Non-archimedean Analysis

No Cover Image

Use `\coverimage{filename}` to add an image

阿巴阿巴!

Contents

1 Valuation fields 1
1.1 Absolute values and completion . . . 1
1.2 Non-archimedean fields . . . 3
2 Ultra-metric spaces 4
3 Residue fields and reductions 5
3.1 Recover non-archimedean complete fields algebraically . . . 5
3.2 Hensel's Lemma . . . 6
3.3 Newton polygons . . . 6
4 Finite field extensions 7
4.1 Finite-dimensional vector space . . . 7
4.2 Finite field extensions . . . 7
5 Analytic functions 7
5.1 Failure of continuous and differentiable functions . . . 7
5.2 Power series . . . 7
5.3 Analytic functions . . . 8
6 Example: p-adic fields 8
6.1 p-adic fields . . . 8
6.2 Completion . . . 9

1 Valuation fields

1.1 Absolute values and completion

Definition 1.1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 1.2. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;

Date: November 2, 2025, **Author**: Tianle Yang, My Homepage

- $\forall x, y \in \mathbf{k}^\times, v(x+y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0,1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Example 1.3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 1.4. The *(multiplicative) valuation group* of a valuation field $(\mathbf{k}, \|\cdot\|)$ is defined as the subgroup of $\mathbb{R}_{>0}$ given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

Definition 1.5. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c \in (0,1)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$. Moreover, the following lemma shows that the converse is also true.

Lemma 1.6. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) $\Rightarrow$ (b) is obvious. Now we prove (b) $\Rightarrow$ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) $\Rightarrow$ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. Suppose the contrary that $\alpha \neq \beta$. Consider the domain $\Omega \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Ω is non-empty. Pick $(n,m) \in \Omega$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. $\square$

Definition 1.7. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x,y) := \|x-y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 1.8. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Simple analysis. □

Example 1.9. Let $|\cdot|_\infty$ be the usual absolute value on the field $\mathbb{Q}$ of rational numbers. Then $(\mathbb{Q}, |\cdot|_\infty)$ is a valuation field. Its completion is the field $\mathbb{R}$ of real numbers equipped with the usual absolute value.

Example 1.10. Let p be a prime number. For any non-zero rational number $x \in \mathbb{Q}$, we can write it as $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then $(\mathbb{Q}, |\cdot|_p)$ is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of p-adic numbers equipped with the p-adic absolute value; see Yang: to be added..

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 1.11. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 1.12. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

1.2 Non-archimedean fields

Definition 1.13. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x+y\| \le \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \le 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 1.14. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \le 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathscr{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^{\circ}/\mathbf{k}^{\circ\circ}.$$

Definition 1.15. Let $\mathbf{k}$ be a non-archimedean field. The *residue absolute value* on the residue field $\mathscr{k}_{\mathbf{k}}$ is defined as

$$|x| := \inf_{y \in \varphi^{-1}(x)} \|y\|, \quad \forall x \in \mathscr{k}_{\mathbf{k}},$$

where $\varphi : \mathbf{k}^{\circ} \to \mathscr{k}_{\mathbf{k}}$ is the canonical projection.

Proposition 1.16. Let $\mathbf{k}$ be a non-archimedean field. Then the residue absolute value on the residue field $\mathscr{k}_{\mathbf{k}}$ is trivial.

Proof. For any $x \in \mathscr{k}_{\mathbf{k}}$, if $x = 0$, then by definition $|x| = 0$. If $x \neq 0$, then $\forall y \in \varphi^{-1}(x)$, we have $y \in \mathbf{k}^{\circ} \setminus \mathbf{k}^{\circ\circ}$, i.e., $\|y\| = 1$. Thus by definition $|x| = 1$. $\square$

2 Ultra-metric spaces

We will use $B(x, r)$ (resp. $E(x, r)$) to denote the open ball (resp. closed ball) with center x and radius r.

Definition 2.1. A metric space (X, d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.$$

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x, y) := \|x - y\|$ on $\mathbf{k}$ makes $(\mathbf{k}, d)$ an ultra-metric space.

