Normed rings and modules

1 Semi-normed algebraic structures

Definition 1. Let G be an abelian group. A *semi-norm* on G is a function $\|\cdot\| : G \to \mathbb{R}_{\geq 0}$ such that

- $\|0\| = 0$;
- $\forall x, y \in G, \|x + y\| \leq \|x\| + \|y\|$.

Suppose that R is a ring (commutative with unity) and $\|\cdot\|$ is a semi-norm on the underlying abelian group of R. We further require that

- $\|1\| = 1$;
- $\forall x, y \in R, \|xy\| \leq \|x\|\|y\|$.

Suppose that $(M, \|\cdot\|_M)$ is an R-module and $\|\cdot\|_M$ is a semi-norm on the underlying abelian group of M. We further require that

- $\forall a \in R, x \in M, \|ax\|_M \leq \|a\|\|x\|_M$.

Suppose that $(A, \|\cdot\|_A)$ is an R-algebra and $\|\cdot\|_A$ is a semi-norm on the underlying R-module of A. We further require that this semi-norm is a semi-norm on the underlying ring of A.
If we further have $\forall x, \|x\| = 0 \implies x = 0$, then we say $\|\cdot\|$ is a *norm* on the corresponding algebraic structure.
If we replace the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ by the stronger inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$, then we say $\|\cdot\|$ is a *non-archimedean* semi-norm.

Definition 2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semi-norms on an abelian group (or ring, R-module, R-algebra) A. We say $\|\cdot\|_1$ is *bounded* by $\|\cdot\|_2$ if there exists a constant $C > 0$ such that $\forall x \in A, \|x\|_1 \leq C\|x\|_2$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are bounded by each other, we say they are *equivalent*.

Remark 3. Equivalent semi-norms induce the same topology on A. However, the converse is not true in general. Compare with Lemma 35.
Yang: what about on a module?

Definition 4. Let M be a semi-normed abelian group (or R-module) and $N \subseteq M$ be a subgroup (or R-submodule). The *residue semi-norm* on the quotient group M/N is defined as

$$\|x + N\|_{M/N} = \inf_{y \in N} \|x + y\|_M.$$

Unless otherwise specified, we always equip the quotient M/N with the residue semi-norm.

Remark 5. The residue semi-norm is a norm if and only if N is closed in M.

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Definition 6. Let M and N be two semi-normed abelian groups (or rings, R-modules, R-algebras). A homomorphism $f : M \to N$ is called *bounded* if there exists a constant $C > 0$ such that $\forall x \in M, \|f(x)\|_N \le C\|x\|_M$.
A bounded homomorphism $f : M \to N$ is called *admissible* if the induced isomorphism $M/\ker f \to \operatorname{Im} f$ is an isometry, i.e., $\forall x \in M, \|f(x)\|_N = \|x\|_{M/\ker f}$.

Definition 7. A semi-norm $\|\cdot\|$ on a ring R is called *multiplicative* if $\forall x, y \in R, \|xy\| = \|x\|\|y\|$. It is called *power-multiplicative* if $\forall x \in R, \|x^n\| = \|x\|^n$ for all integers $n \ge 1$. A multiplicative norm sometimes is called a *(multiplicative) valuation* or an *absolute value*.

Example 8. Let R be arbitrary ring. The *trivial norm* on R is defined as $\|x\| = 0$ if $x = 0$ and $\|x\| = 1$ if $x \neq 0$. The ring R equipped with the trivial norm is a valuation ring.

Example 9. A valuation field $(\mathbf{k}, |\cdot|)$ can be viewed as a valuation ring.

Example 10. Let $|\cdot| = |\cdot|_\infty$ be the usual absolute value on $\mathbb{Z}$. Then $(\mathbb{Z}, |\cdot|)$ is a valuation ring.

Example 11. Let X be a compact Hausdorff topological space. The ring $C(X, \mathbb{R})$ of continuous real-valued functions on X equipped with the norm $\|f\| = \sup_{x\in X} |f(x)|$ is a normed ring. Its norm is power-multiplicative but not multiplicative in general. It is worth mentioning that the Gelfand-Kolmogorov Theorem saying that we can recover X from the normed ring $C(X, \mathbb{R})$.

Definition 12. A (semi-)norm on an abelian group M induces a (pseudo-)metric $d(x, y) = \|x - y\|$ on M. A (semi-)normed abelian group M is called *complete* if it is complete as a (pseudo-)metric space.

Definition 13. A *banach ring* is a complete normed ring.

Proposition 14. Let R be a banach ring and $I \subseteq R$ be a closed ideal. Then the residue norm on the quotient ring R/I is a norm for rings.

Proof. Yang: To be added. □

Proposition 15. Let R be a banach ring. Then the group of invertible elements $R^\times$ is an open subset of R.

