Ergodic

Definition 1. A measure-preserving transformation $T : X \to X$ on a measure space $(X, \mathcal{B}, \mu)$ is said to be *ergodic* if it has no nontrivial invariant sets, i.e., for every $A \in \mathcal{B}$ such that $T^{-1}A = A$, we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Theorem 2. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure-preserving transformation. Then the following statements are equivalent:

(a) T is ergodic;

(b) for each measurable set A, $m((T^{-1}A)\Delta A) = 0$ iff $\mu(A) = 0$ or $\mu(X \setminus A) = 0$;

(c) for each measurable set A with $\mu(A) > 0$, we have $\mu\left(\bigcup_{n=0}^{\infty} T^{-n}A\right) = 1$;

(d) for each measurable sets A, B with $\mu(A)\mu(B) > 0$, there exists $n \in \mathbb{Z}^+$ such that $\mu(T^{-n}A \cap B) > 0$.

Theorem 3. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) < \infty$ and $T : X \to X$ be a measure-preserving transformation. Then TFAE:

(a) T is ergodic;

(b) for each measurable function f which is T-invariant, f is constant almost everywhere;

(c) for each $f \in L^1(X, \mu)$ which is T-invariant, f is constant almost everywhere;

(d) for each $f \in L^2(X, \mu)$ which is T-invariant, f is constant almost everywhere.

Proof. Yang: To be continued...

We show that (a) $\Rightarrow$ (b). □

Example 4. Let $X = \mathbb{N}$ and $\mathcal{B} = \mathcal{P}(\mathbb{N})$ with the counting measure. Define $T : \mathbb{N} \to \mathbb{N}$ by $Tx = x+1$. Then T is measure-preserving and ergodic.

Example 5. Let $X = \mathbb{N}$ and $\mathcal{B} = \mathcal{P}(\mathbb{N})$ with the counting measure. Define $T : \mathbb{N} \to \mathbb{N}$ by $Tx = x+2$. Then T is measure-preserving but not ergodic.

Example 6. Let $X = \mathbb{R}$ and $\mathcal{B} = \mathcal{B}(\mathbb{R})$ with the Lebesgue measure. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = x+1$. Then T is measure-preserving but not ergodic.

Example 7. Let $X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathcal{B} = \mathcal{B}(S^1)$ with the Lebesgue measure. Define $T : S^1 \to S^1$ by $Tx = \mathrm{e}^{2\pi i\theta}x$ where $\theta \in \mathbb{R}$. Then T is measure-preserving. Moreover, T is ergodic iff θ is irrational.

Proposition 8. In Example 7, if θ is irrational, then T is ergodic.

Proof. Yang: To be continued..., By Fourier series □

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Example 9. Let $X = S^1$ and $\mathcal{B} = \mathcal{B}(S^1)$ with the Lebesgue measure. Define $T : S^1 \to S^1$ by $Tx = x^2$. Then T is measure-preserving and ergodic.

Yang: To be continued..., By Fourier series