Spectrum of branch rings

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1 Semi-normed Rings and Modules

1.1 Semi-normed algebraic structures

Definition 1.1. Let M be an abelian group. A *semi-norm* on M is a function $\|\cdot\| : M \to \mathbb{R}_+$ such that

- $\|0\| = 0$;
- $\forall x, y \in M, \|x + y\| \le \|x\| + \|y\|$.

If we further have $\|x\| = 0 \iff x = 0$, then we say $\|\cdot\|$ is a *norm*. A *semi-normed abelian group* (resp. *normed abelian group*) is an abelian group equipped with a semi-norm (resp. norm).

Definition 1.2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group M. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in M, \|x\|_1 \le C\|x\|_2$.

Remark 1.3. If two semi-norms (resp. norms) on an abelian group M are bounded by each other, then they induce the same topology on M.

Definition 1.4. Let M be a semi-normed abelian group and $N \subseteq M$ be a subgroup. The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Remark 1.5. The residue semi-norm is a norm if and only if N is closed in M.

Definition 1.6. Let M and N be two semi-normed abelian groups. A group homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \le C\|x\|_M$. A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to$

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$\operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \inf_{y \in \ker f} \|x + y\|_M$.

Definition 1.7. Let R be a ring (commutative with unity). A *semi-norm* on R is a semi-norm $\|\cdot\|$ on the underlying abelian group of R such that $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$ and $\|1\| = 1$. A *semi-normed ring* is a ring equipped with a semi-norm.

Definition 1.8. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \geq 1$.

Definition 1.9. Let $(R, \|\cdot\|_R)$ be a normed ring. A *semi-normed R-module* is a pair $(M, \|\cdot\|_M)$ where M is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M such that there exists $C > 0$ with $\forall a \in R, x \in M, \|ax\|_M \leq C\|a\|_R\|x\|_M$.

One can talk about boundedness, admissibility and residue semi-norms in the contexts of semi-normed rings and semi-normed modules similar to those in semi-normed abelian groups.

1.2 banach rings

Definition 1.10. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 1.11. A *banach ring* is a complete normed ring.

Definition 1.12. Let $(A, \|\cdot\|_A)$ be a (semi-)normed algebraic structure, e.g., a (semi-)normed abelian group, a (semi-)normed ring, or a (semi-)normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a (pseudo-)metric space. Since A is dense in its completion, the algebraic operations and (semi-)norms on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M \otimes_R N} = \inf \left\{ \sum_i \|x_i\|_M \|y_i\|_N \, : \, z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N \right\}.$$

Definition 1.13. Let R be a complete normed ring and M, N complete semi-normed R-modules. The *complete tensor product* $M \widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Definition 1.14. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n \to \infty} \|f^n\|^{1/n}.$$

Definition 1.15. A banach ring R is called *uniform* if its norm is power-multiplicative.

Proposition 1.16. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 1.17. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\mathrm{Qnil}(R)$.

Definition 1.18. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Proposition 1.19. Let R be a banach ring. The completion of $R/\mathrm{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

1.3 Complete field

Definition 1.20. A multiplicative norm on a field is also called an *absolute value*. A *valuation field* is a field equipped with an absolute value.

Remark 1.21. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Definition 1.22. A valuation field $(\mathbf{k}, |\cdot|)$ is called *non-Archimedean* if $\forall x, y \in \mathbf{k}, |x + y| \leq \max\{|x|, |y|\}$, i.e., the norm satisfies the ultrametric inequality. Otherwise, it is called *Archimedean*.

Definition 1.23. A *complete field* is a valuation field which is complete as a metric space.

Lemma 1.24. Let $\mathbf{k}$ be a non-Archimedean complete field. Then the set $\mathbf{k}^\circ = \{x \in \mathbf{k} : |x| \leq 1\}$ is a subring of $\mathbf{k}$, which is a local ring. Moreover, the set $\mathbf{k}^{\circ\circ} = \{x \in \mathbf{k} : |x| < 1\}$ is the maximal ideal of $\mathbf{k}^\circ$.

Definition 1.25. Let $\mathbf{k}$ be a non-Archimedean complete field. The subring $\mathbf{k}^\circ$ is called the *ring of integers* of $\mathbf{k}$. The set $\mathbf{k}^{\circ\circ} = \{x \in \mathbf{k} : |x| < 1\}$ is the maximal ideal of $\mathbf{k}^\circ$. The residue field $\mathscr{k}_{\mathbf{k}} = \widetilde{\mathbf{k}} = \kappa_{\mathbf{k}} = \mathbf{k}^\circ/\mathbf{k}^{\circ\circ}$ is called the *residue field* of $\mathbf{k}$. Yang: To be revised.

