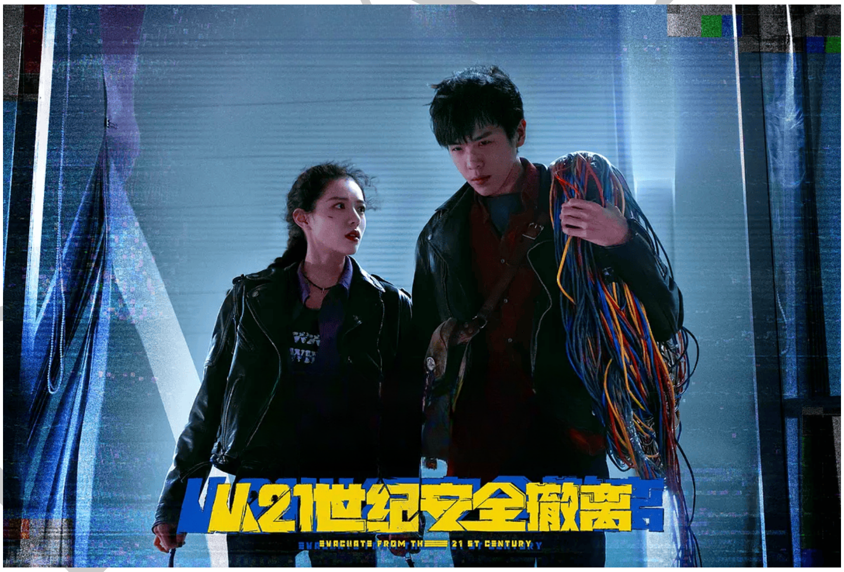
Schemes and Varieties



“宇宙战警行为规范第四十条，遇见违反社会公德的行为，发现违法犯罪的行为，要见义勇为，勇于斗争，善于斗争。第十四条，不说谎，不骗人，不弄虚做假，知错就改，诚实守信，言行一致，答应别人的事要做到。……榴莲，不用跟任何人道歉，你跟这儿的人不一样，你是来让世界变好的。”

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0.1 Sheaves

Definition 0.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

For two open sets $V \subset U \subset X$, the morphism $\mathcal{F}(U) \to \mathcal{F}(V)$, often denoted by res^U_V, is called the *restriction map*.

Example 0.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 0.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U \to \mathbb{R}\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0,1) \cup (2,3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0,1),(2,3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0,1))$ and $s_2 = 2 \in \mathcal{C}((2,3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 0.4. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be presheaves on X with values in the same category (e.g., **Set**, **Ab**, **Ring**, etc.). A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$. In other words, for every open set $U \subset X$, there is a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every inclusion of open sets $V \subset U$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}^U_V"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V). \end{tikzcd}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then φ is called a *morphism of sheaves*.

Fix a topological space X and a category $\mathbf{C}$. The sheaves (resp. presheaves) on X with values in $\mathbf{C}$ form a category, denoted by $\mathbf{Sh}(X, \mathbf{C})$ (resp. $\mathbf{PSh}(X, \mathbf{C})$), where the objects are sheaves (resp. presheaves) on X with values in $\mathbf{C}$ and the morphisms are morphisms of sheaves (resp. presheaves).

Definition 0.5. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in a category $\mathbf{C}$. For a point $x \in X$, the *stalk* of $\mathcal{F}$ at x, denoted by $\mathcal{F}_x$, is defined as the colimit

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the colimit is taken over all open neighborhoods U of x. An element of $\mathcal{F}_x$ is called a *germ* of sections of $\mathcal{F}$ at x.

More concretely, we have

$$\mathcal{F}_x = \{(U, s) : U \in \mathbf{Open}(X), U \ni x, s \in \mathcal{F}(U)\} / \sim,$$

where $(U, s) \sim (V, t)$ if there exists an open neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$.

Definition 0.6. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in **Set** (resp. **Ab**, **Ring**, etc.). A *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}^\dagger$ on X together with a morphism of presheaves $\eta : \mathcal{F} \to \mathcal{F}^+$ such that for every sheaf $\mathcal{G}$ on X and every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \eta$.
In other words, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & \mathcal{F}^\dagger \arrow[d, "\varphi^\dagger"] \\ & \mathcal{G}. \end{tikzcd}$$

Yang: To be checked.

Yang: The concrete describe of sheafification.

Definition 0.7. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The morphism φ is called *injective* (resp. *surjective*) if for every $x \in X$, the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).

Proposition 0.8. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is injective if and only if for every open set $U \subset X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. Yang: To be checked.

Remark 0.9. The surjectivity on stalks cannot imply the surjectivity on sections. A counterexample is given by the exponential map $\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^*$ defined by $\exp(f) = e^f$, where $\mathcal{O}_{\mathbb{C}}$ is the sheaf of holomorphic functions on $\mathbb{C}$ and $\mathcal{O}_{\mathbb{C}}^*$ is the sheaf of non-vanishing holomorphic functions on $\mathbb{C}$. The induced map on stalks $\exp_z : \mathcal{O}_{\mathbb{C},z} \to \mathcal{O}_{\mathbb{C},z}^*$ is surjective for every $z \in \mathbb{C}$ by the existence of logarithm locally. However, the map on global sections $\exp(\mathbb{C}) : \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}^*(\mathbb{C})$ is not surjective since there is no entire function f such that $e^{f(z)} = z$ for all $z \in \mathbb{C}^*$. Yang: To be continued. Yang: This is wrong, need to be revised.

Proposition 0.10. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is an isomorphism if and only if it is injective and surjective.

Yang: Now we consider sheaves with values in an abelian category.

Definition 0.11. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The *kernel* of φ, denoted by $\ker \varphi$, is the sheaf defined by

$$(\ker \varphi)(U) := \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$.
The *cokernel* of φ, denoted by $\operatorname{coker} \varphi$, is the sheafification of the presheaf defined by

$$(\operatorname{coker} \varphi)_{\text{pre}}(U) := \operatorname{coker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$. Yang: To be continued.

Theorem 0.12. Let X be a topological space and $\mathbf{C}$ be an abelian category (e.g., $\mathbf{Ab}$). Then the category of sheaves on X with values in $\mathbf{C}$ is an abelian category.

Proof. Yang: To be continued. □

Yang: To be checked and continuous.

0.2 Locally ringed space

Definition 0.13. Let $f : X \to Y$ be a continuous map between topological spaces. The *push-forward* functor $f_* : \mathbf{Sh}(X, \mathbf{C}) \to \mathbf{Sh}(Y, \mathbf{C})$ is defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

for every open set $V \subset Y$ and sheaf $\mathcal{F} \in \mathbf{Sh}(X, \mathbf{C})$.

Definition 0.14. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 0.15. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Construction 0.16 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i \in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Construction 0.17 (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i \in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i \in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^{\sharp}(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ_i' satisfying the same properties, then by gluing $\psi_i' \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

Definition 0.18. A morphism of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a *closed immersion* (resp. *open immersion*) if f induces a homeomorphism from X to a closed (resp. open) subset of Y and the map $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective (resp. an isomorphism). Yang: To be checked.

0.3 Manifolds as locally ringed spaces

0.4 Vector bundles and $\mathcal{O}_X$-modules

Let $(X, \mathcal{O}_X)$ be a manifold (real or complex) and $(\mathcal{E}, \pi, X)$ a vector bundle over X.

Yang: It can regard as a sheaf on X.

Definition 0.19. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of* $\mathcal{O}_X$*-modules* is a sheaf $\mathcal{F}$ of abelian groups on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\mathrm{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear, where the $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(V)$ is induced by the restriction map $\mathrm{res}_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.
A *morphism of* $\mathcal{O}_X$*-modules* is a morphism of sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ such that for every open set $U \subseteq X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. Yang: To be checked...

Definition 0.20. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is said to be *locally free of rank* r if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_U^r$, where $\mathcal{O}_U^r$ is the direct sum of r copies of $\mathcal{O}_U$. Yang: To be continued.

