Local theory II: maps

Let $\mathbf{k}$ be a complete non-archimedean field.

1 The first properties

Yang: Recall the Runge theorem in complex analysis.

Definition 1. A map $f : (E(0,r) \subset \mathbf{k}^n) \to \mathbf{k}^m$ is called *analytic* if there exists power series $f_1, \dots, f_m \in \mathbf{k}\{\underline{T/r}\}$ such that for any $x \in E(0,r)$, we have

$$f(x) = (f_1(x), \dots, f_m(x)).$$

Yang: To be revised.

Yang: Composition of analytic functions.

Definition 2. A map $f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^m$ is called *analytic* if there exists power series $f_1, \dots, f_m \in \mathbf{k}\{\underline{T/r}\}$ such that for any $x \in E(0,\underline{r})$, we have

$$f(x) = (f_1(x), \dots, f_m(x)).$$

Proposition 3. Let $f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^m$ and $g : (E(0,\underline{s}) \subset \mathbf{k}^m) \to \mathbf{k}^l$ be two analytic maps such that $f(E(0,\underline{r})) \subset E(0,\underline{s})$. Then the composition $g \circ f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^l$ is also analytic. Furthermore, if $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_l)$ with $f_i = \sum_\alpha a_{i,\alpha} \underline{T}^\alpha$ and $g_j = \sum_\beta b_{j,\beta} \underline{T}^\beta$, then the composition $g \circ f = (h_1, \dots, h_l)$ with

$$h_j = \sum_\beta b_{j,\beta} f^\beta = \sum_\beta b_{j,\beta} f_1^{\beta_1} \cdots f_m^{\beta_m}.$$

Yang: To be checked. Yang: To be revised.

Proof. Yang: To be completed. ☐

2 Inverse and implicit function

Definition 4. Let $f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^m$ be an analytic map. The *tangent map* $\mathrm{d}f_0 : \mathbf{k}^n \to \mathbf{k}^m$ of f at $\mathbf{0}$ is defined to be the linear map given by the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial T_j}(0) \right)_{1 \le i \le m, 1 \le j \le n}.$$

Yang: To be checked.

Theorem 5 (Inverse Function Theorem over Non-Archimedean Fields)**.** Let $f : (E(0,\underline{r}) \subset \mathbf{k}^n) \to \mathbf{k}^n$ be an analytic map. Suppose that $f(0) = \mathbf{0}$ and the tangent map $\mathrm{d}f_0 : \mathbf{k}^n \to \mathbf{k}^n$ is an

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isomorphism.
Then there exist $E(0,\underline{r'}) \subset E(0,\underline{r})$, $E(0,\underline{s'}) \subset f(E(0,\underline{r}))$ and an analytic map $g : (E(0,\underline{s'}) \subset \mathbf{k}^n) \to \mathbf{k}^n$ such that

$$f \circ g = \mathrm{id}_{E(0,\underline{s'})}, \quad g \circ f = \mathrm{id}_{E(0,\underline{r'})}.$$

Proof. Yang: To be completed. □

Theorem 6 (Implicit Function Theorem over Non-Archimedean Fields)**.** Let $f : (E(0,\underline{r}) \subset \mathbf{k}^{n+m}) \to \mathbf{k}^m, (x_1, \cdots, x_n, y_1, \dots, y_m) \mapsto f(x,y)$ be an analytic map. Suppose that $f(0) = 0$ and the Jacobian matrix $\left(\partial_j f_i(0)\right)_{1 \le i,j \le m}$ is invertible.
Then there exist $\underline{r'} = (r_1', \dots, r_n')$ with each $r_i' > 0$ and an analytic map $g : (E(0,\underline{r'}) \subset \mathbf{k}^n) \to \mathbf{k}^m$ such that for any $x \in E(0,\underline{r'})$,

$$f(x,y) = 0 \iff y = g(x).$$

Proof. Yang: To be completed. □

Appendix