

Metrics, curvature and connections

1 The first properties

Let X be a complex manifold and $E \rightarrow X$ a holomorphic vector bundle.

Definition 1. A *hermitian metric* on E is a smoothly varying family of hermitian inner products $\langle \cdot, \cdot \rangle_x$ on the fibers E_x for each $x \in X$, i.e.,

$$\langle \cdot, \cdot \rangle_x : E_x \times E_x \rightarrow \mathbb{C}$$

is a hermitian inner product for each x , and for any local smooth sections s, t of E , the function

$$x \mapsto \langle s(x), t(x) \rangle_x$$

is smooth on X . A vector bundle equipped with a hermitian metric is called a *hermitian vector bundle*.

Definition 2. A *hermitian metric* on a complex manifold X is a hermitian metric on its holomorphic tangent bundle TX .

Remark 3. Let h be a hermitian metric on a complex manifold X . Then h induces a Riemannian metric g on the underlying real manifold of X by

$$g(u, v) = \operatorname{Re}(h(u, v))$$

for real tangent vectors $u, v \in T_x X$. **Yang: To be revised.**

Example 4. Let \mathbb{P}^n be the complex projective space. The *Fubini-Study metric* h_{FS} is a hermitian metric on its tangent bundle $T\mathbb{P}^n$ constructed as follows. Recall the **Yang: $\mathbb{P}^n(\mathbb{C}) \cong S^{2n+1}/S^1$** as real manifolds.

On the standard affine chart $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$ with coordinates $z_{j,i} = z_j/z_i$ for $j \neq i$, we know that $T\mathbb{P}^n|_{U_i}$ is spanned by $\{\partial_{j,i} = \partial/\partial z_{j,i}\}_{j \neq i}$. The Fubini-Study metric is given by

$$h_{\text{FS}}(z_{-,i})(\partial_{k,i}, \partial_{l,i}) = \frac{\delta_{kl}}{(1 + \sum_{r \neq i} |z_{r,i}|^2)} - \frac{\overline{z_{k,i}} z_{l,i}}{(1 + \sum_{r \neq i} |z_{r,i}|^2)^2}.$$

On $U_{ij} = U_i \cap U_j$, the differential form transform as

$$dz_{k,i} = z_{j,i} dz_{k,j} - z_{k,i} z_{j,i} dz_{i,j}.$$

It follows that

$$\frac{\partial z_{k,i}}{\partial z_{l,j}} = \begin{cases} z_{j,i}, & l = k, k \neq j, \\ -z_{k,i} z_{j,i}, & l = i, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the tangent vectors transform as

$$\partial_{l,j} = \sum_{k \neq i} \frac{\partial z_{l,j}}{\partial z_{k,i}} \partial_{k,i} = \begin{cases} z_{j,i} \partial_{l,i}, & l \neq i, \\ -\sum_{k \neq i} z_{k,i} z_{j,i} \partial_{k,i}, & l = i. \end{cases}$$

Yang: To be continued.

Example 5. Now let us consider the complex projective plane $\mathbb{P}^2 = \{[X : Y : Z]\}$. On the affine chart $U_Z = \{[X : Y : Z] \mid Z \neq 0\}$ with coordinates $x = X/Z$ and $y = Y/Z$, the Fubini-Study metric h_{FS} on $T\mathbb{P}^2|_{U_Z}$ is given by

$$h_{\text{FS}}(x, y) = \frac{1}{(1 + |x|^2 + |y|^2)^2} \begin{bmatrix} 1 + |y|^2 & -\bar{x}y \\ -x\bar{y} & 1 + |x|^2 \end{bmatrix}.$$

For a tangent vector $\partial = a\partial_x + b\partial_y$, its norm squared is

$$\|\partial\|_{h_{\text{FS}}}^2 = \frac{(1 + |y|^2)|a|^2 + (1 + |x|^2)|b|^2 - 2\operatorname{Re}(\bar{x}y a \bar{b})}{(1 + |x|^2 + |y|^2)^2} = \frac{|a|^2 + |b|^2 + |xb - ya|^2}{(1 + |x|^2 + |y|^2)^2} \geq 0.$$

Definition 6. Let X be a complex manifold and $E \rightarrow X$ a holomorphic vector bundle. A *connection* on E is a \mathbb{C} -linear map between the sheaves of smooth sections

$$\nabla : \mathcal{C}^\infty(-, E) \rightarrow \mathcal{C}^\infty(-, T^*X \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all smooth functions f and smooth sections s of E .

When you choose a vector field $v \in \mathcal{C}^\infty(U, TX)$ on an open set $U \subset X$, the connection ∇ induces an endomorphism

$$\nabla_v : \mathcal{C}^\infty(U, E) \rightarrow \mathcal{C}^\infty(U, E)$$

by applying v on the T^*X component of ∇s for a section s of E . In particular, if $E = TX$ is the tangent bundle, then ∇_v is called a *covariant derivative* along v . Sometimes people call ∇ an *endomorphism-valued 1-form* on X with values in $\operatorname{End}(E)$ by viewing it as a map $v \mapsto \nabla_v$.

Example 7. Let \mathbb{P}^n be the complex projective space and $\mathcal{O}_{\mathbb{P}^n}(1)$ the hyperplane line bundle. The *Chern connection* associated with the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ is a connection defined as follows: For a section s of $\mathcal{O}_{\mathbb{P}^n}(1)$, we define

$$\nabla s = ds + \alpha s,$$

where α is a $(1,0)$ -form determined by the Fubini-Study metric. Yang: To be continued.

Proposition 8. Let X be a complex manifold, $E \rightarrow X$ a holomorphic vector bundle equipped with a hermitian metric h . Then there exists a unique connection ∇ on E that is compatible with both the holomorphic structure and the hermitian metric h . Yang: To be checked.

By the Leibniz rule, the connection ∇ can be extended to act on E -valued differential forms:

$$\nabla : \mathcal{C}^\infty(-, \Lambda^k T^*X \otimes E) \rightarrow \mathcal{C}^\infty(-, \Lambda^{k+1} T^*X \otimes E)$$

for all $k \geq 0$, satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

for $\omega \in \mathcal{C}^\infty(-, \Lambda^k T^*X)$ and $s \in \mathcal{C}^\infty(-, E)$.

Definition 9. Let X be a complex manifold, $E \rightarrow X$ a holomorphic vector bundle, and ∇ a connection on E . The *curvature* of the connection ∇ is defined as the endomorphism-valued 2-form

$$F_\nabla = \nabla^2 : \mathcal{C}^\infty(-, E) \rightarrow \mathcal{C}^\infty(-, \Lambda^2 T^*X \otimes E),$$

where ∇^2 is the composition of ∇ with itself. Yang: To be continued.

When $E = TX$ is the tangent bundle, the curvature F_∇ is a $(3,1)$ -tensor, which is the classical Riemann curvature tensor.

Yang: For a line bundle, everything coincide.

2 Chern-Weil Theory

Theorem 10. Let X be a complex manifold and $E \rightarrow X$ a holomorphic vector bundle equipped with a hermitian metric h . Let ∇ be the unique connection on E compatible with both the holomorphic structure and the hermitian metric h , and let F_∇ be its curvature. Then the Chern classes $c_k(E) \in H^{2k}(X, \mathbb{R})$ can be represented by the differential forms

$$c_k(E) = \left[\frac{1}{(2\pi i)^k} \text{Tr}(F_\nabla^k) \right].$$

Yang: To be checked.

| *Proof.* Yang: To be added. □

Appendix