Commutative branch rings

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1 Semi-normed Rings and Modules

1.1 Semi-normed algebraic structures

Definition 1.1. Let G be an abelian group. A *semi-norm* on G is a function $\|\cdot\| : G \to \mathbb{R}_{\geq 0}$ such that

- $\|0\| = 0$;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$.

Suppose that R is a ring (commutative with unity) and $\|\cdot\|$ is a semi-norm on the underlying abelian group of R. We further require that

- $\|1\| = 1$;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$.

Suppose that $(M, \|\cdot\|_M)$ is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M. We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$.

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Suppose that $(A, \|\cdot\|_A)$ is an R-algebra and $\|\cdot\|_A$ is a semi-norm on the underlying R-module of A. We further require that this semi-norm is a semi-norm on the underlying ring of A.

Definition 1.2. Let A be an abelian group (or ring, R-module, R-algebra) equipped with a semi-norm $\|\cdot\|$. If $\forall x \in A, \|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*.

Yang: Note that this definition of semi-normed module is a little different of [**Ber90**]

Definition 1.3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group (or ring, R-module, R-algebra) A. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in A, \|x\|_1 \leq C\|x\|_2$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are bounded by each other, we say they are *equivalent*.

Remark 1.4. Equivalent semi-norms induce the same topology on A.

Definition 1.5. Let M be a semi-normed abelian group (or ring, R-module, R-algebra) and $N \subseteq M$ be a subgroup (or ideal, R-submodule, ideal). The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Yang: Is this always a semi-norm? In particular, $\|1\| = 1$?

Unless otherwise specified, we always equip the quotient M/N with the residue semi-norm.

Remark 1.6. The residue semi-norm is a norm if and only if N is closed in M.

Definition 1.7. Let M and N be two semi-normed abelian groups (or rings, R-modules, R-algebras). A homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$.

A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$.

Definition 1.8. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$.

1.2 Banach rings

Definition 1.9. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 1.10. A *banach ring* is a complete normed ring.

Definition 1.11. Let $(A, \|\cdot\|_A)$ be a (semi-)normed algebraic structure, e.g., a (semi-)normed abelian group, a (semi-)normed ring, or a (semi-)normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a (pseudo-)metric space. Since A is dense in its completion, the algebraic operations and (semi-)norms on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M\otimes_R N} = \inf\left\{\sum_i \|x_i\|_M \|y_i\|_N \,:\, z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N\right\}.$$

Definition 1.12. Let R be a complete normed ring and M, N complete semi-normed R-modules. The *complete tensor product* $M\widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Definition 1.13. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Yang: Since , $\rho(f)$ exists.

Definition 1.14. A banach ring R is called *uniform* if its norm is power-multiplicative.

Proposition 1.15. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 1.16. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\mathrm{Qnil}(R)$.

Definition 1.17. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Proposition 1.18. Let R be a banach ring. The completion of $R/\mathrm{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

1.3 Complete tensor product

1.4 Examples

Example 1.19. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a normed ring.

Example 1.20. The fields $\mathbb{C}$ and $\mathbb{R}$ equipped with the usual absolute value are complete fields.

Example 1.21. The field $\mathbb{Q}_p$ of p-adic numbers equipped with the p-adic norm is a complete non-Archimedean field.

Example 1.22. Let R be a banach ring and $r > 0$ be a real number. We define the ring of absolutely convergent power series over $\mathbf{k}$ with radius r as

$$R\langle T/r\rangle := \left\{\sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty\right\}.$$

Equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r\rangle$ is a banach ring.

When $R = \mathbf{k}$ is a Yang: To be checked.

