

Valuation fields

DRAFT

No Cover Image

Use \coverimage{filename} to add an image

阿巴阿巴!

Contents

1 Valuation fields	1
1.1 Absolute values and completion	1
1.2 Non-archimedean fields	4
2 Ultra-metric spaces	4
3 Algebraic structures of non-archimedean fields	6
3.1 Recover non-archimedean complete fields algebraically	6
3.2 Hensel's Lemma	8
3.3 Newton polygons	9
4 Finite field extensions	9
4.1 Finite-dimensional vector space	9
4.2 Finite field extensions	11
5 Analytic functions	12
5.1 Tate algebras	13
5.2 Analytic functions on closed polydiscs	16
6 Example: p-adic fields	20
6.1 p-adic fields	20
6.2 Completion	21
6.3 Elementary functions	21
7 Appendix	22
References	22

The main references for this chapter are [Gou97; Rob00; 李文威 18].

1 Valuation fields

1.1 Absolute values and completion

Definition 1.1. Let \mathbf{k} be a field. An *absolute value* on \mathbf{k} is a function $\|\cdot\| : \mathbf{k} \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

- (a) $\|x\| = 0$ if and only if $x = 0$;
- (b) $\|xy\| = \|x\| \cdot \|y\|$;

$$(c) \|x + y\| \leq \|x\| + \|y\|.$$

A field \mathbf{k} equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 1.2. Let \mathbf{k} be a field. Recall that a *valuation* on \mathbf{k} is a function $v : \mathbf{k}^\times \rightarrow \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field \mathbf{k} by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \rightarrow \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Example 1.3. Let \mathbf{k} be a field. The *trivial absolute value* on \mathbf{k} is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 1.4. The *(multiplicative) valuation group* of a valuation field $(\mathbf{k}, \|\cdot\|)$ is defined as the subgroup of $\mathbb{R}_{>0}$ given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation $\sqrt{|\mathbf{k}^\times|}$ to denote the set $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$.

Definition 1.5. Let \mathbf{k} be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on \mathbf{k} are said to be *equivalent* if there exists a real number $c \in (0, 1)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field \mathbf{k} . Moreover, the following lemma shows that the converse is also true.

Lemma 1.6. Let \mathbf{k} be a field and $\|\cdot\|_1, \|\cdot\|_2$ be two absolute values on \mathbf{k} . Then the following statements are equivalent:

- (a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;
- (b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on \mathbf{k} ;
- (c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) \Rightarrow (b) is obvious. Now we prove (b) \Rightarrow (c). For any $x \in D_1$, we have $x^n \rightarrow 0$ as $n \rightarrow \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \rightarrow 0$ as $n \rightarrow \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) \Rightarrow (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. Suppose the

contrary that $\alpha \neq \beta$. Consider the domain $\Omega \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Ω is non-empty. Pick $(n, m) \in \Omega$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. \square

Definition 1.7. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that \mathbf{k} is *complete* if the metric $d(x, y) := \|x - y\|$ makes \mathbf{k} a complete metric space.

Lemma 1.8. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on \mathbf{k} can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing \mathbf{k} as a dense subfield.

Proof. Simple analysis. \square

Example 1.9. Let $|\cdot|_\infty$ be the usual absolute value on the field \mathbb{Q} of rational numbers. Then $(\mathbb{Q}, |\cdot|_\infty)$ is a valuation field. Its completion is the field \mathbb{R} of real numbers equipped with the usual absolute value.

Example 1.10. Let p be a prime number. For any non-zero rational number $x \in \mathbb{Q}$, we can write it as $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p . The p -adic absolute value on \mathbb{Q} is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then $(\mathbb{Q}, |\cdot|_p)$ is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of p -adic numbers equipped with the p -adic absolute value; see Yang: to be added..

Proposition 1.11. Let $(\mathbf{k}, \|\cdot\|)$ be a complete valuation field with non-trivial absolute value. Then \mathbf{k} is uncountable.

Proof. Since the absolute value $\|\cdot\|$ is non-trivial, we can construct a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathbf{k}$ inductively such that $\|x_n\| < \|x_{n-1}\|/2$ for any $n \geq 1$ and $\|x_0\| < 1$. Then there is an injective map from $\mathbb{N}^{\{0,1\}}$ to \mathbf{k} defined by

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n x_n, \quad a_n \in \{0, 1\}.$$

Since $\|x_n\| < 2^{-n}$, the series $\sum_{n=1}^\infty a_n x_n$ converges in \mathbf{k} . Note $\|x_n\| > \|\sum_{m \geq n} x_m\|$ for each n , we have that the map is injective. Thus \mathbf{k} is uncountable. \square

Unlike the real number field \mathbb{R} , even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 1.12. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in \mathbf{k} has a non-empty intersection.

Example 1.13. The field \mathbb{C}_p of p -adic complex numbers is not spherically complete, see Yang: to be added.

1.2 Non-archimedean fields

Definition 1.14. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that \mathbf{k} is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that \mathbf{k} is *archimedean*.

Let \mathbf{k} be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of \mathbf{k} . Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 1.15. Let \mathbf{k} be a non-archimedean field. The *ring of integers* of \mathbf{k} is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of \mathbf{k} is defined as

$$\mathcal{k}_{\mathbf{k}} := \tilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Definition 1.16. Let \mathbf{k} be a non-archimedean field. The *residue absolute value* on the residue field $\mathcal{k}_{\mathbf{k}}$ is defined as

$$|x| := \inf_{y \in \varphi^{-1}(x)} \|y\|, \quad \forall x \in \mathcal{k}_{\mathbf{k}},$$

where $\varphi : \mathbf{k}^\circ \rightarrow \mathcal{k}_{\mathbf{k}}$ is the canonical projection.

Proposition 1.17. Let \mathbf{k} be a non-archimedean field. Then the residue absolute value on the residue field $\mathcal{k}_{\mathbf{k}}$ is trivial.

Proof. For any $x \in \mathcal{k}_{\mathbf{k}}$, if $x = 0$, then by definition $|x| = 0$. If $x \neq 0$, then $\forall y \in \varphi^{-1}(x)$, we have $y \in \mathbf{k}^\circ \setminus \mathbf{k}^{\circ\circ}$, i.e., $\|y\| = 1$. Thus by definition $|x| = 1$. \square

2 Ultra-metric spaces

We will use $B(x, r)$ (resp. $E(x, r)$) to denote the open ball (resp. closed ball) with center x and radius r .

Definition 2.1. A metric space (X, d) is called an *ultra-metric space* if its metric d satisfies the

strong triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.$$

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x, y) := \|x - y\|$ on \mathbf{k} makes (\mathbf{k}, d) an ultra-metric space.

