Complex Manifolds

1 Definition and Examples

Definition 1. A *complex manifold* of complex dimension n is a topological space M such that

(a) M is Hausdorff and second countable;

(b) M is locally homeomorphic to $\mathbb{C}^n$, i.e., for every point $p \in M$, there exists an open neighborhood U of p and a homeomorphism $\varphi : U \to V \subset \mathbb{C}^n$, where V is an open subset of $\mathbb{C}^n$, The pair (U, φ) is called a *chart*;

(c) if (U, φ) and (U', φ') are two charts with $U \cap U' \neq \varnothing$, then the transition map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$$

is holomorphic.

The collection of all charts $\{(U_\alpha, \varphi_\alpha)\}$ that cover M is called an *atlas*. If the atlas is maximal, it is called a *complex structure* on M.

Another way to define complex manifolds is to use the language of ringed spaces.

Definition 2. A *complex manifold* of complex dimension n is a locally ringed space $(M, \mathcal{O}_M)$ such that

(a) M is Hausdorff and second countable;

(b) for every point $p \in M$, there exists an open neighborhood U of p such that $(U, \mathcal{O}_M|_U)$ is isomorphic to $(B, \mathcal{O}_B)$, where B is the unit open ball in $\mathbb{C}^n$ and $\mathcal{O}_B$ is the sheaf of holomorphic functions on B.

Question 3. Given a topological space M that is Hausdorff and second countable, when does it admit a complex structure? Is such a complex structure unique?

For complex dimension 1, the answer is positive and well-known. For higher dimensions, the answer is negative in general. In particular, does the 6-sphere S^6 admit a complex structure? This is a famous open problem in complex geometry.

Question 4. Does the 6-sphere S^6 admit a complex structure?

Definition 5. Let M and N be two complex manifolds. A continuous map $f : M \to N$ is called *holomorphic* if for every point $p \in M$, there exist charts (U, φ) of M around p and (V, ψ) of N around $f(p)$ with $U \subset f^{-1}(V)$ such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

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is holomorphic.

Definition 6. Let M be a complex manifold of complex dimension n. A subset $S \subset M$ is called a *complex submanifold* of complex dimension k if for every point $p \in S$, there exist a chart (U, φ) of M around p such that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{C}^k \times \{0\}) \subset \mathbb{C}^n,$$

where we identify $\mathbb{C}^n$ with $\mathbb{C}^k \times \mathbb{C}^{n-k}$. This gives a chart of S around p. Endowed with the induced topology and the induced complex structure, S is a complex manifold of complex dimension k.

Example 7. Any complex vector space V of complex dimension n is a complex manifold of complex dimension n.

Example 8. The complex projective space $\mathbb{CP}^n := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$ is a complex manifold of complex dimension n. In fact, $\mathbb{CP}^n$ can be covered by $n+1$ charts, each of which is biholomorphic to $\mathbb{C}^n$. For example, the chart $U_0 = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : z_0 \neq 0\}$ is biholomorphic to $\mathbb{C}^n$ via the map

$$[z_0 : z_1 : \cdots : z_n] \mapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right).$$

The other charts are defined similarly.

Proposition 9. Let M and N are complex manifolds of complex dimension n and m respectively, with $n \geq m$. If $f : M \to N$ is a holomorphic map such that p is a regular value of f, i.e., the tangent map $\mathrm{d}f_x$ is surjective for every $x \in f^{-1}(p)$, then $f^{-1}(p)$ is a complex submanifold of M of complex dimension $n - m$.

