

Complex Geometry

No Cover Image

Use \coverimage{filename} to add an image

“abaaba”

Complex Geometry

Author: Tianle Yang

Email: loveandjustice@88.com

Homepage: <https://www.tianleyang.com>

Source code: github.com/MonkeyUnderMountain/Complex_Geometry

Version: 0.1.0

Last updated: October 5, 2025

Copyright © 2025 Tianle Yang

Contents

1	The first properties	1
1.1	Analysis in several complex variables	1
1.1.1	Holomorphic functions	1
1.1.2	Cauchy Integral Formula	3
1.1.3	Zero sets of holomorphic functions	4
1.2	Complex Manifolds	5
1.2.1	Definition and Examples	5
1.2.2	Almost Complex Structures	8

Chapter 1

The first properties

1.1 Analysis in several complex variables

In this section, we introduce some basic concepts and results in complex analysis with multiple variables.

1.1.1 Holomorphic functions

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Definition 1.1.1. A continuous map $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$ is *differentiable* at $p \in \mathbb{R}^{2n}$ if there exists a linear map $df_p : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$ such that

$$f(z) = f(p) + df_p(z - p) + o(|z - p|).$$

A continuous map $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is *holomorphic* at $p \in \mathbb{C}^n$ if it is differentiable at p and df_p is \mathbb{C} -linear, i.e., $df_p(\sqrt{-1}z) = \sqrt{-1}df_p(z)$ for all $z \in \mathbb{C}^n$.

By a “function”, we always mean a complex-valued function, i.e., a map $f : \mathbb{C}^n \rightarrow \mathbb{C}$. Fix a coordinate system $z = (z_1, \dots, z_n)$ on \mathbb{C}^n and write $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Then a differentiable function $f = u + iv : \mathbb{C}^n \rightarrow \mathbb{C}$ is holomorphic at p if and only if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x_i}(p) = \frac{\partial v}{\partial y_i}(p), \quad \frac{\partial u}{\partial y_i}(p) = -\frac{\partial v}{\partial x_i}(p), \quad i = 1, \dots, n.$$

For convenience, we consider the complexified tangent space $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and introduce the following operators.

Definition 1.1.2. The *Wirtinger operators* are defined as

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} - \sqrt{-1} \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} + \sqrt{-1} \frac{\partial}{\partial y_j} \right), \quad j = 1, \dots, n.$$

Then we can rewrite the Cauchy-Riemann equations as

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n.$$

We summarize some properties of Wirtinger operators in the following proposition.

Proposition 1.1.3. The Wirtinger operators satisfy the following properties:

$$(a) \partial_{z_j} z_i = \delta_{ij}, \partial_{z_j} \bar{z}_i = 0, \partial_{z_j} \bar{z}_i = 0, \partial_{z_j} \bar{z}_j = \delta_{ij};$$

$$(b) \overline{\left(\partial_{z_j} f \right)} = \partial_{\bar{z}_j} \bar{f};$$

(c) suppose we have $\mathbb{C}^n \xrightarrow{g} \mathbb{C}^m \xrightarrow{f} \mathbb{C}^l$ and the coordinate on \mathbb{C}^m is $w = (w_1, \dots, w_m)$, then the chain rule holds:

$$\begin{aligned} \frac{\partial(f \circ g)}{\partial z_j} &= \sum_{k=1}^m \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial z_j}(z) + \sum_{k=1}^m \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial z_j}(z), \\ \frac{\partial(f \circ g)}{\partial \bar{z}_j} &= \sum_{k=1}^m \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial \bar{z}_j}(z) + \sum_{k=1}^m \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial \bar{z}_j}(z). \end{aligned}$$

| *Proof.* By direct computation. \square

We can also consider the complexified of derivatives

$$(df_p)_C : T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}.$$

If we take $\{\partial_{z_i}, \partial_{\bar{z}_i}\}_{i=1}^n$ as a basis of $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and $\{\partial_{w_j}, \partial_{\bar{w}_j}\}_{j=1}^m$ as a basis of $T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$, then the matrix representation of $(df_p)_C$ is

$$(df_p)_C = \begin{bmatrix} \frac{\partial_z f(p)}{\partial_{\bar{z}} f(p)} & \frac{\partial_{\bar{z}} f(p)}{\partial_z f(p)} \end{bmatrix}.$$

In particular, if f is holomorphic, then we have $\det(df_p)_C = |\det(\partial_z f)(p)|^2 \geq 0$.

