Non-archimedean Analysis

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Contents

1 Valuation fields

1.1 Absolute values and completion

Definition 1.1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 1.2. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^{\times} \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^{\times}, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^{\times}, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v

Date: November 1, 2025, **Author**: Tianle Yang, My Homepage induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Example 1.3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 1.4. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c \in (0,1)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Lemma 1.5. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) $\Rightarrow$ (b) is obvious. Now we prove (b) $\Rightarrow$ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) $\Rightarrow$ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \in D_1 \setminus \{0\}$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. If $\|x\|_1 = \|y\|_1$, then $x/y, y/x \notin D_1$. Thus $\|x/y\|_2 = 1$ and hence $\|x\|_2 = \|y\|_2$, which implies that $\alpha = \beta$. Hence we can assume that $\|x\|_1 > \|y\|_1$. Yang: To be continued. ☐

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$.

Definition 1.6. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x,y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 1.7. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Yang: To be added. ☐

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 1.8. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 1.9. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

1.2 Non-archimedean fields

Definition 1.10. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 1.11. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathscr{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Yang: Is the valuation on residue field trivial?

Lemma 1.12. Recall that a metric space is *totally bounded* if for every $\varepsilon > 0$, it can be covered by finitely many balls of radius ε. A metric space is compact if and only if it is complete and totally bounded.

Proof. Yang: To be added. □

Definition 1.13. Let $\mathbf{k}$ be a non-archimedean field. The *residue absolute value* on the residue field $\mathscr{k}_{\mathbf{k}}$ is defined as

$$|x| := \inf_{y \in \varphi^{-1}(x)} \|y\|, \quad \forall x \in \mathscr{k}_{\mathbf{k}},$$

where $\varphi : \mathbf{k}^\circ \to \mathscr{k}_{\mathbf{k}}$ is the canonical projection.

Proposition 1.14. Let $\mathbf{k}$ be a non-archimedean field. Then the residue absolute value on the residue field $\mathscr{k}_{\mathbf{k}}$ is trivial.

Proof. For any $x \in \mathscr{k}_{\mathbf{k}}$, if $x = 0$, then by definition $|x| = 0$. If $x \neq 0$, then $\forall y \in \varphi^{-1}(x)$, we have $y \in \mathbf{k}^\circ \setminus \mathbf{k}^{\circ\circ}$, i.e., $\|y\| = 1$. Thus by definition $|x| = 1$. □

Proposition 1.15. Let $\mathbf{k}$ be a non-archimedean field. Set $I_r := \{x \in \mathbf{k} : \|x\| < r\}$ for each $r \in (0, 1)$. They are ideals of the ring of integers $\mathbf{k}^\circ$. Then we have

$$\widehat{\mathbf{k}^\circ} \cong \varprojlim_{r>0} \mathbf{k}^\circ / I_r.$$

Yang: To be checked.

Slogan *Locally compact* $\iff$ *pro-finite.*

Proposition 1.16. Let $\mathbf{k}$ be a non-archimedean field. Then $\mathbf{k}$ is totally bounded iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$.

Proposition 1.17. $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field. and complete. Yang: To be revised.

2 Ultra-metric spaces

Definition 2.1. A metric space (X,d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x,z) \le \max\{d(x,y), d(y,z)\}, \quad \forall x,y,z \in X.$$

Proposition 2.2. Let (X,d) be an ultra-metric space. Then for any $x \in X$ and $r > 0$, the closed ball $B(x,r) := \{y \in X : d(x,y) \le r\}$ satisfies the following properties:

(a) For any $y \in B(x,r)$, we have $B(x,r) = B(y,r)$.

(b) Any two closed balls in X are either disjoint or one is contained in the other.

Yang: To be revised.

We will use $B(x,r)$ to denote the open ball with center x and radius r. We will use $E(x,r)$ to denote the closed ball with center x and radius r.

Proposition 2.3. Let (X,d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the singletons. Yang: To be revised.

3 Residue fields and reductions

Theorem 3.1 (Hensel's lemma)**.** Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $f(T) \in \mathbf{k}^\circ[T]$ be a monic polynomial. Suppose that the reduction $\widetilde{f}(T) \in \mathscr{k}_\mathbf{k}[T]$ of $f(T)$ factors as

$$\widetilde{f}(T) = g(T)h(T),$$

where $g(T), h(T) \in \mathscr{k}_\mathbf{k}[T]$ are monic polynomials that are coprime in $\mathscr{k}_\mathbf{k}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$f(T) = G(T)H(T),$$

and the reductions $\widetilde{G}(T), \widetilde{H}(T) \in \mathscr{k}_\mathbf{k}[T]$ of $G(T), H(T)$ are $g(T), h(T)$ respectively. Yang: To be checked.

4 Finite field extensions

4.1 Finite-dimensional vector space

Proposition 4.1. Let V be a finite-dimensional vector space over a complete non-archimedean field $\mathbf{k}$. Then all norms on V are equivalent. Yang: To be checked.

4.2 Finite field extensions

Proposition 4.2. Let $\mathbf{k}$ be a complete non-archimedean field and ℓ a finite extension of $\mathbf{k}$. Then the absolute value on ℓ is uniquely determined by the absolute value on $\mathbf{k}$. Yang: To be checked.

Proposition 4.3. Let $\mathbf{k}$ be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed. Yang: To be checked.

5 Analytic functions

5.1 Continuous functions

5.2 Power series

Proposition 5.1. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $\sum_{n=0}^{+\infty} a_n$ be a series in $\mathbf{k}$. Then the series $\sum_{n=0}^{+\infty} a_n$ converges if and only if $\lim_{n\to+\infty} a_n = 0$. Yang: To be checked.

5.3 Tate algebras

Definition 5.2. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field.

6 Example: p-adic fields

6.1 p-adic fields

Construction 6.1. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_{\mathfrak{p}}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_{\mathfrak{p}} : K^\times \to \mathbb{Z}$ the corresponding discrete valuation. The *$\mathfrak{p}$-adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the $\mathfrak{p}$-adic absolute value $|\cdot|_{\mathfrak{p}}$ is denoted by $K_{\mathfrak{p}}$, called the *$\mathfrak{p}$-adic field.*

One can just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p.

Example 6.2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Proposition 6.3. The multiplicative group $\mathbb{Q}_p^\times$ of the p-adic field $\mathbb{Q}_p$ admits the following decomposition:

$$\mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^\times,$$

where $p^{\mathbb{Z}} := \{p^n \mid n \in \mathbb{Z}\}$ and $\mathbb{Z}_p^\times := \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$ is the group of units of the ring of p-adic integers $\mathbb{Z}_p$. Yang: To be checked.

Yang: What is the relation between the finite extension of $\mathbb{Q}_p$ and $K_{\mathfrak{p}}$?

6.2 Completion

Proposition 6.4. The algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is not complete with respect to the extension of the p-adic absolute value $|\cdot|_p$.

Construction 6.5. Let p be a prime number. The field $\mathbb{C}_p$ of *p-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$. The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$. Yang: To be completed.

Proposition 6.6. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Construction 6.7. Let p be a prime number. Yang: We construct the *spherically complete p-adic field Ω_p*. Yang: To be completed.