a semi-norm on the underlying abelian group of R . We further require that

- $\|1\| = 1$;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$.

Suppose that $(M, \|\cdot\|_M)$ is an R -module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M . We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$.

Suppose that $(A, \|\cdot\|_A)$ is an R -algebra and $\|\cdot\|_A$ is a semi-norm on the underlying R -module of A . We further require that this semi-norm is a semi-norm on the underlying ring of A .

Definition 2. Let A be an abelian group (or ring, R -module, R -algebra) equipped with a semi-norm $\|\cdot\|$. If $\forall x \in A, \|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*.

Yang: Note that this definition of semi-normed module is a little different of [Ber90]

Definition 3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group (or ring, R -module, R -algebra) A . We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in A, \|x\|_1 \leq C\|x\|_2$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are bounded by each other, we say they are *equivalent*.

Remark 4. Equivalent semi-norms induce the same topology on A .

Definition 5. Let M be a semi-normed abelian group (or ring, R -module, R -algebra) and $N \subseteq M$ be a subgroup (or ideal, R -submodule, ideal). The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Yang: Is this always a semi-norm? In particular, $\|1\| = 1$?

Unless otherwise specified, we always equip the quotient M/N with the residue semi-norm.

Remark 6. The residue semi-norm is a norm if and only if N is closed in M .

Definition 7. Let M and N be two semi-normed abelian groups (or rings, R -modules, R -algebras). A homomorphism $f : M \rightarrow N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$.

A bounded homomorphism $f : M \rightarrow N$ is called *admissible* if the induced isomorphism $M/\ker f \rightarrow \text{Im } f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$.

Definition 8. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$.

2 Banach rings

Definition 9. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M . A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 10. A *banach ring* is a complete normed ring.

Definition 11. Let $(A, \|\cdot\|_A)$ be a (semi-)normed algebraic structure, e.g., a (semi-)normed abelian group, a (semi-)normed ring, or a (semi-)normed module. The *completion* of A , denoted by \widehat{A} , is the completion of A as a (pseudo-)metric space. Since A is dense in its completion, the algebraic operations and (semi-)norms on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R -modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M \otimes_R N} = \inf \left\{ \sum_i \|x_i\|_M \|y_i\|_N : z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N \right\}.$$

Definition 12. Let R be a complete normed ring and M, N complete semi-normed R -modules. The *complete tensor product* $M \widehat{\otimes}_R N$ is defined as the completion of the semi-normed R -module $M \otimes_R N$.

Definition 13. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n \rightarrow \infty} \|f^n\|^{1/n}.$$

Yang: Since, $\rho(f)$ exists.

Definition 14. A banach ring R is called *uniform* if its norm is power-multiplicative.

Proposition 15. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

| *Proof.* Yang: To be continued. □

Definition 16. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\text{Qnil}(R)$.

Definition 17. Let R be a banach ring. The *uniformization* of R , denoted by $R \rightarrow R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. **Yang:** To be continued.

Proposition 18. Let R be a banach ring. The completion of $R/\text{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R .

| *Proof.* Yang: To be continued. □

3 Examples

Example 19. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a normed ring.

Example 20. The fields \mathbb{C} and \mathbb{R} equipped with the usual absolute value are complete fields.

Example 21. The field \mathbb{Q}_p of p -adic numbers equipped with the p -adic norm is a complete non-Archimedean field.

Example 22. Let R be a banach ring and $r > 0$ be a real number. We define the ring of absolutely convergent power series over \mathbf{k} with radius r as

$$R\langle T/r \rangle := \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r \rangle$ is a banach ring.

When $R = \mathbf{k}$ is a **Yang:** To be checked.

Example 23. Let \mathbf{k} be a non-Archimedean complete field. We define

$$\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\} := \left\{ \sum_{I \in \mathbb{N}^n} a_I T^I \in \mathbf{k}[[T_1, \dots, T_n]] : \lim_{|I| \rightarrow \infty} |a_I|r^I = 0 \right\},$$

where $r = (r_1, \dots, r_n)$ is an n-tuple of positive real numbers, $T^I = T_1^{i_1} \cdots T_n^{i_n}$ for $I = (i_1, \dots, i_n)$, and $|I| = i_1 + \cdots + i_n$. Equipped with the norm $\|\sum_{I \in \mathbb{N}^n} a_I T^I\| = \sup_{I \in \mathbb{N}^n} |a_I|r^I$, the affinoid \mathbf{k} -algebra $\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$ is a banach \mathbf{k} -algebra. This is called the *Tate algebra* over \mathbf{k} with polyyradius r equipped with the *Gauss norm*. We will denote $\mathbf{k}\{T/r\} = \mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$ for simplicity.

Yang: To be continued...

Appendix

References

- [Ber90] Vladimir G Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. 33. American Mathematical Soc., 1990 (cit. on p. 1).