Preliminaries

Proposition 1. Let X be a smooth projective variety over $\mathbb{C}$. The Néron-Severi group is a subgroup of $H^2_{\mathrm{sing}}(X, \mathbb{Z})$.

Examples of dynamics on abelian varieties

On this section, fix an algebraically closed field $\Bbbk$ of characteristic zero. Everything is defined over $\Bbbk$ unless otherwise specified.

1 Product of elliptic curves

In this subsection, we consider the dynamics induced by matrices on the product of elliptic curves.

Let E be an elliptic curve without complex multiplication. Consider the abelian variety $X = E \times E$. Let $f_A : X \to X$ be the endomorphism defined by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $[F_1], [F_2], [\Delta]$ be the classes of the fibers of the two projections and the diagonal in $\mathrm{NS}(X)$. It is well-known that they span $\mathrm{NS}(X)$ and the intersection numbers are given by

$$[F_1]^2 = [F_2]^2 = [\Delta]^2 = 0, \quad [F_1] \cdot [F_2] = [F_1] \cdot [\Delta] = [F_2] \cdot [\Delta] = 1;$$

see [Laz04, Section 1.5.B].

We have that $f_A^*[F_1]$ is given by $[a]_E(x) + [b]_E(y) = 0$. Then

$$f_A^*[F_1].[F_1] = b^2, \quad f_A^*[F_1].[F_2] = a^2, \quad f_A^*[F_1].[\Delta] = (a+b)^2.$$

Hence

$$f_A^*[F_1] = (a^2 + ab)[F_1] + (b^2 + ab)[F_2] - ab[\Delta].$$

Similarly, we have

$$\begin{aligned} f_A^*[F_2] &= (c^2 + cd)[F_1] + (d^2 + cd)[F_2] - cd[\Delta], \\ f_A^*[\Delta] &= (a-c)(a+b-c-d)[F_1] + (b-d)(a+b-c-d)[F_2] - (a-c)(b-d)[\Delta]. \end{aligned}$$

Thus, the matrix representation of f_A^* on $\mathrm{NS}(X)$ with respect to the basis $\{[F_1], [F_2], [\Delta]\}$ is

$$\begin{pmatrix} a^2 + ab & c^2 + cd & (a-c)(a+b-c-d) \\ b^2 + ab & d^2 + cd & (b-d)(a+b-c-d) \\ -ab & -cd & -(a-c)(b-d) \end{pmatrix}.$$

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If we take $e_1 = [F_1], e_2 = [F_2], e_3 = [\Delta] - [F_1] - [F_2]$ as a new basis of $\mathrm{NS}(X)$, then the matrix representation of f_A^* on $\mathrm{NS}(X)$ with respect to the basis $\{e_1, e_2, e_3\}$ is

$$M = \begin{pmatrix} a^2 & c^2 & -2ac \\ b^2 & d^2 & -2bd \\ -ab & -cd & ad+bc \end{pmatrix}.$$

The characteristic polynomial of M is given by

$$\chi_{f_A^*}(T) = (T - (ad-bc))(T^2 - (a^2+d^2+2bc)T + (ad-bc)^2).$$

Suppose that the eigenvalues of A are λ, μ. Then the eigenvalues of f_A^* on $\mathrm{NS}(X)$ are given by $\lambda^2, \mu^2, \lambda\nu$. When $a-d, b, c$ are not all zero, $\mathrm{NS}(X)$ has two invariant subspaces of dimension 1 and 2 respectively. They are given by

$$V_1 = \mathbb{Q}\cdot\begin{pmatrix} 2c \\ -2b \\ a-d \end{pmatrix}, \quad V_2 = \mathbb{Q}\cdot\begin{pmatrix} 0 \\ a-d \\ c \end{pmatrix} \oplus \mathbb{Q}\cdot\begin{pmatrix} d-a \\ 0 \\ b \end{pmatrix} = \{(p,q,r) \mid bp - cq + (a-d)r = 0\}.$$

with respect to the basis $\{e_i\}$. One can use Code 1 to check this in SageMathCell.

With respect to the basis $\{e_i\}$, the cones are given by

$$\mathrm{Nef}(X) = \mathrm{Psef}(X) = \{pe_1 + qe_2 + re_3 \mid p, q \geq 0, \quad pq \geq r^2\}.$$

We analyze the intersection of V_1, V_2 with $\mathrm{Psef}(X)$. On the plane $P : p+q = 2$, fix a coordinate system (s,r) with $s = (p-q)/2 = p-1 = 1-q$. The cone $\mathrm{Psef}(X)$ is given by the disk $\{(s,r) \mid s^2 + r^2 \leq 1\}$. The plane V_2 is given by the equation $b(1+s) - c(1-s) + (a-d)r = (b-c) + (b+c)s + (a-d)r = 0$. If $b = c$, then the line V_1 does not intersect the plane P, hence does not intersect the interior of $\mathrm{Psef}(X)$. Otherwise, $V_1 \cap P$ is given by the point $(s,r) = \left(\frac{c+b}{c-b}, \frac{a-d}{c-b}\right)$. There are three cases:

(a) $(a+d)^2 < 4(ad-bc)$: V_1 intersects the interior of $\mathrm{Psef}(X)$, V_2 intersects $\mathrm{Psef}(X)$ at only the origin.

