Sheaves and Bundles on Complex Manifolds

1 Fiber bundles

Definition 1. Let M, F be manifolds. A *fiber bundle* with fiber F over M is a surjective map $\pi : E \to M$ of manifolds such that for each $x \in M$, there exists an open neighborhood U of x and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

where p_1 is the projection onto the first factor.

Given a fiber bundle E over M with fiber F and a covering $\{U_i\}$ of M, for each U_i, U_j and $x \in U_i \cap U_j$, we have two local trivializations

$$\varphi_i|_{E_x}, \varphi_j|_{E_x} : E_x \to \{x\} \times F.$$

They are differed by an automorphism $g_{ij}(x) = \varphi_i|_{E_x} \circ (\varphi_j|_{E_x})^{-1}$ of $\{x\} \times F$ as the following diagram

$$\begin{tikzcd} \{x\} \times F \arrow[rr, "g_{ij}(x)"] & & \{x\} \times F. \\ & E_x \arrow[ul, "\varphi_j|_{E_x}"] \arrow[ur, "\varphi_i|_{E_x}"'] & \end{tikzcd}$$

The map $g_{ij}(x)$ can be identified as an element of $\mathrm{Aut}(F)$. Varying x in $U_i \cap U_j$, we obtain the *transition function*

$$g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$$

which satisfies the *cocycle condition*

$$g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x), \quad \forall x \in U_i \cap U_j \cap U_k,$$

where the multiplication $\cdot$ is given by composition of automorphisms.

There is a natural way to impose smooth (holomorphic) structure on $\mathrm{Aut}(F)$, hence we can talk about smoothness or holomorphicity of transition functions. Set $\Phi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$. Then we have $\Phi_{ij}(x, v) = (x, g_{ij}(x)(v))$ for all $(x, v) \in (U_i \cap U_j) \times F$. Then Φ_{ij} is smooth (holomorphic) if and only if g_{ij} is smooth (holomorphic). Yang: To add ref.

Conversely, given a covering $\{U_i\}$ of M and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$ satisfying the cocycle condition, one can glue the local trivializations $U_i \times F$ via the maps Φ_{ij} to obtain a fiber bundle E over M with fiber F. Therefore, to given a fiber bundle with smooth (holomorphic) structure, it suffices to give a covering $\{U_i\}$ of M and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$ which are smooth (holomorphic) and satisfy the cocycle condition. In general, $\mathrm{Aut}(F)$ might be too large to handle. We can restrict the image of transition functions to a smaller subgroup $G \subset \mathrm{Aut}(F)$. This leads to the notion of structure group.

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Definition 2. Let M, F be manifolds, and $G \subset \mathrm{Aut}(F)$ be a Lie subgroup. A *fiber bundle with structure group* G is a fiber bundle $\pi : E \to M$ given by transition functions $g_{ij} : U_i \cap U_j \to G$.

Example 3. A *(real) vector bundle* of rank r over a manifold M is a fiber bundle with fiber $\mathbb{R}^r$ and structure group $\mathrm{GL}_r(\mathbb{R})$. Similarly, a *complex vector bundle* of rank r over a manifold M is a fiber bundle with fiber $\mathbb{C}^r$ and structure group $\mathrm{GL}_r(\mathbb{C})$.

On a real manifold M of dimension $2n$, an almost complex structure is equivalent to a reduction of the structure group of the tangent bundle TM from $\mathrm{GL}_{2n}(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{C})$.

By the transition functions construction, we can see that

Theorem 4. Let M, F be locally ringed spaces and $G \subset \mathrm{Aut}(F)$ a subgroup. Set $\mathcal{G}$ be the sheaf of "admissible" functions from open subsets of M to G. Then the set of isomorphism classes of fiber bundles over M with fiber F and structure group G is in one-to-one correspondence with the Čech cohomology set $\check{H}^1(M, \mathcal{G})$.

