Valuation fields

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The main references for this chapter are [Gou97; Rob00; 李文威 18].

1 Valuation fields

1.1 Absolute values and completion

Definition 1.1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

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(c) $\|x+y\| \le \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 1.2. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x+y) \ge \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0,1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Example 1.3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 1.4. The *(multiplicative) valuation group* of a valuation field $(\mathbf{k}, \|\cdot\|)$ is defined as the subgroup of $\mathbb{R}_{>0}$ given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation $\sqrt{|\mathbf{k}^\times|}$ to denote the set $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$.

Definition 1.5. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c \in (0,1)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$. Moreover, the following lemma shows that the converse is also true.

Lemma 1.6. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) $\Rightarrow$ (b) is obvious. Now we prove (b) $\Rightarrow$ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) $\Rightarrow$ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. Suppose the contrary that $\alpha \neq \beta$. Consider the domain $\Omega \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Ω is non-empty. Pick $(n, m) \in \Omega$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. □

Definition 1.7. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 1.8. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Simple analysis. □

Example 1.9. Let $|\cdot|_\infty$ be the usual absolute value on the field $\mathbb{Q}$ of rational numbers. Then $(\mathbb{Q}, |\cdot|_\infty)$ is a valuation field. Its completion is the field $\mathbb{R}$ of real numbers equipped with the usual absolute value.

Example 1.10. Let p be a prime number. For any non-zero rational number $x \in \mathbb{Q}$, we can write it as $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then $(\mathbb{Q}, |\cdot|_p)$ is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \ldots, p-1\} \right\}$$

of p-adic numbers equipped with the p-adic absolute value; see Yang: to be added..

Proposition 1.11. Let $(\mathbf{k}, \|\cdot\|)$ be a complete valuation field with non-trivial absolute value. Then $\mathbf{k}$ is uncountable.

Proof. Since the absolute value $\|\cdot\|$ is non-trivial, we can construct a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathbf{k}$ inductively such that $\|x_n\| < \|x_{n-1}\|/2$ for any $n \geq 1$ and $\|x_0\| < 1$. Then there is an injective map from $\mathbb{N}^{\{0,1\}}$ to $\mathbf{k}$ defined by

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n x_n, \quad a_n \in \{0, 1\}.$$

Since $\|x_n\| < 2^{-n}$, the series $\sum_{n=1}^\infty a_n x_n$ converges in $\mathbf{k}$. Note $\|x_n\| > \|\sum_{m \geq n} x_m\|$ for each n, we have that the map is injective. Thus $\mathbf{k}$ is uncountable. □

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 1.12. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 1.13. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

1.2 Non-archimedean fields

Definition 1.14. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} \colon \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} \colon \|x\| < 1\}$.

Definition 1.15. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} \colon \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} \colon \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathscr{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Definition 1.16. Let $\mathbf{k}$ be a non-archimedean field. The *residue absolute value* on the residue field $\mathscr{k}_{\mathbf{k}}$ is defined as

$$|x| := \inf_{y \in \varphi^{-1}(x)} \|y\|, \quad \forall x \in \mathscr{k}_{\mathbf{k}},$$

where $\varphi : \mathbf{k}^\circ \to \mathscr{k}_{\mathbf{k}}$ is the canonical projection.

Proposition 1.17. Let $\mathbf{k}$ be a non-archimedean field. Then the residue absolute value on the residue field $\mathscr{k}_{\mathbf{k}}$ is trivial.

Proof. For any $x \in \mathscr{k}_{\mathbf{k}}$, if $x = 0$, then by definition $|x| = 0$. If $x \neq 0$, then $\forall y \in \varphi^{-1}(x)$, we have $y \in \mathbf{k}^\circ \setminus \mathbf{k}^{\circ\circ}$, i.e., $\|y\| = 1$. Thus by definition $|x| = 1$. □

2 Ultra-metric spaces

We will use $B(x, r)$ (resp. $E(x, r)$) to denote the open ball (resp. closed ball) with center x and radius r.

Definition 2.1. A metric space (X, d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}, \quad \forall x,y,z \in X.$$

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x,y) := \|x-y\|$ on $\mathbf{k}$ makes $(\mathbf{k}, d)$ an ultra-metric space.

Proposition 2.2. Let (X,d) be an ultra-metric space. Then for any $x,y,z \in X$, at least two of the three distances $d(x,y), d(y,z), d(z,x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x,y) \geq d(y,z)$. By the strong triangle inequality, we have

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} = d(x,y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x,y) \leq \max\{d(x,z), d(z,y)\} = \max\{d(x,z), d(y,z)\} \leq d(x,y).$$

This shows that $d(x,y) = \max\{d(x,z), d(y,z)\}$. Thus either $d(x,z) = d(x,y) \geq d(y,z)$ or $d(y,z) = d(x,y) \geq d(x,z)$. □

Proposition 2.3. Let (X,d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1,2$. If $D_1 \cap D_2 \neq \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1,2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1,x_2) \leq \max\{d(x_1,y), d(y,x_2)\}.$$

Without loss of generality, we may assume that $d(x_1,x_2) \leq d(x_1,y)$. It follows that $x_2 \in D_1$ since $d(x_1,y) < r_1$ (or $\leq r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1,z) > d(x_1,x_2)$. Then by Proposition 2.2,

$$r_1 \leq d(x_1,z) = d(x_2,z) \leq r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2,w) \leq \max\{d(x_2,x_1), d(x_1,w)\} \leq r_1 \leq r_2.$$

Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. □

Proposition 2.4. Let (X,d) be an ultra-metric space. Then both $B(x,r)$ and $E(x,r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x,r) := \{y \in X \mid d(x,y) = r\}$ is open in X. Note that if $y \in S(x,r)$, then for any $r' < r$, we have $B(y,r') \cap E(x,r) \neq \varnothing$ and $x \in E(x,r) \setminus B(y,r')$. Thus by Proposition 2.3, we have $B(y,r') \subseteq E(x,r)$. If $B(y,r') \cap B(x,r) \neq \varnothing$, then by Proposition 2.3 again, we have $B(y,r') \subseteq B(x,r)$. However, $y \in B(y,r') \setminus B(x,r)$, a contradiction. Thus $B(y,r') \subseteq E(x,r) \setminus B(x,r) = S(x,r)$. It yields that $S(x,r) = \bigcup_{y \in S(x,r)} B(y,r/2)$ is open in X.

Since $E(x,r) = B(x,r) \cup S(x,r)$ and $B(x,r) = E(x,r) \setminus S(x,r)$, both $B(x,r)$ and $E(x,r)$ are open and closed in X. $\square$

Corollary 2.5. Let (X,d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x,y) > 0$. Consider the open ball $B(x, r/2)$. By Proposition 2.4, $B(x,r/2)$ is both open and closed in X. Thus $B(x,r/2) \cap S$ is both open and closed in S, however, it is non-empty and not equal to S since it contains x but not y. This shows that S is disconnected. $\square$

Proposition 2.6. Let (X,d) be an ultra-metric space. A sequence $\{x_n\}$ in X is cauchy if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Proof. The necessity is true for all metric spaces. Suppose that $d(x_n, x) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \varepsilon$ for all $n \geq N$. For any $m, n \geq N$ with $m < n$, by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \dots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that $\{x_n\}$ is a cauchy sequence. $\square$

3 Algebraic structures of non-archimedean fields

3.1 Recover non-archimedean complete fields algebraically

In this subsection, let $\mathbf{k}$ be a non-archimedean field. Set $I_{r,<} := B(0,r)$ and $I_{r,\leq} := E(0,r)$ for each $r \in [0,1]$.

Proposition 3.1. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers $\mathbf{k}^\circ$. Conversely, any ideal of $\mathbf{k}^\circ$ is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0,1)$.

Proof. Let I be an ideal of $\mathbf{k}^\circ$. Set $r = \sup\{|a| : a \in I\}$ (resp. $r = \max\{|a| : a \in I\}$ when the maximum exists). Then, by definition, we have $I \subset I_{r,<}$ (resp. $I \subset I_{r,\leq}$). For every $x \in \mathbf{k}^\circ$ with $|x| < r$ (resp. $|x| \leq r$), there exists $a \in I$ such that $|x| \leq |a|$. Thus, $|x/a| \leq 1$ and so $x/a \in \mathbf{k}^\circ$. Since I is an ideal, we have $x = (x/a)a \in I$. Therefore, $I_{r,<} \subset I$ (resp. $I_{r,\leq} \subset I$). $\square$

Proposition 3.2. Let I_r be either $I_{r,<}$ or $I_{r,\leq}$ for each $r \in (0,1)$. Suppose $\{r_n \in (0,1)\}_{n\in\mathbb{N}}$ is a decreasing sequence converging to 0. Then the completion $\widehat{\mathbf{k}}$ of $\mathbf{k}$ is isomorphic to the projective limit

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ / I_{r_n}.$$

Proof. For every $x \in \widehat{\mathbf{k}}^\circ$, there exists a cauchy sequence $\{x_m\}_{m\in\mathbb{N}}$ in $\mathbf{k}^\circ$ converging to x. Since $\{r_n\}_{n\in\mathbb{N}}$ converges to 0, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that for all $m, m' \geq M_n$, we have

$|x_m - x_{m'}| < r_n$. Thus, the sequence $\{x_m + I_{r_n}\}_{m\in\mathbb{N}}$ is eventually constant in $\mathbf{k}^\circ/I_{r_n}$. Define a map

$$\Phi : \widehat{\mathbf{k}}^\circ \to \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}, \quad x \mapsto \left(\lim_{m\to\infty} x_m + I_{r_n}\right)_{n\in\mathbb{N}}.$$

It is straightforward to verify that Φ is a well-defined ring homomorphism.

Conversely, for every $(a_n + I_{r_n})_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}$, we can choose a representative $a_n \in \mathbf{k}^\circ$ for each n. We claim that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathbf{k}^\circ$. Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_N < \varepsilon$. For all $m, n \geq N$, since $a_n + I_{r_n}$ maps to $a_m + I_{r_m}$ under the natural projection, we have $|a_n - a_m| < r_N < \varepsilon$. Thus, $\{a_n\}_{n\in\mathbb{N}}$ converges to some $x \in \widehat{\mathbf{k}}^\circ$. Easily see that the limit x is independent of the choice of representatives $\{a_n\}_{n\in\mathbb{N}}$. This gives a map

$$\Psi : \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n} \to \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} a_n.$$

Direct verification shows that $\Psi = \Phi^{-1}$. □

Corollary 3.3. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the residue field $\mathscr{k}_{\widehat{\mathbf{k}}} \cong \mathscr{k}_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$.

Corollary 3.4. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the valuation group $|\widehat{\mathbf{k}}^\times|$ of $\widehat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^\times|$ of $\mathbf{k}$.

