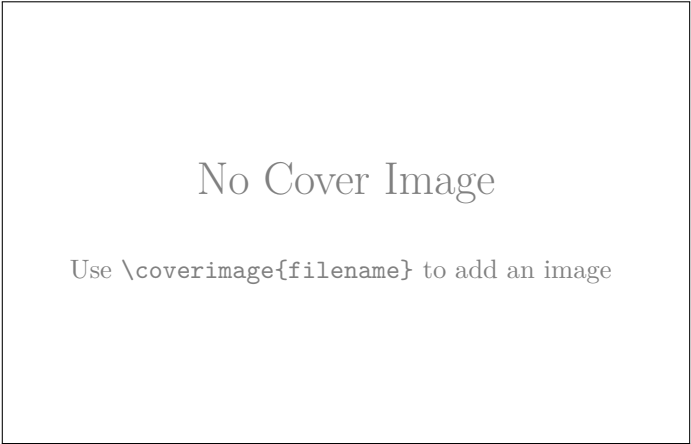


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# *The first properties*



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## 1 Analysis in several complex variables

In this section, we introduce some basic concepts and results in complex analysis with multiple variables.

### 1.1 Holomorphic functions

We identify  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ .

**Definition 1.1.** A continuous map  $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$  is *differentiable* at  $p \in \mathbb{R}^{2n}$  if there exists a linear map  $df_p : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$  such that

$$f(z) = f(p) + df_p(z - p) + o(|z - p|).$$

A continuous map  $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$  is *holomorphic* at  $p \in \mathbb{C}^n$  if it is differentiable at  $p$  and  $df_p$  is  $\mathbb{C}$ -linear, i.e.,  $df_p(\sqrt{-1}z) = \sqrt{-1}df_p(z)$  for all  $z \in \mathbb{C}^n$ .

By a “function”, we always mean a complex-valued function, i.e., a map  $f : \mathbb{C}^n \rightarrow \mathbb{C}$ . Fix a coordinate system  $z = (z_1, \dots, z_n)$  on  $\mathbb{C}^n$  and write  $z_j = x_j + iy_j$  for  $j = 1, \dots, n$ . Then a differentiable function  $f = u + iv : \mathbb{C}^n \rightarrow \mathbb{C}$  is holomorphic at  $p$  if and only if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x_i}(p) = \frac{\partial v}{\partial y_i}(p), \quad \frac{\partial u}{\partial y_i}(p) = -\frac{\partial v}{\partial x_i}(p), \quad i = 1, \dots, n.$$

For convenience, we consider the complexified tangent space  $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$  and introduce the following operators.

**Definition 1.2.** The *Wirtinger operators* are defined as

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} - \sqrt{-1} \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} + \sqrt{-1} \frac{\partial}{\partial y_j} \right), \quad j = 1, \dots, n.$$

Then we can rewrite the Cauchy-Riemann equations as

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n.$$

We summarize some properties of Wirtinger operators in the following proposition.

**Proposition 1.3.** The Wirtinger operators satisfy the following properties:

- (a)  $\partial_{z_j} z_i = \delta_{ij}$ ,  $\partial_{z_j} \bar{z}_i = 0$ ,  $\partial_{\bar{z}_j} z_i = 0$ ,  $\partial_{\bar{z}_j} \bar{z}_i = \delta_{ij}$ ;
- (b)  $\overline{(\partial_{z_j} f)} = \partial_{\bar{z}_j} \bar{f}$ ;
- (c) suppose we have  $\mathbb{C}^n \xrightarrow{g} \mathbb{C}^m \xrightarrow{f} \mathbb{C}^l$  and the coordinate on  $\mathbb{C}^m$  is  $w = (w_1, \dots, w_m)$ , then the chain rule holds:

$$\begin{aligned} \frac{\partial(f \circ g)}{\partial z_j} &= \sum_{k=1}^m \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial z_j}(z) + \sum_{k=1}^m \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial z_j}(z), \\ \frac{\partial(f \circ g)}{\partial \bar{z}_j} &= \sum_{k=1}^m \frac{\partial f}{\partial w_k}(g(z)) \frac{\partial g_k}{\partial \bar{z}_j}(z) + \sum_{k=1}^m \frac{\partial f}{\partial \bar{w}_k}(g(z)) \frac{\partial \bar{g}_k}{\partial \bar{z}_j}(z). \end{aligned}$$

