Application: birational group of varieties of general type

In this section, we apply the results from the previous sections to study the birational automorphism groups of varieties of general type.

Theorem 1. Let X be a projective variety of general type over an algebraically closed field $\Bbbk$ of characteristic zero. Then the group of birational automorphisms $\mathrm{Bir}(X)$ is finite.

Proof. We will prove this theorem in several steps. By replacing X with its resolution of singularities, we may assume that X is smooth.

Step 1. For every $m \geq 1$, $\mathrm{Bir}(X)$ linearly acts on $H^0(X, mK_X)$ via pull-back of functions (as abstract group).

Let $\mathscr{K}(X)$ be the function field of X. Then for every $g \in \mathrm{Bir}(X)$, g induces an automorphism of $\mathscr{K}(X)$ over $\Bbbk$, which we denote by g^*. In particular we know that g^* is injective and $\Bbbk$-linear. By definition, $H^0(X, mK_X) = \{s \in \mathscr{K}(X) \mid \mathrm{div}(s) + mK_X \geq 0\}$. We only need to show that for every $s \in H^0(X, mK_X)$, $g^*(s) \in H^0(X, mK_X)$ since $\dim_{\Bbbk} H^0(X, mK_X) < \infty$. Consider the commutative diagram

$$\begin{tikzcd} \Gamma \arrow[d, "p"'] \arrow[dr, "q"] & \\ X \arrow[r, dashed, "g"] & X \end{tikzcd}$$

with Γ smooth and p, q birational morphisms. Then we have

$$K_\Gamma = p^*K_X + E_p = q^*K_X + E_q,$$

where E_p and E_q are p- and q-exceptional divisors respectively. Moreover, E_p and E_q are effective since X is smooth. For every $s \in H^0(X, mK_X)$, we have

$$\mathrm{div}(q^*s) + mK_\Gamma = q^*(\mathrm{div}(s) + mK_X) + mE_q \geq 0.$$

Then

$$\begin{aligned} \mathrm{div}(g^*s) + mK_X &= p_*p^*(\mathrm{div}(g^*s) + mK_X) \\ &= p_*\left(\mathrm{div}(q^*s) + mK_\Gamma - mE_p\right) \\ &= p_*\left(\mathrm{div}(q^*s) + mK_\Gamma\right) \geq 0. \end{aligned}$$

It follows that $g^*(s) \in H^0(X, mK_X)$.

Note this action $g \mapsto g^*$ is contravariant, i.e., for every $g_1, g_2 \in \mathrm{Bir}(X)$, we have $(g_1 \circ g_2)^* = g_2^* \circ g_1^*$.

Step 2. The group $\mathrm{Bir}(X)$ is a linear algebraic group by identifying it with a closed subgroup of $\mathrm{Aut}(\mathbb{P}(V))$ for some finite-dimensional $\Bbbk$-vector space V (subspace of $H^0(X, mK_X)$ for some $m > 0$). Moreover, its rational action on X is algebraic.

By Theorem 11, there exists an integer $m > 0$ such that the map $\psi : X \dashrightarrow \mathbb{P}(H^0(X, mK_X))$ is birational onto its image Y. Let V be the subspace of $H^0(X, mK_X)$ spanned by the affine cone over Y. Since $\mathrm{Bir}(X)$ linearly acts on $H^0(X, mK_X)$ by Step 1, it also linearly acts on V. we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, dashed, "\psi"] & X \arrow[d, dashed, "\psi"] \\ Y \arrow[r, "\varphi_g|_Y"] \arrow[d, hook] & Y \arrow[d, hook] \\ \mathbb{P}(V) \arrow[r, "\varphi_g"] & \mathbb{P}(V) \end{tikzcd}$$

for every $g \in \mathrm{Bir}(X)$, where φ_g is the induced automorphism of $\mathbb{P}(V)$.

Since ψ is birational, the map $g \mapsto \varphi_g$ defines an injective group homomorphism from $\mathrm{Bir}(X)$ to $\mathrm{Aut}(\mathbb{P}(V))$. Consider the natural algebraic group structure on $\mathrm{Aut}(\mathbb{P}(V))$ and let G be the Zariski closure of the image of $\mathrm{Bir}(X)$ in $\mathrm{Aut}(\mathbb{P}(V))$. Note that $\mathrm{Bir}(X)$ fixes Y. Thus G also fixes Y. Since the affine cone over Y spans V, we conclude that any element $g \in G$ is uniquely determined by its restriction to Y. In particular, we have $G = \mathrm{Bir}(X)$. Note that $\mathrm{Aut}(\mathbb{P}(V))$ is a linear algebraic group and so is its closed subgroup $\mathrm{Bir}(X)$.

Step 3. If $\dim \mathrm{Bir}(X) > 0$, then it contains $\mathbb{G}_a$ or $\mathbb{G}_m$ as a subgroup. We show that the action of $\mathbb{G}_a$ or $\mathbb{G}_m$ on X leads to X being uniruled, which contradicts the assumption that X is of general type.

