

Valuation fields

1 Absolute values and completion

Definition 1. Let \mathbf{k} be a field. An *absolute value* on \mathbf{k} is a function $\|\cdot\| : \mathbf{k} \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

- (a) $\|x\| = 0$ if and only if $x = 0$;
- (b) $\|xy\| = \|x\| \cdot \|y\|$;
- (c) $\|x + y\| \leq \|x\| + \|y\|$.

A field \mathbf{k} equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 2. Let \mathbf{k} be a field. Recall that a *valuation* on \mathbf{k} is a function $v : \mathbf{k}^\times \rightarrow \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field \mathbf{k} by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \rightarrow \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Definition 3. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that \mathbf{k} is *complete* if the metric $d(x, y) := \|x - y\|$ makes \mathbf{k} a complete metric space.

Lemma 4. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. Let $(\widehat{\mathbf{k}}, \|\cdot\|)$ be its completion as a metric space. Then the operations of addition and multiplication on \mathbf{k} can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing \mathbf{k} as a dense subfield.

Proof. Yang: To be added. □

Unlike the real number field \mathbb{R} , even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 5. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in \mathbf{k} has a non-empty intersection.

Example 6. The field \mathbb{C}_p of p -adic complex numbers is not spherically complete, see Yang: to be added.

2 Non-archimedean fields

Definition 7. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that \mathbf{k} is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that \mathbf{k} is *archimedean*.

Let \mathbf{k} be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of \mathbf{k} . Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 8. Let \mathbf{k} be a non-archimedean field. The *ring of integers* of \mathbf{k} is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of \mathbf{k} is defined as

$$\kappa_{\mathbf{k}} := \tilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Yang: Is the valuation on residue field trivial?

Lemma 9. Recall that a metric space is *totally bounded* if for every $\varepsilon > 0$, it can be covered by finitely many balls of radius ε . A metric space is compact if and only if it is complete and totally bounded.

Proof. Yang: To be added. □

Definition 10. Let \mathbf{k} be a non-archimedean field. The *residue absolute value* on the residue field $\kappa_{\mathbf{k}}$ is defined as

$$|x| := \inf_{y \in \varphi^{-1}(x)} \|y\|, \quad \forall x \in \kappa_{\mathbf{k}},$$

where $\varphi : \mathbf{k}^\circ \rightarrow \kappa_{\mathbf{k}}$ is the canonical projection.

Proposition 11. Let \mathbf{k} be a non-archimedean field. Then the residue absolute value on the residue field $\kappa_{\mathbf{k}}$ is trivial.

Proof. For any $x \in \kappa_{\mathbf{k}}$, if $x = 0$, then by definition $|x| = 0$. If $x \neq 0$, then $\forall y \in \varphi^{-1}(x)$, we have $y \in \mathbf{k}^\circ \setminus \mathbf{k}^{\circ\circ}$, i.e., $\|y\| = 1$. Thus by definition $|x| = 1$. □

Slogan *Locally compact* \iff *complete + pro-finite*.