Metrics, curvature and connections

1 The first properties

Let X be a complex manifold and $E \to X$ a holomorphic vector bundle.

Definition 1. A *hermitian metric* on E is a smoothly varying family of hermitian inner products $\langle\cdot,\cdot\rangle_x$ on the fibers E_x for each $x \in X$, i.e.,

$$\langle\cdot,\cdot\rangle_x : E_x \times E_x \to \mathbb{C}$$

is a hermitian inner product for each x, and for any local smooth sections s, t of E, the function

$$x \mapsto \langle s(x), t(x)\rangle_x$$

is smooth on X. A vector bundle equipped with a hermitian metric is called a *hermitian vector bundle.*

Definition 2. A *hermitian metric* on a complex manifold X is a hermitian metric on its holomorphic tangent bundle TX.

Remark 3. Let h be a hermitian metric on a complex manifold X. Then h induces a Riemannian metric g on the underlying real manifold of X by

$$g(u, v) = \mathrm{Re}(h(u, v))$$

for real tangent vectors $u, v \in T_xX$.

Example 4. Let $\mathbb{P}^n$ be the complex projective space. Recall that $\mathbb{P}^n(\mathbb{C}) \cong (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$. We focus on the underlying smooth manifold structure. We have $\mathbb{C}^* \cong S^1 \times \mathbb{R}_{>0}$ and $(\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{R}_{>0} \cong S^{2n+1}$. Hence $\mathbb{P}^n(\mathbb{C}) \cong S^{2n+1}/S^1$. Note that $S^1 \curvearrowright S^{2n+1} \subset \mathbb{C}^{n+1}$ is isometric with respect to the standard hermitian metric on $\mathbb{C}^{n+1}$. Hence the quotient $\mathbb{P}^n(\mathbb{C})$ inherits a natural hermitian metric h_{FS}, called the *Fubini-Study metric.*

On the standard affine chart $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$ with coordinates $z_{j,i} = z_j/z_i$ for $j \neq i$, we know that $T\mathbb{P}^n|_{U_i}$ is spanned by $\{\partial_{j,i} = \partial/\partial z_{j,i}\}_{j\neq i}$. The Fubini-Study metric is given by

$$h_{\mathrm{FS}}(z_{-,i})(\partial_{k,i}, \partial_{l,i}) = \frac{\delta_{kl}}{1 + \sum_{r\neq i} |z_{r,i}|^2} - \frac{\overline{z_{k,i}} z_{l,i}}{(1 + \sum_{r\neq i} |z_{r,i}|^2)^2}.$$

Example 5. Now let us consider the complex projective plane $\mathbb{P}^2 = \{[X : Y : Z]\}$. On the affine chart $U_Z = \{[X : Y : Z] \mid Z \neq 0\}$ with coordinates $x = X/Z$ and $y = Y/Z$, the Fubini-Study metric h_{FS} on $T\mathbb{P}^2|_{U_Z}$ is given by

$$h_{\mathrm{FS}}(x, y) = \frac{1}{(1 + |x|^2 + |y|^2)^2} \begin{bmatrix} 1 + |y|^2 & -\overline{x}y \\ -x\overline{y} & 1 + |x|^2 \end{bmatrix}.$$

Date: January 28, 2026, **Author**: Tianle Yang, My Homepage

For a tangent vector $\partial = a\partial_x + b\partial_y$, its norm squared is

$$\|\partial\|^2_{h_{\mathrm{FS}}} = \frac{(1+|y|^2)|a|^2 + (1+|x|^2)|b|^2 - 2\,\mathrm{Re}(\bar{x}ya\bar{b})}{(1+|x|^2+|y|^2)^2} = \frac{|a|^2+|b|^2+|xb-ya|^2}{(1+|x|^2+|y|^2)^2} \geq 0.$$

Definition 6. Let X be a complex manifold and $E \to X$ a holomorphic vector bundle. A *connection* on E is a $\mathbb{C}$-linear map between the sheaves of smooth sections

$$\nabla : \mathcal{C}^\infty(-, E) \to \mathcal{C}^\infty(-, T^*X \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

for all smooth functions f and smooth sections s of E.