(b) $(a+d)^2 = 4(ad-bc)$: V_1 intersects the boundary of $\mathrm{Psef}(X)$ at a ray, V_2 intersects the boundary of $\mathrm{Psef}(X)$ at a ray.

(c) $(a+d)^2 > 4(ad-bc)$: V_1 intersects $\mathrm{Psef}(X)$ at only the origin, V_2 intersects the interior of $\mathrm{Psef}(X)$.

Note that if $b = c$, we always have $(a+d)^2 > 4(ad-bc)$ under the assumption that $a-d, b, c$ are not all zero. And note that $(a+d)^2 - 4(ad-bc) = \mathrm{disc}(\chi_A)$. Hence, we have the conclusion in table 1.

Now we focus on the case when A has only one eigenvalue, i.e. A is similar to a Jordan block. Assume that A is not a scalar matrix, i.e. b, c are not both zero. Then easily see that $V_1 \subset V_2$ iff $(a+d)^2 = 4(ad-bc)$ iff A has the Jordan normal form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{Z}$. Hence in this case, there is a finite equivariant cover $(X,f) \to (X,g)$ such that g is induced by the Jordan block.

Example 2. Let $f : X \to X$ be the endomorphism defined by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Case	$V_1 \cap \mathrm{Psef}(X)$	$V_2 \cap \mathrm{Psef}(X)$	dimension of minimal invariant subspace intersecting $\mathrm{Psef}(X)^\circ$
A is scalar			
A has only one eigenvalue	Boundary (ray)	Boundary (ray)	3
A has complex eigenvalues	Interior	Origin only	1
A has distinct real eigenvalues	Origin only	Interior	2

Table 1: Intersection of invariant subspaces V_1, V_2 with $\mathrm{Psef}(X)$ in different cases.

Then the matrix representation of f^* on $\mathrm{NS}(X)$ with respect to the basis $\{e_1, e_2, e_3\}$ is

$$M = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & -4 \\ -2 & 0 & 4 \end{pmatrix}.$$

The Jordan normal form of M is

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

The invariant subspaces of M are given by

$$V_1 = \mathbb{R} \cdot \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \quad V_2 = \{(p, q, r) \mid p = 0\}.$$

We have $V_2 \cap \mathrm{Psef}(X) = V_1 \cap \mathrm{Psef}(X) = \mathbb{R}_{\geq 0} \cdot (0, 1, 0)^T$. The action of f^* on $\mathrm{NS}(X)$ induces a dynamics on the plane $P : p + q = 2$; see fig. 1.

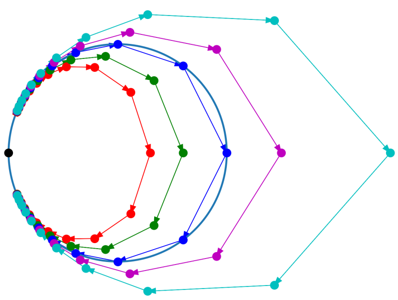

Figure 1: Dynamics on $\mathrm{NS}(X)$ induced by the endomorphism defined by a Jordan block.

2 Appendix

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a, b, c, d = var('a b c d')

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M = matrix([[a^2, c^2, -2*a*c],
            [b^2, d^2, -2*b*d],
            [-a*b, -c*d, a*d+b*c]])

I = identity_matrix(3)
M1 = M - (a*d-b*c)*I
M2 = M^2 - (a^2+d^2+2*b*c)*M + (a*d-b*c)^2*I

v1 = vector([2*c, -2*b, a-d])
v2 = vector([0, a-d, c])
v3 = vector([d-a, 0, b])

print("M1 * v1 =")
print((M1 * v1).simplify_full())
print()
print("M2 * v2 =")
print((M2 * v2).simplify_full())
print()
print("M2 * v3 =")
print((M2 * v3).simplify_full())
print()
```

Listing 1: Test invariant subspaces of $\mathrm{NS}(E \times E)$

References

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: 10.1007/978-3-642-18808-4. URL: https://doi.org/10.1007/978-3-642-18808-4 (cit. on p. 1).