

# *Cohomological Studies*

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## 1 Forms and Currents

### 1.1 Differential forms

Let  $M$  be a complex manifold of complex dimension  $d$ . Recall that we have the decomposition of the cotangent bundle:

$$\Omega_{\text{sm}}^{1,\mathbb{C}} \cong \Omega_{\text{sm}}^{1,0} \oplus \Omega_{\text{sm}}^{0,1} \cong \Omega_{\text{sm}}^1 \oplus \overline{\Omega_{\text{sm}}^1}.$$

Take exterior powers, we have  $\Omega_{\text{sm}}^{k,\mathbb{C}} \cong \bigoplus_{p+q=k} \Omega_{\text{sm}}^{p,q}$ , where  $\Omega_{\text{sm}}^{p,q} = \bigwedge^p \Omega_{\text{sm}}^{1,0} \otimes \bigwedge^q \Omega_{\text{sm}}^{0,1}$ . We also use the notation

$$\mathcal{A}^{p,q} := \Omega_{\text{sm}}^{p,q}, \quad \mathcal{A}^k := \Omega_{\text{sm}}^{k,\mathbb{C}}.$$

A reason to induce this strange sheaf  $\mathcal{A}^k = \Omega_{\text{sm}}^{k,\mathbb{C}}$  is to make sense of integration of top-degree forms. For simplicity, assume that  $M$  is compact. Let  $\omega \in \mathcal{A}^{2d}(M)$  be a smooth complex-valued  $2d$ -form on  $M$ . Then its integration is well-defined in the smooth manifold sense:

$$\int_M \omega \in \mathbb{C}.$$

However, in complex case, it is more natural to “integral” a holomorphic  $d$ -form on a  $d$ -dimensional complex manifold. This does not make sense in the smooth manifold theory. The solution is to associate a holomorphic  $d$ -form  $\eta \in \mathcal{A}^{d,0}(M)$  with a smooth  $2d$ -form (( $d,d$ )-form)  $\omega = \eta \wedge \bar{\eta} \in \mathcal{A}^{d,d}(M) \subset \mathcal{A}^{2d}(M)$ .

Another reason is that  $\bigoplus_k \Omega_{\text{sm}}^k$  is not closed under the exterior derivative  $d$ , while  $\bigoplus_k \Omega_{\text{sm}}^{k,\mathbb{C}}$  is. Suppose that we have local holomorphic coordinates  $(z_1, \dots, z_d)$ . Recall that we have the exterior

derivative

$$d : \Omega_{\text{sm}}^{1,0} \rightarrow \Omega_{\text{sm}}^{2,\mathbb{C}}, \quad f dz_i \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge dz_i + \sum_{i=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_i.$$

On its conjugation, we have

$$d : \Omega_{\text{sm}}^{0,1} \rightarrow \Omega_{\text{sm}}^{2,\mathbb{C}}, \quad f d\bar{z}_i \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge d\bar{z}_i + \sum_{j=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_i.$$

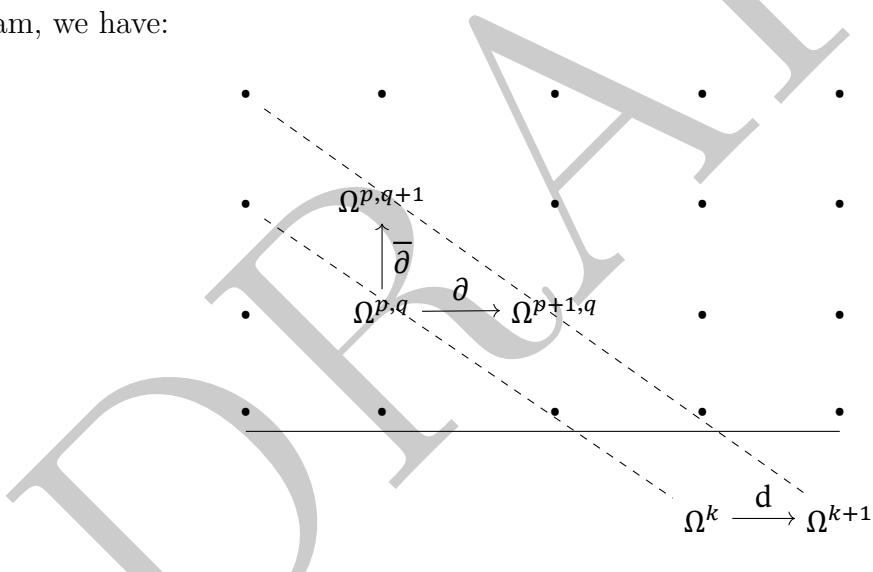
Extending  $d$  by linearity and the Leibniz rule  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$ , we get the exterior derivative

$$d : \mathcal{A}^k = \Omega_{\text{sm}}^{k,\mathbb{C}} \rightarrow \mathcal{A}^{k+1} = \Omega_{\text{sm}}^{k+1,\mathbb{C}},$$

which can be decomposed as  $d = \partial + \bar{\partial}$ , where

$$\begin{aligned} \partial : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{p+1,q}, \quad f dz_I \wedge d\bar{z}_J \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J, \\ \bar{\partial} : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{p,q+1}, \quad f dz_I \wedge d\bar{z}_J \mapsto \sum_{j=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J. \end{aligned}$$

