Non-archimedean valuations

1 Topology: Ultra-metric space

We will use $B(x,r)$ (resp. $E(x,r)$) to denote the open ball (resp. closed ball) with center x and radius r.

Definition 1. A metric space (X,d) is called an *ultra-metric space* if its metric d satisfies the *strong triangle inequality*:

$$d(x,z) \le \max\{d(x,y), d(y,z)\}, \quad \forall x,y,z \in X.$$

Remark 2. The term *ultra-metric space* should be translated into Chinese as “奥特度量空间”. There is no special reason for this translation, except that I insist on using “奥特” to translate “ultra”.

If $(\mathbf{k}, \|\cdot\|)$ is a non-archimedean field, then the metric $d(x,y) := \|x-y\|$ on $\mathbf{k}$ makes $(\mathbf{k}, d)$ an ultra-metric space.

Proposition 3. Let (X,d) be an ultra-metric space. Then for any $x,y,z \in X$, at least two of the three distances $d(x,y), d(y,z), d(z,x)$ are equal. And the third distance is less than or equal to the common value of the other two.

Proof. Suppose that $d(x,y) \ge d(y,z)$. By the strong triangle inequality, we have

$$d(x,z) \le \max\{d(x,y), d(y,z)\} = d(x,y).$$

On the other hand, by the strong triangle inequality again, we have

$$d(x,y) \le \max\{d(x,z), d(z,y)\} = \max\{d(x,z), d(y,z)\} \le d(x,y).$$

This shows that $d(x,y) = \max\{d(x,z), d(y,z)\}$. Thus either $d(x,z) = d(x,y) \ge d(y,z)$ or $d(y,z) = d(x,y) \ge d(x,z)$. □

Proposition 4. Let (X,d) be an ultra-metric space. Let D_i be (open or closed) ball in X for $i = 1,2$. If $D_1 \cap D_2 \neq \varnothing$, then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Proof. Suppose that D_i has center x_i and radius r_i for $i = 1,2$. Let $y \in D_1 \cap D_2$. We have

$$d(x_1,x_2) \le \max\{d(x_1,y), d(y,x_2)\}.$$

Without loss of generality, we may assume that $d(x_1,x_2) \le d(x_1,y)$. It follows that $x_2 \in D_1$ since $d(x_1,y) < r_1$ (or $\le r_1$).

If there exists $z \in D_2 \setminus D_1$, we claim that $D_1 \subseteq D_2$. We have $d(x_1,z) > d(x_1,x_2)$. Then by Proposition 3,

$$r_1 \le d(x_1,z) = d(x_2,z) \le r_2.$$

In particular, if D_2 is an open ball, then we have strict inequality $r_1 < r_2$. For any $w \in D_1$, we have

$$d(x_2,w) \le \max\{d(x_2,x_1), d(x_1,w)\} \le r_1 \le r_2.$$

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Thus $w \in D_2$ whatever D_2 is open or closed, and it shows that $D_1 \subseteq D_2$. □

Proposition 5. Let (X, d) be an ultra-metric space. Then both $B(x, r)$ and $E(x, r)$ are closed and open subsets of X for any $x \in X$ and $r > 0$.

Proof. We show that the sphere $S(x, r) := \{y \in X \mid d(x, y) = r\}$ is open in X. Note that if $y \in S(x, r)$, then for any $r' < r$, we have $B(y, r') \cap E(x, r) \neq \varnothing$ and $x \in E(x, r) \setminus B(y, r')$. Thus by Proposition 4, we have $B(y, r') \subseteq E(x, r)$. If $B(y, r') \cap B(x, r) \neq \varnothing$, then by Proposition 4 again, we have $B(y, r') \subseteq B(x, r)$. However, $y \in B(y, r') \setminus B(x, r)$, a contradiction. Thus $B(y, r') \subseteq E(x, r) \setminus B(x, r) = S(x, r)$. It yields that $S(x, r) = \bigcup_{y \in S(x,r)} B(y, r/2)$ is open in X.

