Complex Manifolds

1 Definition and Examples

Definition 1. A *complex manifold* of complex dimension n is a topological space M such that

(a) M is Hausdorff and second countable;

(b) M is locally homeomorphic to $\mathbb{C}^n$, i.e., for every point $p \in M$, there exists an open neighborhood U of p and a homeomorphism $\varphi : U \to V \subset \mathbb{C}^n$, where V is an open subset of $\mathbb{C}^n$, The pair (U, φ) is called a *chart*;

(c) if (U, φ) and (U', φ') are two charts with $U \cap U' \neq \varnothing$, then the transition map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$$

is holomorphic.

The collection of all charts $\{(U_\alpha, \varphi_\alpha)\}$ that cover M is called an *atlas*. If the atlas is maximal, it is called a *complex structure* on M.

Another way to define complex manifolds is to use the language of ringed spaces.

Definition 2. A *complex manifold* of complex dimension n is a ringed space $(M, \mathcal{O}_M)$ such that

(a) M is Hausdorff and second countable;

(b) for every point $p \in M$, there exists an open neighborhood U of p such that $(U, \mathcal{O}_M|_U)$ is isomorphic to $(B, \mathcal{O}_B)$, where B is the unit open ball in $\mathbb{C}^n$ and $\mathcal{O}_B$ is the sheaf of holomorphic functions on B.

Question 3. Given a topological space M that is Hausdorff and second countable, when does it admit a complex structure? Is such a complex structure unique?

For complex dimension 1, the answer is positive and well-known. For higher dimensions, the answer is negative in general. In particular, does the 6-sphere S^6 admit a complex structure? This is a famous open problem in complex geometry.

Question 4. Does the 6-sphere S^6 admit a complex structure?

Definition 5. Let M and N be two complex manifolds. A continuous map $f : M \to N$ is called *holomorphic* if for every point $p \in M$, there exist charts (U, φ) of M around p and (V, ψ) of N around $f(p)$ such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V)$$

is holomorphic. Yang: To be continued...

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Definition 6. Let M be a complex manifold of complex dimension n. A subset $S \subset M$ is called a *complex submanifold* of complex dimension k if for every point $p \in S$, there exist a chart (U, φ) of M around p such that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{C}^k \times \{0\}) \subset \mathbb{C}^n,$$

where we identify $\mathbb{C}^n$ with $\mathbb{C}^k \times \mathbb{C}^{n-k}$. Yang: To be continued...

Example 7. Any complex vector space V of complex dimension n is a complex manifold of complex dimension n.

Example 8. Let $f : \mathbb{C}^{n+m} \to \mathbb{C}^m$ be a holomorphic function. Suppose that 0 is a regular value of f, i.e., the Jacobian matrix Df_p is surjective for every $p \in f^{-1}(0)$. Then by the holomorphic implicit function theorem (Theorem 16), for every point $p \in f^{-1}(0)$, there exist open neighborhoods U of p and V of 0 such that $f^{-1}(0) \cap U$ is biholomorphic to an open subset of $\mathbb{C}^n$. Thus $f^{-1}(0)$ is a complex manifold of complex dimension n. In particular, any non-singular complex algebraic variety is a complex manifold. Yang: To be continued...

Example 9. The complex projective space $\mathbb{CP}^n := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$ is a complex manifold of complex dimension n. In fact, $\mathbb{CP}^n$ can be covered by $n+1$ charts, each of which is biholomorphic to $\mathbb{C}^n$. For example, the chart $U_0 = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : z_0 \neq 0\}$ is biholomorphic to $\mathbb{C}^n$ via the map

$$[z_0 : z_1 : \cdots : z_n] \mapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \ldots, \frac{z_n}{z_0}\right).$$

The other charts are defined similarly.

Proposition 10. Let $f : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ be a homogeneous polynomial of degree d such that 0 is a regular value of f. Then $f^{-1}(0)/\mathbb{C}^\times$ is a complex submanifold of $\mathbb{CP}^n$ of complex dimension $n-1$.

Proof. Yang: To be continued... □

2 Almost Complex Structures

Let X be a complex manifold of complex dimension n. The tangent bundle TX is a real vector bundle of rank $2n$. There is a natural endomorphism $J : TX \to TX$ induced by the complex structure of X, i.e., for every point $p \in X$, $J_p : T_pX \to T_pX$ is the multiplication by $\sqrt{-1}$. We have $J^2 = -\mathrm{id}$.

Definition 11. Let M be a smooth manifold of real dimension $2n$. An *almost complex structure* on M is a smooth endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$. The pair (M, J) is called an *almost complex manifold*.

Definition 12. Let (M, J) be an almost complex manifold. A smooth function $f : M \to \mathbb{C}$ is called *J-holomorphic* if

$$\mathrm{d}f \circ J = \sqrt{-1} \cdot \mathrm{d}f.$$

Question 13. Given a smooth manifold M of real dimension $2n$, when does it admit an almost complex structure? Is such an almost complex structure unique?

Yang: Giving an almost complex structure J on a smooth manifold M is equivalent to giving M a

GL($n, \mathbb{C}$)-structure. The existence of almost complex structures is a purely topological problem.

Let (M, J) be an almost complex manifold. Then the complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ splits into the direct sum of two complex subbundles

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where

$$T^{1,0}M := \ker(\sqrt{-1}\mathrm{id} - J), \quad T^{0,1}M := \ker(\sqrt{-1}\mathrm{id} + J).$$

We have $\overline{T^{1,0}M} = T^{0,1}M$ and both $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles of rank n. This decomposition induces a decomposition of the complexified cotangent bundle Yang: To be continued...

Proposition 14. There exist differential operators

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \mu : \Omega^{p,q}(M) \to \Omega^{p+2,q-1}(M)$$

such that

$$\mathrm{d} = \partial + \bar{\partial} + \mu + \bar{\mu}.$$

In a diagram:

Proof of Proposition 14. Yang: To be continued... □

Proposition 15. The operators ∂ and μ satisfy the following properties:

(a)

Yang: To be continued...

Requirements

Theorem 16 (Holomorphic implicit function theorem)**.** Let $f : \mathbb{C}^{n+m} \to \mathbb{C}^m$ be a holomorphic function. If the Jacobian determinant $\det(\partial f/\partial w)$ is nonzero at $(z_0, w_0) \in \mathbb{C}^{n+m}$, then there exist open neighborhoods U of z_0 and V of w_0, and a unique holomorphic function $g : U \to V$ such that for any $(z, w) \in U \times V$, $f(z, w) = f(z_0, w_0)$ if and only if $w = g(z)$. Yang: To be continued...