

Algebraic structures of non-archimedean fields

1 Recover non-archimedean complete fields algebraically

In this subsection, let \mathbf{k} be a non-archimedean field. Set $I_{r,<} := B(0, r)$ and $I_{r,\leq} := E(0, r)$ for each $r \in [0, 1]$.

Proposition 1. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers \mathbf{k}° . Conversely, any ideal of \mathbf{k}° is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0, 1)$.

Proof. Let I be an ideal of \mathbf{k}° . Set $r = \sup\{|a| : a \in I\}$ (resp. $r = \max\{|a| : a \in I\}$ when the maximum exists). Then, by definition, we have $I \subset I_{r,<}$ (resp. $I \subset I_{r,\leq}$). For every $x \in \mathbf{k}^\circ$ with $|x| < r$ (resp. $|x| \leq r$), there exists $a \in I$ such that $|x| \leq |a|$. Thus, $|x/a| \leq 1$ and so $x/a \in \mathbf{k}^\circ$. Since I is an ideal, we have $x = (x/a)a \in I$. Therefore, $I_{r,<} \subset I$ (resp. $I_{r,\leq} \subset I$). \square

Proposition 2. Let I_r be either $I_{r,<}$ or $I_{r,\leq}$ for each $r \in (0, 1)$. Suppose $\{r_n \in (0, 1)\}_{n \in \mathbb{N}}$ is a decreasing sequence converging to 0. Then the completion $\hat{\mathbf{k}}$ of \mathbf{k} is isomorphic to the projective limit

$$\hat{\mathbf{k}}^\circ \cong \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}.$$

Proof. For every $x \in \hat{\mathbf{k}}^\circ$, there exists a cauchy sequence $\{x_m\}_{m \in \mathbb{N}}$ in \mathbf{k}° converging to x . Since $\{r_n\}_{n \in \mathbb{N}}$ converges to 0, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that for all $m, m' \geq M_n$, we have $|x_m - x_{m'}| < r_n$. Thus, the sequence $\{x_m + I_{r_n}\}_{m \in \mathbb{N}}$ is eventually constant in $\mathbf{k}^\circ / I_{r_n}$. Define a map

$$\Phi : \hat{\mathbf{k}}^\circ \rightarrow \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}, \quad x \mapsto \left(\lim_{m \rightarrow \infty} x_m + I_{r_n} \right)_{n \in \mathbb{N}}.$$

It is straightforward to verify that Φ is a well-defined ring homomorphism.

Conversely, for every $(a_n + I_{r_n})_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}$, we can choose a representative $a_n \in \mathbf{k}^\circ$ for each n . We claim that the sequence $\{a_n\}_{n \in \mathbb{N}}$ is a cauchy sequence in \mathbf{k}° . Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_N < \varepsilon$. For all $m, n \geq N$, since $a_n + I_{r_n}$ maps to $a_m + I_{r_m}$ under the natural projection, we have $|a_n - a_m| < r_N < \varepsilon$. Thus, $\{a_n\}_{n \in \mathbb{N}}$ converges to some $x \in \hat{\mathbf{k}}^\circ$. Easily see that the limit x is independent of the choice of representatives $\{a_n\}_{n \in \mathbb{N}}$. This gives a map

$$\Psi : \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n} \rightarrow \hat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow \infty} a_n.$$

Direct verification shows that $\Psi = \Phi^{-1}$. \square

Corollary 3. Let \mathbf{k} be a non-archimedean field and $\hat{\mathbf{k}}$ its completion. Then the residue field $\hat{\kappa}_{\hat{\mathbf{k}}} \cong \kappa_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^\circ \hookrightarrow \hat{\mathbf{k}}^\circ$.

Corollary 4. Let \mathbf{k} be a non-archimedean field and $\hat{\mathbf{k}}$ its completion. Then the valuation group $|\hat{\mathbf{k}}^\times|$ of $\hat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^\times|$ of \mathbf{k} .

Proof. Note that

$$\begin{aligned}
 r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\
 &\iff \widehat{\mathbf{k}}^\circ/I_{r,<} \rightarrow \widehat{\mathbf{k}}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\
 &\iff \mathbf{k}^\circ/I_{r,<} \rightarrow \mathbf{k}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\
 &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\
 &\iff r \in |\mathbf{k}^\times|.
 \end{aligned}$$

□

Proposition 5. Let \mathbf{k} be a non-archimedean field. Then \mathbf{k}° is totally bounded iff $\mathbf{k}^\circ/I_{r,<}$ and $\mathbf{k}^\circ/I_{r,\leq}$ are finite for each $r \in [0, 1]$. Moreover, if \mathbf{k} is complete, then it is locally compact iff \mathbf{k}°/I_r is finite for each $r \in (0, 1)$.

Slogan “*Locally compact \iff pro-finite.*”

Proof. We just prove the case for $I_r = I_{r,<}$. The case for $I_r = I_{r,\leq}$ is similar.

