Analysis in several complex variables

In this section, we introduce some basic concepts and results in complex analysis with multiple variables.

1 Holomorphic functions

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Definition 1. A continuous map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is *differentiable* at $p \in \mathbb{R}^{2n}$ if there exists a linear map $\mathrm{d}f_p : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ such that

$$f(z) = f(p) + \mathrm{d}f_p(z - p) + o(|z - p|).$$

A continuous map $f : \mathbb{C}^n \to \mathbb{C}^m$ is *holomorphic* at $p \in \mathbb{C}^n$ if it is differentiable at p and $\mathrm{d}f_p$ is $\mathbb{C}$-linear, i.e., $\mathrm{d}f_p(\sqrt{-1}z) = \sqrt{-1}\mathrm{d}f_p(z)$ for all $z \in \mathbb{C}^n$.

By a "function", we always mean a complex-valued function, i.e., a map $f : \mathbb{C}^n \to \mathbb{C}$. Fix a coordinate system $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n$ and write $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Then a differentiable function $f = u + iv : \mathbb{C}^n \to \mathbb{C}$ is holomorphic at p if and only if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x_i}(p) = \frac{\partial v}{\partial y_i}(p), \quad \frac{\partial u}{\partial y_i}(p) = -\frac{\partial v}{\partial x_i}(p), \quad i = 1, \dots, n.$$

For convenience, we consider the complexified tangent space $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and introduce the following operators.

Definition 2. The *Wirtinger operators* are defined as

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad j = 1, \dots, n.$$

Then we can rewrite the Cauchy-Riemann equations as

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n.$$

We summarize some properties of Wirtinger operators in the following proposition.

Proposition 3. The Wirtinger operators satisfy the following properties:

(a) $\partial_{z_j} z_i = \delta_{ij}$, $\partial_{z_j}\bar{z}_i = 0$, $\partial_{z_j}\bar{z}_i = 0$, $\partial_{z_j}\bar{z}_j = \delta_{ij}$;

(b) $\overline{\left(\partial_{z_j} f\right)} = \partial_{\bar{z}_j}\bar{f}$;

(c) suppose we have $\mathbb{C}^n \xrightarrow{g} \mathbb{C}^m \xrightarrow{f} \mathbb{C}^l$ and the coordinate on $\mathbb{C}^m$ is $w = (w_1, \dots, w_m)$, then the chain

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rule holds:

$$\frac{\partial(f\circ g)}{\partial z_j} = \sum_{k=1}^{m} \frac{\partial f}{\partial w_k}(g(z))\frac{\partial g_k}{\partial z_j}(z) + \sum_{k=1}^{m} \frac{\partial f}{\partial \bar{w}_k}(g(z))\frac{\partial \bar{g}_k}{\partial z_j}(z),$$
$$\frac{\partial(f\circ g)}{\partial \bar{z}_j} = \sum_{k=1}^{m} \frac{\partial f}{\partial w_k}(g(z))\frac{\partial g_k}{\partial \bar{z}_j}(z) + \sum_{k=1}^{m} \frac{\partial f}{\partial \bar{w}_k}(g(z))\frac{\partial \bar{g}_k}{\partial \bar{z}_j}(z).$$

Proof. By direct computation. □

We can also consider the complexified of derivatives

$$(\mathrm{d}f_p)_{\mathbb{C}} : T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \to T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}.$$

If we take $\{\partial_{z_i}, \partial_{\bar{z}_i}\}_{i=1}^{n}$ as a basis of $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and $\{\partial_{w_j}, \partial_{\bar{w}_j}\}_{j=1}^{m}$ as a basis of $T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$, then the matrix representation of $(\mathrm{d}f_p)_{\mathbb{C}}$ is

$$(\mathrm{d}f_p)_{\mathbb{C}} = \begin{bmatrix} \partial_z f(p) & \partial_{\bar{z}} f(p) \\ \overline{\partial_{\bar{z}} f(p)} & \overline{\partial_z f(p)} \end{bmatrix}.$$

In particular, if f is holomorphic, then we have $\det(\mathrm{d}f_p)_{\mathbb{C}} = |\det(\partial_z f)(p)|^2 \geq 0$.

Definition 4. A map $f : \Omega \to \Omega'$ between two open sets $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^m$ is *biholomorphic* if it is a bijection and both f and f^{-1} are holomorphic.