*Proof.* By direct computation. □

We can also consider the complexified of derivatives

$$(df_p)_{\mathbb{C}} : T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}.$$

If we take  $\{\partial_{z_i}, \partial_{\bar{z}_i}\}_{i=1}^n$  as a basis of  $T\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$  and  $\{\partial_{w_j}, \partial_{\bar{w}_j}\}_{j=1}^m$  as a basis of  $T\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$ , then the matrix representation of  $(df_p)_{\mathbb{C}}$  is

$$(df_p)_{\mathbb{C}} = \begin{bmatrix} \partial_z f(p) & \partial_{\bar{z}} f(p) \\ \partial_z f(p) & \partial_{\bar{z}} f(p) \end{bmatrix}.$$

In particular, if  $f$  is holomorphic, then we have  $\det(df_p)_{\mathbb{C}} = |\det(\partial_z f)(p)|^2 \geq 0$ .

**Definition 1.4.** A map  $f : \Omega \rightarrow \Omega'$  between two open sets  $\Omega \subset \mathbb{C}^n$  and  $\Omega' \subset \mathbb{C}^m$  is *biholomorphic* if it is a bijection and both  $f$  and  $f^{-1}$  are holomorphic.

If  $f$  is biholomorphic at  $p$ , then  $m = n$  and  $\det df_p > 0$ .

**Theorem 1.5** (Holomorphic Inverse Function Theorem). Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a holomorphic map. If the Jacobian determinant  $\det df_p$  is nonzero at  $p \in \mathbb{C}^n$ , then there exist open neighborhoods  $U$  of  $p$  and  $V$  of  $f(p)$  such that  $f : U \rightarrow V$  is a biholomorphism.

*Proof.* By the real inverse function theorem, there exist open neighborhoods  $U$  of  $p$  and  $V$  of  $f(p)$  such that  $g = f^{-1} : V \rightarrow U$  is a differentiable map. It suffices to show that  $g$  is holomorphic. By the chain rule (Proposition 1.3), since  $f$  is holomorphic, we have

$$0 = \left( \frac{\partial(f \circ g)_i}{\partial \bar{z}_j} \right)(q) = \left( \frac{\partial f_i}{\partial w_k} \right)(g(q)) \left( \frac{\partial g_k}{\partial \bar{z}_j} \right)(q).$$

Since  $\det(\partial f / \partial w)(f(q)) \neq 0$ , the matrix  $(\partial f_i / \partial w_k)(g(q))$  is invertible, which implies that  $(\partial g_k / \partial \bar{z}_j)(q) = 0$  for all  $k, j$ . Thus  $g$  is holomorphic. □

**Theorem 1.6** (Holomorphic Implicit Function Theorem). Let  $f : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^m$  be a holomorphic map. Write the coordinates of  $\mathbb{C}^{n+m}$  as  $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{C}^n \times \mathbb{C}^m$ . If  $\det(\partial f / \partial w) \neq 0$

at  $(z_0, w_0) \in \mathbb{C}^{n+m}$  with  $f(z_0, w_0) = 0$ , then there exist open neighborhoods  $U$  of  $z_0$  and  $V$  of  $w_0$ , and a unique holomorphic map  $g : U \rightarrow V$  such that for any  $(z, w) \in U \times V$ ,  $f(z, w) = 0$  if and only if  $w = g(z)$ .

*Proof.* By real implicit function theorem, there exist differentiable map  $g : U \rightarrow V$  satisfying the above condition. It suffices to show that  $g$  is holomorphic. Let  $G : U \rightarrow U \times V$  be defined by  $G(z) = (z, g(z))$ . Then we have  $f \circ G = 0$ . By the chain rule, we have

$$0 = \frac{\partial(f \circ G)_i}{\partial \bar{z}_j}(q) = \sum_{k=1}^n \frac{\partial f_i}{\partial w_k}(G(q)) \frac{\partial z_k}{\partial \bar{z}_j}(q) + \sum_{l=1}^m \frac{\partial f_i}{\partial w_l}(G(q)) \frac{\partial g_l}{\partial \bar{z}_j}(q).$$

Since  $\det(\partial f / \partial w)(G(q)) \neq 0$ , the matrix  $(\partial f_i / \partial w_k)(G(q))$  is invertible, which implies that  $(\partial g_l / \partial \bar{z}_j)(q) = 0$  for all  $l, j$ . Thus  $g$  is holomorphic.  $\square$

