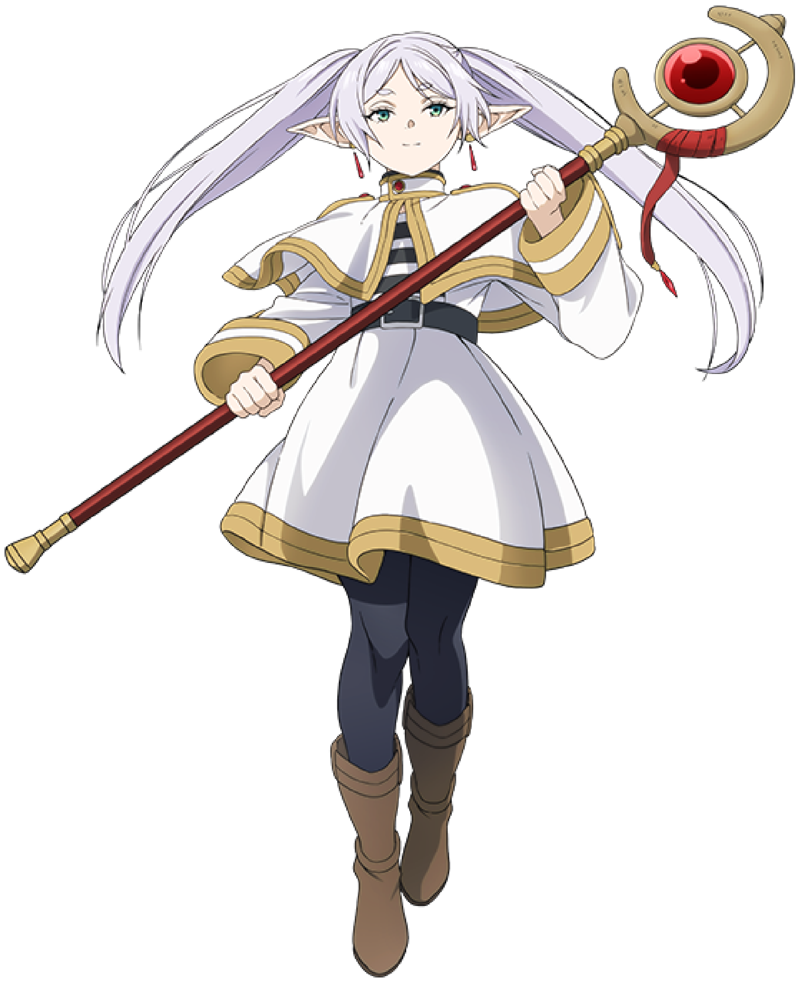
Normal, Cohen-Macaulay and regular schemes



如果是勇者辛美尔，他一定会这么做的！

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1 Height, Depth and Dimension

Krull dimension and height of prime ideals Algebraically, we have the following definitions.

Definition 1. Let A be a noetherian ring. The *height of a prime ideal* $\mathfrak{p}$ in A is defined as the maximum length of chains of prime ideals contained in $\mathfrak{p}$, that is,

$$\operatorname{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as

$$\dim A := \max_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht}(\mathfrak{p}).$$

Geometrically, we have the corresponding definition.

Definition 2. Let X be a noetherian scheme. The *codimension of an irreducible subscheme* Y in X is defined as the length of the longest chain of irreducible closed subsets containing Y, that is,

$$\operatorname{codim}_X(Y) := \sup\{n \mid \exists \text{ a chain of irreducible closed subsets } Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n\}.$$

The *dimension* of X is defined as

$$\dim X := \max_{\xi \in X} \operatorname{codim}_X Z_\xi.$$

For an affine scheme $X = \operatorname{Spec} A$, above two definitions coincide by the correspondence of prime ideals and irreducible closed subsets.

Proposition 3. Let A be a noetherian ring and $\mathfrak{p} \in \operatorname{Spec} A$. Then

$$\operatorname{ht}(\mathfrak{p}) = \operatorname{codim}_{\operatorname{Spec} A} V(\mathfrak{p}) = \dim A_{\mathfrak{p}}.$$

Theorem 4 (Krull's Principal Ideal Theorem)**.** Let A be a noetherian ring. Suppose $f \in A$ is not a unit or a zero divisor. Then $\dim A/(f) \leq \dim A - 1$. Moreover, if A is local or $\dim A_{\mathfrak{m}}$ is constant for all $\mathfrak{m}$, then the equality holds.

Proof. Let $\mathfrak{p} \ni f$ be a minimal prime ideal among those containing f. Note that $A_{\mathfrak{p}}/(f)$ is a local ring of dimension 0. Yang: To be added. □

For "nice" schemes, the Krull dimension behaves well by following proposition.

Theorem 5. Let S be spectrum of a field k or an algebraic integer ring $\mathcal{O}_K$ and X an integral S-variety. Then we have the follows:

(i) For any point $\xi \in X$, $\dim X = \dim \mathcal{O}_{X,\xi} + \operatorname{codim} Z_\xi$.

(ii) For any non-empty open subset $U \subset X$, $\dim U = \dim X$.

(iii) $\dim X = \operatorname{trdeg}(\mathscr{K}(X)/\mathscr{K}(S)) + \dim S$.

Proof. Yang: To be continued. □

Example 6.

Depth For a noetherian local ring $(A, \mathfrak{m})$, we can define the depth of an A-module M. Somehow the Krull dimension is "homological" and the depth is "cohomological".

Definition 7. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A-module. A sequence $t_1, \cdots, t_n \in \mathfrak{m}$ is called an *M-regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \cdots, t_{i-1})M$ for all i.

