Spectrum

Let $\mathbf{k}$ be a spherically complete non-archimedean field which is algebraically closed and $A = \mathbf{k}[T]$. We want to consider the "analytic structure" on $\operatorname{mSpec} A$. However, unlike the complex case, the set $\operatorname{mSpec} A$ is totally disconnected with respect to the topology induced by the absolute value on $\mathbf{k}$ (Corollary 26). To overcome this difficulty, Berkovich uses multiplicative semi-norms to "fill in the gaps" between the points in $\operatorname{mSpec} A$, leading to the notion of the spectrum of a Banach ring.

1 Definition

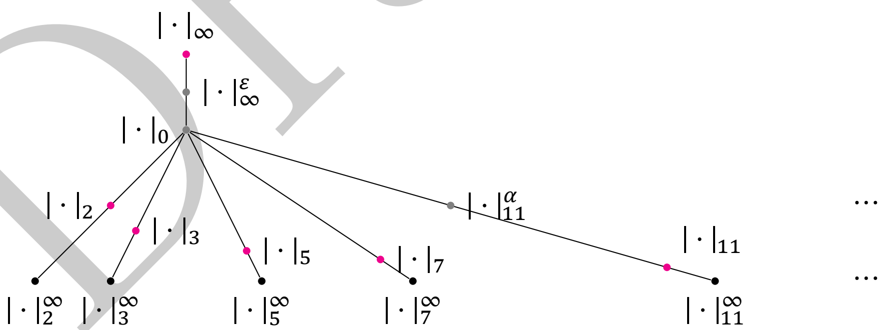
Definition 1. Let R be a Banach ring. The *Berkovich spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$.
We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_{\geq 0}$, defined by $x \mapsto |f|_x =: f(x)$, is continuous.

Example 2. Let $(\mathbf{k}, |\cdot|)$ be a complete valuation field. The Berkovich spectrum $\mathscr{M}(\mathbf{k})$ consists of a single point corresponding to the given absolute value $|\cdot|$ on $\mathbf{k}$.

Example 3. Consider the Banach ring $(\mathbb{Z}, \|\cdot\|)$ with $\|\cdot\| = |\cdot|_\infty$ is the usual absolute value norm on $\mathbb{Z}$. Let $|\cdot|_p$ denote the p-adic norm for each prime number p, i.e., $|n|_p = p^{-v_p(n)}$ for each $n \in \mathbb{Z}$, where $v_p(n)$ is the p-adic valuation of n. The Berkovich spectrum

$$\mathscr{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0,1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0,\infty]\} \cup \{|\cdot|_0\},$$

where $|a|_p^\infty := \lim_{\alpha\to\infty} |a|_p^\alpha$ for each $a \in \mathbb{Z}$ and $|\cdot|_0$ is the trivial norm on $\mathbb{Z}$.



Yang: To be continued.

Theorem 4. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Note that the pullback of residue norm on the residue field $R/\mathfrak{m}$ is bounded with respect to the given norm on R. Replacing R by the completion of $R/\mathfrak{m}$, we may assume that R is a complete field. Consider the set

$$\Sigma = \{\text{norm on } R \text{ bounded by the given norm } \|\cdot\|\}$$

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DRAFT

with the partial order defined by boundedness. Since for a descending chain in Σ, the infimum is a norm, by Zorn's lemma, there exists a minimal element $|\cdot| \in \Sigma$.

We claim that $|\cdot|$ is multiplicative. Since the spectral radius $\rho(f) = \lim_{n\to\infty} |f^n|^{1/n}$ associated to $|\cdot|$ is power-multiplicative and bounded by $|\cdot|$, by minimality of $|\cdot|$, we have $\rho(f) = |f|$ for each $f \in R$. Thus $|\cdot|$ is power-multiplicative. If $|\cdot|$ is not multiplicative, then there exist $a, b \in R \setminus \{0\}$ such that $|ab| < |a||b|$. Then $|b| \leq |a^{-1}||ab| < |a^{-1}||a||b|$, which implies that $|a||a^{-1}| > 1$. Set $r = |a|^{-1} < |a^{-1}|$ and consider $R\langle T/r\rangle$. Since $r \cdot |a| = 1$, we have that

$$\left|\sum_{n=0}^{\infty} a^n T^n\right| = \sum_{n=0}^{\infty} |a^n| r^n = \sum_{n=0}^{\infty} |a|^n r^n = \sum_{n=0}^{\infty} 1 = \infty.$$

