Ultra-metric spaces

We will use $B(x,r)$ (resp. $E(x,r)$) to denote the open ball (resp. closed ball) with center x and radius r.

Definition 1. A metric space (X,d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x,z) \le \max\{d(x,y), d(y,z)\}, \quad \forall x,y,z \in X.$$

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x,y) := \|x-y\|$ on $\mathbf{k}$ makes $(\mathbf{k}, d)$ an ultra-metric space.

Proposition 2. Let (X,d) be an ultra-metric space. Then for any $x,y,z \in X$, at least two of the three distances $d(x,y), d(y,z), d(z,x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x,y) \ge d(y,z)$. By the strong triangle inequality, we have

$$d(x,z) \le \max\{d(x,y), d(y,z)\} = d(x,y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x,y) \le \max\{d(x,z), d(z,y)\} = \max\{d(x,z), d(y,z)\} \le d(x,y).$$

This shows that $d(x,y) = \max\{d(x,z), d(y,z)\}$. Thus either $d(x,z) = d(x,y) \ge d(y,z)$ or $d(y,z) = d(x,y) \ge d(x,z)$. $\square$

Proposition 3. Let (X,d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1,2$. If $D_1 \cap D_2 \neq \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1,2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1,x_2) \le \max\{d(x_1,y), d(y,x_2)\}.$$

Without loss of generality, we may assume that $d(x_1,x_2) \le d(x_1,y)$. It follows that $x_2 \in D_1$ since $d(x_1,y) < r_1$ (or $\le r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1,z) > d(x_1,x_2)$. Then by Proposition 2,

$$r_1 \le d(x_1,z) = d(x_2,z) \le r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2,w) \le \max\{d(x_2,x_1), d(x_1,w)\} \le r_1 \le r_2.$$

Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. $\square$

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Proposition 4. Let (X,d) be an ultra-metric space. Then both $B(x,r)$ and $E(x,r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x,r) := \{y \in X \mid d(x,y) = r\}$ is open in X. Note that if $y \in S(x,r)$, then for any $r' < r$, we have $B(y,r') \cap E(x,r) \neq \varnothing$ and $x \in E(x,r) \setminus B(y,r')$. Thus by Proposition 3, we have $B(y,r') \subseteq E(x,r)$. If $B(y,r') \cap B(x,r) \neq \varnothing$, then by Proposition 3 again, we have $B(y,r') \subseteq B(x,r)$. However, $y \in B(y,r') \setminus B(x,r)$, a contradiction. Thus $B(y,r') \subseteq E(x,r) \setminus B(x,r) = S(x,r)$. It yields that $S(x,r) = \bigcup_{y \in S(x,r)} B(y,r/2)$ is open in X.

Since $E(x,r) = B(x,r) \cup S(x,r)$ and $B(x,r) = E(x,r) \setminus S(x,r)$, both $B(x,r)$ and $E(x,r)$ are open and closed in X. $\square$

Corollary 5. Let (X,d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x,y) > 0$. Consider the open ball $B(x,r/2)$. By Proposition 4, $B(x,r/2)$ is both open and closed in X. Thus $B(x,r/2) \cap S$ is both open and closed in S, however, it is non-empty and not equal to S since it contains x but not y. This shows that S is disconnected. $\square$

Proposition 6. Let (X,d) be an ultra-metric space. A sequence $\{x_n\}$ in X is cauchy if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Proof. The necessity is true for all metric spaces. Suppose that $d(x_n, x) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \varepsilon$ for all $n \geq N$. For any $m, n \geq N$ with $m < n$, by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \dots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that $\{x_n\}$ is a cauchy sequence. $\square$