# Complex Geometry

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# Complex Geometry

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*Source code: github.com/MonkeyUnderMountain/Complex_Geometry*

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# Contents

# Chapter 1

# The first properties

## 1.1 Analysis in several complex variables

In this section, we introduce some basic concepts and results in complex analysis with multiple variables.

### 1.1.1 Holomorphic functions

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

**Definition 1.1.1.** A continuous map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is *differentiable* at $p \in \mathbb{R}^{2n}$ if there exists a linear map $\mathrm{d}f_p : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ such that

$$f(z) = f(p) + \mathrm{d}f_p(z-p) + o(|z-p|).$$

A continuous map $f : \mathbb{C}^n \to \mathbb{C}^m$ is *holomorphic* at $p \in \mathbb{C}^n$ if it is differentiable at $p$ and $\mathrm{d}f_p$ is $\mathbb{C}$-linear, i.e., $\mathrm{d}f_p(\sqrt{-1}z) = \sqrt{-1}\mathrm{d}f_p(z)$ for all $z \in \mathbb{C}^n$.

By a "function", we always mean a complex-valued function, i.e., a map $f : \mathbb{C}^n \to \mathbb{C}$. Fix a coordinate system $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n$ and write $z_j = x_j + iy_j$ for $j = 1, \ldots, n$. Then a differentiable function $f = u + iv : \mathbb{C}^n \to \mathbb{C}$ is holomorphic at $p$ if and only if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x_i}(p) = \frac{\partial v}{\partial y_i}(p), \quad \frac{\partial u}{\partial y_i}(p) = -\frac{\partial v}{\partial x_i}(p), \quad i = 1, \ldots, n.$$

For convenience, we consider the complexified tangent space $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and introduce the following operators.

**Definition 1.1.2.** The *Wirtinger operators* are defined as

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad j = 1, \ldots, n.$$

Then we can rewrite the Cauchy-Riemann equations as

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \ldots, n.$$

We summarize some properties of Wirtinger operators in the following proposition.

**Proposition 1.1.3.** The Wirtinger operators satisfy the following properties:

(a) $\partial_{z_j} z_i = \delta_{ij}$, $\partial_{z_j} \bar{z}_i = 0$, $\partial_{z_j} \bar{z}_i = 0$, $\partial_{z_j} \bar{z}_j = \delta_{ij}$;

(b) $\overline{(\partial_{z_j} f)} = \partial_{\bar{z}_j} \bar{f}$;

(c) suppose we have $\mathbb{C}^n \xrightarrow{g} \mathbb{C}^m \xrightarrow{f} \mathbb{C}^l$ and the coordinate on $\mathbb{C}^m$ is $w = (w_1, \ldots, w_m)$, then the chain rule holds:

$$\begin{aligned}
\frac{\partial (f \circ g)}{\partial z_j} &= \sum_{k=1}^{m} \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial z_j}(z) + \sum_{k=1}^{m} \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial z_j}(z), \\
\frac{\partial (f \circ g)}{\partial \bar{z}_j} &= \sum_{k=1}^{m} \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial \bar{z}_j}(z) + \sum_{k=1}^{m} \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial \bar{z}_j}(z).
\end{aligned}$$

*Proof.* By direct computation. □

We can also consider the complexified of derivatives

$$(\mathrm{d}f_p)_{\mathbb{C}} : T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \to T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}.$$

If we take $\{\partial_{z_i}, \partial_{\bar{z}_i}\}_{i=1}^n$ as a basis of $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and $\{\partial_{w_j}, \partial_{\bar{w}_j}\}_{j=1}^m$ as a basis of $T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$, then the matrix representation of $(\mathrm{d}f_p)_{\mathbb{C}}$ is

$$(\mathrm{d}f_p)_{\mathbb{C}} = \begin{bmatrix} \partial_z f(p) & \partial_{\bar{z}} f(p) \\ \overline{\partial_{\bar{z}} f(p)} & \overline{\partial_z f(p)} \end{bmatrix}.$$

In particular, if $f$ is holomorphic, then we have $\det(\mathrm{d}f_p)_{\mathbb{C}} = |\det(\partial_z f)(p)|^2 \geq 0$.

**Definition 1.1.4.** A map $f : \Omega \to \Omega'$ between two open sets $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^m$ is *biholomorphic* if it is a bijection and both $f$ and $f^{-1}$ are holomorphic.

If $f$ is biholomorphic at $p$, then $m = n$ and $\det \mathrm{d}f_p > 0$.

**Theorem 1.1.5** (Holomorphic Inverse Function Theorem)**.** Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. If the Jacobian determinant $\det \mathrm{d}f_p$ is nonzero at $p \in \mathbb{C}^n$, then there exist open neighborhoods $U$ of $p$ and $V$ of $f(p)$ such that $f : U \to V$ is a biholomorphism.