## 1.2 Cauchy Integral Formula

Recall the Cauchy Integral Formula in one complex variable:

**Theorem 1.7** (Cauchy Integral Formula in one complex variable). Let  $K \subset \mathbb{C}$  be a compact set with piecewise differentiable boundary  $\partial K$ , and let  $f$  be differentiable on a neighborhood of  $K$ . Then for any  $z$  in the interior of  $K$ , we have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial K} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_K \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

*Proof.* Yang: By Stokes' theorem. To be continued...  $\square$

**Theorem 1.8** (Cauchy Integral Formula in several complex variables). Let  $D \subset \mathbb{C}^n$  be a polydisk and  $f$  be holomorphic on a neighborhood of the closure of  $D$ . Then for any  $z \in D$ ,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial D_1 \times \dots \times \partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta_1 \dots d\zeta_n.$$

*Proof.* Yang: To be continued...  $\square$

**Corollary 1.9.** Holomorphic functions are analytic. Yang: To be continued...

**Proposition 1.10.** Holomorphic functions are open mappings. Yang: To be continued...

**Proposition 1.11.** If a holomorphic function  $f : \Omega \rightarrow \mathbb{C}$  on a connected open set  $\Omega \subset \mathbb{C}^n$  attains its maximum at some point in  $\Omega$ , then  $f$  is constant. Yang: To be continued...

**Lemma 1.12.** Let  $D \subset \mathbb{C}^n$  be a polydisk and  $f$  be holomorphic on a neighborhood of the closure of  $D$ . Then for any multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,

$$\max_{z \in D} \left| \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(z) \right| \leq \frac{\alpha!}{r^\alpha} \max_{z \in D} |f(z)|,$$

where  $r = (r_1, \dots, r_n)$  is the radius of the polydisk  $D$ . Yang: To be continued...

**Theorem 1.13** (Generalized Liouville Theorem). A holomorphic function  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  on the whole space  $\mathbb{C}^n$  that satisfies a polynomial growth condition, i.e., there exist constants  $C > 0$  and  $k \geq 0$  such that

$$|f(z)| \leq C(1 + |z|^k), \quad \forall z \in \mathbb{C}^n,$$

must be a polynomial of degree at most  $k$ . Yang: To be continued...

**Theorem 1.14** (Montel's Theorem). A family of holomorphic functions on a domain  $\Omega \subset \mathbb{C}^n$  that is uniformly bounded on compact subsets of  $\Omega$  is a normal family, i.e., every sequence in the family has a subsequence that converges uniformly on compact subsets of  $\Omega$  to a holomorphic function or to infinity. Yang: To be continued...

### 1.3 Zero sets of holomorphic functions

**Theorem 1.15** (Hartogs' Extension Theorem). Let  $D \subset \mathbb{C}^n$  be a domain with  $n \geq 2$ , and let  $K \subset D$  be a compact subset such that  $D \setminus K$  is connected. If  $f : D \setminus K \rightarrow \mathbb{C}$  is a holomorphic function, then there exists a unique holomorphic function  $\tilde{f} : D \rightarrow \mathbb{C}$  such that  $\tilde{f}|_{D \setminus K} = f$ . Yang: To be continued...

*Proof.* Yang: To be checked □

**Corollary 1.16.** In contrast to the one-variable case, isolated singularities do not exist in several complex variables. Specifically, if  $f : D \setminus \{p\} \rightarrow \mathbb{C}$  is a holomorphic function on a domain  $D \subset \mathbb{C}^n$  with  $n \geq 2$  and  $p \in D$ , then  $f$  can be extended to a holomorphic function on the entire domain  $D$ .

*Proof.* This is a direct consequence of Hartogs' Extension Theorem by taking  $K = \{p\}$ . □

**Theorem 1.17** (Weierstrass Preparation Theorem). Let  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  be a holomorphic function in a neighborhood of the origin such that  $f(0) = 0$  and  $f$  is not identically zero. Write the coordinates as  $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$ . Suppose that  $f(0, w)$  has a zero of order  $k$  at  $w = 0$ , i.e.,

$$f(0, w) = a_k w^k + a_{k+1} w^{k+1} + \dots, \quad a_k \neq 0.$$

Then there exists a neighborhood  $U$  of the origin and unique holomorphic functions  $g : U \rightarrow \mathbb{C}$  and  $h_j : U' \rightarrow \mathbb{C}$  for  $j = 1, \dots, k$ , where  $U' \subset \mathbb{C}^n$  is the projection of  $U$  onto the first  $n$  coordinates, such that

$$f(z, w) = (w^k + h_1(z)w^{k-1} + \dots + h_k(z))g(z, w),$$

with  $g(0) \neq 0$  and  $h_j(0) = 0$  for all  $j$ . Yang: To be continued...