If f is biholomorphic at p, then $m = n$ and $\det \mathrm{d}f_p > 0$.

Theorem 5 (Holomorphic Inverse Function Theorem)**.** Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. If the Jacobian determinant $\det \mathrm{d}f_p$ is nonzero at $p \in \mathbb{C}^n$, then there exist open neighborhoods U of p and V of $f(p)$ such that $f : U \to V$ is a biholomorphism.

Proof. By the real inverse function theorem, there exist open neighborhoods U of p and V of $f(p)$ such that $g = f^{-1} : V \to U$ is a differentiable map. It suffices to show that g is holomorphic. By the chain rule (Proposition 3), since f is holomorphic, we have

$$0 = \left(\frac{\partial (f\circ g)_i}{\partial \bar{z}_j}\right)(q) = \left(\frac{\partial f_i}{\partial w_k}\right)(g(q))\left(\frac{\partial g_k}{\partial \bar{z}_j}\right)(q).$$

Since $\det(\partial f/\partial w)(f(q)) \neq 0$, the matrix $(\partial f_i/\partial w_k)(g(q))$ is invertible, which implies that $(\partial g_k/\partial \bar{z}_j)(q) = 0$ for all k, j. Thus g is holomorphic. □

Theorem 6 (Holomorphic Implicit Function Theorem)**.** Let $f : \mathbb{C}^{n+m} \to \mathbb{C}^m$ be a holomorphic map. Write the coordinates of $\mathbb{C}^{n+m}$ as $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{C}^n \times \mathbb{C}^m$. If $\det(\partial f/\partial w) \neq 0$ at $(z_0, w_0) \in \mathbb{C}^{n+m}$ with $f(z_0, w_0) = 0$, then there exist open neighborhoods U of z_0 and V of w_0, and a unique holomorphic map $g : U \to V$ such that for any $(z, w) \in U \times V$, $f(z, w) = 0$ if and only if $w = g(z)$.

Proof. By real implicit function theorem, there exist differentiable map $g : U \to V$ satisfying the above condition. It suffices to show that g is holomorphic. Let $G : U \to U \times V$ be defined by

$G(z) = (z, g(z))$. Then we have $f \circ G = 0$. By the chain rule, we have

$$0 = \frac{\partial (f \circ G)_i}{\partial \bar{z}_j}(q) = \sum_{k=1}^{n} \frac{\partial f_i}{\partial w_k}(G(q)) \frac{\partial z_k}{\partial \bar{z}_j}(q) + \sum_{l=1}^{m} \frac{\partial f_i}{\partial w_l}(G(q)) \frac{\partial g_l}{\partial \bar{z}_j}(q).$$

Since $\det(\partial f/\partial w)(G(q)) \neq 0$, the matrix $(\partial f_i/\partial w_k)(G(q))$ is invertible, which implies that $(\partial g_l/\partial \bar{z}_j)(q) = 0$ for all l, j. Thus g is holomorphic. □

2 Cauchy Integral Formula

Recall the Cauchy Integral Formula in one complex variable:

Theorem 7 (Cauchy Integral Formula in one complex variable)**.** Let $K \subset \mathbb{C}$ be a compact set with piecewise differentiable boundary ∂K, and let f be differentiable on a neighborhood of K. Then for any z in the interior of K, we have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_{K} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

Proof. Yang: By Stokes' theorem. To be continued... □

Theorem 8 (Cauchy Integral Formula in several complex variables)**.** Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D. Then for any $z \in D$,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial D_1 \times \cdots \times \partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n.$$

Proof. Yang: To be continued... □

Corollary 9. Holomorphic functions are analytic. Yang: To be continued...

Proposition 10. Holomorphic functions are open mappings. Yang: To be continued...

Proposition 11. If a holomorphic function $f : \Omega \to \mathbb{C}$ on a connected open set $\Omega \subset \mathbb{C}^n$ attains its maximum at some point in Ω, then f is constant. Yang: To be continued...

Lemma 12. Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D. Then for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\max_{z \in D} \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(z) \right| \leq \frac{\alpha!}{r^\alpha} \max_{z \in D} |f(z)|,$$

where $r = (r_1, \dots, r_n)$ is the radius of the polydisk D. Yang: To be continued...