Proposition 2.2. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x, y) \geq d(y, z)$. By the strong triangle inequality, we have

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(x, y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} = \max\{d(x, z), d(y, z)\} \leq d(x, z).$$

This shows that $d(x, y) = \max\{d(x, z), d(y, z)\}$. Thus either $d(x, z) = d(x, y) \geq d(y, z)$ or $d(y, z) = d(x, y) \geq d(x, z)$. \square

Proposition 2.3. Let (X, d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1, 2$. If $D_1 \cap D_2 \neq \emptyset$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1, 2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1, x_2) \leq \max\{d(x_1, y), d(y, x_2)\}.$$

Without loss of generality, we may assume that $d(x_1, x_2) \leq d(x_1, y)$. It follows that $x_2 \in D_1$ since $d(x_1, y) < r_1$ (or $\leq r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1, z) > d(x_1, x_2)$. Then by [Proposition 2.2](#),

$$r_1 \leq d(x_1, z) = d(x_2, z) \leq r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2, w) \leq \max\{d(x_2, x_1), d(x_1, w)\} \leq r_1 < r_2.$$

Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. \square

Proposition 2.4. Let (X, d) be an ultra-metric space. Then both $B(x, r)$ and $E(x, r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x, r) := \{y \in X \mid d(x, y) = r\}$ is open in X . Note that if $y \in S(x, r)$, then for any $r' < r$, we have $B(y, r') \cap E(x, r) \neq \emptyset$ and $x \in E(x, r) \setminus B(y, r')$. Thus by [Proposition 2.3](#), we have $B(y, r') \subseteq E(x, r)$. If $B(y, r') \cap B(x, r) \neq \emptyset$, then by [Proposition 2.3](#) again, we have $B(y, r') \subseteq B(x, r)$. However, $y \in B(y, r') \setminus B(x, r)$, a contradiction. Thus $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$. It yields that $S(x, r) = \bigcup_{y \in S(x, r)} B(y, r/2)$ is open in X .

Since $E(x, r) = B(x, r) \cup S(x, r)$ and $B(x, r) = E(x, r) \setminus S(x, r)$, both $B(x, r)$ and $E(x, r)$ are open and closed in X . \square

Corollary 2.5. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x, y) > 0$. Consider the open ball $B(x, r/2)$. By Proposition 2.4, $B(x, r/2)$ is both open and closed in X . Thus $B(x, r/2) \cap S$ is both open and closed in S , however, it is non-empty and not equal to S since it contains x but not y . This shows that S is disconnected. \square

Proposition 2.6. Let (X, d) be an ultra-metric space. A sequence $\{x_n\}$ in X is cauchy if and only if $d(x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. The necessity is true for all metric spaces. Suppose that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \varepsilon$ for all $n \geq N$. For any $m, n \geq N$ with $m < n$, by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \dots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that $\{x_n\}$ is a cauchy sequence. \square

3 Algebraic structures of non-archimedean fields

3.1 Recover non-archimedean complete fields algebraically

In this subsection, let \mathbf{k} be a non-archimedean field. Set $I_{r,<} := B(0, r)$ and $I_{r,\leq} := E(0, r)$ for each $r \in [0, 1]$.

Proposition 3.1. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers \mathbf{k}° . Conversely, any ideal of \mathbf{k}° is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0, 1)$.

Proof. Let I be an ideal of \mathbf{k}° . Set $r = \sup\{|a| : a \in I\}$ (resp. $r = \max\{|a| : a \in I\}$ when the maximum exists). Then, by definition, we have $I \subset I_{r,<}$ (resp. $I \subset I_{r,\leq}$). For every $x \in \mathbf{k}^\circ$ with $|x| < r$ (resp. $|x| \leq r$), there exists $a \in I$ such that $|x| \leq |a|$. Thus, $|x/a| \leq 1$ and so $x/a \in \mathbf{k}^\circ$. Since I is an ideal, we have $x = (x/a)a \in I$. Therefore, $I_{r,<} \subset I$ (resp. $I_{r,\leq} \subset I$). \square

Proposition 3.2. Let I_r be either $I_{r,<}$ or $I_{r,\leq}$ for each $r \in (0, 1)$. Suppose $\{r_n \in (0, 1)\}_{n \in \mathbb{N}}$ is a decreasing sequence converging to 0. Then the completion $\widehat{\mathbf{k}}$ of \mathbf{k} is isomorphic to the projective limit

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}.$$

Proof. For every $x \in \widehat{\mathbf{k}}^\circ$, there exists a cauchy sequence $\{x_m\}_{m \in \mathbb{N}}$ in \mathbf{k}° converging to x . Since $\{r_n\}_{n \in \mathbb{N}}$ converges to 0, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that for all $m, m' \geq M_n$, we have

$|x_m - x_{m'}| < r_n$. Thus, the sequence $\{x_m + I_{r_n}\}_{m \in \mathbb{N}}$ is eventually constant in \mathbf{k}°/I_{r_n} . Define a map

$$\Phi : \widehat{\mathbf{k}}^\circ \rightarrow \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ/I_{r_n}, \quad x \mapsto \left(\lim_{m \rightarrow \infty} x_m + I_{r_n} \right)_{n \in \mathbb{N}}.$$

It is straightforward to verify that Φ is a well-defined ring homomorphism.

Conversely, for every $(a_n + I_{r_n})_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ/I_{r_n}$, we can choose a representative $a_n \in \mathbf{k}^\circ$ for each n . We claim that the sequence $\{a_n\}_{n \in \mathbb{N}}$ is a cauchy sequence in \mathbf{k}° . Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_N < \varepsilon$. For all $m, n \geq N$, since $a_n + I_{r_n}$ maps to $a_m + I_{r_m}$ under the natural projection, we have $|a_n - a_m| < r_N < \varepsilon$. Thus, $\{a_n\}_{n \in \mathbb{N}}$ converges to some $x \in \widehat{\mathbf{k}}^\circ$. Easily see that the limit x is independent of the choice of representatives $\{a_n\}_{n \in \mathbb{N}}$. This gives a map

$$\Psi : \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ/I_{r_n} \rightarrow \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow \infty} a_n.$$

Direct verification shows that $\Psi = \Phi^{-1}$. □

Corollary 3.3. Let \mathbf{k} be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the residue field $\kappa_{\widehat{\mathbf{k}}} \cong \kappa_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$.

Corollary 3.4. Let \mathbf{k} be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the valuation group $|\widehat{\mathbf{k}}^\times|$ of $\widehat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^\times|$ of \mathbf{k} .