1 The First Properties of Schemes

If you learn the following content for the first time, it is recommended to skip all the proofs in this section and focus on the examples, remarks and the statements of propositions and theorems.

1.1 Schemes

Let R be a ring. Recall that the *spectrum* of R, denoted by $\mathrm{Spec}\, R$, is the set of all prime ideals of R equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \mathrm{Spec}\, R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set.*

Proposition 1.1. Let R be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

Proof. Yang: To be continued □

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 1.2. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i \in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.
These data form a category, denoted by $\mathbf{Sch}$. If we fix a base scheme S, then an S*-scheme* is a scheme X together with a morphism $X \to S$. The category of S-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

Theorem 1.3. The functor $\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. Yang: To be continued

Definition 1.4. A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if f induces an isomorphism of X onto an open (resp. closed) subscheme of Y. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. Yang: To be continued

Construction 1.5. Let R be a graded ring. The *projective scheme* $\operatorname{Proj} R$ is defined as the scheme associated to the sheaf of rings

$$\mathcal{O}_{\operatorname{Proj} R} = \bigoplus_{d \geq 0} R_d.$$

It can be covered by open affine subschemes of the form $\operatorname{Spec} R_f$ for homogeneous elements $f \in R$. Yang: To be checked.

Construction 1.6 (Glue open subschemes)**.** The construction in Construction 0.17 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Construction 0.17. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

Definition 1.7. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme Z of Y such that f factors through Z. More precisely, if $Y = \operatorname{Spec} A$ is affine, then the scheme theoretic image of f is $\operatorname{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover Y by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of f. Yang: To be checked.

1.2 Fiber product and base change

Definition 1.8. Let C be a category and $X, Y, S \in \mathsf{Obj}(\mathsf{C})$ with morphisms $f : X \to S$ and $g : Y \to S$. A *fiber product* of X and Y over S is an object $X \times_S Y \in \mathsf{Obj}(\mathsf{C})$ together with morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ such that $f \circ p = g \circ q$ and satisfies the universal property that for any object $W \in \mathsf{Obj}(\mathsf{C})$ with morphisms $u : W \to X$ and $v : W \to Y$ such that $f \circ u = g \circ v$, there exists a unique morphism $h = (u, v) : W \to X \times_S Y$ such that $p \circ h = u$ and $q \circ h = v$.

$$\begin{tikzcd}
W \arrow[dr, "h"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& X \times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & S.
\end{tikzcd}$$

Yang: To be checked.

Example 1.9. In the category of sets, the fiber product $X \times_S Y$ is given by

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\},$$

with the projections $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ being the restrictions of the natural projections. Yang: To be checked.

Remark 1.10. If one reverses the arrows in Definition 1.8, one gets the notion of *fiber coproduct*. It is also called the *pushout* or *amalgamated sum* in some literature. We denote the fiber coproduct of X and Y over S by $X \amalg_S Y$. Note that in the category of rings, the fiber coproduct $A \amalg_R B$ of R-algebras A and B over R is given by the tensor product $A \otimes_R B$. Dually, one can expect that fiber products of affine schemes correspond to tensor products of rings.

Theorem 1.11. The category of schemes admits fiber products. Yang: To be continued

Definition 1.12. Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a point. The *scheme theoretic fiber* of f over y is the fiber product $X_y = X \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y)$ is the residue field of the local ring $\mathcal{O}_{Y,y}$. Yang: To be checked.

Definition 1.13. Let X be a scheme and $Z_1, Z_2 \subset X$ be closed subschemes of X with inclusion morphisms $i_1 : Z_1 \to X$ and $i_2 : Z_2 \to X$. The *scheme theoretic intersection* of Z_1 and Z_2 is the fiber product $Z_1 \times_X Z_2$. Yang: To be checked.

1.3 Noetherian schemes and morphisms of finite type

Definition 1.14. A scheme X is *noetherian* if it admits a finite open cover $\{U_i\}_{i=1}^n$ such that each U_i is an affine scheme $\operatorname{Spec} A_i$ with A_i a noetherian ring. Yang: To be checked.

Proposition 1.15. A noetherian scheme is quasi-compact. Yang: To be checked.

Definition 1.16. Let $f : X \to S$ be a morphism of schemes. We say that f is *of finite type*, or X is *of finite type* over S, if there exists a finite affine cover $\{U_i\}_{i=1}^n$ of S such that for each i, $f^{-1}(U_i)$ can be covered by finitely many affine open subsets $\{V_{ij}\}_{j=1}^{m_i}$ with $f(V_{ij}) \subseteq U_i$ and the induced morphism $f|_{V_{ij}} : V_{ij} \to U_i$ corresponds to a finitely generated algebra over the ring of global sections of U_i. Given S, the category consisted of S-scheme of finite type over S, together with morphisms of S-schemes, is denoted by $\mathbf{sch}_S$. Yang: To be checked.

1.4 Integral, reduced and irreducible schemes

Definition 1.17. A topological space X is *irreducible* if it is non-empty and cannot be expressed as the union of two proper closed subsets. Equivalently, every non-empty open subset of X is dense in X. Yang: To be checked.

Proposition 1.18. Let X be a topological space satisfying the descending chain condition on closed subsets. Then X can be written as a finite union of irreducible closed subsets, called the *irreducible components* of X. Moreover, this decomposition is unique up to permutation of the components. Yang: To be checked.

Definition 1.19. A scheme X is *reduced* if its structure sheaf $\mathcal{O}_X$ has no nilpotent elements. Yang: To be checked.

Proposition 1.20. A scheme X is reduced if and only if for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a reduced ring. Yang: To be checked.

Proposition 1.21. Let X be a scheme. There exists a unique closed subscheme X of X such that X is reduced and has the same underlying topological space as X. Moreover, for any morphism of schemes $f : Y \to X$ with Y reduced, f factors uniquely through the inclusion $X \to X$. Yang: To be checked.

Definition 1.22. A scheme X is *integral* if it is both reduced and irreducible. Yang: To be checked.

Proposition 1.23. A scheme X is integral if and only if for every open affine subset $U = \operatorname{Spec} A \subset X$, the ring A is an integral domain. Yang: To be checked.

Corollary 1.24. Let $\Bbbk$ be an algebraically closed field and $n \geq 1$ be an integer. Then the polynomial $\det(x_{ij}) \in k[x_{ij} : 1 \leq i, j \leq n]$ is irreducible. Yang: To be checked.

1.5 Dimension

Definition 1.25. The *Krull dimension* of a topological space X, denoted by $\dim X$, is the supremum of the lengths n of chains of distinct irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X. If no such finite supremum exists, we say that X has infinite dimension. Yang: To be checked.

Definition 1.26. Let $\xi \in X$ be a point in a scheme X. The *local dimension* of X at ξ, denoted by $\dim_\xi X$, is defined as the infimum of the dimensions of all open neighborhoods U of ξ:

$$\dim_\xi X = \inf\{\dim U : U \text{ is an open neighborhood of } \xi\}.$$

Yang: To be checked.

1.6 Separated, proper and projective morphisms

Definition 1.27. A morphism of schemes $f : X \to Y$ is *separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. A scheme X is *separated* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. Yang: To be checked.

Proposition 1.28. Any affine scheme is separated. More generally, any morphism between affine schemes is separated. Yang: To be checked.

Proposition 1.29. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any scheme T and any pair of morphisms $g_1, g_2 : T \to X$ such that $f \circ g_1 = f \circ g_2$, the equalizer of g_1 and g_2 is a closed subscheme of T. Yang: To be checked.

Proposition 1.30. A scheme X is separated if and only if for any pair of affine open subschemes $U, V \subset X$, the intersection $U \cap V$ is also an affine open subscheme. Yang: To be checked.

Proposition 1.31. The composition of separated morphisms is separated. Moreover, separatedness is stable under base change, i.e., if $f : X \to Y$ is a separated morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also separated. Yang: To be checked.

Proposition 1.32. A morphism of schemes $f : X \to Y$ is separated if and only if for every commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[d] \arrow[dr] & \\
\operatorname{Spec} R \arrow[r] \arrow[dr] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists at most one morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. Yang: To be checked.

