Finite field extensions

1 Finite-dimensional vector space

Definition 1. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|ax\| = |a| \cdot \|x\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Example 2. Let $\mathbf{k}$ be a valuation field and V a finite-dimensional vector space over $\mathbf{k}$ with basis $\{e_1, e_2, \dots, e_n\}$. For any $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$, define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then $\|\cdot\|_{\max}$ is a norm on V, called the *maximal norm* with respect to the basis $\{e_1, e_2, \dots, e_n\}$.

Example 3. Setting as in Example 2, for any $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n \in V$, define

$$\|x\|_1 := |a_1| + |a_2| + \cdots + |a_n|.$$

Then $\|\cdot\|_1$ is also a norm on V.

Definition 4. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Lemma 5. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V.

Proof. The sufficiency is clear. Now suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on V. Hence the unit open ball with respect to $\|\cdot\|_1$ contains a unit open ball with respect to $\|\cdot\|_2$. That is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every $x \in V$ with $\|x\|_1 = 1$, we have $\|x\|_2 \geq C = C\|x\|_1$. By scaling, we get that for every $x \in V$,

$$\|x\|_2 \geq C\|x\|_1.$$

Similar for the other direction, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Proposition 6. Let V be a normed finite-dimensional vector space over a complete valuation field $\mathbf{k}$. Then V is complete.

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Proof. Yang: To be added. □

Theorem 7. Let V be a finite-dimensional vector space over a complete field $\mathbf{k}$. Then all norms on V are equivalent.

Proof. Fix a basis $\{e_1, e_2, \ldots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis as in Example 2. Let $\|\cdot\|$ be any norm on V. It suffices to show that $\|\cdot\|$ and $\|\cdot\|_{\max}$ are equivalent. First we have

$$\|y\| \leq \sum_{i=1}^{n} |a_i| \|e_i\| \leq \left(\sum_{i=1}^{n} \|e_i\| \right) \|y\|_{\max}$$

for any $y = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \in V$. It remains to show that there exists a constant $C > 0$ such that for any $y \in V$,

$$\|y\|_{\max} \leq C \|y\|.$$

Yang: To be added. □

Remark 8. If the base field $\mathbf{k}$ is not complete, then Theorem 7 may fail. For example, let $\mathbf{k} = \mathbb{Q}$ with the usual absolute value, and let $V = \mathbb{Q}[\alpha]$ with $\alpha^2 - \alpha - 1 = 0$. There are two embeddings of V into $\mathbb{R}$:

$$\iota_1 : a + b\alpha \mapsto a + b\frac{1+\sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b\frac{1-\sqrt{5}}{2}.$$

Define two norms on V by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where $|\cdot|$ is the usual absolute value on $\mathbb{R}$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent since $\iota_2(\alpha^n) \to 0$ as $n \to \infty$ while $\iota_1(\alpha^n) \to \infty$.

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

Lemma 9. Let $\mathbf{k}$ be a complete field and V a normed finite-dimensional vector space over $\mathbf{k}$. Then

$$\|\cdot\| : \mathrm{End}_{\mathbf{k}}(V) \to \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \mathrm{End}_{\mathbf{k}}(V).$$

Proof. First we show the existence of the supremum, i.e., there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\| \leq C\|x\|$. Fix a basis $\{e_1, e_2, \ldots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis. Since all norms on V are bounded by each other by Theorem 7, we only need to show that there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\|_1 \leq C\|x\|_{\max}$. Write $T(e_i) = \sum_{j=1}^{n} a_{ij} e_j$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^{n} x_i e_i \in V$, we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^{n} \left| \sum_{i=1}^{n} a_{ij} x_i \right| \leq \left(\sum_{1 \leq i,j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of $\|\cdot\|$ are clear. It remains to show the triangle inequality and sub-multiplicativity. For any $A, B \in \mathrm{End}_{\mathbf{k}}(V)$, we have

$$\frac{\|(A+B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all $x \in V \setminus \{0\}$ gives $\|A+B\| \leq \|A\| + \|B\|$. We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$. Taking supremum we get $\|AB\| \leq \|A\| \cdot \|B\|$. □

2 Finite field extensions

Lemma 10. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then there exists an absolute value on $\mathbf{l}$ extending the absolute value on $\mathbf{k}$.

