

# Metrics, curvature and connections

## 1 The first properties

Let  $X$  be a complex manifold and  $E \rightarrow X$  a holomorphic vector bundle.

**Definition 1.** A *Hermitian metric* on  $E$  is a smoothly varying family of Hermitian inner products  $\langle \cdot, \cdot \rangle_x$  on the fibers  $E_x$  for each  $x \in X$ , i.e.,

$$\langle \cdot, \cdot \rangle_x : E_x \times E_x \rightarrow \mathbb{C}$$

is a Hermitian inner product for each  $x$ , and for any local smooth sections  $s, t$  of  $E$ , the function

$$x \mapsto \langle s(x), t(x) \rangle_x$$

is smooth on  $X$ .

**Definition 2.** A *Hermitian metric* on a complex manifold  $X$  is a Hermitian metric on its holomorphic tangent bundle  $TX$ .

**Remark 3.** Let  $h$  be a Hermitian metric on a complex manifold  $X$ . Then  $h$  induces a Riemannian metric  $g$  on the underlying real manifold of  $X$  by

$$g(u, v) = \operatorname{Re}(h(u, v))$$

for real tangent vectors  $u, v \in T_x X$ . Yang: To be revised.

**Example 4.** Let  $\mathbb{P}^n$  be the complex projective space. The *Fubini-Study metric*  $h_{\text{FS}}$  is a Hermitian metric on its tangent bundle  $T\mathbb{P}^n$  defined as follows. On the standard affine chart  $U_i = \{[z_0 : \dots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$  with coordinates  $z_{j,i} = z_j/z_i$  for  $j \neq i$ , we know that  $T\mathbb{P}^n|_{U_i}$  is spanned by  $\{\partial_{j,i} = \partial/\partial z_{j,i}\}_{j \neq i}$ . The Fubini-Study metric is given by

$$h_{\text{FS}}(z_{-,i})(\partial_k, \partial_l) = \frac{\delta_{kl}}{(1 + \sum_{r \neq i} |z_{r,i}|^2)} - \frac{\overline{z_{k,i}}z_{l,i}}{(1 + \sum_{r \neq i} |z_{r,i}|^2)^2}.$$

On  $U_{ij} = U_i \cap U_j$ , the differential form transform as

$$dz_{k,i} = z_{j,i}dz_{k,j} - z_{k,i}z_{j,i}dz_{j,j}.$$

In the matrix form,

$$\begin{bmatrix} dz_{1,i} \\ \vdots \\ dz_{n,i} \end{bmatrix} = \begin{bmatrix} z_{j,i} & 0 & \cdots & -z_{1,i}z_{j,i} & \cdots & 0 \\ 0 & z_{j,i} & \cdots & -z_{2,i}z_{j,i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -z_{n,i}z_{j,i} & \cdots & z_{j,i} \end{bmatrix} \begin{bmatrix} dz_{1,j} \\ \vdots \\ dz_{n,j} \end{bmatrix}.$$

hence the tangent vectors transform as

$$\partial_{k,i} = \frac{\partial}{\partial z_{k,i}} = z_{j,i}\partial_{k,j} \quad \text{for } k \neq j, \quad \text{and}$$

Yang: To be continued.

**Example 5.** Now let us consider the complex projective plane  $\mathbb{P}^2 = \{[X : Y : Z]\}$ . On the affine chart  $U_Z = \{[X : Y : Z] \mid Z \neq 0\}$  with coordinates  $x = X/Z$  and  $y = Y/Z$ , the Fubini-Study metric  $h_{\text{FS}}$  on  $T\mathbb{P}^2|_{U_Z}$  is given by

$$h_{\text{FS}}(x, y) = \frac{1}{(1 + |x|^2 + |y|^2)^2} \begin{bmatrix} 1 + |y|^2 & -\bar{xy} \\ -x\bar{y} & 1 + |x|^2 \end{bmatrix}.$$

For a tangent vector  $\partial = a\partial_x + b\partial_y$ , its norm squared is

$$\|\partial\|_{h_{\text{FS}}}^2 = \frac{(1 + |y|^2)|a|^2 + (1 + |x|^2)|b|^2 - 2 \operatorname{Re}(\bar{xy}ab)}{(1 + |x|^2 + |y|^2)^2} = \frac{|a|^2 + |b|^2 + |xb - ya|^2}{(1 + |x|^2 + |y|^2)^2} \geq 0.$$

**Definition 6.** Let  $X$  be a complex manifold and  $E \rightarrow X$  a holomorphic vector bundle. A *connection* on  $E$  is a  $\mathbb{C}$ -linear map between the sheaves of smooth sections

$$\nabla : \mathcal{C}^\infty(-, E) \rightarrow \mathcal{C}^\infty(-, T^*X \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all smooth functions  $f$  and smooth sections  $s$  of  $E$ .