Proof. Yang: To be added. □

Corollary 16. Let R be a banach ring. Then every maximal ideal of R is closed.

Proof. Yang: To be added. □

Definition 17. Let $(A, \|\cdot\|_A)$ be a normed algebraic structure, e.g., a normed abelian group, a normed ring, or a normed module. The *completion* of A, denoted by $\widehat{A}$, is the completion of A as a metric space. Since A is dense in its completion and the algebraic operations are uniformly continuous, the algebraic operations on A can be uniquely extended to the completion.

Let R be a normed ring and M, N be semi-normed R-modules. There is a natural semi-norm on the tensor product $M \otimes_R N$ defined as

$$\|z\|_{M \otimes_R N} = \inf\left\{\sum_i \|x_i\|_M \|y_i\|_N : z = \sum_i x_i \otimes y_i, x_i \in M, y_i \in N\right\}.$$

Definition 18. Let R be a banach ring and M, N complete semi-normed R-modules. The *complete tensor product* $M \widehat{\otimes}_R N$ is defined as the completion of the semi-normed R-module $M \otimes_R N$.

Construction 19. Let R be a banach ring and $r > 0$ be a real number. We define the *ring of absolutely convergent power series* over $\mathbf{k}$ with radius r as

$$R\langle T/r \rangle := \left\{\sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \sum_{n=0}^{\infty} \|a_n\| r^n < \infty\right\}.$$

Equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| = \sum_{n=0}^{\infty} \|a_n\| r^n$, the ring $R\langle T/r\rangle$ is a banach ring.

When $R = \mathbf{k}$ is a Yang: To be checked.

Example 20. Yang: Example of complete tensor product.

2 Spectral radius

Definition 21. Let R be a banach ring. For each $f \in R$, the *spectral radius* of f is defined as

$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$

Since $\|\cdot\|$ is submultiplicative, the limit defining $\rho(f)$ exists and equals to $\inf_{n\geq 1} \|f^n\|^{1/n}$ by Fekete's Subadditive Lemma.

Proposition 22. Let $(R, \|\cdot\|)$ be a banach ring. The spectral radius $\rho(\cdot)$ defines a power-multiplicative semi-norm on R that is bounded by $\|\cdot\|$.

Proof. Yang: To be continued. □

Definition 23. A banach ring R is called *uniform* if its norm is power-multiplicative.

Definition 24. Let R be a banach ring. The *uniformization* of R, denoted by $R \to R^u$, is the banach ring with the universal property among all bounded homomorphisms from R to uniform banach rings. Yang: To be continued.

Definition 25. Let R be a banach ring. An element $f \in R$ is called *quasi-nilpotent* if $\rho(f) = 0$. All quasi-nilpotent elements of R form an ideal, denoted by $\mathrm{Qnil}(R)$.

Proposition 26. Let R be a banach ring. The completion of $R/\mathrm{Qnil}(R)$ with respect to the spectral radius $\rho(\cdot)$ is the uniformization of R.

Proof. Yang: To be continued. □

Example 27. Let R be a banach ring and $r > 0$ be a real number. Consider the ring of absolutely convergent power series $R\langle T/r\rangle$ defined in Construction 19. For each $f = \sum_{n=0}^{\infty} a_n T^n \in R\langle T/r\rangle$, we

have
$$\rho(f) = \max_{n\geq 0} \|a_n\| r^n.$$
Thus the uniformization of $R\langle T/r\rangle$ is given by the ring
$$R\{T/r\} = \left\{ \sum_{n=0}^{\infty} a_n T^n \in R[[T]] : \lim_{n\to\infty} \|a_n\| r^n = 0 \right\},$$
equipped with the norm $\|\sum_{n=0}^{\infty} a_n T^n\| = \max_{n\geq 0} \|a_n\| r^n$. Yang: To be revised.

Yang: To be continued...

3 Non-archimedean case

Notation 28. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates, $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers, and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. We use the following notations:

- $T^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}$ and $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}$;
- $\underline{T/r} := (T_1/r_1, T_2/r_2, \dots, T_n/r_n)$;
- $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- $\alpha \leq_{\text{total}} \beta$ if and only if for all $i = 1, \dots, n$, we have $\alpha_i \leq \beta_i$;
- Let $\{x_\alpha\}_{\alpha\in\mathbb{N}^n}$ be a set of elements in a metric space X indexed by multi-indices $\alpha \in \mathbb{N}^n$. We say that $\lim_{|\alpha|\to+\infty} x_\alpha = x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| > N$, we have $d(x_\alpha, x) < \varepsilon$.

