Spectrum

1 Definition

Definition 1. Let R be a Banach ring. The *spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$. We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_+$, defined by $x \mapsto |f|_x$, is continuous.

For $x \in \mathscr{M}(R)$, the kernel of the multiplicative semi-norm $|\cdot|_x$ is a closed prime ideal of R, denoted by $\wp_x$. The semi-norm $|\cdot|_x$ induces a multiplicative norm on the residue field $\kappa(x) = \mathrm{Frac}(R/\wp_x)$, denoted by $|\cdot|_x$ as well.

Definition 2. Let R be a Banach ring. A *character* of R is a bounded ring homomorphism $\chi : R \to K$, where K is a complete valued field. Two characters $\chi_1 : R \to K_1$ and $\chi_2 : R \to K_2$ are said to be *equivalent* if there exists an isometric field extension L of both K_1 and K_2 such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\chi_1"] \arrow[d, "\chi_2"'] & K_1 \arrow[d] \\ K_2 \arrow[r] & L \end{tikzcd}$$

Definition 3. Let $f : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $f^* : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $f^*(x) = x \circ f$ for each $x \in \mathscr{M}(S)$. Yang: To be revised.

Proposition 4. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is in bijection with the equivalence classes of characters of R.

Theorem 5. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a nonempty compact Hausdorff space.

Proof. Yang: To be continued. □

2 Examples

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