When you choose a vector field  $v \in \mathcal{C}^\infty(U, TX)$  on an open set  $U \subset X$ , the connection  $\nabla$  induces an endomorphism

$$\nabla_v : \mathcal{C}^\infty(U, E) \rightarrow \mathcal{C}^\infty(U, E)$$

by applying  $v$  on the  $T^*X$  component of  $\nabla s$  for a section  $s$  of  $E$ . In particular, if  $E = TX$  is the tangent bundle, then  $\nabla_v$  is called a *covariant derivative* along  $v$ . Sometimes people call  $\nabla$  an *endomorphism-valued 1-form* on  $X$  with values in  $\operatorname{End}(E)$  by viewing it as a map  $v \mapsto \nabla_v$ .

**Example 7.** Let  $\mathbb{P}^n$  be the complex projective space and  $\mathcal{O}_{\mathbb{P}^n}(1)$  the hyperplane line bundle. The *Chern connection* associated with the Fubini-Study metric on  $\mathcal{O}_{\mathbb{P}^n}(1)$  is a connection defined as follows: For a section  $s$  of  $\mathcal{O}_{\mathbb{P}^n}(1)$ , we define

$$\nabla s = ds + \alpha s,$$

where  $\alpha$  is a  $(1,0)$ -form determined by the Fubini-Study metric. **Yang: To be continued.**

**Proposition 8.** Let  $X$  be a complex manifold,  $E \rightarrow X$  a holomorphic vector bundle equipped with a Hermitian metric  $h$ . Then there exists a unique connection  $\nabla$  on  $E$  that is compatible with both the holomorphic structure and the Hermitian metric  $h$ . **Yang: To be checked.**

By the Leibniz rule, the connection  $\nabla$  can be extended to act on  $E$ -valued differential forms:

$$\nabla : \mathcal{C}^\infty(-, \Lambda^k T^*X \otimes E) \rightarrow \mathcal{C}^\infty(-, \Lambda^{k+1} T^*X \otimes E)$$

for all  $k \geq 0$ , satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

for  $\omega \in \mathcal{C}^\infty(-, \Lambda^k T^*X)$  and  $s \in \mathcal{C}^\infty(-, E)$ .

**Definition 9.** Let  $X$  be a complex manifold,  $E \rightarrow X$  a holomorphic vector bundle, and  $\nabla$  a connection on  $E$ . The *curvature* of the connection  $\nabla$  is defined as the endomorphism-valued 2-form

$$F_\nabla = \nabla^2 : \mathcal{C}^\infty(-, E) \rightarrow \mathcal{C}^\infty(-, \Lambda^2 T^*X \otimes E),$$

where  $\nabla^2$  is the composition of  $\nabla$  with itself. **Yang:** To be continued.

When  $E = TX$  is the tangent bundle, the curvature  $F_\nabla$  is a  $(3, 1)$ -tensor, which is the classical Riemann curvature tensor.

**Yang:** For a line bundle, everything coincide.

## 2 Chern-Weil Theory



**Theorem 10.** Let  $X$  be a complex manifold and  $E \rightarrow X$  a holomorphic vector bundle equipped with a Hermitian metric  $h$ . Let  $\nabla$  be the unique connection on  $E$  compatible with both the holomorphic structure and the Hermitian metric  $h$ , and let  $F_\nabla$  be its curvature. Then the Chern classes  $c_k(E) \in H^{2k}(X, \mathbb{R})$  can be represented by the differential forms

$$c_k(E) = \left[ \frac{1}{(2\pi i)^k} \text{Tr} (F_\nabla^k) \right].$$

**Yang:** To be checked.

| *Proof.* **Yang:** To be added. □

## Appendix

