Local theory I: functions

1 Analytic functions on closed polydisks

Proposition 1. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we can associate a function $F_f : E(0,\underline{r}) \to \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \quad \text{for } x = (x_1, \dots, x_n) \in E(0, \underline{r}).$$

This defines a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T/r}\}$ to the ring of all functions from $E(0,\underline{r})$ to $\mathbf{k}$.

Proof. Given $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \dots, x_n) \in E(0,\underline{r})$, we have

$$\left\| \sum_{|\alpha|=n} a_\alpha x^\alpha \right\| \le \max_{|\alpha|=n} \|a_\alpha\| r^\alpha \to 0 \quad \text{as } n \to +\infty.$$

Hence by ??, the series $F_f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$ converges in $\mathbf{k}$. This defines a function $F_f : E(0,\underline{r}) \to \mathbf{k}$.

Let $g = \sum_{\alpha \in \mathbb{N}^n} b_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. Set

$$A_n = \sum_{|\alpha|<n} a_\alpha x^\alpha, \quad B_n = \sum_{|\beta|<n} b_\beta x^\beta, \quad C_n = \sum_{|\gamma|<n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) x^\gamma.$$

We need to show that $F_f(x) F_g(x) = \lim A_n B_n = \lim C_n = F_{fg}(x)$. Note that

$$A_n B_n - C_n = \sum_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta| \ge n}} a_\alpha b_\beta x^{\alpha+\beta}.$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon / \|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon / \|f\|$. For any $n > 2N$, we have

$$\|A_n B_n - C_n\| \le \max_{\substack{|\alpha|<n, |\beta|<n \\ |\alpha+\beta| \ge n}} \|a_\alpha\| \|b_\beta\| \|x^{\alpha+\beta}\| < \max\left\{ \frac{\varepsilon}{\|g\|} \|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|} \|a_\alpha\| r^\alpha \right\} \le \varepsilon.$$

Thus $F_f(x) F_g(x) = (F_{fg})(x)$. The addition and scalar multiplication can be verified directly. We thus finish the proof. ☐

Proposition 2. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x, y \in E(0,\underline{r})$, we have

$$\|f(y) - f(x)\|_{\mathbf{k}} \le L \cdot \|y - x\|_\infty,$$

where $L = \max_{1 \le i \le n} \|f\|_g / r_i$.

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Proof. Set $y - x = (h_1, \dots, h_n)$ and $x^{(0)} = x$, $x^{(i)} = (x_1 + h_1, \dots, x_i + h_i, x_{i+1}, \dots, x_n)$ for $i = 1, \dots, n$. We have

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{1 \leq i \leq n} \|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}}.$$

We only need to show that for every $i = 1, \dots, n$, we have

$$\|f(x^{(i)}) - f(x^{(i-1)})\|_{\mathbf{k}} \leq \frac{\|f\|_g}{r_i} \|h_i\|.$$

Without loss of generality and for simplicity, we assume that $y = (x_1 + h, x_2, \dots, x_n)$ and $x = (x_1, x_2, \dots, x_n)$. Note that by the strong triangle inequality, we have $\|h\| \leq r_1$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\begin{aligned} f(y) - f(x) &= \sum_{\alpha \in \mathbb{N}^n} a_\alpha \left((x_1 + h)^{\alpha_1} - x_1^{\alpha_1} \right) x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ &= \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{\alpha_1} \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h^k. \end{aligned}$$

Note that

$$\left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| r_1^k \leq \|a_\alpha\| r^\alpha \leq \|f\|_g.$$

It follows that

$$\|f(y) - f(x)\|_{\mathbf{k}} \leq \max_{\alpha \in \mathbb{N}^n} \max_{1 \leq k \leq \alpha_1} \left\{ \left\| \binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1 - k} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right\| \|h\|^k \right\} \leq \max_k \left\{ \|f\|_g \left(\frac{\|h\|}{r_1} \right)^k \right\} \leq \|f\|_g \frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

Lemma 3. Let $\mathbf{k}$ be a complete non-archimedean field. Then we have $\|f(x)\| \leq \|f\|$ for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0, \underline{r})$. In particular, if $f_n \to f$ as $n \to +\infty$ in $\mathbf{k}\{\underline{T/r}\}$, then we have $\|f_n(x) - f(x)\| \to 0$ for every $x \in E(0, \underline{r})$.