Definition 29. Let R be a non-archimedean banach ring. Let $T = (T_1, \dots, T_n)$ be a tuple of n indeterminates and $r = (r_1, \dots, r_n)$ be a tuple of n positive real numbers. The *Tate algebra* (or *ring of restricted power series*) is defined as
$$R\langle \underline{r^{-1}T}\rangle := R\{\underline{r^{-1}T}\} := \left\{ \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \mid a_\alpha \in R, \lim_{|\alpha|\to+\infty} \|a_\alpha\| r^\alpha = 0 \right\}.$$

Proposition 30. Let R be a non-archimedean banach ring. Then the Tate algebra $R\{\underline{T/r}\}$ is a non-archimedean multiplicative banach R-algebra with respect to the *gauss norm*
$$\left\| \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha \right\| := \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha\| r^\alpha = \max_{\alpha\in\mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

Proof. The proof splits into several parts. Every parts is straightforward and standard.

Step 1. We first show that $R\{\underline{T/r}\}$ is a R-algebra.

Easily to see that it is closed under addition and scalar multiplication. Suppose that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$ and $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha T^\alpha$ are two nonzero elements in $R\{\underline{T/r}\}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_\alpha\| r^\alpha < \varepsilon/\|g\|$ and $\|b_\alpha\| r^\alpha < \varepsilon/\|f\|$. For any $|\gamma| > 2N$, we have

$$\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \leq \max_{\alpha+\beta=\gamma} \|a_\alpha\| r^\alpha \cdot \|b_\beta\| r^\beta < \max\left\{\frac{\varepsilon}{\|g\|}\|b_\beta\| r^\beta, \frac{\varepsilon}{\|f\|}\|a_\alpha\| r^\alpha\right\} \leq \varepsilon.$$

Hence $f \cdot g \in R\{\underline{T/r}\}$ and it shows that $R\{\underline{T/r}\}$ is a R-algebra.

Step 2. Show that the gauss norm is a non-archimedean norm on $R\{\underline{T/r}\}$.

The linearity and positive-definiteness of the gauss norm are direct from the definition. We have

$$\|f+g\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha + b_\alpha\| r^\alpha \leq \sup_{\alpha\in\mathbb{N}^n} \max\{\|a_\alpha\| + \|b_\alpha\|\} r^\alpha \leq \max\{\|f\|, \|g\|\}$$

and

$$\begin{aligned} \|f \cdot g\| &= \left\|\sum_{\gamma\in\mathbb{N}^n}\left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) T^\gamma\right\| = \sup_{\gamma\in\mathbb{N}^n}\left\|\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right\| r^\gamma \\ &\leq \sup_{\gamma\in\mathbb{N}^n}\max_{\alpha+\beta=\gamma} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \|a_{\alpha_0}\| r^{\alpha_0} \cdot \|b_{\beta_0}\| r^{\beta_0} \leq \|f\| \cdot \|g\|. \end{aligned}$$

These show that Tate algebra with the gauss norm is a non-archimedean normed $\mathbf{k}$-algebra.

Step 3. Show that the gauss norm is multiplicative.

Suppose that $\|f\| = \|a_{\alpha_1}\| r^{\alpha_1}$ and $\|a_\alpha\| r^\alpha < \|f\|$ for all $\alpha <_{\text{total}} \alpha_1$. Similar to $\|b_{\beta_1}\| r^{\beta_1}$. Then we have

$$\|f\| \cdot \|g\| = \|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1} = \max_{\alpha+\beta=\alpha_1+\beta_1} \|a_\alpha\|\|b_\beta\| r^\alpha r^\beta = \left\|\sum_{\alpha+\beta=\alpha_1+\beta_1} a_\alpha b_\beta\right\| r^{\alpha_1+\beta_1} \leq \|f \cdot g\|,$$

where the third equality holds since (α_1, β_1) is the unique pair such that $\|a_{\alpha_1}\| r^{\alpha_1} \cdot \|b_{\beta_1}\| r^{\beta_1}$ is maximized and by Proposition 36. Thus the gauss norm is multiplicative.

Step 4. Finally show that $R\{\underline{T/r}\}$ is complete with respect to the gauss norm.

Let $\{f_m = \sum a_{\alpha,m} T^\alpha\}$ be a cauchy sequence in $R\{\underline{T/r}\}$. We have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\|.$$

Thus for each $\alpha \in \mathbb{N}^n$, the sequence $\{a_{\alpha,m}\}$ is a cauchy sequence in R. Since R is complete, set $a_\alpha := \lim_{m\to+\infty} a_{\alpha,m}$ and $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha$. Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, l > M$, we have $\|f_m - f_l\| < \varepsilon$. Fixing $m > M$, there exists $N \in \mathbb{N}$ such that for all $|\alpha| > N$, we have $\|a_{\alpha,m}\| r^\alpha < \varepsilon$. Hence for all $|\alpha| > N$ and $l > M$, we have

$$\|a_{\alpha,l}\| r^\alpha \leq \|a_{\alpha,l} - a_{\alpha,m}\| r^\alpha + \|a_{\alpha,m}\| r^\alpha < 2\varepsilon.$$

Taking $l \to +\infty$, we have $\|a_\alpha\| r^\alpha \leq 2\varepsilon$ for all $|\alpha| > N$. It follows that $f \in \mathbf{k}\{\underline{T/r}\}$.