Proof. Note that

$$\begin{aligned} r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\ &\iff \widehat{\mathbf{k}}^\circ/I_{r,<} \rightarrow \widehat{\mathbf{k}}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff \mathbf{k}^\circ/I_{r,<} \rightarrow \mathbf{k}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\ &\iff r \in |\mathbf{k}^\times|. \end{aligned}$$

□

Proposition 3.5. Let \mathbf{k} be a non-archimedean field with non-trivial valuation. Then \mathbf{k}° is totally bounded iff $\mathbf{k}^\circ/I_{r,<}$ and $\mathbf{k}^\circ/I_{r,\leq}$ are finite for each $r \in [0, 1]$. Moreover, if \mathbf{k} is complete, then it is locally compact iff \mathbf{k}°/I_r is finite for each $r \in (0, 1)$.

Slogan “Locally compact \iff pro-finite.”

Proof. We just prove the case for $I_r = I_{r,<}$. The case for $I_r = I_{r,\leq}$ is similar.

Suppose that \mathbf{k}°/I_r is finite for each $r \in [0, 1]$. Then for every $\varepsilon > 0$, there exists $r \in (0, 1)$ such that $r < \varepsilon$ and \mathbf{k}°/I_r is finite. Let $\{a_1 + I_r, \dots, a_n + I_r\}$ be the complete set of representatives of \mathbf{k}°/I_r . Then the balls $B(a_i, r)$ for $i = 1, \dots, n$ cover \mathbf{k}° .

Conversely, suppose that \mathbf{k}°/I_r is infinite for some $r \in [0, 1]$. Then there exists an infinite set $\{a_n\}$ with $|a_n| \in [r, 1]$ such that their images in \mathbf{k}°/I_r are distinct. In particular, for every $m \neq n$, we have $|a_n - a_m| \geq r$. Any subsequence of $\{a_n\}$ is not cauchy. Thus, \mathbf{k}° is not totally bounded. □

Proposition 3.6. The ring \mathbf{k}° is noetherian iff \mathbf{k} is a discrete valuation field.

Proof. Note that $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$ is a multiplicative subgroup. If \mathbf{k} is not a discrete valuation field, then $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. In particular, there exists a strictly ascending sequence $r_n \in |\mathbf{k}^\times| \cap (0, 1)$. Then the ideals $I_{r_n, \leq}$ form a strictly ascending chain of ideals in \mathbf{k}° .

The converse is standard since now \mathbf{k}° is a discrete valuation ring. \square

Proposition 3.7. Let \mathbf{k} be a complete non-archimedean field. Then \mathbf{k} is locally compact iff \mathbf{k} is a discrete valuation field and its residue field $\mathcal{k}_\mathbf{k}$ is finite.

Proof. The necessity follows from Proposition 3.5. For the sufficiency, suppose that \mathbf{k} is a discrete valuation field whose residue field $\mathcal{k}_\mathbf{k}$ is finite. Let $\pi \in \mathbf{k}^\circ$ be a uniformizer. We only need to show that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite for each $n \in \mathbb{N}$. Note that there is an isomorphism

$$\pi^{n-1}\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ \cong \mathcal{k}_\mathbf{k}, \quad x + \pi^n\mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on n , we conclude that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite. \square

3.2 Hensel's Lemma

Theorem 3.8 (Hensel's lemma). Let \mathbf{k} be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathcal{k}_\mathbf{k}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \mathcal{k}_\mathbf{k}[T]$ are monic polynomials that are coprime in $\mathcal{k}_\mathbf{k}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\mathcal{k}_\mathbf{k}[T]$ are $g(T), h(T)$ respectively.

Proof. Since $\gcd(g, h) = 1$ in $\mathcal{k}_\mathbf{k}[T]$, there exist polynomials $u(T), v(T) \in \mathcal{k}_\mathbf{k}[T]$ such that $ug + vh = 1$ and $\deg u < \deg h, \deg v < \deg g$. Choose lifts $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$ of $g(T), h(T), u(T), v(T)$ respectively preserving their degrees such that G_0 and H_0 are monic. Then there exist $r < 1$ such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \pmod{I_r}, \quad F(T) - G_0(T)H_0(T) \equiv 0 \pmod{I_r},$$

where $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$.

We will construct a sequence of monic polynomials $\{G_n(T)\}_{n \in \mathbb{N}}$ and $\{H_n(T)\}_{n \in \mathbb{N}}$ in $\mathbf{k}^\circ[T]$ such that for each $n \in \mathbb{N}$,

$$G_n(T) \equiv G_{n-1}(T) \pmod{I_{rn}}, \quad H_n(T) \equiv H_{n-1}(T) \pmod{I_{rn}},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \pmod{I_{rn+1}}.$$

If we have such sequences, then their coefficients converge in the complete ring \mathbf{k}° . Let $G(T)$ and $H(T)$ be the limits of $\{G_n(T)\}$ and $\{H_n(T)\}$ respectively. Then we have $F(T) = G(T)H(T)$ and the reductions of $G(T), H(T)$ in $\mathcal{K}_\mathbf{k}[T]$ are $g(T), h(T)$ respectively.

The case $n = 0$ is done by the above construction. Now suppose that we have constructed $G_n(T)$ and $H_n(T)$ for some $n \geq 0$. Since $G_n - G_0 \equiv 0 \pmod{I_r}$ and $H_n - H_0 \equiv 0 \pmod{I_r}$, we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \pmod{I_r}.$$

Set $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$ and $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$. Then we have

$$\begin{aligned} (G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T]. \end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

This finishes the induction. \square

Corollary 3.9. Let \mathbf{k} be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathcal{K}_\mathbf{k}[T]$ of $F(T)$ has a simple root $a \in \mathcal{K}_\mathbf{k}$. Then there exists a root $\alpha \in \mathbf{k}^\circ$ of $F(T)$ whose reduction is a .

Proof. Since a is a simple root of $f(T)$, we have the factorization $f(T) = (T - a)h(T)$ for some monic polynomial $h(T) \in \mathcal{K}_\mathbf{k}[T]$ with $h(a) \neq 0$. Then the result follows from [Theorem 3.8](#). \square

3.3 Newton polygons

Yang: To be filled.

4 Finite field extensions

4.1 Finite-dimensional vector space

Definition 4.1. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . A *norm* on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

- (a) $\|x\| = 0$ if and only if $x = 0$;
- (b) $\|ax\| = |a| \cdot \|x\|$;
- (c) $\|x + y\| \leq \|x\| + \|y\|$.