Proof. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $x = (x_1, \dots, x_n) \in E(0, \underline{r})$. We have

$$\left\| \sum_{|\alpha| < N} a_\alpha x^\alpha \right\| \leq \max_{|\alpha| < N} \|a_\alpha\| r^\alpha \leq \|f\|$$

for every $N \in \mathbb{N}$. Taking $N \to +\infty$, we have $\|f(x)\| \leq \|f\|$. □

Let $\mathbf{k}$ be a complete non-archimedean field. Recall that the formal derivative operator $\partial_i : \mathbf{k}[[\underline{T}]] \to \mathbf{k}[[\underline{T}]]$ is defined by

$$\frac{\partial}{\partial T_i} \left(\sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \right) := \sum_{\alpha \in \mathbb{N}^n} \alpha_i a_\alpha T_1^{\alpha_1} \cdots T_i^{\alpha_i - 1} \cdots T_n^{\alpha_n}.$$

Lemma 4. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we have $\partial_i(f) \in \mathbf{k}\{\underline{T/r}\}$.

Proof. Suppose that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$. We have

$$\frac{\partial f}{\partial T_1} = \sum_{\alpha\in\mathbb{N}^n} \alpha_1 a_\alpha T_1^{\alpha_1-1}T_2^{\alpha_2}\cdots T_n^{\alpha_n}.$$

Noting that $\mathbf{k}$ is non-archimedean, we have $\|\alpha_1 a_\alpha\| \le \|a_\alpha\|$. Then

$$\lim_{|\alpha|\to+\infty} \|\alpha_1 a_\alpha\| r_1^{\alpha_1-1}r_2^{\alpha_2}\cdots r_n^{\alpha_n} \le \frac{1}{r_1}\lim_{|\alpha|\to+\infty}\|a_\alpha\| r^\alpha = 0.$$

The conclusion follows. □

Proposition 5. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation, and $\partial_i = \partial/\partial T_i$ be the derivative operator on $\mathbf{k}\{\underline{T/r}\}$ with respect to the indeterminate T_i for $i = 1,\dots,n$. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$ and $x \in E(0,\underline{r})$, we have

$$F_{\partial_i(f)}(x) = \lim_{h\to 0}\frac{F_f(x_1,\dots,x_i+h,\dots,x_n) - F_f(x)}{h}.$$

Proof. Without loss of generality, we can assume that $i = 1$. Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \mathbf{k}\{\underline{T/r}\}$ and $f_n = \sum_{|\alpha|<n} a_\alpha T^\alpha$ for $n \in \mathbb{N}$. Set $x_h = (x_1+h, x_2\dots,x_n)$ and $L_f(h) = (F_f(x_h) - F_f(x))/h$ for $h \in \mathbf{k}^\times$. Note that for fixed h, we have $\lim_{n\to\infty} L_{f_n}(h) = L_f(h)$.

We compute $L_{f_n}(h) - F_{\partial f_n}(x)$ explicitly:

$$\begin{aligned} L_{f_n}(h) - F_{\partial f_n}(x) &= \frac{1}{h}\left(\sum_{|\alpha|<n}\sum_{k=1}^{\alpha_1}\binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k}h^k x_2^{\alpha_2}\cdots x_n^{\alpha_n} - \sum_{|\alpha|<n}\alpha_1 a_\alpha x_1^{\alpha_1-1}h x_2^{\alpha_2}\cdots x_n^{\alpha_n}\right)\\ &= \sum_{|\alpha|<n}\sum_{k=2}^{\alpha_1}\binom{\alpha_1}{k} a_\alpha x_1^{\alpha_1-k}x_2^{\alpha_2}\cdots x_n^{\alpha_n}h^{k-1}. \end{aligned}$$

Note that

$$M = \sup_{\alpha\in\mathbb{N}^n}\|a_\alpha x_1^{\alpha_1-k}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\| r_1^{k-1} \le \|f\|/r_1 < +\infty.$$

Hence

$$\|L_{f_n}(h) - F_{\partial f_n}(x)\| \le \max_{2\le k\le n}\left\{M\frac{\|h\|^{k-1}}{r_1^{k-1}}\right\} \le M\frac{\|h\|}{r_1}$$

for $h \in \mathbf{k}^\times$ with $\|h\| < r_1$. Taking $n \to +\infty$, we have

$$\|L_f(h) - F_{\partial f}(x)\| \le M\frac{\|h\|}{r_1}.$$

Thus the conclusion follows. □

Yang: The following should be a theorem.

