Week 3

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Theorem 1 (Birkhoff Ergodic Theorem)**.** Let $T : X \to X$ be a measure-preserving transformation on a measure space $(X, \mathcal{B}, \mu)$. Let $f \in L^1(X, \mu)$. Then

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$$

converges for almost every $x \in X$. The limit function $\tilde{f}$ is integrable and T-invariant, i.e., $\tilde{f}(Tx) = \tilde{f}(x)$ for almost every $x \in X$. If $\mu(X) < \infty$, then

$$\int_X f \mathrm{d}\mu = \int_X \tilde{f} \mathrm{d}\mu.$$

Proof. We may assume that f is real-valued, since we can apply the theorem to the real and imaginary parts of f separately.

Yang: To be continued... □

Theorem 2 (Maximal Ergodic Theorem)**.** Let $U : L^1_{\mathbb{R}} \to L^1_{\mathbb{R}}$ be a linear operator such that

(a) $\|Uf\|_1 \le \|f\|_1$ for all $f \in L^1$;

(b) $Uf \ge_{a.e.} 0$ if $f \ge_{a.e.} 0$.

Let $N > 0$ and $f \in L^1_{\mathbb{R}}(X, \mu)$. Define $f_0 = 0$ and $f_n = Uf_{n-1} + f$ for $n \ge 1$. Let $F_N = \max_{0 \le n \le N} f_n$ and $E = \{x \in X : F_N(x) > 0\}$. Then

$$\int_E f \mathrm{d}\mu \ge 0.$$

Proof. Yang: To be continued... □

2 Ergodic

Definition 3. A measure-preserving transformation $T : X \to X$ on a measure space $(X, \mathcal{B}, \mu)$ is said to be *ergodic* if it has no nontrivial invariant sets, i.e., for every $A \in \mathcal{B}$ such that $T^{-1}A = A$, we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Theorem 4. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. Then the following statements are equivalent:

(a) T is ergodic;

(b) for each measurable set A, $m((T^{-1}A)\Delta A) = 0$ iff $\mu(A) = 0$ or $\mu(X \setminus A) = 0$;

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(c) for each measurable set A with $\mu(A) > 0$, we have $\mu\left(\bigcup_{n=0}^{\infty} T^{-n}A\right) = 1$;

(d) for each measurable sets A, B with $\mu(A)\mu(B) > 0$, there exists $n \in \mathbb{Z}^+$ such that $\mu(T^{-n}A \cap B) > 0$.

Theorem 5. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) < \infty$ and $T : X \to X$ be a measure-preserving transformation. Then TFAE:

(a) T is ergodic;

(b) for each measurable function f which is T-invariant, f is constant almost everywhere;

(c) for each $f \in L^1(X, \mu)$ which is T-invariant, f is constant almost everywhere;

(d) for each $f \in L^2(X, \mu)$ which is T-invariant, f is constant almost everywhere.

Proof. Yang: To be continued...

We show that (a) $\Rightarrow$ (b). □

Example 6. Let $X = \mathbb{N}$ and $\mathcal{B} = \mathcal{P}(\mathbb{N})$ with the counting measure. Define $T : \mathbb{N} \to \mathbb{N}$ by $Tx = x+1$. Then T is measure-preserving and ergodic.

Example 7. Let $X = \mathbb{N}$ and $\mathcal{B} = \mathcal{P}(\mathbb{N})$ with the counting measure. Define $T : \mathbb{N} \to \mathbb{N}$ by $Tx = x+2$. Then T is measure-preserving but not ergodic.

Example 8. Let $X = \mathbb{R}$ and $\mathcal{B} = \mathcal{B}(\mathbb{R})$ with the Lebesgue measure. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = x+1$. Then T is measure-preserving but not ergodic.

Example 9. Let $X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathcal{B} = \mathcal{B}(S^1)$ with the Lebesgue measure. Define $T : S^1 \to S^1$ by $Tx = \mathrm{e}^{2\pi i\theta}x$ where $\theta \in \mathbb{R}$. Then T is measure-preserving. Moreover, T is ergodic iff θ is irrational.