Valuations on fields

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Contents

1 Valuation fields

1.1 Absolute values and completion

Definition 1.1. Let $\mathbf{k}$ be a field. An *absolute value* on $\mathbf{k}$ is a function $\|\cdot\| : \mathbf{k} \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|xy\| = \|x\| \cdot \|y\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

A field $\mathbf{k}$ equipped with an absolute value $\|\cdot\|$ is called a *valuation field.*

Remark 1.2. Let $\mathbf{k}$ be a field. Recall that a (additive) valuation on $\mathbf{k}$ is a function $v : \mathbf{k}^\times \to \mathbb{R}$ such that

- $\forall x, y \in \mathbf{k}^\times, v(xy) = v(x) + v(y)$;

- $\forall x, y \in \mathbf{k}^\times, v(x + y) \geq \min\{v(x), v(y)\}$.

We can extend v to the whole field $\mathbf{k}$ by defining $v(0) = +\infty$. Fix a real number $\varepsilon \in (0, 1)$. Then v induces an absolute value $|\cdot|_v : \mathbf{k} \to \mathbb{R}_+$ defined by $|x|_v = \varepsilon^{v(x)}$ for each $x \in \mathbf{k}$.

The valuation v defined above is called an *additive valuation*. And an absolute value $|\cdot|$ on $\mathbf{k}$ is called a *multiplicative valuation*. In this note, the term *valuation* may refer to either an additive valuation or a multiplicative valuation, depending on the context.

Example 1.3. Let $\mathbf{k}$ be a field. The *trivial absolute value* on $\mathbf{k}$ is defined as

$$\|x\| := \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

Definition 1.4. The *(multiplicative) valuation group* of a valuation field $(\mathbf{k}, \|\cdot\|)$ is defined as the subgroup of $\mathbb{R}_{>0}$ given by

$$|\mathbf{k}^\times| := \{\|x\| : x \in \mathbf{k}^\times\}.$$

We use the notation $\sqrt{|\mathbf{k}^\times|}$ to denote the set $\{\|x\|^{1/n} : x \in \mathbf{k}^\times, n \in \mathbb{Z}_{>0}\}$.

Note that an absolute value $\|\cdot\|$ is non-trivial if and only if its valuation group $|\mathbf{k}^\times|$ is not equal to $\{1\}$.

Definition 1.5. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *complete* if the metric $d(x, y) := \|x - y\|$ makes $\mathbf{k}$ a complete metric space.

Lemma 1.6. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field and $(\widehat{\mathbf{k}}, \|\cdot\|)$ its completion as a metric space. Then the operations of addition and multiplication on $\mathbf{k}$ can be extended to $\widehat{\mathbf{k}}$ uniquely, making $(\widehat{\mathbf{k}}, \|\cdot\|)$ a complete valuation field containing $\mathbf{k}$ as a dense subfield.

Proof. Simple analysis. $\square$

Proposition 1.7. Let $(\mathbf{k}, \|\cdot\|)$ be a complete valuation field with non-trivial absolute value. Then $\mathbf{k}$ is uncountable.

Proof. Since the absolute value $\|\cdot\|$ is non-trivial, we can construct a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathbf{k}$ inductively such that $\|x_n\| < \|x_{n-1}\|/2$ for any $n \geq 1$ and $\|x_0\| < 1$. Then there is an injective map from $\mathbb{N}^{\{0,1\}}$ to $\mathbf{k}$ defined by

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n x_n, \quad a_n \in \{0, 1\}.$$

Since $\|x_n\| < 2^{-n}$, the series $\sum_{n=1}^\infty a_n x_n$ converges in $\mathbf{k}$. Note $\|x_n\| > \|\sum_{m \geq n} x_m\|$ for each n, we have that the map is injective. Thus $\mathbf{k}$ is uncountable. $\square$

Unlike the real number field $\mathbb{R}$, even a valuation field is complete, we can not expect the theorem of nested intervals to hold.

Definition 1.8. A valuation field $(\mathbf{k}, \|\cdot\|)$ is called *spherically complete* if every decreasing sequence of closed balls in $\mathbf{k}$ has a non-empty intersection.

Example 1.9. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete, see Yang: to be added.

Example 1.10. Let $|\cdot|_\infty$ be the usual absolute value on the field $\mathbb{Q}$ of rational numbers. Then $(\mathbb{Q}, |\cdot|_\infty)$ is a valuation field. Its completion is the field $\mathbb{R}$ of real numbers equipped with the usual absolute value.

Example 1.11. Let p be a prime number. For any non-zero rational number $x \in \mathbb{Q}$, we can write it as $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|x|_p := \begin{cases} 0, & x = 0; \\ p^{-n}, & x = p^n \frac{a}{b} \text{ as above.} \end{cases}$$

Then $(\mathbb{Q}, |\cdot|_p)$ is a valuation field. Its completion is the field

$$\mathbb{Q}_p = \left\{ \sum_{n=k}^{+\infty} a_n p^n : k \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \right\}$$

of p-adic numbers equipped with the p-adic absolute value; see Yang: to be added..

Definition 1.12. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. We say that $\mathbf{k}$ is *non-archimedean* if its absolute value $\|\cdot\|$ satisfies the *strong triangle inequality*:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in \mathbf{k}.$$

Otherwise, we say that $\mathbf{k}$ is *archimedean*.

Proposition 1.13. Let $(\mathbf{k}, \|\cdot\|)$ be a valuation field. Then $\mathbf{k}$ is archimedean if and only if the set $\{\|n \cdot 1\| : n \in \mathbb{Z}\}$ is unbounded.

