Analytic functions

1 Tate algebras

Notation 1. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- $E(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \dots, n\}$ and $B(x, \underline{r}) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \dots, n\}$ for $x = (x_1, \dots, x_n) \in \mathbf{k}^n$.
- Let $\{x_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha| \to +\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

Definition 2. Let $\mathbf{k}$ be a complete non-archimedean field. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as

$$\mathbf{k}\langle \underline{r^{-1}T} \rangle := \mathbf{k}\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{|\alpha| \to +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 3. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ is a non-archimedean multiplicative banach $\mathbf{k}$-algebra with respect to the *gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Yang: For the definition of banach ring, see

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ are two elements in $\mathbf{k}\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon / \|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon / \|f\|$. For any $|\gamma| > 2N$, we have

$$\left\| \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right\| r^\gamma \leq \max_{\alpha + \beta = \gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max \left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \leq \varepsilon.$$

Hence $f \cdot g \in \mathbf{k}\{\underline{T/r}\}$ and it shows that $\mathbf{k}\{\underline{T/r}\}$ is a $\mathbf{k}$-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $\mathbf{k}\{\underline{T/r}\}$.

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The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f+g\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \le \sup_{\alpha\in\mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \le \max\{\|f\|, \|g\|\}$$

and

$$\|f\cdot g\| = \left\| \sum_{\gamma\in\mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) T^\gamma \right\| = \sup_{\gamma\in\mathbb{N}^n} \left\| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right\| r^\gamma$$
$$\le \sup_{\gamma\in\mathbb{N}^n} \max_{\alpha+\beta=\gamma} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \le \|f\|\cdot\|g\|.$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\|\cdot\|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \left\| \sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta \right\| r^{\alpha_1+\beta_1} \le \|f\cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by Proposition 9. Thus the gauss norm is multiplicative.

Step 4. Finally show that $\mathbf{k}\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $\mathbf{k}\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \le \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in $\mathbf{k}$. Since $\mathbf{k}$ is complete, set $a_\alpha := \lim_{m\to+\infty} a_{\alpha,m}$ and $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \le \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \le 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \le \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \le \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \le \varepsilon$$

for all $m > N$. □

Proposition 4. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ can be identified with a subring of the ring of all functions from the closed polydisc $E(0, \underline{r}) \subset \mathbf{k}^n$ to $\mathbf{k}$.

Proof. Given $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \dots, x_n) \in E(0, \underline{r})$, we have

$$\left\| \sum_{|\alpha|=n} a_\alpha x^\alpha \right\| \le \max_{|\alpha|=n} \|a_\alpha\| r^\alpha \to 0 \quad \text{as } n \to +\infty.$$

Hence by Proposition 8, the series $f(x) := \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha$ converges in $\mathbf{k}$. This defines a function $f : E(0, \underline{r}) \to \mathbf{k}$.

Let $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. Set

$$A_n = \sum_{|\alpha|<n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta|<n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma|<n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) x^\gamma.$$

We need to show that $f(x)g(x) = \lim A_n B_n = \lim C_n = (fg)(x)$. Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta|\ge n}} a_\alpha b_\beta x^{\alpha+\beta}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $n > 2N$, we have

$$\|A_n B_n - C_n\| \le \max_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta|\ge n}} \|a_\alpha\| \|b_\beta\| \|x^{\alpha+\beta}\| < \max\left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \le \varepsilon.$$

Thus $f(x)g(x) = (fg)(x)$. The addition and scalar multiplication can be verified directly. Hence above assignments define a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T/r}\}$ to the ring of all functions from $E(0, \underline{r})$ to $\mathbf{k}$.

Finally, Proposition 5 ensures that the homomorphism is injective. □

Proposition 5. Let $\mathbf{k}$ be a complete non-archimedean field. Then the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x\in E(0,\underline{r})} \|f(x)\|_{\mathbf{k}}.$$

Proof. Yang: To be added. □

2 Fundamental properties

Then following shows that analytic functions over non-archimedean fields share some nice properties as in the case of complex analysis. Yang: To be revised.

Theorem 6. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $U \subset \mathbf{k}$ be an open subset. If $f : U \to \mathbf{k}$ is an analytic function, then f is locally Lipschitz continuous on U. Yang: To be checked.

Theorem 7 (Strassman)**.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$ be an analytic function. Suppose that $\|a_N\| > \|a_n\|$ for all $n > N$. Then f has at most N zeros in the closed ball $E(0, r)$. Yang: To be checked.

Proof. Yang: To be add. □

Appendix

Proposition 8. Let (X, d) be an ultra-metric space. A sequence $\{x_n\}$ in X is cauchy if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Proposition 9. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.