Proof. Note that

$$\begin{aligned} r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneqq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\ &\iff \widehat{\mathbf{k}}^\circ/I_{r,<} \to \widehat{\mathbf{k}}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff \mathbf{k}^\circ/I_{r,<} \to \mathbf{k}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff I_{r,<} \subsetneqq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\ &\iff r \in |\mathbf{k}^\times|. \end{aligned}$$

□

Proposition 3.5. Let $\mathbf{k}$ be a non-archimedean field with non-trivial valuation. Then $\mathbf{k}^\circ$ is totally bounded iff $\mathbf{k}^\circ/I_{r,<}$ and $\mathbf{k}^\circ/I_{r,\leq}$ are finite for each $r \in [0,1]$. Moreover, if $\mathbf{k}$ is complete, then it is locally compact iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$.

Slogan *"Locally compact $\iff$ pro-finite."*

Proof. We just prove the case for $I_r = I_{r,<}$. The case for $I_r = I_{r,\leq}$ is similar.

Suppose that $\mathbf{k}^\circ/I_r$ is finite for each $r \in [0,1]$. Then for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that $r < \varepsilon$ and $\mathbf{k}^\circ/I_r$ is finite. Let $\{a_1 + I_r, \ldots, a_n + I_r\}$ be the complete set of representatives of $\mathbf{k}^\circ/I_r$. Then the balls $B(a_i, r)$ for $i = 1, \ldots, n$ cover $\mathbf{k}^\circ$.

Conversely, suppose that $\mathbf{k}^\circ/I_r$ is infinite for some $r \in [0,1]$. Then there exists an infinite set $\{a_n\}$ with $|a_n| \in [r,1]$ such that their images in $\mathbf{k}^\circ/I_r$ are distinct. In particular, for every $m \neq n$, we have $|a_n - a_m| \geq r$. Any subsequence of $\{a_n\}$ is not cauchy. Thus, $\mathbf{k}^\circ$ is not totally bounded. □

Proposition 3.6. The ring $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field.

Proof. Note that $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$ is a multiplicative subgroup. If $\mathbf{k}$ is not a discrete valuation field, then $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. In particular, there exists a strictly ascending sequence $r_n \in |\mathbf{k}^\times| \cap (0,1)$. Then the ideals $I_{r_n,\leq}$ form a strictly ascending chain of ideals in $\mathbf{k}^\circ$.

The converse is standard since now $\mathbf{k}^\circ$ is a discrete valuation ring. $\square$

Proposition 3.7. Let $\mathbf{k}$ be a complete non-archimedean field. Then $\mathbf{k}$ is locally compact iff $\mathbf{k}$ is a discrete valuation field and its residue field $\Bbbk_{\mathbf{k}}$ is finite.

Proof. The necessity follows from Proposition 3.5. For the sufficiency, suppose that $\mathbf{k}$ is a discrete valuation field whose residue field $\Bbbk_{\mathbf{k}}$ is finite. Let $\pi \in \mathbf{k}^\circ$ be a uniformizer. We only need to show that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite for each $n \in \mathbb{N}$. Note that there is an isomorphism

$$\pi^{n-1}\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ \cong \Bbbk_{\mathbf{k}}, \quad x + \pi^n\mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on n, we conclude that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite. $\square$

3.2 Hensel's Lemma

Theorem 3.8 (Hensel's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \Bbbk_{\mathbf{k}}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \Bbbk_{\mathbf{k}}[T]$ are monic polynomials that are coprime in $\Bbbk_{\mathbf{k}}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\Bbbk_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

Proof. Since $\gcd(g,h) = 1$ in $\Bbbk_{\mathbf{k}}[T]$, there exist polynomials $u(T), v(T) \in \Bbbk_{\mathbf{k}}[T]$ such that $ug + vh = 1$ and $\deg u < \deg h, \deg v < \deg g$. Choose lifts $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$ of $g(T), h(T), u(T), v(T)$ respectively preserving their degrees such that G_0 and H_0 are monic. Then there exist $r < 1$ such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \mod I_r, \quad F(T) - G_0(T)H_0(T) \equiv 0 \mod I_r,$$

where $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$.

We will construct a sequence of monic polynomials $\{G_n(T)\}_{n\in\mathbb{N}}$ and $\{H_n(T)\}_{n\in\mathbb{N}}$ in $\mathbf{k}^\circ[T]$ such that for each $n \in \mathbb{N}$,

$$G_n(T) \equiv G_{n-1}(T) \mod I_{r^n}, \quad H_n(T) \equiv H_{n-1}(T) \mod I_{r^n},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \mod I_{r^{n+1}}.$$

If we have such sequences, then their coefficients converge in the complete ring $\mathbf{k}^\circ$. Let $G(T)$ and $H(T)$ be the limits of $\{G_n(T)\}$ and $\{H_n(T)\}$ respectively. Then we have $F(T) = G(T)H(T)$ and the reductions of $G(T), H(T)$ in $\mathcal{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

The case $n = 0$ is done by the above construction. Now suppose that we have constructed $G_n(T)$ and $H_n(T)$ for some $n \geq 0$. Since $G_n - G_0 \equiv 0 \mod I_r$ and $H_n - H_0 \equiv 0 \mod I_r$, we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \mod I_r.$$

Set $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$ and $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$. Then we have

$$\begin{aligned}(G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T].\end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

□

Corollary 3.9. Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathcal{k}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $a \in \mathcal{k}_{\mathbf{k}}$. Then there exists a root $\alpha \in \mathbf{k}^\circ$ of $F(T)$ whose reduction is a.