In a diagram, we have:



**Proposition 1.1.** The operators  $\partial$  and  $\bar{\partial}$  satisfy

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

| *Proof.* Yang: To be added. □

**Proposition 1.2.** Let  $f : M \rightarrow N$  be a holomorphic map between complex manifolds. Then the pull-back of differential forms  $f^* : \mathcal{A}_N^k \rightarrow \mathcal{A}_M^k$  satisfies

$$f^*(\mathcal{A}_N^{p,q}) \subset \mathcal{A}_M^{p,q}, \quad f^* \circ \partial_N = \partial_M \circ f^*, \quad f^* \circ \bar{\partial}_N = \bar{\partial}_M \circ f^*.$$

| *Proof.* Yang: To be added. □

Yang: The following need to checked.

**Topological vector space of forms with compact support** Let  $M$  be a complex manifold of complex dimension  $d$ . Given a differential form  $\omega \in \mathcal{A}^k(M)$  with compact support, for any compact subset  $K \subset M$  and non-negative integer  $m$ , we can define a seminorm

$$p_{K,m}(\omega) = \sup_{x \in K} \max_{|\alpha| \leq m} |D^\alpha \omega(x)|.$$

Here,  $\alpha = (\alpha_1, \dots, \alpha_{2d})$  is a multi-index, and  $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_{2d}^{\alpha_{2d}}}$  in local real coordinates  $(x_1, \dots, x_{2d})$ . The collection of these seminorms endows the space of compactly supported smooth  $k$ -forms on  $M$  with a locally convex topology.

**Definition 1.3.** A differential form  $\omega \in \mathcal{A}^k(M)$  is said to have *compact support* if there exists a compact subset  $K \subset M$  such that  $\omega|_{M \setminus K} = 0$ . The space of smooth complex-valued  $k$ -forms with compact support on  $M$  is denoted by  $\mathcal{A}_c^k(M)$  or  $\mathcal{D}^k(M)$ . On this vector space, we give it the weak topology induced by the family of seminorms

$$p_{K,m}(\omega) = \sup_{x \in K} \max_{|\alpha| \leq m} |D^\alpha \omega(x)|,$$

where  $K$  runs over all compact subsets of  $M$  and  $m$  runs over all non-negative integers.

## 1.2 Currents

**Definition 1.4.** A *current* of degree  $k$  on a complex manifold  $M$  is a continuous linear functional on the space of compactly supported smooth  $(2d - k)$ -forms on  $M$ :

$$T : \mathcal{A}_c^{2d-k}(M) \rightarrow \mathbb{C}.$$

The space of currents of degree  $k$  on  $M$  is denoted by  $\mathcal{D}_k(M)$ . Yang: To be revised.

# 2 Cohomology Theories in Complex Geometry

## 2.1 Various cohomology theories

There are several cohomology theories for complex manifolds.

**Definition 2.1.** Let  $M$  be a complex manifold. The *singular cohomology* of  $M$  with coefficients in a ring  $R$  is defined to be the singular cohomology of the underlying topological space  $|M|$  of  $M$ :

$$H_{\text{sing}}^k(M; R) := H_{\text{sing}}^k(|M|; R).$$

**Definition 2.2.** Let  $M$  be a complex manifold. The *de Rham cohomology* of  $M$  is defined to be the de Rham cohomology of the underlying smooth manifold of  $M$ :

$$H_{\text{dR}}^k(M) := \frac{\text{Ker}(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M))}{\text{Im}(d : \Omega^{k-1}(M) \rightarrow \Omega^k(M))}.$$

Yang: Smooth section or holomorphic section?

**Definition 2.3.** Let  $M$  be a complex manifold. The *Dolbeault cohomology* of  $M$  is defined to be

$$H_{\bar{\partial}}^{p,q}(M) := \frac{\text{Ker}(\bar{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M))}{\text{Im}(\bar{\partial} : \Omega^{p,q-1}(M) \rightarrow \Omega^{p,q}(M))}.$$

**Proposition 2.4.** Let  $\Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \dots, n\}$  be the unit polydisc in  $\mathbb{C}^n$ . Then

$$H_{\bar{\partial}}^{p,q}(\Delta^n) = \begin{cases} \Omega_{\text{hol}}^p(\Delta^n), & q = 0, \\ 0, & q > 0. \end{cases}$$

Yang: To be checked...