Example 1.23. Let $\mathbf{k}$ be a non-Archimedean complete field. We define

$$\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\} := \left\{\sum_{I\in\mathbb{N}^n} a_I T^I \in \mathbf{k}[[T_1, \dots, T_n]] : \lim_{|I|\to\infty} |a_I| r^I = 0\right\},$$

where $r = (r_1, \dots, r_n)$ is an n-tuple of positive real numbers, $T^I = T_1^{i_1} \cdots T_n^{i_n}$ for $I = (i_1, \dots, i_n)$, and $|I| = i_1 + \cdots + i_n$. Equipped with the norm $\|\sum_{I\in\mathbb{N}^n} a_I T^I\| = \sup_{I\in\mathbb{N}^n} |a_I| r^I$, the affinoid $\mathbf{k}$-algebra $\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$ is a banach $\mathbf{k}$-algebra. This is called the *Tate algebra* over $\mathbf{k}$ with polyradius r equipped with the *Gauss norm*. We will denote $\mathbf{k}\{\underline{T/r}\} = \mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$ for simplicity.

Yang: To be continued...

2 Completion of polynomial rings

Definition 2.1. Yang: To be added.

3 Spectrum

Let $\mathbf{k}$ be a spherically complete non-archimedean field which is algebraically closed and $A = \mathbf{k}[T]$. We want to consider the "analytic structure" on $\operatorname{mSpec} A$. However, unlike the complex case, the set $\operatorname{mSpec} A$ is totally disconnected with respect to the topology induced by the absolute value on $\mathbf{k}$ (??). To overcome this difficulty, Berkovich uses multiplicative semi-norms to "fill in the gaps" between the points in $\operatorname{mSpec} A$, leading to the notion of the spectrum of a Banach ring.

We first consider the local model. Hence we should consider the Tate algebra $\mathbf{k}\{T\}$ instead of the polynomial ring $\mathbf{k}[T]$. Yang: The maximal ideal of $\mathbf{k}\{T\}$ corresponding to the point in the disk $\{a \in \mathbf{k} : a \leq 1\}$. Yang: Closed or open disk?

3.1 Definition

Definition 3.1. Let R be a Banach ring. The *spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$. We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_{\geq 0}$, defined by $x \mapsto |f|_x =: f(x)$, is continuous.

Definition 3.2. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $\mathscr{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$ for each $x \in \mathscr{M}(S)$.

Proposition 3.3. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, let $\wp_x$ be the kernel of the multiplicative semi-norm $|\cdot|_x$. Then $\wp_x$ is a closed prime ideal of R, and $x \mapsto \wp_x$ defines a continuous map from $\mathscr{M}(R)$ to $\mathsf{Spec}(R)$ equipped with the Zariski topology.

Proof. Yang: To be completed □

Definition 3.4. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, the *completed residue field* at the point x is defined as the completion of the residue field $\kappa(x) = \mathsf{Frac}(R/\wp_x)$ with respect to the multiplicative norm induced by the semi-norm $|\cdot|_x$, denoted by $\mathscr{H}(x)$.

Definition 3.5. Let R be a Banach ring. The *Gel'fand transform* of R is the bounded ring homomorphism

$$\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x), \quad f \mapsto (f(x))_{x \in \mathscr{M}(R)},$$

where the norm on the product $\prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is given by the supremum norm.

Proposition 3.6. The Gel'fand transform $\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ of a Banach ring R factors through the uniformization R^u of R, and the induced map $R^u \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is an isometric embedding. Yang: To be checked.

Theorem 3.7. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

Proof. Yang: To be continued. □

Lemma 3.8. Let $\{K_i\}_{i \in I}$ be a family of completed fields. Consider the Banach ring $R = \prod_{i \in I} K_i$ equipped with the product norm. The spectrum $\mathscr{M}(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I.

Remark 3.9. The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

Theorem 3.10. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a compact Hausdorff space.

