Sheaves and Bundles on Complex Manifolds

1 Fiber bundles

Definition 1. Let M, F be manifolds. A *fiber bundle* with fiber F over M is a surjective map $\pi : E \to M$ of manifolds such that for each $x \in M$, there exists an open neighborhood U of x and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

where p_1 is the projection onto the first factor.

Given a fiber bundle E over M with fiber F and a covering $\{U_i\}$ of M, for each U_i, U_j and $x \in U_i \cap U_j$, we have two local trivializations

$$\varphi_i|_{E_x}, \varphi_j|_{E_x} : E_x \to \{x\} \times F.$$

They are differed by an automorphism $g_{ij}(x) = \varphi_i|_{E_x} \circ (\varphi_j|_{E_x})^{-1}$ of $\{x\} \times F$ as the following diagram

$$\begin{tikzcd} \{x\} \times F \arrow[rr, "g_{ij}(x)"] & & \{x\} \times F. \\ & E_x \arrow[ul, "\varphi_j|_{E_x}"] \arrow[ur, "\varphi_i|_{E_x}"'] & \end{tikzcd}$$

The map $g_{ij}(x)$ can be identified as an element of $\mathrm{Aut}(F)$. Varying x in $U_i \cap U_j$, we obtain the *transition function*

$$g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$$

which satisfies the *cocycle condition*

$$g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x), \quad \forall x \in U_i \cap U_j \cap U_k,$$

where the multiplication $\cdot$ is given by composition of automorphisms.

There is a natural way to impose smooth (holomorphic) structure on $\mathrm{Aut}(F)$, hence we can talk about smoothness or holomorphicity of transition functions. Set $\Phi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$. Then we have $\Phi_{ij}(x, v) = (x, g_{ij}(x)(v))$ for all $(x, v) \in (U_i \cap U_j) \times F$. Then Φ_{ij} is smooth (holomorphic) if and only if g_{ij} is smooth (holomorphic). Yang: To add ref.

Conversely, given a covering $\{U_i\}$ of M and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$ satisfying the cocycle condition, one can glue the local trivializations $U_i \times F$ via the maps Φ_{ij} to obtain a fiber bundle E over M with fiber F. Therefore, to given a fiber bundle with smooth (holomorphic) structure, it suffices to give a covering $\{U_i\}$ of M and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$ which are smooth (holomorphic) and satisfy the cocycle condition. In general, $\mathrm{Aut}(F)$ might be too large to handle. We can restrict the image of transition functions to a smaller subgroup $G \subset \mathrm{Aut}(F)$. This leads to the notion of structure group.

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Definition 2. Let M, F be manifolds, and $G \subset \mathrm{Aut}(F)$ be a Lie subgroup. A *fiber bundle with structure group* G is a fiber bundle $\pi : E \to M$ given by transition functions $g_{ij} : U_i \cap U_j \to G$.

Example 3. A *(real) vector bundle* of rank r over a manifold M is a fiber bundle with fiber $\mathbb{R}^r$ and structure group $\mathrm{GL}_r(\mathbb{R})$. Similarly, a *complex vector bundle* of rank r over a manifold M is a fiber bundle with fiber $\mathbb{C}^r$ and structure group $\mathrm{GL}_r(\mathbb{C})$.

On a real manifold M of dimension $2n$, an almost complex structure is equivalent to a reduction of the structure group of the tangent bundle TM from $\mathrm{GL}_{2n}(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{C})$.

By the transition functions construction, we can see that

Theorem 4. Let M, F be locally ringed spaces and $G \subset \mathrm{Aut}(F)$ a subgroup. Set $\mathcal{G}$ be the sheaf of "admissible" functions from open subsets of M to G. Then the set of isomorphism classes of fiber bundles over M with fiber F and structure group G is in one-to-one correspondence with the Čech cohomology set $\check{H}^1(M, \mathcal{G})$.