Example 4.2. Let \mathbf{k} be a valuation field and V a finite-dimensional vector space over \mathbf{k} with basis $\{e_1, e_2, \dots, e_n\}$. For any $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$, define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then $\|\cdot\|_{\max}$ is a norm on V , called the *maximal norm* with respect to the basis $\{e_1, e_2, \dots, e_n\}$.

Example 4.3. Setting as in [Example 4.2](#), for any $x = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$, define

$$\|x\|_1 := |a_1| + |a_2| + \dots + |a_n|.$$

Then $\|\cdot\|_1$ is also a norm on V .

Definition 4.4. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Lemma 4.5. Let \mathbf{k} be a valuation field and V a vector space over \mathbf{k} . Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V .

Proof. The sufficiency is clear. Now suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on V . Hence the unit open ball with respect to $\|\cdot\|_1$ contains a unit open ball with respect to $\|\cdot\|_2$. That is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every $x \in V$ with $\|x\|_1 = 1$, we have $\|x\|_2 \geq C = C\|x\|_1$. By scaling, we get that for every $x \in V$,

$$\|x\|_2 \geq C\|x\|_1.$$

Similar for the other direction, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. \square

Proposition 4.6. Let V be a normed finite-dimensional vector space over a complete valuation field \mathbf{k} . Then V is complete.

Proof. Yang: To be added. \square

Theorem 4.7. Let V be a finite-dimensional vector space over a complete field \mathbf{k} . Then all norms on V are equivalent.

Proof. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis as in [Example 4.2](#). Let $\|\cdot\|$ be any norm on V . It suffices to show that $\|\cdot\|$ and $\|\cdot\|_{\max}$ are equivalent. First we have

$$\|y\| \leq \sum_{i=1}^n |a_i| \|e_i\| \leq \left(\sum_{i=1}^n \|e_i\| \right) \|y\|_{\max}$$

for any $y = a_1e_1 + a_2e_2 + \dots + a_ne_n \in V$. It remains to show that there exists a constant $C > 0$ such that for any $y \in V$,

$$\|y\|_{\max} \leq C\|y\|.$$

Yang: To be added. \square

Remark 4.8. If the base field \mathbf{k} is not complete, then [Theorem 4.7](#) may fail. For example, let $\mathbf{k} = \mathbb{Q}$ with the usual absolute value, and let $V = \mathbb{Q}[\alpha]$ with $\alpha^2 - \alpha - 1 = 0$. There are two embeddings of V into \mathbb{R} :

$$\iota_1 : a + b\alpha \mapsto a + b\frac{1 + \sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b\frac{1 - \sqrt{5}}{2}.$$

Define two norms on V by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where $|\cdot|$ is the usual absolute value on \mathbb{R} . Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent since $\iota_2(\alpha^n) \rightarrow 0$ as $n \rightarrow \infty$ while $\iota_1(\alpha^n) \rightarrow \infty$.

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

Lemma 4.9. Let \mathbf{k} be a complete field and V a normed finite-dimensional vector space over \mathbf{k} . Then

$$\|\cdot\| : \text{End}_{\mathbf{k}}(V) \rightarrow \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the \mathbf{k} -vector space $\text{End}_{\mathbf{k}}(V)$ satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \text{End}_{\mathbf{k}}(V).$$

Proof. First we show the existence of the supremum, i.e., there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\| \leq C\|x\|$. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis. Since all norms on V are bounded by each other by [Theorem 4.7](#), we only need to show that there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\|_1 \leq C\|x\|_{\max}$. Write $T(e_i) = \sum_{j=1}^n a_{ij} e_j$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^n x_i e_i \in V$, we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \left(\sum_{1 \leq i, j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of $\|\cdot\|$ are clear. It remains to show the triangle inequality and sub-multiplicativity. For any $A, B \in \text{End}_{\mathbf{k}}(V)$, we have

$$\frac{\|(A + B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all $x \in V \setminus \{0\}$ gives $\|A + B\| \leq \|A\| + \|B\|$. We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$. Taking supremum we get $\|AB\| \leq \|A\| \cdot \|B\|$. \square

4.2 Finite field extensions

Lemma 4.10. Let \mathbf{k} be a complete field and \mathbf{l} a finite extension of \mathbf{k} . Then there exists an absolute value on \mathbf{l} extending the absolute value on \mathbf{k} .

Proof. Fix a norm $\|\cdot\|_V$ on the \mathbf{k} -vector space $V = \mathbf{l}$. The norm $\|\cdot\|_V$ induces an operator norm $\|\cdot\|_{\text{op}}$ on the \mathbf{k} -vector space $\text{End}_{\mathbf{k}}(V)$ as in [Lemma 4.9](#). For any $a \in \mathbf{l}$, let $\mu_a \in \text{End}_{\mathbf{k}}(V)$ be the \mathbf{k} -linear map defined by multiplication by a . Note that $a \mapsto \mu_a$ gives a embedding of \mathbf{k} -algebras and if $a \in \mathbf{k}$, $\|\mu_a\|_{\text{op}} = \|a\|_{\mathbf{k}}$. Thus the restriction of $\|\cdot\|_{\text{op}}$ to \mathbf{l} gives an norm on \mathbf{l} extending that

on \mathbf{k} . The normed ring $(\mathbf{l}, \|\cdot\|_{\text{op}})$ is a Banach ring since it is a finite-dimensional vector space over the complete field \mathbf{k} . By [Theorem 7.1](#), there exists a multiplicative seminorm $\|\cdot\|_{\mathbf{l}}$ on \mathbf{l} bounded by $\|\cdot\|_{\text{op}}$. In particular, $\|\cdot\|_{\mathbf{l}}$ is bounded by $\|\cdot\|_{\mathbf{k}}$ on \mathbf{k} . On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus $\|\cdot\|_{\mathbf{l}}$ extends the absolute value on \mathbf{k} . \square

Theorem 4.11. Let \mathbf{k} be a complete field and \mathbf{l} a finite extension of \mathbf{k} . Then the absolute value on \mathbf{l} which extends the absolute value on \mathbf{k} is uniquely determined by the absolute value on \mathbf{k} . Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where $n = [\mathbf{l} : \mathbf{k}]$ and $N_{\mathbf{l}/\mathbf{k}}$ is the norm map from \mathbf{l} to \mathbf{k} .

Proof. Let $\|\cdot\|_{\mathbf{l}}$ be arbitrary absolute value on \mathbf{l} extending that on \mathbf{k} . We will show that $\|\cdot\|_{\mathbf{l}}$ must be equal to $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$. For any $a \in \mathbf{l}$, set $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$. Then $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that $\|b\|_{\mathbf{l}} = 1$ whenever $N_{\mathbf{l}/\mathbf{k}}(b) = 1$.

