Semi-normed Rings and Modules

1 Semi-normed algebraic structures

Definition 1. Let M be an abelian group. A *semi-norm* on M is a function $\|\cdot\| : M \to \mathbb{R}_+$ such that

- $\|0\| = 0$;
- $\forall x, y \in M, \|x + y\| \le \|x\| + \|y\|$.

If we further have $\|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*. A *semi-normed abelian group* (resp. *normed abelian group*) is an abelian group equipped with a semi-norm (resp. norm).

Definition 2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group M. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in M, \|x\|_1 \le C\|x\|_2$.

Remark 3. If two semi-norms (resp. norms) on an abelian group M are bounded by each other, then they induce the same topology on M.

Definition 4. Let M be a semi-normed abelian group and $N \subseteq M$ be a subgroup. The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Remark 5. The residue semi-norm is a norm if and only if N is closed in M.

Definition 6. Let M and N be two semi-normed abelian groups. A group homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \le C\|x\|_M$.
A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \inf_{y \in \ker f} \|x + y\|_M$.

Definition 7. Let R be a ring (commutative with unity). A *semi-norm* on R is a semi-norm $\|\cdot\|$ on the underlying abelian group of R such that $\forall x, y \in R, \|xy\| \le \|x\|\|y\|$ and $\|1\| = 1$. A *semi-normed ring* is a ring equipped with a semi-norm.

Definition 8. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \ge 1$. A power-multiplicative semi-norm is also called *uniform*.

Remark 9. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k} \to \mathbb{R} \cup \{\infty\}$ such that

- (non-degeneracy) $v(x) = \infty \iff x = 0$;
- (normalization) $v(1) = 0$;

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- (additivity) $\forall x, y \in \mathbf{k}, v(xy) = v(x) + v(y)$;
- (triangle inequality) $\forall x, y \in \mathbf{k}, v(x+y) \geq \min\{v(x), v(y)\}$.

Yang: To be checked.

Definition 10. Let $(R, \|\cdot\|_R)$ be a normed ring. A *semi-normed R-module* is a pair $(M, \|\cdot\|_M)$ where M is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M such that there exists $C > 0$ with $\forall a \in R, x \in M, \|ax\|_M \leq C\|a\|_R\|x\|_M$.

Yang: To be continued...

2 Banach rings

Definition 11. A semi-norm (resp. norm) on an abelian group M induces a pseudo-metric (resp. metric) $d(x, y) = \|x - y\|$ on M. A semi-normed (resp. normed) abelian group M is called *complete* if it is complete as a pseudo-metric (resp. metric) space.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M\otimes_R N} = \inf\left\{\sum_i \|x_i\|_M\|y_i\|_N \,:\, z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N\right\}.$$

Definition 12. Let R be a complete normed ring and M, N complete semi-normed R-modules. The *complete tensor product* $M\widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Definition 13. A *Banach ring* is a complete normed ring.

Definition 14. Let $(A, \|\cdot\|_A)$ be a normed algebraic structure (e.g., a normed vector space, a normed ring, etc.). The *completion* of A is the smallest complete normed algebraic structure A^c such that A is isometrically embedded in A^c. Yang: To be continued.

Definition 15. Let R be a Banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Proposition 16. Let R be a Banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by the given norm on R.

Definition 17. Let R be a Banach ring. The *uniformization* of R is the Banach ring with the universal property among all bounded morphisms from R to uniform Banach rings. Yang: To be continued.

Proposition 18. Let R be a Banach ring. The completion of R with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Yang: To be continued...

3 Examples

Example 19. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a normed ring.

Example 20. The fields $\mathbb{C}$ and $\mathbb{R}$ equipped with the usual absolute value are complete fields.

Example 21. The field $\mathbb{Q}_p$ of p-adic numbers equipped with the p-adic norm is a complete non-Archimedean field.

Example 22. Let $\mathbf{k}$ be a complete field. The ring of formal power series

Yang: To be completed.

Yang: To be continued...

Appendix