*Proof.* By the real inverse function theorem, there exist open neighborhoods $U$ of $p$ and $V$ of $f(p)$ such that $g = f^{-1} : V \to U$ is a differentiable map. It suffices to show that $g$ is holomorphic. By the chain rule (Proposition 1.1.3), since $f$ is holomorphic, we have

$$0 = \left(\frac{\partial (f \circ g)_i}{\partial \bar{z}_j}\right)(q) = \left(\frac{\partial f_i}{\partial w_k}\right)(g(q)) \left(\frac{\partial g_k}{\partial \bar{z}_j}\right)(q).$$

Since $\det(\partial f / \partial w)(f(q)) \neq 0$, the matrix $(\partial f_i / \partial w_k)(g(q))$ is invertible, which implies that $(\partial g_k / \partial \bar{z}_j)(q) = 0$ for all $k, j$. Thus $g$ is holomorphic. □

**Theorem 1.1.6** (Holomorphic Implicit Function Theorem)**.** Let $f : \mathbb{C}^{n+m} \to \mathbb{C}^m$ be a holomorphic map. Write the coordinates of $\mathbb{C}^{n+m}$ as $(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_m) \in \mathbb{C}^n \times \mathbb{C}^m$. If $\det(\partial f / \partial w) \neq 0$

at $(z_0, w_0) \in \mathbb{C}^{n+m}$ with $f(z_0, w_0) = 0$, then there exist open neighborhoods $U$ of $z_0$ and $V$ of $w_0$, and a unique holomorphic map $g : U \to V$ such that for any $(z, w) \in U \times V$, $f(z, w) = 0$ if and only if $w = g(z)$.

*Proof.* By real implicit function theorem, there exist differentiable map $g : U \to V$ satisfying the above condition. It suffices to show that $g$ is holomorphic. Let $G : U \to U \times V$ be defined by $G(z) = (z, g(z))$. Then we have $f \circ G = 0$. By the chain rule, we have

$$0 = \frac{\partial (f \circ G)_i}{\partial \bar{z}_j}(q) = \sum_{k=1}^{n} \frac{\partial f_i}{\partial w_k}(G(q))\frac{\partial z_k}{\partial \bar{z}_j}(q) + \sum_{l=1}^{m} \frac{\partial f_i}{\partial w_l}(G(q))\frac{\partial g_l}{\partial \bar{z}_j}(q).$$

Since $\det(\partial f/\partial w)(G(q)) \neq 0$, the matrix $(\partial f_i/\partial w_k)(G(q))$ is invertible, which implies that $(\partial g_l/\partial \bar{z}_j)(q) = 0$ for all $l, j$. Thus $g$ is holomorphic. □

### 1.1.2 Cauchy Integral Formula

Recall the Cauchy Integral Formula in one complex variable:

**Theorem 1.1.7** (Cauchy Integral Formula in one complex variable)**.** Let $K \subset \mathbb{C}$ be a compact set with piecewise differentiable boundary $\partial K$, and let $f$ be differentiable on a neighborhood of $K$. Then for any $z$ in the interior of $K$, we have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_K \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

*Proof.* Yang: By Stokes' theorem. To be continued... □

**Theorem 1.1.8** (Cauchy Integral Formula in several complex variables)**.** Let $D \subset \mathbb{C}^n$ be a polydisk and $f$ be holomorphic on a neighborhood of the closure of $D$. Then for any $z \in D$,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial D_1 \times \cdots \times \partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n.$$

*Proof.* Yang: To be continued... □

**Corollary 1.1.9.** Holomorphic functions are analytic. Yang: To be continued...

**Proposition 1.1.10.** Holomorphic functions are open mappings. Yang: To be continued...

**Proposition 1.1.11.** If a holomorphic function $f : \Omega \to \mathbb{C}$ on a connected open set $\Omega \subset \mathbb{C}^n$ attains its maximum at some point in $\Omega$, then $f$ is constant. Yang: To be continued...

**Lemma 1.1.12.** Let $D \subset \mathbb{C}^n$ be a polydisk and $f$ be holomorphic on a neighborhood of the closure of $D$. Then for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\max_{z \in D} \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(z) \right| \leq \frac{\alpha!}{r^\alpha} \max_{z \in D} |f(z)|,$$

where $r = (r_1, \dots, r_n)$ is the radius of the polydisk $D$. Yang: To be continued...

**Theorem 1.1.13** (Generalized Liouville Theorem)**.** A holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ on the whole space $\mathbb{C}^n$ that satisfies a polynomial growth condition, i.e., there exist constants $C > 0$ and $k \geq 0$ such that

$$|f(z)| \leq C(1 + |z|^k), \quad \forall z \in \mathbb{C}^n,$$

must be a polynomial of degree at most $k$. Yang: To be continued...

**Theorem 1.1.14** (Montel's Theorem)**.** A family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ that is uniformly bounded on compact subsets of $\Omega$ is a normal family, i.e., every sequence in the family has a subsequence that converges uniformly on compact subsets of $\Omega$ to a holomorphic function or to infinity. Yang: To be continued...

