Complex Geometry

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Complex Geometry

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Contents

Chapter 1

The first properties

1.1 Analysis in several complex variables

In this section, we introduce some basic concepts and results in complex analysis with multiple variables.

1.1.1 Holomorphic functions

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Definition 1.1.1. A continuous map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is *differentiable* at $p \in \mathbb{R}^{2n}$ if there exists a linear map $\mathrm{d}f_p : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ such that

$$f(z) = f(p) + \mathrm{d}f_p(z-p) + o(|z-p|).$$

A continuous map $f : \mathbb{C}^n \to \mathbb{C}^m$ is *holomorphic* at $p \in \mathbb{C}^n$ if it is differentiable at p and $\mathrm{d}f_p$ is $\mathbb{C}$-linear, i.e., $\mathrm{d}f_p(\sqrt{-1}z) = \sqrt{-1}\mathrm{d}f_p(z)$ for all $z \in \mathbb{C}^n$.

By a "function", we always mean a complex-valued function, i.e., a map $f : \mathbb{C}^n \to \mathbb{C}$. Fix a coordinate system $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n$ and write $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Then a differentiable function $f = u + iv : \mathbb{C}^n \to \mathbb{C}$ is holomorphic at p if and only if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x_i}(p) = \frac{\partial v}{\partial y_i}(p), \quad \frac{\partial u}{\partial y_i}(p) = -\frac{\partial v}{\partial x_i}(p), \quad i = 1, \dots, n.$$

For convenience, we consider the complexified tangent space $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and introduce the following operators.

Definition 1.1.2. The *Wirtinger operators* are defined as

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad j = 1, \dots, n.$$

Then we can rewrite the Cauchy-Riemann equations as

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n.$$

We summarize some properties of Wirtinger operators in the following proposition.

Proposition 1.1.3. The Wirtinger operators satisfy the following properties:

(a) $\partial_{z_j} z_i = \delta_{ij}$, $\partial_{z_j} \bar{z}_i = 0$, $\partial_{z_j} \bar{z}_i = 0$, $\partial_{z_j} \bar{z}_j = \delta_{ij}$;

(b) $\overline{(\partial_{z_j} f)} = \partial_{\bar{z}_j} \bar{f}$;

(c) suppose we have $\mathbb{C}^n \xrightarrow{g} \mathbb{C}^m \xrightarrow{f} \mathbb{C}^l$ and the coordinate on $\mathbb{C}^m$ is $w = (w_1, \dots, w_m)$, then the chain rule holds:

$$\frac{\partial (f \circ g)}{\partial z_j} = \sum_{k=1}^m \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial z_j}(z) + \sum_{k=1}^m \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial z_j}(z),$$
$$\frac{\partial (f \circ g)}{\partial \bar{z}_j} = \sum_{k=1}^m \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial \bar{z}_j}(z) + \sum_{k=1}^m \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial \bar{z}_j}(z).$$

Proof. By direct computation. □

We can also consider the complexified of derivatives

$$(\mathrm{d}f_p)_{\mathbb{C}} : T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \to T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}.$$

If we take $\{\partial_{z_i}, \partial_{\bar{z}_i}\}_{i=1}^n$ as a basis of $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and $\{\partial_{w_j}, \partial_{\bar{w}_j}\}_{j=1}^m$ as a basis of $T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$, then the matrix representation of $(\mathrm{d}f_p)_{\mathbb{C}}$ is

$$(\mathrm{d}f_p)_{\mathbb{C}} = \begin{bmatrix} \partial_z f(p) & \partial_{\bar{z}} f(p) \\ \overline{\partial_{\bar{z}} f(p)} & \overline{\partial_z f(p)} \end{bmatrix}.$$

In particular, if f is holomorphic, then we have $\det(\mathrm{d}f_p)_{\mathbb{C}} = |\det(\partial_z f)(p)|^2 \geq 0$.

Definition 1.1.4. A map $f : \Omega \to \Omega'$ between two open sets $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^m$ is *biholomorphic* if it is a bijection and both f and f^{-1} are holomorphic.

If f is biholomorphic at p, then $m = n$ and $\det \mathrm{d}f_p > 0$.

Theorem 1.1.5 (Holomorphic Inverse Function Theorem)**.** Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. If the Jacobian determinant $\det \mathrm{d}f_p$ is nonzero at $p \in \mathbb{C}^n$, then there exist open neighborhoods U of p and V of $f(p)$ such that $f : U \to V$ is a biholomorphism.

Proof. By the real inverse function theorem, there exist open neighborhoods U of p and V of $f(p)$ such that $g = f^{-1} : V \to U$ is a differentiable map. It suffices to show that g is holomorphic. By the chain rule (Proposition 1.1.3), since f is holomorphic, we have

$$0 = \left(\frac{\partial (f \circ g)_i}{\partial \bar{z}_j}\right)(q) = \left(\frac{\partial f_i}{\partial w_k}\right)(g(q)) \left(\frac{\partial g_k}{\partial \bar{z}_j}\right)(q).$$

Since $\det(\partial f / \partial w)(f(q)) \neq 0$, the matrix $(\partial f_i / \partial w_k)(g(q))$ is invertible, which implies that $(\partial g_k / \partial \bar{z}_j)(q) = 0$ for all k, j. Thus g is holomorphic. □

Theorem 1.1.6 (Holomorphic Implicit Function Theorem)**.** Let $f : \mathbb{C}^{n+m} \to \mathbb{C}^m$ be a holomorphic map. Write the coordinates of $\mathbb{C}^{n+m}$ as $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{C}^n \times \mathbb{C}^m$. If $\det(\partial f / \partial w) \neq 0$

at $(z_0, w_0) \in \mathbb{C}^{n+m}$ with $f(z_0, w_0) = 0$, then there exist open neighborhoods U of z_0 and V of w_0, and a unique holomorphic map $g : U \to V$ such that for any $(z, w) \in U \times V$, $f(z, w) = 0$ if and only if $w = g(z)$.

Proof. By real implicit function theorem, there exist differentiable map $g : U \to V$ satisfying the above condition. It suffices to show that g is holomorphic. Let $G : U \to U \times V$ be defined by $G(z) = (z, g(z))$. Then we have $f \circ G = 0$. By the chain rule, we have

$$0 = \frac{\partial (f \circ G)_i}{\partial \bar{z}_j}(q) = \sum_{k=1}^{n} \frac{\partial f_i}{\partial w_k}(G(q)) \frac{\partial z_k}{\partial \bar{z}_j}(q) + \sum_{l=1}^{m} \frac{\partial f_i}{\partial w_l}(G(q)) \frac{\partial g_l}{\partial \bar{z}_j}(q).$$

Since $\det(\partial f / \partial w)(G(q)) \neq 0$, the matrix $(\partial f_i / \partial w_k)(G(q))$ is invertible, which implies that $(\partial g_l / \partial \bar{z}_j)(q) = 0$ for all l, j. Thus g is holomorphic. □

1.1.2 Cauchy Integral Formula

Recall the Cauchy Integral Formula in one complex variable:

Theorem 1.1.7 (Cauchy Integral Formula in one complex variable)**.** Let $K \subset \mathbb{C}$ be a compact set with piecewise differentiable boundary ∂K, and let f be differentiable on a neighborhood of K. Then for any z in the interior of K, we have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_K \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

Proof. Yang: By Stokes' theorem. To be continued... □

Theorem 1.1.8 (Cauchy Integral Formula in several complex variables)**.** Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D. Then for any $z \in D$,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial D_1 \times \cdots \times \partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n.$$

Proof. Yang: To be continued... □

Corollary 1.1.9. Holomorphic functions are analytic. Yang: To be continued...