Definition 1.1.4. A map $f : \Omega \rightarrow \Omega'$ between two open sets $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^m$ is *biholomorphic* if it is a bijection and both f and f^{-1} are holomorphic.

If f is biholomorphic at p , then $m = n$ and $\det df_p > 0$.

Theorem 1.1.5 (Holomorphic Inverse Function Theorem). Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a holomorphic map. If the Jacobian determinant $\det df_p$ is nonzero at $p \in \mathbb{C}^n$, then there exist open neighborhoods U of p and V of $f(p)$ such that $f : U \rightarrow V$ is a biholomorphism.

| *Proof.* By the real inverse function theorem, there exist open neighborhoods U of p and V of $f(p)$ such that $g = f^{-1} : V \rightarrow U$ is a differentiable map. It suffices to show that g is holomorphic. By the chain rule (Proposition 1.1.3), since f is holomorphic, we have

$$0 = \left(\frac{\partial(f \circ g)_i}{\partial \bar{z}_j} \right)(q) = \left(\frac{\partial f_i}{\partial w_k} \right)(g(q)) \left(\frac{\partial g_k}{\partial \bar{z}_j} \right)(q).$$

Since $\det(\partial f / \partial w)(f(q)) \neq 0$, the matrix $(\partial f_i / \partial w_k)(g(q))$ is invertible, which implies that $(\partial g_k / \partial \bar{z}_j)(q) = 0$ for all k, j . Thus g is holomorphic. \square

Theorem 1.1.6 (Holomorphic Implicit Function Theorem). Let $f : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^m$ be a holomorphic map. Write the coordinates of \mathbb{C}^{n+m} as $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{C}^n \times \mathbb{C}^m$. If $\det(\partial f / \partial w) \neq 0$

at $(z_0, w_0) \in \mathbb{C}^{n+m}$ with $f(z_0, w_0) = 0$, then there exist open neighborhoods U of z_0 and V of w_0 , and a unique holomorphic map $g : U \rightarrow V$ such that for any $(z, w) \in U \times V$, $f(z, w) = 0$ if and only if $w = g(z)$.

Proof. By real implicit function theorem, there exist differentiable map $g : U \rightarrow V$ satisfying the above condition. It suffices to show that g is holomorphic. Let $G : U \rightarrow U \times V$ be defined by $G(z) = (z, g(z))$. Then we have $f \circ G = 0$. By the chain rule, we have

$$0 = \frac{\partial(f \circ G)_i}{\partial \bar{z}_j}(q) = \sum_{k=1}^n \frac{\partial f_i}{\partial w_k}(G(q)) \frac{\partial z_k}{\partial \bar{z}_j}(q) + \sum_{l=1}^m \frac{\partial f_i}{\partial w_l}(G(q)) \frac{\partial g_l}{\partial \bar{z}_j}(q).$$

Since $\det(\partial f / \partial w)(G(q)) \neq 0$, the matrix $(\partial f_i / \partial w_k)(G(q))$ is invertible, which implies that $(\partial g_l / \partial \bar{z}_j)(q) = 0$ for all l, j . Thus g is holomorphic. \square

1.1.2 Cauchy Integral Formula

Recall the Cauchy Integral Formula in one complex variable:

Theorem 1.1.7 (Cauchy Integral Formula in one complex variable). Let $K \subset \mathbb{C}$ be a compact set with piecewise differentiable boundary ∂K , and let f be differentiable on a neighborhood of K . Then for any z in the interior of K , we have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_K \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

Proof. Yang: By Stokes' theorem. To be continued... \square

Theorem 1.1.8 (Cauchy Integral Formula in several complex variables). Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D . Then for any $z \in D$,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial D_1 \times \dots \times \partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n.$$

Proof. Yang: To be continued... \square

Corollary 1.1.9. Holomorphic functions are analytic. Yang: To be continued...

Proposition 1.1.10. Holomorphic functions are open mappings. Yang: To be continued...

Proposition 1.1.11. If a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ on a connected open set $\Omega \subset \mathbb{C}^n$ attains its maximum at some point in Ω , then f is constant. Yang: To be continued...