*Proof.* Yang: To be continued... Yang: Use the Cauchy Integral Formula to check the holomorphicity of  $g$  and  $h_j$ . □

**Definition 1.18.** Let  $\Omega \subset \mathbb{C}^n$  be an open set. The *sheaf of holomorphic functions* on  $\Omega$ , denoted by  $\mathcal{O}_\Omega$ , is the assignment that to each open subset  $U \subset \Omega$  assigns the ring  $\mathcal{O}_\Omega(U)$  of all holomorphic functions on  $U$ , and set the restriction as the usual restriction of functions.

A fundamental property of the sheaf of holomorphic functions is its coherence.

**Theorem 1.19** (Oka's Coherence Theorem). The sheaf of holomorphic functions  $\mathcal{O}_\Omega$  on an open set  $\Omega \subset \mathbb{C}^n$  is a coherent sheaf. Yang: To be continued...

In general,  $\mathcal{O}_\Omega(U)$  is not a Noetherian ring for an open set  $U \subset \Omega$ . However, its stalks  $\mathcal{O}_{\Omega,p}$  at points  $p \in \Omega$  are Noetherian rings. Yang: To be checked

**Example 1.20.** Yang: To be continued...

**Proposition 1.21.** For any point  $p \in \Omega$ , the stalk  $\mathcal{O}_{\Omega,p}$  of the sheaf of holomorphic functions at  $p$  is a Noetherian ring. Yang: To be continued...

**Remark 1.22.** The sheaf of holomorphic functions  $\mathcal{O}_\Omega$  is a sheaf of topological rings, where the topology on  $\mathcal{O}_\Omega(U)$  for an open set  $U \subset \Omega$  is given by the compact-open topology. Yang: To be continued...

**Definition 1.23.** A subset  $A \subset \Omega$  of an open set  $\Omega \subset \mathbb{C}^n$  is called an *analytic subset* if for every point  $p \in \Omega$ , there exists a neighborhood  $U$  of  $p$  and finitely many holomorphic functions  $f_1, \dots, f_k \in \mathcal{O}_\Omega(U)$  such that

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = \dots = f_k(z) = 0\}.$$

Yang: To be continued...

## 2 Complex Manifolds

### 2.1 Definition and Examples

**Definition 2.1.** A *complex manifold* of complex dimension  $n$  is a topological space  $M$  such that

- (a)  $M$  is Hausdorff and second countable;
- (b)  $M$  is locally homeomorphic to  $\mathbb{C}^n$ , i.e., for every point  $p \in M$ , there exists an open neighborhood  $U$  of  $p$  and a homeomorphism  $\varphi : U \rightarrow V \subset \mathbb{C}^n$ , where  $V$  is an open subset of  $\mathbb{C}^n$ . The pair  $(U, \varphi)$  is called a *chart*;
- (c) if  $(U, \varphi)$  and  $(U', \varphi')$  are two charts with  $U \cap U' \neq \emptyset$ , then the transition map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \rightarrow \varphi'(U \cap U')$$

is holomorphic.

The collection of all charts  $\{(U_\alpha, \varphi_\alpha)\}$  that cover  $M$  is called an *atlas*. If the atlas is maximal, it is called a *complex structure* on  $M$ .

Another way to define complex manifolds is to use the language of ringed spaces.

**Definition 2.2.** A *complex manifold* of complex dimension  $n$  is a locally ringed space  $(M, \mathcal{O}_M)$  such that

- (a)  $M$  is Hausdorff and second countable;
- (b) for every point  $p \in M$ , there exists an open neighborhood  $U$  of  $p$  such that  $(U, \mathcal{O}_M|_U)$  is isomorphic to  $(B, \mathcal{O}_B)$ , where  $B$  is the unit open ball in  $\mathbb{C}^n$  and  $\mathcal{O}_B$  is the sheaf of holomorphic functions on  $B$ .

