
Cohomological Studies

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1 Forms and Currents

1.1 Differential forms

Let M be a complex manifold of complex dimension d . Recall that we have the decomposition of the cotangent bundle:

$$\Omega_{\text{sm}}^{1,\mathbb{C}} \cong \Omega_{\text{sm}}^{1,0} \oplus \Omega_{\text{sm}}^{0,1} \cong \Omega_{\text{sm}}^1 \oplus \overline{\Omega_{\text{sm}}^1}.$$

Take exterior powers, we have $\Omega_{\text{sm}}^{k,\mathbb{C}} \cong \bigoplus_{p+q=k} \Omega_{\text{sm}}^{p,q}$, where $\Omega_{\text{sm}}^{p,q} = \bigwedge^p \Omega_{\text{sm}}^{1,0} \otimes \bigwedge^q \Omega_{\text{sm}}^{0,1}$. We also use the notation

$$\mathcal{A}^{p,q} := \Omega_{\text{sm}}^{p,q}, \quad \mathcal{A}^k := \Omega_{\text{sm}}^{k,\mathbb{C}}.$$

A reason to induce this strange sheaf $\mathcal{A}^k = \Omega_{\text{sm}}^{k,\mathbb{C}}$ is to make sense of integration of top-degree forms. For simplicity, assume that M is compact. Let $\omega \in \mathcal{A}^{2d}(M)$ be a smooth complex-valued $2d$ -form on M . Then its integration is well-defined in the smooth manifold sense:

$$\int_M \omega \in \mathbb{C}.$$

However, in complex case, it is more natural to “integral” a holomorphic d -form on a d -dimensional complex manifold. This does not make sense in the smooth manifold theory. The solution is to associate a holomorphic d -form $\eta \in \mathcal{A}^{d,0}(M)$ with a smooth $2d$ -form ((d,d)-form) $\omega = \eta \wedge \bar{\eta} \in \mathcal{A}^{d,d}(M) \subset \mathcal{A}^{2d}(M)$.

Another reason is that $\bigoplus_k \Omega_{\text{sm}}^k$ is not closed under the exterior derivative d , while $\bigoplus_k \Omega_{\text{sm}}^{k,\mathbb{C}}$ is. Suppose that we have local holomorphic coordinates (z_1, \dots, z_d) . Recall that we have the exterior

Topological vector space of forms with compact support Let M be a complex manifold of complex dimension d . Given a differential form $\omega \in \mathcal{A}^k(M)$ with compact support, for any compact subset $K \subset M$ and non-negative integer m , we can define a seminorm

$$p_{K,m}(\omega) = \sup_{x \in K} \max_{|\alpha| \leq m} |D^\alpha \omega(x)|.$$

Here, $\alpha = (\alpha_1, \dots, \alpha_{2d})$ is a multi-index, and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_{2d}^{\alpha_{2d}}}$ in local real coordinates (x_1, \dots, x_{2d}) . The collection of these seminorms endows the space of compactly supported smooth k -forms on M with a locally convex topology.

Definition 1.3. A differential form $\omega \in \mathcal{A}^k(M)$ is said to have *compact support* if there exists a compact subset $K \subset M$ such that $\omega|_{M \setminus K} = 0$. The space of smooth complex-valued k -forms with compact support on M is denoted by $\mathcal{A}_c^k(M)$ or $\mathcal{D}^k(M)$. On this vector space, we give it the weak topology induced by the family of seminorms

$$p_{K,m}(\omega) = \sup_{x \in K} \max_{|\alpha| \leq m} |D^\alpha \omega(x)|,$$

where K runs over all compact subsets of M and m runs over all non-negative integers.

1.2 Currents

Definition 1.4. A *current* of degree k on a complex manifold M is a continuous linear functional on the space of compactly supported smooth $(2d - k)$ -forms on M :

$$T : \mathcal{A}_c^{2d-k}(M) \rightarrow \mathbb{C}.$$

The space of currents of degree k on M is denoted by $\mathcal{D}_k(M)$. Yang: To be revised.

2 Cohomology Theories in Complex Geometry

2.1 Various cohomology theories

There are several cohomology theories for complex manifolds.

Definition 2.1. Let M be a complex manifold. The *singular cohomology* of M with coefficients in a ring R is defined to be the singular cohomology of the underlying topological space $|M|$ of M :

$$H_{\text{sing}}^k(M; R) := H_{\text{sing}}^k(|M|; R).$$

Definition 2.2. Let M be a complex manifold. The *de Rham cohomology* of M is defined to be the de Rham cohomology of the underlying smooth manifold of M :

$$H_{\text{dR}}^k(M) := \frac{\text{Ker}(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M))}{\text{Im}(d : \Omega^{k-1}(M) \rightarrow \Omega^k(M))}.$$

Yang: Smooth section or holomorphic section?

