Metrics, curvature and connections

1 The first properties

Let X be a complex manifold and $\mathcal{E} \to X$ a holomorphic vector bundle.

Definition 1. A *Hermitian metric* on $\mathcal{E}$ is a smoothly varying family of Hermitian inner products $\langle\cdot,\cdot\rangle_x$ on the fibers $\mathcal{E}_x$ for each $x \in X$, i.e., Yang: To be continued.

Definition 2. A *Hermitian metric* on a complex manifold X is a Hermitian metric on its holomorphic tangent bundle $\mathcal{T}_X$. It induces a Riemannian metric g on the underlying real manifold of X by

$$g(u,v) = \operatorname{Re}\langle u,v\rangle$$

for tangent vectors $u,v \in \mathcal{T}_{X,x}$. The associated $(1,1)$-form ω is defined by

$$\omega(u,v) = g(Ju,v)$$

where J is the almost complex structure on X. Yang: To be continued.

Example 3. Let $\mathbb{P}^n$ be the complex projective space. The *Fubini-Study metric* h_{FS} is a Hermitian metric on its tangent bundle $\mathcal{T}\mathbb{P}^n$ defined as follows. On the standard affine chart $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$ with coordinates $z_{j,i} = z_j/z_i$ for $j \neq i$, we know that $\mathcal{T}\mathbb{P}^n|_{U_i}$ is spanned by $\{\partial_{j,i} = \partial/\partial z_{j,i}\}_{j\neq i}$. The Fubini-Study metric is given by

$$h_{\mathrm{FS}}(z_{-,i})(\partial_k,\partial_l) = \frac{\delta_{kl}}{(1+\sum_{r\neq i}|z_{r,i}|^2)} - \frac{\overline{z_{k,i}}z_{l,i}}{(1+\sum_{r\neq i}|z_{r,i}|^2)^2}.$$

On $U_{ij} = U_i \cap U_j$, the tangent vectors transform as

$$\partial_{k,j} = \begin{cases} z_{i,j}\partial_{k,i}, & k \neq i \\ -z_{k,j}^2\partial_{i,i}, & k = i \end{cases}.$$

One can check that the above definition is consistent on the overlaps U_{ij}. Yang: To be continued.

Example 4. Now let us consider the complex projective plane $\mathbb{P}^2 = \{[X : Y : Z]\}$. On the affine chart $U_Z = \{[X : Y : Z] \mid Z \neq 0\}$ with coordinates $x = X/Z$ and $y = Y/Z$, the Fubini-Study metric h_{FS} on $\mathcal{T}\mathbb{P}^2|_{U_Z}$ is given by

$$h_{\mathrm{FS}}(x,y) = \frac{1}{(1+|x|^2+|y|^2)^2}\begin{bmatrix} 1+|y|^2 & -\overline{x}y \\ -x\overline{y} & 1+|x|^2 \end{bmatrix}.$$

For a tangent vector $\partial = a\partial_x + b\partial_y$, its norm squared is

$$\|\partial\|_{h_{\mathrm{FS}}}^2 = \frac{(1+|y|^2)|a|^2 + (1+|x|^2)|b|^2 - 2\operatorname{Re}(\overline{x}ya\overline{b})}{(1+|x|^2+|y|^2)^2} = \frac{|a|^2+|b|^2+|xb-ya|^2}{(1+|x|^2+|y|^2)^2} \geq 0.$$

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Definition 5. Let X be a complex manifold and $\mathcal{E} \to X$ a holomorphic vector bundle. A *connection* on $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \Gamma(X, \mathcal{E}) \to \Gamma(X, T^*X \otimes \mathcal{E})$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all smooth functions f on X and sections s of E. Yang: To be continued.

Example 6. Let $\mathbb{P}^n$ be the complex projective space and $\mathcal{O}_{\mathbb{P}^n}(1)$ the hyperplane line bundle. The *Chern connection* associated with the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ is a connection defined as follows: For a section s of $\mathcal{O}_{\mathbb{P}^n}(1)$, we define

$$\nabla s = ds + \alpha s,$$

where α is a (1,0)-form determined by the Fubini-Study metric. Yang: To be continued.

Proposition 7. Let X be a complex manifold, $E \to X$ a holomorphic vector bundle equipped with a Hermitian metric h. Then there exists a unique connection ∇ on E that is compatible with both the holomorphic structure and the Hermitian metric h. Yang: To be checked.

Definition 8. Let X be a complex manifold, $E \to X$ a holomorphic vector bundle, and ∇ a connection on E. The *curvature* of the connection ∇ is defined as the $\mathcal{O}_X$-linear map

$$F_\nabla : \Gamma(X, E) \to \Gamma(X, \Lambda^2 T^*X \otimes E)$$

given by

$$F_\nabla(s) = \nabla^2 s = \nabla(\nabla s)$$

for all sections s of E. Yang: To be continued.

2 Chern-Weil Theory

Theorem 9. Let X be a complex manifold and $E \to X$ a holomorphic vector bundle equipped with a Hermitian metric h. Let ∇ be the unique connection on E compatible with both the holomorphic structure and the Hermitian metric h, and let F_∇ be its curvature. Then the Chern classes $c_k(E) \in H^{2k}(X, \mathbb{R})$ can be represented by the differential forms

$$c_k(E) = \left[\frac{1}{(2\pi i)^k} \operatorname{Tr}\left(F_\nabla^k\right)\right].$$

Yang: To be checked.

3 the Kähler condition

Definition 10. A *Kähler manifold* is a complex manifold X equipped with a Hermitian metric h whose associated (1,1)-form ω is closed, i.e., $d\omega = 0$. Yang: To be checked.

Appendix