**Question 2.3.** Given a topological space  $M$  that is Hausdorff and second countable, when does it admit a complex structure? Is such a complex structure unique?

For complex dimension 1, the answer is positive and well-known. For higher dimensions, the answer is negative in general. In particular, does the 6-sphere  $S^6$  admit a complex structure? This is a famous open problem in complex geometry.

**Question 2.4.** Does the 6-sphere  $S^6$  admit a complex structure?

**Definition 2.5.** Let  $M$  and  $N$  be two complex manifolds. A continuous map  $f : M \rightarrow N$  is called *holomorphic* if for every point  $p \in M$ , there exist charts  $(U, \varphi)$  of  $M$  around  $p$  and  $(V, \psi)$  of  $N$  around  $f(p)$  with  $U \subset f^{-1}(V)$  such that

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$$

is holomorphic.

**Definition 2.6.** Let  $M$  be a complex manifold of complex dimension  $n$ . A subset  $S \subset M$  is called a *complex submanifold* of complex dimension  $k$  if for every point  $p \in S$ , there exist a chart  $(U, \varphi)$  of  $M$  around  $p$  such that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{C}^k \times \{0\}) \subset \mathbb{C}^n,$$

where we identify  $\mathbb{C}^n$  with  $\mathbb{C}^k \times \mathbb{C}^{n-k}$ . This gives a chart of  $S$  around  $p$ . Endowed with the induced topology and the induced complex structure,  $S$  is a complex manifold of complex dimension  $k$ .

**Example 2.7.** Any complex vector space  $V$  of complex dimension  $n$  is a complex manifold of complex dimension  $n$ .

**Example 2.8.** The complex projective space  $\mathbb{CP}^n := \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^\times$  is a complex manifold of complex dimension  $n$ . In fact,  $\mathbb{CP}^n$  can be covered by  $n+1$  charts, each of which is biholomorphic to  $\mathbb{C}^n$ . For example, the chart  $U_0 = \{[z_0 : z_1 : \dots : z_n] \in \mathbb{CP}^n : z_0 \neq 0\}$  is biholomorphic to  $\mathbb{C}^n$  via the map

$$[z_0 : z_1 : \dots : z_n] \mapsto \left( \frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0} \right).$$

The other charts are defined similarly.

**Proposition 2.9.** Let  $M$  and  $N$  be complex manifolds of complex dimension  $n$  and  $m$  respectively, with  $n \geq m$ . If  $f : M \rightarrow N$  is a holomorphic map such that  $p$  is a regular value of  $f$ , i.e., the tangent map  $df_x$  is surjective for every  $x \in f^{-1}(p)$ , then  $f^{-1}(p)$  is a complex submanifold of  $M$  of complex

dimension  $n - m$ .

*Proof.* For every point  $q \in f^{-1}(p)$ , choose charts  $(U, \varphi)$  of  $M$  around  $q$  and  $(V, \psi)$  of  $N$  around  $p$  such that  $f(U) \subset V$ . By changing coordinates if necessary, we may assume that  $\det(\partial f / \partial w)(q) \neq 0$ , where we write the coordinates of  $\varphi(U)$  as  $(z, w) = (z_1, \dots, z_{n-m}, w_1, \dots, w_m) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$ . Then by the Holomorphic Implicit Function Theorem (Theorem 1.6), there exist open neighborhoods  $U'$  of  $q$  such that  $f^{-1}(p) \cap U'$  is biholomorphic to an open subset of  $\mathbb{C}^{n-m}$ .  $\square$

**Example 2.10.** Let  $X \subset \mathbb{C}^n$  be a complex algebraic variety defined by the vanishing of polynomials  $f_1, \dots, f_m \in \mathbb{C}[z_1, \dots, z_n]$ . Suppose that  $X$  is non-singular, i.e., for every point  $p \in X$ , the Jacobian matrix  $(\partial_{z_j} f_i(p))_{i,j}$  has maximal rank  $r$ . Then  $X$  is a complex submanifold of  $\mathbb{C}^n$  of complex dimension  $n - r$ .