Note that the norm map $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \rightarrow \mathbf{k}$ is the determinant of the \mathbf{k} -linear map $\mu_b \in \text{End}_{\mathbf{k}}(V)$ defined by multiplication by b . Hence it is continuous on \mathbf{l} (since it is a polynomial in the entries of the matrix representation). If $\|b\|_{\mathbf{l}} < 1$, then $\|b^m\|_{\mathbf{l}} \rightarrow 0$ as $m \rightarrow \infty$. Thus $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \rightarrow 0$ as $m \rightarrow \infty$, contradicting the fact that $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$ for all m . Similarly, if $\|b\|_{\mathbf{l}} > 1$, then just consider b^{-1} . \square

Proposition 4.12. Let \mathbf{k} be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

Proof. Let $f \in \widehat{\mathbf{k}}[X]$ be a non-constant polynomial. We will show that f has a root in $\widehat{\mathbf{k}}$. Take a sequence of polynomials $\{f_n\}_{n \in \mathbb{N}}$ in $\mathbf{k}[X]$ converging to f coefficient-wisely. Since \mathbf{k} is algebraically closed, each f_n splits completely in \mathbf{k} and hence in $\widehat{\mathbf{k}}$. Write $f_n(X) = \prod_{i=1}^d (X - \alpha_{n,i})$ with $\alpha_{n,i} \in \widehat{\mathbf{k}}$.

Let \mathbf{l} be a finite extension of $\widehat{\mathbf{k}}$ such that f has a root α in \mathbf{l} . For every $\varepsilon > 0$, if there are infinitely many n such that $\alpha_{n,i} \notin B(\alpha, \varepsilon)$ for all $1 \leq i \leq d$, then we have $|f_n(\alpha)| \geq \varepsilon^d$ for infinitely many n , contradicting the fact that $f_n(\alpha) \rightarrow f(\alpha) = 0$. Thus for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$, there exists $1 \leq i \leq d$ with $\alpha_{n,i} \in B(\alpha, \varepsilon)$. That is, we can find a sequence $\alpha_{n,i_n} \in \mathbf{k}$ converging to α . Since $\widehat{\mathbf{k}}$ is complete, we have $\alpha \in \widehat{\mathbf{k}}$. \square

5 Analytic functions

Let \mathbf{k} be a complete non-archimedean field with non-trivial valuation. The following example shows that continuous or differentiable functions over \mathbf{k} may behave very worse than over archimedean fields. As a substitute, we will focus on convergent power series on a closed polydisc over \mathbf{k} .

Example 5.1. Let \mathbf{k} be a non-archimedean field with non-trivial valuation. Then there exists a function $f : \mathbf{k} \rightarrow \mathbf{k}$ that is differentiable everywhere with $f'(x) = 0$ for all $x \in \mathbf{k}$, but f is not

locally constant.

Fix $r \in (0, 1)$. Consider a descending sequence of open ball $\{B(0, r^n)\}$ and $a_n \in \mathbf{k}$ with $\|a_n\| = r^{2n}$. Define

$$f : \mathbf{k} \rightarrow \mathbf{k}, \quad x \mapsto \begin{cases} a_n, & x \in B(0, r^n) \setminus B(0, r^{n+1}) \\ 0, & x = 0 \end{cases}$$

Then

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{n \rightarrow \infty} \frac{a_n - 0}{x_n - 0}$$

for any sequence $x_n \rightarrow 0$ with $x_n \in B(0, r^n) \setminus B(0, r^{n+1})$. Since $\|x_n\| \geq r^{n+1}$, we have

$$\left\| \frac{a_n}{x_n} \right\| \leq \frac{r^{2n}}{r^{n+1}} = r^{n-1} \rightarrow 0$$

as $n \rightarrow \infty$. Thus $f'(0) = 0$ and then $f'(x) = 0$ for all $x \in \mathbf{k}$. However, f is not locally constant near 0.

5.1 Tate algebras

Notation 5.2. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T}/\underline{r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \dots, n$, we have $\alpha_i \leq \beta_i$;
- $E(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \dots, n\}$ and $B(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \dots, n\}$ for $x = (x_1, \dots, x_n) \in \mathbf{k}^n$;
- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \rightarrow +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

Definition 5.3. Let \mathbf{k} be a complete non-archimedean field. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$\mathbf{k}\langle \underline{T} \rangle := \mathbf{k}\{\underline{T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{|\alpha| \rightarrow +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 5.4. Let \mathbf{k} be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T}\}$ is a non-archimedean multiplicative banach \mathbf{k} -algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Yang: For the definition of banach ring, see

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $\mathbf{k}\{\underline{T}/r\}$ is a \mathbf{k} -algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ are two elements in $\mathbf{k}\{\underline{T}/r\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\|r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\|r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\|r^\alpha \cdot \|b_\beta\|r^\beta < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\|r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\|r^\alpha \right\} \leq \varepsilon.$$

Hence $f \cdot g \in \mathbf{k}\{\underline{T}/r\}$ and it shows that $\mathbf{k}\{\underline{T}/r\}$ is a \mathbf{k} -algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $\mathbf{k}\{\underline{T}/r\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f + g\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \leq \sup_{\alpha \in \mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \leq \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned} \|f \cdot g\| &= \left\| \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma \right\| = \sup_{\gamma \in \mathbb{N}^n} \left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma \\ &\leq \sup_{\gamma \in \mathbb{N}^n} \max_{\alpha+\beta=\gamma} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \leq \|f\| \cdot \|g\|. \end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed \mathbf{k} -algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f \cdot g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\| \|b_\beta\| r^\alpha r^\beta = \left\| \sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta \right\| r^{\alpha_1+\beta_1} \leq \|f \cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by [Proposition 2.2](#). Thus the gauss norm is multiplicative.

Step 4. Finally show that $\mathbf{k}\{\underline{T}/r\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $\mathbf{k}\{\underline{T}/r\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in \mathbf{k} . Since \mathbf{k} is complete, set $a_\alpha := \lim_{m \rightarrow +\infty} a_{\alpha,m}$ and $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \rightarrow +\infty$, we have $\|a_\alpha\| r^\alpha \leq 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T}/r\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\|r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \rightarrow +\infty$, we have $\|a_{\alpha,m} - a_\alpha\|r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\|r^\alpha \leq \varepsilon$$

for all $m > N$. \square

Proposition 5.5. Let \mathbf{k} be a complete non-archimedean field. An element $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{T/r\}$ is invertible if and only if $\|a_0\| > \|a_\alpha\|r^\alpha$ for all $\alpha \neq 0$.

