Semi-normed Rings and Modules

1 Semi-normed algebraic structures

Definition 1. Let M be an abelian group. A *semi-norm* on M is a function $\|\cdot\| : M \to \mathbb{R}_+$ such that

- $\|0\| = 0$;
- $\forall x, y \in M, \|x + y\| \le \|x\| + \|y\|$.

If we further have $\|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*. A *semi-normed abelian group* (resp. *normed abelian group*) is an abelian group equipped with a semi-norm (resp. norm).

Definition 2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group M. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in M, \|x\|_1 \le C\|x\|_2$.

Remark 3. If two semi-norms (resp. norms) on an abelian group M are bounded by each other, then they induce the same topology on M.

Definition 4. Let M be a semi-normed abelian group and $N \subseteq M$ be a subgroup. The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Remark 5. The residue semi-norm is a norm if and only if N is closed in M.

Definition 6. Let M and N be two semi-normed abelian groups. A group homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \le C\|x\|_M$.
A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \inf_{y \in \ker f} \|x + y\|_M$.

Definition 7. Let R be a ring (commutative with unity). A *semi-norm* on R is a semi-norm $\|\cdot\|$ on the underlying abelian group of R such that $\forall x, y \in R, \|xy\| \le \|x\|\|y\|$ and $\|1\| = 1$. A *semi-normed ring* is a ring equipped with a semi-norm.

Definition 8. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \ge 1$.

Definition 9. Let $(R, \|\cdot\|_R)$ be a normed ring. A *semi-normed R-module* is a pair $(M, \|\cdot\|_M)$ where M is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M such that there exists $C > 0$ with $\forall a \in R, x \in M, \|ax\|_M \le C\|a\|_R\|x\|_M$.

One can talk about boundedness, admissibility and residue semi-norms in the contexts of semi-normed rings and semi-normed modules similar to those in semi-normed abelian groups.

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2 banach rings

Definition 10. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 11. A *banach ring* is a complete normed ring.

Definition 12. Let $(A, \|\cdot\|_A)$ be a (semi-)normed algebraic structure, e.g., a (semi-)normed abelian group, a (semi-)normed ring, or a (semi-)normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a (pseudo-)metric space. Since A is dense in its completion, the algebraic operations and (semi-)norms on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M\otimes_R N} = \inf\left\{\sum_i \|x_i\|_M \|y_i\|_N \,:\, z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N\right\}.$$

Definition 13. Let R be a complete normed ring and M, N complete semi-normed R-modules. The *complete tensor product* $M\widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Definition 14. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Definition 15. A banach ring R is called *uniform* if its norm is power-multiplicative.

Proposition 16. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 17. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\mathrm{Qnil}(R)$.

Definition 18. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Proposition 19. Let R be a banach ring. The completion of $R/\mathrm{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

3 Complete field

Definition 20. A multiplicative norm on a field is also called an *absolute value*. A *valuation field* is a field equipped with an absolute value.

Remark 21. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x+y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0,1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Definition 22. A valuation field $(\mathbf{k}, |\cdot|)$ is called *non-Archimedean* if $\forall x, y \in \mathbf{k}, |x+y| \leq \max\{|x|, |y|\}$, i.e., the norm satisfies the ultrametric inequality. Otherwise, it is called *Archimedean*.

Definition 23. A *complete field* is a valuation field which is complete as a metric space.

Lemma 24. Let $\mathbf{k}$ be a non-Archimedean complete field. Then the set $\mathbf{k}^\circ = \{x \in \mathbf{k} : |x| \leq 1\}$ is a subring of $\mathbf{k}$, which is a local ring. Moreover, the set $\mathbf{k}^{\circ\circ} = \{x \in \mathbf{k} : |x| < 1\}$ is the maximal ideal of $\mathbf{k}^\circ$.

Definition 25. Let $\mathbf{k}$ be a non-Archimedean complete field. The subring $\mathbf{k}^\circ$ is called the *ring of integers* of $\mathbf{k}$. The set $\mathbf{k}^{\circ\circ} = \{x \in \mathbf{k} : |x| < 1\}$ is the maximal ideal of $\mathbf{k}^\circ$. The residue field $\mathscr{k}_{\mathbf{k}} = \widetilde{\mathbf{k}} = \kappa_{\mathbf{k}} = \mathbf{k}^\circ/\mathbf{k}^{\circ\circ}$ is called the *residue field* of $\mathbf{k}$. Yang: To be revised.

Notation test $\mathscr{k}_{\mathbf{k}}$ or $\widetilde{\mathbf{k}}$ or $\kappa_{\mathbf{k}}$ for the residue field of $\mathbf{k}$. $\mathscr{k}_{\mathbb{Q}_p}$

4 Examples

Example 26. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a normed ring.

Example 27. The fields $\mathbb{C}$ and $\mathbb{R}$ equipped with the usual absolute value are complete fields.

Example 28. The field $\mathbb{Q}_p$ of p-adic numbers equipped with the p-adic norm is a complete non-Archimedean field.

Example 29. Let R be a banach ring and $r > 0$ be a real number. We define the ring of absolutely convergent power series over $\mathbf{k}$ with radius r as

$$R\langle T/r\rangle := \left\{\sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty\right\}.$$

Equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r\rangle$ is a banach ring.

When $R = \mathbf{k}$ is a Yang: To be checked.

Yang: To be continued...

Appendix