The power series is not convergent in $R\langle T/r\rangle$ and hence $1 - aT$ is not invertible in $R\langle T/r\rangle$. Let $\mathfrak{n}$ be a maximal ideal of $R\langle T/r\rangle$ containing $1 - aT$. Consider $R \to R\langle T/r\rangle \to R\langle T/r\rangle/\mathfrak{n}$. Since R is a field, the composition is injective. The residue norm on $R\langle T/r\rangle/\mathfrak{n}$ induces a norm $|\cdot|'$ on R bounded by $|\cdot|$. Note that $|a^{-1}|' \leq |T| = r = |a|^{-1} < |a^{-1}|$, contradicting the minimality of $|\cdot|$. □

Definition 5. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback* map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $\mathscr{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$ for each $x \in \mathscr{M}(S)$.

Note that $\mathscr{M}(\varphi)(f^{-1}(V)) = \varphi(f)^{-1}(V)$ for each $f \in R$ and open subset $V \subset \mathbb{R}_{\geq 0}$. Hence the pullback map is continuous.

Notation 6. Let R be a Banach ring and $x \in \mathscr{M}(R)$. We denote by $|\cdot|_x$ the multiplicative semi-norm on R corresponding to the point x. Its kernel $\{f \in R : |f|_x = 0\}$ is a closed prime ideal of R, denoted by $\wp_x$.

Definition 7. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, the *completed residue field* at the point x is defined as the completion of the residue field $\kappa(x) = \mathrm{Frac}(R/\wp_x)$ with respect to the multiplicative norm induced by the semi-norm $|\cdot|_x$, denoted by $\mathscr{H}(x)$.

Example 8. Consider the Banach ring $(\mathbb{Z}, |\cdot|_\infty)$ as in Example 3. We have

- $x = |\cdot|_\infty^\varepsilon$ for some $\varepsilon \in (0,1]$: $\wp_x = (0)$ and $\mathscr{H}(x) \cong \mathbb{R}$ with the absolute value norm raised to the power ε;
- $x = |\cdot|_0$: $\wp_x = (0)$ and $\mathscr{H}(x) \cong \mathbb{Q}$ with the trivial norm;
- $x = |\cdot|_p^\alpha$ for some prime number p and $\alpha \in (0,\infty)$: $\wp_x = (0)$ and $\mathscr{H}(x) \cong \mathbb{Q}_p$ with the p-adic norm raised to the power α;
- $x = |\cdot|_p^\infty$ for some prime number p: $\wp_x = (p)$ and $\mathscr{H}(x) \cong \mathbb{F}_p$ with the trivial norm.

Definition 9. Let R be a Banach ring. The *Gel'fand transform* of R is the bounded ring homomorphism

$$\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x), \quad f \mapsto (f(x))_{x \in \mathscr{M}(R)},$$

where the norm on the product $\prod_{x\in\mathscr{M}(R)} \mathscr{H}(x)$ is given by the supremum norm.

DRAFT

Proposition 10. The Gel'fand transform $\Gamma : R \to \prod_{x\in\mathscr{M}(R)}\mathscr{H}(x)$ of a Banach ring R factors through the uniformization R^u of R, and the induced map $R^u \to \prod_{x\in\mathscr{M}(R)}\mathscr{H}(x)$ is an isometric embedding. Yang: To be checked.

Proof. Yang: To be added. □

Lemma 11. Let $\{K_i\}_{i\in I}$ be a family of completed fields. Consider the Banach ring $R = \prod_{i\in I} K_i$ equipped with the product norm. The spectrum $\mathscr{M}(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I.

Proof. Yang: To be added. □

Remark 12. The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

Theorem 13. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a compact Hausdorff space.

Proof. Yang: To be added. □

Proposition 14. Let K/k be a Galois extension of complete fields, and let R be a Banach k-algebra. The Galois group $\mathrm{Gal}(K/k)$ acts on the spectrum $\mathscr{M}(R\widehat{\otimes}_k K)$ via

$$g\cdot x : f \mapsto |(1\otimes g^{-1})(f)|_x$$

for each $g \in \mathrm{Gal}(K/k)$, $x\in\mathscr{M}(R\widehat{\otimes}_k K)$ and $f\in R\widehat{\otimes}_k K$. Moreover, the natural map $\mathscr{M}(R\widehat{\otimes}_k K)\to\mathscr{M}(R)$ induces a homeomorphism

$$\mathscr{M}(R\widehat{\otimes}_k K)/\,\mathrm{Gal}(K/k)\xrightarrow{\sim}\mathscr{M}(R).$$

Yang: To be checked.