Proof. Yang: To be added. $\qquad\square$

1.2 Places on a field

Definition 1.14. Let $\mathbf{k}$ be a field. Two absolute values $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbf{k}$ are said to be *equivalent* if there exists a real number $c \in (0, \infty)$ such that

$$\|x\|_1 = \|x\|_2^c, \quad \forall x \in \mathbf{k}.$$

Note that equivalent absolute values induce the same topology on the field $\mathbf{k}$. Moreover, the following lemma shows that the converse is also true.

Lemma 1.15. Let $\mathbf{k}$ be a field and $\|\cdot\|_1, \|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

  (a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} \colon \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} \colon \|x\|_2 < 1\}$ are the same.

Proof. The implications (a) $\Rightarrow$ (b) is obvious. Now we prove (b) $\Rightarrow$ (c). For any $x \in D_1$, we have $x^n \to 0$ as $n \to \infty$ under the absolute value $\|\cdot\|_1$ and thus under $\|\cdot\|_2$. Therefore, $\|x\|_2^n \to 0$ as $n \to \infty$, which implies that $\|x\|_2 < 1$, i.e., $x \in D_2$. Similarly, we can prove that $D_2 \subseteq D_1$.

Finally, we prove (c) $\Rightarrow$ (a). If $\|\cdot\|_1$ is trivial, then $D_1 = \{0\}$ and thus $\|\cdot\|_2$ is also trivial. In this case, they are equivalent. Suppose that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-trivial. Pick any $x, y \notin D_1 = D_2$. Then there exist real numbers $\alpha, \beta > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ and $\|y\|_1 = \|y\|_2^\beta$. Suppose the contrary that $\alpha \neq \beta$. Consider the domain $\Lambda \subseteq \mathbb{Z}^2$ defined by

$$\begin{cases} n \log \|x\|_2 < m \log \|y\|_2; \\ n\alpha \log \|x\|_2 > m\beta \log \|y\|_2. \end{cases}$$

Since $\alpha \neq \beta$, the two lines defined by the equalities are not parallel. Thus Λ is non-empty. Pick $(n, m) \in \Lambda$ and set $z := x^n y^{-m}$. Then we have $\|z\|_2 < 1$ and $\|z\|_1 > 1$, a contradiction. $\square$

Definition 1.16. Let $\mathbf{k}$ be a field. A *place* on $\mathbf{k}$ is an equivalence class of absolute values on $\mathbf{k}$. We denote the set of all places on $\mathbf{k}$ by $\mathrm{Pl}_{\mathbf{k}}$.

Theorem 1.17. Let $(\mathbf{k}, \|\cdot\|)$ be an archimedean complete valuation field. Then $\mathbf{k}$ is isomorphic to either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$ equipped with the usual absolute value. Yang: To be revised.

Proof. Yang: To be added. $\square$

Theorem 1.18 (Ostrowski's theorem)**.** Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to either the usual absolute value $|\cdot|_\infty$ or a p-adic absolute value $|\cdot|_p$ for some prime number p. Yang: To be revised.

Proof. Yang: To be added. $\square$

2 Non-archimedean valuations

2.1 Topology: Ultra-metric space

We will use $B(x, r)$ (resp. $E(x, r)$) to denote the open ball (resp. closed ball) with center x and radius r.

Definition 2.1. A metric space (X, d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x, z) \le \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.$$

Remark 2.2. The term *ultra-metric space* should be translated into Chinese as "奥特度量空间". There is no special reason for this translation, except that I insist on using "奥特" to translate

> "ultra".

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x,y) := \|x - y\|$ on $\mathbf{k}$ makes $(\mathbf{k}, d)$ an ultra-metric space.

Proposition 2.3. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x,y), d(y,z), d(z,x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x,y) \geq d(y,z)$. By the strong triangle inequality, we have

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} = d(x,y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x,y) \leq \max\{d(x,z), d(z,y)\} = \max\{d(x,z), d(y,z)\} \leq d(x,y).$$

This shows that $d(x,y) = \max\{d(x,z), d(y,z)\}$. Thus either $d(x,z) = d(x,y) \geq d(y,z)$ or $d(y,z) = d(x,y) \geq d(x,z)$. $\qquad\square$

Proposition 2.4. Let (X, d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1, 2$. If $D_1 \cap D_2 \neq \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1, 2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1, x_2) \leq \max\{d(x_1, y), d(y, x_2)\}.$$

Without loss of generality, we may assume that $d(x_1, x_2) \leq d(x_1, y)$. It follows that $x_2 \in D_1$ since $d(x_1, y) < r_1$ (or $\leq r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1, z) > d(x_1, x_2)$. Then by Proposition 2.3,

$$r_1 \leq d(x_1, z) = d(x_2, z) \leq r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2, w) \leq \max\{d(x_2, x_1), d(x_1, w)\} \leq r_1 \leq r_2.$$

Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. $\qquad\square$

Proposition 2.5. Let (X, d) be an ultra-metric space. Then both $B(x, r)$ and $E(x, r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x, r) := \{y \in X \mid d(x, y) = r\}$ is open in X. Note that if $y \in S(x, r)$, then for any $r' < r$, we have $B(y, r') \cap E(x, r) \neq \varnothing$ and $x \in E(x, r) \setminus B(y, r')$. Thus by Proposition 2.4, we have $B(y, r') \subseteq E(x, r)$. If $B(y, r') \cap B(x, r) \neq \varnothing$, then by Proposition 2.4 again, we have $B(y, r') \subseteq B(x, r)$. However, $y \in B(y, r') \setminus B(x, r)$, a contradiction. Thus $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$. It yields that $S(x, r) = \bigcup_{y \in S(x, r)} B(y, r/2)$ is open in X.