### 1.1.3 Zero sets of holomorphic functions

**Theorem 1.1.15** (Hartogs' Extension Theorem)**.** Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $K \subset D$ be a compact subset such that $D \setminus K$ is connected. If $f : D \setminus K \to \mathbb{C}$ is a holomorphic function, then there exists a unique holomorphic function $\tilde{f} : D \to \mathbb{C}$ such that $\tilde{f}|_{D\setminus K} = f$. Yang: To be continued...

*Proof.* Yang: To be checked □

**Corollary 1.1.16.** In contrast to the one-variable case, isolated singularities do not exist in several complex variables. Specifically, if $f : D \setminus \{p\} \to \mathbb{C}$ is a holomorphic function on a domain $D \subset \mathbb{C}^n$ with $n \geq 2$ and $p \in D$, then $f$ can be extended to a holomorphic function on the entire domain $D$.

*Proof.* This is a direct consequence of Hartogs' Extension Theorem by taking $K = \{p\}$. □

**Theorem 1.1.17** (Weierstrass Preparation Theorem)**.** Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function in a neighborhood of the origin such that $f(0) = 0$ and $f$ is not identically zero. Write the coordinates as $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that $f(0, w)$ has a zero of order $k$ at $w = 0$, i.e.,

$$f(0, w) = a_k w^k + a_{k+1} w^{k+1} + \cdots, \quad a_k \neq 0.$$

Then there exists a neighborhood $U$ of the origin and unique holomorphic functions $g : U \to \mathbb{C}$ and $h_j : U' \to \mathbb{C}$ for $j = 1, \dots, k$, where $U' \subset \mathbb{C}^n$ is the projection of $U$ onto the first $n$ coordinates, such that

$$f(z, w) = (w^k + h_1(z) w^{k-1} + \cdots + h_k(z)) g(z, w),$$

with $g(0) \neq 0$ and $h_j(0) = 0$ for all $j$. Yang: To be continued...

*Proof.* Yang: To be continued... Yang: Use the Cauchy Integral Formula to check the holomorphicity of $g$ and $h_j$. □

**Definition 1.1.18.** Let $\Omega \subset \mathbb{C}^n$ be an open set. The *sheaf of holomorphic functions* on $\Omega$, denoted by $\mathcal{O}_\Omega$, is the assignment that to each open subset $U \subset \Omega$ assigns the ring $\mathcal{O}_\Omega(U)$ of all holomorphic functions on $U$, and set the restriction as the usual restriction of functions.

A fundamental property of the sheaf of holomorphic functions is its coherence.

**Theorem 1.1.19** (Oka's Coherence Theorem)**.** The sheaf of holomorphic functions $\mathcal{O}_\Omega$ on an open set $\Omega \subset \mathbb{C}^n$ is a coherent sheaf. Yang: To be continued...

In general, $\mathcal{O}_\Omega(U)$ is not a Noetherian ring for an open set $U \subset \Omega$. However, its stalks $\mathcal{O}_{\Omega,p}$ at points $p \in \Omega$ are Noetherian rings. Yang: To be checked

**Example 1.1.20.** Yang: To be continued...

**Proposition 1.1.21.** For any point $p \in \Omega$, the stalk $\mathcal{O}_{\Omega,p}$ of the sheaf of holomorphic functions at $p$ is a Noetherian ring. Yang: To be continued...

**Remark 1.1.22.** The sheaf of holomorphic functions $\mathcal{O}_\Omega$ is a sheaf of topological rings, where the topology on $\mathcal{O}_\Omega(U)$ for an open set $U \subset \Omega$ is given by the compact-open topology. Yang: To be continued...

**Definition 1.1.23.** A subset $A \subset \Omega$ of an open set $\Omega \subset \mathbb{C}^n$ is called an *analytic subset* if for every point $p \in \Omega$, there exists a neighborhood $U$ of $p$ and finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_\Omega(U)$ such that

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = \cdots = f_k(z) = 0\}.$$

Yang: To be continued...

## 1.2 Complex Manifolds

### 1.2.1 Definition and Examples

**Definition 1.2.1.** A *complex manifold* of complex dimension $n$ is a topological space $M$ such that

(a) $M$ is Hausdorff and second countable;

(b) $M$ is locally homeomorphic to $\mathbb{C}^n$, i.e., for every point $p \in M$, there exists an open neighborhood $U$ of $p$ and a homeomorphism $\varphi : U \to V \subset \mathbb{C}^n$, where $V$ is an open subset of $\mathbb{C}^n$, The pair $(U, \varphi)$ is called a *chart*;

(c) if $(U, \varphi)$ and $(U', \varphi')$ are two charts with $U \cap U' \neq \emptyset$, then the transition map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$$

is holomorphic.

The collection of all charts $\{(U_\alpha, \varphi_\alpha)\}$ that cover $M$ is called an *atlas*. If the atlas is maximal, it is called a *complex structure* on $M$.

