Non-archimedean Analysis

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1 Valuation fields

1.1 Absolute values and completion

Definition 1.1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot\| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field.*

Remark 1.2. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^{\times} \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^{\times}, v(xy) = v(x) + v(y)$;

- $\forall x, y \in \mathbf{k}^{\times}, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Example 1.3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 1.4. The *(multiplicative) valuation group* of a valuation field $(\mathbf{k}, \|\cdot\|)$ is defined as the subgroup of $\mathbb{R}_{>0}$ given by

$$|\mathbf{k}^{\times}| := \{\|x\| : x \in \mathbf{k}^{\times}\}.$$

Definition 1.5. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c \in (0, 1)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$. Moreover, the following lemma shows that the converse is also true.

Lemma 1.6. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) $\Rightarrow$ (b) is obvious. Now we prove (b) $\Rightarrow$ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) $\Rightarrow$ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^{\alpha}$ and $\|y\|_1 = \|y\|_2^{\beta}$. Suppose the contrary that $\alpha \neq \beta$. Consider the domain $\Omega \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Ω is non-empty. Pick $(n, m) \in \Omega$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. $\square$

Definition 1.7. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 1.8. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Simple analysis. $\square$

Example 1.9. Let $|\cdot|_\infty$ be the usual absolute value on the field $\mathbb{Q}$ of rational numbers. Then $(\mathbb{Q}, |\cdot|_\infty)$ is a valuation field. Its completion is the field $\mathbb{R}$ of real numbers equipped with the usual absolute value.

Example 1.10. Let p be a prime number. For any non-zero rational number $x \in \mathbb{Q}$, we can write it as $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then $(\mathbb{Q}, |\cdot|_p)$ is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of p-adic numbers equipped with the p-adic absolute value; see Yang: to be added..

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 1.11. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 1.12. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

1.2   Non-archimedean fields

Definition 1.13. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x + y\| \le \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean.*

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \le 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 1.14. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \le 1\}.$$

Its maximal ideal is
$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as
$$\mathscr{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^{\circ}/\mathbf{k}^{\circ\circ}.$$

Definition 1.15. Let $\mathbf{k}$ be a non-archimedean field. The *residue absolute value* on the residue field $\mathscr{k}_{\mathbf{k}}$ is defined as
$$|x| := \inf_{y \in \varphi^{-1}(x)} \|y\|, \quad \forall x \in \mathscr{k}_{\mathbf{k}},$$
where $\varphi : \mathbf{k}^{\circ} \to \mathscr{k}_{\mathbf{k}}$ is the canonical projection.

Proposition 1.16. Let $\mathbf{k}$ be a non-archimedean field. Then the residue absolute value on the residue field $\mathscr{k}_{\mathbf{k}}$ is trivial.

Proof. For any $x \in \mathscr{k}_{\mathbf{k}}$, if $x = 0$, then by definition $|x| = 0$. If $x \neq 0$, then $\forall y \in \varphi^{-1}(x)$, we have $y \in \mathbf{k}^{\circ} \setminus \mathbf{k}^{\circ\circ}$, i.e., $\|y\| = 1$. Thus by definition $|x| = 1$. $\qquad\square$

2 Ultra-metric spaces

We will use $B(x, r)$ (resp. $E(x, r)$) to denote the open ball (resp. closed ball) with center x and radius r.

Definition 2.1. A metric space (X, d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:
$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.$$

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x, y) := \|x - y\|$ on $\mathbf{k}$ makes $(\mathbf{k}, d)$ an ultra-metric space.

Proposition 2.2. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x, y) \geq d(y, z)$. By the strong triangle inequality, we have
$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(x, y).$$

On the other hand, by the strong triangle inequality again, we have
$$d(x, y) \leq \max\{d(x, z), d(z, y)\} = \max\{d(x, z), d(y, z)\} \leq d(x, y).$$

This shows that $d(x, y) = \max\{d(x, z), d(y, z)\}$. Thus either $d(x, z) = d(x, y) \geq d(y, z)$ or $d(y, z) = d(x, y) \geq d(x, z)$. $\qquad\square$

Proposition 2.3. Let (X, d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1, 2$. If $D_1 \cap D_2 \neq \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1, 2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1, x_2) \leq \max\{d(x_1, y), d(y, x_2)\}.$$