Since $E(x, r) = B(x, r) \cup S(x, r)$ and $B(x, r) = E(x, r) \setminus S(x, r)$, both $B(x, r)$ and $E(x, r)$ are open and closed in X. □

Corollary 6. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

Proof. Suppose that $S \subset X$ has at least two distinct points $x, y \in S$. Let $r := d(x, y) > 0$. Consider the open ball $B(x, r/2)$. By Proposition 5, $B(x, r/2)$ is both open and closed in X. Thus $B(x, r/2) \cap S$ is both open and closed in S, however, it is non-empty and not equal to S since it contains x but not y. This shows that S is disconnected. □

Proposition 7. Let (X, d) be an ultra-metric space. A sequence $\{x_n\}$ in X is cauchy if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Proof. The necessity is true for all metric spaces. Suppose that $d(x_n, x) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \varepsilon$ for all $n \geq N$. For any $m, n \geq N$ with $m < n$, by the strong triangle inequality, we have

$$d(x_n, x_m) \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_m)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_{n-2}), \ldots, d(x_{m+1}, x_m)\} < \varepsilon.$$

This shows that $\{x_n\}$ is a cauchy sequence. □

2 Algebra: ring of integers and residue field

Let $\mathbf{k}$ be a non-archimedean field. Then easily see that $\{x \in \mathbf{k} : \|x\| \leq 1\}$ is a subring of $\mathbf{k}$. Moreover, it is a local ring whose maximal ideal is $\{x \in \mathbf{k} : \|x\| < 1\}$.

Definition 8. Let $\mathbf{k}$ be a non-archimedean field. The *ring of integers* of $\mathbf{k}$ is defined as

$$\mathbf{k}^\circ := \{x \in \mathbf{k} : \|x\| \leq 1\}.$$

Its maximal ideal is

$$\mathbf{k}^{\circ\circ} := \{x \in \mathbf{k} : \|x\| < 1\}.$$

The *residue field* of $\mathbf{k}$ is defined as

$$\Bbbk_{\mathbf{k}} := \widetilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}.$$

Set $I_{r,<} := B(0, r)$ and $I_{r,\leq} := E(0, r)$ for each $r \in [0, 1]$.

Proposition 9. The sets $I_{r,<}$ and $I_{r,\leq}$ are ideals of the ring of integers $\mathbf{k}^\circ$. Conversely, any ideal of $\mathbf{k}^\circ$ is of the form $I_{r,<}$ or $I_{r,\leq}$ for some $r \in (0,1)$.

Proof. Let I be an ideal of $\mathbf{k}^\circ$. Set $r = \sup\{|a| : a \in I\}$ (resp. $r = \max\{|a| : a \in I\}$ when the maximum exists). Then, by definition, we have $I \subset I_{r,<}$ (resp. $I \subset I_{r,\leq}$). For every $x \in \mathbf{k}^\circ$ with $|x| < r$ (resp. $|x| \leq r$), there exists $a \in I$ such that $|x| \leq |a|$. Thus, $|x/a| \leq 1$ and so $x/a \in \mathbf{k}^\circ$. Since I is an ideal, we have $x = (x/a)a \in I$. Therefore, $I_{r,<} \subset I$ (resp. $I_{r,\leq} \subset I$). $\square$

Proposition 10. Let I_r be either $I_{r,<}$ or $I_{r,\leq}$ for each $r \in (0,1)$. Suppose $\{r_n \in (0,1)\}_{n\in\mathbb{N}}$ is a decreasing sequence converging to 0. Then the completion $\widehat{\mathbf{k}}$ of $\mathbf{k}$ is isomorphic to the projective limit

$$\widehat{\mathbf{k}}^\circ \cong \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}.$$

Proof. For every $x \in \widehat{\mathbf{k}}^\circ$, there exists a cauchy sequence $\{x_m\}_{m\in\mathbb{N}}$ in $\mathbf{k}^\circ$ converging to x. Since $\{r_n\}_{n\in\mathbb{N}}$ converges to 0, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{N}$ such that for all $m, m' \geq M_n$, we have $|x_m - x_{m'}| < r_n$. Thus, the sequence $\{x_m + I_{r_n}\}_{m\in\mathbb{N}}$ is eventually constant in $\mathbf{k}^\circ/I_{r_n}$. Define a map

$$\Phi : \widehat{\mathbf{k}}^\circ \to \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}, \quad x \mapsto \left(\lim_{m\to\infty} x_m + I_{r_n}\right)_{n\in\mathbb{N}}.$$

It is straightforward to verify that Φ is a well-defined ring homomorphism.

Conversely, for every $(a_n + I_{r_n})_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n}$, we can choose a representative $a_n \in \mathbf{k}^\circ$ for each n. We claim that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathbf{k}^\circ$. Indeed, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_N < \varepsilon$. For all $m, n \geq N$, since $a_n + I_{r_n}$ maps to $a_m + I_{r_m}$ under the natural projection, we have $|a_n - a_m| < r_N < \varepsilon$. Thus, $\{a_n\}_{n\in\mathbb{N}}$ converges to some $x \in \widehat{\mathbf{k}}^\circ$. Easily see that the limit x is independent of the choice of representatives $\{a_n\}_{n\in\mathbb{N}}$. This gives a map

$$\Psi : \varprojlim_{n\in\mathbb{N}} \mathbf{k}^\circ/I_{r_n} \to \widehat{\mathbf{k}}^\circ, \quad (a_n + I_{r_n})_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} a_n.$$

Direct verification shows that $\Psi = \Phi^{-1}$. $\square$

Corollary 11. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the residue field $\mathscr{k}_{\widehat{\mathbf{k}}} \cong \mathscr{k}_{\mathbf{k}}$ under the natural embedding $\mathbf{k}^\circ \hookrightarrow \widehat{\mathbf{k}}^\circ$.