Definition 1.33. A morphism of schemes $f : X \to Y$ is *universally closed* if for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map. Yang: To be checked.

Definition 1.34. A morphism of schemes $f : X \to Y$ is *proper* if it is of finite type, separated, and universally closed. A scheme X is *proper* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is proper. Yang: To be checked.

Theorem 1.35. Any projective morphism is proper. In particular, any projective scheme is proper. Yang: To be checked.

Proposition 1.36. The composition of proper morphisms is proper. Moreover, properness is stable under base change, i.e., if $f : X \to Y$ is a proper morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also proper. Yang: To be checked.

Proposition 1.37. A morphism of schemes $f : X \to Y$ is proper if and only if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists a unique morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. Yang: To be checked.

1.7 Varieties

Definition 1.38. Let $\Bbbk$ be a algebraically closed field. A *variety over* $\Bbbk$ is an integral scheme of finite type over $\operatorname{Spec} \Bbbk$. The category of varieties over $\Bbbk$ is denoted by $\mathbf{Var}_{\Bbbk}$. Yang: To be checked.

Let X be a variety over $\Bbbk$. The closed points $X(\Bbbk)$ is a locally ringed subspace of X with the induced topology and structure sheaf. We denote the category of such locally ringed spaces by $\mathbf{ClaVar}_{\Bbbk}$, meaning the category of *classical varieties* over $\Bbbk$.

Theorem 1.39. Let X be a variety over $\Bbbk$. Then there is an equivalence of categories between $\mathbf{Var}_{\Bbbk}$ and $\mathbf{ClaVar}_{\Bbbk}$.

Slogan *Closed points determine varieties.*

Proof. Yang: To be continued. □

2 Category of sheaves of modules

Mostly results in this section fits into the context of ringed spaces.

2.1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets.

Definition 2.2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that there exists a surjective morphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

Remark 2.3. There are many versions of "local" properties for sheaves of $\mathcal{O}_X$-modules. Yang: To be continued.

Definition 2.4. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it is finitely generated, and for every open set $U \subseteq X$ and every morphism of sheaves of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of φ is finitely generated.

Slogan

$$\mathbf{Sh}_X(\mathbf{Ab}) \supseteq \mathbf{Mod}_{\mathcal{O}_X} \supseteq \mathbf{QCoh}_X \supseteq \mathbf{Coh}_X.$$

Definition 2.5. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a coherent sheaf on X. The *support* of $\mathcal{F}$ is defined to be the set

$$\mathrm{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\},$$

where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x. Yang: To be checked.

2.2 As abelian categories

Theorem 2.6. Let $(X, \mathcal{O}_X)$ be a ringed space. All of $\mathbf{Sh}_X(\mathbf{Ab}), \mathbf{Mod}(\mathcal{O}_X), \mathbf{QCoh}_X, \mathbf{Coh}_X$ are abelian categories.

Theorem 2.7. Let $(X, \mathcal{O}_X)$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules has enough injectives. Yang: To be checked.

Remark 2.8. The category of sheaves of $\mathcal{O}_X$-modules generally does not have enough projectives. Yang: To be checked.

Theorem 2.9. Let X be a noetherian, integral, separated, regular scheme. Then every coherent sheaf on X admits a finite resolution by locally free sheaves.

Proof. Yang: To be continued. □

2.3 Relevant functors

Definition 2.10. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *sheaf Hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf of abelian groups defined as follows: for an open set $U \subseteq X$, we define

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of morphisms of sheaves of $\mathcal{O}_U$-modules from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. For an inclusion of open sets $V \subseteq U$, the restriction map

$$\mathrm{res}_{UV} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(V)$$

is defined by sending a morphism $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ to its restriction $\varphi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. Yang: To be continued.

Definition 2.11. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *dual sheaf* $\mathcal{F}^\vee$ is defined to be

$$\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$$

Yang: To be continued.

Definition 2.12. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules defined as follows: for an open set $U \subseteq X$, we define

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

where $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is the tensor product of $\mathcal{O}_X(U)$-modules. For an inclusion of open sets $V \subseteq U$, the restriction map
Yang: To be continued.

Definition 2.13. Let $f : X \to Y$ be a morphism of ringed spaces. The *pull-back functor* $f^* : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$ is defined as follows: for an $\mathcal{O}_Y$-module $\mathcal{F}$, we define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

where $f^{-1}\mathcal{F}$ is the inverse image sheaf of $\mathcal{F}$. For a morphism of $\mathcal{O}_Y$-modules $\varphi : \mathcal{F} \to \mathcal{G}$, we define

$$f^*\varphi : f^*\mathcal{F} \to f^*\mathcal{G}$$

to be the morphism induced by the morphism of sheaves of abelian groups $f^{-1}\varphi : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$.
Yang: To be continued.

Definition 2.14. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $Z \subseteq X$ be a closed subset. The *functor of sections with support in* Z is defined as follows: for an $\mathcal{O}_X$-module $\mathcal{F}$, we define

$$\Gamma_Z(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) \mid \mathrm{Supp}(s) \subseteq Z\},$$

where $\mathrm{Supp}(s)$ is the support of the section s. Yang: To be checked.

2.4 Cohomological theory

Definition 2.15. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *sheaf cohomology* $H^i(X, \mathcal{F})$ is defined as the i-th right derived functor of the global section functor $\Gamma(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ applied to $\mathcal{F}$, i.e.,

$$H^i(X, \mathcal{F}) := \mathrm{R}^i\Gamma(X, \mathcal{F}).$$

Yang: To be checked.

Definition 2.16. Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *i-th higher direct image* $\mathrm{R}^i f_* \mathcal{F}$ is defined as the i-th right derived functor of the direct image functor $f_* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ applied to $\mathcal{F}$, i.e.,

$$\mathrm{R}^i f_* \mathcal{F} := \mathrm{R}^i(f_* \mathcal{F}).$$

Yang: To be checked.

Definition 2.17. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *i-th sheaf Ext functor* $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is defined as the i-th right derived functor of the sheaf Hom functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$ applied to $\mathcal{G}$, i.e.,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \mathrm{R}^i \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Yang: To be checked.

Proposition 2.18. Let $f : X \to Y$ be a morphism of ringed spaces, and let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. Then there are long exact sequences of $\mathcal{O}_Y$-modules

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to \mathrm{R}^1 f_*\mathcal{F} \to \mathrm{R}^1 f_*\mathcal{G} \to \mathrm{R}^1 f_*\mathcal{H} \to \mathrm{R}^2 f_*\mathcal{F} \to \cdots$$

Yang: To be checked.

Theorem 2.19 (Affine criterion by Serre)**.** Let X be a scheme. Then X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i > 0$. Yang: To be checked.

Theorem 2.20 (Leray spectral sequence)**.** Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there exists a spectral sequence

$$E_2^{p,q} = H^p(Y, \mathrm{R}^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Yang: To be checked.

3 Normal, Cohen-Macaulay, and regular schemes

3.1 Normal schemes

Definition 3.1. A scheme X is called *normal* if for every open affine subset $U = \operatorname{Spec} A$ of X, the ring A is an integrally closed domain. Yang: To be checked.

Definition 3.2. The *normalization* of a scheme X is a normal scheme $\widetilde{X}$ together with a finite birational morphism $\pi : \widetilde{X} \to X$ such that for every normal scheme Y and every birational morphism $f : Y \to X$, there exists a unique morphism $g : Y \to \widetilde{X}$ such that $f = \pi \circ g$. Yang: To be checked.

Theorem 3.3. Let X be a scheme. Then there exists a normalization $\widetilde{X}$ of X.

Theorem 3.4 (Hartog's phenomenon)**.** Let X be a normal integral scheme, and let $U \subseteq X$ be an open subset whose complement has codimension at least 2. Then every regular function on U extends uniquely to a regular function on X, i.e., the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is an isomorphism. Yang: To be checked.

Proposition 3.5. Let X be a normal scheme, and let $x \in X$ be a point with codimension 1. Then X is regular at x.

