The first definition

1 Setup

Definition 1. Let $(X, \mathcal{B}, \mu)$ be a measure space. A function $T : X \to X$ is called a measure transformation if for every measurable set $A \in \mathcal{B}$, the preimage $T^{-1}(A)$ is also in $\mathcal{B}$. Yang: To be checked

Example 2. Let $X = [0, 1)$ with the Borel σ-algebra $\mathcal{B}$ and the Lebesgue measure μ. Define $T : X \to X$ by $T(x) = 2x \mod 1$. Then T is a measure-preserving transformation but not invertible. Indeed, for any measurable set $A \in \mathcal{B}$, we have

$$\mu(T^{-1}(A)) = \mu(\{x \in [0, 1) : 2x \mod 1 \in A\}) = \frac{1}{2}\mu(A) + \frac{1}{2}\mu(A) = \mu(A).$$

However, T is not invertible since, for example, both $x = 0.1$ and $x = 0.6$ map to $T(x) = 0.2$. Yang: To be checked

Definition 3. Let $(X, \mathcal{B}, \mu)$ be a measure space and $T : X \to X$ be a measure transformation. Let $A \in \mathcal{B}$ be a measurable set. A point $x \in A$ is called *recurrent* if $T^n(x) \in A$ for some integer $n > 0$.

Theorem 4 (Poincaré Recurrence Theorem)**.** Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$, and let $T : X \to X$ be a measure-preserving transformation, i.e., for all $A \in \mathcal{B}$, $\mu(T^{-1}(A)) = \mu(A)$. Then for any measurable set A with $\mu(A) > 0$, the almost every point in A is recurrent.

Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$, and let $T : X \to X$ be a measure transformation. Let $E \in \mathcal{B}$ be a measurable set.

Given a point $x \in X$ and given $N \in \mathbb{N}$, for the ratio η the number of these points $\{n \in 1, ..., N : T^n(x) \in A\}$ to $N + 1$, we want to find the limit of η as $N \to \infty$. Yang: To be checked

Let f denote the characteristic function of E, i.e.,

$$f(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Then the ratio η_N can be expressed as

$$\eta_N = \frac{1}{N + 1} \sum_{n=0}^{N} f(T^n(x)).$$

We are interested in the limit

$$\lim_{N\to\infty} \eta_N = \lim_{N\to\infty} \frac{1}{N + 1} \sum_{n=0}^{N} f(T^n(x)).$$

Yang: By "function on X" I mean a map $X \to \mathbb{C}$. To be checked

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Let f be a measurable function on X and $T : X \to X$ be a measure transformation. Let $g : X \to \mathbb{C}$ be defined by $g(x) = f(T(x))$. We usually write $g = U_f$. More precisely, for every function f on X, we define $U_f = f \circ T$. This gives a linear operator $U : f \mapsto U_f$ on the space of measurable functions on X.

Proposition 5. Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$, and let $T : X \to X$ be a measure-preserving transformation. Let

$$L^1(X, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \int_X |f| d\mu < \infty\}.$$

Then U maps $L^1(X, \mu)$ to itself and is an isometry, i.e., for every $f \in L^1(X, \mu)$, we have $U_f \in L^1(X, \mu)$ and $\|U_f\|_1 = \|f\|_1$.

Proof. Yang: To be completed □

More generally, for $1 \leq p \leq \infty$, let

$$L^p(X, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\},$$

where

$$\|f\|_p = \begin{cases} \left(\int_X |f|^p d\mu\right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{x \in X} |f(x)|, & p = \infty. \end{cases}$$

Then $L^p(X, \mu)$ is a Banach space with the norm $\|\cdot\|_p$. Proposition 5 can be generalized to L^p.

Special case: $p = 2$. On $L^2(X, \mu)$, we can define an inner product by

$$\langle f, g \rangle = \int_X f \overline{g} d\mu.$$

Then $L^2(X, \mu)$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$.

Definition 6. Let $\mathcal{H}$ be a Hilbert space. A linear operator $U : \mathcal{H} \to \mathcal{H}$ is called an *isometry* if for every $x \in \mathcal{H}$, $\|Ux\| = \|x\|$. The operator U is called *unitary* if $< Ux, Uy >=< x, y >$ for every $x, y \in \mathcal{H}$.

Proposition 7. Let $U : \mathcal{H} \to \mathcal{H}$ be a linear operator on a Hilbert space $\mathcal{H}$. Then U is isometry if and only if $U^*U = \mathrm{id}_{\mathcal{H}}$, where U^* is the adjoint operator of U. Moreover, U is unitary if and only if $U^*U = UU^* = \mathrm{id}_{\mathcal{H}}$. In particular, U is unitary if and only if U is an isometry and surjective.

Proposition 8. Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure $\mu(X) < \infty$, and let $T : X \to X$ be a measure-preserving transformation which is also invertible. Then the operator $U : L^2(X, \mu) \to L^2(X, \mu)$ defined by $U_f = f \circ T$ is unitary.

Proof. Yang: To be completed □

Recall the limit we want to find:

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} f(T^n(x)).$$

This can be rewritten as

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}U^n f(x).$$

So let us focus on the limit of

$$\frac{1}{N+1}\sum_{n=0}^{N}U^n.$$

In the case $\mathcal{H}=\mathbb{C}$, the unitary operator $U:\mathbb{C}\to\mathbb{C}$ is just a multiplication by a complex number $e^{i\theta}$ with $\theta\in\mathbb{R}$. Then the limit

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}U^n$$

converges for all θ and the limit is given by

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}U^n=\begin{cases}1, & \exp(i\theta)=1,\\ 0, & \text{otherwise.}\end{cases}$$

In the case $\mathcal{H}=\mathbb{C}^n$, the unitary operator $U:\mathbb{C}^n\to\mathbb{C}^n$ can be represented by a unitary matrix. Then the limit

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}U^n=P$$

is the orthogonal projection onto the subspace of $\mathbb{C}^n$ spanned by the eigenvectors of U corresponding to the eigenvalue 1.

Theorem 9 (von Neumann Mean Ergodic Theorem)**.** Let $\mathcal{H}$ be a Hilbert space and $U:\mathcal{H}\to\mathcal{H}$ be an isometry and P the orthogonal projection onto the subspace of $\mathcal{H}$ consisting of all vectors y such that $Uy=y$. Then the limit

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}U^n$$

converges to P. Yang: To be checked Yang: What is the projection exactly?

Proof. Yang: To be completed Yang: Hint: $\mathcal{H}=\ker(U-I)\oplus\overline{\operatorname{Im}(U-I)}$ and this is the orthogonal decomposition. □

Lemma 10. If $U:\mathcal{H}\to\mathcal{H}$ is an isometry, then $U\xi=\xi$ if and only if $U^*\xi=\xi$.

Proof. Yang: To be completed □