Proof. For every point $q \in f^{-1}(p)$, choose charts (U, φ) of M around q and (V, ψ) of N around p such that $f(U) \subset V$. By changing coordinates if necessary, we may assume that $\det(\partial f/\partial w)(q) \neq 0$, where we write the coordinates of $\varphi(U)$ as $(z, w) = (z_1, \dots, z_{n-m}, w_1, \dots, w_m) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$. Then by the Holomorphic Implicit Function Theorem (Theorem 32), there exist open neighborhoods U' of q such that $f^{-1}(p) \cap U'$ is biholomorphic to an open subset of $\mathbb{C}^{n-m}$. □

Example 10. Let $X \subset \mathbb{C}^n$ be a complex algebraic variety defined by the vanishing of polynomials $f_1, \dots, f_m \in \mathbb{C}[z_1, \dots, z_n]$. Suppose that X is non-singular, i.e., for every point $p \in X$, the Jacobian matrix $\left(\partial_{z_j} f_i(p)\right)_{i,j}$ has maximal rank r. Then X is a complex submanifold of $\mathbb{C}^n$ of complex dimension $n - r$.

Example 11. A *hypersurface* H in $\mathbb{CP}^n$ is the zero locus of a homogeneous polynomial $f \in \mathbb{C}[z_0, z_1, \dots, z_n]$. Suppose 0 is a regular value of $f : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$. On each chart $U_i \cong \mathbb{C}^n$ of $\mathbb{CP}^n$, it defines a holomorphic function $f_i : U_i \to \mathbb{C}, [z] \mapsto z = (z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n) \mapsto f(z)$. The regularity condition implies that 0 is a regular value of each f_i. Hence $H \cap U_i = f_i^{-1}(0)$ is a complex submanifold of U_i of complex dimension $n-1$ by Proposition 9. Gluing these local pieces together, we see that H is a complex submanifold of $\mathbb{CP}^n$ of complex dimension $n-1$.

Proposition 12. Let M be a complex manifold and let G be a discrete group acting on M by holomorphic automorphisms. If the action is free and properly discontinuous, then the quotient space M/G is a complex manifold and the quotient map $\pi : M \to M/G$ is a holomorphic covering map.

Proof. For every point $p \in M/G$, choose a point $q \in M$ such that $\pi(q) = p$. Since the action is free and properly discontinuous (see Remark 13), there exists an open neighborhood U of q such that $gU \cap U = \varnothing$ for all $g \in G \setminus \{e\}$. Then $\pi|_U : U \to \pi(U)$ is a homeomorphism. This gives a chart of M/G around p. If we have two such charts $(\pi(U), \varphi)$ and $(\pi(U'), \varphi')$ of M/G whose intersection is non-empty, WLOG, assume that $U \cap U' \neq \varnothing$. Then $\pi^{-1}(\pi(U) \cap \pi(U')) = \bigsqcup_{g \in G} g(U \cap U')$. The transition map of U and U' gives the transition map of $\pi(U)$ and $\pi(U')$. Since the action of G is by holomorphic automorphisms, the transition maps are holomorphic. □

Remark 13. Recall that an action of a group G on a topological space X is said to be *properly discontinuous* if for every compact subset $K \subset X$, the set $\{g \in G : gK \cap K \neq \varnothing\}$ is finite. If G is a discrete group acting on a manifold M by diffeomorphisms, then the action is properly discontinuous and free if and only if for every point $p \in M$, there exists an open neighborhood U of p such that $gU \cap U = \varnothing$ for all $g \in G \setminus \{e\}$.

Example 14. Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e., a discrete subgroup of $\mathbb{C}$ generated by two $\mathbb{R}$-linearly independent complex numbers. Then Λ is isomorphic to $\mathbb{Z}^2$ as an abstract group and acts on $\mathbb{C}$ by translations, which are holomorphic automorphisms of $\mathbb{C}$. Then the quotient space $\mathbb{C}/\Lambda$ is a complex manifold of complex dimension 1 by Proposition 12. Such a complex manifold is called an *elliptic curve*. As real manifolds, it is diffeomorphic to $S^1 \times S^1$.

Example 15. Fix $\alpha \in \mathbb{C}^\times$ with $|\alpha| \neq 1$. Let $\mathbb{Z}$ act on $\mathbb{C}^n \setminus \{0\}$ by $k \cdot z = \alpha^k z$ for every $k \in \mathbb{Z}$ and $z \in \mathbb{C}^n \setminus \{0\}$. This action is free and properly discontinuous. Then the quotient space $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is a complex manifold of complex dimension n by Proposition 12. Such a complex manifold is called a *Hopf manifold*.