3.2 Cohen-Macaulay schemes

Definition 3.6. Let X be a scheme, and let $Z \subseteq X$ be a closed subset. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the *local cohomology* $H^i_Z(X, \mathcal{F})$ is defined as the i-th right derived functor of the functor $\Gamma_Z(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ that sends a sheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ to the abelian group of sections of $\mathcal{G}$ with support in Z, i.e.,

$$H^i_Z(X, \mathcal{F}) := \mathrm{R}^i\Gamma_Z(X, \mathcal{F}).$$

Yang: To be checked.

Definition 3.7. Let X be a scheme, $\xi \in X$ a point, and n a non-negative integer. The scheme X is said to satisfy *Serre's condition S_n at ξ* if the depth of the local ring $\mathcal{O}_{X,\xi}$ is at least $\min\{n, \dim \mathcal{O}_{X,\xi}\}$. The scheme X is said to satisfy *Serre's condition S_n* if it satisfies Serre's condition S_n at every point $\xi \in X$. Yang: To be checked.

Definition 3.8. A scheme X is called *Cohen-Macaulay* if for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a Cohen-Macaulay ring. Yang: To be checked.

Theorem 3.9. Let X be a Cohen-Macaulay scheme, and let $Z \subseteq X$ be a closed subset of codimension at least 2. Then for every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the local cohomology $H^i_Z(X, \mathcal{F}) = 0$ for every $i < 2$. Yang: To be checked.

Definition 3.10. Let X be a scheme, and let $Y \subseteq X$ be a closed subscheme defined by a sheaf of ideals $\mathcal{I}$. The closed immersion $i : Y \to X$ is called a *regular immersion* if for every point $y \in Y$, the ideal $\mathcal{I}_y \subseteq \mathcal{O}_{X,y}$ can be generated by a regular sequence. Yang: To be checked.

Definition 3.11. A scheme $X \to S$ is called a *locally complete intersection* over S if the morphism $X \to S$ can be factored as a regular immersion $X \to Y$ followed by a smooth morphism $Y \to S$. Yang: To be completed.

3.3 Regular schemes

We first define the tangent space of a scheme at a point.

There are many descriptions of the tangent space of a scheme at a point. Here we give one of them. Let X be a scheme over a field $\mathbf{k}$, and let $x \in X(\mathbf{k})$.

Proposition 3.12. Let $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$ be the spectrum of the ring of dual numbers over $\mathbf{k}$ with point $* : \operatorname{Spec} \mathbf{k} \to \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$. The tangent space $T_x X$ is naturally isomorphic to the set of morphisms $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X$ that send $*$ to x, i.e.

$$T_x X \cong \{f : \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X \mid f(*) = x\}.$$

Proof. Yang: To be filled. □

4 Line bundles and divisors

4.1 Cartier divisors

Definition 4.1. Let X be a scheme. A *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of X. Equivalently, a Cartier divisor D can be represented by an open covering $\{U_i\}$ of X and a collection of rational functions $f_i \in \mathcal{K}_X^*(U_i)$ such that for any i, j, the function $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. We denote a Cartier divisor by $D = \{(U_i, f_i)\}$.

4.2 Line bundles and Picard group

Definition 4.2. Let X be a scheme. A *line bundle* on X is a locally free sheaf of $\mathcal{O}_X$-modules of rank 1.

Example 4.3. Let $X = \mathbb{P}_A^n = \operatorname{Proj} A[T_0, T_1, \dots, T_n] = \operatorname{Proj} B$ be the projective n-space over a ring A. For each integer $d \in \mathbb{Z}$, the sheaf $\mathcal{O}_X(d)$, defined by

$$\{f \neq 0\} \mapsto B(d)_{(f)},$$

is a line bundle on X, called the Yang: *twisted line bundle* of degree d. Recall that here $B(d)_{(f)}$ is the degree-zero part of the localization of the shifted graded ring $B(d)$ at the multiplicative set generated by f, and $B(d)$ is defined by $B(d)_m = B_{m+d}$ for all $m \in \mathbb{Z}$.

Let us verify this by direct computation. On the standard open subset $U_i = D_+(T_i) = \operatorname{Spec} B_i$, where $B_i = A[T_0/T_i, \dots, T_n/T_i]$, write $t_{j,i} = T_j/T_i$. We have

$$\mathcal{O}_X(d)(U_i) = B(d)^0_{(T_i)} = \left\{ \frac{f}{T_i^k} \middle| f \in B, \deg f = k + d \right\} = B_i \cdot T_i^d =: B_i \cdot e_i,$$

where we denote $e_i = T_i^d$. Hence $\mathcal{O}_X(d)(U_i)$ is a free B_i-module of rank 1 and thus $\mathcal{O}_X(d)$ is locally free of rank 1.

In the language of bundles, on $U_{ij} = U_i \cap U_j$, we have

$$e_i = t_{i,j}^d \cdot e_j.$$

Thus the transition functions of $\mathcal{O}_X(d)$ are given by $\{(U_{ij},\ t_{i,j}^d : U_{ij} \to \mathbb{G}_m)\}$.

Proposition 4.4. Let X be a scheme and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. Then

(a) the tensor product $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is also a line bundle on X;

(b) the dual $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also a line bundle on X;

(c) there is a natural isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$.

Proof. □

Definition 4.5. Let X be a scheme. The *Picard group* of X is defined to be the group of isomorphism classes of line bundles on X with the group operation given by the tensor product. It is denoted by $\mathsf{Pic}(X)$.

Definition 4.6. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a Yang: non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X \times \{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X \times \{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\text{alg}} \mathcal{L}'$. Yang: To be checked.

4.3 Weil divisors and reflexive sheaves

To talk about Weil divisors, we need to work with normal schemes.

Definition 4.7. Let X be a normal integral scheme. A *Weil divisor* on X is a formal sum

$$D = \sum_Z n_Z Z,$$

where the sum runs over all prime divisors Z of X (i.e., integral closed subschemes of codimension 1) and $n_Z \in \mathbb{Z}$, such that for any affine open subset $U = \mathsf{Spec}\, A \subseteq X$, only finitely many Z intersecting U have nonzero coefficients n_Z. The group of Weil divisors on X is denoted by $\mathrm{WDiv}(X)$.

Definition 4.8. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. The sheaf $\mathcal{F}$ is called *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

Proposition 4.9. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. If $\mathcal{F}$ is reflexive, then it is determined by its restriction to any open subset $U \subseteq X$ whose complement has codimension at least 2, i.e., $\mathcal{F} \cong i_*(\mathcal{F}|_U)$, where $i : U \hookrightarrow X$ is the inclusion map. Yang: To be checked.

Proof. Yang: To be continued. □

Theorem 4.10. Let X be a normal integral scheme. There is a one-to-one correspondence between the set of isomorphism classes of reflexive sheaves of rank 1 on X and the Yang: Weil divisor class group $\mathrm{WDiv}(X)$ of X. Under this correspondence, a Weil divisor D corresponds to the reflexive sheaf $\mathcal{O}_X(D)$. Yang: To be checked.

Proof. Yang: To be continued. □

4.4 The first Chern class

Definition 4.11. Let X be a normal scheme and $\mathcal{L}$ a vector bundle on X. The *first Chern class* of $\mathcal{L}$, denoted by $c_1(\mathcal{L})$, is a Weil divisor class defined as follows:
Yang: To be completed.

Definition 4.12. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. On X_{reg}, the regular locus of X, $\mathcal{F}|_{X_{\mathrm{reg}}}$ admits a finite resolution by vector bundles

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F}|_{X_{\mathrm{reg}}} \to 0.$$

The *first Chern class* of $\mathcal{F}$, denoted by $c_1(\mathcal{F})$, is defined to be

$$c_1(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i c_1(\mathcal{E}_i).$$

Yang: To be revised.

Proposition 4.13. Let X be a normal scheme and $\mathcal{F}$ a torsion sheaf on X. Then

$$c_1(\mathcal{F}) = \sum_Z \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{F}_Z) \cdot Z,$$

where the sum runs over all prime divisors Z of X and $\mathcal{F}_Z$ is the stalk of $\mathcal{F}$ at the generic point of Z. Yang: To be checked.