Proposition 2.2. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x, y) \geq d(y, z)$. By the strong triangle inequality, we have

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(x, y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} = \max\{d(x, z), d(y, z)\} \leq d(x, y).$$

This shows that $d(x, y) = \max\{d(x, z), d(y, z)\}$. Thus either $d(x, z) = d(x, y) \geq d(y, z)$ or $d(y, z) = d(x, y) \geq d(x, z)$. $\square$

Proposition 2.3. Let (X, d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1, 2$. If $D_1 \cap D_2 \neq \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1, 2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1, x_2) \leq \max\{d(x_1, y), d(y, x_2)\}.$$

Without loss of generality, we may assume that $d(x_1, x_2) \leq d(x_1, y)$. It follows that $x_2 \in D_1$ since $d(x_1, y) < r_1$ (or $\leq r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1, z) > d(x_1, x_2)$. Then by Proposition 2.2,

$$r_1 \leq d(x_1, z) = d(x_2, z) \leq r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2, w) \leq \max\{d(x_2, x_1), d(x_1, w)\} \leq r_1 \leq r_2.$$

Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. □

Proposition 2.4. Let (X, d) be an ultra-metric space. Then both $B(x, r)$ and $E(x, r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x, r) := \{y \in X \mid d(x, y) = r\}$ is open in X. Note that if $y \in S(x, r)$, then for any $r' < r$, we have $B(y, r') \cap E(x, r) \neq \varnothing$ and $x \in E(x, r) \setminus B(y, r')$. Thus by Proposition 2.3, we have $B(y, r') \subseteq E(x, r)$. If $B(y, r') \cap B(x, r) \neq \varnothing$, then by Proposition 2.3 again, we have $B(y, r') \subseteq B(x, r)$. However, $y \in B(y, r') \setminus B(x, r)$, a contradiction. Thus $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$. It yields that $S(x, r) = \bigcup_{y \in S(x,r)} B(y, r/2)$ is open in X.

Since $E(x, r) = B(x, r) \cup S(x, r)$ and $B(x, r) = E(x, r) \setminus S(x, r)$, both $B(x, r)$ and $E(x, r)$ are open and closed in X. □

Corollary 2.5. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x, y) > 0$. Consider the open ball $B(x, r/2)$. By Proposition 2.4, $B(x, r/2)$ is both open and closed in X. Thus $B(x, r/2) \cap S$ is both open and closed in S, however, it is non-empty and not equal to S since it contains x but not y. This shows that S is disconnected. □

3 Residue fields and reductions

3.1 Recover non-archimedean complete fields algebraically

In this subsection, let $\mathbf{k}$ be a non-archimedean field. Set $I_{r,<} := \{x \in \mathbf{k} : \|x\| < r\}$ and $I_{r,\leq} := \{x \in \mathbf{k} : \|x\| \leq r\}$ for each $r \in (0, 1)$.

Proposition 3.1. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers $\mathbf{k}^\circ$. Conversely, any ideal of $\mathbf{k}^\circ$ is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0, 1)$. Yang: To be checked.

Proof. Yang: To be checked. □

Proposition 3.2. We have
$$\widehat{\mathbf{k}}^{\circ} \cong \varprojlim_{r\in(0,1)} \mathbf{k}^{\circ}/I_r.$$
Yang: To be checked.

Proposition 3.3. Let $\mathbf{k}$ be a non-archimedean field. Then $\mathbf{k}$ is totally bounded iff $\mathbf{k}^{\circ}/I_r$ is finite for each $r \in (0,1)$. Moreover, if $\mathbf{k}$ is complete, then it is locally compact iff $\mathbf{k}^{\circ}/I_r$ is finite for each $r \in (0,1)$. Yang: To be checked.

Slogan *"Locally compact $\iff$ pro-finite."*

Proof. □

Proposition 3.4. The ring $\mathbf{k}^{\circ}$ is noetherian iff $\mathbf{k}$ is a discrete valuation field. Yang: To be revised.

Proposition 3.5. Let $\mathbf{k}$ be a complete non-archimedean field. Then $\mathbf{k}$ is locally compact iff $\mathbf{k}$ is a discrete valuation field and its residue field $\mathscr{k}_{\mathbf{k}}$ is finite. Yang: To be checked.