Example 16. Let

$$M = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, z_1, z_2, z_3 \in \mathbb{C} \right\}$$

be the complex Heisenberg group, which is biholomorphic to $\mathbb{C}^3$. Let $\Gamma := M \cap \mathrm{GL}(3, \mathbb{Z}[\sqrt{-1}])$. Then Γ is a discrete subgroup of M and acts on M by left multiplication, which are holomorphic automorphisms of M. The action is free and properly discontinuous. Then the quotient space M/Γ is a complex manifold of complex dimension 3 by Proposition 12. It is called the *Iwasawa manifold*. One can replace Γ by other cocompact discrete subgroups of M.

2 Almost Complex Structures

Let X be a complex manifold of complex dimension n. The tangent bundle TX is a real vector bundle of rank $2n$. There is a natural endomorphism $J : TX \to TX$ induced by the complex structure of X, i.e., for every point $p \in X$, $J_p : T_pX \to T_pX$ is the multiplication by $\sqrt{-1}$. We have $J^2 = -\mathrm{id}$.

Definition 17. Let M be a smooth manifold of real dimension $2n$. An *almost complex structure* on M is a smooth endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$. The pair (M, J) is called an *almost complex manifold*.

Question 18. Given a smooth manifold M of real dimension $2n$, when does it admit an almost complex structure? Is such an almost complex structure unique?

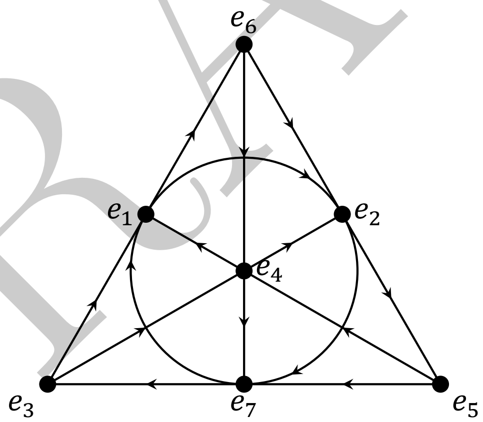
Giving an almost complex structure J on a smooth manifold M is equivalent to giving the tangent bundle TM the structure of a complex vector bundle. Hence the existence of almost complex structures is a purely topological problem. Note that to find a complex structure on M needs to solve some non-linear partial differential equations, which is much harder.

Example 19. The 6-sphere S^6 admits an almost complex structure. In fact, S^6 can be identified with the unit sphere in the imaginary octonions $\operatorname{Im}\mathbb{O}$ (see Remark 20). Denote by $m(x, y)$ the octonionic multiplication of $x, y \in \mathbb{O}$. For every point $p \in S^6$, the tangent space T_pS^6 can be identified with the orthogonal complement of $\mathbb{R}p$ in $\operatorname{Im}\mathbb{O}$. Define $J_p : T_pS^6 \to T_pS^6$ by $J_p(v) = m(p, v)$. Then $J_p^2(v) = p(pv) = -v$ for every $v \in T_pS^6$. Thus we get an almost complex structure on S^6.

Remark 20. Recall some fundamental facts about the octonions $\mathbb{O}$:

(a) $\mathbb{O}$ is an 8-dimensional normed vector space over $\mathbb{R}$ with an orthogonal basis $\{1\}\cup\{e_i | i = 1, \dots, 7\}$. The subspace spanned by $\{e_i\}$ is called the space of imaginary octonions and denoted by $\operatorname{Im}\mathbb{O}$.