Corollary 6. Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation of characteristic zero. Then the assignment $f \mapsto F_f$ in Proposition 1 is injective.

Proof. Note that if $F_f = 0$, then for every $i = 1,\dots,n$, we have $F_{\partial_i(f)} = 0$ by Proposition 5. By taking repeated derivatives, we have $F_{\partial^\alpha f} = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Note that $F_{\partial^\alpha f}(0) = \alpha! a_\alpha$. It follows that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n$ and thus $f = 0$. □

Remark 7. Corollary 6 holds for non-archimedean fields of positive characteristic as well. The proof uses **??** and induction on the number of variables. The readers can try this as an exercise.

From now on, we will identify an element $f \in \mathbf{k}\{\underline{T/r}\}$ with the associated function $F_f : E(0,\underline{r}) \to \mathbf{k}$ as in Proposition 1.

Proposition 8. Let $\Bbbk$ be a complete, non-archimedean and algebraically closed field. Then the gauss norm on the Tate algebra $\Bbbk\{\underline{T/r}\}$ coincides with the supremum norm

$$\|f\|_{\sup} := \sup_{x\in E(0,\underline{r})} \|f(x)\|_{\Bbbk}.$$

Proof. Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \in \Bbbk\{\underline{T/r}\}$. We write $f = g+h$ with $g = \sum_{\alpha\in S} a_\alpha T^\alpha$ and $h = \sum_{\alpha\notin S} a_\alpha T^\alpha$, where

$$S = \{\alpha \in \mathbb{N}^n : \|a_\alpha\| r^\alpha = \|f\|\}.$$

Note that S is a non-empty finite set and $\|h\| < \|f\|$. By Lemma 3, we have $\|h(x)\| < \|f\|$ for every $x \in E(0,\underline{r})$. It suffices to show that $\|g\|_{\sup} = \|g\|$.

Since $\Bbbk$ is algebraically closed, $|\Bbbk^\times|$ is dense in $\mathbb{R}_{>0}$. For every pair $\alpha,\beta \in S$ with $\alpha \neq \beta$, the set $\{t \in \mathbb{R}^n_{>0} : \|a_\alpha\| t^\alpha = \|a_\beta\| t^\beta\}$ is a proper closed subset of $\mathbb{R}^n_{>0}$. Thus we can find $t_m \in |\Bbbk^\times|^n$ such that $t_m < r$, $t_m \to r$ as $m \to +\infty$ and for every $\alpha,\beta \in S$ with $\alpha \neq \beta$, we have $\|a_\alpha\| t_m^\alpha \neq \|a_\beta\| t_m^\beta$ for all m. For each m, we can find $x_m \in E(0,\underline{r})$ such that $\|x_m^\alpha\| = t_m^\alpha$ for every $\alpha \in S$ since $t_m \in |\Bbbk^\times|^n$. It follows that

$$\|g(x_m)\| = \max_{\alpha\in S} \|a_\alpha\| \|x_m^\alpha\| = \max_{\alpha\in S} \|a_\alpha\| t_m^\alpha \to \|g\| \quad \text{as } m \to +\infty.$$

Thus $\|g\|_{\sup} = \|g\|$. □

Remark 9. If $\mathbf{k}$ is locally compact (hence not algebraically closed), the gauss norm on the Tate algebra $\mathbf{k}\{\underline{T/r}\}$ do not coincide with the supremum norm. For example, consider the Tate algebra $\mathbb{Q}_p\{T\}$. The element $f = T^p - T$ has gauss norm $\|f\| = 1$. However, for every $x \in E(0,1) = \mathbb{Z}_p$, we have $f(x) = x^p - x \equiv 0 \mod p$. Thus $\|f(x)\|_p \leq 1/p$ and $\|f\|_{\sup} \leq 1/p < 1 = \|f\|$.

Remark 10. Recall that in classical complex analysis, the closure of the polynomial ring $\mathbb{C}[T_1,\dots,T_n]$ with respect to the supremum norm on a closed polydisc $E(0,\underline{r}) \subset \mathbb{C}^n$ is the ring of all complex-valued continuous functions which are analytic on its interior $B(0,\underline{r})$.

Yang: Invertiblity of a function

Appendix