

Preliminaries

Proposition 1. Let X be a smooth projective variety over \mathbb{C} . The Néron-Severi group is a subgroup of $H_{\text{sing}}^2(X, \mathbb{Z})$.

Examples of dynamics on abelian varieties

On this section, fix an algebraically closed field \mathbb{k} of characteristic zero. Everything is defined over \mathbb{k} unless otherwise specified.

1 Product of elliptic curves

In this subsection, we consider the dynamics induced by matrices on the product of elliptic curves.

Example 2. Let E be an elliptic curve without complex multiplication. Consider the abelian variety $X = E \times E$. Let $f_A : X \rightarrow X$ be the endomorphism defined by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $[F_1], [F_2], [\Delta]$ be the classes of the fibers of the two projections and the diagonal in $\text{NS}(X)$. It is well-known that they span $\text{NS}(X)$ and the intersection numbers are given by

$$[F_1]^2 = [F_2]^2 = [\Delta]^2 = 0, \quad [F_1] \cdot [F_2] = [F_1] \cdot [\Delta] = [F_2] \cdot [\Delta] = 1;$$

see [Laz04, Section 1.5.B].

We have that $f_A^*[F_1]$ is given by $[a]_E(x) + [b]_E(y) = 0$. Then

$$f_A^*[F_1] \cdot [F_1] = b^2, \quad f_A^*[F_1] \cdot [F_2] = a^2, \quad f_A^*[F_1] \cdot [\Delta] = (a + b)^2.$$

Hence

$$f_A^*[F_1] = (a^2 + ab)[F_1] + (b^2 + ab)[F_2] - ab[\Delta].$$

Similarly, we have

$$f_A^*[F_2] = (c^2 + cd)[F_1] + (d^2 + cd)[F_2] - cd[\Delta],$$

$$f_A^*[\Delta] = (a - c)(a + b - c - d)[F_1] + (b - d)(a + b - c - d)[F_2] - (a - c)(b - d)[\Delta].$$

Thus, the matrix representation of f_A^* on $\text{NS}(X)$ with respect to the basis $\{[F_1], [F_2], [\Delta]\}$ is

$$\begin{pmatrix} a^2 + ab & c^2 + cd & (a - c)(a + b - c - d) \\ b^2 + ab & d^2 + cd & (b - d)(a + b - c - d) \\ -ab & -cd & -(a - c)(b - d) \end{pmatrix}.$$

If we take $e_1 = [F_1], e_2 = [F_2], e_3 = [\Delta] - [F_1] - [F_2]$ as a new basis of $\text{NS}(X)$, then the matrix representation of f_A^* on $\text{NS}(X)$ with respect to the basis $\{e_1, e_2, e_3\}$ is

$$M = \begin{pmatrix} a^2 & c^2 & -2ac \\ b^2 & d^2 & -2bd \\ -ab & -cd & ad + bc \end{pmatrix}.$$

The characteristic polynomial of M is given by

$$\chi_{f_A^*}(T) = (T - (ad - bc))(T^2 - (a^2 + d^2 + 2bc)T + (ad - bc)^2).$$

Suppose that the eigenvalues of A are λ, μ . Then the eigenvalues of f_A^* on $\text{NS}(X)$ are given by $\lambda^2, \mu^2, \lambda\mu$. When $a - d, b, c$ are not all zero, $\text{NS}(X)$ has two invariant subspaces of dimension 1 and 2 respectively. They are given by

$$V_1 = \mathbb{Q} \cdot \begin{pmatrix} 2c \\ -2b \\ a-d \end{pmatrix}, \quad V_2 = \mathbb{Q} \cdot \begin{pmatrix} 0 \\ a-d \\ c \end{pmatrix} \oplus \mathbb{Q} \cdot \begin{pmatrix} d-a \\ 0 \\ b \end{pmatrix} = \{(p, q, r) \mid bp - cq + (a-d)r = 0\}.$$

with respect to the basis $\{e_i\}$. One can use [Code 1](#) to check this in [SageMathCell](#).

With respect to the basis $\{e_i\}$, the cones are given by

$$\text{Nef}(X) = \text{Psef}(X) = \{pe_1 + qe_2 + re_3 \mid p, q \geq 0, \quad pq \geq r^2\}.$$

Yang: By direct computation, one can check that V_2 always intersects the interior of $\text{Psef}(X)$. **Yang:** This is wrong.

Question 3. Let X be a smooth projective variety of dimension n over \mathbb{k} and $f : X \rightarrow X$ be a surjective endomorphism. Is there always an f^* -invariant subspace of $\text{NS}(X)_{\mathbb{Q}}$ of dimension $\leq n$ which intersects the interior of $\text{Psef}(X)$?

2 Appendix

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1 a, b, c, d = var('a b c d')
2
3 M = matrix([[a^2, c^2, -2*a*c],
4             [b^2, d^2, -2*b*d],
5             [-a*b, -c*d, a*d+b*c]])
6
7 I = identity_matrix(3)
8 M1 = M - (a*d-b*c)*I
9 M2 = M^2 - (a^2+d^2+2*b*c)*M + (a*d-b*c)^2*I
10
11 v1 = vector([2*c, -2*b, a-d])
12 v2 = vector([0, a-d, c])
13 v3 = vector([d-a, 0, b])
14
15 print("M1 * v1 =")
16 print((M1 * v1).simplify_full())
17 print()
18 print("M2 * v2 =")
19 print((M2 * v2).simplify_full())
20 print()
21 print("M2 * v3 =")
22 print((M2 * v3).simplify_full())
23 print()

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Listing 1: Test invariant subspaces of $\mathrm{NS}(E \times E)$

References

- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: [10.1007/978-3-642-18808-4](https://doi.org/10.1007/978-3-642-18808-4). URL: <https://doi.org/10.1007/978-3-642-18808-4> (cit. on p. 1).
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