Examples of dynamic systems on ruled surfaces

In this section, fix an algebraically closed field $\Bbbk$ of characteristic zero.

1 Automorphism groups

The main reference is [Mar71].

$$
\begin{aligned}
H_r &= \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & \alpha \end{pmatrix}, \cdots, \begin{pmatrix} \alpha & t_r \\ 0 & \alpha \end{pmatrix} \right) \in GL(2,k) \times \cdots \times GL(2,k) \;\middle|\; \begin{matrix} \alpha \in G_m \\ t_i \in k \end{matrix} \right\}, \\
H'_r &= \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & \beta \end{pmatrix}, \cdots, \begin{pmatrix} \alpha & t_r \\ 0 & \beta \end{pmatrix} \right) \in GL(2,k) \times \cdots \times GL(2,k) \;\middle|\; \begin{matrix} \alpha, \beta \in G_m \\ t_i \in k \end{matrix} \right\}, \\
\overline{H}'_r &= \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} \alpha & t_r \\ 0 & 1 \end{pmatrix} \right) \in GL(2,k) \times \cdots \times GL(2,k) \;\middle|\; \begin{matrix} \alpha \in G_m \\ t_i \in k \end{matrix} \right\}.
\end{aligned}
$$

Theorem 1. Let $\pi : X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e)) \to C = \mathbb{P}^1$ be a rational ruled surface. We have an exact sequence of algebraic groups

$$1 \to \overline{H}'_{e+1} \to \mathrm{Aut}(X) \to \mathrm{PGL}(2, \Bbbk) \to 1.$$

2 Polarized induced by base

Let $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ be a rank 2 vector bundle on a smooth projective curve C of genus $g \leq 1$, where $\mathcal{L}$ is a line bundle of degree $-e$ on C. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to C$ be the associated ruled surface. Let $g : C \to C$ be an endomorphism of degree q such that $g^*\mathcal{L} \cong \mathcal{L}^q$. Fix the isomorphism $g^*\mathcal{L} \cong \mathcal{L}^q$, we have

$$g^* : \mathcal{E} \to \mathcal{O}_C \oplus \mathcal{L}^q \hookrightarrow \mathrm{Sym}^q \mathcal{E}.$$

References

[Mar71] Masaki Maruyama. “On automorphism groups of ruled surfaces”. In: *Journal of Mathematics of Kyoto University* 11.1 (1971), pp. 89–112 (cit. on p. 1).