Proof. Since a is a simple root of $f(T)$, we have the factorization $f(T) = (T - a)h(T)$ for some monic polynomial $h(T) \in \mathcal{k}_{\mathbf{k}}[T]$ with $h(a) \neq 0$. Then the result follows from Theorem 3.8. □

3.3 Newton polygons

Yang: To be filled.

4 Finite field extensions

4.1 Finite-dimensional vector space

Definition 4.1. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|ax\| = |a| \cdot \|x\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Example 4.2. Let $\mathbf{k}$ be a valuation field and V a finite-dimensional vector space over $\mathbf{k}$ with basis $\{e_1, e_2, \ldots, e_n\}$. For any $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$, define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then $\|\cdot\|_{\max}$ is a norm on V, called the *maximal norm* with respect to the basis $\{e_1, e_2, \ldots, e_n\}$.

Example 4.3. Setting as in Example 4.2, for any $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$, define

$$\|x\|_1 := |a_1| + |a_2| + \cdots + |a_n|.$$

Then $\|\cdot\|_1$ is also a norm on V.

Definition 4.4. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \le \|x\|_2 \le C_2\|x\|_1.$$

Lemma 4.5. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V.

Proof. The sufficiency is clear. Now suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on V. Hence the unit open ball with respect to $\|\cdot\|_1$ contains a unit open ball with respect to $\|\cdot\|_2$. That is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every $x \in V$ with $\|x\|_1 = 1$, we have $\|x\|_2 \ge C = C\|x\|_1$. By scaling, we get that for every $x \in V$,

$$\|x\|_2 \ge C\|x\|_1.$$

Similar for the other direction, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Proposition 4.6. Let V be a normed finite-dimensional vector space over a complete valuation field $\mathbf{k}$. Then V is complete.

Proof. Yang: To be added. □

Theorem 4.7. Let V be a finite-dimensional vector space over a complete field $\mathbf{k}$. Then all norms on V are equivalent.

Proof. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis as in Example 4.2. Let $\|\cdot\|$ be any norm on V. It suffices to show that $\|\cdot\|$ and $\|\cdot\|_{\max}$ are equivalent. First we have

$$\|y\| \le \sum_{i=1}^n |a_i|\|e_i\| \le \left(\sum_{i=1}^n \|e_i\|\right)\|y\|_{\max}$$

for any $y = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$. It remains to show that there exists a constant $C > 0$ such that for any $y \in V$,

$$\|y\|_{\max} \le C\|y\|.$$

Yang: To be added. □

Remark 4.8. If the base field $\mathbf{k}$ is not complete, then Theorem 4.7 may fail. For example, let $\mathbf{k} = \mathbb{Q}$ with the usual absolute value, and let $V = \mathbb{Q}[\alpha]$ with $\alpha^2 - \alpha - 1 = 0$. There are two embeddings of V into $\mathbb{R}$:

$$\iota_1 : a + b\alpha \mapsto a + b\frac{1+\sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b\frac{1-\sqrt{5}}{2}.$$

Define two norms on V by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where $|\cdot|$ is the usual absolute value on $\mathbb{R}$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent since $\iota_2(\alpha^n) \to 0$ as $n \to \infty$ while $\iota_1(\alpha^n) \to \infty$.

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

Lemma 4.9. Let $\mathbf{k}$ be a complete field and V a normed finite-dimensional vector space over $\mathbf{k}$. Then

$$\|\cdot\| : \mathrm{End}_{\mathbf{k}}(V) \to \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \mathrm{End}_{\mathbf{k}}(V).$$

Proof. First we show the existence of the supremum, i.e., there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\| \leq C\|x\|$. Fix a basis $\{e_1, e_2, \ldots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis. Since all norms on V are bounded by each other by Theorem 4.7, we only need to show that there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\|_1 \leq C\|x\|_{\max}$. Write $T(e_i) = \sum_{j=1}^n a_{ij} e_j$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^n x_i e_i \in V$, we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^n \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \left(\sum_{1 \leq i,j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of $\|\cdot\|$ are clear. It remains to show the triangle inequality and sub-multiplicativity. For any $A, B \in \mathrm{End}_{\mathbf{k}}(V)$, we have

$$\frac{\|(A+B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all $x \in V \setminus \{0\}$ gives $\|A + B\| \leq \|A\| + \|B\|$. We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$. Taking supremum we get $\|AB\| \leq \|A\| \cdot \|B\|$. □

4.2 Finite field extensions

Lemma 4.10. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then there exists an absolute value on $\mathbf{l}$ extending the absolute value on $\mathbf{k}$.

Proof. Fix a norm $\|\cdot\|_V$ on the $\mathbf{k}$-vector space $V = \mathbf{l}$. The norm $\|\cdot\|_V$ induces an operator norm $\|\cdot\|_{\mathrm{op}}$ on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ as in Lemma 4.9. For any $a \in \mathbf{l}$, let $\mu_a \in \mathrm{End}_{\mathbf{k}}(V)$ be the $\mathbf{k}$-linear map defined by multiplication by a. Note that $a \mapsto \mu_a$ gives a embedding of $\mathbf{k}$-algebras and if $a \in \mathbf{k}$, $\|\mu_a\|_{\mathrm{op}} = \|a\|_{\mathbf{k}}$. Thus the restriction of $\|\cdot\|_{\mathrm{op}}$ to $\mathbf{l}$ gives an norm on $\mathbf{l}$ extending that on $\mathbf{k}$. The normed ring $(\mathbf{l}, \|\cdot\|_{\mathrm{op}})$ is a Banach ring since it is a finite-dimensional vector space over the complete field $\mathbf{k}$. By Theorem 7.1, there exists a multiplicative seminorm $\|\cdot\|_{\mathbf{l}}$ on $\mathbf{l}$ bounded by $\|\cdot\|_{\mathrm{op}}$. In particular, $\|\cdot\|_{\mathbf{l}}$ is bounded by $\|\cdot\|_{\mathbf{k}}$ on $\mathbf{k}$. On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus $\|\cdot\|_{\mathbf{l}}$ extends the absolute value on $\mathbf{k}$. □

