Elementary results in commutative algebra

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Elementary Results

1 Nakayama's Lemma

Theorem 1 (Nakayama's Lemma)**.** Let A be a ring and $\mathfrak{M}$ be its Jacobi radical. Suppose M is a finitely generated A-module. If $\mathfrak{a}M = M$ for $\mathfrak{a} \subset \mathfrak{M}$, then $M = 0$.

Proof. Suppose M is generated by $x_1, \cdots, x_n$. Since $M = \mathfrak{a}M$, formally we have $(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(\mathfrak{a})$. Then $(\Phi - \mathrm{id})(x_1, \cdots, x_n)^T = 0$. Note that $\det(\Phi - \mathrm{id}) = 1 + a$ for $a \in \mathfrak{a} \subset \mathfrak{M}$. Then $\Phi - \mathrm{id}$ is invertible and then $M = 0$. □

Proposition 2 (Geometric form of Nakayama's Lemma)**.** Let $X = \operatorname{Spec} A$ be an affine scheme, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proof. Yang: To be completed. □

Corollary 3.

Proof. Yang: To be completed. □

2 Nullstellensatz

Theorem 4 (Noether's Normalization Lemma)**.** Let A be a k-algebra of finite type. Then there is an injection $\mathsf{k}[T_1, \cdots, T_d] \hookrightarrow A$ such that A is finite over $\mathsf{k}[T_1, \cdots, T_d]$.

Remark 5. Here A does not need to be integral. For example,

Theorem 6 (Hilbert's Nullstellensatz)**.** Let A be a