5 Morphisms by line bundles and ampleness

The main references for this section are [Har77] and [Laz04].

5.1 Globally generated line bundles

Definition 5.1. Let X be a scheme over a ring A and $\mathcal{F}$ a quasi-coherent sheaf on X. We say that $\mathcal{F}$ is *globally generated* or *generated by global sections* if the natural map $\Gamma(X, \mathcal{F}) \otimes_A \mathcal{O}_X \to \mathcal{F}$ is surjective.

Proposition 5.2. Let X be a scheme over a ring A and $\mathcal{F}$, $\mathcal{G}$ quasi-coherent sheaves on X. Then we have the following:

(a) if $\mathcal{F}$ is globally generated, then for any morphism $f : Y \to X$ over A, the pullback $f^*\mathcal{F}$ is globally generated on Y;

(b) if both $\mathcal{F}$ and $\mathcal{G}$ are globally generated, then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Yang: To be revised.

The story begins with the following theorem, which uses global sections of a globally generated line bundle to construct a morphism to projective space.

Theorem 5.3. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$. Yang: We need a more "functorial" expression.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \qquad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Example 5.4. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the d*-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 5.5. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_iS_j = \pi_1^*T_i \otimes \pi_2^*S_j$ for $0 \le i \le m$ and $0 \le j \le n$.

They induce a morphism $f : X \to \mathbb{P}_A^{(m+1)(n+1)-1}$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\ldots : x_i y_j : \ldots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Proposition 5.6. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}_{\mathbf{k}}^n$ and $g : X \to \mathbb{P}_{\mathbf{k}}^m$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}_{\mathbf{k}}^n \dashrightarrow \mathbb{P}_{\mathbf{k}}^m$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. Yang: To be continued. □

5.2 Ample line bundles

Definition 5.7. Let X be a scheme over a field $\mathbf{k}$. A line bundle $\mathcal{L}$ on a X is called *very ample* if there exists a closed embedding $i : X \to \mathbb{P}_{\mathbf{k}}^n$ such that $\mathcal{L} \cong i^* \mathcal{O}(1)$.

The following lemma due to Serre gives a good description of very ample line bundles.

Lemma 5.8. Let X be a scheme over a ring A and $\mathcal{L}$ a very ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that for all $n \ge N$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated.

Proof. Yang: To be added. □

By Lemma 5.8, we have a more intrinsic definition.

Definition 5.9. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

Theorem 5.10. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

Proof. Yang: To be continued. □

Remark 5.11. By Theorem 5.10, a scheme X which is proper over a field $\mathbf{k}$ is projective if and only if it admits an ample line bundle. More intrinsically, we will use the definition that a *projective scheme* over a field $\mathbf{k}$ is a scheme proper over $\mathbf{k}$ which admits an ample line bundle. And the ample line bundle is often denoted by $\mathcal{O}_X(1)$. Once fix the ample line bundle $\mathcal{O}_X(1)$, for any coherent sheaf $\mathcal{F}$ on X, we denote $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n)$ for any integer n.

Proposition 5.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

Yang: To be continued.

Proof. Yang: To be continued. ☐

Theorem 5.13 (Serre Vanishing)**.** Let X be a projective scheme over a field k and $\mathcal{L}$ a very ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that for all $n \geq N$, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

Corollary 5.14. Let X be a projective variety over a field $\mathbf{k}$ and $\mathcal{L}$ an ample line bundle on X. Then for any non-zero global section $s \in \Gamma(X, \mathcal{L})$, the support of the effective Cartier divisor $\mathbf{div}(s)$ is connected.

Definition 5.15. Let $(X, \mathcal{O}_X(1))$ be a projective variety over a field $\mathbf{k}$ and $\mathcal{F}$ a coherent sheaf on X. The *Hilbert polynomial* of $\mathcal{F}$ with respect to $\mathcal{O}_X(1)$ is the polynomial

$$P_{\mathcal{F}}(n) = \chi(X, \mathcal{F}(n)) = \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathcal{F}(n)).$$

Let $Z \subseteq X$ be a closed subscheme with structure sheaf $\mathcal{O}_Z$. The *Hilbert polynomial* of Z with respect to $\mathcal{O}_X(1)$ is defined as $P_Z(n) = P_{\mathcal{O}_Z}(n)$. Yang: To be revised.

Note that the Euler characteristic $\chi(X, \mathcal{F}(n))$ is additive on short exact sequences of coherent sheaves. Fix an hypersurface $H \subseteq X$ defined by a global section of $\mathcal{O}_X(1)$. Then we have $\mathcal{O}_H \cong \mathcal{O}_X/\mathcal{O}_X(-1)$. Thus by the exact sequence

$$0 \to \mathcal{F}(n-1) \to \mathcal{F}(n) \to \mathcal{F}(n)|_H \to 0,$$

we have

$$P_{\mathcal{F}}(n) - P_{\mathcal{F}}(n-1) = P_{\mathcal{F}|_H}(n).$$

Inductively, Yang: ...

By Theorem 5.13, we have

$$P_{\mathcal{F}}(n) = h^0(X, \mathcal{F} \otimes \mathcal{O}_X(n)), \quad \text{for } n \gg 0.$$

Example 5.16. Let $Z \subseteq \mathbb{P}^r_{\mathbf{k}}$ be a hypersurface of degree d. Note that $h^0(\mathbb{P}^r_{\mathbf{k}}, \mathcal{O}_{\mathbb{P}^r}(n)) = C_r^{n+r}$. Then the Hilbert polynomial of Z with respect to $\mathcal{O}_{\mathbb{P}^r}(1)$ is

$$P_Z(n) = P_{\mathcal{O}}(n) - P_{\mathcal{O}(-d)}(n) = \binom{n+r}{r} - \binom{n+r-d}{r} = \frac{d}{(r-1)!}n^{r-1} + \text{lower degree terms.}$$

Yang: To be checked.

5.3 Linear systems

In this subsection, when work over a field $\mathbf{k}$, we give a more geometric interpretation of previous subsections using the language of linear systems.

Definition 5.17. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

Definition 5.18. Let $\mathcal{L}$ be a line bundle on a scheme X. Yang: To be continued.

Yang: ref for the follow is [Laz04]

The following theorem is a version of Bertini's theorem on irreducibility.

Theorem 5.19 (Bertini)**.** Let X be a quasi-projective variety over a field $\mathbf{k}$ and $\mathcal{L}$ a globally generated line bundle on X.

Lemma 5.20. Let $X \to Y$ be a dominant morphism of varieties over a field $\mathbf{k}$. If there exists a section $s : Y \to X$ such that the image $s(Y)$ is not contained in the singular locus of X, then for a general point $y \in Y(\mathbb{k})$, the fiber X_y is irreducible.

Proof. Yang: □

Corollary 5.21. Let X be a quasi-projective variety over a field $\mathbf{k}$ of dimension ≥ 2. Given any finitely many closed points $x_1, x_2, \ldots, x_r \in X(\mathbb{k})$, there exists an irreducible curve $C \subseteq X$ passing through all the given points.

Proof. Yang: □

6 Relative objects

6.1 Relative schemes

Definition 6.1. Let X be a scheme. An $\mathcal{O}_X$-algebra is a sheaf . Yang: To be continued...

Definition 6.2. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. The relative Spec of $\mathcal{A}$, denoted by $\mathcal{S}pec_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Spec} \mathcal{A}(U) \to U$ for all affine open subsets $U \subset X$. Yang: To be continued...

Proposition 6.3. Let X be a scheme and $\mathcal{E}$ be a locally free sheaf of finite rank on X. Then the relative Spec of the symmetric algebra of $\mathcal{E}$, denoted by $\mathbb{V}(\mathcal{E}) = \operatorname{Spec}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}$, is called the geometric vector bundle associated to $\mathcal{E}$. The projection morphism $\pi : \mathbb{V}(\mathcal{E}) \to X$ is affine and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Spec} \operatorname{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$. Yang: To be continued... Yang: To be revised, need to take dual.