Proposition 1.1.10. Holomorphic functions are open mappings. Yang: To be continued...

Proposition 1.1.11. If a holomorphic function $f : \Omega \to \mathbb{C}$ on a connected open set $\Omega \subset \mathbb{C}^n$ attains its maximum at some point in Ω, then f is constant. Yang: To be continued...

Lemma 1.1.12. Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D. Then for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\max_{z \in D} \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(z) \right| \leq \frac{\alpha!}{r^\alpha} \max_{z \in D} |f(z)|,$$

where $r = (r_1, \dots, r_n)$ is the radius of the polydisk D. Yang: To be continued...

Theorem 1.1.13 (Generalized Liouville Theorem)**.** A holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ on the whole space $\mathbb{C}^n$ that satisfies a polynomial growth condition, i.e., there exist constants $C > 0$ and $k \geq 0$ such that

$$|f(z)| \leq C(1 + |z|^k), \quad \forall z \in \mathbb{C}^n,$$

must be a polynomial of degree at most k. Yang: To be continued...

Theorem 1.1.14 (Montel's Theorem)**.** A family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ that is uniformly bounded on compact subsets of Ω is a normal family, i.e., every sequence in the family has a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function or to infinity. Yang: To be continued...

1.1.3 Zero sets of holomorphic functions

Theorem 1.1.15 (Hartogs' Extension Theorem)**.** Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $K \subset D$ be a compact subset such that $D \setminus K$ is connected. If $f : D \setminus K \to \mathbb{C}$ is a holomorphic function, then there exists a unique holomorphic function $\tilde{f} : D \to \mathbb{C}$ such that $\tilde{f}|_{D \setminus K} = f$. Yang: To be continued...

Proof. Yang: To be checked □

Corollary 1.1.16. In contrast to the one-variable case, isolated singularities do not exist in several complex variables. Specifically, if $f : D \setminus \{p\} \to \mathbb{C}$ is a holomorphic function on a domain $D \subset \mathbb{C}^n$ with $n \geq 2$ and $p \in D$, then f can be extended to a holomorphic function on the entire domain D.

Proof. This is a direct consequence of Hartogs' Extension Theorem by taking $K = \{p\}$. □

Theorem 1.1.17 (Weierstrass Preparation Theorem)**.** Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function in a neighborhood of the origin such that $f(0) = 0$ and f is not identically zero. Write the coordinates as $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that $f(0, w)$ has a zero of order k at $w = 0$, i.e.,

$$f(0, w) = a_k w^k + a_{k+1} w^{k+1} + \cdots, \quad a_k \neq 0.$$

Then there exists a neighborhood U of the origin and unique holomorphic functions $g : U \to \mathbb{C}$ and $h_j : U' \to \mathbb{C}$ for $j = 1, \dots, k$, where $U' \subset \mathbb{C}^n$ is the projection of U onto the first n coordinates, such that

$$f(z, w) = (w^k + h_1(z) w^{k-1} + \cdots + h_k(z)) g(z, w),$$

with $g(0) \neq 0$ and $h_j(0) = 0$ for all j. Yang: To be continued...

Proof. Yang: To be continued... Yang: Use the Cauchy Integral Formula to check the holomorphicity of g and h_j. □

Definition 1.1.18. Let $\Omega \subset \mathbb{C}^n$ be an open set. The *sheaf of holomorphic functions* on Ω, denoted by $\mathcal{O}_\Omega$, is the assignment that to each open subset $U \subset \Omega$ assigns the ring $\mathcal{O}_\Omega(U)$ of all holomorphic functions on U, and set the restriction as the usual restriction of functions.

A fundamental property of the sheaf of holomorphic functions is its coherence.

Theorem 1.1.19 (Oka's Coherence Theorem)**.** The sheaf of holomorphic functions $\mathcal{O}_\Omega$ on an open set $\Omega \subset \mathbb{C}^n$ is a coherent sheaf. Yang: To be continued...

In general, $\mathcal{O}_\Omega(U)$ is not a Noetherian ring for an open set $U \subset \Omega$. However, its stalks $\mathcal{O}_{\Omega,p}$ at points $p \in \Omega$ are Noetherian rings. Yang: To be checked

Example 1.1.20. Yang: To be continued...

Proposition 1.1.21. For any point $p \in \Omega$, the stalk $\mathcal{O}_{\Omega,p}$ of the sheaf of holomorphic functions at p is a Noetherian ring. Yang: To be continued...

Remark 1.1.22. The sheaf of holomorphic functions $\mathcal{O}_\Omega$ is a sheaf of topological rings, where the topology on $\mathcal{O}_\Omega(U)$ for an open set $U \subset \Omega$ is given by the compact-open topology. Yang: To be continued...

Definition 1.1.23. A subset $A \subset \Omega$ of an open set $\Omega \subset \mathbb{C}^n$ is called an *analytic subset* if for every point $p \in \Omega$, there exists a neighborhood U of p and finitely many holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}_\Omega(U)$ such that

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = \cdots = f_k(z) = 0\}.$$

Yang: To be continued...

1.2 Complex Manifolds

1.2.1 Definition and Examples

Definition 1.2.1. A *complex manifold* of complex dimension n is a topological space M such that

(a) M is Hausdorff and second countable;

(b) M is locally homeomorphic to $\mathbb{C}^n$, i.e., for every point $p \in M$, there exists an open neighborhood U of p and a homeomorphism $\varphi : U \to V \subset \mathbb{C}^n$, where V is an open subset of $\mathbb{C}^n$, The pair (U, φ) is called a *chart*;

(c) if (U, φ) and (U', φ') are two charts with $U \cap U' \neq \emptyset$, then the transition map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$$

is holomorphic.

The collection of all charts $\{(U_\alpha, \varphi_\alpha)\}$ that cover M is called an *atlas*. If the atlas is maximal, it is called a *complex structure* on M.

Another way to define complex manifolds is to use the language of ringed spaces.

Definition 1.2.2. A *complex manifold* of complex dimension n is a locally ringed space $(M, \mathcal{O}_M)$ such that

(a) M is Hausdorff and second countable;

(b) for every point $p \in M$, there exists an open neighborhood U of p such that $(U, \mathcal{O}_M|_U)$ is isomorphic to $(B, \mathcal{O}_B)$, where B is the unit open ball in $\mathbb{C}^n$ and $\mathcal{O}_B$ is the sheaf of holomorphic functions on B.

Question 1.2.3. Given a topological space M that is Hausdorff and second countable, when does it admit a complex structure? Is such a complex structure unique?

For complex dimension 1, the answer is positive and well-known. For higher dimensions, the answer is negative in general. In particular, does the 6-sphere S^6 admit a complex structure? This is a famous open problem in complex geometry.

Question 1.2.4. Does the 6-sphere S^6 admit a complex structure?

Definition 1.2.5. Let M and N be two complex manifolds. A continuous map $f : M \to N$ is called *holomorphic* if for every point $p \in M$, there exist charts (U, φ) of M around p and (V, ψ) of N around $f(p)$ with $U \subset f^{-1}(V)$ such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is holomorphic.

Definition 1.2.6. Let M be a complex manifold of complex dimension n. A subset $S \subset M$ is called a *complex submanifold* of complex dimension k if for every point $p \in S$, there exist a chart (U, φ) of M around p such that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{C}^k \times \{0\}) \subset \mathbb{C}^n,$$

where we identify $\mathbb{C}^n$ with $\mathbb{C}^k \times \mathbb{C}^{n-k}$. This gives a chart of S around p. Endowed with the induced topology and the induced complex structure, S is a complex manifold of complex dimension k.