Another way to define complex manifolds is to use the language of ringed spaces.

**Definition 1.2.2.** A *complex manifold* of complex dimension $n$ is a locally ringed space $(M, \mathcal{O}_M)$ such that

(a) $M$ is Hausdorff and second countable;

(b) for every point $p \in M$, there exists an open neighborhood $U$ of $p$ such that $(U, \mathcal{O}_M|_U)$ is isomorphic to $(B, \mathcal{O}_B)$, where $B$ is the unit open ball in $\mathbb{C}^n$ and $\mathcal{O}_B$ is the sheaf of holomorphic functions on $B$.

**Question 1.2.3.** Given a topological space $M$ that is Hausdorff and second countable, when does it admit a complex structure? Is such a complex structure unique?

For complex dimension $1$, the answer is positive and well-known. For higher dimensions, the answer is negative in general. In particular, does the $6$-sphere $S^6$ admit a complex structure? This is a famous open problem in complex geometry.

**Question 1.2.4.** Does the $6$-sphere $S^6$ admit a complex structure?

**Definition 1.2.5.** Let $M$ and $N$ be two complex manifolds. A continuous map $f : M \to N$ is called *holomorphic* if for every point $p \in M$, there exist charts $(U, \varphi)$ of $M$ around $p$ and $(V, \psi)$ of $N$ around $f(p)$ with $U \subset f^{-1}(V)$ such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is holomorphic.

**Definition 1.2.6.** Let $M$ be a complex manifold of complex dimension $n$. A subset $S \subset M$ is called a *complex submanifold* of complex dimension $k$ if for every point $p \in S$, there exist a chart $(U, \varphi)$ of $M$ around $p$ such that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{C}^k \times \{0\}) \subset \mathbb{C}^n,$$

where we identify $\mathbb{C}^n$ with $\mathbb{C}^k \times \mathbb{C}^{n-k}$. This gives a chart of $S$ around $p$. Endowed with the induced topology and the induced complex structure, $S$ is a complex manifold of complex dimension $k$.

**Example 1.2.7.** Any complex vector space $V$ of complex dimension $n$ is a complex manifold of complex dimension $n$.

**Example 1.2.8.** The complex projective space $\mathbb{CP}^n := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$ is a complex manifold of complex dimension $n$. In fact, $\mathbb{CP}^n$ can be covered by $n+1$ charts, each of which is biholomorphic to $\mathbb{C}^n$. For example, the chart $U_0 = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : z_0 \neq 0\}$ is biholomorphic to $\mathbb{C}^n$ via the map

$$[z_0 : z_1 : \cdots : z_n] \mapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right).$$

The other charts are defined similarly.

**Proposition 1.2.9.** Let $M$ and $N$ are complex manifolds of complex dimension $n$ and $m$ respectively, with $n \geq m$. If $f : M \to N$ is a holomorphic map such that $p$ is a regular value of $f$, i.e., the tangent map $\mathrm{d}f_x$ is surjective for every $x \in f^{-1}(p)$, then $f^{-1}(p)$ is a complex submanifold of $M$ of complex dimension $n-m$.

*Proof.* For every point $q \in f^{-1}(p)$, choose charts $(U,\varphi)$ of $M$ around $q$ and $(V,\psi)$ of $N$ around $p$ such that $f(U) \subset V$. By changing coordinates if necessary, we may assume that $\det(\partial f/\partial w)(q) \neq 0$, where we write the coordinates of $\varphi(U)$ as $(z,w) = (z_1,\dots,z_{n-m},w_1,\dots,w_m) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$. Then by the Holomorphic Implicit Function Theorem (Theorem 1.1.6), there exist open neighborhoods $U'$ of $q$ such that $f^{-1}(p) \cap U'$ is biholomorphic to an open subset of $\mathbb{C}^{n-m}$. □

**Example 1.2.10.** Let $X \subset \mathbb{C}^n$ be a complex algebraic variety defined by the vanishing of polynomials $f_1,\dots,f_m \in \mathbb{C}[z_1,\dots,z_n]$. Suppose that $X$ is non-singular, i.e., for every point $p \in X$, the Jacobian matrix $\left(\partial_{z_j} f_i(p)\right)_{i,j}$ has maximal rank $r$. Then $X$ is a complex submanifold of $\mathbb{C}^n$ of complex dimension $n-r$.

**Example 1.2.11.** A *hypersurface* $H$ in $\mathbb{CP}^n$ is the zero locus of a homogeneous polynomial $f \in \mathbb{C}[z_0,z_1,\dots,z_n]$. Suppose $0$ is a regular value of $f : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$. On each chart $U_i \cong \mathbb{C}^n$ of $\mathbb{CP}^n$, it defines a holomorphic function $f_i : U_i \to \mathbb{C}, [z] \mapsto z = (z_1,\dots,z_{i-1},1,z_{i+1},\dots,z_n) \mapsto f(z)$. The regularity condition implies that $0$ is a regular value of each $f_i$. Hence $H \cap U_i = f_i^{-1}(0)$ is a complex submanifold of $U_i$ of complex dimension $n-1$ by Proposition 1.2.9. Gluing these local pieces together, we see that $H$ is a complex submanifold of $\mathbb{CP}^n$ of complex dimension $n-1$.

