Example: p-adic fields

1 p-adic fields

Construction 1. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_{\mathfrak{p}}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_{\mathfrak{p}} : K^\times \to \mathbb{Z}$ the corresponding discrete valuation. The $\mathfrak{p}$-*adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the $\mathfrak{p}$-adic absolute value $|\cdot|_{\mathfrak{p}}$ is denoted by $K_{\mathfrak{p}}$, called the $\mathfrak{p}$-*adic field*.

We just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p in the following.

Example 2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The p-*adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Unlike the field of real numbers $\mathbb{R}$, the p-adic field $\mathbb{Q}_p$ has many finite extensions.

Proposition 3. There are infinitely many irreducible polynomials over the p-adic field $\mathbb{Q}_p$.

Proof. Since there are infinitely many irreducible monic polynomials over the finite field $\mathbb{F}_p$, consider any lift of such an irreducible monic polynomial to a monic polynomial with coefficients in $\mathbb{Z}_p$. If the lift is not irreducible over $\mathbb{Q}_p$, then the factorization of the lift gives a nontrivial factorization of its reduction modulo p since the factors can be chosen to be monic and have coefficients in $\mathbb{Z}_p$, which contradicts the irreducibility of the original polynomial over $\mathbb{F}_p$. Thus, the lift is irreducible over $\mathbb{Q}_p$.

On the other hand, note that $|\mathbb{Q}_p^\times|_p = p^{\mathbb{Z}}$. It follows that $f(T) = T^n - p$ is irreducible over $\mathbb{Q}_p$ for every integer $n \geq 1$. Otherwise, suppose we have a monic factorization $f(T) = g(T)h(T)$ with $g(T), h(T) \in \mathbb{Z}_p[T]$ and $\deg g, \deg h < n$. Then by considering the reduction modulo p, we have $g(0), h(0) \equiv 0 \mod p$. It follows that $|f(0)|_p = |g(0)h(0)|_p \leq p^{-2}$, which contradicts $|f(0)|_p = |p|_p = p^{-1}$. $\square$

Date: November 12, 2025, **Author**: Tianle Yang, My Homepage

2 Completion

Proposition 4. The algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is not complete with respect to the extension of the p-adic absolute value $|\cdot|_p$.

Proof. Yang: To be completed. □

Construction 5. Let p be a prime number. The *field* $\mathbb{C}_p$ *of* p*-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$.

The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$ by Proposition 9. By Corollaries 10 and 11, we have

$$|\mathbb{C}_p^\times|_p = |\overline{\mathbb{Q}_p}^\times|_p = p^{\mathbb{Q}}, \quad \Bbbk_{\mathbb{C}_p} \cong \Bbbk_{\overline{\mathbb{Q}_p}} \cong \overline{\mathbb{F}_p}.$$

Proposition 6. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Proof. Yang: To be completed. □

Construction 7. Let p be a prime number. Yang: We construct the *spherically complete* p*-adic field* Ω_p. Yang: To be completed.

3 Elementary functions

Yang: Exponential, logarithmic, and the interpolation functions.

Fix a prime number p in the following and consider $\mathbf{k} = \mathbb{Q}_p, \mathbb{C}_p$, or Ω_p. Let $r_p := p^{-1/(p-1)}$.

Construction 8. The *exponential function* $\exp : \mathbf{k} \to \mathbf{k}$ is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The radius of convergence of $\exp(x)$ is $+\infty$ if $p = 2$ and $p^{-1/(p-1)}$ if $p > 2$.

The *logarithmic function* $\log : 1 + \mathbf{k}^{\circ\circ} \to \mathbf{k}$ is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

The radius of convergence of $\log(1+x)$ is 1.

Moreover, for every x in the domain of convergence of exp and every y in the domain of convergence of log, we have

$$\log(\exp(x)) = x, \quad \exp(\log(y)) = y.$$

Yang: To be checked.

Appendix

Proposition 9. Let $\mathbf{k}$ be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

Corollary 10. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the residue field $\mathscr{k}_{\widehat{\mathbf{k}}} \cong \mathscr{k}_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^{\circ} \hookrightarrow \widehat{\mathbf{k}}^{\circ}$.

Corollary 11. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the valuation group $|\widehat{\mathbf{k}}^{\times}|$ of $\widehat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^{\times}|$ of $\mathbf{k}$.