

Cohomology Theories in Complex Geometry

1 Differential forms

Let M be a complex manifold.

Denote by $\Omega_{\text{sm}}^k(M)$ the space of smooth differential k -forms on M and by $\Omega_{\text{sm}}^{p,q}(M)$ the space of smooth (p,q) -forms on M . Then $\Omega_{\text{sm}}^k(M) = \bigoplus_{p+q=k} \Omega_{\text{sm}}^{p,q}(M)$. Denote by $\Omega_{\text{hol}}^k(M)$ the space of holomorphic differential k -forms on M . Then we have $\Omega_{\text{sm}}^{k,0}(M) = \Omega_{\text{hol}}^k(M) \otimes_{\mathcal{O}_M^{\text{hol}}} \mathcal{O}_M^{\text{sm}}$.

Recall that we have the exterior derivative

$$d : \Omega_{\text{sm}}^k(M) \rightarrow \Omega_{\text{sm}}^{k+1}(M),$$

which can be decomposed as

$$d = \partial + \bar{\partial},$$

where

$$\partial : \Omega_{\text{sm}}^{p,q}(M) \rightarrow \Omega_{\text{sm}}^{p+1,q}(M), \quad \bar{\partial} : \Omega_{\text{sm}}^{p,q}(M) \rightarrow \Omega_{\text{sm}}^{p,q+1}(M).$$

In a diagram, we have:

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$\Omega^{p,q} \xrightarrow{\partial} \Omega^{p+1,q}$
 $\Omega^{p,q} \xrightarrow{\bar{\partial}} \Omega^{p,q+1}$
 $\Omega^k \xrightarrow{d} \Omega^{k+1}$

2 Various cohomology theories

There are several cohomology theories for complex manifolds.

Definition 1. Let M be a complex manifold. The *singular cohomology* of M with coefficients in a ring R is defined to be the singular cohomology of the underlying topological space $|M|$ of M :

$$H_{\text{sing}}^k(M; R) := H_{\text{sing}}^k(|M|; R).$$

Definition 2. Let M be a complex manifold. The *de Rham cohomology* of M is defined to be the de Rham cohomology of the underlying smooth manifold of M :

$$H_{\text{dR}}^k(M) := \frac{\text{Ker}(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M))}{\text{Im}(d : \Omega^{k-1}(M) \rightarrow \Omega^k(M))}.$$

Yang: Smooth section or holomorphic section?

Definition 3. Let M be a complex manifold. The *Dolbeault cohomology* of M is defined to be

$$H_{\bar{\partial}}^{p,q}(M) := \frac{\text{Ker}(\bar{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M))}{\text{Im}(\bar{\partial} : \Omega^{p,q-1}(M) \rightarrow \Omega^{p,q}(M))}.$$

Proposition 4. Let $\Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \dots, n\}$ be the unit polydisc in \mathbb{C}^n . Then

$$H_{\bar{\partial}}^{p,q}(\Delta^n) = \begin{cases} \Omega_{\text{hol}}^p(\Delta^n), & q = 0, \\ 0, & q > 0. \end{cases}$$

Yang: To be checked...

Appendix