For example, if $F = \mathbb{C}$ and $G = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$, consider the holomorphic line bundles over a complex manifold M. The sheaf $\mathcal{G}$ is equal to $\mathcal{O}_M^*$, the sheaf of nowhere vanishing holomorphic functions on M. Therefore, by Theorem 4, we get the classic result $\mathrm{Pic}(M) \cong \check{H}^1(M, \mathcal{O}_M^*)$.

Slogan *For a fiber bundle E over M, we care about*

- *fiber F,*
- *structure group $G \subset \mathrm{Aut}(F)$,*
- *"admissible" functions class of transition functions $g_{ij} : U_i \cap U_j \to G$ (e.g. continuous, smooth, holomorphic).*

2 Sheaves

Construction 5. Let M be a manifold and $\pi : E \to M$ be a fiber bundle with fiber F. For each open subset $U \subset M$, we can consider the set of "admissible" sections of E over U:

$$\Gamma(U, E) = \{s : U \to E \mid \pi \circ s = \mathrm{id}_U, s \text{ is admissible}\}.$$

Here "admissible" means continuous, smooth, holomorphic, etc., depending on the context. The assignment $U \mapsto \Gamma(U, E)$ defines a sheaf of sets (or groups, modules, etc. if F has additional structure) on M, called the *sheaf of sections* of the bundle E.

Example 6. Let M be a complex manifold. We explain how to view the tangent bundle TM and the cotangent bundle T^*M as sheaves. There are two important classes of admissible sections of these bundles, namely holomorphic and smooth sections. We denote the sheaf of holomorphic (respectively smooth) sections of TM by $\mathcal{T}_{M,\mathrm{hol}}$ (respectively $\mathcal{T}_{M,\mathrm{sm}}$).

Correspondingly, we denote the sheaf of holomorphic (respectively smooth) sections of T^*M by $\Omega^1_{M,\mathrm{hol}}$ (respectively $\Omega^1_{M,\mathrm{sm}}$). Sometime we omit the subscript M if there is no confusion.

The elements in $\mathcal{T}_{M,\mathrm{hol}}(U)$ (respectively $\mathcal{T}_{M,\mathrm{sm}}(U)$) are holomorphic (respectively smooth) vector fields on U, and holomorphic (respectively smooth) 1-forms on U for $\Omega^1_{M,\mathrm{hol}}(U)$ (respectively

$\Omega^1_{M,\mathrm{sm}}(U)$). As sheaves, we have

$$\Omega^1_{M,\mathrm{hol}} \cong \mathcal{H}om_{\mathcal{O}_M}(\mathcal{T}_{M,\mathrm{hol}}, \mathcal{O}_M) \quad \text{and} \quad \Omega^1_{M,\mathrm{sm}} \cong \mathcal{H}om_{\mathcal{C}^\infty_M}(\mathcal{T}_{M,\mathrm{sm}}, \mathcal{C}^\infty_M),$$

where $\mathcal{C}^\infty_M$ is the sheaf of smooth complex-valued functions on M.

For $T^*M^{\mathbb{C}} := T^*M \otimes_{\mathbb{R}} \mathbb{C}$, this is also a holomorphic complex vector bundle over M. Hence we have the sheaf of holomorphic sections $\Omega^{\mathbb{C}}_{M,\mathrm{hol}}$. However, in general $T^*M^{0,1}$ is not a holomorphic bundle since its transition functions are given by the complex conjugates of those of $T^*M^{1,0} \cong T^*M$. Hence for the decomposition $T^*M^{\mathbb{C}} = T^*M^{1,0} \oplus T^*M^{0,1}$, we only have the decomposition of sheaves of smooth sections

$$\Omega^{1,\mathbb{C}}_{\mathrm{sm}} \cong \Omega^{1,0}_{\mathrm{sm}} \oplus \Omega^{0,1}_{\mathrm{sm}} \cong \Omega^1_{\mathrm{sm}} \oplus \overline{\Omega^1_{\mathrm{sm}}}.$$

Appendix