Theorem 4.11. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the absolute value on $\mathbf{l}$ which extends the absolute value on $\mathbf{k}$ is uniquely determined by the absolute value on $\mathbf{k}$. Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where $n = [\mathbf{l} : \mathbf{k}]$ and $N_{\mathbf{l}/\mathbf{k}}$ is the norm map from $\mathbf{l}$ to $\mathbf{k}$.

Proof. Let $\|\cdot\|_{\mathbf{l}}$ be arbitrary absolute value on $\mathbf{l}$ extending that on $\mathbf{k}$. We will show that $\|\cdot\|_{\mathbf{l}}$ must be equal to $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$. For any $a \in \mathbf{l}$, set $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$. Then $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that $\|b\|_{\mathbf{l}} = 1$ whenever $N_{\mathbf{l}/\mathbf{k}}(b) = 1$.

Note that the norm map $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \to \mathbf{k}$ is the determinant of the $\mathbf{k}$-linear map $\mu_b \in \mathrm{End}_{\mathbf{k}}(V)$ defined by multiplication by b. Hence it is continuous on $\mathbf{l}$ (since it is a polynomial in the entries of the matrix representation). If $\|b\|_{\mathbf{l}} < 1$, then $\|b^m\|_{\mathbf{l}} \to 0$ as $m \to \infty$. Thus $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \to 0$ as $m \to \infty$, contradicting the fact that $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$ for all m. Similarly, if $\|b\|_{\mathbf{l}} > 1$, then just consider b^{-1}.

□

Proposition 4.12. Let $\mathbf{k}$ be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

Proof. Let $f \in \widehat{\mathbf{k}}[X]$ be a non-constant polynomial. We will show that f has a root in $\widehat{\mathbf{k}}$. Take a sequence of polynomials $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbf{k}[X]$ converging to f coefficient-wisely. Since $\mathbf{k}$ is algebraically closed, each f_n splits completely in $\mathbf{k}$ and hence in $\widehat{\mathbf{k}}$. Write $f_n(X) = \prod_{i=1}^d (X - \alpha_{n,i})$ with $\alpha_{n,i} \in \widehat{\mathbf{k}}$.

Let $\mathbf{l}$ be a finite extension of $\widehat{\mathbf{k}}$ such that f has a root α in $\mathbf{l}$. For every $\varepsilon > 0$, if there are infinitely many n such that $\alpha_{n,i} \notin B(\alpha, \varepsilon)$ for all $1 \le i \le d$, then we have $|f_n(\alpha)| \ge \varepsilon^d$ for infinitely many n, contradicting the fact that $f_n(\alpha) \to f(\alpha) = 0$. Thus for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \ge N$, there exists $1 \le i \le d$ with $\alpha_{n,i} \in B(\alpha, \varepsilon)$. That is, we can find a sequence $\alpha_{n,i_n} \in \mathbf{k}$ converging to α. Since $\widehat{\mathbf{k}}$ is complete, we have $\alpha \in \widehat{\mathbf{k}}$. □

5 Analytic functions

Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation. The following example shows that continuous or differentiable functions over $\mathbf{k}$ may behave very worse than over archimedean fields. As a substitute, we will focus on convergent power series on a closed polydisc over $\mathbf{k}$.

Example 5.1. Let $\mathbf{k}$ be a non-archimedean field with non-trivial valuation. Then there exists a function $f : \mathbf{k} \to \mathbf{k}$ that is differentiable everywhere with $f'(x) = 0$ for all $x \in \mathbf{k}$, but f is not locally constant.

Fix $r \in (0,1)$. Consider a descending sequence of open ball $\{B(0,r^n)\}$ and $a_n \in \mathbf{k}$ with $\|a_n\| = r^{2n}$. Define

$$f : \mathbf{k} \to \mathbf{k}, \quad x \mapsto \begin{cases} a_n, & x \in B(0,r^n) \setminus B(0,r^{n+1}) \\ 0, & x = 0 \end{cases}$$

Then

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{n \to \infty} \frac{a_n - 0}{x_n - 0}$$

for any sequence $x_n \to 0$ with $x_n \in B(0,r^n) \setminus B(0,r^{n+1})$. Since $\|x_n\| \geq r^{n+1}$, we have

$$\left\| \frac{a_n}{x_n} \right\| \leq \frac{r^{2n}}{r^{n+1}} = r^{n-1} \to 0$$

as $n \to \infty$. Thus $f'(0) = 0$ and then $f'(x) = 0$ for all $x \in \mathbf{k}$. However, f is not locally constant near 0.

