Berkovich Space

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1 Spectrum

Let $\mathbf{k}$ be a spherically complete non-archimedean field which is algebraically closed and $A = \mathbf{k}[T]$. We want to consider the “analytic structure” on $\operatorname{mSpec} A$. However, unlike the complex case, the set $\operatorname{mSpec} A$ is totally disconnected with respect to the topology induced by the absolute value on $\mathbf{k}$ (**??**). To overcome this difficulty, Berkovich uses multiplicative semi-norms to “fill in the gaps” between the points in $\operatorname{mSpec} A$, leading to the notion of the spectrum of a Banach ring.

We first consider the local model. Hence we should consider the Tate algebra $\mathbf{k}\{T\}$ instead of the polynomial ring $\mathbf{k}[T]$. Yang: The maximal ideal of $\mathbf{k}\{T\}$ corresponding to the point in the disk $\{a \in \mathbf{k} : a \leq 1\}$. Yang: Closed or open disk?

1.1 Definition

Definition 1.1. Let R be a Banach ring. The *spectrum* $\mathscr{M}(R)$ of R is defined as the set of all multiplicative semi-norms on R that are bounded with respect to the given norm on R. For every point $x \in \mathscr{M}(R)$, we denote the corresponding multiplicative semi-norm by $|\cdot|_x$.
We equip $\mathscr{M}(R)$ with the weakest topology such that for each $f \in R$, the evaluation map $\mathscr{M}(R) \to \mathbb{R}_{\geq 0}$, defined by $x \mapsto |f|_x =: f(x)$, is continuous.

Example 1.2. Let $(\mathbf{k}, |\cdot|)$ be a complete valuation field. The spectrum $\mathscr{M}(\mathbf{k})$ consists of a single point corresponding to the given absolute value $|\cdot|$ on $\mathbf{k}$. Yang: To be checked.

Example 1.3. Consider the Banach ring $(\mathbb{Z}, \|\cdot\|)$ with $\|\cdot\| = |\cdot|_\infty$ is the usual absolute value norm on $\mathbb{Z}$. Let $|\cdot|_p$ denote the p-adic norm for each prime number p, i.e., $|n|_p = p^{-v_p(n)}$ for each $n \in \mathbb{Z}$, where $v_p(n)$ is the p-adic valuation of n. The spectrum

$$\mathscr{M}(\mathbb{Z}) = \{|\cdot|_\infty^\varepsilon : \varepsilon \in (0,1]\} \cup \{|\cdot|_p^\alpha : p \text{ is prime}, \alpha \in (0,\infty]\} \cup \{|\cdot|_0\},$$

where $|a|_p^\infty := \lim_{\alpha\to\infty} |a|_p^\alpha$ for each $a \in \mathbb{Z}$ and $|\cdot|_0$ is the trivial norm on $\mathbb{Z}$. Yang: To be checked.

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Definition 1.4. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The *pullback map* $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is defined by $\mathscr{M}(\varphi)(x) = x \circ \varphi : f \mapsto |\varphi(f)|_x$ for each $x \in \mathscr{M}(S)$.

Proposition 1.5. Let $\varphi : R \to S$ be a bounded ring homomorphism of Banach rings. The pullback map $\mathscr{M}(\varphi) : \mathscr{M}(S) \to \mathscr{M}(R)$ is continuous.

Proof. Yang: To be completed. □

Definition 1.6. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, the *completed residue field* at the point x is defined as the completion of the residue field $\kappa(x) = \mathrm{Frac}(R/\wp_x)$ with respect to the multiplicative norm induced by the semi-norm $|\cdot|_x$, denoted by $\mathscr{H}(x)$.

Definition 1.7. Let R be a Banach ring. The *Gel'fand transform* of R is the bounded ring homomorphism

$$\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x), \quad f \mapsto (f(x))_{x \in \mathscr{M}(R)},$$

where the norm on the product $\prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is given by the supremum norm.

Proposition 1.8. The Gel'fand transform $\Gamma : R \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ of a Banach ring R factors through the uniformization R^u of R, and the induced map $R^u \to \prod_{x \in \mathscr{M}(R)} \mathscr{H}(x)$ is an isometric embedding. Yang: To be checked.

Theorem 1.9. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is nonempty.

Proof. Yang: To be continued. □

Lemma 1.10. Let $\{K_i\}_{i \in I}$ be a family of completed fields. Consider the Banach ring $R = \prod_{i \in I} K_i$ equipped with the product norm. The spectrum $\mathscr{M}(R)$ is homeomorphic to the Stone-Čech compactification of the discrete space I.

Remark 1.11. The Stone-Čech compactification of a discrete space is the largest compact Hausdorff space in which the original space can be densely embedded. Yang: To be checked.

Theorem 1.12. Let R be a Banach ring. The spectrum $\mathscr{M}(R)$ is a compact Hausdorff space.