Proof. Multiplying by a_0^{-1} , we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $\mathbf{k}[[T]]$. Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = - \sum_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\|r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order \leq_{total} , we have

$$\|b_\gamma\|r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\|r^\alpha \cdot \|b_\beta\|r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\|r^\beta \leq 1.$$

Suppose that the claim holds for m . There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\|r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\|r^\gamma \leq \max_{\substack{\alpha+\beta=\gamma \\ \alpha \neq 0}} \|a_\alpha\|r^\alpha \cdot \|b_\beta\|r^\beta \leq \max\{A^{m+1}, A \cdot A^m\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\|r^\alpha \rightarrow 0$ as $|\alpha| \rightarrow +\infty$. It follows that $g \in \mathbf{k}\{T/r\}$. \square

Let \mathbf{k} be a complete non-archimedean field. Recall that a derivative operator $\partial : \mathbf{k}\{T/r\} \rightarrow \mathbf{k}\{T/r\}$ is defined as the \mathbf{k} -linear map such that for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we have **Yang: To be revised.**

Proposition 5.6. Let \mathbf{k} be a complete non-archimedean field, and ∂ be a derivative operator on $\mathbf{k}\{T/r\}$. Then for every $f \in \mathbf{k}\{T/r\}$, we have $\partial(f) \in \mathbf{k}\{T/r\}$.

Proof. **Yang:** We only need to check the case $\partial = \partial/\partial T_1$. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{T/r\}$.

We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha \in \mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1-1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}.$$

Noting that \mathbf{k} is non-archimedean, we have $\|\alpha_1 a_\alpha\| \leq \|a_\alpha\|$. Then

$$\lim_{|\alpha| \rightarrow +\infty} \|\alpha_1 a_\alpha\| r_1^{\alpha_1-1} r_2^{\alpha_2} \cdots r_n^{\alpha_n} \leq \frac{1}{r_1} \lim_{|\alpha| \rightarrow +\infty} \|a_\alpha\| r^\alpha = 0.$$

The conclusion follows. \square

5.2 Analytic functions on closed polydiscs

Proposition 5.7. Let \mathbf{k} be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{T/r\}$, we can associate a function $F_f : E(0, \underline{r}) \rightarrow \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \quad \text{for } x = (x_1, \dots, x_n) \in E(0, \underline{r}).$$

This defines a homomorphism of \mathbf{k} -algebras from $\mathbf{k}\{T/r\}$ to the ring of all functions from $E(0, \underline{r})$ to \mathbf{k} .

Proof. Given $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{T/r\}$ and $x = (x_1, \dots, x_n) \in E(0, \underline{r})$, we have

$$\left\| \sum_{|\alpha|=n} a_\alpha x^\alpha \right\| \leq \max_{|\alpha|=n} \|a_\alpha\| r^\alpha \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Hence by Proposition 2.6, the series $F_f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$ converges in \mathbf{k} . This defines a function $F_f : E(0, \underline{r}) \rightarrow \mathbf{k}$.

Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{T/r\}$. Set

$$A_n = \sum_{|\alpha| < n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta| < n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma| < n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) x^\gamma.$$

We need to show that $F_f(x)F_g(x) = \lim A_n B_n = \lim C_n = F_{fg}(x)$. Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha| < n, |\beta| < n \\ |\alpha+\beta| \geq n}} a_\alpha b_\beta x^{\alpha+\beta}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $n > 2N$, we have

$$\|A_n B_n - C_n\| \leq \max_{\substack{|\alpha| < n, |\beta| < n \\ |\alpha+\beta| \geq n}} \|a_\alpha\| \|b_\beta\| \|x^{\alpha+\beta}\| < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Thus $F_f(x)F_g(x) = (F_{fg})(x)$. The addition and scalar multiplication can be verified directly. We thus finish the proof. \square

Proposition 5.8. Let \mathbf{k} be a complete non-archimedean field with non-trivial valuation. Then for every $f \in \mathbf{k}\{T/r\}$ and $x, y \in E(0, r)$, we have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq L \cdot \|y - x\|_{\infty},$$

where $L = \max_{1 \leq i \leq n} \|f\|_g / r_i$.

Proof. Set $y - x = (h_1, \dots, h_n)$ and $x^{(0)} = x$, $x^{(i)} = (x_1 + h_1, \dots, x_i + h_i, x_{i+1}, \dots, x_n)$ for $i = 1, \dots, n$. We have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{1 \leq i \leq n} \|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}}.$$

We only need to show that for every $i = 1, \dots, n$, we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}} \leq \frac{\|f\|_g}{r_i} \|h_i\|.$$

Without loss of generality and for simplicity, we assume that $y = (x_1 + h, x_2, \dots, x_n)$ and $x = (x_1, x_2, \dots, x_n)$. Note that by the strong triangle inequality, we have $\|h\| \leq r_1$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} T^{\alpha} \in \mathbf{k}\{T/r\}$. We have

$$\begin{aligned} f(y) - f(x) &= \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} ((x_1 + h)^{\alpha_1} - x_1^{\alpha_1}) x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ &= \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_{\alpha} x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k. \end{aligned}$$

Note that

$$\left\| \binom{\alpha_1}{k} a_{\alpha} x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| r_1^k \leq \|a_{\alpha}\| r^{\alpha} \leq \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{\alpha \in \mathbb{N}^n} \max_{1 \leq k \leq \alpha_1} \left\{ \left\| \binom{\alpha_1}{k} a_{\alpha} x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| \|h\|^k \right\} \leq \max_k \left\{ \|f\|_g \left(\frac{\|h\|}{r_1} \right)^k \right\} \leq \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. \square

Lemma 5.9. Let \mathbf{k} be a complete non-archimedean field. Then we have $\|f(x)\| \leq \|f\|$ for every $f \in \mathbf{k}\{T/r\}$ and $x \in E(0, r)$. In particular, if $f_n \rightarrow f$ as $n \rightarrow +\infty$ in $\mathbf{k}\{T/r\}$, then we have $\|f_n(x) - f(x)\| \rightarrow 0$ for every $x \in E(0, r)$.

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} T^{\alpha} \in \mathbf{k}\{T/r\}$ and $x = (x_1, \dots, x_n) \in E(0, r)$. We have

$$\left\| \sum_{|\alpha| < N} a_{\alpha} x^{\alpha} \right\| \leq \max_{|\alpha| < N} \|a_{\alpha}\| r^{\alpha} \leq \|f\|$$

for every $N \in \mathbb{N}$. Taking $N \rightarrow +\infty$, we have $\|f(x)\| \leq \|f\|$. \square

Proposition 5.10. Let \mathbf{k} be a complete non-archimedean field with non-trivial valuation, and $\partial_i = \partial/\partial T_i$ be the derivative operator on $\mathbf{k}\{T/r\}$ with respect to the indeterminate T_i for $i = 1, \dots, n$.

