

Analysis in several complex variables

In this section, we introduce some basic concepts and results in complex analysis with multiple variables.

1 Holomorphic functions

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Definition 1. A continuous map $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$ is *differentiable* at $p \in \mathbb{R}^{2n}$ if there exists a linear map $Df_p : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$ such that

$$f(z) = f(p) + Df_p(z - p) + o(|z - p|).$$

A continuous map $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is *holomorphic* at $p \in \mathbb{C}^n$ if it is differentiable at p and Df_p is \mathbb{C} -linear, i.e., $Df_p(\sqrt{-1}z) = \sqrt{-1}Df_p(z)$ for all $z \in \mathbb{C}^n$.

By a “function”, we always mean a complex-valued function, i.e., a map $f : \mathbb{C}^n \rightarrow \mathbb{C}$. Fix a coordinate system $z = (z_1, \dots, z_n)$ on \mathbb{C}^n and write $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Then a differentiable function $f = u + iv : \mathbb{C}^n \rightarrow \mathbb{C}$ is holomorphic at p if and only if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x_i}(p) = \frac{\partial v}{\partial y_i}(p), \quad \frac{\partial u}{\partial y_i}(p) = -\frac{\partial v}{\partial x_i}(p), \quad i = 1, \dots, n.$$

For convenience, we consider the complexified tangent space $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and introduce the following operators.

Definition 2. The *Wirtinger operators* are defined as

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} - \sqrt{-1} \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} + \sqrt{-1} \frac{\partial}{\partial y_j} \right), \quad j = 1, \dots, n.$$

Then we can rewrite the Cauchy-Riemann equations as

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n.$$

We summarize some properties of Wirtinger operators in the following proposition.

Proposition 3. The Wirtinger operators satisfy the following properties:

- (a) $\partial_{z_j} z_i = \delta_{ij}$, $\partial_{z_j} \bar{z}_i = 0$, $\partial_{\bar{z}_j} z_i = 0$, $\partial_{\bar{z}_j} \bar{z}_i = \delta_{ij}$;
- (b) $\overline{(\partial_{z_j} f)} = \partial_{\bar{z}_j} \bar{f}$;
- (c) suppose we have $\mathbb{C}^n \xrightarrow{g} \mathbb{C}^m \xrightarrow{f} \mathbb{C}^l$ and the coordinate on \mathbb{C}^m is $w = (w_1, \dots, w_m)$, then the chain

rule holds:

$$\begin{aligned}\frac{\partial(f \circ g)}{\partial z_j} &= \sum_{k=1}^m \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial z_j}(z) + \sum_{k=1}^m \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial z_j}(z), \\ \frac{\partial(f \circ g)}{\partial \bar{z}_j} &= \sum_{k=1}^m \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial \bar{z}_j}(z) + \sum_{k=1}^m \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial \bar{z}_j}(z).\end{aligned}$$

Proof. Yang: By computation. □

We can also consider the complexified of derivatives

$$Df_p^{\mathbb{C}} : T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}.$$

If we take $\{\partial_{z_i}, \partial_{\bar{z}_i}\}_{i=1}^n$ as a basis of $T^*\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and $\{\partial_{w_j}, \partial_{\bar{w}_j}\}_{j=1}^m$ as a basis of $T^*\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$, then the matrix representation of $Df_p^{\mathbb{C}}$ is

$$Df_p^{\mathbb{C}} = \begin{pmatrix} \frac{\partial f}{\partial z}(p) & \frac{\partial f}{\partial \bar{z}}(p) \\ \frac{\partial f}{\partial \bar{z}}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix}.$$

Yang: To be checked In particular, if f is holomorphic, then we have $\det Df_p^{\mathbb{C}} = |\det(\partial_z f)(p)|^2 \geq 0$.

Definition 4. A map $f : \Omega \rightarrow \Omega'$ between two open sets $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^m$ is *biholomorphic* if it is a bijection and both f and f^{-1} are holomorphic.

If f is biholomorphic at p , then $m = n$ and $\det Df_p > 0$.

Theorem 5 (Holomorphic inverse function theorem). Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a holomorphic function. If the Jacobian determinant $\det Df_p$ is nonzero at $p \in \mathbb{C}^n$, then there exist open neighborhoods U of p and V of $f(p)$ such that $f : U \rightarrow V$ is a biholomorphism.

Proof. Yang: To be continued... □

Theorem 6 (Holomorphic implicit function theorem). Let $f : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^m$ be a holomorphic function. If the Jacobian determinant $\det(\partial f / \partial w)$ is nonzero at $(z_0, w_0) \in \mathbb{C}^{n+m}$, then there exist open neighborhoods U of z_0 and V of w_0 , and a unique holomorphic function $g : U \rightarrow V$ such that for any $(z, w) \in U \times V$, $f(z, w) = f(z_0, w_0)$ if and only if $w = g(z)$. Yang: To be continued...

