Valuation fields

1 Absolute values and completion

Definition 1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 2. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Definition 3. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c > 0$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$.

Definition 4. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 5. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Yang: To be added. □

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

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Definition 6. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 7. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

2 Non-archimedean fields

Definition 8. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} \colon \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} \colon \|x\| < 1\}$.

Definition 9. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^{\circ} := \{x \in \mathbf{k} \colon \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} \colon \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathcal{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^{\circ}/\mathbf{k}^{\circ\circ}.$$

Yang: Is the valuation on residue field trivial?

Lemma 10. Recall that a metric space is *totally bounded* if for every $\varepsilon > 0$, it can be covered by finitely many balls of radius ε. A metric space is compact if and only if it is complete and totally bounded.

Proof. Yang: To be added. ☐

Definition 11. Let $\mathbf{k}$ be a non-archimedean field. The *residue absolute value* on the residue field $\mathcal{k}_{\mathbf{k}}$ is defined as

$$|x| := \inf_{y \in \varphi^{-1}(x)} \|y\|, \quad \forall x \in \mathcal{k}_{\mathbf{k}},$$

where $\varphi : \mathbf{k}^{\circ} \to \mathcal{k}_{\mathbf{k}}$ is the canonical projection.

Proposition 12. Let $\mathbf{k}$ be a non-archimedean field. Then the residue absolute value on the residue field $\mathcal{k}_{\mathbf{k}}$ is trivial.

Proof. For any $x \in \mathcal{k}_{\mathbf{k}}$, if $x = 0$, then by definition $|x| = 0$. If $x \neq 0$, then $\forall y \in \varphi^{-1}(x)$, we have $y \in \mathbf{k}^{\circ} \setminus \mathbf{k}^{\circ\circ}$, i.e., $\|y\| = 1$. Thus by definition $|x| = 1$. ☐

Proposition 13. Let $\mathbf{k}$ be a non-archimedean field. Set $I_r := \{x \in \mathbf{k} : \|x\| < r\}$ for each $r \in (0,1)$. They are ideals of the ring of integers $\mathbf{k}^\circ$. Then we have

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{r>0} \mathbf{k}^\circ / I_r.$$

Yang: To be checked.

Slogan *Locally compact* $\iff$ *pro-finite.*

Proposition 14. Let $\mathbf{k}$ be a non-archimedean field. Then $\mathbf{k}$ is totally bounded iff $\mathbf{k}^\circ / I_r$ is finite for each $r \in (0,1)$.