

# Algebraic structures of non-archimedean fields

## 1 Recover non-archimedean complete fields algebraically

In this subsection, let  $\mathbf{k}$  be a non-archimedean field. Set  $I_{r,<} := B(0, r)$  and  $I_{r,\leq} := E(0, r)$  for each  $r \in [0, 1]$ .

**Proposition 1.** The sets  $I_{r,<}$  and  $I_{r,\leq}$  are ideals of the ring of integers  $\mathbf{k}^\circ$ . Conversely, any ideal of  $\mathbf{k}^\circ$  is of the form  $I_{r,<}$  or  $I_{r,\leq}$  for some  $r \in (0, 1)$ .

*Proof.* Let  $I$  be an ideal of  $\mathbf{k}^\circ$ . Set  $r = \sup\{|a| : a \in I\}$  (resp.  $r = \max\{|a| : a \in I\}$  when the maximum exists). Then, by definition, we have  $I \subset I_{r,<}$  (resp.  $I \subset I_{r,\leq}$ ). For every  $x \in \mathbf{k}^\circ$  with  $|x| < r$  (resp.  $|x| \leq r$ ), there exists  $a \in I$  such that  $|x| \leq |a|$ . Thus,  $|x/a| \leq 1$  and so  $x/a \in \mathbf{k}^\circ$ . Since  $I$  is an ideal, we have  $x = (x/a)a \in I$ . Therefore,  $I_{r,<} \subset I$  (resp.  $I_{r,\leq} \subset I$ ).  $\square$

**Proposition 2.** Let  $I_r$  be either  $I_{r,<}$  or  $I_{r,\leq}$  for each  $r \in (0, 1)$ . Suppose  $\{r_n \in (0, 1)\}_{n \in \mathbb{N}}$  is a decreasing sequence converging to 0. Then the completion  $\widehat{\mathbf{k}}$  of  $\mathbf{k}$  is isomorphic to the projective limit

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}.$$

*Proof.* For every  $x \in \widehat{\mathbf{k}}^\circ$ , there exists a cauchy sequence  $\{x_m\}_{m \in \mathbb{N}}$  in  $\mathbf{k}^\circ$  converging to  $x$ . Since  $\{r_n\}_{n \in \mathbb{N}}$  converges to 0, for each  $n \in \mathbb{N}$ , there exists  $M_n \in \mathbb{N}$  such that for all  $m, m' \geq M_n$ , we have  $|x_m - x_{m'}| < r_n$ . Thus, the sequence  $\{x_m + I_{r_n}\}_{m \in \mathbb{N}}$  is eventually constant in  $\mathbf{k}^\circ / I_{r_n}$ . Define a map

$$\Phi : \widehat{\mathbf{k}}^\circ \rightarrow \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}, \quad x \mapsto \left( \lim_{m \rightarrow \infty} x_m + I_{r_n} \right)_{n \in \mathbb{N}}.$$

It is straightforward to verify that  $\Phi$  is a well-defined ring homomorphism.

Conversely, for every  $(a_n + I_{r_n})_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}$ , we can choose a representative  $a_n \in \mathbf{k}^\circ$  for each  $n$ . We claim that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  is a cauchy sequence in  $\mathbf{k}^\circ$ . Indeed, for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $r_N < \varepsilon$ . For all  $m, n \geq N$ , since  $a_n + I_{r_n}$  maps to  $a_m + I_{r_m}$  under the natural projection, we have  $|a_n - a_m| < r_N < \varepsilon$ . Thus,  $\{a_n\}_{n \in \mathbb{N}}$  converges to some  $x \in \widehat{\mathbf{k}}^\circ$ . Easily see that the limit  $x$  is independent of the choice of representatives  $\{a_n\}_{n \in \mathbb{N}}$ . This gives a map

$$\Psi : \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n} \rightarrow \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow \infty} a_n.$$

Direct verification shows that  $\Psi = \Phi^{-1}$ .  $\square$

**Corollary 3.** Let  $\mathbf{k}$  be a non-archimedean field and  $\widehat{\mathbf{k}}$  its completion. Then the residue field  $\kappa_{\widehat{\mathbf{k}}} \cong \kappa_{\mathbf{k}}$  under the natural embedding  $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$ .

