Complex analysis with multiple variables

In this section, we introduce some basic concepts and results in complex analysis with multiple variables.

1 Holomorphic functions

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Definition 1. A continuous map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is *differentiable* at $p \in \mathbb{R}^{2n}$ if there exists a linear map $Df_p : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ such that

$$f(z) = f(p) + Df_p(z-p) + o(|z-p|).$$

A continuous map $f : \mathbb{C}^n \to \mathbb{C}^m$ is *holomorphic* at $p \in \mathbb{C}^n$ if it is differentiable at p and Df_p is $\mathbb{C}$-linear, i.e., $Df_p(\sqrt{-1}z) = \sqrt{-1}Df_p(z)$ for all $z \in \mathbb{C}^n$.

By a "function", we always mean a complex-valued function, i.e., a map $f : \mathbb{C}^n \to \mathbb{C}$. Fix a coordinate system $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n$ and write $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Then a differentiable function $f = u + iv : \mathbb{C}^n \to \mathbb{C}$ is holomorphic at p if and only if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x_i}(p) = \frac{\partial v}{\partial y_i}(p), \quad \frac{\partial u}{\partial y_i}(p) = -\frac{\partial v}{\partial x_i}(p), \quad i = 1, \dots, n.$$

For convenience, we consider the complexified tangent space $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and introduce the following operators.

Definition 2. The *Wirtinger operators* are defined as

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad j = 1, \dots, n.$$

Then we can rewrite the Cauchy-Riemann equations as

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n.$$

We summarize some properties of Wirtinger operators in the following proposition.

Proposition 3. The Wirtinger operators satisfy the following properties:

(a) $\partial_{z_j} z_i = \delta_{ij}$, $\partial_{z_j} \bar{z}_i = 0$, $\partial_{z_j} \bar{z}_i = 0$, $\partial_{z_j} \bar{z}_j = \delta_{ij}$;

(b) $\overline{\left(\partial_{z_j} f\right)} = \partial_{\bar{z}_j} \bar{f}$;

(c) suppose we have $\mathbb{C}^n \xrightarrow{g} \mathbb{C}^m \xrightarrow{f} \mathbb{C}^l$ and the coordinate on $\mathbb{C}^m$ is $w = (w_1, \dots, w_m)$, then the chain

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rule holds:

$$\frac{\partial(f\circ g)}{\partial z_j} = \sum_{k=1}^{m} \frac{\partial f}{\partial w_k}(g(z))\frac{\partial g_k}{\partial z_j}(z) + \sum_{k=1}^{m} \frac{\partial f}{\partial \bar{w}_k}(g(z))\frac{\partial \bar{g}_k}{\partial z_j}(z),$$

$$\frac{\partial(f\circ g)}{\partial \bar{z}_j} = \sum_{k=1}^{m} \frac{\partial f}{\partial w_k}(g(z))\frac{\partial g_k}{\partial \bar{z}_j}(z) + \sum_{k=1}^{m} \frac{\partial f}{\partial \bar{w}_k}(g(z))\frac{\partial \bar{g}_k}{\partial \bar{z}_j}(z).$$

Proof. Yang: By computation. □

We can also consider the complexified of derivatives

$$Df_p^{\mathbb{C}} : T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \to T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}.$$

If we take $\{\partial_{z_i}, \partial_{\bar{z}_i}\}_{i=1}^{n}$ as a basis of $T^*\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ and $\{\partial_{w_j}, \partial_{\bar{w}_j}\}_{j=1}^{m}$ as a basis of $T^*\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$, then the matrix representation of $Df_p^{\mathbb{C}}$ is

$$Df_p^{\mathbb{C}} = \begin{pmatrix} \frac{\partial f}{\partial z}(p) & \frac{\partial f}{\partial \bar{z}}(p) \\ \frac{\partial f}{\partial \bar{z}}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix}.$$

Yang: To be checked In particular, if f is holomorphic, then we have $\det Df_p^{\mathbb{C}} = |\det(\partial_z f)(p)|^2 \geq 0$.

