Elementary functions

1 Exponential and logarithmic functions

Fix a prime number p in the following and consider $\mathbf{k}$ being a complete non-archimedean field with $|p| = p^{-1}$. Let $r_p := p^{-1/(p-1)}$.

Construction 1. The *exponential function* exp is defined by the power series

$$\exp(x) := \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

The *logarithmic function* log is defined by the power series

$$\log(1+x) := \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Proposition 2. We have the following properties:

(a) the exponential function exp converges on the open disk $B(0, r_p)$;

(b) the logarithmic function log converges on the open disk $B(1, 1)$;

(c) $|\exp(x) - 1| = |x|$ and $|\log(1+x)| = |x|$ for all $x \in B(0, r_p)$ or $x \in B(1, r_p)$ respectively;

(d) endow $B(0, r_p)$ with the group structure induced by addition in $\mathbf{k}$ and $B(1, r_p)$ with the group structure induced by multiplication in $\mathbf{k}$, then $\exp : B(0, r_p) \to B(1, r_p)$ is an isometric group isomorphism with inverse $\log : B(1, r_p) \to B(0, r_p)$.

Proof. For the convergent radius of exponential function, by Lemma 3, noting that

$$\liminf_{n \to +\infty} \frac{s_n}{n} = 0,$$

we have

$$\limsup_{n \to +\infty} |n!|_p^{-1/n} = \limsup_{n \to +\infty} p^{v_p(n!)/n} = p^{\limsup_{n\to+\infty}(1-(s_n/n))/(p-1)} = p^{1/(p-1)}.$$

That is, the convergent radius of the exponential function is $r_p = p^{-1/(p-1)}$. Considering $n = p^m$, we have

$$|p^m!|_p r_p^n = p^{(p^m-1)/(p-1)} \cdot p^{-p^m/(p-1)} = p^{-1/(p-1)} \neq 0.$$

Hence the convergent domain of the exponential function is $B(0, r_p)$.

For the logarithmic function, we have

$$\limsup_{n \to +\infty} |n|_p^{-1/n} = \limsup_{n \to +\infty} p^{v_p(n)/n} = p^0 = 1.$$

And $|1/(np+1)|_p = 1$ for all $n \in \mathbb{N}$. Thus, the convergent domain of the logarithmic function is $B(1, 1)$.

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For $x \in B(0, r_p)$, we have

$$\left|\frac{x^{n-1}}{n!}\right|_p < r_p^{n-1} \cdot p^{v_p(n!)} = p^{v_p(n!)-(n-1)/(p-1)} \leq 1.$$

Hence $|x^n/n!|_p < |x|_p$ for all $n \geq 2$ and thus

$$|\exp(x) - 1|_p = \left|\sum_{n=1}^{+\infty} \frac{x^n}{n!}\right|_p = |x|_p.$$

For $x + 1 \in B(1, r_p)$, setting $|x|_p = p^{-t}$ with $t \geq 1/(p-1)$, we have

$$\left|\frac{x^{n-1}}{n}\right|_p = p^{v_p(n)-t(n-1)} \leq p^{v_p(n!)-t(n-1)} \leq p^{(1/(p-1)-t)(n-1)} \leq 1, \quad \forall n \geq 2.$$

Similarly, we have $|x^n/n|_p < |x|_p$ and hence $|\log(1+x)|_p = |x|_p$.

The identities

$$\begin{aligned} \exp(X+Y) &= \exp(X) \cdot \exp(Y), \\ \log((1+X)(1+Y)) &= \log(1+X) + \log(1+Y), \\ \exp(\log(1+X)) &= 1 + X, \\ \log(\exp(X)) &= X \end{aligned}$$

are purely formal and holds for indeterminates X and Y. Easy to check that $\exp(X+Y), \log(1+X) + \log(1+Y) \in \mathbf{k}\{X/r_p, Y/r_p\}$. Thus, the assertion (d) follows from (c) and Proposition 10. □

Recall the following useful lemma regarding the p-adic valuation of factorials.

Lemma 3. Let p be a prime number and $n \in \mathbb{N}$, write $n = \sum_{k=0}^m a_k p^k$ in the p-adic expansion and set $s_n := \sum_{k=0}^m a_k$. Then

$$v_p(n!) = \frac{n - s_n}{p - 1}.$$

Proof. Yang: To be added. □

Corollary 4. Let $\mathbf{k}$ be a complete non-archimedean field with $|p| = p^{-1}$. The multiplication group

$$\mathbf{k}^\times \cong |\mathbf{k}^\times| \times k_{\mathbf{k}}^\times \times \mathbf{k}^{\circ\circ}$$

where $k_{\mathbf{k}}$ is the residue field of $\mathbf{k}$. Yang: To be revised.

Proof. Yang: To be added. □

Proposition 5. Suppose that $\mathbf{k} = \Bbbk$ is algebraically closed. The logarithmic function log defines a surjective group homomorphism $1 + \Bbbk^{\circ\circ} \to \Bbbk$ with kernel the group μ_{p^∞} of all p-power roots of unity. Yang: To be checked.

Proof. □

Yang: continuation of exponential and logarithmic

2 Mahler series

Notation 6. We use $\binom{x}{n}$ to denote the *binomial polynomial* defined by

$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}.$$

Definition 7. Fix a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbf{k}$. The *Mahler series* associated to $\{a_n\}$ is defined to be the formal series

$$f(x) := \sum_{n=0}^{+\infty} a_n \binom{x}{n}.$$

Yang: To be checked.

Proposition 8.

Theorem 9. The series converges.

Preliminaries

Appendix

Proposition 10. Let $\mathbf{k}$ be a complete non-archimedean field. Then for every $f \in \mathbf{k}\{\underline{T/r}\}$, we can associate a function $F_f : E(0, \underline{r}) \to \mathbf{k}$ defined by

$$F_f(x) := \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha \quad \text{for } x = (x_1, \dots, x_n) \in E(0, \underline{r}).$$

This defines a homomorphism of $\mathbf{k}$-algebras from $\mathbf{k}\{\underline{T/r}\}$ to the ring of all functions from $E(0, \underline{r})$ to $\mathbf{k}$.