Corollary 12. Let $\mathbf{k}$ be a non-archimedean field and $\widehat{\mathbf{k}}$ its completion. Then the valuation group $|\widehat{\mathbf{k}}^\times|$ of $\widehat{\mathbf{k}}$ is equal to the valuation group $|\mathbf{k}^\times|$ of $\mathbf{k}$.

Proof. Note that

$$\begin{aligned} r \in |\widehat{\mathbf{k}}^\times| &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \widehat{\mathbf{k}}^\circ \\ &\iff \widehat{\mathbf{k}}^\circ/I_{r,<} \to \widehat{\mathbf{k}}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff \mathbf{k}^\circ/I_{r,<} \to \mathbf{k}^\circ/I_{r,\leq} \text{ is not an isomorphism} \\ &\iff I_{r,<} \subsetneq I_{r,\leq} \text{ in } \mathbf{k}^\circ \\ &\iff r \in |\mathbf{k}^\times|. \end{aligned}$$

$\square$

Proposition 13. Let $\mathbf{k}$ be a non-archimedean field with non-trivial valuation. Then $\mathbf{k}^\circ$ is totally bounded iff $\mathbf{k}^\circ/I_{r,<}$ and $\mathbf{k}^\circ/I_{r,\leq}$ are finite for each $r \in [0,1]$. Moreover, if $\mathbf{k}$ is complete, then it is locally compact iff $\mathbf{k}^\circ/I_r$ is finite for each $r \in (0,1)$.

Slogan *"Locally compact $\iff$ pro-finite."*

Proof. We just prove the case for $I_r = I_{r,<}$. The case for $I_r = I_{r,\leq}$ is similar.

Suppose that $\mathbf{k}^\circ/I_r$ is finite for each $r \in [0,1]$. Then for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that $r < \varepsilon$ and $\mathbf{k}^\circ/I_r$ is finite. Let $\{a_1 + I_r, \dots, a_n + I_r\}$ be the complete set of representatives of $\mathbf{k}^\circ/I_r$. Then the balls $B(a_i, r)$ for $i = 1, \dots, n$ cover $\mathbf{k}^\circ$.

Conversely, suppose that $\mathbf{k}^\circ/I_r$ is infinite for some $r \in [0,1]$. Then there exists an infinite set $\{a_n\}$ with $|a_n| \in [r,1]$ such that their images in $\mathbf{k}^\circ/I_r$ are distinct. In particular, for every $m \neq n$, we have $|a_n - a_m| \geq r$. Any subsequence of $\{a_n\}$ is not cauchy. Thus, $\mathbf{k}^\circ$ is not totally bounded. $\square$

Proposition 14. The ring $\mathbf{k}^\circ$ is noetherian iff $\mathbf{k}$ is a discrete valuation field.

Proof. Note that $|\mathbf{k}^\times| \subset \mathbb{R}_{>0}$ is a multiplicative subgroup. If $\mathbf{k}$ is not a discrete valuation field, then $|\mathbf{k}^\times|$ is dense in $\mathbb{R}_{>0}$. In particular, there exists a strictly ascending sequence $r_n \in |\mathbf{k}^\times| \cap (0,1)$. Then the ideals $I_{r_n,\leq}$ form a strictly ascending chain of ideals in $\mathbf{k}^\circ$.

The converse is standard since now $\mathbf{k}^\circ$ is a discrete valuation ring. $\square$

Proposition 15. Let $\mathbf{k}$ be a complete non-archimedean field. Then $\mathbf{k}$ is locally compact iff $\mathbf{k}$ is a discrete valuation field and its residue field $\Bbbk_{\mathbf{k}}$ is finite.

