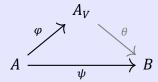
## Affinoid domains

Consider  $X = \mathcal{M}(A)$  with  $A = \mathbf{k}\{T_1, ..., T_n\}$ . Yang: Not every open subset of X gives an affinoid space, that is, the completion of the ring of analytic functions on that open subset is not necessarily an affinoid algebra. Yang: Right? example?

## 1 Definition

**Definition 1.** Let A be a **k**-affinoid algebra, and let  $X = \mathcal{M}(A)$  be the associated affinoid space. A closed subset  $V \subseteq X$  is called an *affinoid domain* if there exists a **k**-affinoid algebra  $A_V$  and a morphism of **k**-affinoid algebras  $\varphi: A \to A_V$  satisfying the following universal property: for every bounded homomorphism of **k**-affinoid algebras  $\psi: A \to B$  such that the induced map on spectra  $\mathcal{M}(\psi): \mathcal{M}(B) \to X$  has its image contained in V, there exists a unique bounded homomorphism  $\theta: A_V \to B$  such that the following diagram commutes:



In this case, we say that V is represented by the affinoid algebra  $A_V$ .

**Slogan** A closed subset  $V \subset X$  is an affinoid domain if the functor "Mor(-,V)" is representable.

Yang: Why we consider closed subset rather that open subset?

Construction 2. Let  $f = (f_1, ..., f_n)$  be a tuple of elements in A and  $r = (r_1, ..., r_n)$  be a tuple of positive real numbers. Consider the closed subset of X:

$$X(f/r) := \{x \in X : |f_i(x)| \le r_i, 1 \le i \le n\}.$$

Such a closed subset is called a Weierstrass domain of X. Moreover, we can define a  $\mathbf{k}$ -affinoid algebra

$$A\left\{ \frac{f/r}{r} \right\} := A\left\{ f_1/r_1, \dots, f_n/r_n \right\}.$$

Yang: The domain  $X(\underline{f/r})$  is represented by  $A\{\underline{f/r}\}$ .

Construction 3. Let  $f = (f_1, ..., f_n), g = (g_1, ..., g_m)$  be two tuples of elements in A and  $r = (r_1, ..., r_n), s = (s_1, ..., s_m)$  be two tuples of positive real numbers. Consider the following closed subset of X:

$$X(f/r;g/s^{-1}) := \{x \in X : |f_i(x)| \le r_i, |g_j(x)| \ge s_j, 1 \le i \le n, 1 \le j \le m\}.$$

Such a closed subset is called a Laurent domain of X. Moreover, we can define a  $\mathbf{k}$ -affinoid algebra

$$A\left\{\underline{f/r};\underline{g/s}^{-1}\right\} := A\left\{f_1/r_1,\ldots,f_n/r_n,g_1^{-1}/s_1,\ldots,g_m^{-1}/s_m\right\}.$$

Date: October 26, 2025, Author: Tianle Yang, My Homepage

Yang: The domain  $X(\underline{f/r}; \underline{g/s}^{-1})$  is represented by  $A\{\underline{f/r}; \underline{g/s}^{-1}\}$ .

Construction 4. Let  $f = (f_1, ..., f_n), g$  be elements in A such that the ideal generated by them is the whole algebra A. Set  $p = (p_1, ..., p_n)$  be a tuple of positive real numbers. We define the following closed subset of X:

$$X(f/p,g) := \{x \in X : |f_i(x)| \le p_i|g(x)|, 1 \le i \le n\}.$$

Such a closed subset is called a  $rational\ domain$  of X. Moreover, we can define a  $\mathbf{k}$ -affinoid algebra

$$A\left\langle \underline{f/p}, g^{-1}\right\rangle := A\left\langle \frac{f_1}{p_1 g}, \dots, \frac{f_n}{p_n g}\right\rangle,$$

which is the quotient of the Tate algebra

$$A\langle T_1,\ldots,T_n\rangle$$

by the ideal generated by the elements  $p_igT_i - f_i$  for  $1 \le i \le n$ . There is a natural bounded homomorphism  $\varphi: A \to A\langle \underline{f/p}, g^{-1}\rangle$  induced by the inclusion. It can be shown that the closed subset  $X(\underline{f/p}, g)$  is an affinoid domain represented by the affinoid algebra  $A\langle \underline{f/p}, g^{-1}\rangle$ . Yang: To be checked

Yang: We have a sequence of inclusion:

 $\{\text{Weierstrass domains}\}\subseteq \{\text{Laurent domains}\}\subseteq \{\text{Rational domains}\}\subseteq \{\text{Affinoid domains}\}.$ 

**Proposition 5.** Let A be a **k**-affinoid algebra, and let  $X = \mathcal{M}(A)$  be the associated affinoid space. Let  $V \subseteq X$  be an affinoid domain represented by the **k**-affinoid algebra  $A_V$ . Then the natural bounded homomorphism  $\varphi : A \to A_V$  is flat.

We have  $\mathcal{M}(A_V) \cong V$ .

## 2 The Grothendieck topology of affinoid domains

## Appendix

