

Forms and Currents

Let M be a complex manifold of complex dimension d .

1 Differential forms

Recall that we have the decomposition of the cotangent bundle:

$$\Omega_{\text{sm}}^{1,\mathbb{C}} \cong \Omega_{\text{sm}}^{1,0} \oplus \Omega_{\text{sm}}^{0,1} \cong \Omega_{\text{sm}}^1 \oplus \overline{\Omega_{\text{sm}}^1}.$$

Take exterior powers, we have $\Omega_{\text{sm}}^{k,\mathbb{C}} \cong \bigoplus_{p+q=k} \Omega_{\text{sm}}^{p,q}$, where $\Omega_{\text{sm}}^{p,q} = \bigwedge^p \Omega_{\text{sm}}^{1,0} \otimes \bigwedge^q \Omega_{\text{sm}}^{0,1}$. We also use the notation

$$\mathcal{A}^{p,q} := \Omega_{\text{sm}}^{p,q}, \quad \mathcal{A}^k := \Omega_{\text{sm}}^{k,\mathbb{C}}.$$

A reason to induce this strange sheaf $\mathcal{A}^k = \Omega_{\text{sm}}^{k,\mathbb{C}}$ is to make sense of integration of top-degree forms. For simplicity, assume that M is compact. Let $\omega \in \mathcal{A}^{2d}(M)$ be a smooth complex-valued $2d$ -form on M . Then its integration is well-defined in the smooth manifold sense:

$$\int_M \omega \in \mathbb{C}.$$

However, in complex case, it is more natural to “integral” a holomorphic d -form on a d -dimensional complex manifold. This does not make sense in the smooth manifold theory. The solution is to associate a holomorphic d -form $\eta \in \mathcal{A}^{d,0}(M)$ with a smooth $2d$ -form ((d, d)-form) $\omega = \eta \wedge \bar{\eta} \in \mathcal{A}^{d,d}(M) \subset \mathcal{A}^{2d}(M)$.

Another reason is that $\bigoplus_k \Omega_{\text{sm}}^k$ is not closed under the exterior derivative d , while $\bigoplus_k \Omega_{\text{sm}}^{k,\mathbb{C}}$ is. Suppose that we have local holomorphic coordinates (z_1, \dots, z_d) . Recall that we have the exterior derivative

$$d : \Omega_{\text{sm}}^{1,0} \rightarrow \Omega_{\text{sm}}^{2,\mathbb{C}}, \quad f dz_i \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge dz_i + \sum_{i=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_i.$$

On its conjugation, we have

$$d : \Omega_{\text{sm}}^{0,1} \rightarrow \Omega_{\text{sm}}^{2,\mathbb{C}}, \quad f d\bar{z}_i \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge d\bar{z}_j + \sum_{j=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_j.$$

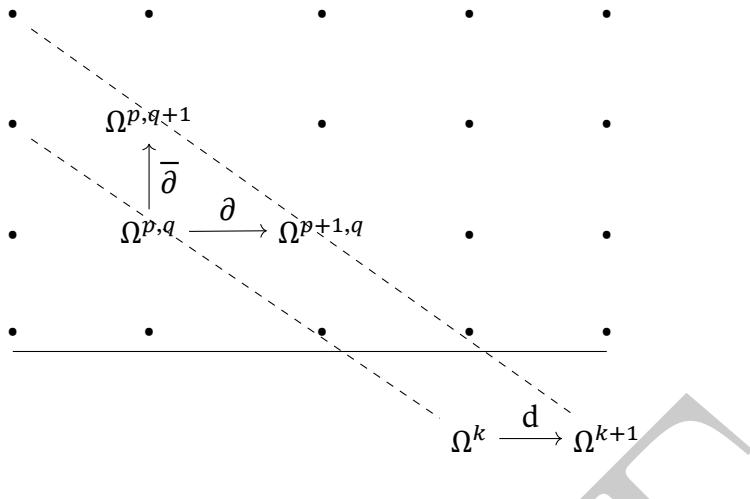
Extending d by linearity and the Leibniz rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$, we get the exterior derivative

$$d : \mathcal{A}^k = \Omega_{\text{sm}}^{k,\mathbb{C}} \rightarrow \mathcal{A}^{k+1} = \Omega_{\text{sm}}^{k+1,\mathbb{C}},$$

which can be decomposed as $d = \partial + \bar{\partial}$, where

$$\begin{aligned} \partial : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{p+1,q}, \quad f dz_I \wedge d\bar{z}_J \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J, \\ \bar{\partial} : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{p,q+1}, \quad f dz_I \wedge d\bar{z}_J \mapsto \sum_{j=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J. \end{aligned}$$

In a diagram, we have:



Proposition 1. The operators ∂ and $\bar{\partial}$ satisfy

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Proof. Yang: To be added. □

Proposition 2. Let $f : M \rightarrow N$ be a holomorphic map between complex manifolds. Then the pull-back of differential forms $f^* : \mathcal{A}_N^k \rightarrow \mathcal{A}_M^k$ satisfies

$$f^*(\mathcal{A}_N^{p,q}) \subset \mathcal{A}_M^{p,q}, \quad f^* \circ \partial_N = \partial_M \circ f^*, \quad f^* \circ \bar{\partial}_N = \bar{\partial}_M \circ f^*.$$

Proof. Yang: To be added. □

Construction 3. To define a topology on $\mathcal{A}^k(M)$, we use the collection of seminorms as follows. Let $U \subset M$ be a coordinate chart with local real coordinates (x_1, \dots, x_{2d}) . Given $\omega \in \mathcal{A}^k(M)$, we can write $\omega|_U$ as $\omega|_U = \sum_I f_I dx_I$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_{2d})$, we can define the partial derivative

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_{2d}^{\alpha_{2d}}}.$$

Then for any compact subset $K \subset U$ and non-negative integer m , we can define a seminorm on $\mathcal{A}^k(M)$ by

$$p_{U,K,m}(\omega) = \sup_{x \in K} \max_I \max_{|\alpha| \leq m} |D^\alpha f_I(x)|,$$

where m is a non-negative integer.

The topology on $\mathcal{A}^k(M)$ is defined to be the weakest topology such that all seminorms $p_{U,K,m}$ are continuous, where U runs over all coordinate charts of M , K runs over all compact subsets of U , and m runs over all non-negative integers.

Proposition 4. Let $f : M \rightarrow N$ be a holomorphic map between complex manifolds. Then the pull-back of differential forms $f^* : \mathcal{A}^k(N) \rightarrow \mathcal{A}^k(M)$ is a continuous linear map with respect to the topologies defined above. *Yang: To be checked*

Proposition 5. The space $\mathcal{A}^k(M)$ equipped with the topology defined above is a Fréchet space, i.e., a complete metrizable locally convex topological vector space. *Yang: To be checked*

2 Currents

We denote by $\mathcal{A}_c^k(M)$ (resp. $\mathcal{A}_c^{p,q}(M)$) the space of smooth complex-valued k -forms (resp. (p, q) -forms) on M with compact support. It is also denoted by $\mathcal{D}^k(M)$.

Definition 6. A *current* of dimension k or degree $(2d - k)$ on M is a continuous linear functional

$$T : \mathcal{D}^k(M) \rightarrow \mathbb{C}.$$

We denote the space of currents of dimension k on M by $\mathcal{D}'_k(M)$ or $\mathcal{D}'^{2d-k}(M)$. **Yang:** To be revised.

Appendix

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