### 3 Metrics, curvature and connections

#### 3.1 The first properties

Let  $X$  be a complex manifold and  $E \rightarrow X$  a holomorphic vector bundle.

**Definition 3.1.** A *hermitian metric* on  $E$  is a smoothly varying family of hermitian inner products  $\langle \cdot, \cdot \rangle_x$  on the fibers  $E_x$  for each  $x \in X$ , i.e.,

$$\langle \cdot, \cdot \rangle_x : E_x \times E_x \rightarrow \mathbb{C}$$

is a hermitian inner product for each  $x$ , and for any local smooth sections  $s, t$  of  $E$ , the function

$$x \mapsto \langle s(x), t(x) \rangle_x$$

is smooth on  $X$ . A vector bundle equipped with a hermitian metric is called a *hermitian vector bundle*.

**Definition 3.2.** A *hermitian metric* on a complex manifold  $X$  is a hermitian metric on its holomorphic tangent bundle  $TX$ .

**Remark 3.3.** Let  $h$  be a hermitian metric on a complex manifold  $X$ . Then  $h$  induces a Riemannian metric  $g$  on the underlying real manifold of  $X$  by

$$g(u, v) = \text{Re}(h(u, v))$$

for real tangent vectors  $u, v \in T_x X$ .

**Example 3.4.** Let  $\mathbb{P}^n$  be the complex projective space. Recall that  $\mathbb{P}^n(\mathbb{C}) \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ . We focus on the underlying smooth manifold structure. We have  $\mathbb{C}^* \cong S^1 \times \mathbb{R}_{>0}$  and  $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{R}_{>0} \cong S^{2n+1}$ . Hence  $\mathbb{P}^n(\mathbb{C}) \cong S^{2n+1}/S^1$ . Note that  $S^1 \curvearrowright S^{2n+1} \subset \mathbb{C}^{n+1}$  is isometric with respect to the standard hermitian metric on  $\mathbb{C}^{n+1}$ . Hence the quotient  $\mathbb{P}^n(\mathbb{C})$  inherits a natural hermitian metric  $h_{\text{FS}}$ , called the *Fubini-Study metric*.

On the standard affine chart  $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$  with coordinates  $z_{j,i} = z_j/z_i$  for  $j \neq i$ , we know that  $T\mathbb{P}^n|_{U_i}$  is spanned by  $\{\partial_{j,i} = \partial/\partial z_{j,i}\}_{j \neq i}$ . The Fubini-Study metric is given by

$$h_{\text{FS}}(z_{-,i})(\partial_{k,i}, \partial_{l,i}) = \frac{\delta_{kl}}{1 + \sum_{r \neq i} |z_{r,i}|^2} - \frac{\overline{z_{k,i}} z_{l,i}}{(1 + \sum_{r \neq i} |z_{r,i}|^2)^2}.$$

**Example 3.5.** Now let us consider the complex projective plane  $\mathbb{P}^2 = \{[X : Y : Z]\}$ . On the affine chart  $U_Z = \{[X : Y : Z] \mid Z \neq 0\}$  with coordinates  $x = X/Z$  and  $y = Y/Z$ , the Fubini-Study metric  $h_{\text{FS}}$  on  $T\mathbb{P}^2|_{U_Z}$  is given by

$$h_{\text{FS}}(x, y) = \frac{1}{(1 + |x|^2 + |y|^2)^2} \begin{bmatrix} 1 + |y|^2 & -\bar{x}y \\ -x\bar{y} & 1 + |x|^2 \end{bmatrix}.$$

For a tangent vector  $\partial = a\partial_x + b\partial_y$ , its norm squared is

$$\|\partial\|_{h_{\text{FS}}}^2 = \frac{(1 + |y|^2)|a|^2 + (1 + |x|^2)|b|^2 - 2 \operatorname{Re}(\bar{x}yab)}{(1 + |x|^2 + |y|^2)^2} = \frac{|a|^2 + |b|^2 + |xb - ya|^2}{(1 + |x|^2 + |y|^2)^2} \geq 0.$$

**Definition 3.6.** Let  $X$  be a complex manifold and  $E \rightarrow X$  a holomorphic vector bundle. A *connection* on  $E$  is a  $\mathbb{C}$ -linear map between the sheaves of smooth sections

$$\nabla : \mathcal{C}^\infty(-, E) \rightarrow \mathcal{C}^\infty(-, T^*X \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all smooth functions  $f$  and smooth sections  $s$  of  $E$ .

