Meromorphic functions and Siegel theorem on function fields

1 Meromorphic functions

Definition 1. Let M be a complex manifold. A *meromorphic function* on M is a holomorphic map $f : M \to \mathbb{CP}^1$.
The set of meromorphic functions on M is denoted by $\mathrm{Mer}(M)$ or $\mathscr{K}(M)$.

Proposition 2. Let M be a complex manifold. Then there is a natural inclusion $\mathrm{Hol}(M) \hookrightarrow \mathrm{Mer}(M)$. Moreover, we have $\mathrm{Mer}(M) = \mathrm{Frac}(\mathrm{Hol}(M))$, i.e., every meromorphic function can be expressed as a quotient of two holomorphic functions. Yang: to be checked.

Proposition 3. Let M be a complex manifold. The set of meromorphic functions on M forms a field under the usual addition and multiplication of functions.

Yang: To be complemented and revised.

2 Siegel theorem

Proposition 4. Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset $U \subset \mathbb{C}^n$. Suppose that f has order k at a point $x \in U$. Then there exists a neighborhood $\overline{B(x,r)} \subset U$ of x such that

$$|f(z)| \leq C|z-x|^k, \quad \forall z \in \overline{B(x,r)},$$

where $C = \sup_{z \in \partial \overline{B(x,r)}} |f(z)|$. Yang: To be revised.

Theorem 5 (Siegel theorem on function fields)**.** Let X be a connected and compact complex manifold of dimension n. Then the field of meromorphic functions on X satisfies

$$\mathrm{trdeg}_{\mathbb{C}}\, \mathscr{K}(X) \leq n.$$

Proof. Let $\{f_1, f_2, \dots, f_{n+1}\} \subset \mathscr{K}(X)$ be meromorphic functions on X. We want to find $P \in \mathbb{C}[x_1, x_2, \dots, x_{n+1}] \setminus \{0\}$ such that

$$P(f_1, f_2, \dots, f_{n+1}) = 0.$$

Step 1. Let $z \in X$, there exists $g_1, g_2, \dots, g_{n+1}, h \in \mathrm{Hol}(X)$ such that $f_i = g_i/h$ for each $1 \leq i \leq n+1$.

Yang: To be revised and complemented. □

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Appendix