Lemma 1.1.12. Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D . Then for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\max_{z \in D} \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(z) \right| \leq \frac{\alpha!}{r^\alpha} \max_{z \in D} |f(z)|,$$

where $r = (r_1, \dots, r_n)$ is the radius of the polydisk D . Yang: To be continued...

Theorem 1.1.13 (Generalized Liouville Theorem). A holomorphic function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ on the whole space \mathbb{C}^n that satisfies a polynomial growth condition, i.e., there exist constants $C > 0$ and $k \geq 0$ such that

$$|f(z)| \leq C(1 + |z|^k), \quad \forall z \in \mathbb{C}^n,$$

must be a polynomial of degree at most k . **Yang:** To be continued...

Theorem 1.1.14 (Montel's Theorem). A family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ that is uniformly bounded on compact subsets of Ω is a normal family, i.e., every sequence in the family has a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function or to infinity. **Yang:** To be continued...

1.1.3 Zero sets of holomorphic functions

Theorem 1.1.15 (Hartogs' Extension Theorem). Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $K \subset D$ be a compact subset such that $D \setminus K$ is connected. If $f : D \setminus K \rightarrow \mathbb{C}$ is a holomorphic function, then there exists a unique holomorphic function $\tilde{f} : D \rightarrow \mathbb{C}$ such that $\tilde{f}|_{D \setminus K} = f$. **Yang:** To be continued...

| *Proof.* **Yang:** To be checked □

Corollary 1.1.16. In contrast to the one-variable case, isolated singularities do not exist in several complex variables. Specifically, if $f : D \setminus \{p\} \rightarrow \mathbb{C}$ is a holomorphic function on a domain $D \subset \mathbb{C}^n$ with $n \geq 2$ and $p \in D$, then f can be extended to a holomorphic function on the entire domain D .

| *Proof.* This is a direct consequence of Hartogs' Extension Theorem by taking $K = \{p\}$. □

Theorem 1.1.17 (Weierstrass Preparation Theorem). Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic function in a neighborhood of the origin such that $f(0) = 0$ and f is not identically zero. Write the coordinates as $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that $f(0, w)$ has a zero of order k at $w = 0$, i.e.,

$$f(0, w) = a_k w^k + a_{k+1} w^{k+1} + \dots, \quad a_k \neq 0.$$

Then there exists a neighborhood U of the origin and unique holomorphic functions $g : U \rightarrow \mathbb{C}$ and $h_j : U' \rightarrow \mathbb{C}$ for $j = 1, \dots, k$, where $U' \subset \mathbb{C}^n$ is the projection of U onto the first n coordinates, such that

$$f(z, w) = (w^k + h_1(z)w^{k-1} + \dots + h_k(z))g(z, w),$$

with $g(0) \neq 0$ and $h_j(0) = 0$ for all j . **Yang:** To be continued...

| *Proof.* **Yang:** To be continued... **Yang:** Use the Cauchy Integral Formula to check the holomorphicity of g and h_j . □

Definition 1.1.18. Let $\Omega \subset \mathbb{C}^n$ be an open set. The *sheaf of holomorphic functions* on Ω , denoted by \mathcal{O}_Ω , is the assignment that to each open subset $U \subset \Omega$ assigns the ring $\mathcal{O}_\Omega(U)$ of all holomorphic functions on U , and set the restriction as the usual restriction of functions.

A fundamental property of the sheaf of holomorphic functions is its coherence.

Theorem 1.1.19 (Oka's Coherence Theorem). The sheaf of holomorphic functions \mathcal{O}_Ω on an open set $\Omega \subset \mathbb{C}^n$ is a coherent sheaf. **Yang:** To be continued...

In general, $\mathcal{O}_\Omega(U)$ is not a Noetherian ring for an open set $U \subset \Omega$. However, its stalks $\mathcal{O}_{\Omega,p}$ at points $p \in \Omega$ are Noetherian rings. **Yang:** To be checked

Example 1.1.20. **Yang:** To be continued...

Proposition 1.1.21. For any point $p \in \Omega$, the stalk $\mathcal{O}_{\Omega,p}$ of the sheaf of holomorphic functions at p is a Noetherian ring. **Yang:** To be continued...

Remark 1.1.22. The sheaf of holomorphic functions \mathcal{O}_Ω is a sheaf of topological rings, where the topology on $\mathcal{O}_\Omega(U)$ for an open set $U \subset \Omega$ is given by the compact-open topology. **Yang:** To be continued...