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, l > N$, we have $\|f_m - f_l\| < \varepsilon$. Thus for all $\alpha \in \mathbb{N}^n$ and $m, l > N$, we have

$$\|a_{\alpha,m} - a_{\alpha,l}\| r^\alpha \leq \|f_m - f_l\| < \varepsilon.$$

Taking $l \to +\infty$, we have $\|a_{\alpha,m} - a_\alpha\| r^\alpha \leq \varepsilon$ for all $m > N$. It follows that

$$\|f - f_m\| = \sup_{\alpha\in\mathbb{N}^n} \|a_\alpha - a_{\alpha,m}\| r^\alpha \leq \varepsilon$$

for all $m > N$. □

Definition 31. Let R be a non-archimedean banach ring. We define

$$R^{\circ} = \{f \in R : \rho(f) \leq 1\}, \quad R^{\circ\circ} = \{f \in R : \rho(f) < 1\}.$$

The *reduction* of R is defined as the quotient ring

$$\widetilde{R} = R^{\circ}/R^{\circ\circ}.$$

For a non-archimedean field $\mathbf{k}$, its reduction ring $\widetilde{\mathbf{k}} = \mathscr{k}_{\mathbf{k}}$ is just the residue field of its valuation ring.

Example 32. Let R be a ring equipped with the trivial norm. Then we have $R^{\circ} = R$ and $R^{\circ\circ} = \mathsf{nil}(R)$.

Example 33. Let R be a non-archimedean banach ring and $A = R\{T\}$ be the Tate algebra in one variable over R. Then we have

$$A^{\circ} = \left\{ \sum_{n \geq 0} a_n T^n : |a_n| \leq 1 \text{ for all } n \in \mathbb{N} \right\} = R^{\circ}\{T\},$$

and

$$A^{\circ\circ} = \left\{ \sum_{n \geq 0} a_n T^n : |a_n| < 1 \text{ for all } n \in \mathbb{N} \right\} = R^{\circ\circ}\{T\}.$$

Since the norm of items in a restricted power series will tend to 0, we have

$$\widetilde{A} = \widetilde{R}[\underline{T}].$$

Example 34. Let R is a multiplicative non-archimedean banach ring. Set

$$\sqrt{|R|^{-1}} = \{r \in \mathbb{R}_{>0} : r^{-n} \in |R| \text{ for some } n \in \mathbb{N}_{>0}\}.$$

Fix $r \in \mathbb{R}^n_{>0}$, consider the Tate algebra $A = R\{T/r\}$.

Suppose that $r \in \sqrt{|R|^{-1}}$. Let n be the minimal positive integer such that $r^n \in |R|^{-1}$ and

$$\widetilde{M}_k := \{a \in R : |a| = r^{-nk}\}/\{a \in R : |a| < r^{-nk}\}.$$

For $a_m T^m$ with $n \nmid m$, we have$\|a_m T^m\| = |a_m| r^m \leq 1 \implies |a_m| r^m < 1$. Hence

$$\widetilde{R\{T/r\}} = \widetilde{R} \oplus \widetilde{M}_1 T^n \oplus \widetilde{M}_2 T^{2n} \oplus \widetilde{M}_3 T^{3n} \oplus \cdots.$$

In case $R = \mathbf{k}$ is a non-archimedean field, we have $\widetilde{M}_k \cong \widetilde{\mathbf{k}}$ by choosing an element $c \in \mathbf{k}$ with $|c| = r^{-n}$. Hence

$$\widetilde{\mathbf{k}\{T/r\}} \cong \mathscr{k}_{\mathbf{k}}[T^n].$$

Suppose that $r \notin \sqrt{|R|^{-1}}$. Then for every $\|a_n T^n\| = a_n r^n \leq 1$, we have $|a_n| < 1$. It follows that

$$\widetilde{R\{T/r\}} = \widetilde{R}.$$

Appendix

Lemma 35. Let $\mathbf{k}$ be a field and $\|\cdot\|_1$, $\|\cdot\|_2$ be two absolute values on $\mathbf{k}$. Then the following statements are equivalent:

(a) $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent;

(b) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on $\mathbf{k}$;

(c) The unit disks $D_1 = \{x \in \mathbf{k} : \|x\|_1 < 1\}$ and $D_2 = \{x \in \mathbf{k} : \|x\|_2 < 1\}$ are the same.

Proposition 36. Let (X, d) be an ultra-metric space. Then for any $x, y, z \in X$, at least two of the three distances $d(x, y), d(y, z), d(z, x)$ are equal. And the third distance is less than or equal to the common value of the other two.