**Proposition 1.2.12.** Let $M$ be a complex manifold and let $G$ be a discrete group acting on $M$ by holomorphic automorphisms. If the action is free and properly discontinuous, then the quotient space $M/G$ is a complex manifold and the quotient map $\pi : M \to M/G$ is a holomorphic covering map.

*Proof.* For every point $p \in M/G$, choose a point $q \in M$ such that $\pi(q) = p$. Since the action is free and properly discontinuous (see Remark 1.2.13), there exists an open neighborhood $U$ of $q$ such that $gU \cap U = \varnothing$ for all $g \in G \setminus \{e\}$. Then $\pi|_U : U \to \pi(U)$ is a homeomorphism. This gives a chart of $M/G$ around $p$. If we have two such charts $(\pi(U),\varphi)$ and $(\pi(U'),\varphi')$ of $M/G$ whose intersection is non-empty, WLOG, assume that $U \cap U' \neq \varnothing$. Then $\pi^{-1}(\pi(U) \cap \pi(U')) = \bigsqcup_{g\in G} g(U \cap U')$. The transition map of $U$ and $U'$ gives the transition map of $\pi(U)$ and $\pi(U')$. Since the action of $G$ is by holomorphic automorphisms, the transition maps are holomorphic. □

**Remark 1.2.13.** Recall that an action of a group $G$ on a topological space $X$ is said to be *properly discontinuous* if for every compact subset $K \subset X$, the set $\{g \in G : gK \cap K \neq \varnothing\}$ is finite. If $G$ is a discrete group acting on a manifold $M$ by diffeomorphisms, then the action is properly discontinuous and free if and only if for every point $p \in M$, there exists an open neighborhood $U$ of $p$ such that $gU \cap U = \varnothing$ for all $g \in G \setminus \{e\}$.

**Example 1.2.14.** Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e., a discrete subgroup of $\mathbb{C}$ generated by two $\mathbb{R}$-linearly independent complex numbers. Then $\Lambda$ is isomorphic to $\mathbb{Z}^2$ as an abstract group and acts on $\mathbb{C}$ by translations, which are holomorphic automorphisms of $\mathbb{C}$. Then the quotient space $\mathbb{C}/\Lambda$ is a complex manifold of complex dimension $1$ by Proposition 1.2.12. Such a complex manifold is called an *elliptic curve*. As real manifolds, it is diffeomorphic to $S^1 \times S^1$.

**Example 1.2.15.** Fix $\alpha \in \mathbb{C}^\times$ with $|\alpha| \neq 1$. Let $\mathbb{Z}$ act on $\mathbb{C}^n \setminus \{0\}$ by $k \cdot z = \alpha^k z$ for every $k \in \mathbb{Z}$ and $z \in \mathbb{C}^n \setminus \{0\}$. This action is free and properly discontinuous. Then the quotient space $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is a complex manifold of complex dimension $n$ by Proposition 1.2.12. Such a complex manifold is called a *Hopf manifold*.

**Example 1.2.16.** Let
$$M = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_1, z_2, z_3 \in \mathbb{C} \right\}$$
be the complex Heisenberg group, which is biholomorphic to $\mathbb{C}^3$. Let $\Gamma := M \cap \mathrm{GL}(3, \mathbb{Z}[\sqrt{-1}])$. Then $\Gamma$ is a discrete subgroup of $M$ and acts on $M$ by left multiplication, which are holomorphic automorphisms of $M$. The action is free and properly discontinuous. Then the quotient space $M/\Gamma$ is a complex manifold of complex dimension $3$ by Proposition 1.2.12. It is called the *Iwasawa manifold*. One can replace $\Gamma$ by other cocompact discrete subgroups of $M$.

### 1.2.2 Almost Complex Structures

Let $X$ be a complex manifold of complex dimension $n$. The tangent bundle $TX$ is a real vector bundle of rank $2n$. There is a natural endomorphism $J : TX \to TX$ induced by the complex structure of $X$, i.e., for every point $p \in X$, $J_p : T_pX \to T_pX$ is the multiplication by $\sqrt{-1}$. We have $J^2 = -\mathrm{id}$.

**Definition 1.2.17.** Let $M$ be a smooth manifold of real dimension $2n$. An *almost complex structure* on $M$ is a smooth endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$. The pair $(M, J)$ is called an *almost complex manifold*.

**Question 1.2.18.** Given a smooth manifold $M$ of real dimension $2n$, when does it admit an almost complex structure? Is such an almost complex structure unique?