Without loss of generality, we may assume that $d(x_1, x_2) \leq d(x_1, y)$. It follows that $x_2 \in D_1$ since $d(x_1, y) < r_1$ (or $\leq r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1, z) > d(x_1, x_2)$. Then by Proposition 2.2,

$$r_1 \leq d(x_1, z) = d(x_2, z) \leq r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2, w) \leq \max\{d(x_2, x_1), d(x_1, w)\} \leq r_1 \leq r_2.$$

Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. $\square$

Proposition 2.4. Let (X, d) be an ultra-metric space. Then both $B(x, r)$ and $E(x, r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x, r) := \{y \in X \mid d(x, y) = r\}$ is open in X. Note that if $y \in S(x, r)$, then for any $r' < r$, we have $B(y, r') \cap E(x, r) \neq \emptyset$ and $x \in E(x, r) \setminus B(y, r')$. Thus by Proposition 2.3, we have $B(y, r') \subseteq E(x, r)$. If $B(y, r') \cap B(x, r) \neq \emptyset$, then by Proposition 2.3 again, we have $B(y, r') \subseteq B(x, r)$. However, $y \in B(y, r') \setminus B(x, r)$, a contradiction. Thus $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$. It yields that $S(x, r) = \bigcup_{y \in S(x, r)} B(y, r/2)$ is open in X.

Since $E(x, r) = B(x, r) \cup S(x, r)$ and $B(x, r) = E(x, r) \setminus S(x, r)$, both $B(x, r)$ and $E(x, r)$ are open and closed in X. $\square$

Corollary 2.5. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x, y) > 0$. Consider the open ball $B(x, r/2)$. By Proposition 2.4, $B(x, r/2)$ is both open and closed in X. Thus $B(x, r/2) \cap S$ is both open and closed in S, however, it is non-empty and not equal to S since it contains x but not y. This shows that S is disconnected. $\square$

3 Algebraic structures of non-archimedean fields

3.1 Recover non-archimedean complete fields algebraically

In this subsection, let **k** be a non-archimedean field. Set $I_{r,<} := B(0, r)$ and $I_{r,\leq} := E(0, r)$ for each $r \in [0, 1]$.

Proposition 3.1. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers $\mathbf{k}^\circ$. Conversely, any ideal of $\mathbf{k}^\circ$ is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0, 1)$.

Proof. Let I be an ideal of $\mathbf{k}^\circ$. Set $r = \sup\{|a| : a \in I\}$ (resp. $r = \max\{|a| : a \in I\}$ when the maximum exists). Then, by definition, we have $I \subset I_{r,<}$ (resp. $I \subset I_{r,\leq}$). For every $x \in \mathbf{k}^\circ$ with

$|x| < r$ (resp. $|x| \leq r$), there exists $a \in I$ such that $|x| \leq |a|$. Thus, $|x/a| \leq 1$ and so $x/a \in \mathbf{k}^\circ$. Since I is an ideal, we have $x = (x/a)a \in I$. Therefore, $I_{r,<} \subset I$ (resp. $I_{r,\leq} \subset I$). $\qquad\square$

Proposition 3.2. Let I_r be either $I_{r,<}$ or $I_{r,\leq}$ for each $r \in (0,1)$. Suppose $\{r_n \in (0,1)\}_{n\in\mathbb{N}}$ is a decreasing sequence converging to 0. Then the completion $\widehat{\mathbf{k}}$ of $\mathbf{k}$ is isomorphic to the projective limit

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}.$$

Proof. For every $x \in \widehat{\mathbf{k}}^\circ$, there exists a cauchy sequence $\{x_m\}_{m\in\mathbb{N}}$ in $\mathbf{k}^\circ$ converging to x. Since $\{r_n\}_{n\in\mathbb{N}}$ converges to 0, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that for all $m, m' \geq M_n$, we have $|x_m - x_{m'}| < r_n$. Thus, the sequence $\{x_m + I_{r_n}\}_{m\in\mathbb{N}}$ is eventually constant in $\mathbf{k}^\circ/I_{r_n}$. Define a map

$$\Phi : \widehat{\mathbf{k}}^\circ \to \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}, \quad x \mapsto \left(\lim_{m\to\infty} x_m + I_{r_n}\right)_{n\in\mathbb{N}}.$$

It is straightforward to verify that Φ is a well-defined ring homomorphism.