(b) The multiplication $m : \mathbb{O}\times\mathbb{O} \to \mathbb{O}$ is a bilinear map and satisfies the distributive law and the norm multiplicative law $\|xy\| = \|x\|\|y\|$ for all $x, y \in \mathbb{O}$. It is given by the following Fano plane $\mathbb{P}^2(\mathbb{F}_2)$:



If $e_i \to e_j \to e_k$ is a directed line in the Fano plane, then $e_ie_j = e_k$, $e_je_k = e_i$, and $e_ke_i = e_j$. The multiplication is anti-commutative, i.e., $e_ie_j = -e_je_i$ for all $i \neq j$. And we have $e_i^2 = -1$ for all i.

Yang: To be checked...

Let (M, J) be an almost complex manifold. Then the complexified tangent bundle $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$ splits into the direct sum of two complex subbundles

$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M,$$

where

$$T^{1,0}M := \ker(\sqrt{-1}\mathrm{id} - J), \quad T^{0,1}M := \ker(\sqrt{-1}\mathrm{id} + J).$$

We have $\overline{T^{1,0}M} = T^{0,1}M$ and both $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles of rank n. This decomposition induces a decomposition of the complexified cotangent bundle

$$\Omega^1(M) := (TM_{\mathbb{C}})^* = (T^{1,0}M)^* \oplus (T^{0,1}M)^* =: \Omega^{1,0}(M) \oplus \Omega^{0,1}(M).$$

More generally, for every $p, q \geq 0$, define

$$\Omega^{p,q}(M) := \wedge^p (T^{1,0}M)^* \otimes \wedge^q (T^{0,1}M)^* \subset \wedge^{p+q}\Omega^1(M).$$

Then we have the decomposition

$$\Omega^k(M) := \wedge^k \Omega^1(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

The elements of $\Omega^{p,q}(M)$ are called *differential forms of type* (p,q) or (p,q)*-forms* for short.

Recall the *exterior differential operator* $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ is locally given by

$$\mathrm{d}\left(\sum_I f_I \mathrm{d}x_I\right) = \sum_I \sum_{j=1}^{2n} \frac{\partial f_I}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_I,$$

where I runs over all multi-indices with $|I| = k$ and $x_1, \dots, x_{2n}$ are local coordinates on M.

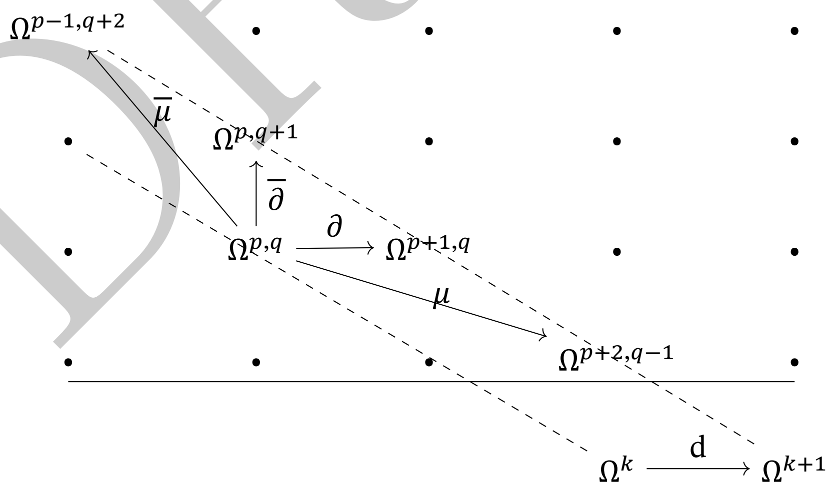
Proposition 21. There exist differential operators

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \mu : \Omega^{p,q}(M) \to \Omega^{p+2,q-1}(M)$$

such that

$$\mathrm{d} = \partial + \overline{\partial} + \mu + \overline{\mu}.$$

In a diagram:



Proof of Proposition 21. Yang: To be continued... □

Definition 22. The operator μ in Proposition 21 is called the *Nijenhuis operator* of the almost complex structure J. If $\mu = 0$, then J is called *integrable*. In this case, we have $\mathrm{d} = \partial + \overline{\partial}$.

