The first properties

1 Holomorphic functions

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Definition 1. A function $f : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is *differentiable* at $p \in \mathbb{R}^{2n}$ if there exists a linear map $L : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ such that

$$f(z) = f(p) + L(z-p) + o(|z-p|).$$

A function $f : \mathbb{C}^n \to \mathbb{C}^m$ is *holomorphic* at $p \in \mathbb{C}^n$ if it is differentiable at p and Yang: To be continued...

Definition 2. The *Wirtinger operators* are defined as

$$\partial z_j = \frac{1}{2}\left(\partial x_j - i\partial y_j\right), \quad \partial \bar{z}_j = \frac{1}{2}\left(\partial x_j + i\partial y_j\right).$$

Then we can rewrite the Cauchy-Riemann equations as

$$\partial \bar{z}_j f = 0, \quad j = 1, \dots, n.$$

We summarize some properties of Wirtinger operators in the following proposition.

Proposition 3. The Wirtinger operators satisfy the following properties:

(a) ∂z_j and $\partial \bar{z}_j$ are linear operators.

(b) $\partial z_j z_k = \delta_{jk}$, $\partial z_j \bar{z}_k = 0$, $\partial \bar{z}_j z_k = 0$, and $\partial \bar{z}_j \bar{z}_k = \delta_{jk}$.

(c) The Leibniz rule holds: for any two functions $f, g : \mathbb{C}^n \to \mathbb{C}$,

$$\partial z_j(fg) = (\partial z_j f)g + f(\partial z_j g), \quad \partial \bar{z}_j(fg) = (\partial \bar{z}_j f)g + f(\partial \bar{z}_j g).$$

(d) The operators commute: for any j, k,

$$\partial z_j \partial \bar{z}_k = \partial \bar{z}_k \partial z_j, \quad \partial z_j \partial z_k = \partial z_k \partial z_j, \quad \partial \bar{z}_j \partial \bar{z}_k = \partial \bar{z}_k \partial \bar{z}_j.$$

Yang: To be continued...

Consider the complexified differential $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. We can extend the Wirtinger operators to complexified tangent vectors by linearity. Then ∂_{z_j} and $\partial \bar{z}_j$ form a basis of $T^*\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. If f is holomorphic, under this basis, its differential df can be written as

$$df = \begin{pmatrix} \frac{\partial f}{\partial z} & \\ & \overline{\frac{\partial f}{\partial z}} \end{pmatrix}$$

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Theorem 4 (Holomorphic inverse function theorem). Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic function. If the Jacobian determinant $\det(\partial f/\partial z)$ is nonzero at $p \in \mathbb{C}^n$, then there exist open neighborhoods U of p and V of $f(p)$ such that $f : U \to V$ is a biholomorphism. Yang: To be continued...

Theorem 5 (Holomorphic implicit function theorem). Let $f : \mathbb{C}^{n+m} \to \mathbb{C}^m$ be a holomorphic function. If the Jacobian determinant $\det(\partial f/\partial w)$ is nonzero at $(z_0, w_0) \in \mathbb{C}^{n+m}$, then there exist open neighborhoods U of z_0 and V of w_0, and a unique holomorphic function $g : U \to V$ such that for any $(z, w) \in U \times V$, $f(z, w) = f(z_0, w_0)$ if and only if $w = g(z)$. Yang: To be continued...

2 Cauchy Integral Formula

Recall the Cauchy Integral Formula in one complex variable:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in D,$$

where D is a disk in $\mathbb{C}$ and f is holomorphic on a neighborhood of the closure of D. Yang: Need to check

Theorem 6 (Cauchy Integral Formula in one complex variable). Let $D \subset \mathbb{C}$ be a disk and f be holomorphic on a neighborhood of the closure of D. Then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Yang: To be continued...

Theorem 7 (Cauchy Integral Formula in several complex variables). Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D. Then for any $z \in D$,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D_1 \times \cdots \times \partial D_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n.$$

Yang: To be continued...

Corollary 8. Holomorphic functions are analytic. Yang: To be continued...

Proposition 9. Holomorphic functions are open mappings. Yang: To be continued...

Proposition 10. If a holomorphic function $f : \Omega \to \mathbb{C}$ on a connected open set $\Omega \subset \mathbb{C}^n$ attains its maximum at some point in Ω, then f is constant. Yang: To be continued...

Proposition 11. Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D. Then for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$\max_{z \in D} \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(z) \right| \leq \frac{\alpha!}{r^\alpha} \max_{z \in D} |f(z)|,$$

where $r = (r_1, \ldots, r_n)$ is the radius of the polydisk D. Yang: To be continued...

Theorem 12 (Generalized Liouville Theorem)**.** A holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ on the whole space $\mathbb{C}^n$ that satisfies a polynomial growth condition, i.e., there exist constants $C > 0$ and $k \geq 0$ such that

$$|f(z)| \leq C(1 + |z|^k), \quad \forall z \in \mathbb{C}^n,$$

must be a polynomial of degree at most k. Yang: To be continued...

Theorem 13 (Montel's Theorem)**.** A family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ that is uniformly bounded on compact subsets of Ω is a normal family, i.e., every sequence in the family has a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function or to infinity. Yang: To be continued...

3 Hartogs' phenomenon

Theorem 14 (Hartogs' Extension Theorem)**.** Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $K \subset D$ be a compact subset such that $D \setminus K$ is connected. If $f : D \setminus K \to \mathbb{C}$ is a holomorphic function, then there exists a unique holomorphic function $F : D \to \mathbb{C}$ such that $F|_{D \setminus K} = f$. Yang: To be continued...

Theorem 15 (Hartogs' Separate Analyticity Theorem)**.** Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $f : D \to \mathbb{C}$ be a function such that for each fixed $z' = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$, the function $f(z', z_j)$ is holomorphic in z_j for all $j = 1, \dots, n$. Then f is holomorphic on D. Yang: To be continued...