Then for every $f \in \mathbf{k}\{T/r\}$ and $x \in E(0, r)$, we have

$$F_{\partial_i(f)}(x) = \lim_{h \rightarrow 0} \frac{F_f(x_1, \dots, x_i + h, \dots, x_n) - F_f(x)}{h}.$$

Proof. Without loss of generality, we can assume that $i = 1$. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{T/r\}$ and $f_n = \sum_{|\alpha| < n} a_\alpha T^\alpha$ for $n \in \mathbb{N}$. Set $x_h = (x_1 + h, x_2, \dots, x_n)$ and $L_f(h) = (F_f(x_h) - F_f(x))/h$ for $h \in \mathbf{k}^\times$. Note that for fixed h , we have $\lim_{n \rightarrow \infty} L_{f_n}(h) = L_f(h)$.

We compute $L_{f_n}(h) - F_{\partial f_n}(x)$ explicitly:

$$\begin{aligned} L_{f_n}(h) - F_{\partial f_n}(x) &= \frac{1}{h} \left(\sum_{|\alpha| < n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} h^k x_2^{\alpha_2} \cdots x_n^{\alpha_n} - \sum_{|\alpha| < n} \alpha_1 a_\alpha x_1^{\alpha_1-1} h x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right) \\ &= \sum_{|\alpha| < n} \sum_{k=2}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^{k-1}. \end{aligned}$$

Note that

$$M = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\| r_1^{k-1} \leq \|f\|/r_1 < +\infty.$$

Hence

$$\|L_{f_n}(h) - F_{\partial f_n}(x)\| \leq \max_{2 \leq k \leq n} \left\{ M \frac{\|h\|^{k-1}}{r_1^{k-1}} \right\} \leq M \frac{\|h\|}{r_1}$$

for $h \in \mathbf{k}^\times$ with $\|h\| < r_1$. Taking $n \rightarrow +\infty$, we have

$$\|L_f(h) - F_{\partial f}(x)\| \leq M \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. \square

Corollary 5.11. Let \mathbf{k} be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment $f \mapsto F_f$ in [Proposition 5.7](#) is injective.

Proof. Note that if $F_f = 0$, then for every $i = 1, \dots, n$, we have $F_{\partial_i(f)} = 0$ by [Proposition 5.10](#). By taking repeated derivatives, we have $F_{\partial^\alpha f} = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Note that $F_{\partial^\alpha f}(0) = \alpha! a_\alpha$. It follows that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n$ and thus $f = 0$. \square

Remark 5.12. [Corollary 5.11](#) holds for non-archimedean fields of positive characteristic as well. The proof uses [Theorem 5.16](#) and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element $f \in \mathbf{k}\{T/r\}$ with the associated function $F_f : E(0, r) \rightarrow \mathbf{k}$ as in [Proposition 5.7](#).

Proposition 5.13. Let \mathbf{k} be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra $\mathbf{k}\{T/r\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in E(0, r)} \|f(x)\|_{\mathbf{k}}.$$

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/r\}$. We write $f = g+h$ with $g = \sum_{\alpha \in S} a_\alpha T^\alpha$ and $h = \sum_{\alpha \notin S} a_\alpha T^\alpha$, where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that S is a non-empty finite set and $\|h\| < \|f\|$. By Lemma 5.9, we have $\|h(x)\| < \|f\|$ for every $x \in E(0, \underline{r})$. It suffices to show that $\|g\|_{\sup} = \|g\|$.

Since \mathbf{k} is algebraically closed, $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. For every pair $\alpha, \beta \in S$ with $\alpha \neq \beta$, the set $\{t \in \mathbb{R}_{>0}^n : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$ is a proper closed subset of $\mathbb{R}_{>0}^n$. Thus we can find $t_m \in |\mathbf{k}^\times|^n$ such that $t_m < r$, $t_m \rightarrow r$ as $m \rightarrow +\infty$ and for every $\alpha, \beta \in S$ with $\alpha \neq \beta$, we have $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$ for all m . For each m , we can find $x_m \in E(0, \underline{r})$ such that $\|x_m^\alpha\| = t_m^\alpha$ for every $\alpha \in S$ since $t_m \in |\mathbf{k}^\times|^n$. It follows that

$$\|g(x_m)\| = \max_{\alpha \in S} \|a_\alpha\| \|x_m^\alpha\| = \max_{\alpha \in S} \|a_\alpha\| t_m^\alpha \rightarrow \|g\| \quad \text{as } m \rightarrow +\infty.$$

Thus $\|g\|_{\sup} = \|g\|$. □

Remark 5.14. If \mathbf{k} is not algebraically closed, the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T}/r\}$ may not coincide with the supremum norm. For example, consider the Tate algebra $\mathbb{Q}_p\{T\}$. The element $f = T^p - T$ has gauss norm $\|f\| = 1$. However, for every $x \in E(0, 1) = \mathbb{Z}_p$, we have $f(x) = x^p - x \equiv 0 \pmod{p}$. Thus $\|f(x)\|_p \leq 1/p$ and $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$.

Remark 5.15. Recall the Weierstrass-Stone theorem in classical analysis which states that the closure of the polynomial ring $\mathbb{C}[T_1, \dots, T_n]$ with respect to the supremum norm on a closed polydisc $E \subset \mathbb{C}^n$ is the ring of all complex-valued continuous functions on E .

In the context of non-archimedean analysis, Proposition 5.13 can be viewed as an analogue of this theorem. It states that the closure of the polynomial ring $\mathbf{k}[T_1, \dots, T_n]$ with respect to the supremum norm on a closed polydisc $E(0, \underline{r}) \subset \mathbf{k}^n$ is the Tate algebra $\mathbf{k}\{\underline{T}/r\}$.

From this perspective, the Tate algebra can be viewed as the “correct” analogue of the ring of continuous functions on a closed polydisc in non-archimedean analysis.

Theorem 5.16 (Strassman). Let \mathbf{k} be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{\underline{T}/r\}$ be an analytic function. Suppose that $\|a_N\| r^N > \|a_n\| r^n$ for all $n > N$. Then f has at most N zeros in the closed ball $E(0, \underline{r})$.

Proof. We induct on N . The case $N = 0$ is direct from Proposition 5.5. Suppose that the conclusion holds for $N - 1$. Let x be a zero of f in $E(0, \underline{r})$. Set

$$g(T) = \frac{f(T) - f(x)}{T - x} = \sum_{k=0}^{+\infty} \left(\sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n \geq k+1} \|a_n\| r^{n-1} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

It follows that $g(T) \in \mathbf{k}\{\underline{T}/r\}$.