**Example 2.11.** A *hypersurface*  $H$  in  $\mathbb{CP}^n$  is the zero locus of a homogeneous polynomial  $f \in \mathbb{C}[z_0, z_1, \dots, z_n]$ . Suppose  $0$  is a regular value of  $f : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}$ . On each chart  $U_i \cong \mathbb{C}^n$  of  $\mathbb{CP}^n$ , it defines a holomorphic function  $f_i : U_i \rightarrow \mathbb{C}, [z] \mapsto z = (z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n) \mapsto f(z)$ . The regularity condition implies that  $0$  is a regular value of each  $f_i$ . Hence  $H \cap U_i = f_i^{-1}(0)$  is a complex submanifold of  $U_i$  of complex dimension  $n - 1$  by Proposition 2.9. Gluing these local pieces together, we see that  $H$  is a complex submanifold of  $\mathbb{CP}^n$  of complex dimension  $n - 1$ .

**Proposition 2.12.** Let  $M$  be a complex manifold and let  $G$  be a discrete group acting on  $M$  by holomorphic automorphisms. If the action is free and properly discontinuous, then the quotient space  $M/G$  is a complex manifold and the quotient map  $\pi : M \rightarrow M/G$  is a holomorphic covering map.

*Proof.* For every point  $p \in M/G$ , choose a point  $q \in M$  such that  $\pi(q) = p$ . Since the action is free and properly discontinuous (see Remark 2.13), there exists an open neighborhood  $U$  of  $q$  such that  $gU \cap U = \emptyset$  for all  $g \in G \setminus \{e\}$ . Then  $\pi|_U : U \rightarrow \pi(U)$  is a homeomorphism. This gives a chart of  $M/G$  around  $p$ . If we have two such charts  $(\pi(U), \varphi)$  and  $(\pi(U'), \varphi')$  of  $M/G$  whose intersection is non-empty, WLOG, assume that  $U \cap U' \neq \emptyset$ . Then  $\pi^{-1}(\pi(U) \cap \pi(U')) = \bigsqcup_{g \in G} g(U \cap U')$ . The transition map of  $U$  and  $U'$  gives the transition map of  $\pi(U)$  and  $\pi(U')$ . Since the action of  $G$  is by holomorphic automorphisms, the transition maps are holomorphic.  $\square$

**Remark 2.13.** Recall that an action of a group  $G$  on a topological space  $X$  is said to be *properly discontinuous* if for every compact subset  $K \subset X$ , the set  $\{g \in G : gK \cap K \neq \emptyset\}$  is finite. If  $G$  is a discrete group acting on a manifold  $M$  by diffeomorphisms, then the action is properly discontinuous and free if and only if for every point  $p \in M$ , there exists an open neighborhood  $U$  of  $p$  such that  $gU \cap U = \emptyset$  for all  $g \in G \setminus \{e\}$ .

**Example 2.14.** Let  $\Lambda \subset \mathbb{C}$  be a lattice, i.e., a discrete subgroup of  $\mathbb{C}$  generated by two  $\mathbb{R}$ -linearly independent complex numbers. Then  $\Lambda$  is isomorphic to  $\mathbb{Z}^2$  as an abstract group and acts on  $\mathbb{C}$  by translations, which are holomorphic automorphisms of  $\mathbb{C}$ . Then the quotient space  $\mathbb{C}/\Lambda$  is a complex manifold of complex dimension 1 by Proposition 2.12. Such a complex manifold is called an *elliptic curve*. As real manifolds, it is diffeomorphic to  $S^1 \times S^1$ .

**Example 2.15.** Fix  $\alpha \in \mathbb{C}^\times$  with  $|\alpha| \neq 1$ . Let  $\mathbb{Z}$  act on  $\mathbb{C}^n \setminus \{0\}$  by  $k \cdot z = \alpha^k z$  for every  $k \in \mathbb{Z}$  and  $z \in \mathbb{C}^n \setminus \{0\}$ . This action is free and properly discontinuous. Then the quotient space  $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$  is a complex manifold of complex dimension  $n$  by Proposition 2.12. Such a complex manifold is called



| a *Hopf manifold*.

**Example 2.16.** Let

$$M = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$$

be the complex Heisenberg group, which is biholomorphic to  $\mathbb{C}^3$ . Let  $\Gamma := M \cap \mathrm{GL}(3, \mathbb{Z}[\sqrt{-1}])$ . Then  $\Gamma$  is a discrete subgroup of  $M$  and acts on  $M$  by left multiplication, which are holomorphic automorphisms of  $M$ . The action is free and properly discontinuous. Then the quotient space  $M/\Gamma$  is a complex manifold of complex dimension 3 by [Proposition 2.12](#). It is called the *Iwasawa manifold*. One can replace  $\Gamma$  by other cocompact discrete subgroups of  $M$ .