Notation test $\mathscr{k}_{\mathbf{k}}$ or $\widetilde{\mathbf{k}}$ or $\kappa_{\mathbf{k}}$ for the residue field of $\mathbf{k}$. $\mathscr{k}_{\mathbb{Q}_p}$

1.4 Examples

Example 1.26. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a normed ring.

Example 1.27. The fields $\mathbb{C}$ and $\mathbb{R}$ equipped with the usual absolute value are complete fields.

Example 1.28. The field $\mathbb{Q}_p$ of p-adic numbers equipped with the p-adic norm is a complete non-Archimedean field.

Example 1.29. Let R be a banach ring and $r > 0$ be a real number. We define the ring of absolutely convergent power series over $\mathbf{k}$ with radius r as

$$R\langle T/r\rangle := \left\{\sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty\right\}.$$

Equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r\rangle$ is a banach ring.

When $R = \mathbf{k}$ is a Yang: To be checked.

Yang: To be continued...

2 Spectrum

2.1 Definition

Definition 2.1. Let R be a Banach ring. The *spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$. We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_{\geq 0}$, defined by $x \mapsto |f|_x =: f(x)$, is continuous.

Definition 2.2. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $\mathscr{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$ for each $x \in \mathscr{M}(S)$.

Definition 2.3. Let R be a Banach ring. A *character* of R is a bounded ring homomorphism $\chi : R \to K$, where K is a completed field. Two characters $\chi_1 : R \to K_1$ and $\chi_2 : R \to K_2$ are said to be *equivalent* if there exists a commutative diagram of bounded ring homomorphisms

$$\begin{tikzcd} & R \arrow[dl, "\chi_1"'] \arrow[d] \arrow[dr, "\chi_2"] & \\ K_1 & K \arrow[l, hook'] \arrow[r, hook] & K_2 \end{tikzcd}$$

for some completed field K.

Proposition 2.4. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is in bijection with the equivalence classes of characters of R.

Proof. Yang: To be completed □

Proposition 2.5. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, let $\wp_x$ be the kernel of the multiplicative semi-norm $|\cdot|_x$. Then $\wp_x$ is a closed prime ideal of R, and $x \mapsto \wp_x$ defines a continuous map from $\mathscr{M}(R)$ to $\mathsf{Spec}(R)$ equipped with the Zariski topology.

Proof. Yang: To be completed □

Definition 2.6. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, the *completed residue field* at the point x is defined as the completion of the residue field $\kappa(x) = \mathsf{Frac}(R/\wp_x)$ with respect to the multiplicative norm induced by the semi-norm $|\cdot|_x$, denoted by $\mathscr{H}(x)$.

Definition 2.7. Let R be a Banach ring. The *Gel'fand transform* of R is the bounded ring homomorphism

$$\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x), \quad f \mapsto (f(x))_{x \in \mathscr{M}(R)},$$

where the norm on the product $\prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is given by the supremum norm.

Proposition 2.8. The Gel'fand transform $\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ of a Banach ring R factors through the uniformization R^u of R, and the induced map $R^u \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is an isometric embedding. Yang: To be checked.

Theorem 2.9. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a nonempty compact Hausdorff space.

Proof. Yang: To be continued. □

Lemma 2.10. Let $\{K_i\}_{i \in I}$ be a family of completed fields. Consider the Banach ring $R = \prod_{i \in I} K_i$ equipped with the product norm. The spectrum $\mathscr{M}(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I.

Remark 2.11. The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

Proposition 2.12. Let K/k be a Galois extension of complete fields, and let R be a Banach k-algebra. The Galois group $\mathsf{Gal}(K/k)$ acts on the spectrum $\mathscr{M}(R \widehat{\otimes}_k K)$ via

$$g \cdot x : f \mapsto |(1 \otimes g^{-1})(f)|_x$$

for each $g \in \mathsf{Gal}(K/k)$, $x \in \mathscr{M}(R \widehat{\otimes}_k K)$ and $f \in R \widehat{\otimes}_k K$. Moreover, the natural map $\mathscr{M}(R \widehat{\otimes}_k K) \to \mathscr{M}(R)$ induces a homeomorphism

$$\mathscr{M}(R \widehat{\otimes}_k K)/\,\mathsf{Gal}(K/k) \xrightarrow{\sim} \mathscr{M}(R).$$

Yang: To be checked.