Proof. Yang: To be added. □

Proposition 1.13. Let K/k be a Galois extension of complete fields, and let R be a Banach k-algebra. The Galois group $\mathrm{Gal}(K/k)$ acts on the spectrum $\mathscr{M}(R \widehat{\otimes}_k K)$ via

$$g \cdot x : f \mapsto |(1 \otimes g^{-1})(f)|_x$$

for each $g \in \mathrm{Gal}(K/k)$, $x \in \mathscr{M}(R \widehat{\otimes}_k K)$ and $f \in R \widehat{\otimes}_k K$. Moreover, the natural map $\mathscr{M}(R \widehat{\otimes}_k K) \to \mathscr{M}(R)$ induces a homeomorphism

$$\mathscr{M}(R \widehat{\otimes}_k K) / \mathrm{Gal}(K/k) \xrightarrow{\sim} \mathscr{M}(R).$$

Yang: To be checked.

1.2 Reduction map and kernel map

Proposition 1.14. Let R be a Banach ring. For each $x \in \mathscr{M}(R)$, let $\wp_x$ be the kernel of the multiplicative semi-norm $|\cdot|_x$. Then $\wp_x$ is a closed prime ideal of R, and $x \mapsto \wp_x$ defines a continuous map from $\mathscr{M}(R)$ to $\mathsf{Spec}(R)$ equipped with the Zariski topology.

Proof. Yang: To be completed □

Suppose that R is a non-archimedean Banach ring with valuation subring R° and maximal ideal $R^{\circ\circ}$. There is a natural reduction map from the spectrum $\mathscr{M}(R)$ to the spectrum $\mathsf{Spec}(\widetilde{R} = R^\circ/R^{\circ\circ})$ of the residue ring $\widetilde{R}$.

Yang: To be continued.

1.3 Spectrum of Tate algebras

Spectrum of Tate algebra in one variable Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{T/r\}$. We list some types of points in the spectrum $\mathscr{M}(A)$.

For each $a \in \mathbf{k}$ with $|a| \leq r$, we have the *type I* point x_a corresponding to the evaluation at a, i.e., $|f|_{x_a} := |f(a)|$ for each $f \in A$. For each closed disk $E = E(a, s) := \{b \in \mathbf{k} : |b - a| \leq s\}$ with center $a \in \mathbf{k}$ and radius $s \leq r$, we have the point $x_{a,s}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_E} := \sup_{b \in E(a,s)} |f(b)|$$

for each $f \in A$. If $s \in |\mathbf{k}^\times|$, then the point x_E is called a *type II* point; otherwise, it is called a *type III* point.

Let $\{E^{(s)}\}_s$ be a family of closed disks in $\mathbf{k}$ such that $E^{(s)}$ is of radius s, $E^{(s_1)} \subsetneq E^{(s_2)}$ for any $s_1 < s_2$ and $\bigcap_s E^{(s)} = \varnothing$. Then we have the point $x_{\{E^{(s)}\}}$ corresponding to the multiplicative semi-norm defined by

$$|f|_{x_{\{E^{(s)}\}}} := \inf_s |f|_{x_{E^{(s)}}}$$

for each $f \in A$. Such a point is called a *type IV* point.

Yang: To be completed.

Proposition 1.15. The points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ can be classified into four types as described above. Yang: To be checked

Proof. Yang: To be completed. □

Proposition 1.16. The completed residue fields of the four types of points in the spectrum $\mathscr{M}(\mathbf{k}\{r^{-1}T\})$ are described as follows:

- type I point x_a: $\mathscr{H}(x_a)$ is isomorphic to $\mathbf{k}$;
- type II point $x_{a,s}$: $\mathscr{H}(x_{a,s}) \cong \mathscr{k}_{\mathbf{k}}((t))$;
- type III point $x_{a,s}$: $\mathscr{k}_{\mathscr{H}(x_{a,s})} \cong \mathscr{k}_{\mathbf{k}}$ and the value group $|\mathscr{H}(x_{a,s})^\times|$ is generated by $|\mathbf{k}^\times|$ and s;
- type IV point $x_{\{E^{(s)}\}}$: $\mathscr{H}(x_{\{E^{(s)}\}})$ is an immediate extension of $\mathbf{k}$.

Yang: To be checked.

Example 1.17. The completed residue field $\mathscr{H}(x_a)$ for a type I point x_a with $a \in \mathbf{k}$ and $|a| \leq r$ is isomorphic to $\mathbf{k}$. Yang: To be complete.

Spectrum of Tate algebra in servel variables Let $\mathbf{k}$ be a complete non-archimedean field, and let $A = \mathbf{k}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$. We can consider the spectrum $\mathscr{M}(A)$ similarly.

2 Affinoid domains

Consider $X = \mathscr{M}(A)$ with $A = \mathbf{k}\{T_1, \ldots, T_n\}$. Yang: Not every open subset of X gives an affinoid space, that is, the completion of the ring of analytic functions on that open subset is not necessarily an affinoid algebra. Yang: Right? example?