Definition 1.1.23. A subset $A \subset \Omega$ of an open set $\Omega \subset \mathbb{C}^n$ is called an *analytic subset* if for every point $p \in A$, there exists a neighborhood U of p and finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_\Omega(U)$ such that

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = \dots = f_k(z) = 0\}.$$

Yang: To be continued...

1.2 Complex Manifolds

1.2.1 Definition and Examples

Definition 1.2.1. A *complex manifold* of complex dimension n is a topological space M such that

- (a) M is Hausdorff and second countable;
- (b) M is locally homeomorphic to \mathbb{C}^n , i.e., for every point $p \in M$, there exists an open neighborhood U of p and a homeomorphism $\varphi : U \rightarrow V \subset \mathbb{C}^n$, where V is an open subset of \mathbb{C}^n . The pair (U, φ) is called a *chart*;
- (c) if (U, φ) and (U', φ') are two charts with $U \cap U' \neq \emptyset$, then the transition map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \rightarrow \varphi'(U \cap U')$$

is holomorphic.

The collection of all charts $\{(U_\alpha, \varphi_\alpha)\}$ that cover M is called an *atlas*. If the atlas is maximal, it is called a *complex structure* on M .

Another way to define complex manifolds is to use the language of ringed spaces.

Definition 1.2.2. A *complex manifold* of complex dimension n is a locally ringed space (M, \mathcal{O}_M) such that

- (a) M is Hausdorff and second countable;
- (b) for every point $p \in M$, there exists an open neighborhood U of p such that $(U, \mathcal{O}_M|_U)$ is isomorphic to (B, \mathcal{O}_B) , where B is the unit open ball in \mathbb{C}^n and \mathcal{O}_B is the sheaf of holomorphic functions on B .

Question 1.2.3. Given a topological space M that is Hausdorff and second countable, when does it admit a complex structure? Is such a complex structure unique?

For complex dimension 1, the answer is positive and well-known. For higher dimensions, the answer is negative in general. In particular, does the 6-sphere S^6 admit a complex structure? This is a famous open problem in complex geometry.

Question 1.2.4. Does the 6-sphere S^6 admit a complex structure?

Definition 1.2.5. Let M and N be two complex manifolds. A continuous map $f : M \rightarrow N$ is called *holomorphic* if for every point $p \in M$, there exist charts (U, φ) of M around p and (V, ψ) of N around $f(p)$ with $U \subset f^{-1}(V)$ such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$$

is holomorphic.

Definition 1.2.6. Let M be a complex manifold of complex dimension n . A subset $S \subset M$ is called a *complex submanifold* of complex dimension k if for every point $p \in S$, there exist a chart (U, φ) of M around p such that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{C}^k \times \{0\}) \subset \mathbb{C}^n,$$

where we identify \mathbb{C}^n with $\mathbb{C}^k \times \mathbb{C}^{n-k}$. This gives a chart of S around p . Endowed with the induced topology and the induced complex structure, S is a complex manifold of complex dimension k .

Example 1.2.7. Any complex vector space V of complex dimension n is a complex manifold of complex dimension n .

Example 1.2.8. The complex projective space $\mathbb{CP}^n := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$ is a complex manifold of complex dimension n . In fact, \mathbb{CP}^n can be covered by $n + 1$ charts, each of which is biholomorphic to \mathbb{C}^n . For example, the chart $U_0 = \{[z_0 : z_1 : \dots : z_n] \in \mathbb{CP}^n : z_0 \neq 0\}$ is biholomorphic to \mathbb{C}^n via the map

$$[z_0 : z_1 : \dots : z_n] \mapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0} \right).$$

The other charts are defined similarly.

Proposition 1.2.9. Let M and N be complex manifolds of complex dimension n and m respectively, with $n \geq m$. If $f : M \rightarrow N$ is a holomorphic map such that p is a regular value of f , i.e., the tangent map df_x is surjective for every $x \in f^{-1}(p)$, then $f^{-1}(p)$ is a complex submanifold of M of complex

dimension $n - m$.