Since $E(x, r) = B(x, r) \cup S(x, r)$ and $B(x, r) = E(x, r) \setminus S(x, r)$, both $B(x, r)$ and $E(x, r)$ are open and closed in X. $\qquad\square$

Corollary 2.6. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x, y) > 0$. Consider the open ball $B(x, r/2)$. By Proposition 2.5, $B(x, r/2)$ is both open and closed in X. Thus $B(x, r/2) \cap S$ is both open and closed in S, however, it is non-empty and not equal to S since it contains x but not y. This shows that S is disconnected. $\square$

Proposition 2.7. Let (X, d) be an ultra-metric space. A sequence $\{x_n\}$ in X is cauchy if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Proof. The necessity is true for all metric spaces. Suppose that $d(x_n, x) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \varepsilon$ for all $n \geq N$. For any $m, n \geq N$ with $m < n$, by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \ldots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that $\{x_n\}$ is a cauchy sequence. $\square$

2.2 Algebra: ring of integers and residue field

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 2.8. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\mathscr{k}_\mathbf{k} := \widetilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Set $I_{r,<} := B(0, r)$ and $I_{r,\leq} := E(0, r)$ for each $r \in [0, 1]$.

Proposition 2.9. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers $\mathbf{k}^\circ$. Conversely, any ideal of $\mathbf{k}^\circ$ is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0, 1)$.

Proof. Let I be an ideal of $\mathbf{k}^\circ$. Set $r = \sup\{|a| : a \in I\}$ (resp. $r = \max\{|a| : a \in I\}$ when the maximum exists). Then, by definition, we have $I \subset I_{r,<}$ (resp. $I \subset I_{r,\leq}$). For every $x \in \mathbf{k}^\circ$ with $|x| < r$ (resp. $|x| \leq r$), there exists $a \in I$ such that $|x| \leq |a|$. Thus, $|x/a| \leq 1$ and so $x/a \in \mathbf{k}^\circ$. Since I is an ideal, we have $x = (x/a)a \in I$. Therefore, $I_{r,<} \subset I$ (resp. $I_{r,\leq} \subset I$). $\square$

Proposition 2.10. Let I_r be either $I_{r,<}$ or $I_{r,\leq}$ for each $r \in (0, 1)$. Suppose $\{r_n \in (0, 1)\}_{n \in \mathbb{N}}$ is a decreasing sequence converging to 0. Then the completion $\widehat{\mathbf{k}}$ of $\mathbf{k}$ is isomorphic to the projective

limit
$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}.$$

Proof. For every $x \in \widehat{\mathbf{k}}^\circ$, there exists a cauchy sequence $\{x_m\}_{m \in \mathbb{N}}$ in $\mathbf{k}^\circ$ converging to x. Since $\{r_n\}_{n \in \mathbb{N}}$ converges to 0, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that for all $m, m' \geq M_n$, we have $|x_m - x_{m'}| < r_n$. Thus, the sequence $\{x_m + I_{r_n}\}_{m \in \mathbb{N}}$ is eventually constant in $\mathbf{k}^\circ / I_{r_n}$. Define a map

$$\Phi : \widehat{\mathbf{k}}^\circ \to \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}, \quad x \mapsto \left(\lim_{m \to \infty} x_m + I_{r_n} \right)_{n \in \mathbb{N}}.$$

It is straightforward to verify that Φ is a well-defined ring homomorphism.

Conversely, for every $(a_n + I_{r_n})_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n}$, we can choose a representative $a_n \in \mathbf{k}^\circ$ for each n. We claim that the sequence $\{a_n\}_{n \in \mathbb{N}}$ is a cauchy sequence in $\mathbf{k}^\circ$. Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_N < \varepsilon$. For all $m, n \geq N$, since $a_n + I_{r_n}$ maps to $a_m + I_{r_m}$ under the natural projection, we have $|a_n - a_m| < r_N < \varepsilon$. Thus, $\{a_n\}_{n \in \mathbb{N}}$ converges to some $x \in \widehat{\mathbf{k}}^\circ$. Easily see that the limit x is independent of the choice of representatives $\{a_n\}_{n \in \mathbb{N}}$. This gives a map

$$\Psi : \varprojlim_{n \in \mathbb{N}} \mathbf{k}^\circ / I_{r_n} \to \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n \in \mathbb{N}} \mapsto \lim_{n \to \infty} a_n.$$

Direct verification shows that $\Psi = \Phi^{-1}$. $\qquad\square$

Corollary 2.11. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the residue field $\Bbbk_{\widehat{\mathbf{k}}} \cong \Bbbk_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$.

Corollary 2.12. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the valuation group $|\widehat{\mathbf{k}}^\times|$ of $\widehat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^\times|$ of $\mathbf{k}$.