**Corollary 4.** Let  $\mathbf{k}$  be a non-archimedean field and  $\widehat{\mathbf{k}}$  its completion. Then the valuation group  $|\widehat{\mathbf{k}}^\times|$  of  $\widehat{\mathbf{k}}$  is equal to the valuation group  $|\mathbf{k}^\times|$  of  $\mathbf{k}$ .

*Proof.* Note that

$$\begin{aligned}
 r \in |\hat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \hat{\mathbf{k}}^\circ \\
 &\iff \hat{\mathbf{k}}^\circ/I_{r,<} \rightarrow \hat{\mathbf{k}}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\
 &\iff \mathbf{k}^\circ/I_{r,<} \rightarrow \mathbf{k}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\
 &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\
 &\iff r \in |\mathbf{k}^\times|.
 \end{aligned}$$

□

**Proposition 5.** Let  $\mathbf{k}$  be a non-archimedean field. Then  $\mathbf{k}^\circ$  is totally bounded iff  $\mathbf{k}^\circ/I_{r,<}$  and  $\mathbf{k}^\circ/I_{r,\leq}$  are finite for each  $r \in [0, 1]$ . Moreover, if  $\mathbf{k}$  is complete, then it is locally compact iff  $\mathbf{k}^\circ/I_r$  is finite for each  $r \in (0, 1)$ .

**Slogan** “Locally compact  $\iff$  pro-finite.”

*Proof.* We just prove the case for  $I_r = I_{r,<}$ . The case for  $I_r = I_{r,\leq}$  is similar.

Suppose that  $\mathbf{k}^\circ/I_r$  is finite for each  $r \in [0, 1]$ . Then for every  $\varepsilon > 0$ , there exists  $r \in (0, 1)$  such that  $r < \varepsilon$  and  $\mathbf{k}^\circ/I_r$  is finite. Let  $\{a_1 + I_r, \dots, a_n + I_r\}$  be the complete set of representatives of  $\mathbf{k}^\circ/I_r$ . Then the balls  $B(a_i, r)$  for  $i = 1, \dots, n$  cover  $\mathbf{k}^\circ$ .

Conversely, suppose that  $\mathbf{k}^\circ/I_r$  is infinite for some  $r \in [0, 1]$ . Then there exists an infinite set  $\{a_n\}$  with  $|a_n| \in [r, 1]$  such that their images in  $\mathbf{k}^\circ/I_r$  are distinct. In particular, for every  $m \neq n$ , we have  $|a_n - a_m| \geq r$ . Any subsequence of  $\{a_n\}$  is not cauchy. Thus,  $\mathbf{k}^\circ$  is not totally bounded. □

**Proposition 6.** The ring  $\mathbf{k}^\circ$  is noetherian iff  $\mathbf{k}$  is a discrete valuation field.

*Proof.* Note that  $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$  is a multiplicative subgroup. If  $\mathbf{k}$  is not a discrete valuation field, then  $|\mathbf{k}^\times|$  is dense in  $\mathbb{R}_{>0}$ . In particular, there exists a strictly ascending sequence  $r_n \in |\mathbf{k}^\times| \cap (0, 1)$ . Then the ideals  $I_{r_n,\leq}$  form a strictly ascending chain of ideals in  $\mathbf{k}^\circ$ .

The converse is standard since now  $\mathbf{k}^\circ$  is a discrete valuation ring. □

**Proposition 7.** Let  $\mathbf{k}$  be a complete non-archimedean field. Then  $\mathbf{k}$  is locally compact iff  $\mathbf{k}$  is a discrete valuation field and its residue field  $\mathcal{K}_\mathbf{k}$  is finite.