Example 1.2.7. Any complex vector space V of complex dimension n is a complex manifold of complex dimension n.

Example 1.2.8. The complex projective space $\mathbb{CP}^n := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$ is a complex manifold of complex dimension n. In fact, $\mathbb{CP}^n$ can be covered by $n+1$ charts, each of which is biholomorphic to $\mathbb{C}^n$. For example, the chart $U_0 = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : z_0 \neq 0\}$ is biholomorphic to $\mathbb{C}^n$ via the map

$$[z_0 : z_1 : \cdots : z_n] \mapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right).$$

The other charts are defined similarly.

Proposition 1.2.9. Let M and N are complex manifolds of complex dimension n and m respectively, with $n \geq m$. If $f : M \to N$ is a holomorphic map such that p is a regular value of f, i.e., the tangent map $\mathrm{d}f_x$ is surjective for every $x \in f^{-1}(p)$, then $f^{-1}(p)$ is a complex submanifold of M of complex dimension $n - m$.

Proof. For every point $q \in f^{-1}(p)$, choose charts (U, φ) of M around q and (V, ψ) of N around p such that $f(U) \subset V$. By changing coordinates if necessary, we may assume that $\det(\partial f/\partial w)(q) \neq 0$, where we write the coordinates of $\varphi(U)$ as $(z, w) = (z_1, \dots, z_{n-m}, w_1, \dots, w_m) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$. Then by the Holomorphic Implicit Function Theorem (Theorem 1.1.6), there exist open neighborhoods U' of q such that $f^{-1}(p) \cap U'$ is biholomorphic to an open subset of $\mathbb{C}^{n-m}$. □

Example 1.2.10. Let $X \subset \mathbb{C}^n$ be a complex algebraic variety defined by the vanishing of polynomials $f_1, \dots, f_m \in \mathbb{C}[z_1, \dots, z_n]$. Suppose that X is non-singular, i.e., for every point $p \in X$, the Jacobian matrix $\left(\partial_{z_j} f_i(p)\right)_{i,j}$ has maximal rank r. Then X is a complex submanifold of $\mathbb{C}^n$ of complex dimension $n - r$.

Example 1.2.11. A *hypersurface* H in $\mathbb{CP}^n$ is the zero locus of a homogeneous polynomial $f \in \mathbb{C}[z_0, z_1, \dots, z_n]$. Suppose 0 is a regular value of $f : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$. On each chart $U_i \cong \mathbb{C}^n$ of $\mathbb{CP}^n$, it defines a holomorphic function $f_i : U_i \to \mathbb{C}, [z] \mapsto z = (z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n) \mapsto f(z)$. The regularity condition implies that 0 is a regular value of each f_i. Hence $H \cap U_i = f_i^{-1}(0)$ is a complex submanifold of U_i of complex dimension $n - 1$ by Proposition 1.2.9. Gluing these local pieces together, we see that H is a complex submanifold of $\mathbb{CP}^n$ of complex dimension $n - 1$.

Proposition 1.2.12. Let M be a complex manifold and let G be a discrete group acting on M by holomorphic automorphisms. If the action is free and properly discontinuous, then the quotient space M/G is a complex manifold and the quotient map $\pi : M \to M/G$ is a holomorphic covering map.

Proof. For every point $p \in M/G$, choose a point $q \in M$ such that $\pi(q) = p$. Since the action is free and properly discontinuous (see Remark 1.2.13), there exists an open neighborhood U of q such that $gU \cap U = \varnothing$ for all $g \in G \setminus \{e\}$. Then $\pi|_U : U \to \pi(U)$ is a homeomorphism. This gives a chart of M/G around p. If we have two such charts $(\pi(U), \varphi)$ and $(\pi(U'), \varphi')$ of M/G whose intersection is non-empty, WLOG, assume that $U \cap U' \neq \varnothing$. Then $\pi^{-1}(\pi(U) \cap \pi(U')) = \bigsqcup_{g \in G} g(U \cap U')$. The transition map of U and U' gives the transition map of $\pi(U)$ and $\pi(U')$. Since the action of G is by holomorphic automorphisms, the transition maps are holomorphic. □

Remark 1.2.13. Recall that an action of a group G on a topological space X is said to be *properly discontinuous* if for every compact subset $K \subset X$, the set $\{g \in G : gK \cap K \neq \varnothing\}$ is finite. If G is a discrete group acting on a manifold M by diffeomorphisms, then the action is properly discontinuous and free if and only if for every point $p \in M$, there exists an open neighborhood U of p such that $gU \cap U = \varnothing$ for all $g \in G \setminus \{e\}$.

Example 1.2.14. Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e., a discrete subgroup of $\mathbb{C}$ generated by two $\mathbb{R}$-linearly independent complex numbers. Then Λ is isomorphic to $\mathbb{Z}^2$ as an abstract group and acts on $\mathbb{C}$ by translations, which are holomorphic automorphisms of $\mathbb{C}$. Then the quotient space $\mathbb{C}/\Lambda$ is a complex manifold of complex dimension 1 by Proposition 1.2.12. Such a complex manifold is called an *elliptic curve*. As real manifolds, it is diffeomorphic to $S^1 \times S^1$.

Example 1.2.15. Fix $\alpha \in \mathbb{C}^\times$ with $|\alpha| \neq 1$. Let $\mathbb{Z}$ act on $\mathbb{C}^n \setminus \{0\}$ by $k \cdot z = \alpha^k z$ for every $k \in \mathbb{Z}$ and $z \in \mathbb{C}^n \setminus \{0\}$. This action is free and properly discontinuous. Then the quotient space $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is a complex manifold of complex dimension n by Proposition 1.2.12. Such a complex manifold is called a *Hopf manifold*.

Example 1.2.16. Let

$$M = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_1, z_2, z_3 \in \mathbb{C} \right\}$$

be the complex Heisenberg group, which is biholomorphic to $\mathbb{C}^3$. Let $\Gamma := M \cap \mathrm{GL}(3, \mathbb{Z}[\sqrt{-1}])$. Then Γ is a discrete subgroup of M and acts on M by left multiplication, which are holomorphic automorphisms of M. The action is free and properly discontinuous. Then the quotient space M/Γ is a complex manifold of complex dimension 3 by Proposition 1.2.12. It is called the *Iwasawa manifold*. One can replace Γ by other cocompact discrete subgroups of M.

1.2.2 Almost Complex Structures

Let X be a complex manifold of complex dimension n. The tangent bundle TX is a real vector bundle of rank $2n$. There is a natural endomorphism $J : TX \to TX$ induced by the complex structure of X, i.e., for every point $p \in X$, $J_p : T_pX \to T_pX$ is the multiplication by $\sqrt{-1}$. We have $J^2 = -\mathrm{id}$.

Definition 1.2.17. Let M be a smooth manifold of real dimension $2n$. An *almost complex structure* on M is a smooth endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$. The pair (M, J) is called an *almost complex manifold*.

Question 1.2.18. Given a smooth manifold M of real dimension $2n$, when does it admit an almost complex structure? Is such an almost complex structure unique?