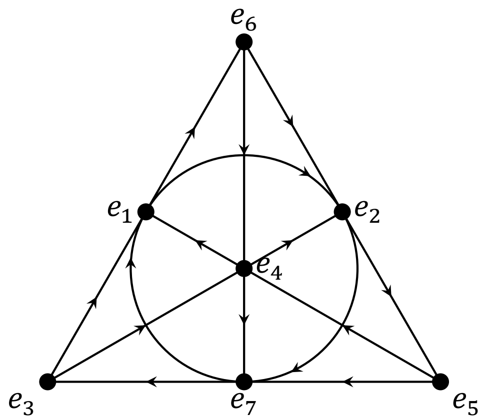
Giving an almost complex structure $J$ on a smooth manifold $M$ is equivalent to giving the tangent bundle $TM$ the structure of a complex vector bundle. Hence the existence of almost complex structures is a purely topological problem. Note that to find a complex structure on $M$ needs to solve some non-linear partial differential equations, which is much harder.

**Example 1.2.19.** The $6$-sphere $S^6$ admits an almost complex structure. In fact, $S^6$ can be identified with the unit sphere in the imaginary octonions $\mathrm{Im}\,\mathbb{O}$ (see Remark 1.2.20). Denote by $m(x, y)$ the octonionic multiplication of $x, y \in \mathbb{O}$. For every point $p \in S^6$, the tangent space $T_pS^6$ can be identified with the orthogonal complement of $\mathbb{R}p$ in $\mathrm{Im}\,\mathbb{O}$. Define $J_p : T_pS^6 \to T_pS^6$ by $J_p(v) = m(p, v)$. Then $J_p^2(v) = p(pv) = -v$ for every $v \in T_pS^6$. Thus we get an almost complex structure on $S^6$.

**Remark 1.2.20.** Recall some fundamental facts about the octonions $\mathbb{O}$:

(a) $\mathbb{O}$ is an $8$-dimensional normed vector space over $\mathbb{R}$ with an orthogonal basis $\{1\} \cup \{e_i | i = 1, \dots, 7\}$. The subspace spanned by $\{e_i\}$ is called the space of imaginary octonions and denoted by $\mathrm{Im}\,\mathbb{O}$.

(b) The multiplication $m : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ is a bilinear map and satisfies the distributive law and the norm multiplicative law $\|xy\| = \|x\|\|y\|$ for all $x, y \in \mathbb{O}$. It is given by the following Fano plane $\mathbb{P}^2(\mathbb{F}_2)$:




If $e_i \to e_j \to e_k$ is a directed line in the Fano plane, then $e_i e_j = e_k$, $e_j e_k = e_i$, and $e_k e_i = e_j$. The multiplication is anti-commutative, i.e., $e_i e_j = -e_j e_i$ for all $i \neq j$. And we have $e_i^2 = -1$ for all $i$.

Yang: To be checked...

Let $(M, J)$ be an almost complex manifold. Then the complexified tangent bundle $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$ splits into the direct sum of two complex subbundles

$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M,$$

where

$$T^{1,0}M := \ker(\sqrt{-1}\mathrm{id} - J), \quad T^{0,1}M := \ker(\sqrt{-1}\mathrm{id} + J).$$

We have $\overline{T^{1,0}M} = T^{0,1}M$ and both $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles of rank $n$. This decomposition induces a decomposition of the complexified cotangent bundle

$$\Omega^1(M) := (TM_{\mathbb{C}})^* = (T^{1,0}M)^* \oplus (T^{0,1}M)^* =: \Omega^{1,0}(M) \oplus \Omega^{0,1}(M).$$

More generally, for every $p, q \geq 0$, define

$$\Omega^{p,q}(M) := \wedge^p (T^{1,0}M)^* \otimes \wedge^q (T^{0,1}M)^* \subset \wedge^{p+q}\Omega^1(M).$$

Then we have the decomposition

$$\Omega^k(M) := \wedge^k \Omega^1(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

The elements of $\Omega^{p,q}(M)$ are called *differential forms of type* $(p, q)$ or $(p, q)$*-forms* for short.

Recall the *exterior differential operator* $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ is locally given by

$$\mathrm{d}\left(\sum_I f_I \mathrm{d}x_I\right) = \sum_I \sum_{j=1}^{2n} \frac{\partial f_I}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_I,$$

where $I$ runs over all multi-indices with $|I| = k$ and $x_1, \ldots, x_{2n}$ are local coordinates on $M$.

