

# Forms and Currents

Let  $M$  be a complex manifold of complex dimension  $d$ .

## 1 Differential forms

Recall that we have the decomposition of the cotangent bundle:

$$\Omega_{\text{sm}}^{1,\mathbb{C}} \cong \Omega_{\text{sm}}^{1,0} \oplus \Omega_{\text{sm}}^{0,1} \cong \Omega_{\text{sm}}^1 \oplus \overline{\Omega_{\text{sm}}^1}.$$

Take exterior powers, we have  $\Omega_{\text{sm}}^{k,\mathbb{C}} \cong \bigoplus_{p+q=k} \Omega_{\text{sm}}^{p,q}$ , where  $\Omega_{\text{sm}}^{p,q} = \bigwedge^p \Omega_{\text{sm}}^{1,0} \otimes \bigwedge^q \Omega_{\text{sm}}^{0,1}$ . We also use the notation

$$\mathcal{A}^{p,q} := \Omega_{\text{sm}}^{p,q}, \quad \mathcal{A}^k := \Omega_{\text{sm}}^{k,\mathbb{C}}.$$

A reason to induce this strange sheaf  $\mathcal{A}^k = \Omega_{\text{sm}}^{k,\mathbb{C}}$  is to make sense of integration of top-degree forms. For simplicity, assume that  $M$  is compact. Let  $\omega \in \mathcal{A}^{2d}(M)$  be a smooth complex-valued  $2d$ -form on  $M$ . Then its integration is well-defined in the smooth manifold sense:

$$\int_M \omega \in \mathbb{C}.$$

However, in complex case, it is more natural to “integral” a holomorphic  $d$ -form on a  $d$ -dimensional complex manifold. This does not make sense in the smooth manifold theory. The solution is to associate a holomorphic  $d$ -form  $\eta \in \mathcal{A}^{d,0}(M)$  with a smooth  $2d$ -form (( $d,d$ )-form)  $\omega = \eta \wedge \bar{\eta} \in \mathcal{A}^{d,d}(M) \subset \mathcal{A}^{2d}(M)$ .

Another reason is that  $\bigoplus_k \Omega_{\text{sm}}^{k,\mathbb{C}}$  is not closed under the exterior derivative  $d$ , while  $\bigoplus_k \Omega_{\text{sm}}^{k,\mathbb{C}}$  is. Suppose that we have local holomorphic coordinates  $(z_1, \dots, z_d)$ . Recall that we have the exterior derivative

$$d : \Omega_{\text{sm}}^{1,0} \rightarrow \Omega_{\text{sm}}^{2,\mathbb{C}}, \quad f dz_i \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge dz_i + \sum_{i=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_i.$$

On its conjugation, we have

$$d : \Omega_{\text{sm}}^{0,1} \rightarrow \Omega_{\text{sm}}^{2,\mathbb{C}}, \quad f d\bar{z}_i \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge d\bar{z}_i + \sum_{j=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_i.$$

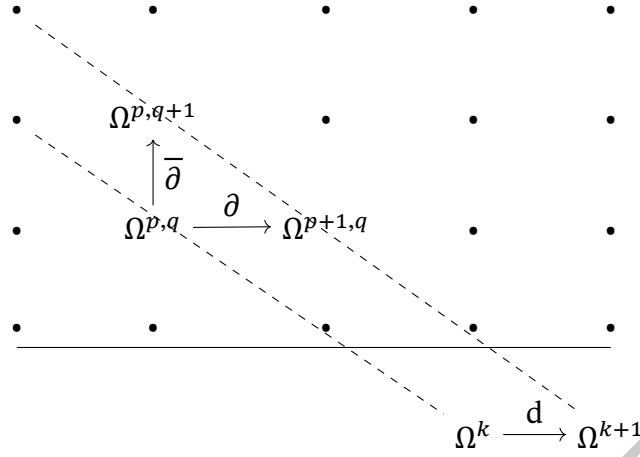
Extending  $d$  by linearity and the Leibniz rule  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$ , we get the exterior derivative