## 2.2 Almost Complex Structures

Let  $X$  be a complex manifold of complex dimension  $n$ . The tangent bundle  $TX$  is a real vector bundle of rank  $2n$ . There is a natural endomorphism  $J : TX \rightarrow TX$  induced by the complex structure of  $X$ , i.e., for every point  $p \in X$ ,  $J_p : T_p X \rightarrow T_p X$  is the multiplication by  $\sqrt{-1}$ . We have  $J^2 = -\mathrm{id}$ .

**Definition 2.17.** Let  $M$  be a smooth manifold of real dimension  $2n$ . An *almost complex structure* on  $M$  is a smooth endomorphism  $J : TM \rightarrow TM$  such that  $J^2 = -\mathrm{id}$ . The pair  $(M, J)$  is called an *almost complex manifold*.

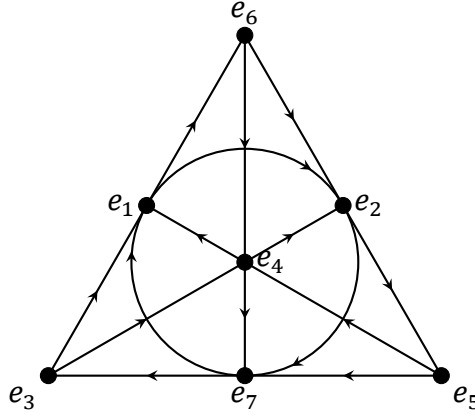
**Question 2.18.** Given a smooth manifold  $M$  of real dimension  $2n$ , when does it admit an almost complex structure? Is such an almost complex structure unique?

Giving an almost complex structure  $J$  on a smooth manifold  $M$  is equivalent to giving the tangent bundle  $TM$  the structure of a complex vector bundle. Hence the existence of almost complex structures is a purely topological problem. Note that to find a complex structure on  $M$  needs to solve some non-linear partial differential equations, which is much harder.

**Example 2.19.** The 6-sphere  $S^6$  admits an almost complex structure. In fact,  $S^6$  can be identified with the unit sphere in the imaginary octonions  $\mathrm{Im} \mathbb{O}$  (see [Remark 2.20](#)). Denote by  $m(x, y)$  the octonionic multiplication of  $x, y \in \mathbb{O}$ . For every point  $p \in S^6$ , the tangent space  $T_p S^6$  can be identified with the orthogonal complement of  $\mathbb{R}p$  in  $\mathrm{Im} \mathbb{O}$ . Define  $J_p : T_p S^6 \rightarrow T_p S^6$  by  $J_p(v) = m(p, v)$ . Then  $J_p^2(v) = p(pv) = -v$  for every  $v \in T_p S^6$ . Thus we get an almost complex structure on  $S^6$ .

**Remark 2.20.** Recall some fundamental facts about the octonions  $\mathbb{O}$ :

- (a)  $\mathbb{O}$  is an 8-dimensional normed vector space over  $\mathbb{R}$  with an orthogonal basis  $\{1\} \cup \{e_i \mid i = 1, \dots, 7\}$ . The subspace spanned by  $\{e_i\}$  is called the space of imaginary octonions and denoted by  $\mathrm{Im} \mathbb{O}$ .
- (b) The multiplication  $m : \mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$  is a bilinear map and satisfies the distributive law and the norm multiplicative law  $\|xy\| = \|x\|\|y\|$  for all  $x, y \in \mathbb{O}$ . It is given by the following Fano plane  $\mathbb{P}^2(\mathbb{F}_2)$ :



If  $e_i \rightarrow e_j \rightarrow e_k$  is a directed line in the Fano plane, then  $e_i e_j = e_k$ ,  $e_j e_k = e_i$ , and  $e_k e_i = e_j$ . The multiplication is anti-commutative, i.e.,  $e_i e_j = -e_j e_i$  for all  $i \neq j$ . And we have  $e_i^2 = -1$  for all  $i$ .

Yang: To be checked...