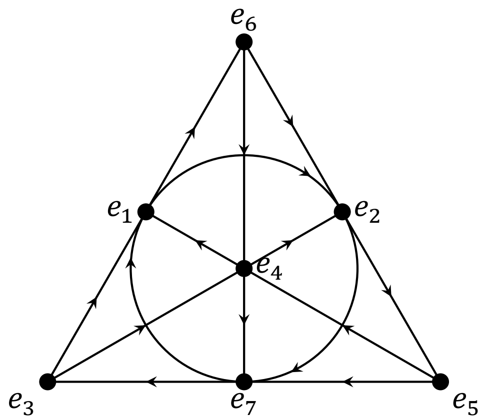
Giving an almost complex structure J on a smooth manifold M is equivalent to giving the tangent bundle TM the structure of a complex vector bundle. Hence the existence of almost complex structures is a purely topological problem. Note that to find a complex structure on M needs to solve some non-linear partial differential equations, which is much harder.

Example 1.2.19. The 6-sphere S^6 admits an almost complex structure. In fact, S^6 can be identified with the unit sphere in the imaginary octonions $\mathrm{Im}\,\mathbb{O}$ (see Remark 1.2.20). Denote by $m(x, y)$ the octonionic multiplication of $x, y \in \mathbb{O}$. For every point $p \in S^6$, the tangent space T_pS^6 can be identified with the orthogonal complement of $\mathbb{R}p$ in $\mathrm{Im}\,\mathbb{O}$. Define $J_p : T_pS^6 \to T_pS^6$ by $J_p(v) = m(p, v)$. Then $J_p^2(v) = p(pv) = -v$ for every $v \in T_pS^6$. Thus we get an almost complex structure on S^6.

Remark 1.2.20. Recall some fundamental facts about the octonions $\mathbb{O}$:

(a) $\mathbb{O}$ is an 8-dimensional normed vector space over $\mathbb{R}$ with an orthogonal basis $\{1\} \cup \{e_i | i = 1, \dots, 7\}$. The subspace spanned by $\{e_i\}$ is called the space of imaginary octonions and denoted by $\mathrm{Im}\,\mathbb{O}$.

(b) The multiplication $m : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ is a bilinear map and satisfies the distributive law and the norm multiplicative law $\|xy\| = \|x\|\|y\|$ for all $x, y \in \mathbb{O}$. It is given by the following Fano plane $\mathbb{P}^2(\mathbb{F}_2)$:



If $e_i \to e_j \to e_k$ is a directed line in the Fano plane, then $e_i e_j = e_k$, $e_j e_k = e_i$, and $e_k e_i = e_j$. The multiplication is anti-commutative, i.e., $e_i e_j = -e_j e_i$ for all $i \neq j$. And we have $e_i^2 = -1$ for all i.

Yang: To be checked...

Let (M, J) be an almost complex manifold. Then the complexified tangent bundle $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$ splits into the direct sum of two complex subbundles

$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M,$$

where

$$T^{1,0}M := \ker(\sqrt{-1}\mathrm{id} - J), \quad T^{0,1}M := \ker(\sqrt{-1}\mathrm{id} + J).$$

We have $\overline{T^{1,0}M} = T^{0,1}M$ and both $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles of rank n. This decomposition induces a decomposition of the complexified cotangent bundle

$$\Omega^1(M) := (TM_{\mathbb{C}})^* = (T^{1,0}M)^* \oplus (T^{0,1}M)^* =: \Omega^{1,0}(M) \oplus \Omega^{0,1}(M).$$

More generally, for every $p, q \geq 0$, define

$$\Omega^{p,q}(M) := \wedge^p (T^{1,0}M)^* \otimes \wedge^q (T^{0,1}M)^* \subset \wedge^{p+q}\Omega^1(M).$$

Then we have the decomposition

$$\Omega^k(M) := \wedge^k \Omega^1(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

The elements of $\Omega^{p,q}(M)$ are called *differential forms of type* (p, q) or (p, q)*-forms* for short.

Recall the *exterior differential operator* $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ is locally given by

$$\mathrm{d}\left(\sum_I f_I \mathrm{d}x_I\right) = \sum_I \sum_{j=1}^{2n} \frac{\partial f_I}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_I,$$

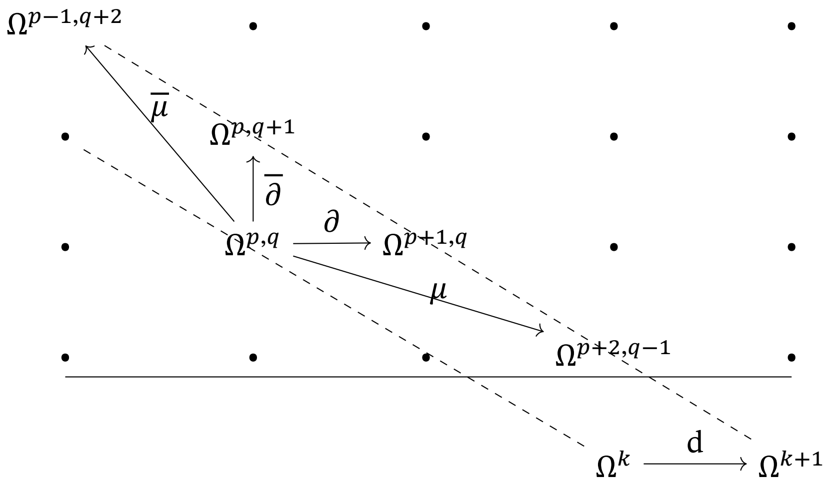
where I runs over all multi-indices with $|I| = k$ and $x_1, \ldots, x_{2n}$ are local coordinates on M.

Proposition 1.2.21. There exist differential operators

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \mu : \Omega^{p,q}(M) \to \Omega^{p+2,q-1}(M)$$

such that
$$\mathrm{d} = \partial + \overline{\partial} + \mu + \overline{\mu}.$$

In a diagram:



Proof of Proposition 1.2.21. Yang: To be continued... □

Definition 1.2.22. The operator μ in Proposition 1.2.21 is called the *Nijenhuis operator* of the almost complex structure J. If $\mu = 0$, then J is called *integrable*. In this case, we have $\mathrm{d} = \partial + \overline{\partial}$.

Example 1.2.23. Let J be the almost complex structure on S^6 defined in Example 1.2.19.

Yang: To be continued...

Proposition 1.2.24. Let (M, J) be an almost complex manifold. If J is induced by a complex structure on M, then J is integrable, i.e., the Nijenhuis operator $\mu = 0$.

Proof. Yang: To be continued... □

The converse of Proposition 1.2.24 is also true, which is the famous Newlander-Nirenberg theorem. Yang: To add reference...

Theorem 1.2.25 (Newlander-Nirenberg Theorem)**.** Let (M, J) be an almost complex manifold of real dimension $2n$. If $\mu = 0$, then J is induced by a complex structure on M.

Proposition 1.2.26. Let (M, J) be an almost complex manifold. Then J is integrable if and only if $\partial^2 = 0$.

1.3 Meromorphic functions

1.3.1 Meromorphic functions

Definition 1.3.1. Let M be a complex manifold. A *meromorphic function* on M is a holomorphic map $f : M \to \mathbb{CP}^1$.

The set of meromorphic functions on M is denoted by $\mathrm{Mer}(M)$ or $\mathscr{K}(M)$.

Proposition 1.3.2. Let M be a complex manifold. Then there is a natural inclusion $\mathsf{Hol}(M) \hookrightarrow \mathsf{Mer}(M)$. Moreover, we have $\mathsf{Mer}(M) = \mathsf{Frac}(\mathsf{Hol}(M))$, i.e., every meromorphic function can be expressed as a quotient of two holomorphic functions. Yang: to be checked.

Proposition 1.3.3. Let M be a complex manifold. The set of meromorphic functions on M forms a field under the usual addition and multiplication of functions.

Yang: To be complemented and revised.

