Recurrence in Ergodic Theory

Let $(X, \mathcal{B}, \mu)$ be a measure space. We always assume that μ is σ-finite, i.e., $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is measurable and $\mu(X_i) < \infty$. By "function on X" I mean a map $X \to \mathbb{C}$.

1 Measurable transformations

Definition 1. Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be two measure spaces. A map $T : X \to Y$ is called a *measurable transformation* if for every measurable set $B \in \mathcal{B}_Y$, the preimage $T^{-1}(B) \in \mathcal{B}_X$ is measurable.
A measurable transformation $T : X \to Y$ is called *measure-preserving* if for every measurable set $B \in \mathcal{B}_Y$, we have $\mu_X(T^{-1}(B)) = \mu_Y(B)$.
A measurable transformation $T : X \to Y$ is called *invertible* if there exists a measurable transformation $S : Y \to X$ such that $S \circ T = \mathrm{id}_X$ and $T \circ S = \mathrm{id}_Y$.

Note that an invertible measurable transformation is measure-preserving if and only if its inverse is measure-preserving.

Example 2. Let $X = \mathbb{R}$ with the Borel σ-algebra and the Lebesgue measure. Define $T : X \to X$ by $T(x) = 2x$. Then T is an invertible measurable transformation, but it is not measure-preserving.

Example 3. Let $X = S^1$ be the unit circle with the Borel σ-algebra and the Lebesgue measure. Define $T : X \to X$ by $T(z) = z^2$. Then T is a measure-preserving transformation but not invertible.

Indeed, for any connected subset $A \subseteq S^1$ with arc length θ, the preimage $T^{-1}(A)$ consists of two connected subsets, each with arc length $\theta/2$.

Example 4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (2x, y/2)$ and endow $\mathbb{R}^2$ with Lebesgue measure and the Borel σ-algebra. Then T is an invertible measure-preserving transformation since it preserves the area of rectangles.

2 Poincaré Recurrence Theorem

In this subsection, we focus on $T : X \to X$ where $(X, \mathcal{B}, \mu)$ is a measure space.

Definition 5. Let $(X, \mathcal{B}, \mu)$ be a measure space and $T : X \to X$ be a measure transformation. Let $A \in \mathcal{B}$ be a measurable set. A point $x \in A$ is called *recurrent* if $T^n(x) \in A$ for some integer $n > 0$.

Theorem 6 (Poincaré Recurrence Theorem)**.** Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$, and let $T : X \to X$ be a measure-preserving transformation, i.e., for all $A \in \mathcal{B}$, $\mu(T^{-1}(A)) = \mu(A)$. Then for any measurable set A with $\mu(A) > 0$, the for almost every point $x \in A$, there exist infinitely many integers $n > 0$ such that $T^n(x) \in A$.

Proof. Yang: To be continued. □

Let f be the characteristic function of A, then Poincaré Recurrence Theorem states that for almost every $x \in A$, the sequence $\sum_{n=1}^{\infty} f(T^n(x))$ diverges.

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Hence we have a generalization.

Theorem 7. Let $T : X \to X$ be a measure-preserving transformation on a measure space $(X, \mathcal{B}, \mu)$ with $\mu(X) < \infty$. Let $f : X \to [0, \infty)$ be a non-negative measurable function. Then for almost every point $x \in \{y | f(y) > 0\}$, the sequence $\sum_{n=1}^{\infty} f(T^n(x))$ diverges. Yang: To be checked

Proof. Yang: To be completed. □

3 Mean Convergence

Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$, and let $T : X \to X$ be a measure transformation. Let $E \in \mathcal{B}$ be a measurable set.

Given a point $x \in X$ and given $N \in \mathbb{N}$, for the ratio η the number of these points $\{n \in 1, \ldots, N : T^n(x) \in A\}$ to $N+1$, we want to find the limit of η as $N \to \infty$. Yang: To be checked

Let f denote the characteristic function of E, i.e.,

$$f(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Then the ratio η_N can be expressed as

$$\eta_N = \frac{1}{N+1} \sum_{n=0}^{N} f(T^n(x)).$$

We are interested in the limit

$$\lim_{N \to \infty} \eta_N = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} f(T^n(x)).$$

Yang: By "function on X" I mean a map $X \to \mathbb{C}$. To be checked

Let f be a measurable function on X and $T : X \to X$ be a measure transformation. Let $g : X \to \mathbb{C}$ be defined by $g(x) = f(T(x))$. We usually write $g = U_f$. More precisely, for every function f on X, we define $U_f = f \circ T$. This gives a linear operator $U : f \mapsto U_f$ on the space of measurable functions on X.

Proposition 8. Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$, and let $T : X \to X$ be a measure-preserving transformation. Let

$$L^1(X, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \int_X |f| d\mu < \infty\}.$$

Then U maps $L^1(X, \mu)$ to itself and is an isometry, i.e., for every $f \in L^1(X, \mu)$, we have $U_f \in L^1(X, \mu)$ and $\|U_f\|_1 = \|f\|_1$.