Proof. For every point $q \in f^{-1}(p)$, choose charts (U, φ) of M around q and (V, ψ) of N around p such that $f(U) \subset V$. By changing coordinates if necessary, we may assume that $\det(\partial f / \partial w)(q) \neq 0$, where we write the coordinates of $\varphi(U)$ as $(z, w) = (z_1, \dots, z_{n-m}, w_1, \dots, w_m) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$. Then by the Holomorphic Implicit Function Theorem (Theorem 1.1.6), there exist open neighborhoods U' of q such that $f^{-1}(p) \cap U'$ is biholomorphic to an open subset of \mathbb{C}^{n-m} . \square

Example 1.2.10. Let $X \subset \mathbb{C}^n$ be a complex algebraic variety defined by the vanishing of polynomials $f_1, \dots, f_m \in \mathbb{C}[z_1, \dots, z_n]$. Suppose that X is non-singular, i.e., for every point $p \in X$, the Jacobian matrix $(\partial_{z_j} f_i(p))_{i,j}$ has maximal rank r . Then X is a complex submanifold of \mathbb{C}^n of complex dimension $n - r$.

Example 1.2.11. A *hypersurface* H in \mathbb{CP}^n is the zero locus of a homogeneous polynomial $f \in \mathbb{C}[z_0, z_1, \dots, z_n]$. Suppose 0 is a regular value of $f : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}$. On each chart $U_i \cong \mathbb{C}^n$ of \mathbb{CP}^n , it defines a holomorphic function $f_i : U_i \rightarrow \mathbb{C}, [z] \mapsto z = (z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n) \mapsto f(z)$. The regularity condition implies that 0 is a regular value of each f_i . Hence $H \cap U_i = f_i^{-1}(0)$ is a complex submanifold of U_i of complex dimension $n - 1$ by Proposition 1.2.9. Gluing these local pieces together, we see that H is a complex submanifold of \mathbb{CP}^n of complex dimension $n - 1$.

Proposition 1.2.12. Let M be a complex manifold and let G be a discrete group acting on M by holomorphic automorphisms. If the action is free and properly discontinuous, then the quotient space M/G is a complex manifold and the quotient map $\pi : M \rightarrow M/G$ is a holomorphic covering map.

Proof. For every point $p \in M/G$, choose a point $q \in M$ such that $\pi(q) = p$. Since the action is free and properly discontinuous (see Remark 1.2.13), there exists an open neighborhood U of q such that $gU \cap U = \emptyset$ for all $g \in G \setminus \{e\}$. Then $\pi|_U : U \rightarrow \pi(U)$ is a homeomorphism. This gives a chart of M/G around p . If we have two such charts $(\pi(U), \varphi)$ and $(\pi(U'), \varphi')$ of M/G whose intersection is non-empty, WLOG, assume that $U \cap U' \neq \emptyset$. Then $\pi^{-1}(\pi(U) \cap \pi(U')) = \bigsqcup_{g \in G} g(U \cap U')$. The transition map of U and U' gives the transition map of $\pi(U)$ and $\pi(U')$. Since the action of G is by holomorphic automorphisms, the transition maps are holomorphic. \square

Remark 1.2.13. Recall that an action of a group G on a topological space X is said to be *properly discontinuous* if for every compact subset $K \subset X$, the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite. If G is a discrete group acting on a manifold M by diffeomorphisms, then the action is properly discontinuous and free if and only if for every point $p \in M$, there exists an open neighborhood U of p such that $gU \cap U = \emptyset$ for all $g \in G \setminus \{e\}$.

Example 1.2.14. Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e., a discrete subgroup of \mathbb{C} generated by two \mathbb{R} -linearly independent complex numbers. Then Λ is isomorphic to \mathbb{Z}^2 as an abstract group and acts on \mathbb{C} by translations, which are holomorphic automorphisms of \mathbb{C} . Then the quotient space \mathbb{C}/Λ is a complex manifold of complex dimension 1 by Proposition 1.2.12. Such a complex manifold is called an *elliptic curve*. As real manifolds, it is diffeomorphic to $S^1 \times S^1$.

Example 1.2.15. Fix $\alpha \in \mathbb{C}^\times$ with $|\alpha| \neq 1$. Let \mathbb{Z} act on $\mathbb{C}^n \setminus \{0\}$ by $k \cdot z = \alpha^k z$ for every $k \in \mathbb{Z}$ and $z \in \mathbb{C}^n \setminus \{0\}$. This action is free and properly discontinuous. Then the quotient space $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is a complex manifold of complex dimension n by Proposition 1.2.12. Such a complex manifold is

called a *Hopf manifold*.