1.3.2 Siegel theorem

Proposition 1.3.4. Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset $U \subset \mathbb{C}^n$. Suppose that f has order k at a point $x \in U$. Then there exists a neighborhood $\overline{B(x,r)} \subset U$ of x such that

$$|f(z)| \leq C|z-x|^k, \quad \forall z \in \overline{B(x,r)},$$

where $C = \sup_{z \in \partial \overline{B(x,r)}} |f(z)|$. Yang: To be revised.

Theorem 1.3.5 (Siegel theorem on function fields)**.** Let X be a connected and compact complex manifold of dimension n. Then the field of meromorphic functions on X satisfies

$$\operatorname{trdeg}_{\mathbb{C}} \mathscr{K}(X) \leq n.$$

Proof. Let $\{f_1, f_2, \dots, f_{n+1}\} \subset \mathscr{K}(X)$ be meromorphic functions on X. We want to find $P \in \mathbb{C}[x_1, x_2, \dots, x_{n+1}] \setminus \{0\}$ such that

$$P(f_1, f_2, \dots, f_{n+1}) = 0.$$

Step 1. Let $z \in X$, there exists $g_1, g_2, \dots, g_{n+1}, h \in \mathsf{Hol}(X)$ such that $f_i = g_i/h$ for each $1 \leq i \leq n+1$.

Yang: To be revised and complemented. □

Yang: To be revised and complemented.

1.4 Sheaves and Bundles on Complex Manifolds

1.4.1 Fiber bundles

Definition 1.4.1. Let M, F be manifolds. A *fiber bundle* with fiber F over M is a surjective map $\pi : E \to M$ of manifolds such that for each $x \in M$, there exists an open neighborhood U of x and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

where p_1 is the projection onto the first factor.

Given a fiber bundle E over M with fiber F and a covering $\{U_i\}$ of M, for each U_i, U_j and $x \in U_i \cap U_j$, we have two local trivializations

$$\varphi_i|_{E_x}, \varphi_j|_{E_x} : E_x \to \{x\} \times F.$$

They are differed by an automorphism $g_{ij}(x) = \varphi_i|_{E_x} \circ (\varphi_j|_{E_x})^{-1}$ of $\{x\} \times F$ as the following diagram

$$\begin{tikzcd}
\{x\} \times F \arrow[rr, "g_{ij}(x)"] & & \{x\} \times F. \\
& E_x \arrow[ul, "\varphi_j|_{E_x}"] \arrow[ur, "\varphi_i|_{E_x}"'] &
\end{tikzcd}$$

The map $g_{ij}(x)$ can be identified as an element of $\mathrm{Aut}(F)$. Varying x in $U_i \cap U_j$, we obtain the *transition function*

$$g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$$

which satisfies the *cocycle condition*

$$g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x), \quad \forall x \in U_i \cap U_j \cap U_k,$$

where the multiplication $\cdot$ is given by composition of automorphisms.

There is a natural way to impose smooth (holomorphic) structure on $\mathrm{Aut}(F)$, hence we can talk about smoothness or holomorphicity of transition functions. Set $\Phi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$. Then we have $\Phi_{ij}(x, v) = (x, g_{ij}(x)(v))$ for all $(x, v) \in (U_i \cap U_j) \times F$. Then Φ_{ij} is smooth (holomorphic) if and only if g_{ij} is smooth (holomorphic). Yang: To add ref.

Conversely, given a covering $\{U_i\}$ of M and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$ satisfying the cocycle condition, one can glue the local trivializations $U_i \times F$ via the maps Φ_{ij} to obtain a fiber bundle E over M with fiber F. Therefore, to given a fiber bundle with smooth (holomorphic) structure, it suffices to give a covering $\{U_i\}$ of M and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F)$ which are smooth (holomorphic) and satisfy the cocycle condition. In general, $\mathrm{Aut}(F)$ might be too large to handle. We can restrict the image of transition functions to a smaller subgroup $G \subset \mathrm{Aut}(F)$. This leads to the notion of structure group.

Definition 1.4.2. Let M, F be manifolds, and $G \subset \mathrm{Aut}(F)$ be a Lie subgroup. A *fiber bundle with structure group* G is a fiber bundle $\pi : E \to M$ given by transition functions $g_{ij} : U_i \cap U_j \to G$.

Example 1.4.3. A *(real) vector bundle* of rank r over a manifold M is a fiber bundle with fiber $\mathbb{R}^r$ and structure group $\mathrm{GL}_r(\mathbb{R})$. Similarly, a *complex vector bundle* of rank r over a manifold M is a fiber bundle with fiber $\mathbb{C}^r$ and structure group $\mathrm{GL}_r(\mathbb{C})$.

On a real manifold M of dimension $2n$, an almost complex structure is equivalent to a reduction of the structure group of the tangent bundle TM from $\mathrm{GL}_{2n}(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{C})$.

By the transition functions construction, we can see that

Theorem 1.4.4. Let M, F be locally ringed spaces and $G \subset \mathrm{Aut}(F)$ a subgroup. Set $\mathcal{G}$ be the sheaf of "admissible" functions from open subsets of M to G. Then the set of isomorphism classes of fiber bundles over M with fiber F and structure group G is in one-to-one correspondence with the Čech cohomology set $\check{H}^1(M, \mathcal{G})$.

Remark 1.4.5. Let us clarify the meaning of $\check{H}^1(M, \mathcal{G})$ when $\mathcal{G}$ is a sheaf of (not necessarily abelian) groups. Given an open covering $\mathcal{U} = \{U_i\}$ of M, we have a "complex" of groups

$$\prod_i \mathcal{G}(U_i) \xrightarrow{\delta^0} \prod_{i,j} \mathcal{G}(U_i \cap U_j) \xrightarrow{\delta^1} \prod_{i,j,k} \mathcal{G}(U_i \cap U_j \cap U_k),$$

where the maps δ^0 and δ^1 are defined by

$$\delta^0((g_i)_i) = (g_i|_{U_i \cap U_j} \cdot (g_j|_{U_i \cap U_j})^{-1})_{i,j},$$

$$\delta^1((g_{ij})_{i,j}) = (g_{ij}|_{U_i \cap U_j \cap U_k} \cdot g_{jk}|_{U_i \cap U_j \cap U_k} \cdot (g_{ik}|_{U_i \cap U_j \cap U_k})^{-1})_{i,j,k}.$$

Note that $\delta^1 \circ \delta^0$ is the constant map to the identity element. We define

- the set of 1-cocycles $Z^1(\mathcal{U}, \mathcal{G}) = \ker(\delta^1)$,
- the set of 1-coboundaries $B^1(\mathcal{U}, \mathcal{G}) = \sqrt{-1}(\delta^0)$.

The set $\check{H}^1(\mathcal{U}, \mathcal{G}) = Z^1(\mathcal{U}, \mathcal{G})/B^1(\mathcal{U}, \mathcal{G})$ is defined as the set of orbits of the action of $\prod_i \mathcal{G}(U_i)$ on $Z^1(\mathcal{U}, \mathcal{G})$ given by

$$((g_i)_i, (g_{ij})_{i,j}) \mapsto (g_i|_{U_i \cap U_j} \cdot g_{ij} \cdot (g_j|_{U_i \cap U_j})^{-1})_{i,j}.$$

Finally, we define $\check{H}^1(M, \mathcal{G}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{G})$ where the limit is taken over all open coverings of M. Yang: To be revised.