Proof. Yang: To be added. □

DRAFT

2 Reduction map and kernel map

Proposition 15. Let R be a Banach ring. The kernel map $\mathscr{M}(R)\to\mathrm{Spec}(R), x\mapsto\wp_x$ is continuous with respect to the Zariski topology on $\mathrm{Spec}(R)$.

Proof. Let $D(f)=\{f\neq 0\}\subset\mathrm{Spec}(R)$ be a principal open subset for some $f\in R$. The preimage of $D(f)$ under the kernel map is just the set $\{x\in\mathscr{M}(R) : |f|_x>0\}=f^{-1}(\mathbb{R}_{>0})$, which is open in $\mathscr{M}(R)$ by definition of the topology on $\mathscr{M}(R)$. □

Example 16. Let us consider the spectrum $\mathscr{M}(\mathbb{Z})$ in Example 3. Under the kernel map $\mathscr{M}(\mathbb{Z})\to\mathrm{Spec}(\mathbb{Z})$, the points $|\cdot|_p^\infty$ for each prime number p are mapped to the prime ideal (p), the other above points are all mapped to the zero ideal (0).

Yang: Is this surjective? what is its fiber?

Proposition 17. Yang: To be added.

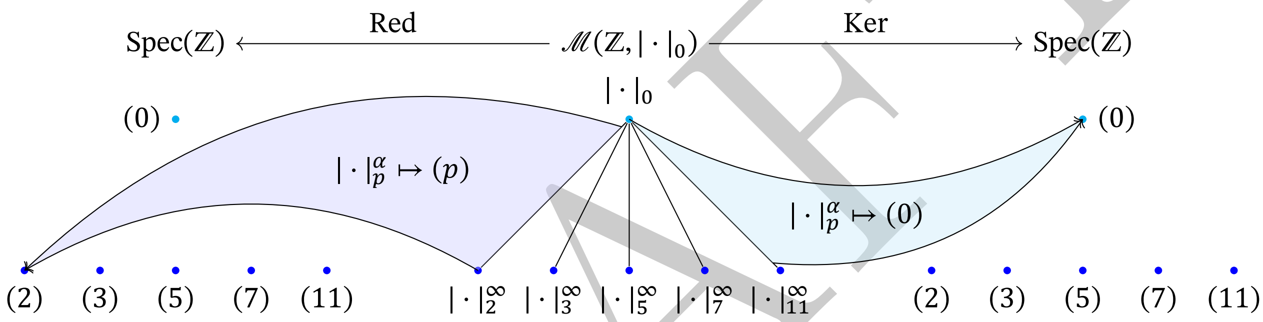
Construction 18. Suppose that R is a non-archimedean Banach ring with valuation subring R° and maximal ideal $R^{\circ\circ}$. For each $x \in \mathscr{M}(R)$, there is an induced homomorphism $R^\circ \to \mathscr{H}(x)^\circ$ between the valuation subrings. Furthermore, we have an induced homomorphism between the residue rings $\widetilde{R} = R^\circ/R^{\circ\circ} \to \mathcal{k}_{\mathscr{H}(x)}$. This gives rise to the *reduction map*

$$\mathsf{Red} : \mathscr{M}(R) \to \mathsf{Spec}(\widetilde{R}), \quad x \mapsto \ker(\widetilde{R} \to \mathcal{k}_{\mathscr{H}(x)}).$$

Example 19. Let $(\mathbb{Z}, |\cdot|_0)$ be the Banach ring with the trivial norm. The reduction ring is $\widetilde{\mathbb{Z}} = \mathbb{Z}$.

- $x = |\cdot|_p^\alpha$ for some prime number p and $\alpha \in (0,\infty]$: $\mathcal{k}_{\mathscr{H}(x)} \cong \mathbb{F}_p$ and the induced homomorphism $\widetilde{\mathbb{Z}} = \mathbb{Z} \to \mathcal{k}_{\mathscr{H}(x)} = \mathbb{F}_p$ is the natural projection $\mathbb{Z} \to \mathbb{F}_p$;
- $x = |\cdot|_0$: $\mathcal{k}_{\mathscr{H}(x)} \cong \mathbb{Q}$ and the induced homomorphism $\widetilde{\mathbb{Z}} = \mathbb{Z} \to \mathcal{k}_{\mathscr{H}(x)} = \mathbb{Q}$ is the natural inclusion $\mathbb{Z} \to \mathbb{Q}$.