5.1 Tate algebras

Notation 5.2. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \dots, n$, we have $\alpha_i \leq \beta_i$;
- $E(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \dots, n\}$ and $B(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \dots, n\}$ for $x = (x_1, \dots, x_n) \in \mathbf{k}^n$;
- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

Definition 5.3. Let $\mathbf{k}$ be a complete non-archimedean field. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$\mathbf{k}\langle \underline{r^{-1}T} \rangle := \mathbf{k}\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 5.4. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ is a non-archimedean multiplicative banach $\mathbf{k}$-algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Yang: For the definition of banach ring, see

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha$ are two elements in $\mathbf{k}\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \le \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{\frac{\varepsilon}{\|g\|}\|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|}\|a_\alpha\| r^\alpha\right\} \le \varepsilon.$$

Hence $f \cdot g \in \mathbf{k}\{\underline{T/r}\}$ and it shows that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $\mathbf{k}\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f+g\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \le \sup_{\alpha\in\mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \le \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned}\|f\cdot g\| &= \left\|\sum_{\gamma\in\mathbb{N}^n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) T^\gamma\right\| = \sup_{\gamma\in\mathbb{N}^n}\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \\ &\le \sup_{\gamma\in\mathbb{N}^n}\max_{\alpha+\beta=\gamma} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \le \|f\|\cdot\|g\|.\end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\|\cdot\|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \left\|\sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta\right\| r^{\alpha_1+\beta_1} \le \|f\cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by Proposition 2.2. Thus the gauss norm is multiplicative.

Step 4. Finally show that $\mathbf{k}\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $\mathbf{k}\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \le \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in $\mathbf{k}$. Since $\mathbf{k}$ is complete, set $a_\alpha := \lim_{m\to+\infty} a_{\alpha,m}$ and $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \le \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \le 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \leq \varepsilon$$

for all $m > N$. □

Proposition 5.5. Let $\mathbf{k}$ be a complete non-archimedean field. An element $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/r\}$ is invertible if and only if $\|a_0\| > \|a_\alpha\| r^\alpha$ for all $\alpha \neq 0$.

Proof. Multiplying by a_0^{-1}, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $\mathbf{k}[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = - \sum_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\| r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for m. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\| r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\{A^{m+1}, A \cdot A^m\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in \mathbf{k}\{\underline{T}/r\}$. □

Let $\mathbf{k}$ be a complete non-archimedean field. Recall that a derivative operator $\partial : \mathbf{k}\{\underline{T}/r\} \to \mathbf{k}\{\underline{T}/r\}$ is defined as the $\mathbf{k}$-linear map such that for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we have Yang: To be revised.

Proposition 5.6. Let $\mathbf{k}$ be a complete non-archimedean field, and ∂ be a derivative operator on $\mathbf{k}\{\underline{T}/r\}$. Then for every $f \in \mathbf{k}\{\underline{T}/r\}$, we have $\partial(f) \in \mathbf{k}\{\underline{T}/r\}$.

Proof. Yang: We only need to check the case $\partial = \partial/\partial T_1$. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T}/r\}$.

We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha \in \mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1 - 1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}.$$

Noting that $\mathbf{k}$ is non-archimedean, we have $\|\alpha_1 a_\alpha\| \leq \|a_\alpha\|$. Then

$$\lim_{|\alpha| \to +\infty} \|\alpha_1 a_\alpha\| r_1^{\alpha_1 - 1} r_2^{\alpha_2} \cdots r_n^{\alpha_n} \leq \frac{1}{r_1} \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0.$$

The conclusion follows. □

5.2 Analytic functions on closed polydiscs

Proposition 5.7. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we can associate a function $F_f : E(0, \underline{r}) \to \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \quad \text{for } x = (x_1, \ldots, x_n) \in E(0, \underline{r}).$$

This defines a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T/r}\}$ to the ring of all functions from $E(0, \underline{r})$ to $\mathbf{k}$.

Proof. Given $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \ldots, x_n) \in E(0, \underline{r})$, we have

$$\left\| \sum_{|\alpha| = n} a_\alpha x^\alpha \right\| \leq \max_{|\alpha| = n} \|a_\alpha\| r^\alpha \to 0 \quad \text{as } n \to +\infty.$$

Hence by Proposition 2.6, the series $F_f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$ converges in $\mathbf{k}$. This defines a function $F_f : E(0, \underline{r}) \to \mathbf{k}$.

Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. Set

$$A_n = \sum_{|\alpha| < n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta| < n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma| < n} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) x^\gamma.$$

We need to show that $F_f(x) F_g(x) = \lim A_n B_n = \lim C_n = F_{fg}(x)$. Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha| < n, |\beta| < n \\ |\alpha + \beta| \geq n}} a_\alpha b_\beta x^{\alpha + \beta}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon / \|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon / \|f\|$. For any $n > 2N$, we have

$$\|A_n B_n - C_n\| \leq \max_{\substack{|\alpha| < n, |\beta| < n \\ |\alpha + \beta| \geq n}} \|a_\alpha\| \|b_\beta\| \|x^{\alpha + \beta}\| < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Thus $F_f(x) F_g(x) = (F_{fg})(x)$. The addition and scalar multiplication can be verified directly. We thus finish the proof. □

Proposition 5.8. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x, y \in E(0, \underline{r})$, we have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq L \cdot \|y - x\|_{\infty},$$

where $L = \max_{1 \leq i \leq n} \|f\|_g / r_i$.