Definition 2.3. Let M be a complex manifold. The *Dolbeault cohomology* of M is defined to be

$$H_{\bar{\partial}}^{p,q}(M) := \frac{\text{Ker}(\bar{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M))}{\text{Im}(\bar{\partial} : \Omega^{p,q-1}(M) \rightarrow \Omega^{p,q}(M))}.$$

Proposition 2.4. Let $\Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \dots, n\}$ be the unit polydisc in \mathbb{C}^n . Then

$$H_{\bar{\partial}}^{p,q}(\Delta^n) = \begin{cases} \Omega_{\text{hol}}^p(\Delta^n), & q = 0, \\ 0, & q > 0. \end{cases}$$

Yang: To be checked...

3 Metrics, curvature and connections

3.1 The first properties

Let X be a complex manifold and $E \rightarrow X$ a holomorphic vector bundle.

Definition 3.1. A *hermitian metric* on E is a smoothly varying family of hermitian inner products $\langle \cdot, \cdot \rangle_x$ on the fibers E_x for each $x \in X$, i.e.,

$$\langle \cdot, \cdot \rangle_x : E_x \times E_x \rightarrow \mathbb{C}$$

is a hermitian inner product for each x , and for any local smooth sections s, t of E , the function

$$x \mapsto \langle s(x), t(x) \rangle_x$$

is smooth on X . A vector bundle equipped with a hermitian metric is called a *hermitian vector bundle*.

Definition 3.2. A *hermitian metric* on a complex manifold X is a hermitian metric on its holomorphic tangent bundle TX .

Remark 3.3. Let h be a hermitian metric on a complex manifold X . Then h induces a Riemannian metric g on the underlying real manifold of X by

$$g(u, v) = \text{Re}(h(u, v))$$

for real tangent vectors $u, v \in T_x X$.

Example 3.4. Let \mathbb{P}^n be the complex projective space. Recall that $\mathbb{P}^n(\mathbb{C}) \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. We focus on the underlying smooth manifold structure. We have $\mathbb{C}^* \cong S^1 \times \mathbb{R}_{>0}$ and $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{R}_{>0} \cong S^{2n+1}$. Hence $\mathbb{P}^n(\mathbb{C}) \cong S^{2n+1}/S^1$. Note that $S^1 \curvearrowright S^{2n+1} \subset \mathbb{C}^{n+1}$ is isometric with respect to the standard hermitian metric on \mathbb{C}^{n+1} . Hence the quotient $\mathbb{P}^n(\mathbb{C})$ inherits a natural hermitian metric h_{FS} , called the *Fubini-Study metric*.

On the standard affine chart $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$ with coordinates $z_{j,i} = z_j/z_i$ for $j \neq i$, we know that $T\mathbb{P}^n|_{U_i}$ is spanned by $\{\partial_{j,i} = \partial/\partial z_{j,i} \mid j \neq i\}$. The Fubini-Study metric is given by

$$h_{\text{FS}}(z_{-,i})(\partial_{k,i}, \partial_{l,i}) = \frac{\delta_{kl}}{1 + \sum_{r \neq i} |z_{r,i}|^2} - \frac{\overline{z_{k,i}} z_{l,i}}{(1 + \sum_{r \neq i} |z_{r,i}|^2)^2}.$$

Example 3.5. Now let us consider the complex projective plane $\mathbb{P}^2 = \{[X : Y : Z]\}$. On the affine chart $U_Z = \{[X : Y : Z] \mid Z \neq 0\}$ with coordinates $x = X/Z$ and $y = Y/Z$, the Fubini-Study metric h_{FS} on $T\mathbb{P}^2|_{U_Z}$ is given by

$$h_{\text{FS}}(x, y) = \frac{1}{(1 + |x|^2 + |y|^2)^2} \begin{bmatrix} 1 + |y|^2 & -\bar{x}y \\ -x\bar{y} & 1 + |x|^2 \end{bmatrix}.$$

For a tangent vector $\partial = a\partial_x + b\partial_y$, its norm squared is

$$\|\partial\|_{h_{\text{FS}}}^2 = \frac{(1 + |y|^2)|a|^2 + (1 + |x|^2)|b|^2 - 2 \operatorname{Re}(\bar{x}y a \bar{b})}{(1 + |x|^2 + |y|^2)^2} = \frac{|a|^2 + |b|^2 + |xb - ya|^2}{(1 + |x|^2 + |y|^2)^2} \geq 0.$$

Definition 3.6. Let X be a complex manifold and $E \rightarrow X$ a holomorphic vector bundle. A *connection* on E is a \mathbb{C} -linear map between the sheaves of smooth sections

$$\nabla : \mathcal{C}^\infty(-, E) \rightarrow \mathcal{C}^\infty(-, T^*X \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all smooth functions f and smooth sections s of E .

