# Hodge Index Theorem by Linear Algebra

**Proposition 1.** Let $V$ be a real vector space of dimension $n$ with a non-degenerated symmetric bilinear form $\langle -,-\rangle$. Consider the subset

$$S = \{x \in V \mid \langle x,x\rangle > 0\}.$$

Suppose that $S \neq \varnothing$ and there exists $z \in V$ such that $z^\perp \cap S = \varnothing$, then the signature of $\langle -,-\rangle$ is of type $(1, n-1)$, where $z^\perp = \{v \in V \mid \langle z,v\rangle = 0\}$.

*Proof.* Choose $h \in S$. We claim that the restriction of $\langle -,-\rangle$ on $h^\perp$ is negative definite. First, the restriction is non-degenerated. Otherwise, note that $h$ and $h^\perp$ generate $V$. If there exists $0 \neq x \in h^\perp$ such that $\langle x,y\rangle = 0$ for all $y \in h^\perp$, then in particular $\langle x,h\rangle = 0$, thus $x \in h^\perp$ and $x \in V^\perp = \{0\}$, which is a contradiction.

Hence if the restriction is not negative definite, then there exists $x \in h^\perp$ such that $\langle x,x\rangle > 0$. Then consider the subspace $V_0$ generated by $h$ and $x$. We have $h^2, x^2 > 0$ and $h \cdot x = 0$, thus the restriction of $\langle -,-\rangle$ on $V_0$ is of type $(2,0)$. Hence $z^\perp \cap V_0 = \{0\}$. However, consider the dimension count

$$\dim(z^\perp + V_0) = \dim(z^\perp) + \dim(V_0) - \dim(z^\perp \cap V_0) = (n-1) + 2 - 0 = n + 1 > n = \dim(V),$$

which is a contradiction. □

**Remark 2.** Geometrically, we have the following equivalent statement:

(a) $S \neq \varnothing$ and there exists $z \in V$ such that $z^\perp \cap S = \varnothing$;

(b) the signature of $\langle -,-\rangle$ is of type $(1, n-1)$;

(c) the set $S$ has two connected components.

We have shown $(a) \Rightarrow (b)$ in Proposition 1. If the signature of $\langle -,-\rangle$ is of type $(1, n-1)$, then we can choose a basis such that the matrix of $\langle -,-\rangle$ is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

Then the set $S$ is given by the equation

$$x_1^2 - x_2^2 - \cdots - x_n^2 > 0,$$

which has two connected components, thus $(b) \Rightarrow (c)$. Finally, if $S$ has two connected components, then for any $z \in S$, we claim that $z^\perp \cap S = \varnothing$. Otherwise, there exists $x \in z^\perp \cap S$. Considering on the subspace $V_0$ generated by $z$ and $x$, we see that $z$ and $-z$ lie in the same connected component of $S$. For every $y \in S$, assume $z.y > 0$ (otherwise replace $z$ by $-z$), then the line segment $tz + (1-t)y$ for $t \in [0,1]$ connects $z$ and $y$ in $S$. Hence $S$ is path connected, which is a contradiction.

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**Lemma 3** (Riemann-Roch theorem for surfaces)**.** Let $X$ be a smooth projective surface over an algebraically closed field $\Bbbk$ and $D$ a divisor on $X$. Then we have

$$h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(K_X - D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D \cdot (D - K_X),$$

where $K_X$ is the canonical divisor of $X$.

**Lemma 4.** Let $X$ be a smooth projective surface over an algebraically closed field $\Bbbk$ and $D$ a divisor on $X$. If $D^2 > 0$, then at least one of $D$ and $-D$ is pseudo-effective.

*Proof.* Suppose for contradiction that both $D$ and $-D$ are not pseudo-effective. In particular, we have $h^0(\mathcal{O}_X(mD)) = 0$ for all $m > 0$. By Lemma 3, we have

$$h^0(\mathcal{O}_X(K_X - mD)) \geq \chi(\mathcal{O}_X) + \frac{1}{2}mD \cdot (mD + K_X) > 0 \text{ for all } m \gg 0.$$

Hence there exist effective divisors $E_m \sim K_X - mD$ for all $m \gg 0$. We have $-D \sim_{\mathbb{Q}} \frac{1}{m}(E_m - K_X)$ is pseudo-effective, which is a contradiction. □

**Theorem 5.** Let $X$ be a smooth projective surface over an algebraically closed field $\Bbbk$. Then the intersection form on $\mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is of type $(1, \rho(X) - 1)$, where $\rho(X) = \dim_{\mathbb{R}} \mathrm{NS}(X)_{\mathbb{R}}$ is the Picard number of $X$.

*Proof.* Note that both $\mathrm{Psef}(X) \setminus \{0\}$ and $-\mathrm{Psef}(X) \setminus \{0\}$ are convex cones in $\mathrm{NS}(X)_{\mathbb{R}}$ and they are disjoint. Hence there exists a hyperplane $H$ in $\mathrm{NS}(X)_{\mathbb{R}}$ such that $H \cap \mathrm{Psef}(X) = \{0\}$ by the geometric form of Hahn-Banach theorem. By Lemma 4,

$$H \cap \{D \in \mathrm{NS}(X)_{\mathbb{R}} \mid D^2 > 0\} = \varnothing.$$

Then the conclusion follows from Proposition 1. □