For example, if $F = \mathbb{C}$ and $G = \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$, consider the holomorphic line bundles over a complex manifold M. The sheaf $\mathcal{G}$ is equal to $\mathcal{O}_M^*$, the sheaf of nowhere vanishing holomorphic functions on M. Therefore, by Theorem 1.4.4, we get the classic result $\mathsf{Pic}(M) \cong \check{H}^1(M, \mathcal{O}_M^*)$.

Slogan *For a fiber bundle E over M, we care about*

- *fiber F,*
- *structure group $G \subset \mathrm{Aut}(F)$,*
- *"admissible" functions class of transition functions $g_{ij} : U_i \cap U_j \to G$ (e.g. continuous, smooth, holomorphic).*

1.4.2 Sheaves

Construction 1.4.6. Let M be a manifold and $\pi : E \to M$ be a fiber bundle with fiber F. For each open subset $U \subset M$, we can consider the set of "admissible" sections of E over U:

$$\Gamma(U, E) = \{s : U \to E \mid \pi \circ s = \mathrm{id}_U, s \text{ is admissible}\}.$$

Here "admissible" means continuous, smooth, holomorphic, etc., depending on the context. The assignment $U \mapsto \Gamma(U, E)$ defines a sheaf of sets (or groups, modules, etc. if F has additional structure) on M, called the *sheaf of sections* of the bundle E.

Example 1.4.7. Let M be a complex manifold. We explain how to view the tangent bundle TM and the cotangent bundle T^*M as sheaves. There are two important classes of admissible sections of these bundles, namely holomorphic and smooth sections. We denote the sheaf of holomorphic (respectively smooth) sections of TM by $\mathcal{T}_{M,\mathrm{hol}}$ (respectively $\mathcal{T}_{M,\mathrm{sm}}$).

Correspondingly, we denote the sheaf of holomorphic (respectively smooth) sections of T^*M by $\Omega^1_{M,\mathrm{hol}}$ (respectively $\Omega^1_{M,\mathrm{sm}}$). Sometime we omit the subscript M if there is no confusion.

The elements in $\mathcal{T}_{M,\mathrm{hol}}(U)$ (respectively $\mathcal{T}_{M,\mathrm{sm}}(U)$) are holomorphic (respectively smooth) vector fields on U, and holomorphic (respectively smooth) 1-forms on U for $\Omega^1_{M,\mathrm{hol}}(U)$ (respectively $\Omega^1_{M,\mathrm{sm}}(U)$). As sheaves, we have

$$\Omega^1_{M,\mathrm{hol}} \cong \mathcal{H}om_{\mathcal{O}_M}(\mathcal{T}_{M,\mathrm{hol}}, \mathcal{O}_M) \quad \text{and} \quad \Omega^1_{M,\mathrm{sm}} \cong \mathcal{H}om_{\mathcal{C}^\infty_M}(\mathcal{T}_{M,\mathrm{sm}}, \mathcal{C}^\infty_M),$$

where $\mathcal{C}^\infty_M$ is the sheaf of smooth complex-valued functions on M.

Example 1.4.8. Let M be a complex manifold. Consider the trivial real vector bundle $\mathbb{C}\times M \to M$. Its sheaf of holomorphic sections is just the structure sheaf $\mathcal{O}_M$, while its sheaf of smooth sections is the sheaf $\mathcal{C}^\infty_M = \mathcal{C}^\infty_M(-,\mathbb{C})$ of smooth complex-valued functions on M. Similarly, we have the trivial real vector bundle $\mathbb{R}\times M \to M$ whose sheaf of smooth sections is $\mathcal{C}^\infty_M(-,\mathbb{R})$.

Hence, the complexification of a holomorphic vector bundle E over M, i.e. the fiber bundle $E^{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$, has sheaf of smooth sections given by $\mathcal{E}^{\mathbb{C}} := \mathcal{E}_{\mathrm{sm}} \otimes_{\mathcal{C}^\infty_M(-,\mathbb{R})} \mathcal{C}^\infty_M(-,\mathbb{C}) \cong \mathcal{E}_{\mathrm{sm}} \oplus \overline{\mathcal{E}_{\mathrm{sm}}}$, where $\mathcal{E}_{\mathrm{sm}}$ is the sheaf of smooth sections of E and $\overline{\mathcal{E}_{\mathrm{sm}}}$ is its complex conjugate sheaf, i.e. $\overline{\mathcal{E}_{\mathrm{sm}}}(U) = \{\overline{s} \mid s \in \mathcal{E}_{\mathrm{sm}}(U)\}$. These sheaves are $\mathcal{C}^\infty_M$-modules. Yang: Note that the action of $\mathcal{C}^\infty_M$ on $\overline{\mathcal{E}_{\mathrm{sm}}}$ is given by

$$(f,v) \mapsto \overline{f}\cdot v, \quad \forall f \in \mathcal{C}^\infty_M(U), v \in \overline{\mathcal{E}_{\mathrm{sm}}}(U).$$

Let us return to the cotangent bundle T^*M of a complex manifold M. By the almost complex structure on M, we have the decomposition of complexified cotangent bundle

$$T^*M^{\mathbb{C}} := T^*M \otimes_{\mathbb{R}} \mathbb{C} \cong T^*M^{1,0} \oplus T^*M^{0,1},$$

This gives a decomposition of sheaves of smooth sections

$$\Omega^{1,\mathbb{C}}_{\mathrm{sm}} \cong \Omega^{1,0}_{\mathrm{sm}} \oplus \Omega^{0,1}_{\mathrm{sm}} \cong \Omega^1_{\mathrm{sm}} \oplus \overline{\Omega^1_{\mathrm{sm}}}.$$

Chapter 2

Cohomological studies

2.1 Forms and Currents

2.1.1 Differential forms

Let M be a complex manifold of complex dimension d. Recall that we have the decomposition of the cotangent bundle:

$$\Omega^{1,\mathbb{C}}_{\text{sm}} \cong \Omega^{1,0}_{\text{sm}} \oplus \Omega^{0,1}_{\text{sm}} \cong \Omega^{1}_{\text{sm}} \oplus \overline{\Omega^{1}_{\text{sm}}}.$$

Take exterior powers, we have$\Omega^{k,\mathbb{C}}_{\text{sm}} \cong \bigoplus_{p+q=k} \Omega^{p,q}_{\text{sm}}$, where $\Omega^{p,q}_{\text{sm}} = \bigwedge^p \Omega^{1,0}_{\text{sm}} \otimes \bigwedge^q \Omega^{0,1}_{\text{sm}}$. We also use the notation

$$\mathcal{A}^{p,q} := \Omega^{p,q}_{\text{sm}}, \quad \mathcal{A}^{k} := \Omega^{k,\mathbb{C}}_{\text{sm}}.$$

A reason to induce this strange sheaf $\mathcal{A}^k = \Omega^{k,\mathbb{C}}_{\text{sm}}$ is to make sense of integration of top-degree forms. For simplicity, assume that M is compact. Let $\omega \in \mathcal{A}^{2d}(M)$ be a smooth complex-valued $2d$-form on M. Then its integration is well-defined in the smooth manifold sense:

$$\int_M \omega \in \mathbb{C}.$$

However, in complex case, it is more natural to “integral” a holomorphic d-form on a d-dimensional complex manifold. This does not make sense in the smooth manifold theory. The solution is to associate a holomorphic d-form $\eta \in \mathcal{A}^{d,0}(M)$ with a smooth $2d$-form ((d,d)-form) $\omega = \eta \wedge \overline{\eta} \in \mathcal{A}^{d,d}(M) \subset \mathcal{A}^{2d}(M)$.

