Forms and Currents

1   Differential forms

Let M be a complex manifold of complex dimension d. Recall that we have the decomposition of the cotangent bundle:

$$\Omega_{\text{sm}}^{1,\mathbb{C}} \cong \Omega_{\text{sm}}^{1,0} \oplus \Omega_{\text{sm}}^{0,1} \cong \Omega_{\text{sm}}^{1} \oplus \overline{\Omega_{\text{sm}}^{1}}.$$

Take exterior powers, we have $\Omega_{\text{sm}}^{k,\mathbb{C}} \cong \bigoplus_{p+q=k} \Omega_{\text{sm}}^{p,q}$, where $\Omega_{\text{sm}}^{p,q} = \bigwedge^{p} \Omega_{\text{sm}}^{1,0} \otimes \bigwedge^{q} \Omega_{\text{sm}}^{0,1}$. We also use the notation

$$\mathcal{A}^{p,q} := \Omega_{\text{sm}}^{p,q}, \quad \mathcal{A}^{k} := \Omega_{\text{sm}}^{k,\mathbb{C}}.$$

A reason to induce this strange sheaf $\mathcal{A}^{k} = \Omega_{\text{sm}}^{k,\mathbb{C}}$ is to make sense of integration of top-degree forms. For simplicity, assume that M is compact. Let $\omega \in \mathcal{A}^{2d}(M)$ be a smooth complex-valued $2d$-form on M. Then its integration is well-defined in the smooth manifold sense:

$$\int_{M} \omega \in \mathbb{C}.$$

However, in complex case, it is more natural to "integral" a holomorphic d-form on a d-dimensional complex manifold. This does not make sense in the smooth manifold theory. The solution is to associate a holomorphic d-form $\eta \in \mathcal{A}^{d,0}(M)$ with a smooth $2d$-form ((d,d)-form) $\omega = \eta \wedge \overline{\eta} \in \mathcal{A}^{d,d}(M) \subset \mathcal{A}^{2d}(M)$.

Another reason is that $\bigoplus_{k} \Omega_{\text{sm}}^{k}$ is not closed under the exterior derivative d, while $\bigoplus_{k} \Omega_{\text{sm}}^{k,\mathbb{C}}$ is. Suppose that we have local holomorphic coordinates $(z_1, \ldots, z_d)$. Recall that we have the exterior derivative

$$\text{d} : \Omega_{\text{sm}}^{1,0} \to \Omega_{\text{sm}}^{2,\mathbb{C}}, \quad f \, dz_i \mapsto \sum_{j=1}^{d} \frac{\partial f}{\partial z_j} dz_j \wedge dz_i + \sum_{i=1}^{d} \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j} \wedge dz_i.$$

On its conjugation, we have

$$\text{d} : \Omega_{\text{sm}}^{0,1} \to \Omega_{\text{sm}}^{2,\mathbb{C}}, \quad f \, d\overline{z_i} \mapsto \sum_{j=1}^{d} \frac{\partial f}{\partial z_j} dz_j \wedge d\overline{z_j} + \sum_{j=1}^{d} \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j} \wedge d\overline{z_j}.$$
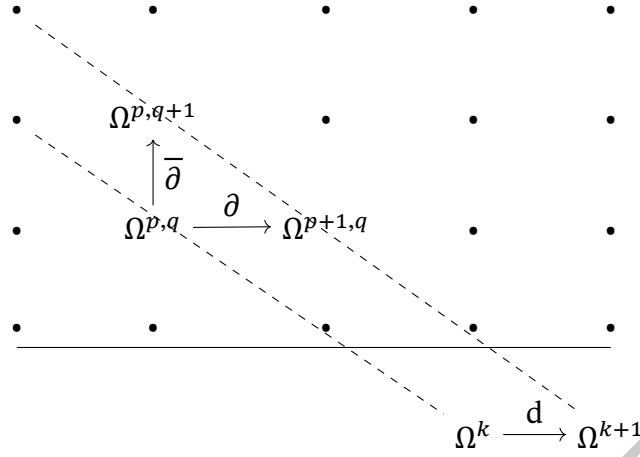
Extending d by linearity and the Leibniz rule $\text{d}(\alpha \wedge \beta) = \text{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \text{d}\beta$, we get the exterior derivative

$$\text{d} : \mathcal{A}^{k} = \Omega_{\text{sm}}^{k,\mathbb{C}} \to \mathcal{A}^{k+1} = \Omega_{\text{sm}}^{k+1,\mathbb{C}},$$

which can be decomposed as $\text{d} = \partial + \overline{\partial}$, where

$$\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}, \quad f \, dz_I \wedge d\overline{z_J} \mapsto \sum_{j=1}^{d} \frac{\partial f}{\partial z_j} dz_j \wedge dz_I \wedge d\overline{z_J},$$

$$\overline{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}, \quad f \, dz_I \wedge d\overline{z_J} \mapsto \sum_{j=1}^{d} \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j} \wedge dz_I \wedge d\overline{z_J}.$$

In a diagram, we have:

$$\Omega^{p,q+1}$$
$$\uparrow \overline{\partial}$$
$$\Omega^{p,q} \xrightarrow{\partial} \Omega^{p+1,q}$$

$$\Omega^k \xrightarrow{d} \Omega^{k+1}$$

Proposition 1. The operators ∂ and $\overline{\partial}$ satisfy

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Proof. Yang: To be added. □

Proposition 2. Let $f : M \to N$ be a holomorphic map between complex manifolds. Then the pull-back of differential forms $f^* : \mathcal{A}_N^k \to \mathcal{A}_M^k$ satisfies

$$f^*(\mathcal{A}_N^{p,q}) \subset \mathcal{A}_M^{p,q}, \quad f^* \circ \partial_N = \partial_M \circ f^*, \quad f^* \circ \overline{\partial}_N = \overline{\partial}_M \circ f^*.$$

Proof. Yang: To be added. □

Yang: The following need to checked.

Topological vector space of forms with compact support Let M be a complex manifold of complex dimension d. Given a differential form $\omega \in \mathcal{A}^k(M)$ with compact support, for any compact subset $K \subset M$ and non-negative integer m, we can define a seminorm

$$p_{K,m}(\omega) = \sup_{x \in K} \max_{|\alpha| \leq m} |D^\alpha \omega(x)|.$$

Here, $\alpha = (\alpha_1, \dots, \alpha_{2d})$ is a multi-index, and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_{2d}^{\alpha_{2d}}}$ in local real coordinates $(x_1, \dots, x_{2d})$. The collection of these seminorms endows the space of compactly supported smooth k-forms on M with a locally convex topology.

Definition 3. A differential form $\omega \in \mathcal{A}^k(M)$ is said to have *compact support* if there exists a compact subset $K \subset M$ such that $\omega|_{M \setminus K} = 0$. The space of smooth complex-valued k-forms with compact support on M is denoted by $\mathcal{A}_c^k(M)$ or $\mathcal{D}^k(M)$. On this vector space, we give it the weak topology induced by the family of seminorms

$$p_{K,m}(\omega) = \sup_{x \in K} \max_{|\alpha| \leq m} |D^\alpha \omega(x)|,$$

where K runs over all compact subsets of M and m runs over all non-negative integers.

2 Currents

Definition 4. A *current* of degree k on a complex manifold M is a continuous linear functional on the space of compactly supported smooth $(2d-k)$-forms on M:

$$T : \mathcal{A}_c^{2d-k}(M) \to \mathbb{C}.$$

The space of currents of degree k on M is denoted by $\mathcal{D}_k(M)$. Yang: To be revised.

Appendix