The following diagram illustrates the reduction map and the kernel map for the spectrum $\mathscr{M}(\mathbb{Z}, |\cdot|_0)$:



Proposition 20. Let R be a non-archimedean Banach ring and $\widetilde{U} \subset \mathsf{Spec}(\widetilde{R})$ be a Zariski open subset. Then the preimage $\mathsf{Red}^{-1}(\widetilde{U})$ is a closed subset of $\mathscr{M}(R)$.

Proof. Yang: To be completed. □

3 Spectrum of Tate algebras

Spectrum of Tate algebra in one variable Let $\Bbbk$ be an algebraically closed complete non-archimedean field, and let $A = \Bbbk\{T/r\}$. We list some types of points in the spectrum $\mathscr{M}(A)$.

Construction 21. For each $a \in \Bbbk$ with $|a| \le r$, we have the *type I* point x_a corresponding to the evaluation at a, i.e., $|f|_{x_a} := |f(a)|$ for each $f \in A$.

For each closed disk $E = E(a,s) := \{b \in \Bbbk : |b-a| \le s\}$ with center $a \in \Bbbk$ and radius $s \le r$, we have the point $x_E = x_{a,s}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} = |f|_{x_{a,s}} := \sup_{b \in E(a,s)} |f(b)|$$

for each $f \in A$. If $s \in |\Bbbk^\times|$, then the point x_E is called a *type II* point; otherwise, it is called a *type III* point.

Let $E_n = E(a_n, s_n)$ be a sequence of closed disks in $\Bbbk$ such that $E_{n+1} \subsetneq E_n$ and $\bigcap_n E_n = \varnothing$. Then we have the point $x_{\{E_n\}} = x_{\{a_n,s_n\}}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E_n\}}} = |f|_{x_{\{a_n,s_n\}}} := \inf_n |f|_{x_{E_n}}$$

DRAFT

for each $f \in A$. Such a point is called a *type IV* point. Yang: To be completed. Check the definition of type IV points.

Proposition 22. The points in the spectrum $\mathscr{M}(\Bbbk\{r^{-1}T\})$ can be classified into four types as described above.

Proof. Fix $x \in \mathscr{M}(\Bbbk\{r^{-1}T\})$, set

$$s = \inf_{a\in\Bbbk} |T-a|_x \le r, \quad E = \{a \in \Bbbk : |T-a|_x = s\} \subset E(0,r).$$

Case 1. $E \neq \varnothing$ and $s = 0$.

By assumption, there exists $a \in E$ such that $|T-a|_x = 0$. Note that if $f(a) = 0$, then $T - a \mid f$ in $\Bbbk\{r^{-1}T\}$ and hence $|f|_x = |T-a|_x|g|_x = 0$. Then we have

$$|f(a)| = ||f(a)| - |f(a) - f|_x| \le |f|_x \le |f(a)| + |f(a) - f|_x = |f(a)|$$

for each $f \in \Bbbk\{r^{-1}T\}$, which implies that $|f|_x = |f(a)|$. Thus x is a type I point x_a.

Case 2. $E \neq \varnothing$ and $s > 0$.

Let $a \in E$. Note that for every $b \in \Bbbk$, we have

$$|a-b| \le \max\{|T-a|_x, |T-b|_x\} = |T-b|_x.$$

First we show that $E = E(a,s)$. For each $b \in E(a,s)$, we have $|T-b|_x \le \max\{|T-a|_x, |a-b|\} = s$, which implies that $b \in E$. Conversely, for each $b \in E$, we have $|a-b| \le \max\{|T-a|_x, |T-b|_x\} = s$.

