Valuation fields

1 Absolute values and completion

Definition 1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot|:\mathbf{k}\to\mathbb{R}_{\geq 0}$ satisfying the following properties for all $x,y\in\mathbf{k}$:

(a) $\|x\|=0$ if and only if $x=0$;

(b) $\|xy\|=\|x\|\cdot\|y\|$;

(c) $\|x+y\|\leq\|x\|+\|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 2. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v:\mathbf{k}^\times\to\mathbb{R}$ such that

- $\forall x,y\in\mathbf{k}^\times, v(xy)=v(x)+v(y)$;
- $\forall x,y\in\mathbf{k}^\times, v(x+y)\geq\min\{v(x),v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0)=+\infty$. Fix a real number $\varepsilon\in(0,1)$. Then v induces an absolute value $|\cdot|_v:\mathbf{k}\to\mathbb{R}_+$ defined by $|x|_v=\varepsilon^{v(x)}$ for each $x\in\mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Example 3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\|:=\begin{cases}0, & x=0;\\ 1, & x\neq 0.\end{cases}$$

Definition 4. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c\in(0,1)$ such that

$$\|x\|_1=\|x\|_2^c,\quad\forall x\in\mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$. Moreover, the following lemma shows that the converse is also true.

Lemma 5. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1=\{x\in\mathbf{k}:\|x\|_1<1\}$ and $D_2=\{x\in\mathbf{k}:\|x\|_2<1\}$ are the same.

Date: November 1, 2025, **Author**: Tianle Yang, My Homepage

Proof. The implications (a) ⇒ (b) is obvious. Now we prove (b) ⇒ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) ⇒ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. Suppose the contrary that $\alpha \neq \beta$. Consider the domain $\Omega \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Ω is non-empty. Pick $(n, m) \in \Omega$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. □

Definition 6. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 7. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Recall that

$$\widehat{\mathbf{k}} := \{\text{Cauchy sequences in } \mathbf{k}\}/\sim,$$

where $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} \|x_n - y_n\| = 0$. For any two elements $x = [(x_n)], y = [(y_n)] \in \widehat{\mathbf{k}}$, we set

$$x + y := [(x_n + y_n)], \quad xy := [(x_n y_n)], \quad -x := [(-x_n)], \quad x^{-1} := [(x_n^{-1})],$$

where the last one is defined only when $x \neq 0$. We have

$$\begin{aligned} \|(x_n + y_n) - (x_m + y_m)\| &\leq \|x_n - x_m\| + \|y_n - y_m\|; \\ \|-x_n - (-x_m)\| &= \|x_n - x_m\|; \\ \|x_n y_n - x_m y_m\| &\leq \|x_n\|\|y_n - y_m\| + \|y_m\|\|x_n - x_m\|; \\ \|x_n^{-1} - x_m^{-1}\| & \frac{1}{\|x_n\|\|x_m\|} (\|x_n - x_m\|). \end{aligned}$$

Hence

Yang: To be added. □

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 8. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 9. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

2 Non-archimedean fields

Definition 10. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 11. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathscr{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^\circ/\mathbf{k}^{\circ\circ}.$$

Yang: Is the valuation on residue field trivial?

Lemma 12. Recall that a metric space is *totally bounded* if for every $\varepsilon > 0$, it can be covered by finitely many balls of radius ε. A metric space is compact if and only if it is complete and totally bounded.

Proof. Yang: To be added. □

Definition 13. Let $\mathbf{k}$ be a non-archimedean field. The *residue absolute value* on the residue field $\mathscr{k}_{\mathbf{k}}$ is defined as

$$|x| := \inf_{y \in \varphi^{-1}(x)} \|y\|, \quad \forall x \in \mathscr{k}_{\mathbf{k}},$$

where $\varphi : \mathbf{k}^\circ \to \mathscr{k}_{\mathbf{k}}$ is the canonical projection.

Proposition 14. Let $\mathbf{k}$ be a non-archimedean field. Then the residue absolute value on the residue field $\mathscr{k}_{\mathbf{k}}$ is trivial.

Proof. For any $x \in \mathscr{k}_{\mathbf{k}}$, if $x = 0$, then by definition $|x| = 0$. If $x \neq 0$, then $\forall y \in \varphi^{-1}(x)$, we have $y \in \mathbf{k}^\circ \setminus \mathbf{k}^{\circ\circ}$, i.e., $\|y\| = 1$. Thus by definition $|x| = 1$. □

Proposition 15. Let $\mathbf{k}$ be a non-archimedean field. Set $I_r := \{x \in \mathbf{k} : \|x\| < r\}$ for each $r \in (0, 1)$. They are ideals of the ring of integers $\mathbf{k}^\circ$. Then we have

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{r>0} \mathbf{k}^\circ/I_r.$$

Yang: To be checked.

Slogan *Locally compact* $\iff$ *pro-finite.*

Proposition 16. Let $\mathbf{k}$ be a non-archimedean field. Then $\mathbf{k}$ is totally bounded iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$.

Proposition 17. $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field. and complete. Yang: To be revised.