Proof. Note that

$$
\begin{aligned}
r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\
&\iff \widehat{\mathbf{k}}^\circ / I_{r,<} \to \widehat{\mathbf{k}}^\circ / I_{r,\leq} \text{ is not an isomorphism} \\
&\iff \mathbf{k}^\circ / I_{r,<} \to \mathbf{k}^\circ / I_{r,\leq} \text{ is not an isomorphism} \\
&\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\
&\iff r \in |\mathbf{k}^\times|.
\end{aligned}
$$

$\qquad\square$

Proposition 2.13. Let $\mathbf{k}$ be a non-archimedean field with non-trivial valuation. Then $\mathbf{k}^\circ$ is totally bounded iff $\mathbf{k}^\circ / I_{r,<}$ and $\mathbf{k}^\circ / I_{r,\leq}$ are finite for each $r \in [0, 1]$. Moreover, if $\mathbf{k}$ is complete, then it is locally compact iff $\mathbf{k}^\circ / I_r$ is finite for each $r \in (0, 1)$.

Slogan *"Locally compact $\iff$ pro-finite."*

Proof. We just prove the case for $I_r = I_{r,<}$. The case for $I_r = I_{r,\leq}$ is similar.

Suppose that $\mathbf{k}^\circ/I_r$ is finite for each $r \in [0,1]$. Then for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that $r < \varepsilon$ and $\mathbf{k}^\circ/I_r$ is finite. Let $\{a_1 + I_r, \ldots, a_n + I_r\}$ be the complete set of representatives of $\mathbf{k}^\circ/I_r$. Then the balls $B(a_i, r)$ for $i = 1, \ldots, n$ cover $\mathbf{k}^\circ$.

Conversely, suppose that $\mathbf{k}^\circ/I_r$ is infinite for some $r \in [0,1]$. Then there exists an infinite set $\{a_n\}$ with $|a_n| \in [r,1]$ such that their images in $\mathbf{k}^\circ/I_r$ are distinct. In particular, for every $m \neq n$, we have $|a_n - a_m| \geq r$. Any subsequence of $\{a_n\}$ is not cauchy. Thus, $\mathbf{k}^\circ$ is not totally bounded. $\square$

Proposition 2.14. The ring $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field.

Proof. Note that $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$ is a multiplicative subgroup. If $\mathbf{k}$ is not a discrete valuation field, then $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. In particular, there exists a strictly ascending sequence $r_n \in |\mathbf{k}^\times| \cap (0,1)$. Then the ideals $I_{r_n,\leq}$ form a strictly ascending chain of ideals in $\mathbf{k}^\circ$.

The converse is standard since now $\mathbf{k}^\circ$ is a discrete valuation ring. $\square$

Proposition 2.15. Let $\mathbf{k}$ be a complete non-archimedean field. Then $\mathbf{k}$ is locally compact iff $\mathbf{k}$ is a discrete valuation field and its residue field $\mathscr{k}_\mathbf{k}$ is finite.

Proof. The necessity follows from Proposition 2.13. For the sufficiency, suppose that $\mathbf{k}$ is a discrete valuation field whose residue field $\mathscr{k}_\mathbf{k}$ is finite. Let $\pi \in \mathbf{k}^\circ$ be a uniformizer. We only need to show that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite for each $n \in \mathbb{N}$. Note that there is an isomorphism

$$\pi^{n-1}\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ \cong \mathscr{k}_\mathbf{k}, \quad x + \pi^n\mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on n, we conclude that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite. $\square$

2.3 Hensel's Lemma

Theorem 2.16 (Hensel's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_\mathbf{k}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \mathscr{k}_\mathbf{k}[T]$ are monic polynomials that are coprime in $\mathscr{k}_\mathbf{k}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\mathscr{k}_\mathbf{k}[T]$ are $g(T), h(T)$ respectively.

Proof. Since $\gcd(g,h) = 1$ in $\mathscr{k}_\mathbf{k}[T]$, there exist polynomials $u(T), v(T) \in \mathscr{k}_\mathbf{k}[T]$ such that $ug + vh = 1$ and $\deg u < \deg h, \deg v < \deg g$. Choose lifts $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$ of $g(T), h(T), u(T), v(T)$ respectively preserving their degrees such that G_0 and H_0 are monic. Then there exist $r < 1$ such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \mod I_r, \quad F(T) - G_0(T)H_0(T) \equiv 0 \mod I_r,$$

where $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$.

We will construct a sequence of monic polynomials $\{G_n(T)\}_{n\in\mathbb{N}}$ and $\{H_n(T)\}_{n\in\mathbb{N}}$ in $\mathbf{k}^\circ[T]$ such that for each $n \in \mathbb{N}$,

$$G_n(T) \equiv G_{n-1}(T) \mod I_{r^n}, \quad H_n(T) \equiv H_{n-1}(T) \mod I_{r^n},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \mod I_{r^{n+1}}.$$

If we have such sequences, then their coefficients converge in the complete ring $\mathbf{k}^\circ$. Let $G(T)$ and $H(T)$ be the limits of $\{G_n(T)\}$ and $\{H_n(T)\}$ respectively. Then we have $F(T) = G(T)H(T)$ and the reductions of $G(T), H(T)$ in $\mathscr{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

The case $n = 0$ is done by the above construction. Now suppose that we have constructed $G_n(T)$ and $H_n(T)$ for some $n \geq 0$. Since $G_n - G_0 \equiv 0 \mod I_r$ and $H_n - H_0 \equiv 0 \mod I_r$, we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \mod I_r.$$

Set $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$ and $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$. Then we have

$$(G_n + \epsilon_n)(H_n + \delta_n) - F_n = G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n$$
$$= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T].$$

Thus, we can set
$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

This finishes the induction. $\qquad\square$

Corollary 2.17. Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $a \in \mathscr{k}_{\mathbf{k}}$. Then there exists a root $\alpha \in \mathbf{k}^\circ$ of $F(T)$ whose reduction is a.

