Analytic functions

1 Strictly differentiable functions

Recalling the definition of differentiable functions over valuation fields.

Definition 1. Let $\mathbf{k}$ be a valuation field and $U \subset \mathbf{k}$ be an open subset. A function $f : U \to \mathbf{k}$ is said to be *differentiable* at a point $a \in U$ if the limit

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists in $\mathbf{k}$. If f is differentiable at every point in U, we say that f is differentiable on U.

Unlike the case of real or complex analysis, the differentiable functions over non-archimedean fields may behave very differently. There exists differentiable functions with zero derivative that are not locally constant.

Proposition 2. Let $\mathbf{k}$ be a non-archimedean field. Then there exists a function $f : \mathbf{k} \to \mathbf{k}$ that is differentiable everywhere with $f'(x) = 0$ for all $x \in \mathbf{k}$, but f is not locally constant.

Proof. Fix $r \in (0, 1)$. Consider a descending sequence of open ball $\{B(0, r^n)\}$ and $a_n \in \mathbf{k}$ with $\|a_n\| = r^{2n}$. Define

$$f : \mathbf{k} \to \mathbf{k}, \quad x \mapsto \begin{cases} a_n, & x \in B(0, r^n) \setminus B(0, r^{n+1}) \\ 0, & x = 0 \end{cases}$$

Then

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{n \to \infty} \frac{a_n - 0}{x_n - 0}$$

for any sequence $x_n \to 0$ with $x_n \in B(0, r^n) \setminus B(0, r^{n+1})$. Since $\|x_n\| \geq r^{n+1}$, we have

$$\left\| \frac{a_n}{x_n} \right\| \leq \frac{r^{2n}}{r^{n+1}} = r^{n-1} \to 0$$

as $n \to \infty$. Thus $f'(0) = 0$ and then $f'(x) = 0$ for all $x \in \mathbf{k}$. However, f is not locally constant near 0. □

Definition 3. Let $\mathbf{k}$ be a valuation field and $U \subset \mathbf{k}$ be an open subset. A function $f : U \to \mathbf{k}$ is said to be *strictly differentiable* at a point $a \in U$ if the limit

$$f'(a) := \lim_{\substack{(x,y) \to (a,a) \\ x \neq y}} \frac{f(x) - f(y)}{x - y}$$

exists in $\mathbf{k}$. If f is strictly differentiable at every point in U, we say that f is strictly differentiable on U.

Remark 4. If $\mathbf{k}$ is a complete archimedean field (i.e., $\mathbb{R}$ or $\mathbb{C}$), then a function $f : U \to \mathbf{k}$ is strictly differentiable at a point $a \in U$ if and only if f is differentiable at a and the derivative f' is continuous

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Proposition 5. Let $\mathbf{k}$ be a non-archimedean complete field and $U \subset \mathbf{k}$ be an open subset. Suppose that $f : U \to \mathbf{k}$ is strictly differentiable and $f'(a) \neq 0$ for some $a \in U$. There exists an open neighborhood $V \subset U$ of a such that $x \mapsto f(x)/f'(a)$ is an isometry on V.

Proof. Yang: To be added. □

2 Tate algebras

Lemma 6. Let $\mathbf{k}$ be a non-archimedean field and $\sum_{n=0}^{+\infty} a_n$ be a series in $\mathbf{k}$. Then the series $\sum_{n=0}^{+\infty} a_n$ converges if and only if $\lim_{n\to+\infty} a_n = 0$.

Proof. The necessity is clear and true for all fields. Suppose that $\lim_{n\to+\infty} a_n = 0$.

Yang: To be added. □

Proposition 7. Let $\mathbf{k}$ be a non-archimedean field and $f = \sum_{n=0}^{+\infty} a_n x^n \in \mathbf{k}[[x]]$. Set

$$R := \frac{1}{\limsup_{n\to+\infty} \|a_n\|^{1/n}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Then we have

(a) if $R = 0$, then the series $f(x)$ converges only at $x = 0$;

(b) if $R = +\infty$, then the series $f(x)$ converges for all $x \in \mathbf{k}$;

(c) if $0 < R < +\infty$, then the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$ and diverges for all $x \in \mathbf{k}$ with $\|x\| > R$.

Suppose that $0 < R < +\infty$. Then the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| = R$ if and only if $\lim_{n\to+\infty} \|a_n\| R^n = 0$.

Proof. Yang: To be added. □

Notation 8. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $E(r,x) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| \leq r_i, i = 1, \dots, n\}$ and $B(r,x) = \{y \in \mathbf{k}^n \mid \|y_i - x_i\| < r_i, i = 1, \dots, n\}$ for $x = (x_1, \dots, x_n) \in \mathbf{k}^n$.

Definition 9. Let $\mathbf{k}$ be a complete non-archimedean field. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *restricted power series*) is defined as

$$\mathbf{k}\langle \underline{r^{-1}T} \rangle := \mathbf{k}\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in \mathbf{k}, \lim_{\|\alpha\| \to +\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 10. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\langle \underline{T/r} \rangle$ is a Banach $\mathbf{k}$-algebra with respect to the *Gauss norm*

$$\left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Proof. Yang: To be added. □

Proposition 11. Let $\mathbf{k}$ be a complete non-archimedean field. Then the Tate algebra $\mathbf{k}\langle \underline{T/r} \rangle$ can be identified with a subring of the ring of all functions from the closed polydisc $E(0, r)$ to $\mathbf{k}$.

Proof. Yang: To be added. □

Proposition 12. Let $\mathbf{k}$ be a complete non-archimedean field. Then the gauss norm on the Tate algebra $\mathbf{k}\langle x_1, \dots, x_n \rangle$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in D^n} \|f(x)\|.$$

Proof. Yang: To be added. □

3 Fundamental properties

Then following shows that analytic functions over non-archimedean fields share some nice properties as in the case of complex analysis. Yang: To be revised.

Theorem 13. Let $(\mathbf{k}, \|\cdot\|)$ be a complete non-archimedean field and $U \subset \mathbf{k}$ be an open subset. If $f : U \to \mathbf{k}$ is an analytic function, then f is locally Lipschitz continuous on U. Yang: To be checked.

Theorem 14 (Strassman)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ be an analytic function on the closed unit disc in $\mathbf{k}$. Then f has only finitely many zeros in the closed unit disc unless f is identically zero. Yang: To be checked.