Example 23. Let J be the almost complex structure on S^6 defined in Example 19.

Yang: To be continued...

Proposition 24. Let (M,J) be an almost complex manifold. If J is induced by a complex structure on M, then J is integrable, i.e., the Nijenhuis operator $\mu = 0$.

Proof. Yang: To be continued... □

The converse of Proposition 24 is also true, which is the famous Newlander-Nirenberg theorem. Yang: To add reference...

Theorem 25 (Newlander-Nirenberg Theorem)**.** Let (M,J) be an almost complex manifold of real dimension $2n$. If $\mu = 0$, then J is induced by a complex structure on M.

Proposition 26. Let (M,J) be an almost complex manifold. Then J is integrable if and only if $\partial^2 = 0$.

3 Cohomology in complex manifolds

Let M be a complex manifold. Denote by $\Omega^k_{\mathrm{sm}}(M)$ the space of smooth differential k-forms on M and by $\Omega^{p,q}_{\mathrm{sm}}(M)$ the space of smooth (p,q)-forms on M. Then $\Omega^k_{\mathrm{sm}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}_{\mathrm{sm}}(M)$. Denote by $\Omega^k_{\mathrm{hol}}(M)$ the space of holomorphic differential k-forms on M. Then we have $\Omega^{k,0}_{\mathrm{sm}}(M) = \Omega^k_{\mathrm{hol}}(M) \otimes_{\mathcal{O}^{\mathrm{hol}}_M} \mathcal{O}^{\mathrm{sm}}_M$.

There are several cohomology theories for complex manifolds.

Definition 27. Let M be a complex manifold. The *de Rham cohomology* of M is defined to be the de Rham cohomology of the underlying smooth manifold of M:

$$H^k_{\mathrm{dR}}(M) := \frac{\mathrm{Ker}(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

Definition 28. Let M be a complex manifold. The *Dolbeault cohomology* of M is defined to be

$$H^{p,q}_{\overline{\partial}}(M) := \frac{\mathrm{Ker}(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Definition 29. Let M be a complex manifold. The *Bott-Chern cohomology* of M is defined to be

$$H^{p,q}_{\mathrm{BC}}(M) := \frac{\mathrm{Ker}(\mathrm{d} : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M))}{\mathrm{Im}(\partial\overline{\partial} : \Omega^{p-1,q-1}(M) \to \Omega^{p,q}(M))}.$$

Yang: To be checked...

Definition 30. Let M be a complex manifold. The *Aeppli cohomology* of M is defined to be

$$H^{p,q}_{\mathrm{A}}(M) := \frac{\mathrm{Ker}(\partial\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M))}{\mathrm{Im}(\partial : \Omega^{p-1,q}(M) \to \Omega^{p,q}(M)) + \mathrm{Im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Yang: To be checked...

There are natural maps between these cohomology theories. Yang: To be continued...

Proposition 31. Let $\Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \dots, n\}$ be the unit polydisc in $\mathbb{C}^n$. Then

$$H^{p,q}_{\bar\partial}(\Delta^n) = \begin{cases} \Omega^p_{\mathrm{hol}}(\Delta^n), & q = 0, \\ 0, & q > 0. \end{cases}$$

Yang: To be checked...

Appendix

Theorem 32 (Holomorphic Implicit Function Theorem)**.** Let $f : \mathbb{C}^{n+m} \to \mathbb{C}^m$ be a holomorphic map. Write the coordinates of $\mathbb{C}^{n+m}$ as $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{C}^n \times \mathbb{C}^m$. If $\det(\partial f/\partial w) \neq 0$ at $(z_0, w_0) \in \mathbb{C}^{n+m}$ with $f(z_0, w_0) = 0$, then there exist open neighborhoods U of z_0 and V of w_0, and a unique holomorphic map $g : U \to V$ such that for any $(z, w) \in U \times V$, $f(z, w) = 0$ if and only if $w = g(z)$.