Proof. Since a is a simple root of $f(T)$, we have the factorization $f(T) = (T-a)h(T)$ for some monic polynomial $h(T) \in \mathscr{k}_{\mathbf{k}}[T]$ with $h(a) \neq 0$. Then the result follows from Theorem 2.16. $\qquad\square$

2.4 Newton polygons

Yang: To be filled.

3 Finite field extensions

3.1 Finite-dimensional vector space

Definition 3.1. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x, y \in V$ and $a \in \mathbf{k}$:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|ax\| = |a| \cdot \|x\|$;

(c) $\|x + y\| \leq \|x\| + \|y\|$.

Example 3.2. Let $\mathbf{k}$ be a valuation field and V a finite-dimensional vector space over $\mathbf{k}$ with basis $\{e_1, e_2, \ldots, e_n\}$. For any $x = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \in V$, define

$$\|x\|_{\max} := \max_{1 \leq i \leq n} |a_i|.$$

Then $\|\cdot\|_{\max}$ is a norm on V, called the *maximal norm* with respect to the basis $\{e_1, e_2, \ldots, e_n\}$.

Example 3.3. Setting as in Example 3.2, for any $x = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \in V$, define

$$\|x\|_1 := |a_1| + |a_2| + \cdots + |a_n|.$$

Then $\|\cdot\|_1$ is also a norm on V.

Definition 3.4. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be *equivalent* if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in V$,

$$C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1.$$

Lemma 3.5. Let $\mathbf{k}$ be a valuation field and V a vector space over $\mathbf{k}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if and only if they induce the same topology on V.

Proof. The sufficiency is clear. Now suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on V. Hence the unit open ball with respect to $\|\cdot\|_1$ contains a unit open ball with respect to $\|\cdot\|_2$. That is,

$$\{x \in V : \|x\|_1 < 1\} \supseteq \{x \in V : \|x\|_2 < C\}.$$

Then for every $x \in V$ with $\|x\|_1 = 1$, we have $\|x\|_2 \geq C = C\|x\|_1$. By scaling, we get that for every $x \in V$,

$$\|x\|_2 \geq C\|x\|_1.$$

Similar for the other direction, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. $\square$

Proposition 3.6. Let V be a normed finite-dimensional vector space over a complete valuation field $\mathbf{k}$. Then V is complete.

Proof. Yang: To be added. $\square$

Theorem 3.7. Let V be a finite-dimensional vector space over a complete field $\mathbf{k}$. Then all norms on V are equivalent.

Proof. Fix a basis $\{e_1, e_2, \ldots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis as in Example 3.2. Let $\|\cdot\|$ be any norm on V. It suffices to show that $\|\cdot\|$ and $\|\cdot\|_{\max}$ are equivalent. First we have

$$\|y\| \leq \sum_{i=1}^{n} |a_i| \|e_i\| \leq \left(\sum_{i=1}^{n} \|e_i\| \right) \|y\|_{\max}$$

for any $y = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \in V$. It remains to show that there exists a constant $C > 0$ such

that for any $y \in V$,

$$\|y\|_{\max} \leq C\|y\|.$$

Yang: To be added.   □

Remark 3.8. If the base field $\mathbf{k}$ is not complete, then Theorem 3.7 may fail. For example, let $\mathbf{k} = \mathbb{Q}$ with the usual absolute value, and let $V = \mathbb{Q}[\alpha]$ with $\alpha^2 - \alpha - 1 = 0$. There are two embeddings of V into $\mathbb{R}$:

$$\iota_1 : a + b\alpha \mapsto a + b\frac{1 + \sqrt{5}}{2}, \quad \iota_2 : a + b\alpha \mapsto a + b\frac{1 - \sqrt{5}}{2}.$$

Define two norms on V by

$$\|x\|_1 := |\iota_1(x)|, \quad \|x\|_2 := |\iota_2(x)|,$$

where $|\cdot|$ is the usual absolute value on $\mathbb{R}$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent since $\iota_2(\alpha^n) \to 0$ as $n \to \infty$ while $\iota_1(\alpha^n) \to \infty$.

The following lemma is a classical result in functional analysis, which will be used in the next subsection.

Lemma 3.9. Let $\mathbf{k}$ be a complete field and V a normed finite-dimensional vector space over $\mathbf{k}$. Then

$$\|\cdot\| : \mathrm{End}_{\mathbf{k}}(V) \to \mathbb{R}_{\geq 0}, \quad T \mapsto \sup_{x \in V \setminus \{0\}} \frac{\|T(x)\|}{\|x\|}$$

defines a norm on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ satisfying

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad \forall A, B \in \mathrm{End}_{\mathbf{k}}(V).$$

Proof. First we show the existence of the supremum, i.e., there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\| \leq C\|x\|$. Fix a basis $\{e_1, e_2, \ldots, e_n\}$ of V and let $\|\cdot\|_{\max}$ be the maximal norm with respect to this basis. Since all norms on V are bounded by each other by Theorem 3.7, we only need to show that there exists $C > 0$ such that for all $x \in V \setminus \{0\}$, $\|T(x)\|_1 \leq C\|x\|_{\max}$. Write $T(e_i) = \sum_{j=1}^{n} a_{ij} e_j$ for $1 \leq i \leq n$. For any $x = \sum_{i=1}^{n} x_i e_i \in V$, we have

$$\|T(x)\|_1 = \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} x_i \right) e_j \right\|_1 = \sum_{j=1}^{n} \left| \sum_{i=1}^{n} a_{ij} x_i \right| \leq \left(\sum_{1 \leq i, j \leq n} |a_{ij}| \right) \|x\|_{\max}.$$

Thus the supremum is finite.

