Commutative branch rings

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1 Semi-normed Rings and Modules

1.1 Semi-normed algebraic structures

Definition 1.1. Let M be an abelian group. A *semi-norm* on M is a function $\|\cdot\| : M \to \mathbb{R}_+$ such that

- $\|0\| = 0$;
- $\forall x, y \in M, \|x + y\| \leq \|x\| + \|y\|$.

If we further have $\|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*. A *semi-normed abelian group* (resp. *normed abelian group*) is an abelian group equipped with a semi-norm (resp. norm).

Definition 1.2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group M. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in M, \|x\|_1 \leq C\|x\|_2$.

Remark 1.3. If two semi-norms (resp. norms) on an abelian group M are bounded by each other, then they induce the same topology on M.

Definition 1.4. Let M be a semi-normed abelian group and $N \subseteq M$ be a subgroup. The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Remark 1.5. The residue semi-norm is a norm if and only if N is closed in M.

Definition 1.6. Let M and N be two semi-normed abelian groups. A group homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$.
A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to$

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$\operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \inf_{y \in \ker f} \|x + y\|_M$.

Definition 1.7. Let R be a ring (commutative with unity). A *semi-norm* on R is a semi-norm $\|\cdot\|$ on the underlying abelian group of R such that $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$ and $\|1\| = 1$. A *semi-normed ring* is a ring equipped with a semi-norm.

Definition 1.8. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$.

Definition 1.9. Let $(R, \|\cdot\|_R)$ be a normed ring. A *semi-normed R-module* is a pair $(M, \|\cdot\|_M)$ where M is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M such that there exists $C > 0$ with $\forall a \in R, x \in M, \|ax\|_M \leq C\|a\|_R\|x\|_M$.

One can talk about boundedness, admissibility and residue semi-norms in the contexts of semi-normed rings and semi-normed modules similar to those in semi-normed abelian groups.

1.2 banach rings

Definition 1.10. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 1.11. A *banach ring* is a complete normed ring.

Definition 1.12. Let $(A, \|\cdot\|_A)$ be a (semi-)normed algebraic structure, e.g., a (semi-)normed abelian group, a (semi-)normed ring, or a (semi-)normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a (pseudo-)metric space. Since A is dense in its completion, the algebraic operations and (semi-)norms on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M \otimes_R N} = \inf \left\{ \sum_i \|x_i\|_M \|y_i\|_N \,:\, z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N \right\}.$$

Definition 1.13. Let R be a complete normed ring and M, N complete semi-normed R-modules. The *complete tensor product* $M \widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Definition 1.14. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n \to \infty} \|f^n\|^{1/n}.$$

Yang: Since , $\rho(f)$ exists.

Definition 1.15. A banach ring R is called *uniform* if its norm is power-multiplicative.

Proposition 1.16. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 1.17. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\mathrm{Qnil}(R)$.

Definition 1.18. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Proposition 1.19. Let R be a banach ring. The completion of $R/\mathrm{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

1.3 Examples

Example 1.20. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a normed ring.

Example 1.21. The fields $\mathbb{C}$ and $\mathbb{R}$ equipped with the usual absolute value are complete fields.

Example 1.22. The field $\mathbb{Q}_p$ of p-adic numbers equipped with the p-adic norm is a complete non-Archimedean field.

Example 1.23. Let R be a banach ring and $r > 0$ be a real number. We define the ring of absolutely convergent power series over $\mathbf{k}$ with radius r as

$$R\langle T/r\rangle := \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r\rangle$ is a banach ring.

When $R = \mathbf{k}$ is a Yang: To be checked.

Example 1.24. Let $\mathbf{k}$ be a non-Archimedean complete field. We define

$$\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\} := \left\{ \sum_{I \in \mathbb{N}^n} a_I T^I \in \mathbf{k}[[T_1, \dots, T_n]] : \lim_{|I| \to \infty} |a_I| r^I = 0 \right\},$$

where $r = (r_1, \dots, r_n)$ is an n-tuple of positive real numbers, $T^I = T_1^{i_1} \cdots T_n^{i_n}$ for $I = (i_1, \dots, i_n)$, and $|I| = i_1 + \cdots + i_n$. Equipped with the norm $\|\sum_{I \in \mathbb{N}^n} a_I T^I\| = \sup_{I \in \mathbb{N}^n} |a_I| r^I$, the affinoid $\mathbf{k}$-algebra $\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$ is a banach $\mathbf{k}$-algebra. This is called the *Tate algebra* over $\mathbf{k}$ with polyradius r equipped with the *Gauss norm*. We will denote $\mathbf{k}\{\underline{T/r}\} = \mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$ for simplicity.

Yang: To be continued...

2 Affinoid algebras

2.1 The first properties

Definition 2.1. Let $\mathbf{k}$ be a non-archimedean field. A banach $\mathbf{k}$-algebra A is called a *affinoid* $\mathbf{k}$*-algebra* if there exists an admissible surjective homomorphism

$$\varphi : \mathbf{k}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \twoheadrightarrow A$$

for some $n \in \mathbb{N}$ and $r_1, \dots, r_n \in \mathbb{R}_{>0}$.
If one can choose $r_1 = \dots = r_n = 1$, then we say that A is a *strict affinoid* $\mathbf{k}$*-algebra*.

Definition 2.2. Let $\mathbf{k}$ be a non-archimedean field. We define the *ring of restricted Laurent series* over $\mathbf{k}$ as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} = \left\{ \sum_{n\in\mathbb{Z}} a_n T^n : a_n \in \mathbf{k}, \lim_{|n|\to\infty} |a_n| r^n = 0 \right\}$$

equipped with the norm

$$\|f\| = \sup_{n\in\mathbb{Z}} |a_n| r^n.$$

Yang: Is $\mathbf{K}_r$ always a field? Yang: Do we have $\mathbf{L}_{\mathbf{k},r} = \mathrm{Frac}(\mathbf{k}\{T/r\})$?

Proposition 2.3. Let $\mathbf{k}$ be a non-archimedean field. If $r \notin \sqrt{|\mathbf{k}^\times|}$, then $\mathbf{K}_r$ is a complete non-archimedean field with non-trivial absolute value extending that of $\mathbf{k}$.

Proposition 2.4. Let A be an affinoid $\mathbf{k}$-algebra. Then A is noetherian, and every ideal of A is closed.

Proposition 2.5. Let A be an affinoid $\mathbf{k}$-algebra. Then there exists a constant $C > 0$ and $N > 0$ such that for all $f \in A$ and $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

Proposition 2.6. Let A be an affinoid $\mathbf{k}$-algebra. If and only if $\rho(f) \in \sqrt{|\mathbf{k}|}$ for all $f \in A$, then A is strict. Yang: To be complete.

3 Finite modules

3.1 Finite banach module

There are three different categories of finite modules over an affinoid algebra A:

- The category $\mathbf{Banmod}_A$ of finite banach A-modules with A-linear maps as morphisms.
- The category $\mathbf{Banmod}_A^b$ of finite banach A-modules with bounded A-linear maps as morphisms.

- The category $\mathbf{mod}_A$ of finite A-modules with all A-linear maps as morphisms.

Theorem 3.1. Let A be an affinoid $\mathbf{k}$-algebra. Then the category of finite banach A-modules with bounded A-linear maps as morphisms is equivalent to the category of finite A-modules with A-linear maps as morphisms. Yang: To be revised.

For simplicity, we will just write mod_A to denote the category of finite banach A-modules with bounded A-linear maps as morphisms.