Proof. Yang: To be added. □

3.2 Hensel's Lemma

Theorem 3.6 (Hensel's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^{\circ}[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ factors as
$$f(T) = g(T)h(T),$$
where $g(T), h(T) \in \mathscr{k}_{\mathbf{k}}[T]$ are monic polynomials that are coprime in $\mathscr{k}_{\mathbf{k}}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^{\circ}[T]$ such that
$$F(T) = G(T)H(T),$$
and the reductions of $G(T), H(T)$ in $\mathscr{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively. Yang: To be checked.

Proof. Yang: To be added. □

Corollary 3.7. Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^{\circ}[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $\alpha \in \mathscr{k}_{\mathbf{k}}$. Then there exists a root $a \in \mathbf{k}^{\circ}$ of $F(T)$ whose reduction is α. Yang: To be revised.

Proof. Yang: To be added. □

3.3 Newton polygons

Yang: To be filled.

4 Finite field extensions

4.1 Finite-dimensional vector space

Proposition 4.1. Let V be a finite-dimensional vector space over a complete non-archimedean field $\mathbf{k}$. Then all norms on V are equivalent. Yang: To be checked.

4.2 Finite field extensions

Proposition 4.2. Let $\mathbf{k}$ be a complete non-archimedean field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the absolute value on $\mathbf{l}$ is uniquely determined by the absolute value on $\mathbf{k}$. Yang: To be checked.

Proposition 4.3. Let $\mathbf{k}$ be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed. Yang: To be checked.

5 Analytic functions

5.1 Failure of continuous and differentiable functions

Definition 5.1. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field and $U \subset \mathbf{k}$ be an open subset. A function $f : U \to \mathbf{k}$ is said to be *differentiable* at a point $a \in U$ if the limit

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists in $\mathbf{k}$. If f is differentiable at every point in U, we say that f is differentiable on U. Yang: to be revised.

Proposition 5.2. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field. Then there exists a continuous function $f : \mathbf{k} \to \mathbf{k}$ such that for any $x, y \in \mathbf{k}$ with $x \neq y$, we have

$$\frac{f(x) - f(y)}{x - y} = 0.$$

5.2 Power series

Proposition 5.3. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $\sum_{n=0}^{+\infty} a_n$ be a series in $\mathbf{k}$. Then the series $\sum_{n=0}^{+\infty} a_n$ converges if and only if $\lim_{n \to +\infty} a_n = 0$. Yang: To be checked.

Definition 5.4. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field.

5.3 Analytic functions

As in the case of real analysis, we can define analytic functions over non-archimedean fields using power series.

6 Example: p-adic fields

6.1 p-adic fields

Construction 6.1. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_{\mathfrak{p}}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_{\mathfrak{p}} : K^\times \to \mathbb{Z}$ the corresponding discrete valuation. The $\mathfrak{p}$*-adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the $\mathfrak{p}$-adic absolute value $|\cdot|_{\mathfrak{p}}$ is denoted by $K_{\mathfrak{p}}$, called the $\mathfrak{p}$*-adic field*.

One can just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p.

Example 6.2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The p*-adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Proposition 6.3. The multiplicative group $\mathbb{Q}_p^\times$ of the p-adic field $\mathbb{Q}_p$ admits the following decomposition:

$$\mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^\times,$$

where $p^{\mathbb{Z}} := \{p^n \mid n \in \mathbb{Z}\}$ and $\mathbb{Z}_p^\times := \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$ is the group of units of the ring of p-adic integers $\mathbb{Z}_p$. Yang: To be checked.

Yang: What is the relation between the finite extension of $\mathbb{Q}_p$ and $K_{\mathfrak{p}}$?

6.2 Completion

Proposition 6.4. The algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is not complete with respect to the extension of the p-adic absolute value $|\cdot|_p$.

Construction 6.5. Let p be a prime number. The field $\mathbb{C}_p$ of *p-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$. The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$. Yang: To be completed.

Proposition 6.6. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Construction 6.7. Let p be a prime number. Yang: We construct the *spherically complete p-adic field* Ω_p. Yang: To be completed.