Proof. Yang: To be added. □

Proposition 3.11. Let K/k be a Galois extension of complete fields, and let R be a Banach k-algebra. The Galois group $\mathsf{Gal}(K/k)$ acts on the spectrum $\mathscr{M}(R \widehat{\otimes}_k K)$ via

$$g \cdot x : f \mapsto |(1 \otimes g^{-1})(f)|_x$$

for each $g \in \mathsf{Gal}(K/k)$, $x \in \mathscr{M}(R \widehat{\otimes}_k K)$ and $f \in R \widehat{\otimes}_k K$. Moreover, the natural map $\mathscr{M}(R \widehat{\otimes}_k K) \to \mathscr{M}(R)$ induces a homeomorphism

$$\mathscr{M}(R \widehat{\otimes}_k K)/\mathsf{Gal}(K/k) \xrightarrow{\sim} \mathscr{M}(R).$$

Yang: To be checked.

3.2 Examples

Example 3.12. Let $(\mathbf{k}, |\cdot|)$ be a complete valuation field. The spectrum $\mathscr{M}(\mathbf{k})$ consists of a single point corresponding to the given absolute value $|\cdot|$ on $\mathbf{k}$. Yang: To be checked.

Example 3.13. Consider the Banach ring $(\mathbb{Z}, \|\cdot\|)$ with $\|\cdot\| = |\cdot|_\infty$ is the usual absolute value norm on $\mathbb{Z}$. Let $|\cdot|_p$ denote the p-adic norm for each prime number p, i.e., $|n|_p = p^{-v_p(n)}$ for each $n \in \mathbb{Z}$, where $v_p(n)$ is the p-adic valuation of n. The spectrum

$$\mathscr{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0,1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0,\infty]\} \cup \{|\cdot|_0\},$$

where $|a|_p^\infty := \lim_{\alpha\to\infty} |a|_p^\alpha$ for each $a \in \mathbb{Z}$ and $|\cdot|_0$ is the trivial norm on $\mathbb{Z}$. Yang: To be checked.

Spectrum of Tate algebra in one variable Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{T/r\}$. We list some types of points in the spectrum $\mathscr{M}(A)$.

For each $a \in \mathbf{k}$ with $|a| \leq r$, we have the *type I* point x_a corresponding to the evaluation at a, i.e., $|f|_{x_a} := |f(a)|$ for each $f \in A$. For each closed disk $E = E(a,s) := \{b \in \mathbf{k} : |b-a| \leq s\}$ with center $a \in \mathbf{k}$ and radius $s \leq r$, we have the point $x_{a,s}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} := \sup_{b\in E(a,s)} |f(b)|$$

for each $f \in A$. If $s \in |\mathbf{k}^\times|$, then the point x_E is called a *type II* point; otherwise, it is called a *type III* point.

Let $\{E^{(s)}\}_s$ be a family of closed disks in $\mathbf{k}$ such that $E^{(s)}$ is of radius s, $E^{(s_1)} \subsetneq E^{(s_2)}$ for any $s_1 < s_2$ and $\bigcap_s E^{(s)} = \varnothing$. Then we have the point $x_{\{E^{(s)}\}}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E^{(s)}\}}} := \inf_s |f|_{x_{E^{(s)}}}$$

for each $f \in A$. Such a point is called a *type IV* point.

Yang: To be completed.

Proposition 3.14. Let $\mathbf{k}$ be a complete non-archimedean field, and let $r > 0$ be a positive real number. Consider the Tate algebra $\mathbf{k}\{r^{-1}T\}$ equipped with the Gauss norm. The points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ can be classified into four types as described above. Yang: To be checked

Proof. Yang: To be completed. □

Proposition 3.15. Let $\mathbf{k}$ be a complete non-archimedean field, and let $r > 0$ be a positive real number. Consider the Tate algebra $\mathbf{k}\{r^{-1}T\}$ equipped with the Gauss norm. The completed residue fields of the four types of points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ are described as follows:

- For a type I point x_a with $a \in \mathbf{k}$ and $|a| \leq r$, the completed residue field $\mathscr{H}(x_a)$ is isomorphic to $\mathbf{k}$.
- For a type II point $x_{a,s}$ with $a \in \mathbf{k}$ and $s \in |\mathbf{k}^\times|$, the completed residue field $\mathscr{H}(x_{a,s})$ is isomorphic to the field of Laurent series over the residue field $\mathscr{k}_{\mathbf{k}}$, i.e., $\mathscr{k}_{\mathbf{k}}((t))$.
- For a type III point $x_{a,s}$ with $a \in \mathbf{k}$ and $s \notin |\mathbf{k}^\times|$, the completed residue field $\mathscr{H}(x_{a,s})$ is

isomorphic to a transcendental extension of $\mathscr{k}_{\mathbf{k}}$ of degree one.