Suppose that \mathbf{k}°/I_r is finite for each $r \in [0, 1]$. Then for every $\varepsilon > 0$, there exists $r \in (0, 1)$ such that $r < \varepsilon$ and \mathbf{k}°/I_r is finite. Let $\{a_1 + I_r, \dots, a_n + I_r\}$ be the complete set of representatives of \mathbf{k}°/I_r . Then the balls $B(a_i, r)$ for $i = 1, \dots, n$ cover \mathbf{k}° .

Conversely, suppose that \mathbf{k}°/I_r is infinite for some $r \in [0, 1]$. Then there exists an infinite set $\{a_n\}$ with $|a_n| \in [r, 1]$ such that their images in \mathbf{k}°/I_r are distinct. In particular, for every $m \neq n$, we have $|a_n - a_m| \geq r$. Any subsequence of $\{a_n\}$ is not Cauchy. Thus, \mathbf{k}° is not totally bounded. □

Proposition 6. The ring \mathbf{k}° is noetherian iff \mathbf{k} is a discrete valuation field.

Proof. Note that $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$ is a multiplicative subgroup. If \mathbf{k} is not a discrete valuation field, then $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. In particular, there exists a strictly ascending sequence $r_n \in |\mathbf{k}^\times| \cap (0, 1)$. Then the ideals $I_{r_n,\leq}$ form a strictly ascending chain of ideals in \mathbf{k}° .

The converse is standard since now \mathbf{k}° is a discrete valuation ring. □

Proposition 7. Let \mathbf{k} be a complete non-archimedean field. Then \mathbf{k} is locally compact iff \mathbf{k} is a discrete valuation field and its residue field $\mathcal{K}_{\mathbf{k}}$ is finite.

Proof. The necessity follows from Proposition 5. For the sufficiency, suppose that \mathbf{k} is a discrete valuation field whose residue field $\mathcal{K}_{\mathbf{k}}$ is finite. Let $\pi \in \mathbf{k}^\circ$ be a uniformizer. We only need to show that $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$ is finite for each $n \in \mathbb{N}$. Note that there is an isomorphism

$$\pi^{n-1} \mathbf{k}^\circ / \pi^n \mathbf{k}^\circ \cong \mathcal{K}_{\mathbf{k}}, \quad x + \pi^n \mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on n , we conclude that $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$ is finite. □

2 Hensel's Lemma

Theorem 8 (Hensel's lemma). Let \mathbf{k} be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathcal{K}_{\mathbf{k}}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \mathcal{K}_{\mathbf{k}}[T]$ are monic polynomials that are coprime in $\mathcal{K}_{\mathbf{k}}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\mathcal{K}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

Proof. Since $\gcd(g, h) = 1$ in $\mathcal{K}_{\mathbf{k}}[T]$, there exist polynomials $u(T), v(T) \in \mathcal{K}_{\mathbf{k}}[T]$ such that $ug + vh = 1$ and $\deg u < \deg h, \deg v < \deg g$. Choose lifts $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$ of $g(T), h(T), u(T), v(T)$ respectively preserving their degrees such that G_0 and H_0 are monic. Then there exist $r < 1$ such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \pmod{I_r}, \quad F(T) - G_0(T)H_0(T) \equiv 0 \pmod{I_r},$$

where $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$.

We will construct a sequence of monic polynomials $\{G_n(T)\}_{n \in \mathbb{N}}$ and $\{H_n(T)\}_{n \in \mathbb{N}}$ in $\mathbf{k}^\circ[T]$ such that for each $n \in \mathbb{N}$,

$$G_n(T) \equiv G_{n-1}(T) \pmod{I_{r^n}}, \quad H_n(T) \equiv H_{n-1}(T) \pmod{I_{r^n}},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \pmod{I_{r^{n+1}}}.$$

If we have such sequences, then their coefficients converge in the complete ring \mathbf{k}° . Let $G(T)$ and $H(T)$ be the limits of $\{G_n(T)\}$ and $\{H_n(T)\}$ respectively. Then we have $F(T) = G(T)H(T)$ and the reductions of $G(T), H(T)$ in $\mathcal{K}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

The case $n = 0$ is done by the above construction. Now suppose that we have constructed $G_n(T)$ and $H_n(T)$ for some $n \geq 0$. Since $G_n - G_0 \equiv 0 \pmod{I_r}$ and $H_n - H_0 \equiv 0 \pmod{I_r}$, we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \pmod{I_r}.$$

Set $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$ and $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$. Then we have

$$\begin{aligned} (G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T]. \end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

□

Corollary 9. Let \mathbf{k} be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathcal{K}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $a \in \mathcal{K}_{\mathbf{k}}$. Then there exists a root $\alpha \in \mathbf{k}^\circ$ of $F(T)$ whose reduction is a .

Proof. Since a is a simple root of $f(T)$, we have the factorization $f(T) = (T - a)h(T)$ for some monic polynomial $h(T) \in \mathcal{K}_{\mathbf{k}}[T]$ with $h(a) \neq 0$. Then the result follows from [Theorem 8](#). \square

3 Newton polygons

Yang: To be filled.

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