When you choose a vector field  $v \in \mathcal{C}^\infty(U, TX)$  on an open set  $U \subset X$ , the connection  $\nabla$  induces an endomorphism

$$\nabla_v : \mathcal{C}^\infty(U, E) \rightarrow \mathcal{C}^\infty(U, E)$$

by applying  $v$  on the  $T^*X$  component of  $\nabla s$  for a section  $s$  of  $E$ . In particular, if  $E = TX$  is the tangent bundle, then  $\nabla_v$  is called a *covariant derivative* along  $v$ . Sometimes people call  $\nabla$  an *endomorphism-valued 1-form* on  $X$  with values in  $\operatorname{End}(E)$  by viewing it as a map  $v \mapsto \nabla_v$ .

**Proposition 3.7.** Let  $X$  be a complex manifold,  $E \rightarrow X$  a holomorphic vector bundle equipped with a hermitian metric  $h$ . Then there exists a unique connection  $\nabla$  on  $E$  that is compatible with both the holomorphic structure and the hermitian metric  $h$ . Yang: To be checked.

| *Proof.* Yang: to be added.  $\square$

**Example 3.8.** Let  $\mathbb{P}^n$  be the complex projective space and  $\mathcal{O}_{\mathbb{P}^n}(1)$  the hyperplane line bundle. The *Chern connection* associated with the Fubini-Study metric on  $\mathcal{O}_{\mathbb{P}^n}(1)$  is a connection defined as follows: For a section  $s$  of  $\mathcal{O}_{\mathbb{P}^n}(1)$ , we define

$$\nabla s = ds + \alpha s,$$

where  $\alpha$  is a  $(1,0)$ -form determined by the Fubini-Study metric. Yang: To be continued.

By the Leibniz rule, the connection  $\nabla$  can be extended to act on  $E$ -valued differential forms:

$$\nabla : \mathcal{C}^\infty(-, \Lambda^k T^*X \otimes E) \rightarrow \mathcal{C}^\infty(-, \Lambda^{k+1} T^*X \otimes E)$$

for all  $k \geq 0$ , satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

for  $\omega \in \mathcal{C}^\infty(-, \Lambda^k T^*X)$  and  $s \in \mathcal{C}^\infty(-, E)$ .

**Definition 3.9.** Let  $X$  be a complex manifold,  $E \rightarrow X$  a holomorphic vector bundle, and  $\nabla$  a connection on  $E$ . The *curvature* of the connection  $\nabla$  is defined as the endomorphism-valued 2-form

$$F_\nabla = \nabla^2 : \mathcal{C}^\infty(-, E) \rightarrow \mathcal{C}^\infty(-, \Lambda^2 T^*X \otimes E),$$

where  $\nabla^2$  is the composition of  $\nabla$  with itself.

When  $E = TX$  is the tangent bundle, the curvature  $F_\nabla$  is a  $(3, 1)$ -tensor, which is the classical Riemann curvature tensor.

| **Example 3.10.** Yang: To be added.

Yang: For a line bundle, everything coincide.

## 3.2 On line bundles

Let  $X$  be a complex manifold and  $L \rightarrow X$  a holomorphic line bundle.

**Proposition 3.11.** Let  $h_1, h_2$  be two hermitian metrics on  $L$ . Then there exists a smooth function  $\varphi : X \rightarrow \mathbb{R}$  such that

$$h_2(s, t) = \exp(\varphi) \cdot h_1(s, t)$$

for all local smooth sections  $s, t$  of  $L$ .

| *Proof.* Yang: to be added. □

**Proposition 3.12.** There is a one-to-one correspondence between hermitian metrics on  $L$  and real  $(1, 1)$ -forms representing the first Chern class  $c_1(L) \in H^{1,1}(X, \mathbb{R})$ . More precisely, given a hermitian metric  $h$  on  $L$ , there exists a unique real  $(1, 1)$ -form  $\omega_h$  such that for any local holomorphic non-vanishing section  $s$  of  $L$ ,

$$\omega_h = -\frac{i}{2\pi} \partial \bar{\partial} \log h(s, s).$$

Conversely, given a real  $(1, 1)$ -form  $\omega$  representing  $c_1(L)$ , there exists a hermitian metric  $h$  on  $L$  such that  $\omega = \omega_h$ . Yang: To be checked.

Yang: Green functions?

### 3.3 Chern-Weil Theory

**Theorem 3.13.** Let  $X$  be a complex manifold and  $E \rightarrow X$  a holomorphic vector bundle equipped with a hermitian metric  $h$ . Let  $\nabla$  be the unique connection on  $E$  compatible with both the holomorphic structure and the hermitian metric  $h$ , and let  $F_\nabla$  be its curvature. Then the Chern classes  $c_k(E) \in H^{2k}(X, \mathbb{R})$  can be represented by the differential forms

$$c_k(E) = \left[ \frac{1}{(2\pi i)^k} \text{Tr} (F_\nabla^k) \right].$$

Yang: To be checked.

| Proof. Yang: To be added. □

## 4 Kähler manifolds

## 5 Hodge star and harmonic forms

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