Proof. Set $y - x = (h_1, \dots, h_n)$ and $x^{(0)} = x$, $x^{(i)} = (x_1 + h_1, \dots, x_i + h_i, x_{i+1}, \dots, x_n)$ for $i = 1, \dots, n$. We have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{1 \leq i \leq n} \|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}}.$$

We only need to show that for every $i = 1, \dots, n$, we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}} \leq \frac{\|f\|_g}{r_i} \|h_i\|.$$

Without loss of generality and for simplicity, we assume that $y = (x_1 + h, x_2, \dots, x_n)$ and $x = (x_1, x_2, \dots, x_n)$. Note that by the strong triangle inequality, we have $\|h\| \leq r_1$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\begin{aligned} f(y) - f(x) &= \sum_{\alpha \in \mathbb{N}^n} a_\alpha \left((x_1 + h)^{\alpha_1} - x_1^{\alpha_1}\right) x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ &= \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k. \end{aligned}$$

Note that

$$\left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| r_1^k \leq \|a_\alpha\| r^\alpha \leq \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{\alpha \in \mathbb{N}^n} \max_{1 \leq k \leq \alpha_1} \left\{ \left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| \|h\|^k \right\} \leq \max_k \left\{ \|f\|_g \left(\frac{\|h\|}{r_1} \right)^k \right\} \leq \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

Lemma 5.9. Let $\mathbf{k}$ be a complete non-archimedean field. Then we have $\|f(x)\| \leq \|f\|$ for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$. In particular, if $f_n \to f$ as $n \to +\infty$ in $\mathbf{k}\{\underline{T/r}\}$, then we have $\|f_n(x) - f(x)\| \to 0$ for every $x \in E(0, \underline{r})$.

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \dots, x_n) \in E(0, \underline{r})$. We have

$$\left\| \sum_{|\alpha| < N} a_\alpha x^\alpha \right\| \leq \max_{|\alpha| < N} \|a_\alpha\| r^\alpha \leq \|f\|$$

for every $N \in \mathbb{N}$. Taking $N \to +\infty$, we have $\|f(x)\| \leq \|f\|$. □

Proposition 5.10. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation, and $\partial_i = \partial / \partial T_i$ be the derivative operator on $\mathbf{k}\{\underline{T/r}\}$ with respect to the indeterminate T_i for $i = 1, \dots, n$.

Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$, we have

$$F_{\partial_i(f)}(x) = \lim_{h\to 0} \frac{F_f(x_1, \dots, x_i + h, \dots, x_n) - F_f(x)}{h}.$$

Proof. Without loss of generality, we can assume that $i = 1$. Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $f_n = \sum_{|\alpha|<n} a_\alpha T^\alpha$ for $n \in \mathbb{N}$. Set $x_h = (x_1 + h, x_2 \dots, x_n)$ and $L_f(h) = (F_f(x_h) - F_f(x))/h$ for $h \in \mathbf{k}^\times$. Note that for fixed h, we have $\lim_{n\to\infty} L_{f_n}(h) = L_f(h)$.

We compute $L_{f_n}(h) - F_{\partial f_n}(x)$ explicitly:

$$\begin{aligned} L_{f_n}(h) - F_{\partial f_n}(x) &= \frac{1}{h}\left(\sum_{|\alpha|<n}\sum_{k=1}^{\alpha_1}\binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} h^k x_2^{\alpha_2}\cdots x_n^{\alpha_n} - \sum_{|\alpha|<n} \alpha_1 a_\alpha x_1^{\alpha_1-1} h x_2^{\alpha_2}\cdots x_n^{\alpha_n}\right) \\ &= \sum_{|\alpha|<n}\sum_{k=2}^{\alpha_1}\binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2}\cdots x_n^{\alpha_n} h^{k-1}. \end{aligned}$$

Note that

$$M = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha x_1^{\alpha_1-k} x_2^{\alpha_2}\cdots x_n^{\alpha_n}\| r_1^{k-1} \le \|f\|/r_1 < +\infty.$$

Hence

$$\|L_{f_n}(h) - F_{\partial f_n}(x)\| \le \max_{2\le k\le n}\left\{M\frac{\|h\|^{k-1}}{r_1^{k-1}}\right\} \le M\frac{\|h\|}{r_1}$$

for $h \in \mathbf{k}^\times$ with $\|h\| < r_1$. Taking $n \to +\infty$, we have

$$\|L_f(h) - F_{\partial f}(x)\| \le M\frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

Corollary 5.11. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment $f \mapsto F_f$ in Proposition 5.7 is injective.

Proof. Note that if $F_f = 0$, then for every $i = 1, \dots, n$, we have $F_{\partial_i(f)} = 0$ by Proposition 5.10. By taking repeated derivatives, we have $F_{\partial^\alpha f} = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Note that $F_{\partial^\alpha f}(0) = \alpha! a_\alpha$. It follows that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n$ and thus $f = 0$. □

Remark 5.12. Corollary 5.11 holds for non-archimedean fields of positive characteristic as well. The proof uses Theorem 5.16 and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element $f \in \mathbf{k}\{\underline{T/r}\}$ with the associated function $F_f : E(0, \underline{r}) \to \mathbf{k}$ as in Proposition 5.7.

Proposition 5.13. Let $\Bbbk$ be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra $\Bbbk\{\underline{T/r}\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x\in E(0,\underline{r})} \|f(x)\|_{\Bbbk}.$$

Proof. Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \Bbbk\{\underline{T/r}\}$. We write $f = g+h$ with $g = \sum_{\alpha\in S} a_\alpha T^\alpha$ and $h = \sum_{\alpha\notin S} a_\alpha T^\alpha$, where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that S is a non-empty finite set and $\|h\| < \|f\|$. By Lemma 5.9, we have $\|h(x)\| < \|f\|$ for every $x \in E(0,\underline{r})$. It suffices to show that $\|g\|_{\sup} = \|g\|$.