Definition 4. A map $f : \Omega \to \Omega'$ between two open sets $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^m$ is *biholomorphic* if it is a bijection and both f and f^{-1} are holomorphic.

If f is biholomorphic at p, then $m = n$ and $\det Df_p > 0$.

Theorem 5 (Holomorphic inverse function theorem)**.** Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic function. If the Jacobian determinant $\det Df_p$ is nonzero at $p \in \mathbb{C}^n$, then there exist open neighborhoods U of p and V of $f(p)$ such that $f : U \to V$ is a biholomorphism.

Proof. Yang: To be continued... □

Theorem 6 (Holomorphic implicit function theorem)**.** Let $f : \mathbb{C}^{n+m} \to \mathbb{C}^m$ be a holomorphic function. If the Jacobian determinant $\det(\partial f/\partial w)$ is nonzero at $(z_0, w_0) \in \mathbb{C}^{n+m}$, then there exist open neighborhoods U of z_0 and V of w_0, and a unique holomorphic function $g : U \to V$ such that for any $(z, w) \in U \times V$, $f(z, w) = f(z_0, w_0)$ if and only if $w = g(z)$. Yang: To be continued...

2 Cauchy Integral Formula

Recall the Cauchy Integral Formula in one complex variable:

Theorem 7 (Cauchy Integral Formula in one complex variable)**.** Let $K \subset \mathbb{C}$ be a compact set with piecewise differentiable boundary ∂K, and let f be differentiable on a neighborhood of K. Then for any z in the interior of K, we have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_{K} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

Proof. Yang: By Stokes' theorem. To be continued... □

Theorem 8 (Cauchy Integral Formula in several complex variables). Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D. Then for any $z \in D$,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial D_1 \times \cdots \times \partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n.$$

Proof. Yang: To be continued... □

Corollary 9. Holomorphic functions are analytic. Yang: To be continued...

Proposition 10. Holomorphic functions are open mappings. Yang: To be continued...

Proposition 11. If a holomorphic function $f : \Omega \to \mathbb{C}$ on a connected open set $\Omega \subset \mathbb{C}^n$ attains its maximum at some point in Ω, then f is constant. Yang: To be continued...

Proposition 12. Let $D \subset \mathbb{C}^n$ be a polydisk and f be holomorphic on a neighborhood of the closure of D. Then for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\max_{z \in D} \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(z) \right| \leq \frac{\alpha!}{r^\alpha} \max_{z \in D} |f(z)|,$$

where $r = (r_1, \dots, r_n)$ is the radius of the polydisk D. Yang: To be continued...

Theorem 13 (Generalized Liouville Theorem). A holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ on the whole space $\mathbb{C}^n$ that satisfies a polynomial growth condition, i.e., there exist constants $C > 0$ and $k \geq 0$ such that

$$|f(z)| \leq C(1 + |z|^k), \quad \forall z \in \mathbb{C}^n,$$

must be a polynomial of degree at most k. Yang: To be continued...

Theorem 14 (Montel's Theorem). A family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ that is uniformly bounded on compact subsets of Ω is a normal family, i.e., every sequence in the family has a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function or to infinity. Yang: To be continued...

3 Hartogs' phenomenon

Theorem 15 (Hartogs' Extension Theorem). Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $K \subset D$ be a compact subset such that $D \setminus K$ is connected. If $f : D \setminus K \to \mathbb{C}$ is a holomorphic function, then there exists a unique holomorphic function $F : D \to \mathbb{C}$ such that $F|_{D \setminus K} = f$. Yang: To be continued...

Theorem 16 (Hartogs' Separate Analyticity Theorem). Let $D \subset \mathbb{C}^n$ be a domain with $n \geq 2$, and let $f : D \to \mathbb{C}$ be a function such that for each fixed $z' = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$, the function $f(z', z_j)$ is holomorphic in z_j for all $j = 1, \dots, n$. Then f is holomorphic on D. Yang: To be continued...