2.2 Examples

Example 2.13. Let $(\mathbf{k}, |\cdot|)$ be a complete valuation field. The spectrum $\mathscr{M}(\mathbf{k})$ consists of a single point corresponding to the given absolute value $|\cdot|$ on $\mathbf{k}$. Yang: To be checked.

Example 2.14. Consider the Banach ring $(\mathbb{Z}, \|\cdot\|)$ with $\|\cdot\| = |\cdot|_\infty$ is the usual absolute value norm on $\mathbb{Z}$. Let $|\cdot|_p$ denote the p-adic norm for each prime number p, i.e., $|n|_p = p^{-v_p(n)}$ for each $n \in \mathbb{Z}$, where $v_p(n)$ is the p-adic valuation of n. The spectrum

$$\mathscr{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0,1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0,\infty]\} \cup \{|\cdot|_0\},$$

where $|a|_p^\infty := \lim_{\alpha\to\infty} |a|_p^\alpha$ for each $a \in \mathbb{Z}$ and $|\cdot|_0$ is the trivial norm on $\mathbb{Z}$. Yang: To be checked.

Spectrum of Tate algebra in one variable Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{T/r\}$. We list some types of points in the spectrum $\mathscr{M}(A)$.

For each $a \in \mathbf{k}$ with $|a| \le r$, we have the *type I* point x_a corresponding to the evaluation at a, i.e., $|f|_{x_a} := |f(a)|$ for each $f \in A$. For each closed disk $E = E(a,s) := \{b \in \mathbf{k} : |b-a| \le s\}$ with center $a \in \mathbf{k}$ and radius $s \le r$, we have the point $x_{a,s}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} := \sup_{b\in E(a,s)} |f(b)|$$

for each $f \in A$. If $s \in |\mathbf{k}^\times|$, then the point x_E is called a *type II* point; otherwise, it is called a *type III* point.

Let $\{E^{(s)}\}_s$ be a family of closed disks in $\mathbf{k}$ such that $E^{(s)}$ is of radius s, $E^{(s_1)} \subsetneq E^{(s_2)}$ for any $s_1 < s_2$ and $\bigcap_s E^{(s)} = \emptyset$. Then we have the point $x_{\{E^{(s)}\}}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E^{(s)}\}}} := \inf_s |f|_{x_{E^{(s)}}}$$

for each $f \in A$. Such a point is called a *type IV* point.

Yang: To be completed.

Proposition 2.15. Let $\mathbf{k}$ be a complete non-archimedean field, and let $r > 0$ be a positive real number. Consider the Tate algebra $\mathbf{k}\{r^{-1}T\}$ equipped with the Gauss norm. The points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ can be classified into four types as described above. Yang: To be checked

Proof. Yang: To be completed. □

Proposition 2.16. Let $\mathbf{k}$ be a complete non-archimedean field, and let $r > 0$ be a positive real number. Consider the Tate algebra $\mathbf{k}\{r^{-1}T\}$ equipped with the Gauss norm. The completed residue fields of the four types of points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ are described as follows:

- For a type I point x_a with $a \in \mathbf{k}$ and $|a| \le r$, the completed residue field $\mathscr{H}(x_a)$ is isomorphic to $\mathbf{k}$.
- For a type II point $x_{a,s}$ with $a \in \mathbf{k}$ and $s \in |\mathbf{k}^\times|$, the completed residue field $\mathscr{H}(x_{a,s})$ is isomorphic to the field of Laurent series over the residue field $\mathscr{k}_{\mathbf{k}}$, i.e., $\mathscr{k}_{\mathbf{k}}((t))$.
- For a type III point $x_{a,s}$ with $a \in \mathbf{k}$ and $s \notin |\mathbf{k}^\times|$, the completed residue field $\mathscr{H}(x_{a,s})$ is

isomorphic to a transcendental extension of $\mathscr{k}_{\mathbf{k}}$ of degree one.

- For a type IV point $x_{\{E^{(s)}\}}$, the completed residue field $\mathscr{H}(x_{\{E^{(s)}\}})$ is isomorphic to a transcendental extension of $\mathscr{k}_{\mathbf{k}}$ of infinite degree.

Yang: To be checked.

Example 2.17. The completed residue field $\mathscr{H}(x_a)$ for a type I point x_a with $a \in \mathbf{k}$ and $|a| \leq r$ is isomorphic to $\mathbf{k}$. Yang: To be complete.