*Proof.* The necessity follows from [Proposition 5](#). For the sufficiency, suppose that  $\mathbf{k}$  is a discrete valuation field whose residue field  $\mathcal{K}_\mathbf{k}$  is finite. Let  $\pi \in \mathbf{k}^\circ$  be a uniformizer. We only need to show that  $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$  is finite for each  $n \in \mathbb{N}$ . Note that there is an isomorphism

$$\pi^{n-1} \mathbf{k}^\circ / \pi^n \mathbf{k}^\circ \cong \mathcal{K}_\mathbf{k}, \quad x + \pi^n \mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on  $n$ , we conclude that  $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$  is finite. □

## 2 Hensel's Lemma

**Theorem 8** (Hensel's lemma). Let  $\mathbf{k}$  be a complete non-archimedean field and  $F(T) \in \mathbf{k}^\circ[T]$  a monic polynomial. Suppose that the reduction  $f(T) \in \mathcal{K}_\mathbf{k}[T]$  of  $F(T)$  factors as

$$f(T) = g(T)h(T),$$

where  $g(T), h(T) \in \mathcal{K}_\mathbf{k}[T]$  are monic polynomials that are coprime in  $\mathcal{K}_\mathbf{k}[T]$ . Then there exist monic polynomials  $G(T), H(T) \in \mathbf{k}^\circ[T]$  such that

$$F(T) = G(T)H(T),$$

and the reductions of  $G(T), H(T)$  in  $\mathcal{K}_\mathbf{k}[T]$  are  $g(T), h(T)$  respectively.

*Proof.* Since  $\gcd(g, h) = 1$  in  $\mathcal{K}_\mathbf{k}[T]$ , there exist polynomials  $u(T), v(T) \in \mathcal{K}_\mathbf{k}[T]$  such that  $ug + vh = 1$  and  $\deg u < \deg h, \deg v < \deg g$ . Choose lifts  $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$  of  $g(T), h(T), u(T), v(T)$  respectively preserving their degrees such that  $G_0$  and  $H_0$  are monic. Then there exist  $r < 1$  such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \pmod{I_r}, \quad F(T) - G_0(T)H_0(T) \equiv 0 \pmod{I_r},$$

where  $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$ .

We will construct a sequence of monic polynomials  $\{G_n(T)\}_{n \in \mathbb{N}}$  and  $\{H_n(T)\}_{n \in \mathbb{N}}$  in  $\mathbf{k}^\circ[T]$  such that for each  $n \in \mathbb{N}$ ,

$$G_n(T) \equiv G_{n-1}(T) \pmod{I_{r^n}}, \quad H_n(T) \equiv H_{n-1}(T) \pmod{I_{r^n}},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \pmod{I_{r^{n+1}}}.$$

If we have such sequences, then their coefficients converge in the complete ring  $\mathbf{k}^\circ$ . Let  $G(T)$  and  $H(T)$  be the limits of  $\{G_n(T)\}$  and  $\{H_n(T)\}$  respectively. Then we have  $F(T) = G(T)H(T)$  and the reductions of  $G(T), H(T)$  in  $\mathcal{K}_\mathbf{k}[T]$  are  $g(T), h(T)$  respectively.

The case  $n = 0$  is done by the above construction. Now suppose that we have constructed  $G_n(T)$  and  $H_n(T)$  for some  $n \geq 0$ . Since  $G_n - G_0 \equiv 0 \pmod{I_r}$  and  $H_n - H_0 \equiv 0 \pmod{I_r}$ , we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \pmod{I_r}.$$

Set  $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$  and  $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$ . Then we have

$$\begin{aligned} (G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T]. \end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

□

**Corollary 9.** Let  $\mathbf{k}$  be a complete non-archimedean field and  $F(T) \in \mathbf{k}^\circ[T]$  a monic polynomial. Suppose that the reduction  $f(T) \in \mathcal{K}_\mathbf{k}[T]$  of  $F(T)$  has a simple root  $a \in \mathcal{K}_\mathbf{k}$ . Then there exists a root  $\alpha \in \mathbf{k}^\circ$  of  $F(T)$  whose reduction is  $a$ .

*Proof.* Since  $a$  is a simple root of  $f(T)$ , we have the factorization  $f(T) = (T - a)h(T)$  for some monic polynomial  $h(T) \in \mathcal{K}_\mathbf{k}[T]$  with  $h(a) \neq 0$ . Then the result follows from [Theorem 8](#).  $\square$

### 3 Newton polygons

Yang: To be filled.

DRAFT