Affinoid algebras

1 The first properties

Definition 1. Let $\mathbf{k}$ be a non-archimedean field. A banach $\mathbf{k}$-algebra A is called a *affinoid* $\mathbf{k}$*-algebra* if there exists an admissible surjective homomorphism

$$\varphi : \mathbf{k}\{\underline{T/r}\} \twoheadrightarrow A$$

for some $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$.
If one can choose $r_1 = \cdots = r_n = 1$, then we say that A is a *strict affinoid* $\mathbf{k}$*-algebra*.

Definition 2. Let $\mathbf{k}$ be a non-archimedean field. We define the *ring of restricted Laurent series* over $\mathbf{k}$ as

$$\mathbf{K}_r = \mathbf{L}_{\mathbf{k},r} = \left\{ \sum_{n\in\mathbb{Z}} a_n T^n : a_n \in \mathbf{k}, \lim_{|n|\to\infty} |a_n| r^n = 0 \right\}$$

equipped with the norm

$$\|f\| = \sup_{n\in\mathbb{Z}} |a_n| r^n.$$

Yang: Is $\mathbf{K}_r$ always a field? Yang: Do we have $\mathbf{L}_{\mathbf{k},r} = \mathrm{Frac}(\mathbf{k}\{T/r\})$?

Proposition 3. Let $\mathbf{k}$ be a non-archimedean field. If $r \notin \sqrt{|\mathbf{k}^\times|}$, then $\mathbf{K}_r$ is a complete non-archimedean field with non-trivial absolute value extending that of $\mathbf{k}$.

Proposition 4. Let A be an affinoid $\mathbf{k}$-algebra. Then A is noetherian, and every ideal of A is closed.

Proof. Yang: To be completed. □

Proposition 5. Let A be an affinoid $\mathbf{k}$-algebra. Then there exists a constant $C > 0$ and $N > 0$ such that for all $f \in A$ and $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

Proof. Yang: To be completed. □

Proposition 6. Let A be an affinoid $\mathbf{k}$-algebra. If and only if $\rho(f) \in \sqrt{|\mathbf{k}|}$ for all $f \in A$, then A is strict. Yang: To be complete.

Proof. Yang: To be completed. □

Date: January 17, 2026, **Author**: Tianle Yang, My Homepage

2 Noetherian normalization theorem

Theorem 7. Let A be an affinoid $\mathbf{k}$-algebra. Then there exists a finite injective homomorphism

$$\varphi : \mathbf{k}\{r_1^{-1}T_1, \dots, r_d^{-1}T_d\} \hookrightarrow A$$

for some $d \in \mathbb{N}$ and $r_1, \dots, r_d \in \mathbb{R}_{>0}$. Yang: To be checked.

3 Tate algebras and Weierstrass division

Definition 8. Let R be a non-archimedean banach ring and $r \in \mathbb{R}_{>0}$. A restricted power series $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in R\{\underline{T/r}\}$ is said to be *distinguished in the variable* T_n *of degree* d if

- $a_\alpha \in R$ is a unit for $\alpha = (0, \dots, 0, d)$;
- $\|a_\alpha\| r^\alpha < \|a_{(0,\dots,0,d)}\| r_n^d$ for all $\alpha_n < d$.

Yang: To be revised.

Proposition 9. Let R be a non-archimedean banach ring. An element $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in R\{\underline{T/r}\}$ is invertible if and only if a_0 is invertible in R and $\|a_0\| > \|a_\alpha\| r^\alpha$ for all $\alpha \neq 0$.

Proof. Multiplying by a_0^{-1}, we can reduce to the case $a_0 = 1$. Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha$ be the inverse of f in $R[[\underline{T}]]$. Then we have

$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \cdot \sum_{\beta \in \mathbb{N}^n} b_\beta T^\beta = \sum_{\gamma \in \mathbb{N}^n} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) T^\gamma = 1.$$

That is, for every $\gamma \neq 0 \in \mathbb{N}^n$,

$$b_\gamma = - \sum_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} a_\alpha b_\beta.$$

Let $A = \|f - 1\| < 1$. We show that for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq C_m$, we have $\|b_\alpha\| r^\alpha \leq A^m$. For $m = 0$, note that $b_0 = 1$. By induction on γ with respect to the total order $\leq_{\text{total}}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq A \max_{\beta <_{\text{total}} \gamma} \|b_\beta\| r^\beta \leq 1.$$

Suppose that the claim holds for m. There exists $D_{m+1} \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq D_{m+1}$, we have $\|a_\alpha\| r^\alpha \leq A^{m+1}$. Set $C_{m+1} = C_m + D_{m+1} + 1$. For any $\gamma \in \mathbb{N}^n$ with $|\gamma| \geq C_{m+1}$, we have

$$\|b_\gamma\| r^\gamma \leq \max_{\substack{\alpha + \beta = \gamma \\ \alpha \neq 0}} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta \leq \max\left\{A^{m+1}, A \cdot A^m\right\} = A^{m+1}$$

since either $|\alpha| \geq D_{m+1}$ or $|\beta| \geq C_m$. Thus by induction, we have $\|b_\alpha\| r^\alpha \to 0$ as $|\alpha| \to +\infty$. It follows that $g \in R\{\underline{T/r}\}$. $\square$

Theorem 10 (Weierstrass preparation theorem)**.** Let $\mathbf{k}$ be a complete non-archimedean field. Let $f \in \mathbf{k}\{\underline{T/r}\}$ be a restricted power series that is distinguished in the variable T_n of degree d, i.e.,

$$f = \sum_{\alpha \in \mathbb{N}^{n-1}} a_\alpha T^\alpha + \sum_{\alpha_n \geq 1} a_\alpha T^\alpha$$

with $a_{(0,\ldots,0,d)}$ being a unit in $\mathbf{k}\{\underline{T/r}\}$ and $\|a_\alpha\| r^\alpha < \|a_{(0,\ldots,0,d)}\| r_n^d$ for all $\alpha_n < d$. Then there exists a unique monic polynomial $P \in \mathbf{k}\{\underline{T/r}\}[T_n]$ of degree d in T_n and a unique unit $U \in \mathbf{k}\{\underline{T/r}\}$ such that

$$f = P \cdot U.$$

Yang: To be checked.

Theorem 11 (Weierstrass division theorem)**.** Let $\mathbf{k}$ be a complete non-archimedean field. Let $f \in \mathbf{k}\{\underline{T/r}\}$ be a restricted power series that is distinguished in the variable T_n of degree d. Then for every $g \in \mathbf{k}\{\underline{T/r}\}$, there exists a unique $Q \in \mathbf{k}\{\underline{T/r}\}$ and a unique polynomial $R \in \mathbf{k}\{\underline{T/r}\}[T_n]$ of degree less than d in T_n such that

$$g = Q \cdot f + R.$$

Yang: To be checked.

Proposition 12. Let $\mathbf{k}$ be a complete non-archimedean field and $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$. Then

$$\operatorname{Spec} \mathbf{k}\{\underline{T/r}\} = \{\},$$

where

Appendix