For every $n > N$, we have

$$\|a_N\| > \|a_n\|r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\| \sum_{n=N}^{N+m} a_n x^{n-N} \right\| = \|a_N\|$$

for every $m \in \mathbb{N}$ by [Proposition 2.2](#). Take $m \rightarrow +\infty$, we have $\|b_{N-1}\| = \|a_N\|$. For every $k > N - 1$, we have

$$\|b_k\|r^k = \max_{n \geq k+1} \|a_n\|r^{n-1} \leq \max_{n > N} \|a_n\|r^{n-1} < \|a_N\|r^{N-1} = \|b_{N-1}\|r^{N-1}.$$

By the induction hypothesis, g has at most $N - 1$ zeros in $E(0, r)$. It follows that f has at most N zeros in $E(0, r)$ since $f(T) = (T - x) \cdot g(T)$. \square

6 Example: p -adic fields

6.1 p -adic fields

Construction 6.1. Let K be a number field and \mathfrak{p} be a prime ideal of the ring of integers \mathcal{O}_K of K . Considering the localization $(\mathcal{O}_K)_{\mathfrak{p}}$ of \mathcal{O}_K at \mathfrak{p} , which is a discrete valuation ring, denote by $v_{\mathfrak{p}} : K^\times \rightarrow \mathbb{Z}$ the corresponding discrete valuation. The p -adic absolute value on K associated to \mathfrak{p} is defined as

$$|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of \mathfrak{p} .

The completion of K with respect to the p -adic absolute value $|\cdot|_{\mathfrak{p}}$ is denoted by $K_{\mathfrak{p}}$, called the \mathfrak{p} -adic field.

We just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p in the following.

Example 6.2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p . The p -adic absolute value on \mathbb{Q} is defined as

$$|r|_p := p^{-n}.$$

The p -adic field \mathbb{Q}_p can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \mid n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p -adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on \mathbb{Q}_p are defined similarly as those on decimal expansions.

Unlike the field of real numbers \mathbb{R} , the p -adic field \mathbb{Q}_p has many finite extensions.

Proposition 6.3. There are infinitely many irreducible polynomials over the p -adic field \mathbb{Q}_p .

Proof. Since there are infinitely many irreducible monic polynomials over the finite field \mathbb{F}_p , consider any lift of such an irreducible monic polynomial to a monic polynomial with coefficients in \mathbb{Z}_p . If the lift is not irreducible over \mathbb{Q}_p , then the factorization of the lift gives a nontrivial factorization of its reduction modulo p since the factors can be chosen to be monic and have coefficients in \mathbb{Z}_p , which contradicts the irreducibility of the original polynomial over \mathbb{F}_p . Thus, the lift is irreducible over \mathbb{Q}_p .

On the other hand, note that $|\mathbb{Q}_p^\times|_p = p^\mathbb{Z}$. It follows that $f(T) = T^n - p$ is irreducible over \mathbb{Q}_p for every integer $n \geq 1$. Otherwise, suppose we have a monic factorization $f(T) = g(T)h(T)$ with $g(T), h(T) \in \mathbb{Z}_p[T]$ and $\deg g, \deg h < n$. Then by considering the reduction modulo p , we have $g(0), h(0) \equiv 0 \pmod{p}$. It follows that $|f(0)|_p = |g(0)h(0)|_p \leq p^{-2}$, which contradicts $|f(0)|_p = |p|_p = p^{-1}$. \square

6.2 Completion

Proposition 6.4. The algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p is not complete with respect to the extension of the p -adic absolute value $|\cdot|_p$.

Proof. Yang: To be completed. \square

Construction 6.5. Let p be a prime number. The *field \mathbb{C}_p of p -adic complex numbers* is defined as the completion of the algebraic closure of \mathbb{Q}_p with respect to the unique extension of the p -adic absolute value $|\cdot|_p$ on \mathbb{Q}_p .

The field \mathbb{C}_p is algebraically closed and complete with respect to $|\cdot|_p$ by Proposition 4.12. By Corollaries 3.3 and 3.4, we have

$$|\mathbb{C}_p^\times|_p = |\overline{\mathbb{Q}_p}^\times|_p = p^\mathbb{Q}, \quad k_{\mathbb{C}_p} \cong k_{\overline{\mathbb{Q}_p}} \cong \overline{\mathbb{F}_p}.$$

Proposition 6.6. The field \mathbb{C}_p of p -adic complex numbers is not spherically complete.

Proof. Yang: To be completed. \square

Construction 6.7. Let p be a prime number. Yang: We construct the *spherically complete p -adic field Ω_p* . Yang: To be completed.

6.3 Elementary functions

Yang: Exponential, logarithmic, and the interpolation functions.

Fix a prime number p in the following and consider $\mathbf{k} = \mathbb{Q}_p, \mathbb{C}_p$, or Ω_p . Let $r_p := p^{-1/(p-1)}$.

Construction 6.8. The *exponential function* $\exp : \mathbf{k} \rightarrow \mathbf{k}$ is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The radius of convergence of $\exp(x)$ is $+\infty$ if $p = 2$ and $p^{-1/(p-1)}$ if $p > 2$.

The *logarithmic function* $\log : 1 + \mathbf{k}^\circ \rightarrow \mathbf{k}$ is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

The radius of convergence of $\log(1 + x)$ is 1.

Moreover, for every x in the domain of convergence of \exp and every y in the domain of convergence of \log , we have

$$\log(\exp(x)) = x, \quad \exp(\log(y)) = y.$$

Yang: To be checked.

7 Appendix

Theorem 7.1. Let R be a Banach ring. The spectrum $\mathcal{M}(R)$ is nonempty.

References

- [Gou97] Fernando Q. Gouvêa. *p-adic Numbers: An Introduction*. 2nd ed. Universitext. Berlin, Heidelberg: Springer, 1997. ISBN: 978-3-642-59058-0. DOI: [10.1007/978-3-642-59058-0](https://doi.org/10.1007/978-3-642-59058-0). URL: <https://link.springer.com/book/10.1007/978-3-642-59058-0> (cit. on p. 1).
- [Rob00] Alain M Robert. *A course in p-adic analysis*. Vol. 198. Graduate Texts in Mathematics. Springer Science & Business Media, 2000 (cit. on p. 1).
- [李文威 18] 李文威. 代数学方法 (第一卷) 基础架构. 现代数学基础. 版面字数: 612 千字, 全书页数: 448 页. 北京: 高等教育出版社, 2018. ISBN: 978-7-04-050725-6 (cit. on p. 1).