Conversely, for every $(a_n + I_{r_n})_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}$, we can choose a representative $a_n \in \mathbf{k}^\circ$ for each n. We claim that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathbf{k}^\circ$. Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_N < \varepsilon$. For all $m, n \geq N$, since $a_n + I_{r_n}$ maps to $a_m + I_{r_m}$ under the natural projection, we have $|a_n - a_m| < r_N < \varepsilon$. Thus, $\{a_n\}_{n\in\mathbb{N}}$ converges to some $x \in \widehat{\mathbf{k}}^\circ$. Easily see that the limit x is independent of the choice of representatives $\{a_n\}_{n\in\mathbb{N}}$. This gives a map

$$\Psi : \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n} \to \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} a_n.$$

Direct verification shows that $\Psi = \Phi^{-1}$. $\qquad\square$

Proposition 3.3. Let $\mathbf{k}$ be a non-archimedean field. Then $\mathbf{k}^\circ$ is totally bounded iff $\mathbf{k}^\circ/I_{r,<}$ and $\mathbf{k}^\circ/I_{r,\leq}$ are finite for each $r \in [0,1]$. Moreover, if $\mathbf{k}$ is complete, then it is locally compact iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$.

Slogan *"Locally compact $\iff$ pro-finite."*

Proof. We just prove the case for $I_r = I_{r,<}$. The case for $I_r = I_{r,\leq}$ is similar.

Suppose that $\mathbf{k}^\circ/I_r$ is finite for each $r \in [0,1]$. Then for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that $r < \varepsilon$ and $\mathbf{k}^\circ/I_r$ is finite. Let $\{a_1 + I_r, \ldots, a_n + I_r\}$ be the complete set of representatives of $\mathbf{k}^\circ/I_r$. Then the balls $B(a_i, r)$ for $i = 1, \ldots, n$ cover $\mathbf{k}^\circ$.

Conversely, suppose that $\mathbf{k}^\circ/I_r$ is infinite for some $r \in [0,1]$. Then there exists an infinite set $\{a_n\}$ with $|a_n| \in [r,1]$ such that their images in $\mathbf{k}^\circ/I_r$ are distinct. In particular, for every $m \neq n$, we have $|a_n - a_m| \geq r$. Any subsequence of $\{a_n\}$ is not cauchy. Thus, $\mathbf{k}^\circ$ is not totally bounded. $\qquad\square$

Proposition 3.4. The ring $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field.

Proof. Note that $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$ is a multiplicative subgroup. If $\mathbf{k}$ is not a discrete valuation field, then $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. In particular, there exists a strictly ascending sequence $r_n \in |\mathbf{k}^\times| \cap (0,1)$. Then the ideals $I_{r_n,\leq}$ form a strictly ascending chain of ideals in $\mathbf{k}^\circ$.

The converse is standard since now $\mathbf{k}^\circ$ is a discrete valuation ring. $\qquad\square$

Proposition 3.5. Let $\mathbf{k}$ be a complete non-archimedean field. Then $\mathbf{k}$ is locally compact iff $\mathbf{k}$ is a discrete valuation field and its residue field $\mathscr{k}_{\mathbf{k}}$ is finite.

Proof. The necessity follows from Proposition 3.3. For the sufficiency, suppose that $\mathbf{k}$ is a discrete valuation field whose residue field $\mathscr{k}_{\mathbf{k}}$ is finite. Let $\pi \in \mathbf{k}^\circ$ be a uniformizer. We only need to show that $\mathbf{k}^\circ/\pi^n \mathbf{k}^\circ$ is finite for each $n \in \mathbb{N}$. Note that there is an isomorphism

$$\pi^{n-1}\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ \cong \mathscr{k}_{\mathbf{k}}, \quad x + \pi^n\mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on n, we conclude that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite. □

3.2 Hensel's Lemma

Theorem 3.6 (Hensel's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \mathscr{k}_{\mathbf{k}}[T]$ are monic polynomials that are coprime in $\mathscr{k}_{\mathbf{k}}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\mathscr{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

Proof. Since $\gcd(g, h) = 1$ in $\mathscr{k}_{\mathbf{k}}[T]$, there exist polynomials $u(T), v(T) \in \mathscr{k}_{\mathbf{k}}[T]$ such that $ug + vh = 1$ and $\deg u < \deg h, \deg v < \deg g$. Choose lifts $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$ of $g(T), h(T), u(T), v(T)$ respectively preserving their degrees such that G_0 and H_0 are monic. Then there exist $r < 1$ such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \mod I_r, \quad F(T) - G_0(T)H_0(T) \equiv 0 \mod I_r,$$

where $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$.