Example 1.2.16. Let

$$M = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$$

be the complex Heisenberg group, which is biholomorphic to \mathbb{C}^3 . Let $\Gamma := M \cap \mathrm{GL}(3, \mathbb{Z}[\sqrt{-1}])$. Then Γ is a discrete subgroup of M and acts on M by left multiplication, which are holomorphic automorphisms of M . The action is free and properly discontinuous. Then the quotient space M/Γ is a complex manifold of complex dimension 3 by [Proposition 1.2.12](#). It is called the *Iwasawa manifold*. One can replace Γ by other cocompact discrete subgroups of M .

1.2.2 Almost Complex Structures

Let X be a complex manifold of complex dimension n . The tangent bundle TX is a real vector bundle of rank $2n$. There is a natural endomorphism $J : TX \rightarrow TX$ induced by the complex structure of X , i.e., for every point $p \in X$, $J_p : T_p X \rightarrow T_p X$ is the multiplication by $\sqrt{-1}$. We have $J^2 = -\mathrm{id}$.

Definition 1.2.17. Let M be a smooth manifold of real dimension $2n$. An *almost complex structure* on M is a smooth endomorphism $J : TM \rightarrow TM$ such that $J^2 = -\mathrm{id}$. The pair (M, J) is called an *almost complex manifold*.

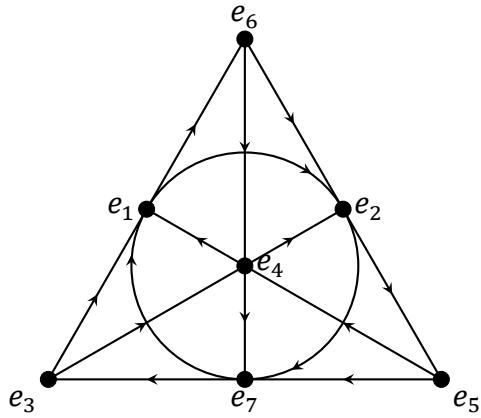
Question 1.2.18. Given a smooth manifold M of real dimension $2n$, when does it admit an almost complex structure? Is such an almost complex structure unique?

Giving an almost complex structure J on a smooth manifold M is equivalent to giving the tangent bundle TM the structure of a complex vector bundle. Hence the existence of almost complex structures is a purely topological problem. Note that to find a complex structure on M needs to solve some non-linear partial differential equations, which is much harder.

Example 1.2.19. The 6-sphere S^6 admits an almost complex structure. In fact, S^6 can be identified with the unit sphere in the imaginary octonions $\mathrm{Im}\, \mathbb{O}$ (see [Remark 1.2.20](#)). Denote by $m(x, y)$ the octonionic multiplication of $x, y \in \mathbb{O}$. For every point $p \in S^6$, the tangent space $T_p S^6$ can be identified with the orthogonal complement of Rp in $\mathrm{Im}\, \mathbb{O}$. Define $J_p : T_p S^6 \rightarrow T_p S^6$ by $J_p(v) = m(p, v)$. Then $J_p^2(v) = p(pv) = -v$ for every $v \in T_p S^6$. Thus we get an almost complex structure on S^6 .

Remark 1.2.20. Recall some fundamental facts about the octonions \mathbb{O} :

- (a) \mathbb{O} is an 8-dimensional normed vector space over \mathbb{R} with an orthogonal basis $\{1\} \cup \{e_i | i = 1, \dots, 7\}$. The subspace spanned by $\{e_i\}$ is called the space of imaginary octonions and denoted by $\mathrm{Im}\, \mathbb{O}$.
- (b) The multiplication $m : \mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$ is a bilinear map and satisfies the distributive law and the norm multiplicative law $\|xy\| = \|x\|\|y\|$ for all $x, y \in \mathbb{O}$. It is given by the following Fano plane $\mathbb{P}^2(\mathbb{F}_2)$:



If $e_i \rightarrow e_j \rightarrow e_k$ is a directed line in the Fano plane, then $e_i e_j = e_k$, $e_j e_k = e_i$, and $e_k e_i = e_j$. The multiplication is anti-commutative, i.e., $e_i e_j = -e_j e_i$ for all $i \neq j$. And we have $e_i^2 = -1$ for all i .

