Ergodic

Definition 1. A measure-preserving transformation $T : X \to X$ on a measure space $(X, \mathcal{B}, \mu)$ is said to be *ergodic* if it has no nontrivial invariant sets, i.e., for every $A \in \mathcal{B}$ such that $T^{-1}A = A$, we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Theorem 2. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. Then the following statements are equivalent:

(a) T is ergodic;

(b) for each measurable set A, $m((T^{-1}A)\Delta A) = 0$ iff $\mu(A) = 0$ or $\mu(X \setminus A) = 0$;

(c) for each measurable set A with $\mu(A) > 0$, we have $\mu\left(\bigcup_{n=0}^{\infty} T^{-n}A\right) = 1$;

(d) for each measurable sets A, B with $\mu(A)\mu(B) > 0$, there exists $n \in \mathbb{Z}^+$ such that $\mu(T^{-n}A \cap B) > 0$.

Theorem 3. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) < \infty$ and $T : X \to X$ be a measure-preserving transformation. Then TFAE:

(a) T is ergodic;

(b) for each measurable function f which is T-invariant, f is constant almost everywhere;

(c) for each $f \in L^1(X, \mu)$ which is T-invariant, f is constant almost everywhere;

(d) for each $f \in L^2(X, \mu)$ which is T-invariant, f is constant almost everywhere.

Proof. Yang: To be continued...

We show that (a) $\Rightarrow$ (b). □

Example 4. Let $X = \mathbb{N}$ and $\mathcal{B} = \mathcal{P}(\mathbb{N})$ with the counting measure. Define $T : \mathbb{N} \to \mathbb{N}$ by $Tx = x+1$. Then T is measure-preserving and ergodic.

Example 5. Let $X = \mathbb{N}$ and $\mathcal{B} = \mathcal{P}(\mathbb{N})$ with the counting measure. Define $T : \mathbb{N} \to \mathbb{N}$ by $Tx = x+2$. Then T is measure-preserving but not ergodic.

Example 6. Let $X = \mathbb{R}$ and $\mathcal{B} = \mathcal{B}(\mathbb{R})$ with the Lebesgue measure. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = x+1$. Then T is measure-preserving but not ergodic.

Example 7. Let $X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathcal{B} = \mathcal{B}(S^1)$ with the Lebesgue measure. Define $T : S^1 \to S^1$ by $Tx = \mathrm{e}^{2\pi i\theta}x$ where $\theta \in \mathbb{R}$. Then T is measure-preserving. Moreover, T is ergodic iff θ is irrational.

Proposition 8. In Example 7, if θ is irrational, then T is ergodic.

Proof. Yang: To be continued..., By Fourier series □

Date: October 21, 2025, **Author**: Tianle Yang, My Homepage

Example 9. Let $X = S^1$ and $\mathcal{B} = \mathcal{B}(S^1)$ with the Lebesgue measure. Define $T : S^1 \to S^1$ by $Tx = x^2$. Then T is measure-preserving and ergodic.

Yang: To be continued..., By Fourier series

Proposition 10. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. Then T is ergodic iff for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B).$$

Theorem 11. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. Then T is ergodic iff for all $f, g \in L^2(X, \mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_X f(T^k x) g(x) \,\mathrm{d}\mu(x) = \int_X f(x) \,\mathrm{d}\mu(x) \int_X g(x) \,\mathrm{d}\mu(x).$$

In the language of operators, Theorem 11 can be restated as follows.

Corollary 12. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. Define the operator $U_T : L^2(X, \mu) \to L^2(X, \mu)$ by $U_T f = f \circ T$. Then T is ergodic iff for all $f, g \in L^2(X, \mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle U_T^k f, g\rangle = \langle f, 1\rangle\langle 1, g\rangle.$$

Definition 13. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. We say that T is *mixing* if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Lemma 14. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. If T is mixing, then T is ergodic.

Theorem 15. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. TFAE:

(a) T is mixing;

(b) for all $f, g \in L^2(X, \mu)$, $\lim_{n\to\infty} \langle U_T^n f, g\rangle = \langle f, 1\rangle\langle 1, g\rangle$;

(c) for all $f \in L^2(X, \mu)$, $\lim_{n\to\infty} \langle U_T^n f, f\rangle = |\langle f, 1\rangle|^2$.

Example 16. Let $X = S^1$ and $\mathcal{B} = \mathcal{B}(S^1)$ with the Lebesgue measure. Define $T : S^1 \to S^1$ by $Tx = cx$ where $|c| = 1$ and c is not a root of unity. Then T is ergodic but not mixing.

Example 17. Let $X = S^1$ and $\mathcal{B} = \mathcal{B}(S^1)$ with the Lebesgue measure. Define $T : S^1 \to S^1$ by $Tx = x^2$. Then T is mixing. Yang: To be continued.

Definition 18. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. We say that T is *weakly mixing* if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(T^{-k}A \cap B) - \mu(A)\mu(B)| = 0.$$

Proposition 19. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. TFAE:

(a) T is weakly mixing;

(b) for all $f, g \in L^2(X, \mu)$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle - \langle f, 1\rangle\langle 1, g\rangle| = 0$;

(c) for all $f \in L^2(X, \mu)$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f, f\rangle - |\langle f, 1\rangle|^2| = 0$.

(d) for all $f \in L^2(X, \mu)$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f, f\rangle - |\langle f, 1\rangle|^2|^2 = 0$.

Yang: To be checked.