Spectrum

1 Definition

Definition 1. Let R be a Banach ring. The *spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$. We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_{\geq 0}$, defined by $x \mapsto |f|_x =: f(x)$, is continuous.

Definition 2. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $\mathscr{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$ for each $x \in \mathscr{M}(S)$.

Definition 3. Let R be a Banach ring. A *character* of R is a bounded ring homomorphism $\chi : R \to K$, where K is a completed field. Two characters $\chi_1 : R \to K_1$ and $\chi_2 : R \to K_2$ are said to be *equivalent* if there exists a commutative diagram of bounded ring homomorphisms

$$\begin{tikzcd}
& R \arrow[dl, "\chi_1"'] \arrow[d] \arrow[dr, "\chi_2"] & \\
K_1 & K \arrow[l, hook'] \arrow[r, hook] & K_2
\end{tikzcd}$$

for some completed field K.

Proposition 4. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is in bijection with the equivalence classes of characters of R.

Proof. Yang: To be completed □

Proposition 5. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, let $\wp_x$ be the kernel of the multiplicative semi-norm $|\cdot|_x$. Then $\wp_x$ is a closed prime ideal of R, and $x \mapsto \wp_x$ defines a continuous map from $\mathscr{M}(R)$ to $\mathrm{Spec}(R)$ equipped with the Zariski topology.

Proof. Yang: To be completed □

Definition 6. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, the *completed residue field* at the point x is defined as the completion of the residue field $\kappa(x) = \mathrm{Frac}(R/\wp_x)$ with respect to the multiplicative norm induced by the semi-norm $|\cdot|_x$, denoted by $\mathscr{H}(x)$.

Definition 7. Let R be a Banach ring. The *Gel'fand transform* of R is the bounded ring homomorphism

$$\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x), \quad f \mapsto (f(x))_{x \in \mathscr{M}(R)},$$

where the norm on the product $\prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is given by the supremum norm.

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Proposition 8. The Gel'fand transform $\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ of a Banach ring R factors through the uniformization R^u of R, and the induced map $R^u \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is an isometric embedding. Yang: To be checked.

Theorem 9. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a nonempty compact Hausdorff space.

Proof. Yang: To be continued. □

Lemma 10. Let $\{K_i\}_{i \in I}$ be a family of completed fields. Consider the Banach ring $R = \prod_{i \in I} K_i$ equipped with the product norm. The spectrum $\mathscr{M}(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I.

Remark 11. The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

2 Examples

Example 12. Let $(\mathbf{k}, |\cdot|)$ be a complete valuation field. The spectrum $\mathscr{M}(\mathbf{k})$ consists of a single point corresponding to the given absolute value $|\cdot|$ on $\mathbf{k}$. Yang: To be checked.

Example 13. Consider the Banach ring $(\mathbb{Z}, \|\cdot\|)$ with $\|\cdot\| = |\cdot|_\infty$ is the usual absolute value norm on $\mathbb{Z}$. Let $|\cdot|_p$ denote the p-adic norm for each prime number p, i.e., $|n|_p = p^{-v_p(n)}$ for each $n \in \mathbb{Z}$, where $v_p(n)$ is the p-adic valuation of n. The spectrum

$$\mathscr{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0,1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0,\infty]\} \cup \{|\cdot|_0\},$$

where $|a|_p^\infty := \lim_{\alpha \to \infty} |a|_p^\alpha$ for each $a \in \mathbb{Z}$ and $|\cdot|_0$ is the trivial norm on $\mathbb{Z}$. Yang: To be checked.

Spectrum of Tate algebra in one variable Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{T/r\}$. We list some types of points in the spectrum $\mathscr{M}(A)$.

For each $a \in \mathbf{k}$ with $|a| \le r$, we have the *type I* point x_a corresponding to the evaluation at a, i.e., $|f|_{x_a} := |f(a)|$ for each $f \in A$. For each closed disk $D(a,s) = \{b \in \mathbf{k} : |b - a| \le s\}$ with center $a \in \mathbf{k}$ and radius $s \le r$, we have the point $x_{a,s}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{a,s}} := \sup_{b \in D(a,s)} |f(b)|$$

for each $f \in A$.

Yang: To be completed.

Proposition 14. Let $\mathbf{k}$ be a complete non-archimedean field, and let $r > 0$ be a positive real number. Consider the Tate algebra $\mathbf{k}\{r^{-1}T\}$ equipped with the Gauss norm. The points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ can be classified into four types:

- Type I: Points corresponding to evaluation at $\mathbf{k}$-rational points in the closed disk of radius r.
- Type II: Points corresponding to multiplicative semi-norms associated with closed disks of radius in $|\mathbf{k}^\times|$ centered at points in the closed disk of radius r.
- Type III: Points corresponding to multiplicative semi-norms associated with closed disks of

radius not in $|\mathbf{k}^{\times}|$ centered at points in the closed disk of radius r.

- Type IV: Points corresponding to multiplicative semi-norms that are not associated with any disk, often arising as limits of Type II or Type III points.

Yang: To be checked

Proof. Yang: To be completed. □