Definition 6.4. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_0 = \mathcal{O}_X$ and $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{O}_X$-algebra. The relative Proj of $\mathcal{A}$, denoted by $\mathcal{P}roj_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Proj} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. The projection morphism $\pi : \mathcal{P}roj_X \mathcal{A} \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{A}(U)$. Yang: To be continued...

Let X be a scheme and $\mathcal{F}$ be a quasi-coherent sheaf on X. The *projective bundle* associated to $\mathcal{F}$ is defined to be $\mathbb{P}(\mathcal{F}) = \mathcal{P}roj_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{F}$. The *vector bundle* associated to $\mathcal{F}$ is defined to be $\mathbb{V}(\mathcal{F}) = \mathcal{S}pec_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{F}$.

6.2 Relative ampleness and projective morphisms

Definition 6.5. Let X be an S-scheme via a morphism $f : X \to S$. A line bundle $\mathcal{L}$ on X is called *relatively ample* or *f-ample* if for every affine open subset $U \subset S$, the restriction $\mathcal{L}|_{f^{-1}(U)}$ is an ample line bundle on the scheme $f^{-1}(U)$. Yang: To be revised.

Definition 6.6. Let $f : X \to S$ be a morphism of schemes. The morphism f is called *projective* if there exists a coherent sheaf $\mathcal{F}$ on S and a closed immersion $i : X \to \mathbb{P}(\mathcal{F})$ over S.

Remark 6.7. There may be three different definitions of projective morphisms in the literature:

- (Grothendieck) this is the definition in the sense of Definition 6.6;
- (Altman and Kleiman) this need the coherent sheaf $\mathcal{F}$ to be locally free of finite rank in Definition 6.6;
- (Hartshorne) this need the coherent sheaf $\mathcal{F}$ to be free of finite rank in Definition 6.6.

For more details, see [FGA05, Section 5.5].

Definition 6.8. Let $f : X \to S$ be a proper morphism of schemes. A line bundle $\mathcal{L}$ on X is called *relatively very ample* or *f-very ample* if there exists a closed immersion $i : X \to \mathbb{P}(\mathcal{F})$ over S for some coherent sheaf $\mathcal{F}$ on S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.

Theorem 6.9. Let $f : X \to S$ be a proper morphism of schemes and $\mathcal{L}$ be a line bundle on X. The following are equivalent:

(a) $\mathcal{L}$ is f-ample;

(b) for every coherent sheaf $\mathcal{F}$ on X, there exists an integer $n_0 > 0$ such that for all integers $n \geq n_0$,

$$f^*f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

is surjective;

(c) $\mathcal{L}^{\otimes n}$ is f-very ample for some integer $n > 0$;

Moreover, the morphism f is projective if and only if there exists an f-ample line bundle on X. Yang: To be continued...

Theorem 6.10 (Relative Serre Vanishing)**.** Let $f : X \to S$ be a projective morphism of Noetherian schemes and $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is f-ample if and only if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer $n_0 > 0$ such that for all integers $n \geq n_0$ and all $i > 0$, we have

$$R^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0.$$

Yang: To be continued...

Theorem 6.11 (Openness of ample locus)**.** Let $f : X \to S$ be a projective morphism of varieties over $\mathbf{k}$ and $\mathcal{L}$ be a line bundle on X. Then the set

$$U = \{s \in S(\mathbb{k}) : \mathcal{L}|_{X_s} \text{ is an ample line bundle on the fibre } X_s = f^{-1}(s)\}$$

is an open subset of $S(\mathbb{k})$. Yang: To be continued...

Theorem 6.12 (Fibrewise ampleness)**.** Let $f : X \to S$ be a projective morphism of varieties over $\mathbf{k}$ and $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is f-ample if and only if for every point $s \in S(\mathbb{k})$, the restriction $\mathcal{L}|_{X_s}$ is an ample line bundle on the fibre $X_s = f^{-1}(s)$. Yang: To be continued...

6.3 Blowing up

Definition 6.13. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blow up of X along $\mathcal{I}$, denoted by $\mathrm{Bl}_{\mathcal{I}} X$, is defined to be the relative Proj of the Rees algebra of $\mathcal{I}$:

$$\mathrm{Bl}_{\mathcal{I}} X = \mathcal{P}roj_X \bigoplus_{n=0}^{\infty} \mathcal{I}^n.$$

Yang: To be continued...

Proposition 6.14. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blow up morphism $\pi : \mathrm{Bl}_{\mathcal{I}} X \to X$ is projective. Moreover, if the support of $\mathcal{O}_X/\mathcal{I}$ is nowhere dense in X, then π is birational. Yang: To be continued...

Proposition 6.15. Let X be a regular scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then the blow up $\mathrm{Bl}_{\mathcal{I}} X$ is also a regular scheme. Yang: To be continued...

Proposition 6.16. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The exceptional locus of the blow up morphism $\pi : \mathrm{Bl}_{\mathcal{I}} X \to X$ is equal to the inverse image of the support of $\mathcal{O}_X/\mathcal{I}$:

$$\mathrm{Exc}(\pi) = \pi^{-1}(\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})).$$

Yang: To be continued...

7 Differentials and duality

Let S be a base noetherian scheme, $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over S and all varieties are assumed to be defined over $\Bbbk$.

7.1 The sheaves of differentials

Definition 7.1. Let $f : X \to S$ be an S-scheme. The *sheaf of differentials* of X over S, denoted by $\Omega_{X/S}$, is the $\mathcal{O}_X$-module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$.

Proposition 7.2. Let X and T be S-schemes and $X_T := X \times_S T$ be the base change of X along $T \to S$. Let $p : X_T \to X$ be the projection morphism. Then there is a natural isomorphism of $\mathcal{O}_{X_T}$-modules

$$\Omega_{X_T/T} \cong p^*\Omega_{X/S}.$$

Proof. Given by algebras, see Yang: ref. Yang: To be continued. □

Proposition 7.3. Let X be an S-scheme and $U \subseteq X$ be an open subscheme. Then there is a natural isomorphism of $\mathcal{O}_U$-modules

$$\Omega_{U/S} \cong \Omega_{X/S}|_U.$$

Furthermore, let $\xi \in X$, then there is a natural isomorphism of $\mathcal{O}_{X,\xi}$-modules

$$\Omega_{X/S,\xi} \cong \Omega_{\mathcal{O}_{X,\xi}/\mathcal{O}_{S,f(\xi)}}.$$

Yang: To be checked.

Proof. Yang: To be continued. ☐

Proposition 7.4. Let X be a regular variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a locally free sheaf of rank n.

Proof. Yang: To be continued. ☐

Proposition 7.5. Let X be a normal variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a reflexive sheaf of rank n.

Proof. Yang: To be continued. ☐

Definition 7.6. Let X be a normal variety over $\Bbbk$. The *canonical divisor* K_X of X is defined to be the Weil divisor class $c_1(\Omega_{X/\Bbbk})$.

Theorem 7.7 (Euler sequence for projective bundle)**.** Let X be a normal variety over $\Bbbk$ and $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on X. Let $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. Then there is an exact sequence of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$-modules

$$0 \to \Omega_{\mathbb{P}_X(\mathcal{E})/X} \xrightarrow{\phi} \pi^*\mathcal{E}(-1) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \to 0.$$

Here $\pi^*\mathcal{E}(-1)$ is twisted by the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-1)$.

Proof.

Step 1. First assume that $X = \operatorname{Spec} A$ is affine and $\mathcal{E}$ is free. Under this assumption, find expressions for ϕ and ψ.

Fix a basis $T_0, \dots, T_r$ of the free A-module $\mathcal{E}(X)$. On the standard open subset $U_i = \{T_i \neq 0\} = \operatorname{Spec} B_i \subseteq \mathbb{P}_X(\mathcal{E})$, we have coordinates $t_{j,i} := T_j/T_i$ for $j \neq i$. The exact sequence becomes

$$0 \to \bigoplus_{k\neq i} B_i \mathrm{d}t_{k,i} \xrightarrow{\phi} \bigoplus_{k=0}^{r} B_i e_i \cdot T_k \xrightarrow{\psi} B_i \to 0.$$

Here e_i is the local generator of $\mathcal{O}_{\mathbb{P}_A(\mathcal{E})}(-1)$ on U_i, symbolically satisfying $e_i T_i = 1$.