2 Cauchy Integral Formula

Recall the Cauchy Integral Formula in one complex variable:

Theorem 7 (Cauchy Integral Formula in one complex variable). Let $K \subset \mathbb{C}$ be a compact set with piecewise differentiable boundary ∂K , and let f be differentiable on a neighborhood of K . Then for any z in the interior of K , we have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_K \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

Proof. Yang: By Stokes' theorem. To be continued... □

Theorem 8 (Cauchy Integral Formula in several complex variables). Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D . Then for any $z \in D$,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial D_1 \times \dots \times \partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta_1 \dots d\zeta_n.$$

Proof. Yang: To be continued... □

Corollary 9. Holomorphic functions are analytic. Yang: To be continued...

Proposition 10. Holomorphic functions are open mappings. Yang: To be continued...

Proposition 11. If a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ on a connected open set $\Omega \subset \mathbb{C}^n$ attains its maximum at some point in Ω , then f is constant. Yang: To be continued...

Proposition 12. Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D . Then for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\max_{z \in D} \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(z) \right| \leq \frac{\alpha!}{r^\alpha} \max_{z \in D} |f(z)|,$$

where $r = (r_1, \dots, r_n)$ is the radius of the polydisk D . Yang: To be continued...

Theorem 13 (Generalized Liouville Theorem). A holomorphic function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ on the whole space \mathbb{C}^n that satisfies a polynomial growth condition, i.e., there exist constants $C > 0$ and $k \geq 0$ such that

$$|f(z)| \leq C(1 + |z|^k), \quad \forall z \in \mathbb{C}^n,$$

must be a polynomial of degree at most k . Yang: To be continued...

Theorem 14 (Montel's Theorem). A family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ that is uniformly bounded on compact subsets of Ω is a normal family, i.e., every sequence in the family has a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function or to infinity. Yang: To be continued...

3 Zero sets of holomorphic functions

Theorem 15 (Hartogs' Extension Theorem). Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $K \subset D$ be a compact subset such that $D \setminus K$ is connected. If $f : D \setminus K \rightarrow \mathbb{C}$ is a holomorphic function, then there exists a unique holomorphic function $\tilde{f} : D \rightarrow \mathbb{C}$ such that $\tilde{f}|_{D \setminus K} = f$. Yang: To be continued...

Proof. Yang: To be checked □

Corollary 16. In contrast to the one-variable case, isolated singularities do not exist in several complex variables. Specifically, if $f : D \setminus \{p\} \rightarrow \mathbb{C}$ is a holomorphic function on a domain $D \subset \mathbb{C}^n$ with $n \geq 2$ and $p \in D$, then f can be extended to a holomorphic function on the entire domain D .

Proof. This is a direct consequence of Hartogs' Extension Theorem by taking $K = \{p\}$. \square

Theorem 17 (Weierstrass Preparation Theorem). Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic function in a neighborhood of the origin such that $f(0) = 0$ and f is not identically zero. Write the coordinates as $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that $f(0, w)$ has a zero of order k at $w = 0$, i.e.,

$$f(0, w) = a_k w^k + a_{k+1} w^{k+1} + \dots, \quad a_k \neq 0.$$

Then there exists a neighborhood U of the origin and unique holomorphic functions $g : U \rightarrow \mathbb{C}$ and $h_j : U' \rightarrow \mathbb{C}$ for $j = 1, \dots, k$, where $U' \subset \mathbb{C}^n$ is the projection of U onto the first n coordinates, such that

$$f(z, w) = (w^k + h_1(z)w^{k-1} + \dots + h_k(z))g(z, w),$$

with $g(0) \neq 0$ and $h_j(0) = 0$ for all j . Yang: To be continued...

Proof. Yang: To be continued... Yang: Use the Cauchy Integral Formula to check the holomorphicity of g and h_j . \square

Definition 18. Let $\Omega \subset \mathbb{C}^n$ be an open set. The *sheaf of holomorphic functions* on Ω , denoted by \mathcal{O}_Ω , is the assignment that to each open subset $U \subset \Omega$ assigns the ring $\mathcal{O}_\Omega(U)$ of all holomorphic functions on U , and set the restriction as the usual restriction of functions.

A fundamental property of the sheaf of holomorphic functions is its coherence.

Theorem 19 (Oka's Coherence Theorem). The sheaf of holomorphic functions \mathcal{O}_Ω on an open set $\Omega \subset \mathbb{C}^n$ is a coherent sheaf. Yang: To be continued...

In general, $\mathcal{O}_\Omega(U)$ is not a Noetherian ring for an open set $U \subset \Omega$. However, its stalks $\mathcal{O}_{\Omega, p}$ at points $p \in \Omega$ are Noetherian rings. Yang: To be checked

Example 20. Yang: To be continued...

Proposition 21. For any point $p \in \Omega$, the stalk $\mathcal{O}_{\Omega, p}$ of the sheaf of holomorphic functions at p is a Noetherian ring. Yang: To be continued...

Remark 22. The sheaf of holomorphic functions \mathcal{O}_Ω is a sheaf of topological rings, where the topology on $\mathcal{O}_\Omega(U)$ for an open set $U \subset \Omega$ is given by the compact-open topology. Yang: To be continued...

Definition 23. A subset $A \subset \Omega$ of an open set $\Omega \subset \mathbb{C}^n$ is called an *analytic subset* if for every point $p \in \Omega$, there exists a neighborhood U of p and finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_\Omega(U)$ such that

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = \dots = f_k(z) = 0\}.$$

Yang: To be continued...