Remark 5. Let us clarify the meaning of $\check{H}^1(M, \mathcal{G})$ when $\mathcal{G}$ is a sheaf of (not necessarily abelian) groups. Given an open covering $\mathcal{U} = \{U_i\}$ of M, we have a "complex" of groups

$$\prod_i \mathcal{G}(U_i) \xrightarrow{\delta^0} \prod_{i,j} \mathcal{G}(U_i \cap U_j) \xrightarrow{\delta^1} \prod_{i,j,k} \mathcal{G}(U_i \cap U_j \cap U_k),$$

where the maps δ^0 and δ^1 are defined by

$$\delta^0((g_i)_i) = (g_i|_{U_i \cap U_j} \cdot (g_j|_{U_i \cap U_j})^{-1})_{i,j},$$

$$\delta^1((g_{ij})_{i,j}) = (g_{ij}|_{U_i \cap U_j \cap U_k} \cdot g_{jk}|_{U_i \cap U_j \cap U_k} \cdot (g_{ik}|_{U_i \cap U_j \cap U_k})^{-1})_{i,j,k}.$$

Note that $\delta^1 \circ \delta^0$ is the constant map to the identity element. We define

- the set of 1-cocycles $Z^1(\mathcal{U}, \mathcal{G}) = \ker(\delta^1)$,
- the set of 1-coboundaries $B^1(\mathcal{U}, \mathcal{G}) = \sqrt{-1}(\delta^0)$.

The set $\check{H}^1(\mathcal{U}, \mathcal{G}) = Z^1(\mathcal{U}, \mathcal{G})/B^1(\mathcal{U}, \mathcal{G})$ is defined as the set of orbits of the action of $\prod_i \mathcal{G}(U_i)$ on $Z^1(\mathcal{U}, \mathcal{G})$ given by

$$((g_i)_i, (g_{ij})_{i,j}) \mapsto (g_i|_{U_i \cap U_j} \cdot g_{ij} \cdot (g_j|_{U_i \cap U_j})^{-1})_{i,j}.$$

Finally, we define $\check{H}^1(M, \mathcal{G}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{G})$ where the limit is taken over all open coverings of M.
Yang: To be revised.

For example, if $F = \mathbb{C}$ and $G = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$, consider the holomorphic line bundles over a complex manifold M. The sheaf $\mathcal{G}$ is equal to $\mathcal{O}_M^*$, the sheaf of nowhere vanishing holomorphic functions on M. Therefore, by Theorem 4, we get the classic result $\mathrm{Pic}(M) \cong \check{H}^1(M, \mathcal{O}_M^*)$.

Slogan *For a fiber bundle E over M, we care about*

- *fiber F,*
- *structure group $G \subset \mathrm{Aut}(F)$,*

- *"admissible" functions class of transition functions $g_{ij} : U_i \cap U_j \to G$ (e.g. continuous, smooth, holomorphic).*

2 Sheaves

Construction 6. Let M be a manifold and $\pi : E \to M$ be a fiber bundle with fiber F. For each open subset $U \subset M$, we can consider the set of "admissible" sections of E over U:

$$\Gamma(U, E) = \{s : U \to E \mid \pi \circ s = \mathrm{id}_U, s \text{ is admissible}\}.$$

Here "admissible" means continuous, smooth, holomorphic, etc., depending on the context. The assignment $U \mapsto \Gamma(U, E)$ defines a sheaf of sets (or groups, modules, etc. if F has additional structure) on M, called the *sheaf of sections* of the bundle E.

Example 7. Let M be a complex manifold. We explain how to view the tangent bundle TM and the cotangent bundle T^*M as sheaves. There are two important classes of admissible sections of these bundles, namely holomorphic and smooth sections. We denote the sheaf of holomorphic (respectively smooth) sections of TM by $\mathcal{T}_{M,\mathrm{hol}}$ (respectively $\mathcal{T}_{M,\mathrm{sm}}$).

