Semi-normed Rings and Modules

Definition 1. Let M be an abelian group. A *semi-norm* on M is a function $\|\cdot\| : M \to \mathbb{R}_+$ such that

- $\|0\| = 0$;
- $\forall x, y \in M, \|x + y\| \leq \|x\| + \|y\|$.

If we further have $\|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*. A *semi-normed abelian group* (resp. *normed abelian group*) is an abelian group equipped with a semi-norm (resp. norm).

Definition 2. A semi-norm (resp. norm) on an abelian group M induces a pseudo-metric (resp. metric) $d(x, y) = \|x - y\|$ on M. A semi-normed (resp. normed) abelian group M is called *complete* if it is complete as a pseudo-metric (resp. metric) space.

Definition 3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group M. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in M, \|x\|_1 \leq C\|x\|_2$.

Remark 4. If two semi-norms (resp. norms) on an abelian group M are bounded by each other, then they induce the same topology on M.

Yang: To be continued...

Appendix

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