By Lemma 9 and Theorem 8, if $\dim \mathrm{Bir}(X) > 0$, then $\mathrm{Bir}(X)$ contains either $\mathbb{G}_a$ or $\mathbb{G}_m$ as a subgroup. Note that both $\mathbb{G}_a$ and $\mathbb{G}_m$ are rational varieties, without loss of generality, we may assume that $\mathrm{Bir}(X)$ contains $\mathbb{G}_m$ as a subgroup. Then we have a rational map

$$\Phi : \mathbb{G}_m \times X \dashrightarrow X.$$

Fix $x \in X$ such that $\Phi|_{\mathbb{G}_m \times \{x\}} : \mathbb{G}_m \to X$ is not constant. Choose $Z \subset X$ a closed subvariety of codimension 1 passing through x such that $\mathbb{G}_m.x \not\subseteq Z$. Then the closure of $\Phi(\mathbb{G}_m \times Z)$ in X has dimension at least $\dim Z + 1 = \dim X$. Hence we have a dominant rational map

$$\Phi : \mathbb{P}^1 \times Z \dashrightarrow X.$$

This contradicts Theorem 7 and the assumption that X is of general type. Therefore, we must have $\dim \mathrm{Bir}(X) = 0$, i.e., $\mathrm{Bir}(X)$ is finite. □

Remark 2. In the proof of Theorem 1, by $\mathbb{P}(V)$ we mean the projective space associated to the vector space V in the sense of Grothendieck, i.e., $\mathbb{P}(V) = \mathrm{Proj}(\bigoplus_{k \geq 0} \mathrm{Sym}^k V)$. Hence if one have a linear map $f : V \to W$ between two finite-dimensional $\Bbbk$-vector spaces, then it induces a morphism $\mathbb{P}(W) \to \mathbb{P}(V)$ (not $\mathbb{P}(V) \to \mathbb{P}(W)$).

Corollary 3. Let X be a projective variety of general type over an algebraically closed field $\Bbbk$ of characteristic zero. Then there exists a projective variety Y birational to X such that $\mathrm{Bir}(Y) = \mathrm{Aut}(Y)$.

Corollary 4. Let X be a smooth projective Fano variety over an algebraically closed field $\Bbbk$ of characteristic zero. Then the group of automorphisms $\mathrm{Aut}(X)$ is a linear algebraic group.

Proof. Note that for every $g \in \mathrm{Aut}(X)$, g induces an automorphism of $H^0(X, -mK_X)$ for every integer $m \geq 1$ via pull-back of functions. Then the same argument as in Step 2 shows that $\mathrm{Aut}(X)$ is a linear algebraic group. □

Appendix

Definition 5. A projective variety X is called *of general type* if its canonical divisor K_X is big.

Definition 6. A projective variety X is called *uniruled* if there exists a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$ for some variety Y with $\dim Y = \dim X - 1$.

Theorem 7 (ref. [BDPP12, Corollary 0.3])**.** Let X be a smooth projective variety over an algebraically closed field $\Bbbk$ of characteristic zero. Then the canonical divisor K_X is not pseudo-effective if and only if X is uniruled.

Theorem 8. Let G be a linear algebraic group of dimension 1 over an algebraically closed field $\Bbbk$. Then G is isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$.

Proof. Yang: To be proved. □

Lemma 9. Let G be a linear algebraic group over an algebraically closed field $\Bbbk$. Then G has a one-dimensional algebraic subgroup.

Proof. Yang: To be proved. □

Definition 10. Let X be a normal variety over $\Bbbk$ of dimension n. If X is smooth, then the *canonical divisor* K_X is defined to be $c_1(\omega_X)$. In general, let $U \subseteq X$ be the smooth locus of X and $i : U \hookrightarrow X$ be the inclusion map. Then the *canonical divisor* K_X is defined to be any Weil divisor on X such that $\mathcal{O}_X(K_X) \cong i_*\omega_U$. Note that U is big in X since X is normal, so such a Weil divisor always exists and is unique up to linear equivalence.

Theorem 11 (Iitaka fibration, ref. [Laz04, Theorem2.1.33])**.** Let X be a normal projective variety, and L a line bundle on X such that $\kappa(X, L) > 0$. Then for all sufficiently large $k \in N(X, L)$, the rational mappings $\phi_k : X \to Y_k$ are birationally equivalent to a fixed algebraic fibre space

$$\phi_\infty : X_\infty \to Y_\infty$$

of normal varieties, and the restriction of L to a very general fibre of ϕ_∞ has Iitaka dimension $= 0$. More specifically, there exists for large $k \in N(X, L)$ a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "u_\infty"] \arrow[d, "\phi_k"'] & X_\infty \arrow[d, "\phi_\infty"] \\ Y_k \arrow[r, dashed, "\nu_k"'] & Y_\infty \end{tikzcd}$$

of rational maps and morphisms, where the horizontal maps are birational and u_∞ is a morphism. One has $\dim Y_\infty = \kappa(X, L)$. Moreover, if we set $L_\infty = u_\infty^* L$, and take $F \subseteq X_\infty$ to be a very general fibre of ϕ_∞, then

$$\kappa(F, L_\infty|F) = 0.$$

More precisely, the assertion is that the last displayed formula holds for the fibres of ϕ_∞ over all points in the complement of the union of countably many proper subvarieties of Y_∞.

References

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