Another reason is that $\bigoplus_k \Omega^k_{\text{sm}}$ is not closed under the exterior derivative d, while $\bigoplus_k \Omega^{k,\mathbb{C}}_{\text{sm}}$ is. Suppose that we have local holomorphic coordinates $(z_1, \dots, z_d)$. Recall that we have the exterior derivative

$$\mathrm{d} : \Omega^{1,0}_{\text{sm}} \to \Omega^{2,\mathbb{C}}_{\text{sm}}, \quad f dz_i \mapsto \sum_{j=1}^{d} \frac{\partial f}{\partial z_j} dz_j \wedge dz_i + \sum_{i=1}^{d} \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j} \wedge dz_i.$$

On its conjugation, we have

$$\mathrm{d} : \Omega^{0,1}_{\text{sm}} \to \Omega^{2,\mathbb{C}}_{\text{sm}}, \quad f d\overline{z_i} \mapsto \sum_{j=1}^{d} \frac{\partial f}{\partial z_j} dz_j \wedge d\overline{z_j} + \sum_{j=1}^{d} \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j} \wedge d\overline{z_j}.$$

Extending d by linearity and the Leibniz rule $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$, we get the exterior derivative

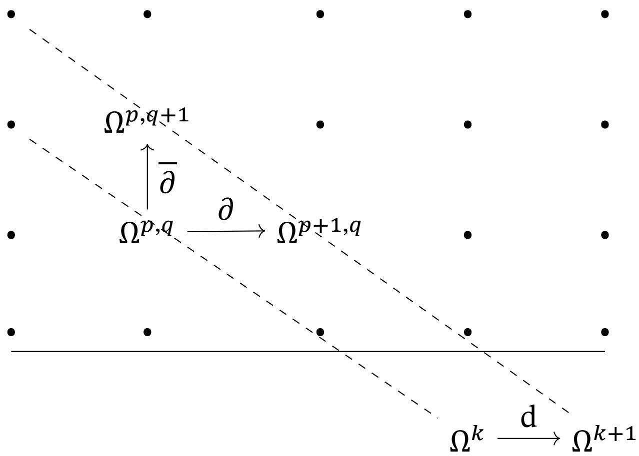
$$\mathrm{d} : \mathcal{A}^k = \Omega^{k,\mathbb{C}}_{\mathrm{sm}} \to \mathcal{A}^{k+1} = \Omega^{k+1,\mathbb{C}}_{\mathrm{sm}},$$

which can be decomposed as $\mathrm{d} = \partial + \overline{\partial}$, where

$$\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}, \quad f dz_I \wedge d\overline{z_J} \mapsto \sum_{j=1}^{d} \frac{\partial f}{\partial z_j} dz_j \wedge dz_I \wedge d\overline{z_J},$$

$$\overline{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}, \quad f dz_I \wedge d\overline{z_J} \mapsto \sum_{j=1}^{d} \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j} \wedge dz_I \wedge d\overline{z_J}.$$

In a diagram, we have:



Proposition 2.1.1. The operators ∂ and $\overline{\partial}$ satisfy

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Proof. Yang: To be added. □

Proposition 2.1.2. Let $f : M \to N$ be a holomorphic map between complex manifolds. Then the pull-back of differential forms $f^* : \mathcal{A}^k_N \to \mathcal{A}^k_M$ satisfies

$$f^*(\mathcal{A}^{p,q}_N) \subset \mathcal{A}^{p,q}_M, \quad f^* \circ \partial_N = \partial_M \circ f^*, \quad f^* \circ \overline{\partial}_N = \overline{\partial}_M \circ f^*.$$

Proof. Yang: To be added. □

Yang: The following need to checked.

Topological vector space of forms with compact support Let M be a complex manifold of complex dimension d. Given a differential form $\omega \in \mathcal{A}^k(M)$ with compact support, for any compact subset $K \subset M$ and non-negative integer m, we can define a seminorm

$$p_{K,m}(\omega) = \sup_{x \in K} \max_{|\alpha| \le m} |D^\alpha \omega(x)|.$$

Here, $\alpha = (\alpha_1, \dots, \alpha_{2d})$ is a multi-index, and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_{2d}^{\alpha_{2d}}}$ in local real coordinates $(x_1, \dots, x_{2d})$. The collection of these seminorms endows the space of compactly supported smooth k-forms on M with a locally convex topology.

Definition 2.1.3. A differential form $\omega \in \mathcal{A}^k(M)$ is said to have *compact support* if there exists a compact subset $K \subset M$ such that $\omega|_{M\setminus K} = 0$. The space of smooth complex-valued k-forms with compact support on M is denoted by $\mathcal{A}_c^k(M)$ or $\mathcal{D}^k(M)$. On this vector space, we give it the weak topology induced by the family of seminorms

$$p_{K,m}(\omega) = \sup_{x\in K} \max_{|\alpha|\leq m} |D^\alpha \omega(x)|,$$

where K runs over all compact subsets of M and m runs over all non-negative integers.

2.1.2 Currents

Definition 2.1.4. A *current* of degree k on a complex manifold M is a continuous linear functional on the space of compactly supported smooth $(2d-k)$-forms on M:

$$T : \mathcal{A}_c^{2d-k}(M) \to \mathbb{C}.$$

The space of currents of degree k on M is denoted by $\mathcal{D}_k(M)$. Yang: To be revised.

2.2 Cohomology Theories in Complex Geometry

2.2.1 Various cohomology theories

There are several cohomology theories for complex manifolds.

Definition 2.2.1. Let M be a complex manifold. The *singular cohomology* of M with coefficients in a ring R is defined to be the singular cohomology of the underlying topological space $|M|$ of M:

$$H^k_{\mathrm{sing}}(M;R) := H^k_{\mathrm{sing}}(|M|;R).$$

Definition 2.2.2. Let M be a complex manifold. The *de Rham cohomology* of M is defined to be the de Rham cohomology of the underlying smooth manifold of M:

$$H^k_{\mathrm{dR}}(M) := \frac{\mathrm{Ker}(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

Yang: Smooth section or holomorphic section?

Definition 2.2.3. Let M be a complex manifold. The *Dolbeault cohomology* of M is defined to be

$$H^{p,q}_{\overline{\partial}}(M) := \frac{\mathrm{Ker}(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Proposition 2.2.4. Let $\Delta^n = \{(z_1,\dots,z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1,\dots,n\}$ be the unit polydisc in $\mathbb{C}^n$.

Then

$$H^{p,q}_{\bar\partial}(\Delta^n) = \begin{cases} \Omega^p_{\mathrm{hol}}(\Delta^n), & q = 0, \\ 0, & q > 0. \end{cases}$$

Yang: To be checked...

2.3 Metrics, curvature and connections

2.3.1 The first properties

Let X be a complex manifold and $E \to X$ a holomorphic vector bundle.

Definition 2.3.1. A *hermitian metric* on E is a smoothly varying family of hermitian inner products $\langle\cdot,\cdot\rangle_x$ on the fibers E_x for each $x \in X$, i.e.,

$$\langle\cdot,\cdot\rangle_x : E_x \times E_x \to \mathbb{C}$$

is a hermitian inner product for each x, and for any local smooth sections s, t of E, the function

$$x \mapsto \langle s(x), t(x)\rangle_x$$

is smooth on X. A vector bundle equipped with a hermitian metric is called a *hermitian vector bundle.*

Definition 2.3.2. A *hermitian metric* on a complex manifold X is a hermitian metric on its holomorphic tangent bundle TX.

Remark 2.3.3. Let h be a hermitian metric on a complex manifold X. Then h induces a Riemannian metric g on the underlying real manifold of X by

$$g(u, v) = \mathrm{Re}(h(u, v))$$

for real tangent vectors $u, v \in T_xX$.