- For a type IV point $x_{\{E(s)\}}$, the completed residue field $\mathscr{H}(x_{\{E(s)\}})$ is isomorphic to a transcendental extension of $\mathscr{k}_{\mathbf{k}}$ of infinite degree.

Yang: To be checked.

Example 3.16. The completed residue field $\mathscr{H}(x_a)$ for a type I point x_a with $a \in \mathbf{k}$ and $|a| \leq r$ is isomorphic to $\mathbf{k}$. Yang: To be complete.

Spectrum of Tate algebra in servel variables Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$. We can consider the spectrum $\mathscr{M}(A)$ similarly.

4

5 Affinoid algebras

5.1 The first properties

Definition 5.1. Let $\mathbf{k}$ be a non-archimedean field. A banach $\mathbf{k}$-algebra A is called a *affinoid* $\mathbf{k}$*-algebra* if there exists an admissible surjective homomorphism

$$\varphi : \mathbf{k}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \twoheadrightarrow A$$

for some $n \in \mathbb{N}$ and $r_1, \dots, r_n \in \mathbb{R}_{>0}$.
If one can choose $r_1 = \dots = r_n = 1$, then we say that A is a *strict affinoid* $\mathbf{k}$*-algebra*.

Definition 5.2. Let $\mathbf{k}$ be a non-archimedean field. We define the *ring of restricted Laurent series* over $\mathbf{k}$ as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} = \left\{ \sum_{n\in\mathbb{Z}} a_n T^n : a_n \in \mathbf{k}, \lim_{|n|\to\infty} |a_n| r^n = 0 \right\}$$

equipped with the norm

$$\|f\| = \sup_{n\in\mathbb{Z}} |a_n| r^n.$$

Yang: Is $\mathbf{K}_r$ always a field? Yang: Do we have $\mathbf{L}_{\mathbf{k},r} = \mathrm{Frac}(\mathbf{k}\{T/r\})$?

Proposition 5.3. Let $\mathbf{k}$ be a non-archimedean field. If $r \notin \sqrt{|\mathbf{k}^\times|}$, then $\mathbf{K}_r$ is a complete non-archimedean field with non-trivial absolute value extending that of $\mathbf{k}$.

Proposition 5.4. Let A be an affinoid $\mathbf{k}$-algebra. Then A is noetherian, and every ideal of A is closed.

Proposition 5.5. Let A be an affinoid $\mathbf{k}$-algebra. Then there exists a constant $C > 0$ and $N > 0$ such that for all $f \in A$ and $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

Proposition 5.6. Let A be an affinoid $\mathbf{k}$-algebra. If and only if $\rho(f) \in \sqrt{|\mathbf{k}|}$ for all $f \in A$, then A is strict. Yang: To be complete.

5.2 Noetherian normalization theorem

6 Finite modules

6.1 Finite banach module

There are three different categories of finite modules over an affinoid algebra A:

- The category $\mathbf{Banmod}_A$ of finite banach A-modules with A-linear maps as morphisms.
- The category $\mathbf{Banmod}_A^b$ of finite banach A-modules with bounded A-linear maps as morphisms.
- The category $\mathbf{mod}_A$ of finite A-modules with all A-linear maps as morphisms.

Theorem 6.1. Let A be an affinoid $\mathbf{k}$-algebra. Then the category of finite banach A-modules with bounded A-linear maps as morphisms is equivalent to the category of finite A-modules with A-linear maps as morphisms. Yang: To be revised.

For simplicity, we will just write mod_A to denote the category of finite banach A-modules with bounded A-linear maps as morphisms.