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1 Valuation fields

Definition 1.1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 1.2. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^{\times} \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^{\times}, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^{\times}, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Definition 1.3. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 1.4. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. Let $(\widehat{\mathbf{k}}, \|\cdot\|)$ be its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Definition 1.5. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Date: October 30, 2025, **Author**: Tianle Yang, My Homepage

2 Ultra-metric spaces

Definition 2.1. A metric space (X, d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.$$

Proposition 2.2. Let (X, d) be an ultra-metric space. Then for any $x \in X$ and $r > 0$, the closed ball $B(x, r) := \{y \in X : d(x, y) \leq r\}$ satisfies the following properties:

(a) For any $y \in B(x, r)$, we have $B(x, r) = B(y, r)$.

(b) Any two closed balls in X are either disjoint or one is contained in the other.

Yang: To be revised.

We will use $B(x, r)$ to denote the open ball with center x and radius r. We will use $E(x, r)$ to denote the closed ball with center x and radius r.

Proposition 2.3. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the singletons. Yang: To be revised.