2.1 Definition

Definition 2.1. Let A be a $\mathbf{k}$-affinoid algebra, and let $X = \mathscr{M}(A)$ be the associated affinoid space. A closed subset $V \subseteq X$ is called an *affinoid domain* if there exists a $\mathbf{k}$-affinoid algebra A_V and a morphism of $\mathbf{k}$-affinoid algebras $\varphi : A \to A_V$ satisfying the following universal property: for every bounded homomorphism of $\mathbf{k}$-affinoid algebras $\psi : A \to B$ such that the induced map on spectra $\mathscr{M}(\psi) : \mathscr{M}(B) \to X$ has its image contained in V, there exists a unique bounded homomorphism $\theta : A_V \to B$ such that the following diagram commutes:

$$\begin{tikzcd} & A_V \arrow[dr, "\theta"] & \\ A \arrow[ur, "\varphi"] \arrow[rr, "\psi"'] & & B \end{tikzcd}$$

In this case, we say that V is represented by the affinoid algebra A_V.

Slogan *A closed subset $V \subset X$ is an affinoid domain if the functor "$\mathrm{Mor}(-, V)$" is representable.*

Yang: Why we consider closed subset rather that open subset?

Construction 2.2. Let $f = (f_1, \ldots, f_n)$ be a tuple of elements in A and $r = (r_1, \ldots, r_n)$ be a tuple of positive real numbers. Consider the closed subset of X:

$$X\left(\underline{f/r}\right) := \{x \in X : |f_i(x)| \leq r_i, 1 \leq i \leq n\}.$$

Such a closed subset is called a *Weierstrass domain* of X. Moreover, we can define a $\mathbf{k}$-affinoid algebra

$$A\left\{\underline{f/r}\right\} := A\{f_1/r_1, \ldots, f_n/r_n\}.$$

Yang: The domain $X\left(\underline{f/r}\right)$ is represented by $A\left\{\underline{f/r}\right\}$.

Construction 2.3. Let $f = (f_1, \dots, f_n), g = (g_1, \dots, g_m)$ be two tuples of elements in A and $r = (r_1, \dots, r_n), s = (s_1, \dots, s_m)$ be two tuples of positive real numbers. Consider the following closed subset of X:

$$X\left(\underline{f/r}; \underline{g/s}^{-1}\right) := \left\{x \in X : |f_i(x)| \leq r_i, |g_j(x)| \geq s_j, 1 \leq i \leq n, 1 \leq j \leq m\right\}.$$

Such a closed subset is called a *Laurent domain* of X. Moreover, we can define a **k**-affinoid algebra

$$A\left\{\underline{f/r}; \underline{g/s}^{-1}\right\} := A\left\{f_1/r_1, \dots, f_n/r_n, g_1^{-1}/s_1, \dots, g_m^{-1}/s_m\right\}.$$

Yang: The domain $X\left(\underline{f/r}; \underline{g/s}^{-1}\right)$ is represented by $A\left\{\underline{f/r}; \underline{g/s}^{-1}\right\}$.

Construction 2.4. Let $f = (f_1, \dots, f_n), g$ be elements in A such that the ideal generated by them is the whole algebra A. Set $p = (p_1, \dots, p_n)$ be a tuple of positive real numbers. We define the following closed subset of X:

$$X\left(\underline{f/p}, g\right) := \{x \in X : |f_i(x)| \leq p_i |g(x)|, 1 \leq i \leq n\}.$$

Such a closed subset is called a *rational domain* of X. Moreover, we can define a **k**-affinoid algebra

$$A\left\langle \underline{f/p}, g^{-1}\right\rangle := A\left\langle \frac{f_1}{p_1 g}, \dots, \frac{f_n}{p_n g}\right\rangle,$$

which is the quotient of the Tate algebra

$$A\langle T_1, \dots, T_n\rangle$$

by the ideal generated by the elements $p_i g T_i - f_i$ for $1 \leq i \leq n$. There is a natural bounded homomorphism $\varphi : A \to A\langle \underline{f/p}, g^{-1}\rangle$ induced by the inclusion. It can be shown that the closed subset $X(\underline{f/p}, g)$ is an affinoid domain represented by the affinoid algebra $A\langle \underline{f/p}, g^{-1}\rangle$. Yang: To be checked

Yang: We have a sequence of inclusion:

$$\{\text{Weierstrass domains}\} \subseteq \{\text{Laurent domains}\} \subseteq \{\text{Rational domains}\} \subseteq \{\text{Affinoid domains}\}.$$

Proposition 2.5. Let A be a **k**-affinoid algebra, and let $X = \mathscr{M}(A)$ be the associated affinoid space. Let $V \subseteq X$ be an affinoid domain represented by the **k**-affinoid algebra A_V. Then the natural bounded homomorphism $\varphi : A \to A_V$ is flat.
We have $\mathscr{M}(A_V) \cong V$.

2.2 The Grothendieck topology of affinoid domains