Cohomology Theories in Complex Geometry

1 Differential forms

2 Various cohomology theories

Let M be a complex manifold. Denote by $\Omega^k_{\mathrm{sm}}(M)$ the space of smooth differential k-forms on M and by $\Omega^{p,q}_{\mathrm{sm}}(M)$ the space of smooth (p,q)-forms on M. Then $\Omega^k_{\mathrm{sm}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}_{\mathrm{sm}}(M)$. Denote by $\Omega^k_{\mathrm{hol}}(M)$ the space of holomorphic differential k-forms on M. Then we have $\Omega^{k,0}_{\mathrm{sm}}(M) = \Omega^k_{\mathrm{hol}}(M) \otimes_{\mathcal{O}^{\mathrm{hol}}_M} \mathcal{O}^{\mathrm{sm}}_M$.

There are several cohomology theories for complex manifolds.

Definition 1. Let M be a complex manifold. The *de Rham cohomology* of M is defined to be the de Rham cohomology of the underlying smooth manifold of M:

$$H^k_{\mathrm{dR}}(M) := \frac{\mathrm{Ker}(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

Definition 2. Let M be a complex manifold. The *Dolbeault cohomology* of M is defined to be

$$H^{p,q}_{\overline{\partial}}(M) := \frac{\mathrm{Ker}(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Definition 3. Let M be a complex manifold. The *Bott-Chern cohomology* of M is defined to be

$$H^{p,q}_{\mathrm{BC}}(M) := \frac{\mathrm{Ker}(\mathrm{d} : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M))}{\mathrm{Im}(\partial\overline{\partial} : \Omega^{p-1,q-1}(M) \to \Omega^{p,q}(M))}.$$

Yang: To be checked...

Definition 4. Let M be a complex manifold. The *Aeppli cohomology* of M is defined to be

$$H^{p,q}_{\mathrm{A}}(M) := \frac{\mathrm{Ker}(\partial\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M))}{\mathrm{Im}(\partial : \Omega^{p-1,q}(M) \to \Omega^{p,q}(M)) + \mathrm{Im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Yang: To be checked...

There are natural maps between these cohomology theories. Yang: To be continued...

Proposition 5. Let $\Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \dots, n\}$ be the unit polydisc in $\mathbb{C}^n$. Then

$$H^{p,q}_{\overline{\partial}}(\Delta^n) = \begin{cases} \Omega^p_{\mathrm{hol}}(\Delta^n), & q = 0, \\ 0, & q > 0. \end{cases}$$

Yang: To be checked...

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Appendix