Analytic functions

1 Failure of continuous and differentiable functions

Definition 1. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field and $U \subset \mathbf{k}$ be an open subset. A function $f : U \to \mathbf{k}$ is said to be *differentiable* at a point $a \in U$ if the limit

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists in $\mathbf{k}$. If f is differentiable at every point in U, we say that f is differentiable on U. Yang: to be revised.

Proposition 2. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field. Then there exists a continuous function $f : \mathbf{k} \to \mathbf{k}$ such that for any $x, y \in \mathbf{k}$ with $x \neq y$, we have

$$\frac{f(x) - f(y)}{x - y} = 0.$$

2 Power series

Lemma 3. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field and $\sum_{n=0}^{+\infty} a_n$ be a series in $\mathbf{k}$. Then the series $\sum_{n=0}^{+\infty} a_n$ converges if and only if $\lim_{n \to +\infty} a_n = 0$. Yang: To be checked.

Proposition 4. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field and $\sum_{n=0}^{+\infty} a_n (x - c)^n$ be a power series in $\mathbf{k}$. Then there exists a radius of convergence $R \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that the series converges for all $x \in \mathbf{k}$ with $\|x - c\| < R$ and diverges for all $x \in \mathbf{k}$ with $\|x - c\| > R$. Yang: To be revised.

Proposition 5. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $\sum_{n=0}^{+\infty} a_n$ be a series in $\mathbf{k}$. Then the series $\sum_{n=0}^{+\infty} a_n$ converges if and only if $\lim_{n \to +\infty} a_n = 0$. Yang: To be checked.

Definition 6. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field.

Proposition 7. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field. Then the norm on the Tate algebra $\mathbf{k}\langle x_1, \ldots, x_n \rangle$ coincides with the supremum norm on the closed unit polydisc in $\mathbf{k}^n$. Yang: To be checked.

3 Analytic functions and maps

As in the case of real analysis, we can define analytic functions over non-archimedean fields using power series.

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Definition 8. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $U \subset \mathbf{k}$ be an open subset. A function $f : U \to \mathbf{k}$ is said to be *analytic* at a point $c \in U$ if there exists a power series $\sum_{n=0}^{+\infty} a_n(x-c)^n$ that converges to $f(x)$ for all x in some neighborhood of c. If f is analytic at every point in U, we say that f is analytic on U. Yang: to be revised.

Theorem 9. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $U \subset \mathbf{k}$ be an open subset. If $f : U \to \mathbf{k}$ is an analytic function, then f is locally Lipschitz continuous on U. Yang: To be checked.

Theorem 10. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $U \subset \mathbf{k}$ be an open subset. If $f : U \to \mathbf{k}$ is an analytic function, then f satisfies the maximum modulus principle, i.e., if there exists a point $x_0 \in U$ such that $\|f(x_0)\| \geq \|f(x)\|$ for all $x \in U$, then f is constant on U. Yang: To be checked.