The linearity and positive-definiteness of $\|\cdot\|$ are clear. It remains to show the triangle inequality and sub-multiplicativity. For any $A, B \in \mathrm{End}_{\mathbf{k}}(V)$, we have

$$\frac{\|(A + B)(x)\|}{\|x\|} = \frac{\|A(x)\|}{\|x\|} + \frac{\|B(x)\|}{\|x\|} \leq \|A\| + \|B\|.$$

Taking supremum over all $x \in V \setminus \{0\}$ gives $\|A + B\| \leq \|A\| + \|B\|$. We have

$$\|AB(x)\| \leq \|A\| \cdot \|B(x)\| \leq \|A\| \cdot \|B\| \cdot \|x\|$$

and hence $\|AB(x)\|/\|x\| \leq \|A\| \cdot \|B\|$. Taking supremum we get $\|AB\| \leq \|A\| \cdot \|B\|$.   □

3.2 Finite field extensions

Lemma 3.10. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then there exists an absolute value on $\mathbf{l}$ extending the absolute value on $\mathbf{k}$.

Proof. Fix a norm $\|\cdot\|_V$ on the $\mathbf{k}$-vector space $V = \mathbf{l}$. The norm $\|\cdot\|_V$ induces an operator norm $\|\cdot\|_{\mathrm{op}}$ on the $\mathbf{k}$-vector space $\mathrm{End}_{\mathbf{k}}(V)$ as in Lemma 3.9. For any $a \in \mathbf{l}$, let $\mu_a \in \mathrm{End}_{\mathbf{k}}(V)$ be the $\mathbf{k}$-linear map defined by multiplication by a. Note that $a \mapsto \mu_a$ gives a embedding of $\mathbf{k}$-algebras and if $a \in \mathbf{k}$, $\|\mu_a\|_{\mathrm{op}} = \|a\|_{\mathbf{k}}$. Thus the restriction of $\|\cdot\|_{\mathrm{op}}$ to $\mathbf{l}$ gives an norm on $\mathbf{l}$ extending that on $\mathbf{k}$. The normed ring $(\mathbf{l}, \|\cdot\|_{\mathrm{op}})$ is a Banach ring since it is a finite-dimensional vector space over the complete field $\mathbf{k}$. By **??**, there exists a multiplicative seminorm $\|\cdot\|_{\mathbf{l}}$ on $\mathbf{l}$ bounded by $\|\cdot\|_{\mathrm{op}}$. In particular, $\|\cdot\|_{\mathbf{l}}$ is bounded by $\|\cdot\|_{\mathbf{k}}$ on $\mathbf{k}$. On a field, if one norm is bounded by another norm, then they must be equal (consider the inverse elements). Thus $\|\cdot\|_{\mathbf{l}}$ extends the absolute value on $\mathbf{k}$. $\square$

Theorem 3.11. Let $\mathbf{k}$ be a complete field and $\mathbf{l}$ a finite extension of $\mathbf{k}$. Then the absolute value on $\mathbf{l}$ which extends the absolute value on $\mathbf{k}$ is uniquely determined by the absolute value on $\mathbf{k}$. Furthermore, we have

$$\|\cdot\|_{\mathbf{l}} = \|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n},$$

where $n = [\mathbf{l} : \mathbf{k}]$ and $N_{\mathbf{l}/\mathbf{k}}$ is the norm map from $\mathbf{l}$ to $\mathbf{k}$.

Proof. Let $\|\cdot\|_{\mathbf{l}}$ be arbitrary absolute value on $\mathbf{l}$ extending that on $\mathbf{k}$. We will show that $\|\cdot\|_{\mathbf{l}}$ must be equal to $\|N_{\mathbf{l}/\mathbf{k}}(\cdot)\|_{\mathbf{k}}^{1/n}$. For any $a \in \mathbf{l}$, set $b = a^n/N_{\mathbf{l}/\mathbf{k}}(a) \in \mathbf{l}$. Then $N_{\mathbf{l}/\mathbf{k}}(b) = 1$ and

$$\|b\|_{\mathbf{l}} = \frac{\|a\|_{\mathbf{l}}^n}{\|N_{\mathbf{l}/\mathbf{k}}(a)\|_{\mathbf{k}}}.$$

Thus it suffices to show that $\|b\|_{\mathbf{l}} = 1$ whenever $N_{\mathbf{l}/\mathbf{k}}(b) = 1$.

Note that the norm map $N_{\mathbf{l}/\mathbf{k}} : \mathbf{l} \to \mathbf{k}$ is the determinant of the $\mathbf{k}$-linear map $\mu_b \in \mathrm{End}_{\mathbf{k}}(V)$ defined by multiplication by b. Hence it is continuous on $\mathbf{l}$ (since it is a polynomial in the entries of the matrix representation). If $\|b\|_{\mathbf{l}} < 1$, then $\|b^m\|_{\mathbf{l}} \to 0$ as $m \to \infty$. Thus $N_{\mathbf{l}/\mathbf{k}}(b^m) = \det(\mu_{b^m}) \to 0$ as $m \to \infty$, contradicting the fact that $N_{\mathbf{l}/\mathbf{k}}(b^m) = 1$ for all m. Similarly, if $\|b\|_{\mathbf{l}} > 1$, then just consider b^{-1}. $\square$

Proposition 3.12. Let $\mathbf{k}$ be an algebraically closed valuation field. Then its completion $\widehat{\mathbf{k}}$ is also algebraically closed.