Example 2.3.4. Let $\mathbb{P}^n$ be the complex projective space. Recall that $\mathbb{P}^n(\mathbb{C}) \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. We focus on the underlying smooth manifold structure. We have $\mathbb{C}^* \cong S^1 \times \mathbb{R}_{>0}$ and $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{R}_{>0} \cong S^{2n+1}$. Hence $\mathbb{P}^n(\mathbb{C}) \cong S^{2n+1}/S^1$. Note that $S^1 \curvearrowright S^{2n+1} \subset \mathbb{C}^{n+1}$ is isometric with respect to the standard hermitian metric on $\mathbb{C}^{n+1}$. Hence the quotient $\mathbb{P}^n(\mathbb{C})$ inherits a natural hermitian metric h_{FS}, called the *Fubini-Study metric.*

On the standard affine chart $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$ with coordinates $z_{j,i} = z_j/z_i$ for $j \neq i$, we know that $T\mathbb{P}^n|_{U_i}$ is spanned by $\{\partial_{j,i} = \partial/\partial z_{j,i}\}_{j\neq i}$. The Fubini-Study metric is given by

$$h_{\mathrm{FS}}(z_{-,i})(\partial_{k,i}, \partial_{l,i}) = \frac{\delta_{kl}}{1 + \sum_{r\neq i} |z_{r,i}|^2} - \frac{\overline{z_{k,i}} z_{l,i}}{(1 + \sum_{r\neq i} |z_{r,i}|^2)^2}.$$

Example 2.3.5. Now let us consider the complex projective plane $\mathbb{P}^2 = \{[X : Y : Z]\}$. On the affine chart $U_Z = \{[X : Y : Z] \mid Z \neq 0\}$ with coordinates $x = X/Z$ and $y = Y/Z$, the Fubini-Study metric h_{FS} on $T\mathbb{P}^2|_{U_Z}$ is given by

$$h_{\mathrm{FS}}(x, y) = \frac{1}{(1 + |x|^2 + |y|^2)^2} \begin{bmatrix} 1 + |y|^2 & -\bar{x}y \\ -x\bar{y} & 1 + |x|^2 \end{bmatrix}.$$

For a tangent vector $\partial = a\partial_x + b\partial_y$, its norm squared is

$$\|\partial\|^2_{h_{\mathrm{FS}}} = \frac{(1 + |y|^2)|a|^2 + (1 + |x|^2)|b|^2 - 2\,\mathrm{Re}(\bar{x}ya\bar{b})}{(1 + |x|^2 + |y|^2)^2} = \frac{|a|^2 + |b|^2 + |xb - ya|^2}{(1 + |x|^2 + |y|^2)^2} \geq 0.$$

Definition 2.3.6. Let X be a complex manifold and $E \to X$ a holomorphic vector bundle. A *connection* on E is a $\mathbb{C}$-linear map between the sheaves of smooth sections

$$\nabla : \mathcal{C}^\infty(-, E) \to \mathcal{C}^\infty(-, T^*X \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

for all smooth functions f and smooth sections s of E.

When you choose a vector field $v \in \mathcal{C}^\infty(U, TX)$ on an open set $U \subset X$, the connection ∇ induces an endomorphism

$$\nabla_v : \mathcal{C}^\infty(U, E) \to \mathcal{C}^\infty(U, E)$$

by applying v on the T^*X component of ∇s for a section s of E. In particular, if $E = TX$ is the tangent bundle, then ∇_v is called a *covariant derivative* along v. Sometimes people call ∇ an *endomorphism-valued 1-form* on X with values in $\mathrm{End}(E)$ by viewing it as a map $v \mapsto \nabla_v$.

Proposition 2.3.7. Let X be a complex manifold, $E \to X$ a holomorphic vector bundle equipped with a hermitian metric h. Then there exists a unique connection ∇ on E that is compatible with both the holomorphic structure and the hermitian metric h. Yang: To be checked.

Proof. Yang: to be added. □

Example 2.3.8. Let $\mathbb{P}^n$ be the complex projective space and $\mathcal{O}_{\mathbb{P}^n}(1)$ the hyperplane line bundle. The *Chern connection* associated with the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ is a connection defined as follows: For a section s of $\mathcal{O}_{\mathbb{P}^n}(1)$, we define

$$\nabla s = ds + \alpha s,$$

where α is a (1,0)-form determined by the Fubini-Study metric. Yang: To be continued.

By the Leibniz rule, the connection ∇ can be extended to act on E-valued differential forms:

$$\nabla : \mathcal{C}^\infty(-, \Lambda^k T^*X \otimes E) \to \mathcal{C}^\infty(-, \Lambda^{k+1} T^*X \otimes E)$$

for all $k \geq 0$, satisfying

$$\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

for $\omega \in \mathcal{C}^\infty(-, \Lambda^k T^*X)$ and $s \in \mathcal{C}^\infty(-, E)$.

Definition 2.3.9. Let X be a complex manifold, $E \to X$ a holomorphic vector bundle, and ∇ a connection on E. The *curvature* of the connection ∇ is defined as the endomorphism-valued 2-form

$$F_\nabla = \nabla^2 : \mathcal{C}^\infty(-, E) \to \mathcal{C}^\infty(-, \Lambda^2 T^* X \otimes E),$$

where ∇^2 is the composition of ∇ with itself.

When $E = TX$ is the tangent bundle, the curvature F_∇ is a $(3,1)$-tensor, which is the classical Riemann curvature tensor.

Example 2.3.10. Yang: To be added.

Yang: For a line bundle, everything coincide.

2.3.2 On line bundles

Let X be a complex manifold and $L \to X$ a holomorphic line bundle.

Proposition 2.3.11. Let h_1, h_2 be two hermitian metrics on L. Then there exists a smooth function $\varphi : X \to \mathbb{R}$ such that

$$h_2(s,t) = \exp(\varphi) \cdot h_1(s,t)$$

for all local smooth sections s, t of L.

Proof. Yang: to be added. □

Proposition 2.3.12. There is a one-to-one correspondence between hermitian metrics on L and real $(1,1)$-forms representing the first Chern class $c_1(L) \in H^{1,1}(X, \mathbb{R})$. More precisely, given a hermitian metric h on L, there exists a unique real $(1,1)$-form ω_h such that for any local holomorphic non-vanishing section s of L,

$$\omega_h = -\frac{i}{2\pi} \partial \bar{\partial} \log h(s,s).$$

Conversely, given a real $(1,1)$-form ω representing $c_1(L)$, there exists a hermitian metric h on L such that $\omega = \omega_h$. Yang: To be checked.

Yang: Green functions?

2.3.3 Chern-Weil Theory

Theorem 2.3.13. Let X be a complex manifold and $E \to X$ a holomorphic vector bundle equipped with a hermitian metric h. Let ∇ be the unique connection on E compatible with both the holomorphic structure and the hermitian metric h, and let F_∇ be its curvature. Then the Chern classes $c_k(E) \in H^{2k}(X, \mathbb{R})$ can be represented by the differential forms

$$c_k(E) = \left[\frac{1}{(2\pi i)^k} \operatorname{Tr}\left(F_\nabla^k\right) \right].$$

Yang: To be checked.

Proof. Yang: To be added. □

2.4 Kähler manifolds

2.5 Hodge star and harmonic forms

Chapter 3

Algebraic and analytic geometry

3.1 The Chow Theorem

3.2 GAGA

3.3 Kodaira Embedding Theorem

Chapter 4

Deformation theory