Spectrum of commutative branch rings

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1 Semi-normed Rings and Modules

1.1 Semi-normed algebraic structures

Definition 1.1. Let M be an abelian group. A *semi-norm* on M is a function $\|\cdot\| : M \to \mathbb{R}_+$ such that

- $\|0\| = 0$;
- $\forall x, y \in M, \|x + y\| \leq \|x\| + \|y\|$.

If we further have $\|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*. A *semi-normed abelian group* (resp. *normed abelian group*) is an abelian group equipped with a semi-norm (resp. norm).

Definition 1.2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group M. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in M, \|x\|_1 \leq C\|x\|_2$.

Remark 1.3. If two semi-norms (resp. norms) on an abelian group M are bounded by each other, then they induce the same topology on M.

Definition 1.4. Let M be a semi-normed abelian group and $N \subseteq M$ be a subgroup. The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Remark 1.5. The residue semi-norm is a norm if and only if N is closed in M.

Definition 1.6. Let M and N be two semi-normed abelian groups. A group homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \leq C\|x\|_M$.
A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \inf_{y \in \ker f} \|x + y\|_M$.

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Definition 1.7. Let R be a ring (commutative with unity). A *semi-norm* on R is a semi-norm $\|\cdot\|$ on the underlying abelian group of R such that $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$ and $\|1\| = 1$. A *semi-normed ring* is a ring equipped with a semi-norm.

Definition 1.8. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$. A power-multiplicative semi-norm is also called *uniform*.

Remark 1.9. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k} \to \mathbb{R} \cup \{\infty\}$ such that

- (non-degeneracy) $v(x) = \infty \iff x = 0$;
- (normalization) $v(1) = 0$;
- (additivity) $\forall x, y \in \mathbf{k}, v(xy) = v(x) + v(y)$;
- (triangle inequality) $\forall x, y \in \mathbf{k}, v(x + y) \geq \min\{v(x), v(y)\}$.

Yang: To be checked.

Definition 1.10. Let $(R, \|\cdot\|_R)$ be a normed ring. A *semi-normed R-module* is a pair $(M, \|\cdot\|_M)$ where M is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M such that there exists $C > 0$ with $\forall a \in R, x \in M, \|ax\|_M \leq C\|a\|_R\|x\|_M$.

Yang: To be continued...

1.2 Banach rings

Definition 1.11. A semi-norm (resp. norm) on an abelian group M induces a pseudo-metric (resp. metric) $d(x, y) = \|x - y\|$ on M. A semi-normed (resp. normed) abelian group M is called *complete* if it is complete as a pseudo-metric (resp. metric) space.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M\otimes_R N} = \inf\left\{\sum_i \|x_i\|_M\|y_i\|_N : z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N\right\}.$$

Definition 1.12. Let R be a complete normed ring and M, N complete semi-normed R-modules. The *complete tensor product* $M\widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Definition 1.13. A *Banach ring* is a complete normed ring.

Definition 1.14. Let $(A, \|\cdot\|_A)$ be a normed algebraic structure (e.g., a normed vector space, a normed ring, etc.). The *completion* of A is the smallest complete normed algebraic structure A^c such that A is isometrically embedded in A^c. Yang: To be continued.

Definition 1.15. Let R be a Banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Proposition 1.16. Let R be a Banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by the given norm on R.

Definition 1.17. Let R be a Banach ring. The *uniformization* of R is the Banach ring with the universal property among all bounded morphisms from R to uniform Banach rings. Yang: To be continued.

Proposition 1.18. Let R be a Banach ring. The completion of R with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Yang: To be continued...

1.3 Examples

Example 1.19. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a normed ring.

Example 1.20. The fields $\mathbb{C}$ and $\mathbb{R}$ equipped with the usual absolute value are complete fields.

Example 1.21. The field $\mathbb{Q}_p$ of p-adic numbers equipped with the p-adic norm is a complete non-Archimedean field.

Example 1.22. Let R be a banach ring and $r > 0$ be a real number. We define the ring of absolutely convergent power series over $\mathbf{k}$ with radius r as

$$R < T/r > := \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] \, : \, \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

Equipped with the norm $\| \sum_{n=0}^{\infty} a_n T^n \| = \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R < T/r >$ is a Banach ring. Yang: To be checked.

Example 1.23. Let $\mathbf{k}$ be a non-Archimedean complete field. The *affinoid* $\mathbf{k}$*-algebra* is defined as

$$\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\} := \left\{ \sum_{I\in\mathbb{N}^n} a_I T^I \in \mathbf{k}[[T_1, \dots, T_n]] \, : \, \lim_{|I|\to\infty} |a_I| r^I = 0 \right\},$$

where $r = (r_1, \dots, r_n)$ is an n-tuple of positive real numbers and $T^I = T_1^{i_1} \cdots T_n^{i_n}$ for $I = (i_1, \dots, i_n)$, and $|I| = i_1 + \cdots + i_n$. Equipped with the norm $\| \sum_{I\in\mathbb{N}^n} a_I T^I \| = \sup_{I\in\mathbb{N}^n} |a_I| r^I$, the affinoid $\mathbf{k}$-algebra $\mathbf{k}\{T_1/r_1, \dots, T_n/r_n\}$ is a Banach $\mathbf{k}$-algebra.

Yang: To be continued...

2 Spectrum

2.1 Definition

Definition 2.1. Let R be a Banach ring. The *spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$. We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_+$, defined by $x \mapsto |f|_x$, is continuous.

For $x \in \mathscr{M}(R)$, the kernel of the multiplicative semi-norm $|\cdot|_x$ is a closed prime ideal of R, denoted by $\wp_x$. The semi-norm $|\cdot|_x$ induces a multiplicative norm on the residue field $\kappa(x) = \mathrm{Frac}(R/\wp_x)$, denoted by $|\cdot|_x$ as well.

Definition 2.2. Let R be a Banach ring. A *character* of R is a bounded ring homomorphism $\chi : R \to K$, where K is a complete valued field. Two characters $\chi_1 : R \to K_1$ and $\chi_2 : R \to K_2$ are said to be *equivalent* if there exists an isometric field extension L of both K_1 and K_2 such that the following diagram commutes:

$$\begin{tikzcd}
R \arrow[r, "\chi_1"] \arrow[d, "\chi_2"'] & K_1 \arrow[d] \\
K_2 \arrow[r] & L
\end{tikzcd}$$

Definition 2.3. Let $f : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $f^* : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $f^*(x) = x \circ f$ for each $x \in \mathscr{M}(S)$. Yang: To be revised.

Proposition 2.4. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is in bijection with the equivalence classes of characters of R.

Theorem 2.5. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a nonempty compact Hausdorff space.

Proof. Yang: To be continued. □

2.2 Examples