Proof. Let $f \in \widehat{\mathbf{k}}[X]$ be a non-constant polynomial. We will show that f has a root in $\widehat{\mathbf{k}}$. Take a sequence of polynomials $\{f_n\}_{n \in \mathbb{N}}$ in $\mathbf{k}[X]$ converging to f coefficient-wisely and of the same degree d. Since $\mathbf{k}$ is algebraically closed, each f_n splits completely in $\mathbf{k}$ and hence in $\widehat{\mathbf{k}}$. Write $f_n(X) = \prod_{i=1}^d (X - \alpha_{n,i})$ with $\alpha_{n,i} \in \widehat{\mathbf{k}}$.

Let $\mathbf{l}$ be a finite extension of $\widehat{\mathbf{k}}$ such that f has a root α in $\mathbf{l}$. For every $\varepsilon > 0$, if there are infinitely many n such that $\alpha_{n,i} \notin B(\alpha, \varepsilon)$ for all $1 \le i \le d$, then we have $|f_n(\alpha)| \ge \varepsilon^d$ for infinitely many n, contradicting the fact that $f_n(\alpha) \to f(\alpha) = 0$. Thus for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \ge N$, there exists $1 \le i \le d$ with $\alpha_{n,i} \in B(\alpha, \varepsilon)$. That is, we can find a sequence $\alpha_{n,i_n} \in \mathbf{k}$ converging to α. Since $\widehat{\mathbf{k}}$ is complete, we have $\alpha \in \widehat{\mathbf{k}}$. $\square$

3.3 Ramification and inertia

In this subsection, we study the extensions of absolute values on finite field extensions. Note that we do not assume the base field to be complete.

Definition 3.13. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. We denote by $w|v$ if $w \in M_L$ is an absolute value on L extending v. For each $w|v$, we define the *ramification index* $e(w|v)$ and the *inertia degree* $f(w|v)$ by

$$e(w|v) := [|\hat{L}^\times|_w : |\hat{K}^\times|_v], \quad f(w|v) := \frac{[\hat{L} : \hat{K}]}{e(w|v)},$$

where $\hat{K}$ and $\hat{L}$ are the completions of K and L with respect to v and w, respectively.

Lemma 3.14. Suppose that v is non-archimedean and k_v and ℓ_w are the residue fields of K and L with respect to v and w, respectively. Then we have

$$f(w|v) = [\ell_w : k_v].$$

Remark 3.15. Yang: To be added.

Theorem 3.16. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. Then we have

$$\sum_{w|v} e(w|v)f(w|v) = [L : K].$$

Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. We have

$$L \otimes_K K_v \cong \prod_{w|v} L_w,$$

where the product is taken over all absolute values $w \in M_L$ extending v.

Theorem 3.17. Let $\mathbf{k}$ be a number field. Then

$$M_{\mathbf{k}}^\infty = \{\text{embeddings } \sigma : \mathbf{k} \to \mathbb{C}\}$$

and

$$M_{\mathbf{k}}^{\mathrm{f}} = \{\text{non-zero prime ideals } \mathfrak{p} \subseteq \mathcal{O}_{\mathbf{k}}\}.$$

Yang: To be revised.

Proposition 3.18 (Product formula)**.** Let $\mathbf{k}$ be a number field. For each $x \in \mathbf{k}^\times$, we have

$$\prod_{v \in M_{\mathbf{k}}} |x|_v^{n_v} = 1,$$

where

$$n_v := \begin{cases} [\mathbf{k}_v : \mathbb{R}], & v \in M_{\mathbf{k}}^\infty; \\ 1, & v \in M_{\mathbf{k}}^0. \end{cases}$$

Yang: To be revised.

Remark 3.19. Let L/K be a finite field extension, and $v \in M_K$ an absolute value on K. Suppose that v is non-archimedean. Yang: To be rewritten.

4 Artin-Whaples approximations

Theorem 4.1 (Artin-Whaples approximations)**.** Let K be a field, and let $v_1, v_2, \ldots, v_n$ be pairwise inequivalent nontrivial absolute values on K. For any $a_1, a_2, \ldots, a_n \in K$ and any $\varepsilon > 0$, there exists an element $x \in K$ such that

$$|x - a_i|_{v_i} < \varepsilon$$

for all $1 \leq i \leq n$. Yang: To be checked.

4.1 Geometric version

Theorem 4.2. Let $\mathbf{k}$ be a field with algebraic closure $\Bbbk$. Let X be a normal, projective, geometrically integral variety over $\mathbf{k}$. Let $x_1, x_2, \ldots, x_n \in X(\mathbf{k})$ be closed points lying over pairwise distinct points of X. Let $v_1, v_2, \ldots, v_n \in M_{\mathbf{k}}$ be pairwise inequivalent absolute values on $\mathbf{k}$. For every $i = 1, 2, \ldots, n$, let U_i be an open neighborhood of x_i in $X(\Bbbk)$ with respect to the topology induced by v_i. Then there exists a rational point $x \in X(\mathbf{k})$ such that $x \in U_i$ for all $1 \leq i \leq n$. Yang: To be revsied.