Yang: To be checked...

Let (M, J) be an almost complex manifold. Then the complexified tangent bundle $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$ splits into the direct sum of two complex subbundles

$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M,$$

where

$$T^{1,0}M := \ker(\sqrt{-1}\text{id} - J), \quad T^{0,1}M := \ker(\sqrt{-1}\text{id} + J).$$

We have $\overline{T^{1,0}M} = T^{0,1}M$ and both $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles of rank n . This decomposition induces a decomposition of the complexified cotangent bundle

$$\Omega^1(M) := (TM_{\mathbb{C}})^* = (T^{1,0}M)^* \oplus (T^{0,1}M)^* =: \Omega^{1,0}(M) \oplus \Omega^{0,1}(M).$$

More generally, for every $p, q \geq 0$, define

$$\Omega^{p,q}(M) := \wedge^p(T^{1,0}M)^* \otimes \wedge^q(T^{0,1}M)^* \subset \wedge^{p+q}\Omega^1(M).$$

Then we have the decomposition

$$\Omega^k(M) := \wedge^k \Omega^1(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

The elements of $\Omega^{p,q}(M)$ are called *differential forms of type (p, q)* or (p, q) -forms for short.

Recall the *exterior differential operator* $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ is locally given by

$$d \left(\sum_I f_I dx_I \right) = \sum_I \sum_{j=1}^{2n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I,$$

where I runs over all multi-indices with $|I| = k$ and x_1, \dots, x_{2n} are local coordinates on M .

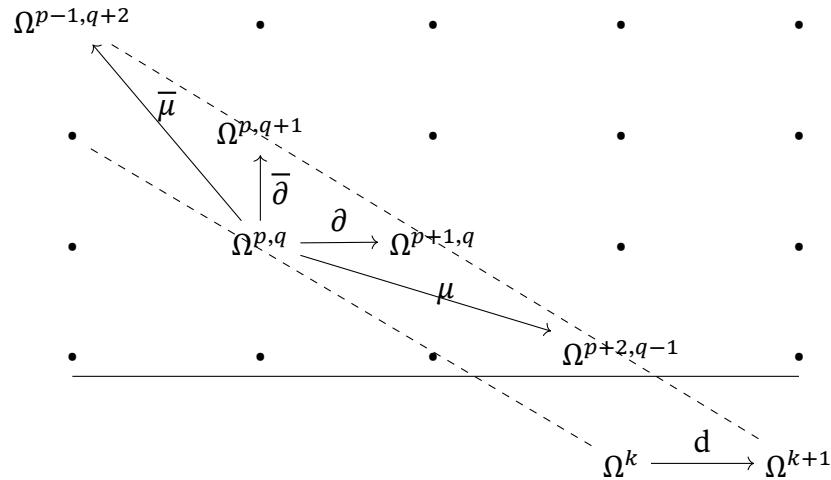
Proposition 1.2.21. There exist differential operators

$$\partial : \Omega^{p,q}(M) \rightarrow \Omega^{p+1,q}(M), \quad \mu : \Omega^{p,q}(M) \rightarrow \Omega^{p+2,q-1}(M)$$

such that

$$d = \partial + \bar{\partial} + \mu + \bar{\mu}.$$

In a diagram:



| Proof of Proposition 1.2.21. Yang: To be continued... □

Definition 1.2.22. The operator μ in Proposition 1.2.21 is called the *Nijenhuis operator* of the almost complex structure J . If $\mu = 0$, then J is called *integrable*. In this case, we have $d = \partial + \bar{\partial}$.
Yang: To be continued...

Example 1.2.23. Let J be the almost complex structure on S^6 defined in Example 1.2.19.

Yang: To be checked...

Proposition 1.2.24. Let M be a smooth manifold of real dimension $2n$ with an almost complex structure J . If J is induced by a complex structure on M , then $\mu = 0$.

| Proof. Yang: To be continued... □

The converse of Proposition 1.2.24 is also true, which is the famous Newlander-Nirenberg theorem.
Yang: To add reference...

Theorem 1.2.25. Let M be a smooth manifold of real dimension $2n$ with an almost complex structure J . If $\mu = 0$, then J is induced by a complex structure on M .