Theorem 13 (Generalized Liouville Theorem)**.** A holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ on the whole space $\mathbb{C}^n$ that satisfies a polynomial growth condition, i.e., there exist constants $C > 0$ and $k \geq 0$ such that

$$|f(z)| \leq C(1 + |z|^k), \quad \forall z \in \mathbb{C}^n,$$

must be a polynomial of degree at most k. Yang: To be continued...

Theorem 14 (Montel's Theorem)**.** A family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ that is uniformly bounded on compact subsets of Ω is a normal family, i.e., every sequence in the family has a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function or to infinity. Yang: To be continued...

3 Zero sets of holomorphic functions

Theorem 15 (Hartogs' Extension Theorem)**.** Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $K \subset D$ be a compact subset such that $D \setminus K$ is connected. If $f : D \setminus K \to \mathbb{C}$ is a holomorphic function, then there exists a unique holomorphic function $\tilde{f} : D \to \mathbb{C}$ such that $\tilde{f}|_{D\setminus K} = f$. Yang: To be continued...

Proof. Yang: To be checked □

Corollary 16. In contrast to the one-variable case, isolated singularities do not exist in several complex variables. Specifically, if $f : D \setminus \{p\} \to \mathbb{C}$ is a holomorphic function on a domain $D \subset \mathbb{C}^n$ with $n \geq 2$ and $p \in D$, then f can be extended to a holomorphic function on the entire domain D.

Proof. This is a direct consequence of Hartogs' Extension Theorem by taking $K = \{p\}$. □

Theorem 17 (Weierstrass Preparation Theorem)**.** Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function in a neighborhood of the origin such that $f(0) = 0$ and f is not identically zero. Write the coordinates as $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that $f(0, w)$ has a zero of order k at $w = 0$, i.e.,

$$f(0, w) = a_k w^k + a_{k+1} w^{k+1} + \cdots, \quad a_k \neq 0.$$

Then there exists a neighborhood U of the origin and unique holomorphic functions $g : U \to \mathbb{C}$ and $h_j : U' \to \mathbb{C}$ for $j = 1, \dots, k$, where $U' \subset \mathbb{C}^n$ is the projection of U onto the first n coordinates, such that

$$f(z, w) = (w^k + h_1(z) w^{k-1} + \cdots + h_k(z)) g(z, w),$$

with $g(0) \neq 0$ and $h_j(0) = 0$ for all j. Yang: To be continued...

Proof. Yang: To be continued... Yang: Use the Cauchy Integral Formula to check the holomorphicity of g and h_j. □

Definition 18. Let $\Omega \subset \mathbb{C}^n$ be an open set. The *sheaf of holomorphic functions* on Ω, denoted by $\mathcal{O}_\Omega$, is the assignment that to each open subset $U \subset \Omega$ assigns the ring $\mathcal{O}_\Omega(U)$ of all holomorphic functions on U, and set the restriction as the usual restriction of functions.

A fundamental property of the sheaf of holomorphic functions is its coherence.

Theorem 19 (Oka's Coherence Theorem)**.** The sheaf of holomorphic functions $\mathcal{O}_\Omega$ on an open set $\Omega \subset \mathbb{C}^n$ is a coherent sheaf. Yang: To be continued...

In general, $\mathcal{O}_\Omega(U)$ is not a Noetherian ring for an open set $U \subset \Omega$. However, its stalks $\mathcal{O}_{\Omega,p}$ at points

$p \in \Omega$ are Noetherian rings. Yang: To be checked

Example 20. Yang: To be continued...

Proposition 21. For any point $p \in \Omega$, the stalk $\mathcal{O}_{\Omega,p}$ of the sheaf of holomorphic functions at p is a Noetherian ring. Yang: To be continued...

Remark 22. The sheaf of holomorphic functions $\mathcal{O}_\Omega$ is a sheaf of topological rings, where the topology on $\mathcal{O}_\Omega(U)$ for an open set $U \subset \Omega$ is given by the compact-open topology. Yang: To be continued...

Definition 23. A subset $A \subset \Omega$ of an open set $\Omega \subset \mathbb{C}^n$ is called an *analytic subset* if for every point $p \in \Omega$, there exists a neighborhood U of p and finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_\Omega(U)$ such that

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = \cdots = f_k(z) = 0\}.$$

Yang: To be continued...