Analytic functions in one variable

Proposition 1. Let $\mathbf{k}$ be a complete non-archimedean field and $f = \sum_{n=0}^{+\infty} a_n T^n \in \mathbf{k}[[T]]$. Set

$$R := \frac{1}{\limsup_{n\to+\infty} \|a_n\|^{1/n}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Then we have

(a) the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$ and diverges for all $x \in \mathbf{k}$ with $\|x\| > R$;

(b) if $R < +\infty$, the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| = R$ if and only if $\lim_{n\to+\infty} \|a_n\| R^n = 0$.

Proof. By ??, we only need to check when the terms $a_n x^n$ tend to zero as $n \to +\infty$. If $\|x\| < R$, there exists $r \in (0,1)$ such that $\|x\| < r^2 R$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\|a_n\|^{1/n} < 1/(rR)$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n < \|a_n\| (r^2 R)^n < (r^2 R)^n \cdot \frac{1}{(rR)^n} = r^n \to 0.$$

Thus the series $f(x)$ converges for all $x \in \mathbf{k}$ with $\|x\| < R$.

Suppose that $\|x\| > R$. There exists $s > 1$ such that $\|x\| > R/s$. By the definition of R, there exist infinitely many $n \in \mathbb{N}$ such that $\|a_n\|^{1/n} > s/R$ and thus

$$\|a_n x^n\| = \|a_n\| \|x\|^n > \|a_n\| \frac{R^n}{s^n} > \left(\frac{s}{R}\right)^n \cdot \frac{R^n}{s^n} = 1.$$

Thus the series $f(x)$ diverges for all $x \in \mathbf{k}$ with $\|x\| > R$.

Finally, the case $\|x\| = R$ is direct from ??. Yang: To be revised. □

Theorem 2 (Strassman)**.** Let $\mathbf{k}$ be a complete non-archimedean field with non-trivial valuation and $f = \sum a_n T^n \in \mathbf{k}\{T/r\}$ be an analytic function. Suppose that $\|a_N\| r^N > \|a_n\| r^n$ for all $n > N$. Then f has at most N zeros in the closed ball $E(0,r)$.

Proof. We induct on N. The case $N = 0$ is direct from ??. Suppose that the conclusion holds for $N-1$. Let x be a zero of f in $E(0,r)$. Set

$$g(T) = \frac{f(T) - f(x)}{T - x} = \sum_{k=0}^{+\infty} \left(\sum_{n=k+1}^{+\infty} a_n x^{n-k-1} \right) T^k = \sum_{n=0}^{+\infty} b_k T^k.$$

That is,

$$b_k = \sum_{n=0}^{\infty} a_{k+1+n} x^n.$$

Hence we have

$$\|b_k\| r^k = \max_{n\geq k+1} \|a_n x^{n-k-1}\| r^k \leq \max_{n \geq k+1} \|a_n\| r^{n-1} \to 0 \quad \text{as } k \to \infty.$$

It follows that $g(T) \in \mathbf{k}\{T/r\}$.

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For every $n > N$, we have

$$\|a_N\| > \|a_n\|r^{n-N} \geq \|a_n x^{n-N}\|.$$

Hence

$$\left\|\sum_{n=N}^{N+m} a_n x^{n-N}\right\| = \|a_N\|$$

for every $m \in \mathbb{N}$ by ??. Take $m \to +\infty$, we have $\|b_{N-1}\| = \|a_N\|$. For every $k > N-1$, we have

$$\|b_k\|r^k = \max_{n \geq k+1} \|a_n\|r^{n-1} \leq \max_{n>N} \|a_n\|r^{n-1} < \|a_N\|r^{N-1} = \|b_{N-1}\|r^{N-1}.$$

By the induction hypothesis, g has at most $N-1$ zeros in $E(0,r)$. It follows that f has at most N zeros in $E(0,r)$ since $f(T) = (T-x) \cdot g(T)$. □

Yang: Does the proof mean that $\mathbf{k}\{T\}$ with $v(f) := n$ such that $a_n = \max a_i$ and $a_n > a_m$ for all $m > n$ is an Euclidean ring?

Yang: There exist $f \in \mathbf{k}\{T\}$ with $f(a) \neq 0$ for all $|a| \leq 1$ but $1/f \notin \mathbf{k}\{T\}$. Yang: Is this right?

1 Entire functions

2 Maximum principle

Appendix