Holomorphic bundles

Definition 1. Let $E \xrightarrow{\pi} X$ be a complex vector bundle over a complex manifold X. We say E is a *holomorphic vector bundle* if there exists an open cover $\{U_\alpha\}$ of X and holomorphic trivializations

$$\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{C}^n$$

such that the transition maps

$$\phi_\beta \circ \phi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{C}^n \to U_\alpha \cap U_\beta \times \mathbb{C}^n$$

are holomorphic for all α, β. Yang: To be checked.

Example 2. The holomorphic tangent bundle $T^{1,0}X$ of a complex manifold X is a holomorphic vector bundle.

Proposition 3. Let $E, F \xrightarrow{\pi} X$ be holomorphic vector bundles over a complex manifold X. Then the following bundles are also holomorphic vector bundles:

(a) The direct sum bundle $E \oplus F$.

(b) The tensor product bundle $E \otimes F$.

(c) The dual bundle E^*.

Yang: To be completed.

1 Exponential sequence

Let X be a complex manifold. Recall that we have an exact sequence of sheaves on X:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0,$$

where $\underline{\mathbb{Z}}$ is the constant sheaf with stalk $\mathbb{Z}$, $\mathcal{O}_X$ is the sheaf of holomorphic functions on X, and $\mathcal{O}_X^*$ is the sheaf of nowhere vanishing holomorphic functions on X.

Theorem 4. The Picard group $\mathrm{Pic}(X)$ of holomorphic line bundles on X is isomorphic to the group of connected components of the sheaf of nowhere vanishing holomorphic functions on X, which is given by the exponential sequence.

Appendix

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