Valuation fields

1 absolute values and completion

Definition 1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot|: \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field*.

Remark 2. Let $\mathbf{k}$ be a field. Recall that a *valuation* on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;
- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

In some literature, the valuation v is called an *additive valuation* and the induced absolute value $|\cdot|_v$ is called a *multiplicative valuation*. In this note, the term *valuation* always refers to the additive valuation.

Definition 3. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 4. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. Let $(\widehat{\mathbf{k}}, \|\cdot\|)$ be its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Definition 5. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

2 Non-archimedean fields and ultra-metric spaces

Definition 6. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Date: October 26, 2025, **Author**: Tianle Yang, My Homepage

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 7. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathscr{k}_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^\circ/\mathbf{k}^{\circ\circ}.$$

Yang: Is the valuation on residue field trivial?

Theorem 8 (Hessel's lemma)**.** Let $\mathbf{k}$ be a non-archimedean field and $\mathscr{k}_{\mathbf{k}}$ be its residue field. For any polynomial $\widetilde{f}(X) \in \mathscr{k}_{\mathbf{k}}[X]$ and any simple root $\widetilde{a} \in \mathscr{k}_{\mathbf{k}}$ of $\widetilde{f}(X)$, there exists a root $a \in \mathbf{k}^\circ$ of $f(X) \in \mathbf{k}^\circ[X]$ such that the image of a in $\mathscr{k}_{\mathbf{k}}$ is $\widetilde{a}$. Yang: To be checked.

Definition 9. A metric space (X, d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}, \quad \forall x,y,z \in X.$$

Proposition 10. Let (X, d) be an ultra-metric space. Then for any $x \in X$ and $r > 0$, the closed ball $B(x,r) := \{y \in X : d(x,y) \leq r\}$ satisfies the following properties:

(a) For any $y \in B(x,r)$, we have $B(x,r) = B(y,r)$.

(b) Any two closed balls in X are either disjoint or one is contained in the other.

Yang: To be revised.

We will use $B(x,r)$ to denote the open ball with center x and radius r. We will use $E(x,r)$ to denote the closed ball with center x and radius r.

Proposition 11. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the singletons. Yang: To be revised.

3 p-adic fields

Construction 12. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_\mathfrak{p}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_\mathfrak{p} : K^\times \to \mathbb{Z}$ the corresponding discrete valuation. The $\mathfrak{p}$*-adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_\mathfrak{p} := N(\mathfrak{p})^{-v_\mathfrak{p}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the $\mathfrak{p}$-adic absolute value $|\cdot|_\mathfrak{p}$ is denoted by $K_\mathfrak{p}$, called the $\mathfrak{p}$*-adic field*.

One can just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p.

Example 13. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The p*-adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Construction 14. Let p be a prime number. The field $\mathbb{C}_p$ of p*-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$. The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$. Yang: To be completed.

Proposition 15. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Construction 16. Let p be a prime number. Yang: We construct the *spherically complete* p*-adic field* Ω_p. Yang: To be completed.

Yang: What is the relation between the finite extension of $\mathbb{Q}_p$ and $K_\mathfrak{p}$?

Appendix