When you choose a vector field $v \in \mathcal{C}^\infty(U, TX)$ on an open set $U \subset X$, the connection ∇ induces an endomorphism

$$\nabla_v : \mathcal{C}^\infty(U, E) \rightarrow \mathcal{C}^\infty(U, E)$$

by applying v on the T^*X component of ∇s for a section s of E . In particular, if $E = TX$ is the tangent bundle, then ∇_v is called a *covariant derivative* along v . Sometimes people call ∇ an *endomorphism-valued 1-form* on X with values in $\operatorname{End}(E)$ by viewing it as a map $v \mapsto \nabla_v$.

Proposition 3.7. Let X be a complex manifold, $E \rightarrow X$ a holomorphic vector bundle equipped with a hermitian metric h . Then there exists a unique connection ∇ on E that is compatible with both the holomorphic structure and the hermitian metric h . **Yang: To be checked.**

Proof. **Yang: to be added.** □

Example 3.8. Let \mathbb{P}^n be the complex projective space and $\mathcal{O}_{\mathbb{P}^n}(1)$ the hyperplane line bundle. The *Chern connection* associated with the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ is a connection defined as follows: For a section s of $\mathcal{O}_{\mathbb{P}^n}(1)$, we define

$$\nabla s = ds + \alpha s,$$

where α is a $(1,0)$ -form determined by the Fubini-Study metric. **Yang: To be continued.**

By the Leibniz rule, the connection ∇ can be extended to act on E -valued differential forms:

$$\nabla : \mathcal{C}^\infty(-, \Lambda^k T^*X \otimes E) \rightarrow \mathcal{C}^\infty(-, \Lambda^{k+1} T^*X \otimes E)$$

for all $k \geq 0$, satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

for $\omega \in \mathcal{C}^\infty(-, \Lambda^k T^*X)$ and $s \in \mathcal{C}^\infty(-, E)$.

Definition 3.9. Let X be a complex manifold, $E \rightarrow X$ a holomorphic vector bundle, and ∇ a connection on E . The *curvature* of the connection ∇ is defined as the endomorphism-valued 2-form

$$F_\nabla = \nabla^2 : \mathcal{C}^\infty(-, E) \rightarrow \mathcal{C}^\infty(-, \Lambda^2 T^*X \otimes E),$$

where ∇^2 is the composition of ∇ with itself.

When $E = TX$ is the tangent bundle, the curvature F_∇ is a $(3,1)$ -tensor, which is the classical Riemann curvature tensor.

Example 3.10. Yang: To be added.

Yang: For a line bundle, everything coincide.

3.2 On line bundles

Let X be a complex manifold and $L \rightarrow X$ a holomorphic line bundle.

Proposition 3.11. Let h_1, h_2 be two hermitian metrics on L . Then there exists a smooth function $\varphi : X \rightarrow \mathbb{R}$ such that

$$h_2(s, t) = \exp(\varphi) \cdot h_1(s, t)$$

for all local smooth sections s, t of L .

Proof. Yang: to be added. □

Proposition 3.12. There is a one-to-one correspondence between hermitian metrics on L and real $(1,1)$ -forms representing the first Chern class $c_1(L) \in H^{1,1}(X, \mathbb{R})$. More precisely, given a hermitian metric h on L , there exists a unique real $(1,1)$ -form ω_h such that for any local holomorphic non-vanishing section s of L ,

$$\omega_h = -\frac{i}{2\pi} \partial \bar{\partial} \log h(s, s).$$

Conversely, given a real $(1,1)$ -form ω representing $c_1(L)$, there exists a hermitian metric h on L such that $\omega = \omega_h$. Yang: To be checked.

Yang: Green functions?

3.3 Chern-Weil Theory

Theorem 3.13. Let X be a complex manifold and $E \rightarrow X$ a holomorphic vector bundle equipped with a hermitian metric h . Let ∇ be the unique connection on E compatible with both the holomorphic structure and the hermitian metric h , and let F_∇ be its curvature. Then the Chern classes $c_k(E) \in H^{2k}(X, \mathbb{R})$ can be represented by the differential forms

$$c_k(E) = \left[\frac{1}{(2\pi i)^k} \text{Tr}(F_\nabla^k) \right].$$

Yang: To be checked.

| *Proof.* Yang: To be added. □

4 Kähler manifolds

5 Hodge star and harmonic forms