Proof. The necessity follows from Proposition 13. For the sufficiency, suppose that $\mathbf{k}$ is a discrete valuation field whose residue field $\Bbbk_{\mathbf{k}}$ is finite. Let $\pi \in \mathbf{k}^\circ$ be a uniformizer. We only need to show that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite for each $n \in \mathbb{N}$. Note that there is an isomorphism

$$\pi^{n-1}\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ \cong \Bbbk_{\mathbf{k}}, \quad x + \pi^n\mathbf{k}^\circ \mapsto \overline{x/\pi^{n-1}}.$$

Thus, by induction on n, we conclude that $\mathbf{k}^\circ/\pi^n\mathbf{k}^\circ$ is finite. $\square$

3 Hensel's Lemma

Theorem 16 (Hensel's lemma)**.** Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \Bbbk_{\mathbf{k}}[T]$ of $F(T)$ factors as

$$f(T) = g(T)h(T),$$

where $g(T), h(T) \in \Bbbk_{\mathbf{k}}[T]$ are monic polynomials that are coprime in $\Bbbk_{\mathbf{k}}[T]$. Then there exist monic polynomials $G(T), H(T) \in \mathbf{k}^\circ[T]$ such that

$$F(T) = G(T)H(T),$$

and the reductions of $G(T), H(T)$ in $\Bbbk_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

Proof. Since $\gcd(g,h) = 1$ in $\mathscr{k}_{\mathbf{k}}[T]$, there exist polynomials $u(T), v(T) \in \mathscr{k}_{\mathbf{k}}[T]$ such that $ug + vh = 1$ and $\deg u < \deg h, \deg v < \deg g$. Choose lifts $G_0(T), H_0(T), U(T), V(T) \in \mathbf{k}^\circ[T]$ of $g(T), h(T), u(T), v(T)$ respectively preserving their degrees such that G_0 and H_0 are monic. Then there exist $r < 1$ such that

$$U(T)G_0(T) + V(T)H_0(T) \equiv 1 \mod I_r, \quad F(T) - G_0(T)H_0(T) \equiv 0 \mod I_r,$$

where $I_r = \{a \in \mathbf{k}^\circ : |a| < r\}$.

We will construct a sequence of monic polynomials $\{G_n(T)\}_{n\in\mathbb{N}}$ and $\{H_n(T)\}_{n\in\mathbb{N}}$ in $\mathbf{k}^\circ[T]$ such that for each $n \in \mathbb{N}$,

$$G_n(T) \equiv G_{n-1}(T) \mod I_{r^n}, \quad H_n(T) \equiv H_{n-1}(T) \mod I_{r^n},$$

and

$$F(T) - G_n(T)H_n(T) \equiv 0 \mod I_{r^{n+1}}.$$

If we have such sequences, then their coefficients converge in the complete ring $\mathbf{k}^\circ$. Let $G(T)$ and $H(T)$ be the limits of $\{G_n(T)\}$ and $\{H_n(T)\}$ respectively. Then we have $F(T) = G(T)H(T)$ and the reductions of $G(T), H(T)$ in $\mathscr{k}_{\mathbf{k}}[T]$ are $g(T), h(T)$ respectively.

The case $n = 0$ is done by the above construction. Now suppose that we have constructed $G_n(T)$ and $H_n(T)$ for some $n \geq 0$. Since $G_n - G_0 \equiv 0 \mod I_r$ and $H_n - H_0 \equiv 0 \mod I_r$, we have

$$UG_n + VH_n = UG_0 + VH_0 + U(G_n - G_0) + V(H_n - H_0) \equiv 1 \mod I_r.$$

Set $\Delta_n(T) = F(T) - G_n(T)H_n(T) \in I_{r^{n+1}}[T]$ and $\epsilon_n = U\Delta_n, \delta_n = V\Delta_n \in I_{r^{n+1}}[T]$. Then we have

$$\begin{aligned}(G_n + \epsilon_n)(H_n + \delta_n) - F_n &= G_nH_n + G_n\delta_n + H_n\epsilon_n + \epsilon_n\delta_n - F_n \\ &= (UG_n + VH_n - 1)\Delta_n + \epsilon_n\delta_n \in I_{r^{n+2}}[T].\end{aligned}$$

Thus, we can set

$$G_{n+1}(T) = G_n(T) + \epsilon_n(T), \quad H_{n+1}(T) = H_n(T) + \delta_n(T).$$

This finishes the induction. □

Corollary 17. Let $\mathbf{k}$ be a complete non-archimedean field and $F(T) \in \mathbf{k}^\circ[T]$ a monic polynomial. Suppose that the reduction $f(T) \in \mathscr{k}_{\mathbf{k}}[T]$ of $F(T)$ has a simple root $a \in \mathscr{k}_{\mathbf{k}}$. Then there exists a root $\alpha \in \mathbf{k}^\circ$ of $F(T)$ whose reduction is a.

Proof. Since a is a simple root of $f(T)$, we have the factorization $f(T) = (T - a)h(T)$ for some monic polynomial $h(T) \in \mathscr{k}_{\mathbf{k}}[T]$ with $h(a) \neq 0$. Then the result follows from Theorem 16. □

4 Newton polygons

Yang: To be filled.

Appendix