Since $\Bbbk$ is algebraically closed, $|\Bbbk^\times|$ is dense in $\mathbb{R}_{>0}$. For every pair $\alpha,\beta \in S$ with $\alpha \neq \beta$, the set $\{t \in \mathbb{R}^n_{>0} : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$ is a proper closed subset of $\mathbb{R}^n_{>0}$. Thus we can find $t_m \in |\Bbbk^\times|^n$ such that $t_m < r$, $t_m \to r$ as $m \to +\infty$ and for every $\alpha,\beta \in S$ with $\alpha \neq \beta$, we have $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$ for all m. For each m, we can find $x_m \in E(0,\underline{r})$ such that $\|x_m^\alpha\| = t_m^\alpha$ for every $\alpha \in S$ since $t_m \in |\Bbbk^\times|^n$. It follows that

$$\|g(x_m)\| = \max_{\alpha\in S} \|a_\alpha\|\|x_m^\alpha\| = \max_{\alpha\in S} \|a_\alpha\| t_m^\alpha \to \|g\| \quad \text{as } m \to +\infty.$$

Thus $\|g\|_{\sup} = \|g\|$. □

Remark 5.14. If $\mathbf{k}$ is not algebraically closed, the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ may not coincide with the supremum norm. For example, consider the Tate algebra $\mathbb{Q}_p\{T\}$. The element $f = T^p - T$ has gauss norm $\|f\| = 1$. However, for every $x \in E(0,1) = \mathbb{Z}_p$, we have $f(x) = x^p - x \equiv 0 \mod p$. Thus $\|f(x)\|_p \leq 1/p$ and $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$.

Remark 5.15. Recall the Weierstrass-Stone theorem in classical analysis which states that the closure of the polynomial ring $\mathbb{C}[T_1,\dots,T_n]$ with respect to the supremum norm on a closed polydisc $E \subset \mathbb{C}^n$ is the ring of all complex-valued continuous functions on E.

In the context of non-archimedean analysis, Proposition 5.13 can be viewed as an analogue of this theorem. It states that the closure of the polynomial ring $\Bbbk[T_1,\dots,T_n]$ with respect to the supremum norm on a closed polydisc $E(0,\underline{r}) \subset \Bbbk^n$ is the Tate algebra $\Bbbk\{\underline{T/r}\}$.

From this perspective, the Tate algebra can be viewed as the "correct" analogue of the ring of continuous functions on a closed polydisc in non-archimedean analysis.

Theorem 5.16 (Strassman)**.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$ be an analytic function. Suppose that $\|a_N\| r^N > \|a_n\| r^n$ for all $n > N$. Then f has at most N zeros in the closed ball $E(0,r)$.

Proof. We induct on N. The case $N = 0$ is direct from Proposition 5.5. Suppose that the conclusion holds for $N-1$. Let x be a zero of f in $E(0,r)$. Set

$$g(T) = \frac{f(T) - f(x)}{T - x} = \sum_{k=0}^{+\infty} \left(\sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n\geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n\geq k+1} \|a_n\| r^{n-1} \to 0 \quad \text{as } k \to \infty.$$

It follows that $g(T) \in \mathbf{k}\{T/r\}$.

For every $n > N$, we have

$$\|a_N\| > \|a_n\| r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\| \sum_{n=N}^{N+m} a_n x^{n-N} \right\| = \|a_N\|$$

for every $m \in \mathbb{N}$ by Proposition 2.2. Take $m \to +\infty$, we have $\|b_{N-1}\| = \|a_N\|$. For every $k > N-1$, we have

$$\|b_k\| r^k = \max_{n \geq k+1} \|a_n\| r^{n-1} \leq \max_{n > N} \|a_n\| r^{n-1} < \|a_N\| r^{N-1} = \|b_{N-1}\| r^{N-1}.$$

By the induction hypothesis, g has at most $N-1$ zeros in $E(0,r)$. It follows that f has at most N zeros in $E(0,r)$ since $f(T) = (T-x) \cdot g(T)$. □

6 Example: p-adic fields

6.1 p-adic fields

Construction 6.1. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_\mathfrak{p}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_\mathfrak{p} : K^\times \to \mathbb{Z}$ the corresponding discrete valuation. The *$\mathfrak{p}$-adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_\mathfrak{p} := N(\mathfrak{p})^{-v_\mathfrak{p}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the $\mathfrak{p}$-adic absolute value $|\cdot|_\mathfrak{p}$ is denoted by $K_\mathfrak{p}$, called the *$\mathfrak{p}$-adic field*.

One can just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p.

Example 6.2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Proposition 6.3. The multiplicative group $\mathbb{Q}_p^\times$ of the p-adic field $\mathbb{Q}_p$ admits the following decomposition:

$$\mathbb{Q}_p^\times \cong p^\mathbb{Z} \times \mathbb{Z}_p^\times,$$

where $p^{\mathbb{Z}} := \{p^n \mid n \in \mathbb{Z}\}$ and $\mathbb{Z}_p^\times := \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$ is the group of units of the ring of p-adic integers $\mathbb{Z}_p$. Yang: To be checked.

Yang: What is the relation between the finite extension of $\mathbb{Q}_p$ and $K_\mathfrak{p}$?

6.2 Completion

Proposition 6.4. The algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is not complete with respect to the extension of the p-adic absolute value $|\cdot|_p$.

Construction 6.5. Let p be a prime number. The field $\mathbb{C}_p$ of *p-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$. The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$. Yang: To be completed.

Proposition 6.6. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Construction 6.7. Let p be a prime number. Yang: We construct the *spherically complete p-adic field* Ω_p. Yang: To be completed.

6.3 Elementary functions

Exponential, logarithmic, and the interpolation functions.

7 Appendix

Theorem 7.1. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

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