Correspondingly, we denote the sheaf of holomorphic (respectively smooth) sections of T^*M by $\Omega^1_{M,\mathrm{hol}}$ (respectively $\Omega^1_{M,\mathrm{sm}}$). Sometime we omit the subscript M if there is no confusion.

The elements in $\mathcal{T}_{M,\mathrm{hol}}(U)$ (respectively $\mathcal{T}_{M,\mathrm{sm}}(U)$) are holomorphic (respectively smooth) vector fields on U, and holomorphic (respectively smooth) 1-forms on U for $\Omega^1_{M,\mathrm{hol}}(U)$ (respectively $\Omega^1_{M,\mathrm{sm}}(U)$). As sheaves, we have

$$\Omega^1_{M,\mathrm{hol}} \cong \mathcal{H}om_{\mathcal{O}_M}(\mathcal{T}_{M,\mathrm{hol}}, \mathcal{O}_M) \quad \text{and} \quad \Omega^1_{M,\mathrm{sm}} \cong \mathcal{H}om_{\mathcal{C}^\infty_M}(\mathcal{T}_{M,\mathrm{sm}}, \mathcal{C}^\infty_M),$$

where $\mathcal{C}^\infty_M$ is the sheaf of smooth complex-valued functions on M.

Example 8. Let M be a complex manifold. Consider the trivial real vector bundle $\mathbb{C} \times M \to M$. Its sheaf of holomorphic sections is just the structure sheaf $\mathcal{O}_M$, while its sheaf of smooth sections is the sheaf $\mathcal{C}^\infty_M = \mathcal{C}^\infty_M(-, \mathbb{C})$ of smooth complex-valued functions on M. Similarly, we have the trivial real vector bundle $\mathbb{R} \times M \to M$ whose sheaf of smooth sections is $\mathcal{C}^\infty_M(-, \mathbb{R})$.

Hence, the complexification of a holomorphic vector bundle E over M, i.e. the fiber bundle $E^{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$, has sheaf of smooth sections given by $\mathcal{E}^{\mathbb{C}} := \mathcal{E}_{\mathrm{sm}} \otimes_{\mathcal{C}^\infty_M(-,\mathbb{R})} \mathcal{C}^\infty_M(-, \mathbb{C}) \cong \mathcal{E}_{\mathrm{sm}} \oplus \overline{\mathcal{E}_{\mathrm{sm}}}$, where $\mathcal{E}_{\mathrm{sm}}$ is the sheaf of smooth sections of E and $\overline{\mathcal{E}_{\mathrm{sm}}}$ is its complex conjugate sheaf, i.e. $\overline{\mathcal{E}_{\mathrm{sm}}}(U) = \{\overline{s} \mid s \in \mathcal{E}_{\mathrm{sm}}(U)\}$. These sheaves are $\mathcal{C}^\infty_M$-modules. Yang: Note that the action of $\mathcal{C}^\infty_M$ on $\overline{\mathcal{E}_{\mathrm{sm}}}$ is given by

$$(f, v) \mapsto \overline{f} \cdot v, \quad \forall f \in \mathcal{C}^\infty_M(U), v \in \overline{\mathcal{E}_{\mathrm{sm}}}(U).$$

Let us return to the cotangent bundle T^*M of a complex manifold M. By the almost complex structure on M, we have the decomposition of complexified cotangent bundle

$$T^*M^{\mathbb{C}} := T^*M \otimes_{\mathbb{R}} \mathbb{C} \cong T^*M^{1,0} \oplus T^*M^{0,1},$$

This gives a decomposition of sheaves of smooth sections

$$\Omega^{1,\mathbb{C}}_{\mathrm{sm}} \cong \Omega^{1,0}_{\mathrm{sm}} \oplus \Omega^{0,1}_{\mathrm{sm}} \cong \Omega^1_{\mathrm{sm}} \oplus \overline{\Omega^1_{\mathrm{sm}}}.$$

Appendix