Recall that on the overlap $U_{ij} = U_i \cap U_j$, the coordinates are related by

$$t_{i,j} e_i = e_j, \quad \mathrm{d}t_{k,i} = t_{j,i} \mathrm{d}t_{k,j} - t_{k,i} t_{j,i} \mathrm{d}t_{i,j}.$$

Here we set $t_{l,l} := 1$ for convenience. Symbolically, we have

$$\text{“ } \mathrm{d}t_{k,i} = \frac{T_i \mathrm{d}T_k - T_k \mathrm{d}T_i}{T_i^2} = e_i \mathrm{d}T_k - t_{k,i} e_i \mathrm{d}T_i \text{ ”}.$$

On the overlap U_{ij}, it transitions as

$$\begin{aligned} \text{“}\mathrm{d}t_{k,i} &= t_{j,i} \mathrm{d}t_{k,j} - t_{k,i} t_{j,i} \mathrm{d}t_{i,j} \\ &= t_{j,i} e_j \mathrm{d}T_k - t_{j,i} t_{k,j} e_j \mathrm{d}T_j - t_{k,i} t_{j,i} \left(e_j \mathrm{d}T_i - t_{i,j} e_j \mathrm{d}T_j\right) \\ &= e_i \mathrm{d}T_k - t_{k,i} e_i \mathrm{d}T_i\text{”}. \end{aligned}$$

To make sense of the above symbolic expressions, we define ϕ and ψ locally on each U_i by

$$\phi(\mathrm{d}t_{k,i}) = e_i T_k - t_{k,i} e_i T_i, \quad \psi(e_i T_k) = t_{k,i}.$$

Step 2. Verify that ϕ and ψ are well-defined and the sequence is exact.

By computations in Step 1, ϕ is well-defined on the overlaps U_{ij}. For ψ, on the overlap U_{ij}, we have

$$\psi(e_j T_k) = \psi(t_{i,j} e_i T_k) = t_{i,j} t_{k,i} = t_{k,j}.$$

Thus ψ is also well-defined. It is clear that $\psi \circ \phi = 0$. Consider the matrix representation of ϕ with respect to the bases $\{\mathrm{d}t_{k,i}\}_{k \neq i}$ and $\{e_i T_k\}_{k=0}^{r}$:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ -t_{0,i} & -t_{1,i} & \cdots & -t_{i-1,i} & -t_{i+1,i} & \cdots & -t_{r,i} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

It has rank r, ϕ is injective and $\ker \psi = \operatorname{Im} \phi$. Thus the sequence is exact.

Step 3. General case: glue the local exact sequences on affine open subsets of X.

In the local case, choose a different basis $S_0, \ldots, S_r$ of $\mathcal{E}(X)$ given by the transition matrix $g \in \mathrm{GL}_{r+1}(A)$. For simplicity, we just look at on the open subset $U = \{T_0 \neq 0, S_0 \neq 0\}$. Set B_U be the localization of $B = A[T_0, \ldots, T_r]$ at the multiplicative set generated by T_0 and S_0. It is still a graded algebra.

Note that ϕ is formally given by differentials in $A[T_0, \ldots, T_r]$ and then sending the symbol $\mathrm{d}T_i$ to T_i and $1/T_0$ to e_0. The differentials are intrinsic and linear over A, and the assignment of $1/T_0$ to e_0 is just a change of notation. Thus ϕ is independent of the choice of basis. For ψ, it is indeed given by multiplying $B_U(-1)$ by the linear part of B and then taking the degree 0 part. It is also independent of the choice of basis.

Therefore, after changing basis, ϕ and ψ remain the same. This allows us to glue the local exact sequences on each affine open subset of X to obtain a global exact sequence. □

Corollary 7.8. Let $\mathbf{k}$ be a field. We have

$$\omega_{\mathbb{P}^n_{\mathbf{k}}/\mathbf{k}} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbf{k}}}(-(n+1)) \quad \text{and} \quad K_{\mathbb{P}^n_{\mathbf{k}}} \sim -(n+1)H,$$

where H is a hyperplane in $\mathbb{P}^n_{\mathbf{k}}$.

7.2 Fundamental sequences

Theorem 7.9 (The first fundamental sequence of differentials)**.** Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Proof. Yang: To be completed. □

Proposition 7.10. Let $f : X \to Y$ be a surjective and generically finite morphism of normal varieties over $\Bbbk$. Then the first fundamental sequence of differentials is exact on the left.

Proof. Yang: To be completed. □

Corollary 7.11 (Ramification formula)**.** Let $f : X \to Y$ be a finite morphism of normal varieties. Then

$$K_X = f^*K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} \left(\operatorname{Mult}_D f^*(f(D)) - 1\right) D$$

is the ramification divisor of f. Yang: To be checked. definition of ramification divisor needs to be checked.

Proof. Yang: To be completed. □

Theorem 7.12 (The second fundamental sequence of differentials)**.** Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{J}/\mathcal{J}^2 \to \Omega_{X/S}|_Z \to \Omega_{Z/S} \to 0.$$

Suppose further that $Z \to X$ is a regular immersion. Then the above sequence is also exact on the left.

Proof. Yang: To be completed. □

Corollary 7.13 (Adjunction formula)**.** Let X be a normal variety and $Z \subseteq X$ be a prime Cartier divisor which is normal as variety. Then

$$K_Z = (K_X + Z)|_Z.$$

Proof. Since both X and Z are normal, they are smooth in codimension 1. Removing the singular locus of X and Z, we may assume that both X and Z are smooth varieties. This is valid since the canonical divisor is determined by the smooth locus.

Since Z is Cartier, it is a local complete intersection in X. By Theorem 7.12, we have the exact sequence

$$0 \to \mathcal{J}_Z/\mathcal{J}_Z^2 \to \Omega_{X/\Bbbk}|_Z \to \Omega_{Z/\Bbbk} \to 0.$$

Note that Z is of codimension 1 in X, so $\mathcal{J}_Z \cong \mathcal{O}_X(-Z)$ and thus $\mathcal{J}_Z/\mathcal{J}_Z^2 \cong \mathcal{O}_X(-Z)|_Z$. Taking c_1, we obtain

$$c_1(\Omega_X)|_Z = c_1(\Omega_Z) + c_1(\mathcal{O}_X(-Z))|_Z.$$

That is,

$$K_X|_Z = K_Z - Z|_Z.$$

Rearranging gives the desired result. Yang: To be revised. restriction of Weil divisors needs to be clarified. □

7.3 Serre duality

Definition 7.14 (Dualizing sheaf)**.** Let X be a proper scheme of dimension n over $\Bbbk$. A *dualizing sheaf* on X is a coherent sheaf ω_X° together with a trace map $\mathrm{tr}_X : H^n(X, \omega_X^\circ) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}_X} \Bbbk$$

induces an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

Theorem 7.15. Let X be a projective scheme of dimension n over $\Bbbk$. Then there exists a dualizing sheaf ω_X° on X up to isomorphism. Moreover, if X is smooth, $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\Bbbk}$.

Proof. Yang: To be completed. □

Theorem 7.16 (Serre duality)**.** Let X be a projective, Cohen-Macaulay variety of dimension n over $\Bbbk$ with dualizing sheaf ω_X°. Then for every coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. Yang: To be completed. □

Yang: When $\mathcal{F}$ is locally free, we have $\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^i(X, \omega_X^\circ \otimes \mathcal{F}^\vee)$.

Corollary 7.17. Let X be a projective, normal and Cohen-Macaulay variety of dimension n over $\Bbbk$. Then for every divisor D, there is a natural isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee.$$

Yang: To be completed.