Let $f \in \Bbbk[T]$ be a polynomial. Write $f = \prod_{i=1}^n (T-c_i)$ for some $c_1, \dots, c_n \in \Bbbk$. Then we have

$$|f|_x = \prod_{i=1}^n |T-c_i|_x \ge \prod_{i=1}^n |b-c_i| = |f(b)|, \quad \forall b \in E.$$

I claim that for every $\varepsilon \in (0,1)$, there exists $b \in E$ such that $\varepsilon|T-c_i|_x < |b-c_i|$ for each $i = 1, \dots, n$. Indeed, if $c_i \notin E$, then $|T-c_i|_x = |b-c_i|$ for each $b \in E$. Hence we only need to consider the case when $c_i \in E$. Since $\Bbbk$ is algebraically closed, $E(a,s) \setminus \bigcup_{i=1}^n E(c_i, \varepsilon s) \neq \varnothing$. Choose b in the set. Then we have

$$|f(b)| \ge \prod_{i=1}^n \varepsilon|T-c_i|_x = \varepsilon^n|f|_x.$$

Thus $|f|_x = \sup_{b\in E}|f(b)|$ for each polynomial $f \in \Bbbk[T]$. Since polynomials are dense in $\Bbbk\{r^{-1}T\}$, we have $|f|_x = \sup_{b\in E}|f(b)|$ for each $f \in \Bbbk\{r^{-1}T\}$. Therefore, x is the point $x_E = x_{a,s}$, which is of type II or type III depending on whether $s \in |\Bbbk^\times|$ or not.

Case 3. $E = \varnothing$.

Set $E_n = \{a \in \Bbbk : |T-a|_x \le s + 1/n\}$ and $a_n \in E_n$ for each $n \in \mathbb{N}$. By the similar argument as in Case 2, we have $E_n = E(a_n, s+1/n)$. Note that E_n is a decreasing sequence of closed disks with $\bigcap_n E_n = E = \varnothing$.

For $c \in \Bbbk$, there exists N such that $\forall n \ge N$, we have

$$c \notin E_n \implies |T-c|_x > |T-a_n|_x \implies |T-c|_x = |a_n - c|.$$

Thus

$$\inf_n |T-c|_{E_n} = \inf_n |a_n - c| = |T-c|_x.$$

By multiplicativity, we have $\inf_n |f|_{E_n} = |f|_x$ for each polynomial $f \in \Bbbk[T]$. And then by density of polynomials, the equality holds for each $f \in \Bbbk\{r^{-1}T\}$. Therefore, $x = x_{\{E_n\}} = x_{\{a_n,s_n\}}$ is of type IV. □

Proposition 23. The completed residue fields of the four types of points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ are described as follows:

- type I point x_a: $\mathscr{H}(x_a)$ is isomorphic to $\Bbbk$;
- type II point $x_{a,s}$: $\mathscr{H}(x_{a,s}) \cong \tilde{k}_{\Bbbk}((t))$;
- type III point $x_{a,s}$: $\tilde{k}_{\mathscr{H}(x_{a,s})} \cong \tilde{k}_{\Bbbk}$ and the value group $|\mathscr{H}(x_{a,s})^\times|$ is generated by $|\Bbbk^\times|$ and s;
- type IV point $x_{\{a_n,s_n\}}$: $\mathscr{H}(x_{\{a_n,s_n\}})$ is an immediate extension of $\Bbbk$.

Yang: To be checked.

Proof. Yang: To be completed. □

Example 24. The completed residue field $\mathscr{H}(x_a)$ for a type I point x_a with $a \in \Bbbk$ and $|a| \leq r$ is isomorphic to $\Bbbk$. Yang: To be complete.

Example 25. Let $\Bbbk$ be a complete algebraically closed non-archimedean field and $A = \Bbbk\{T/r\}$. We have $\widetilde{A} \cong \tilde{k}_{\Bbbk}[T]$. For a point $x_a \in \mathscr{M}(A)$ of type I corresponding to $a \in \Bbbk$ with $|a| \leq r$ (see Construction 21), the induced homomorphism $\widetilde{A} = \tilde{k}_{\Bbbk}[T] \to \tilde{k}_{\mathscr{H}(x_a)} = \tilde{k}_{\Bbbk}$ is given by $T \mapsto a \mod \Bbbk^{\circ\circ}$.

Yang: To be continued.

Spectrum of Tate algebra in servel variables Let $\Bbbk$ be a complete non-archimedean field, and let $A = \Bbbk\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$. We can consider the spectrum $\mathscr{M}(A)$ similarly.

Appendix

Corollary 26. Let (X, d) be an ultra-metric space. Then X is totally disconnected, i.e., the only connected subsets of X are the set with at most one point.

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