Let  $(M, J)$  be an almost complex manifold. Then the complexified tangent bundle  $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$  splits into the direct sum of two complex subbundles

$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M,$$

where

$$T^{1,0}M := \ker(\sqrt{-1}\text{id} - J), \quad T^{0,1}M := \ker(\sqrt{-1}\text{id} + J).$$

We have  $\overline{T^{1,0}M} = T^{0,1}M$  and both  $T^{1,0}M$  and  $T^{0,1}M$  are complex vector bundles of rank  $n$ . This decomposition induces a decomposition of the complexified cotangent bundle

$$\Omega^1(M) := (TM_{\mathbb{C}})^* = (T^{1,0}M)^* \oplus (T^{0,1}M)^* =: \Omega^{1,0}(M) \oplus \Omega^{0,1}(M).$$

More generally, for every  $p, q \geq 0$ , define

$$\Omega^{p,q}(M) := \wedge^p(T^{1,0}M)^* \otimes \wedge^q(T^{0,1}M)^* \subset \wedge^{p+q}\Omega^1(M).$$

Then we have the decomposition

$$\Omega^k(M) := \wedge^k \Omega^1(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

The elements of  $\Omega^{p,q}(M)$  are called *differential forms of type  $(p, q)$*  or  *$(p, q)$ -forms* for short.

Recall the *exterior differential operator*  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  is locally given by

$$d\left(\sum_I f_I dx_I\right) = \sum_I \sum_{j=1}^{2n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I,$$

where  $I$  runs over all multi-indices with  $|I| = k$  and  $x_1, \dots, x_{2n}$  are local coordinates on  $M$ .

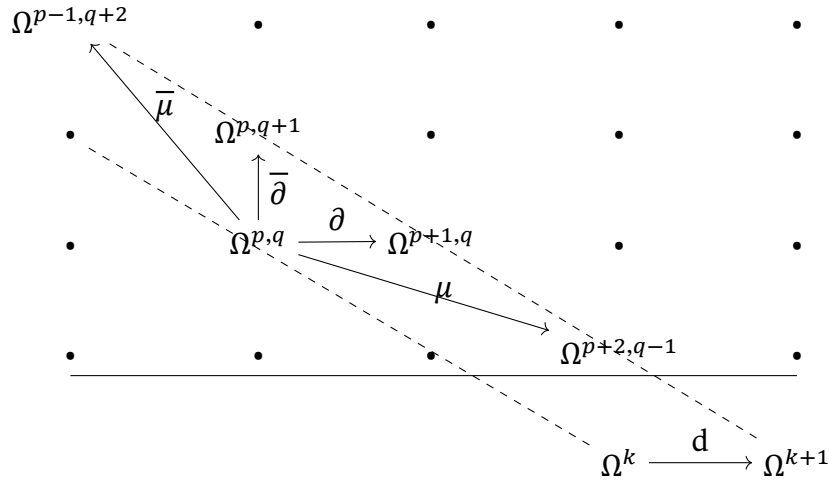
**Proposition 2.21.** There exist differential operators

$$\partial : \Omega^{p,q}(M) \rightarrow \Omega^{p+1,q}(M), \quad \mu : \Omega^{p,q}(M) \rightarrow \Omega^{p+2,q-1}(M)$$

such that

$$d = \partial + \bar{\partial} + \mu + \bar{\mu}.$$

In a diagram:



*Proof of Proposition 2.21.* Yang: To be continued...

□

**Definition 2.22.** The operator  $\mu$  in Proposition 2.21 is called the *Nijenhuis operator* of the almost complex structure  $J$ . If  $\mu = 0$ , then  $J$  is called *integrable*. In this case, we have  $d = \partial + \bar{\partial}$ . Yang: To be continued...

**Example 2.23.** Let  $J$  be the almost complex structure on  $S^6$  defined in Example 2.19.

Yang: To be checked...

**Proposition 2.24.** Let  $M$  be a smooth manifold of real dimension  $2n$  with an almost complex structure  $J$ . If  $J$  is induced by a complex structure on  $M$ , then  $\mu = 0$ .

*Proof.* Yang: To be continued...

□

The converse of Proposition 2.24 is also true, which is the famous Newlander-Nirenberg theorem. Yang: To add reference...

**Theorem 2.25.** Let  $M$  be a smooth manifold of real dimension  $2n$  with an almost complex structure  $J$ . If  $\mu = 0$ , then  $J$  is induced by a complex structure on  $M$ .