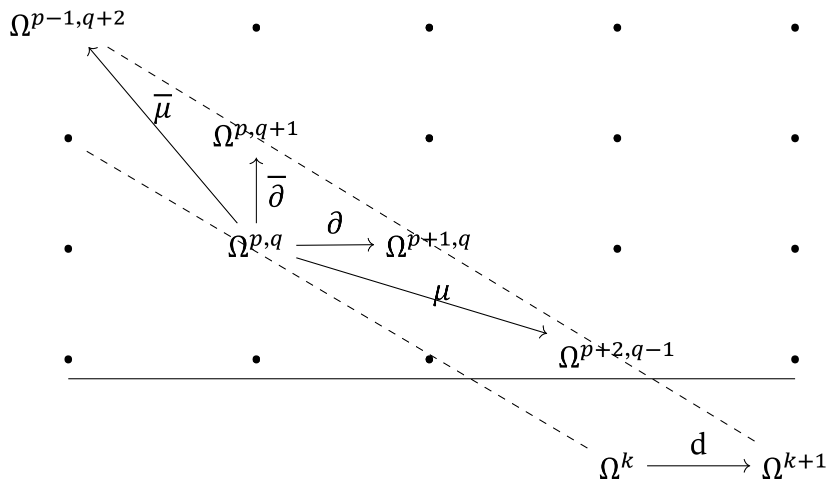
**Proposition 1.2.21.** There exist differential operators

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \mu : \Omega^{p,q}(M) \to \Omega^{p+2,q-1}(M)$$

such that

$$\mathrm{d} = \partial + \overline{\partial} + \mu + \overline{\mu}.$$

In a diagram:




*Proof of Proposition 1.2.21.* Yang: To be continued... ☐

**Definition 1.2.22.** The operator $\mu$ in Proposition 1.2.21 is called the *Nijenhuis operator* of the almost complex structure $J$. If $\mu = 0$, then $J$ is called *integrable*. In this case, we have $\mathrm{d} = \partial + \overline{\partial}$.

**Example 1.2.23.** Let $J$ be the almost complex structure on $S^6$ defined in Example 1.2.19.

Yang: To be continued...

**Proposition 1.2.24.** Let $(M, J)$ be an almost complex manifold. If $J$ is induced by a complex structure on $M$, then $J$ is integrable, i.e., the Nijenhuis operator $\mu = 0$.

*Proof.* Yang: To be continued... ☐

The converse of Proposition 1.2.24 is also true, which is the famous Newlander-Nirenberg theorem. Yang: To add reference...

**Theorem 1.2.25** (Newlander-Nirenberg Theorem)**.** Let $(M, J)$ be an almost complex manifold of real dimension $2n$. If $\mu = 0$, then $J$ is induced by a complex structure on $M$.

**Proposition 1.2.26.** Let $(M, J)$ be an almost complex manifold. Then $J$ is integrable if and only if $\partial^2 = 0$.

### 1.2.3 Cohomology in complex manifolds

Let $M$ be a complex manifold. Denote by $\Omega^k_{\mathrm{sm}}(M)$ the space of smooth differential $k$-forms on $M$ and by $\Omega^{p,q}_{\mathrm{sm}}(M)$ the space of smooth $(p,q)$-forms on $M$. Then $\Omega^k_{\mathrm{sm}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}_{\mathrm{sm}}(M)$. Denote by $\Omega^k_{\mathrm{hol}}(M)$ the space of holomorphic differential $k$-forms on $M$. Then we have $\Omega^{k,0}_{\mathrm{sm}}(M) = \Omega^k_{\mathrm{hol}}(M) \otimes_{\mathcal{O}^{\mathrm{hol}}_M} \mathcal{O}^{\mathrm{sm}}_M$.

There are several cohomology theories for complex manifolds.

**Definition 1.2.27.** Let $M$ be a complex manifold. The *de Rham cohomology* of $M$ is defined to be the de Rham cohomology of the underlying smooth manifold of $M$:

$$H^k_{\mathrm{dR}}(M) := \frac{\mathrm{Ker}(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

**Definition 1.2.28.** Let $M$ be a complex manifold. The *Dolbeault cohomology* of $M$ is defined to be

$$H^{p,q}_{\bar\partial}(M) := \frac{\mathrm{Ker}(\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\bar\partial : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

**Definition 1.2.29.** Let $M$ be a complex manifold. The *Bott-Chern cohomology* of $M$ is defined to be

$$H^{p,q}_{\mathrm{BC}}(M) := \frac{\mathrm{Ker}(\mathrm{d} : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M))}{\mathrm{Im}(\partial\bar\partial : \Omega^{p-1,q-1}(M) \to \Omega^{p,q}(M))}.$$

Yang: To be checked...

**Definition 1.2.30.** Let $M$ be a complex manifold. The *Aeppli cohomology* of $M$ is defined to be

$$H^{p,q}_{\mathrm{A}}(M) := \frac{\mathrm{Ker}(\partial\bar\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M))}{\mathrm{Im}(\partial : \Omega^{p-1,q}(M) \to \Omega^{p,q}(M)) + \mathrm{Im}(\bar\partial : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Yang: To be checked...

There are natural maps between these cohomology theories. Yang: To be continued...