7.4 Logarithm version

8 Flat, smooth and étale morphisms

8.1 Flat families

Definition 8.1. Let $f : X \to Y$ be a morphism of schemes. For a point $\xi \in X$, we say that f is *flat at* ξ if the local ring $\mathcal{O}_{X,\xi}$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that f is *flat* if it is flat at every point $\xi \in X$.

The notation and terminology of flatness can be extended to sheaves of modules over schemes.

Definition 8.2. Let X be Y-scheme via a morphism $f : X \to Y$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *flat over* Y *at* $\xi \in X$ if the stalk $\mathcal{F}_\xi$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map

$f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that $\mathcal{F}$ is *flat over* Y if it is flat over Y at every point $\xi \in X$.

Proposition 8.3. We have the following fundamental properties of flat morphisms:

(a) open immersions are flat;

(b) the composition of flat morphisms is flat;

(c) flatness is preserved under base change;

(d) a coherent sheaf $\mathcal{F}$ on a noetherian scheme X is flat over X iff it is locally free.

Proof. Yang: To be added. □

Proposition 8.4. Let X be a regular integral scheme of dimension 1 and $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is flat over X iff it is torsion-free. Yang: To be checked.

Proposition 8.5. Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field $\mathbf{k}$. Then for every point $\xi \in X$, we have

$$\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}.$$

Yang: To be checked.

Theorem 8.6 (Miracle flatness)**.** Let $f : X \to Y$ be a morphism between noetherian schemes. Suppose that X is Cohen–Macaulay and that Y is regular. Then f is flat at $\xi \in X$ iff $\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}$. Yang: To be checked.

Theorem 8.7. Let X be a projective scheme with relatively ample line bundle $\mathcal{O}_X(1)$ over a noetherian scheme T. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\mathcal{F}$ is flat over T. Then the Hilbert polynomials $P_{X_t,\mathcal{F}_t}(m)$ are independent of $t \in T$. Conversely, suppose that T is reduced, the constant Hilbert polynomial $P_{X_t,\mathcal{F}_t}(m)$ implies that $\mathcal{F}$ is flat over T. Yang: To be checked.

Theorem 8.8. Let S be a integral noetherian scheme, $f : X \to S$ be a morphism of finite type and $\mathcal{F}$ be a coherent sheaf on X. Then there exists a non-empty open subset $U \subseteq S$ such that the restriction $\mathcal{F}|_{f^{-1}(U)}$ is flat over U.

Proof. Yang: To be added. □

8.2 Base change and semicontinuity

Let Y be an integral noetherian scheme, $f : X \to Y$ be a projective morphism, and $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. There is a natural base change map for every point $y \in Y$ and integer $i \geq 0$:

$$\varphi_y^i : (\mathrm{R}^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y).$$

Consider the function

$$h^i : Y \to \mathbb{Z}, \quad y \mapsto \dim_{\kappa(y)} H^i(X_y, \mathcal{F}_y).$$

Theorem 8.9 (Semicontinuity of cohomology)**.** The function h^i is upper semicontinuous on Y.

Yang: To be checked.

Theorem 8.10 (Grauert's theorem)**.** Suppose that for an integer $i \geq 0$, the function $h^i(y)$ is constant for all points $y \in Y$. Then the sheaf $\mathrm{R}^i f_* \mathcal{F}$ is locally free on Y, and for every point $y \in Y$, the base change map φ^i_y is an isomorphism.

Theorem 8.11 (Cohomology and base change)**.** Suppose that for a point $y \in Y$ and an integer $i \geq 0$, the base change map φ^i_y is surjective. Then there exists an open neighborhood U of y in Y such that for every point $y' \in U$, the base change map is an isomorphism. Moreover, the following are equivalent:

(a) the base change map φ^i_y is surjective;

(b) the sheaf $\mathrm{R}^i f_* \mathcal{F}$ is locally free on an open neighborhood of y.

Yang: To be checked.

8.3 Smooth morphisms

Definition 8.12. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. For $\xi \in X$ with image $\zeta = f(\xi) \in Y$, set $\overline{\zeta} : \operatorname{Spec} \overline{\kappa(\zeta)} \to Y$ to be the geometric point over ζ and $X_{\overline{\zeta}}$ be the geometric fiber over ζ. We say that f is *smooth at* ξ if f is flat at ξ and the geometric fiber $X_{\overline{\zeta}}$ is regular over $\overline{\kappa(\zeta)}$ at every point lying over ξ. We say that f is *smooth* if it is smooth at every point $\xi \in X$.

Yang: To be checked.

8.4 Étale morphisms

Definition 8.13. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. We say that f is *étale at* ξ if f is smooth and finite at ξ. We say that f is *étale* if it is étale at every point $\xi \in X$.

Yang: To be checked.

Theorem 8.14. Let $f : X \to Y$ be a morphism of finite type between noetherian integral regular schemes. Suppose that f is étale in codimension 1, i.e., there exists a closed subset $Z \subseteq X$ of codimension at least 2 such that $f|_{X \setminus Z} : X \setminus Z \to Y$ is étale. Then f is étale. Yang: To be checked.

9 Generic principle and general varieties

9.1 Generic principle

9.2 Fibration

Definition 9.1. Let S be an integral excellent scheme and X be an integral scheme over S of finite type via the structure morphism $f : X \to S$. Suppose that f is dominant. We say that f is a *fibration* if $\mathscr{K}(S)$ is algebraically closed in $\mathscr{K}(X)$.

Theorem 9.2. Let $f : X \to S$ be a fibration. Then for a general point $\sigma \in S$, the fiber X_σ is integral.

Proof. Yang: □

Slogan *Fibration is the relatively version of "integral".*

Proposition 9.3. Let S be an integral excellent scheme and X be an integral scheme over S of finite type via the structure morphism $f : X \to S$. Suppose that f is proper surjective and that S is normal. Then f is a fibration if and only if $f_*\mathcal{O}_X = \mathcal{O}_S$.

Proof. Yang: □

Proposition 9.4. Let $f : X \to Y$ be a fibration with Y normal. Then the fibers of f are connected.

Remark 9.5. If we drop the normality assumption in Proposition 9.3, then the condition $f_*\mathcal{O}_X = \mathcal{O}_Y$ is still sufficient to guarantee that f is a fibration. However, the converse may fail.

Yang: To be added.

Theorem 9.6 (Zariski's Main Theorem)**.** Let $f : Y \to X$ be a birational finite type morphism of excellent integral schemes. Suppose that X is normal. Then the fiber of f are connected.

Theorem 9.7 (Stein factorization)**.** Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then there exists a factorization

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where g is a proper morphism with connected fibers and h is a finite morphism. Moreover, this factorization is unique up to isomorphism. Yang: To be checked.

Theorem 9.8 (Rigidity Lemma)**.** Let $f : Y \to X$ be a fibration of noetherian schemes. Let $g : Y \to Z$ be a morphism such that the restriction $g|_{f^{-1}(x)} : f^{-1}(x) \to Z$ is constant for every point $x \in X$. Then there exists a unique morphism $h : X \to Z$ such that $g = h \circ f$. Yang:

9.3 Varieties in general setting

10 Formal schemes

10.1 Definitions and examples

Definition 10.1. Let X be a noetherian scheme, and let $Z \subseteq X$ be a closed subset defined by a sheaf of ideals $\mathcal{I}$. The *formal completion* of X along Z is the ringed space $(Z, \mathcal{O}_X^\wedge)$, where $\mathcal{O}_X^\wedge = \varprojlim \mathcal{O}_X/\mathcal{I}^n$. Yang: To be added.

Definition 10.2. Let X be a noetherian scheme, and let $Z \subseteq X$ be a closed subset. The *formal completion* of X along Z is the ringed space $(Z, \mathcal{O}_X^\wedge)$, where $\mathcal{O}_X^\wedge = \varprojlim \mathcal{O}_X/\mathcal{I}^n$ with $\mathcal{I}$ being the sheaf of ideals defining Z. A *formal scheme* is a ringed space that is locally isomorphic to such a formal completion.

10.2 Theorem on formal functions

Theorem 10.3.

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