Yang: This gives [Xie25, Proposition 3.9]

5 Example: p-adic fields

5.1 p-adic fields

Construction 5.1. Let K be a number field and $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K. Considering the localization $(\mathcal{O}_K)_{\mathfrak{p}}$ of $\mathcal{O}_K$ at $\mathfrak{p}$, which is a discrete valuation ring, denote by $v_{\mathfrak{p}} : K^\times \to \mathbb{Z}$ the corresponding discrete valuation. The *p-adic absolute value* on K associated to $\mathfrak{p}$ is defined as

$$|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}, \quad \forall x \in K,$$

where $N(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$.

The completion of K with respect to the p-adic absolute value $|\cdot|_{\mathfrak{p}}$ is denoted by $K_{\mathfrak{p}}$, called the $\mathfrak{p}$-*adic field.*

We just focus on the case $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ for a prime number p in the following.

Example 5.2. Let p be a prime number. For every $r \in \mathbb{Q}$, we can write r as $r = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are integers not divisible by p. The *p-adic absolute value* on $\mathbb{Q}$ is defined as

$$|r|_p := p^{-n}.$$

The p-adic field $\mathbb{Q}_p$ can be described concretely as follows:

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{+\infty} a_i p^i \,\middle|\, n \in \mathbb{Z}, a_i \in \{0, 1, \dots, p-1\} \right\}.$$

For $x = \sum_{i=n}^{+\infty} a_i p^i \in \mathbb{Q}_p$ with $a_n \neq 0$, its p-adic absolute value is given by $|x|_p = p^{-n}$. The operations of addition and multiplication on $\mathbb{Q}_p$ are defined similarly as those on decimal expansions.

Unlike the field of real numbers $\mathbb{R}$, the p-adic field $\mathbb{Q}_p$ has many finite extensions.

Proposition 5.3. There are infinitely many irreducible polynomials over the p-adic field $\mathbb{Q}_p$.

Proof. Since there are infinitely many irreducible monic polynomials over the finite field $\mathbb{F}_p$, consider any lift of such an irreducible monic polynomial to a monic polynomial with coefficients in $\mathbb{Z}_p$. If the lift is not irreducible over $\mathbb{Q}_p$, then the factorization of the lift gives a nontrivial factorization of its reduction modulo p since the factors can be chosen to be monic and have coefficients in $\mathbb{Z}_p$, which contradicts the irreducibility of the original polynomial over $\mathbb{F}_p$. Thus, the lift is irreducible over $\mathbb{Q}_p$.

On the other hand, note that $|\mathbb{Q}_p^\times|_p = p^{\mathbb{Z}}$. It follows that $f(T) = T^n - p$ is irreducible over $\mathbb{Q}_p$ for every integer $n \geq 1$. Otherwise, suppose we have a monic factorization $f(T) = g(T)h(T)$ with $g(T), h(T) \in \mathbb{Z}_p[T]$ and $\deg g, \deg h < n$. Then by considering the reduction modulo p, we have $g(0), h(0) \equiv 0 \mod p$. It follows that $|f(0)|_p = |g(0)h(0)|_p \leq p^{-2}$, which contradicts $|f(0)|_p = |p|_p = p^{-1}$. $\qquad\square$

5.2   Completion

Proposition 5.4. The algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is not complete with respect to the extension of the p-adic absolute value $|\cdot|_p$.

Proof. Yang: To be completed. $\qquad\square$

Construction 5.5. Let p be a prime number. The *field $\mathbb{C}_p$ of p-adic complex numbers* is defined as the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$.

The field $\mathbb{C}_p$ is algebraically closed and complete with respect to $|\cdot|_p$ by Proposition 3.12. By Corollaries 2.11 and 2.12, we have

$$|\mathbb{C}_p^\times|_p = |\overline{\mathbb{Q}_p}^\times|_p = p^{\mathbb{Q}}, \quad \mathcal{k}_{\mathbb{C}_p} \cong \mathcal{k}_{\overline{\mathbb{Q}_p}} \cong \overline{\mathbb{F}_p}.$$

Proposition 5.6. The field $\mathbb{C}_p$ of p-adic complex numbers is not spherically complete.

Proof. Yang: To be completed. $\qquad\square$

Construction 5.7. Let p be a prime number. Yang: We construct the *spherically complete p-adic field Ω_p*. Yang: To be completed.

5.3 Elementary functions

Lemma 5.8. Let p be a prime number and $n \in \mathbb{N}$. We have $v_p(n!) =$.

Yang: Exponential, logarithmic, and the interpolation functions.

Fix a prime number p in the following and consider $\mathbf{k} = \mathbb{Q}_p, \mathbb{C}_p$, or Ω_p. Let $r_p := p^{-1/(p-1)}$.

Construction 5.9. The *exponential function* $\exp : \mathbf{k} \to \mathbf{k}$ is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The radius of convergence of $\exp(x)$ is $+\infty$ if $p = 2$ and $p^{-1/(p-1)}$ if $p > 2$.

The *logarithmic function* $\log : 1 + \mathbf{k}^{\circ\circ} \to \mathbf{k}$ is defined by the power series

$$\log(1 + x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

The radius of convergence of $\log(1 + x)$ is 1.

Moreover, for every x in the domain of convergence of $\exp$ and every y in the domain of convergence of $\log$, we have

$$\log(\exp(x)) = x, \quad \exp(\log(y)) = y.$$

Yang: To be checked.

Definition 5.10. Let

Theorem 5.11. The series converges.

References

[Xie25] Junyi Xie. "The existence of Zariski dense orbits for endomorphisms of projective surfaces". In: *Journal of the American Mathematical Society* 38.1 (2025), pp. 1–62 (cit. on p. 14).