When you choose a vector field $v \in \mathcal{C}^\infty(U, TX)$ on an open set $U \subset X$, the connection ∇ induces an endomorphism

$$\nabla_v : \mathcal{C}^\infty(U, E) \to \mathcal{C}^\infty(U, E)$$

by applying v on the T^*X component of ∇s for a section s of E. In particular, if $E = TX$ is the tangent bundle, then ∇_v is called a *covariant derivative* along v. Sometimes people call ∇ an *endomorphism-valued 1-form* on X with values in $\mathrm{End}(E)$ by viewing it as a map $v \mapsto \nabla_v$.

Proposition 7. Let X be a complex manifold, $E \to X$ a holomorphic vector bundle equipped with a hermitian metric h. Then there exists a unique connection ∇ on E that is compatible with both the holomorphic structure and the hermitian metric h. Yang: To be checked.

Proof. Yang: to be added. □

Example 8. Let $\mathbb{P}^n$ be the complex projective space and $\mathcal{O}_{\mathbb{P}^n}(1)$ the hyperplane line bundle. The *Chern connection* associated with the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ is a connection defined as follows: For a section s of $\mathcal{O}_{\mathbb{P}^n}(1)$, we define

$$\nabla s = ds + \alpha s,$$

where α is a (1,0)-form determined by the Fubini-Study metric. Yang: To be continued.

By the Leibniz rule, the connection ∇ can be extended to act on E-valued differential forms:

$$\nabla : \mathcal{C}^\infty(-, \Lambda^k T^*X \otimes E) \to \mathcal{C}^\infty(-, \Lambda^{k+1} T^*X \otimes E)$$

for all $k \geq 0$, satisfying

$$\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

for $\omega \in \mathcal{C}^\infty(-, \Lambda^k T^*X)$ and $s \in \mathcal{C}^\infty(-, E)$.

Definition 9. Let X be a complex manifold, $E \to X$ a holomorphic vector bundle, and ∇ a connection on E. The *curvature* of the connection ∇ is defined as the endomorphism-valued 2-form

$$F_\nabla = \nabla^2 : \mathcal{C}^\infty(-, E) \to \mathcal{C}^\infty(-, \Lambda^2 T^*X \otimes E),$$

where ∇^2 is the composition of ∇ with itself.

When $E = TX$ is the tangent bundle, the curvature F_∇ is a $(3,1)$-tensor, which is the classical Riemann curvature tensor.

Example 10. Yang: To be added.

Yang: For a line bundle, everything coincide.

2 On line bundles

Let X be a complex manifold and $L \to X$ a holomorphic line bundle.

Proposition 11. Let h_1, h_2 be two hermitian metrics on L. Then there exists a smooth function $\varphi : X \to \mathbb{R}$ such that

$$h_2(s,t) = \exp(\varphi) \cdot h_1(s,t)$$

for all local smooth sections s, t of L.

Proof. Yang: to be added. □

Proposition 12. There is a one-to-one correspondence between hermitian metrics on L and real $(1,1)$-forms representing the first Chern class $c_1(L) \in H^{1,1}(X, \mathbb{R})$. More precisely, given a hermitian metric h on L, there exists a unique real $(1,1)$-form ω_h such that for any local holomorphic non-vanishing section s of L,

$$\omega_h = -\frac{i}{2\pi}\partial\bar{\partial}\log h(s,s).$$

Conversely, given a real $(1,1)$-form ω representing $c_1(L)$, there exists a hermitian metric h on L such that $\omega = \omega_h$. Yang: To be checked.

Yang: Green functions?

3 Chern-Weil Theory

Theorem 13. Let X be a complex manifold and $E \to X$ a holomorphic vector bundle equipped with a hermitian metric h. Let ∇ be the unique connection on E compatible with both the holomorphic structure and the hermitian metric h, and let F_∇ be its curvature. Then the Chern classes $c_k(E) \in H^{2k}(X, \mathbb{R})$ can be represented by the differential forms

$$c_k(E) = \left[\frac{1}{(2\pi i)^k}\mathrm{Tr}\left(F_\nabla^k\right)\right].$$

Yang: To be checked.

Proof. Yang: To be added. □

Appendix