We will construct a sequence of monic polynomials $\{G_n(T)\}_{n\in\mathbb{N}}$ and $\{H_n(T)\}_{n\in\mathbb{N}}$ in $\mathbf{k}^\circ[T]$ such that for each $n \in \mathbb{N}$,

$$G_n(T) \equiv G_{n-1}(T) \mod I_{r^n}, \quad H_n(T) \equiv H_{n-1}(T) \mod I_{r^n},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \mod I_{r^{n+1}}.$$

If we have such sequences, then their coefficients converge in the complete ring $\mathbf{k}^\circ$. Let $G(T)$ and $H(T)$ be the limits of $\{G_n(T)\}$ and $\{H_n(T)\}$ respectively. Then we have $F(T) = G(T)H(T)$ and the reductions of $G(T), H(T)$ in $\mathscr{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

The case $n = 0$ is done by the above construction. Now suppose that we have constructed $G_n(T)$ and $H_n(T)$ for some $n \geq 0$. Since $G_n - G_0 \equiv 0 \mod I_r$ and $H_n - H_0 \equiv 0 \mod I_r$, we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \mod I_r.$$

Set $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$ and $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$. Then we have

$$(G_n + \epsilon_n)(H_n + \delta_n) - F_n = G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n$$
$$= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T].$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

$\square$

Corollary 3.7. Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}°[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $a \in \mathscr{k}_{\mathbf{k}}$. Then there exists a root $\alpha \in \mathbf{k}°$ of $F(T)$ whose reduction is a.

Proof. Since a is a simple root of $f(T)$, we have the factorization $f(T) = (T - a)h(T)$ for some monic polynomial $h(T) \in \mathscr{k}_{\mathbf{k}}[T]$ with $h(a) \neq 0$. Then the result follows from Theorem 3.6. $\square$

3.3 Newton polygons

Yang: To be filled.

4 Finite field extensions

4.1 Finite-dimensional vector space

Definition 4.1. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

  (a) $\|x\| = 0$ if and only if $x = 0$;

  (b) $\|ax\| = |a| \cdot \|x\|$;

  (c) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 4.2. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Lemma 4.3. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V.

Proof. Yang: To be added. $\square$

Proposition 4.4. Let V be a finite-dimensional vector space over a complete non-archimedean field $\mathbf{k}$. Then all norms on V are equivalent. Yang: To be checked.

Proof. Yang: To be added. $\square$

Proposition 4.5. Let V be a finite-dimensional vector space over a complete valuation field $\mathbf{k}$. Then V is complete with respect to any norm on V.

4.2 Finite field extensions

Construction 4.6. Let $\mathbf{k}$ be a valuation field and $\mathbf{l}$ a finite extension of $\mathbf{k}$ with degree $n = [\mathbf{l} : \mathbf{k}]$. For any $a \in \mathbf{l}$, define
$$|a|_\mathbf{l} := |N_{\mathbf{l}/\mathbf{k}}(a)|_\mathbf{k}^{1/n},$$
where $N_{\mathbf{l}/\mathbf{k}}(a)$ is the norm of a from $\mathbf{l}$ to $\mathbf{k}$. By Lemma 4.7, $|\cdot|_\mathbf{l}$ is an absolute value on $\mathbf{l}$ extending the absolute value $|\cdot|_\mathbf{k}$ on $\mathbf{k}$. Yang: To be checked.

Lemma 4.7. Let $\mathbf{k}$ be a valuation field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the function $|\cdot|_\mathbf{l}$ defined in Construction 4.6 is an absolute value on $\mathbf{l}$ extending the absolute value on $\mathbf{k}$.
Moreover, if $\mathbf{k}$ is non-archimedean, then so is $\mathbf{l}$.

Proof. Yang: To be added. □

Proposition 4.8. Let $\mathbf{k}$ be a complete non-archimedean field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the absolute value on $\mathbf{l}$ is uniquely determined by the absolute value on $\mathbf{k}$. Yang: To be checked.

Proof. Yang: To be added. □

Remark 4.9. Yang: I want to discuss some compatiblity of extension and completion.

Proposition 4.10. Let $\mathbf{k}$ be an algebraically closed non-archimedean field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed. Yang: To be checked.

Proof. Yang: To be added. □

5 Analytic functions

5.1 Failure of continuous and differentiable functions

Definition 5.1. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field and $U \subset \mathbf{k}$ be an open subset. A function $f : U \to \mathbf{k}$ is said to be *differentiable* at a point $a \in U$ if the limit
$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
exists in $\mathbf{k}$. If f is differentiable at every point in U, we say that f is differentiable on U. Yang: to be revised.