Proof. Yang: To be completed □

More generally, for $1 \le p \le \infty$, let

$$L^p(X, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\},$$

where

$$\|f\|_p = \begin{cases} \left(\int_X |f|^p d\mu\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x \in X} |f(x)|, & p = \infty. \end{cases}$$

Then $L^p(X,\mu)$ is a Banach space with the norm $\|\cdot\|_p$. Proposition 8 can be generalized to L^p.

Special case: $p = 2$. On $L^2(X,\mu)$, we can define an inner product by

$$\langle f, g\rangle = \int_X f\overline{g}d\mu.$$

Then $L^2(X,\mu)$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$.

Definition 9. Let $\mathcal{H}$ be a Hilbert space. A linear operator $U : \mathcal{H} \to \mathcal{H}$ is called an *isometry* if for every $x \in \mathcal{H}$, $\|Ux\| = \|x\|$. The operator U is called *unitary* if $< Ux, Uy >=< x, y >$ for every $x, y \in \mathcal{H}$.

Proposition 10. Let $U : \mathcal{H} \to \mathcal{H}$ be a linear operator on a Hilbert space $\mathcal{H}$. Then U is isometry if and only if $U^*U = \mathrm{id}_{\mathcal{H}}$, where U^* is the adjoint operator of U. Moreover, U is unitary if and only if $U^*U = UU^* = \mathrm{id}_{\mathcal{H}}$. In particular, U is unitary if and only if U is an isometry and surjective.

Proposition 11. Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$, and let $T : X \to X$ be a measure-preserving transformation which is also invertible. Then the operator $U : L^2(X,\mu) \to L^2(X,\mu)$ defined by $U_f = f \circ T$ is unitary.

Proof. Yang: To be completed □

Recall the limit we want to find:

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(T^n(x)).$$

This can be rewritten as

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} U^n f(x).$$

So let us focus on the limit of

$$\frac{1}{N+1} \sum_{n=0}^{N} U^n.$$

In the case $\mathcal{H} = \mathbb{C}$, the unitary operator $U : \mathbb{C} \to \mathbb{C}$ is just a multiplication by a complex number $e^{i\theta}$ with $\theta \in \mathbb{R}$. Then the limit

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} U^n$$

converges for all θ and the limit is given by

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} U^n = \begin{cases} 1, & \exp(i\theta) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In the case $\mathcal{H} = \mathbb{C}^n$, the unitary operator $U : \mathbb{C}^n \to \mathbb{C}^n$ can be represented by a unitary matrix. Then the limit

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} U^n = P$$

is the orthogonal projection onto the subspace of $\mathbb{C}^n$ spanned by the eigenvectors of U corresponding to the eigenvalue 1.

Theorem 12 (von Neumann Mean Ergodic Theorem)**.** Let $\mathcal{H}$ be a Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ be an isometry and P the orthogonal projection onto the subspace of $\mathcal{H}$ consisting of all vectors y such that $Uy = y$. Then the limit

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} U^n$$

converges to P. Yang: To be checked Yang: What is the projection exactly?

Proof. Yang: To be completed Yang: Hint: $\mathcal{H} = \ker(U - I) \oplus \overline{\mathrm{Im}(U - I)}$ and this is the orthogonal decomposition. □

Lemma 13. If $U : \mathcal{H} \to \mathcal{H}$ is an isometry, then $U\xi = \xi$ if and only if $U^*\xi = \xi$.

Proof. Yang: To be completed □

Theorem 14 (Birkhoff Ergodic Theorem)**.** Let $T : X \to X$ be a measure-preserving transformation on a measure space $(X, \mathcal{B}, \mu)$. Let $f \in L^1(X, \mu)$. Then

$$\frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

converges for almost every $x \in X$. The limit function $\tilde{f}$ is integrable and T-invariant, i.e., $\tilde{f}(Tx) = \tilde{f}(x)$ for almost every $x \in X$. If $\mu(X) < \infty$, then

$$\int_X f \mathrm{d}\mu = \int_X \tilde{f} \mathrm{d}\mu.$$

Proof. We may assume that f is real-valued, since we can apply the theorem to the real and imaginary parts of f separately.

Yang: To be continued... □

Theorem 15 (Maximal Ergodic Theorem)**.** Let $U : L^1_{\mathbb{R}} \to L^1_{\mathbb{R}}$ be a linear operator such that

(a) $\|Uf\|_1 \le \|f\|_1$ for all $f \in L^1$;

(b) $Uf \ge_{a.e.} 0$ if $f \ge_{a.e.} 0$.

Let $N > 0$ and $f \in L^1_{\mathbb{R}}(X, \mu)$. Define $f_0 = 0$ and $f_n = Uf_{n-1} + f$ for $n \ge 1$. Let $F_N = \max_{0 \le n \le N} f_n$ and $E = \{x \in X : F_N(x) > 0\}$. Then

$$\int_E f \mathrm{d}\mu \ge 0.$$

Proof. Yang: To be continued... □