**Proposition 1.2.31.** Let $\Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \dots, n\}$ be the unit polydisc in $\mathbb{C}^n$. Then

$$H^{p,q}_{\bar\partial}(\Delta^n) = \begin{cases} \Omega^p_{\mathrm{hol}}(\Delta^n), & q = 0, \\ 0, & q > 0. \end{cases}$$

Yang: To be checked...

## 1.3 Meromorphic functions and Siegel theorem on function fields

### 1.3.1 Meromorphic functions

**Definition 1.3.1.** Let $M$ be a complex manifold. A *meromorphic function* on $M$ is a holomorphic map $f : M \to \mathbb{CP}^1$.
The set of meromorphic functions on $M$ is denoted by $\mathrm{Mer}(M)$ or $\mathscr{K}(M)$.

**Proposition 1.3.2.** Let $M$ be a complex manifold. Then there is a natural inclusion $\mathrm{Hol}(M) \hookrightarrow \mathrm{Mer}(M)$. Moreover, we have $\mathrm{Mer}(M) = \mathrm{Frac}(\mathrm{Hol}(M))$, i.e., every meromorphic function can be expressed as a quotient of two holomorphic functions. Yang: to be checked.

**Proposition 1.3.3.** Let $M$ be a complex manifold. The set of meromorphic functions on $M$ forms a field under the usual addition and multiplication of functions.

Yang: To be complemented and revised.

### 1.3.2 Siegel theorem

**Proposition 1.3.4.** Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset $U \subset \mathbb{C}^n$. Suppose that $f$ has order $k$ at a point $x \in U$. Then there exists a neighborhood $\overline{B(x,r)} \subset U$ of $x$ such that

$$|f(z)| \leq C|z-x|^k, \quad \forall z \in \overline{B(x,r)},$$

where $C = \sup_{z \in \partial \overline{B(x,r)}} |f(z)|$. Yang: To be revised.

**Theorem 1.3.5** (Siegel theorem on function fields)**.** Let $X$ be a connected and compact complex manifold of dimension $n$. Then the field of meromorphic functions on $X$ satisfies

$$\operatorname{trdeg}_{\mathbb{C}} \mathscr{K}(X) \leq n.$$

*Proof.* Let $\{f_1, f_2, \dots, f_{n+1}\} \subset \mathscr{K}(X)$ be meromorphic functions on $X$. We want to find $P \in \mathbb{C}[x_1, x_2, \dots, x_{n+1}] \setminus \{0\}$ such that

$$P(f_1, f_2, \dots, f_{n+1}) = 0.$$

**Step 1.** Let $z \in X$, there exists $g_1, g_2, \dots, g_{n+1}, h \in \mathrm{Hol}(X)$ such that $f_i = g_i/h$ for each $1 \leq i \leq n+1$.

Yang: To be revised and complemented. □

Yang: To be revised and complemented.

## 1.4 Holomorphic bundles

**Definition 1.4.1.** Let $E \xrightarrow{\pi} X$ be a complex vector bundle over a complex manifold $X$. We say $E$ is a *holomorphic vector bundle* if there exists an open cover $\{U_\alpha\}$ of $X$ and holomorphic trivializations

$$\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{C}^n$$

such that the transition maps

$$\phi_\beta \circ \phi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{C}^n \to U_\alpha \cap U_\beta \times \mathbb{C}^n$$

are holomorphic for all $\alpha, \beta$. Yang: To be checked.

**Example 1.4.2.** The holomorphic tangent bundle $T^{1,0}X$ of a complex manifold $X$ is a holomorphic vector bundle.

**Proposition 1.4.3.** Let $E, F \xrightarrow{\pi} X$ be holomorphic vector bundles over a complex manifold $X$. Then the following bundles are also holomorphic vector bundles:

(a) The direct sum bundle $E \oplus F$.

(b) The tensor product bundle $E \otimes F$.

(c) The dual bundle $E^*$.

Yang: To be completed.

### 1.4.1 Exponential sequence

Let $X$ be a complex manifold. Recall that we have an exact sequence of sheaves on $X$:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0,$$

where $\underline{\mathbb{Z}}$ is the constant sheaf with stalk $\mathbb{Z}$, $\mathcal{O}_X$ is the sheaf of holomorphic functions on $X$, and $\mathcal{O}_X^*$ is the sheaf of nowhere vanishing holomorphic functions on $X$.

**Theorem 1.4.4.** The Picard group $\mathrm{Pic}(X)$ of holomorphic line bundles on $X$ is isomorphic to the group of connected components of the sheaf of nowhere vanishing holomorphic functions on $X$, which is given by the exponential sequence.

## 1.5 Kähler manifolds

# Chapter 2

# Intersection and positivity

## 2.1 Forms and currents

## 2.2 Kodaira Vanishing Theorem

# Chapter 3

# Hodge theory

# Chapter 4

# Algebraic and analytic geometry

## 4.1 The Chow Theorem

## 4.2 GAGA

## 4.3 Kodaira Embedding Theorem

# Chapter 5

# Deformation theory