$$d : \mathcal{A}^k = \Omega_{\text{sm}}^{k,\mathbb{C}} \rightarrow \mathcal{A}^{k+1} = \Omega_{\text{sm}}^{k+1,\mathbb{C}},$$

which can be decomposed as  $d = \partial + \bar{\partial}$ , where

$$\begin{aligned} \partial : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{p+1,q}, \quad f dz_I \wedge d\bar{z}_J \mapsto \sum_{j=1}^d \frac{\partial f}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J, \\ \bar{\partial} : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{p,q+1}, \quad f dz_I \wedge d\bar{z}_J \mapsto \sum_{j=1}^d \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J. \end{aligned}$$

In a diagram, we have:



**Proposition 1.** The operators  $\partial$  and  $\bar{\partial}$  satisfy

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

*Proof.* Yang: To be added. □

**Proposition 2.** Let  $f : M \rightarrow N$  be a holomorphic map between complex manifolds. Then the pull-back of differential forms  $f^* : \mathcal{A}_N^k \rightarrow \mathcal{A}_M^k$  satisfies

$$f^*(\mathcal{A}_N^{p,q}) \subset \mathcal{A}_M^{p,q}, \quad f^* \circ \partial_N = \partial_M \circ f^*, \quad f^* \circ \bar{\partial}_N = \bar{\partial}_M \circ f^*.$$

*Proof.* Yang: To be added. □

**Construction 3.** To define a topology on  $\mathcal{A}^k(M)$ , we use the collection of seminorms as follows. Let  $U \subset M$  be a coordinate chart with local real coordinates  $(x_1, \dots, x_{2d})$ . Given  $\omega \in \mathcal{A}^k(M)$ , we can write  $\omega|_U$  as  $\omega|_U = \sum_I f_I dx_I$ . For any multi-index  $\alpha = (\alpha_1, \dots, \alpha_{2d})$ , we can define the partial derivative

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_{2d}^{\alpha_{2d}}}.$$

Then for any compact subset  $K \subset U$  and non-negative integer  $m$ , we can define a seminorm on  $\mathcal{A}^k(M)$  by

$$p_{U,K,m}(\omega) = \sup_{x \in K} \max_I \max_{|\alpha| \leq m} |D^\alpha f_I(x)|,$$

where  $m$  is a non-negative integer.

The topology on  $\mathcal{A}^k(M)$  is defined to be the weakest topology such that all seminorms  $p_{U,K,m}$  are continuous, where  $U$  runs over all coordinate charts of  $M$ ,  $K$  runs over all compact subsets of  $U$ , and  $m$  runs over all non-negative integers.

**Proposition 4.** Let  $f : M \rightarrow N$  be a holomorphic map between complex manifolds. Then the pull-back of differential forms  $f^* : \mathcal{A}^k(N) \rightarrow \mathcal{A}^k(M)$  is a continuous linear map with respect to the topologies defined above. Yang: To be checked

**Proposition 5.** The space  $\mathcal{A}^k(M)$  equipped with the topology defined above is a Fréchet space, i.e., a complete metrizable locally convex topological vector space. Yang: To be checked

## 2 Currents

We denote by  $\mathcal{A}_c^k(M)$  (resp.  $\mathcal{A}_c^{p,q}(M)$ ) the space of smooth complex-valued  $k$ -forms (resp.  $(p, q)$ -forms) on  $M$  with compact support. It is also denoted by  $\mathcal{D}^k(M)$ .

**Definition 6.** A *current* of dimension  $k$  or degree  $(2d - k)$  on  $M$  is a continuous linear functional

$$T : \mathcal{D}^k(M) \rightarrow \mathbb{C}.$$

We denote the space of currents of dimension  $k$  on  $M$  by  $\mathcal{D}'_k(M)$  or  $\mathcal{D}'^{2d-k}(M)$ . Yang: To be revised.

## Appendix

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