Proposition 5.2. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field. Then there exists a continuous function $f : \mathbf{k} \to \mathbf{k}$ such that for any $x, y \in \mathbf{k}$ with $x \neq y$, we have
$$\frac{f(x) - f(y)}{x - y} = 0.$$

5.2 Power series

Lemma 5.3. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field and $\sum_{n=0}^{+\infty} a_n$ be a series in $\mathbf{k}$. Then the series $\sum_{n=0}^{+\infty} a_n$ converges if and only if $\lim_{n\to+\infty} a_n = 0$. Yang: To be checked.

Proposition 5.4. Let $(\mathbf{k}, \|\cdot\|)$ be a non-archimedean field and $\sum_{n=0}^{+\infty} a_n(x-c)^n$ be a power series in $\mathbf{k}$. Then there exists a radius of convergence $R \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that the series converges for all $x \in \mathbf{k}$ with $\|x - c\| < R$ and diverges for all $x \in \mathbf{k}$ with $\|x - c\| > R$. Yang: To be revised.

Proposition 5.5. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $\sum_{n=0}^{+\infty} a_n$ be a series in $\mathbf{k}$. Then the series $\sum_{n=0}^{+\infty} a_n$ converges if and only if $\lim_{n\to+\infty} a_n = 0$. Yang: To be checked.

Definition 5.6. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field.

Proposition 5.7. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field. Then the norm on the Tate algebra $\mathbf{k}\langle x_1, \ldots, x_n \rangle$ coincides with the supremum norm on the closed unit polydisc in $\mathbf{k}^n$. Yang: To be checked.

5.3 Analytic functions and maps

As in the case of real analysis, we can define analytic functions over non-archimedean fields using power series.

Definition 5.8. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $U \subset \mathbf{k}$ be an open subset. A function $f : U \to \mathbf{k}$ is said to be *analytic* at a point $c \in U$ if there exists a power series $\sum_{n=0}^{+\infty} a_n(x-c)^n$ that converges to $f(x)$ for all x in some neighborhood of c. If f is analytic at every point in U, we say that f is analytic on U. Yang: to be revised.

Theorem 5.9. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $U \subset \mathbf{k}$ be an open subset. If $f : U \to \mathbf{k}$ is an analytic function, then f is locally Lipschitz continuous on U. Yang: To be checked.

Theorem 5.10. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $U \subset \mathbf{k}$ be an open subset. If $f : U \to \mathbf{k}$ is an analytic function, then f satisfies the maximum modulus principle, i.e., if there exists a point $x_0 \in U$ such that $\|f(x_0)\| \geq \|f(x)\|$ for all $x \in U$, then f is constant on U. Yang: To be checked.

6 Example: p-adic fields

6.1 p-adic fields

Construction 6.1. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_\mathfrak{p}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_\mathfrak{p} : K^\times \to \mathbb{Z}$ the corresponding discrete valuation. The *p-adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_\mathfrak{p} := N(\mathfrak{p})^{-v_\mathfrak{p}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the p-adic absolute value $|\cdot|_\mathfrak{p}$ is denoted by $K_\mathfrak{p}$, called the $\mathfrak{p}$-*adic field.*

One can just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p.

Example 6.2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Proposition 6.3. The multiplicative group $\mathbb{Q}_p^\times$ of the p-adic field $\mathbb{Q}_p$ admits the following decomposition:

$$\mathbb{Q}_p^\times \cong p^\mathbb{Z} \times \mathbb{Z}_p^\times,$$

where $p^\mathbb{Z} := \{p^n \mid n \in \mathbb{Z}\}$ and $\mathbb{Z}_p^\times := \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$ is the group of units of the ring of p-adic integers $\mathbb{Z}_p$. Yang: To be checked.

Yang: What is the relation between the finite extension of $\mathbb{Q}_p$ and $K_\mathfrak{p}$?

6.2 Completion

Proposition 6.4. The algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is not complete with respect to the extension of the p-adic absolute value $|\cdot|_p$.

Construction 6.5. Let p be a prime number. The field $\mathbb{C}_p$ of *p-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$. The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$. Yang: To be completed.

Proposition 6.6. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Construction 6.7. Let p be a prime number. Yang: We construct the *spherically complete p-adic field* Ω_p. Yang: To be completed.