Proof. Fix a norm $\|\cdot\|_V$ on the $\mathbf{k}$-vector space $V = \mathbf{l}$. The norm $\|\cdot\|_V$ induces an operator norm $\|\cdot\|_{\mathrm{op}}$ on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ as in Lemma 9. For any $a \in \mathbf{l}$, let $\mu_a \in \mathrm{End}_{\mathbf{k}}(V)$ be the $\mathbf{k}$-linear map defined by multiplication by a. Note that $a \mapsto \mu_a$ gives a embedding of $\mathbf{k}$-algebras and if $a \in \mathbf{k}$, $\|\mu_a\|_{\mathrm{op}} = \|a\|_{\mathbf{k}}$. Thus the restriction of $\|\cdot\|_{\mathrm{op}}$ to $\mathbf{l}$ gives an norm on $\mathbf{l}$ extending that on $\mathbf{k}$. The normed ring $(\mathbf{l}, \|\cdot\|_{\mathrm{op}})$ is a Banach ring since it is a finite-dimensional vector space over the complete field $\mathbf{k}$. By Theorem 13, there exists a multiplicative seminorm $\|\cdot\|_{\mathbf{l}}$ on $\mathbf{l}$ bounded by $\|\cdot\|_{\mathrm{op}}$. In particular, $\|\cdot\|_{\mathbf{l}}$ is bounded by $\|\cdot\|_{\mathbf{k}}$ on $\mathbf{k}$. On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus $\|\cdot\|_{\mathbf{l}}$ extends the absolute value on $\mathbf{k}$. □

Theorem 11. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the absolute value on $\mathbf{l}$ which extends the absolute value on $\mathbf{k}$ is uniquely determined by the absolute value on $\mathbf{k}$. Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where $n = [\mathbf{l} : \mathbf{k}]$ and $N_{\mathbf{l}/\mathbf{k}}$ is the norm map from $\mathbf{l}$ to $\mathbf{k}$.

Proof. Let $\|\cdot\|_{\mathbf{l}}$ be arbitrary absolute value on $\mathbf{l}$ extending that on $\mathbf{k}$. We will show that $\|\cdot\|_{\mathbf{l}}$ must be equal to $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$. For any $a \in \mathbf{l}$, set $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$. Then $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that $\|b\|_{\mathbf{l}} = 1$ whenever $N_{\mathbf{l}/\mathbf{k}}(b) = 1$.

Note that the norm map $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \to \mathbf{k}$ is the determinant of the $\mathbf{k}$-linear map $\mu_b \in \mathrm{End}_{\mathbf{k}}(V)$ defined by multiplication by b. Hence it is continuous on $\mathbf{l}$ (since it is a polynomial in the entries of the matrix representation). If $\|b\|_{\mathbf{l}} < 1$, then $\|b^m\|_{\mathbf{l}} \to 0$ as $m \to \infty$. Thus $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \to 0$ as $m \to \infty$, contradicting the fact that $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$ for all m. Similarly, if $\|b\|_{\mathbf{l}} > 1$, then just consider b^{-1}.

□

Proposition 12. Let $\mathbf{k}$ be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

Proof. Let $f \in \widehat{\mathbf{k}}[X]$ be a non-constant polynomial. We will show that f has a root in $\widehat{\mathbf{k}}$. Take a sequence of polynomials $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbf{k}[X]$ converging to f coefficient-wisely. Since $\mathbf{k}$ is algebraically closed, each f_n splits completely in $\mathbf{k}$ and hence in $\widehat{\mathbf{k}}$. Write $f_n(X) = \prod_{i=1}^{d}(X - \alpha_{n,i})$ with $\alpha_{n,i} \in \widehat{\mathbf{k}}$.

Let $\mathbf{l}$ be a finite extension of $\widehat{\mathbf{k}}$ such that f has a root α in $\mathbf{l}$. For every $\varepsilon > 0$, if there are infinitely many n such that $\alpha_{n,i} \notin B(\alpha, \varepsilon)$ for all $1 \leq i \leq d$, then we have $|f_n(\alpha)| \geq \varepsilon^d$ for infinitely many n, contradicting the fact that $f_n(\alpha) \to f(\alpha) = 0$. Thus for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$, there exists $1 \leq i \leq d$ with $\alpha_{n,i} \in B(\alpha, \varepsilon)$. That is, we can find a sequence $\alpha_{n,i_n} \in \mathbf{k}$ converging to α. Since $\widehat{\mathbf{k}}$ is complete, we have $\alpha \in \widehat{\mathbf{k}}$. □

Appendix

Theorem 13. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.