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Proposition 8. Suppose A is local and $I = \mathfrak{m}$ is the maximal ideal of A. Then any permutation of an M-regular sequence is M-regular.

Definition 9. The *I-depth* of M is defined as the maximum length of M-regular sequences in I, denoted by $\operatorname{depth}_I M$. When A is a local ring with maximal ideal $\mathfrak{m}$, we write $\operatorname{depth} M$ for $\operatorname{depth}_{\mathfrak{m}} M$.

Regular and Serre's conditions Up to now, there are three numbers measuring the "size" of a local ring $(A, \mathfrak{m})$:

- $\dim A$: the Krull dimension of A.
- $\operatorname{depth} A$: the depth of A.
- $\dim_{\kappa(\mathfrak{m})} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$-vector space.

These three numbers are related by the following inequalities.

Proposition 10. Let $(A, \mathfrak{m})$ be a local noetherian ring with residue field k. Then the following inequalities hold:

$$\operatorname{depth} A \leq \dim A \leq \dim_{\mathsf{k}} T_{A,\mathfrak{m}}.$$

To see these, we need the following well-known theorem.

Theorem 11 (Nakayama's Lemma)**.** Let $(A, \mathfrak{m})$ be a local ring. Suppose M is a finitely generated A-module. If $\mathfrak{m}M = M$, then $M = 0$.

Proof. Yang: To be added. □

Definition 12. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property R_k* or *is regular in codimension k* if $\forall \xi \in X$ with $\operatorname{codim} \overline{\{\xi\}} \leq k$,

$$\dim_{\kappa(\xi)} T_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

We say that X *verifies property S_k* if $\forall \xi \in X$,

$$\operatorname{depth} \mathcal{O}_{X,\xi} \geq \min\{k, \dim \mathcal{O}_{X,\xi}\}.$$

Proposition 13. Let X be a locally noetherian scheme. Then X is reduced if and only if it verifies R_0 and S_1.

2 Normal schemes

Definition 14. A ring A is called *normal* if it is an integral domain and integrally closed in its field of fractions $\operatorname{Frac}(A)$.

Proposition 15. Normality is a local property. That is, TFAE:

(a) A is normal.

(b) For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is normal.

(c) For any maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$, the localization $A_{\mathfrak{m}}$ is normal.

Proof. □

Proposition 16. Let A be a normal ring. Then $A[X]$ is also normal.

Definition 17. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,x}$ is normal for any point $x \in X$.

Example 18.

Definition 19. Let X be a scheme. The *normalization* of X is an X-scheme X^ν with the following universal property: for any normal X-scheme Y with dominant structure morphism, its structure morphism $Y \to X$ factors through X^ν.

Proposition 20. Let X be an integral scheme. Then the normalization X^ν of X exists.

Proposition 21. Let $S = \operatorname{Spec} \mathsf{k}$ or $\operatorname{Spec} \mathcal{O}_K$ and X an S-variety. Then the normalization $X^\nu \to X$ is birational. In particular, $\{\xi \in X : \mathcal{O}_{X,\xi} \text{ is normal}\}$ is open in X.

Proposition 22. Let A be a normal noetherian ring A of dimension ≥ 1. Then

$$A = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A, \operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

Theorem 23. Let X be a normal noetherian scheme. Let $F \subset X$ be a closed subset of codimension ≥ 2. Then the restriction $H^0(X, \mathcal{O}_X) \to H^0(X \setminus F, \mathcal{O}_X)$ is an isomorphism.

Theorem 24. Let X be a normal noetherian S-scheme and Y a proper S-scheme. Let $f : X \dashrightarrow Y$ be a rational map. Then f is defined on an open subset $U \subset X$ whose complement has codimension ≥ 2.

Theorem 25 (Serre's criterion for normality)**.** Let X be a locally noetherian scheme. Then X is normal if and only if it verifies R_1 and S_2.

3 Cohen-Macaulay schemes

Definition 26 (Cohen-Macaulay)**.** A noetherian local ring $(A, \mathfrak{m})$ is called *Cohen-Macaulay* if $\dim A = \operatorname{depth} A$. A locally noetherian scheme X is called *Cohen-Macaulay* if $\mathcal{O}_{X,\xi}$ is Cohen-Macaulay for any point $\xi \in X$.

By definition, it is easy to see that X is Cohen-Macaulay if and only if it verifies S_k for all $k \geq 0$.

Example 27 (Non Cohen-Macaulay rings)**.**

Definition 28. An ideal I of a noetherian ring A is called *unmixed* if

$$\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \operatorname{Ass}(A/I).$$

We say that *the unmixedness theorem holds for a noetherian ring* A if any ideal $I \subset A$ generated by $\operatorname{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme* X if $\mathcal{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Remark 29. Recall that the set of associated primes of a module M is defined as

$$\operatorname{Ass}(M) := \{\mathfrak{p} \in \operatorname{Spec} A : \exists x \in M \text{ such that } \mathfrak{p} = \operatorname{Ann}(x)\}.$$

Theorem 30. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Theorem 31. Let X be a locally noetherian scheme. Suppose that X is Cohen-Macaulay. Let $F \subset X$ be a closed subset of codimension $\geq k$. Then the restriction $H^i(X, \mathcal{O}_X) \to H^i(X \setminus F, \mathcal{O}_X)$ induced by the is an isomorphism.

4 Regular schemes

Proposition 32. Let $(A, \mathfrak{m})$ be a regular local ring. Then A is integral.

Proposition 33. If X verifies R_k, then $\operatorname{codim}_X X_{\mathrm{sing}} \geq k+1$.

Proposition 34. A regular scheme is Cohen-Macaulay.

Corollary 35. A regular scheme is normal.