Applications to Commutative Algebra

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1 Homological dimension

Lemma 1. Let A be a ring and M an A-module. Then

$$\sup_M \operatorname{proj.dim} M = \sup_N \operatorname{inj.dim} N.$$

Proof. Note that

$$\operatorname{proj.dim} M \le n$$

if and only if

$$\operatorname{Ext}^A_{n+1}(M,N) = 0, \quad \forall N.$$

And this is equivalent to

$$\operatorname{inj.dim} N \le n.$$

□

Remark 2. In fact, for fix N, we have

$$\operatorname{inj.dim} N \le n$$

if and only if

$$\operatorname{Ext}^A_{n+1}(A/I,N) = 0, \quad \forall I$$

By Lemma Yang: ?. Hence we have

$$\sup_{M \text{ finite}} \operatorname{proj.dim} M = \sup_M \operatorname{proj.dim} M = \sup_N \operatorname{inj.dim} N.$$

Definition 3. Let A be a ring. The *homological dimension* of A, denoted $\operatorname{hl.dim} A$, is defined as

$$\operatorname{hl.dim} A := \sup_M \operatorname{proj.dim} M = \sup_M \operatorname{inj.dim} M.$$

Lemma 4. Let A be a noetherian ring, B a flat A-algebra and M a finite A-module. Then we have

$$\operatorname{Ext}^i_A(M,N) \otimes B \cong \operatorname{Ext}^i_B(M \otimes B, N \otimes M), \quad \forall N.$$

Proof. Yang: To be completed. □

Proposition 5. Let A be a noetherian ring. Then

$$\operatorname{hl.dim} A = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{hl.dim} A_{\mathfrak{p}}.$$

Proof. Compute homological dimension of A using $\operatorname{Ext}^i_A(M,N)$ for finite M. The conclusion follows from Propostion 5. □

Definition 6. Let $(A,\mathfrak{m},\mathsf{k})$ be a noetherian local ring. We say that a homomorphism of A-modules $f: M \to N$ is *minimal* if the induced map $M \otimes \mathsf{k} \to N \otimes \mathsf{k}$ is an isomorphism. Equivalently, f is minimal if and only if f is surjective and $\operatorname{Ker} f \subset \mathfrak{m}M$.

Definition 7. Let A be a noetherian local ring and M a finite A-module. A *minimal projective resolution* of M is a projective resolution

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

such that each homomorphism $P_i \to \operatorname{Ker} d_{i-1}$ is minimal.

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Proposition 8. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring and M a finite A-module. Then M has a minimal projective resolution. Moreover, any two minimal projective resolutions of M are isomorphic.

Proof. Suppose $M \otimes_A \mathsf{k} = \bigoplus \mathsf{k} \cdot \overline{x_i}$. Lift x_i to elements of M. Then we have a minimal homomorphism $\mathrm{d}_0 : \bigoplus A \cdot x_i \to M$. Similarly choose minimal homomorphisms $\mathrm{d}_k : A^{n_i} \to \operatorname{Ker} \mathrm{d}_{i-1}$ for $i = 1, 2, \cdots$. This gives a minimal projective resolution.
Suppose we have two minimal homomorphism $f, g : A^n \to M$. After tensoring with k, we have isomorphisms between $f \otimes \mathsf{k}$ and $g \otimes \mathsf{k}$. Lifting to A, we get an homomorphism $\varphi : f \to g$. Here homomorphism between f, g means a homomorphism $A^n \to A^n$ such that $f = g \circ \varphi$. The homomorphism φ is represented by a matrix T. We have $\det T \notin \mathfrak{m}$, whence φ is an isomorphism. □

Proposition 9. Let $L_\bullet \to M$ be a minimal projective resolution and $P_\bullet$ be an arbitrary projective resolution of M. Then we have $P_\bullet \cong L_\bullet \oplus P'_\bullet$ for some exact complexes $P'_\bullet$.

Proof. By Propostion **??**, we have homomorphism

$$L_\bullet \xrightarrow{\varphi_\bullet} P_\bullet \xrightarrow{\psi_\bullet} L_\bullet.$$

between complexes. By Propostion **??** again, $T_\bullet := \psi_\bullet \circ \varphi_\bullet$ is homotopic to the identity by $h_\bullet$. Suppose $T_\bullet$ is represented by a matrix. Since $L_\bullet$ is minimal, we have

$$(T - \mathrm{id})(L_n) = (\mathrm{d}_{n+1} \circ h_n + h_{n-1} \circ \mathrm{d}_n)(L_n) \subset \mathfrak{m} L_n.$$

Then $\det T \notin \mathfrak{m}$ and hence $T_\bullet$ is an isomorphism. It follows that $\psi_\bullet$ is surjective, whence it splits $P_\bullet$ into a direct sum $L \oplus P'_\bullet$ since $L_\bullet$ is projective. By the Five Lemma, we see that $P'_\bullet$ is exact. □

Lemma 10. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring and M a finite A-module. Then $\operatorname{proj.dim} M \le n$ if and only if $\operatorname{Tor}^A_{n+1}(M, \mathsf{k}) = 0$.

Proof. The necessity is clear. For the sufficiency, we have a minimal projective resolution

$$\cdots \to P_{n+1} \xrightarrow{\mathrm{d}_{n+1}} P_n \xrightarrow{\mathrm{d}_n} P_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \cdots \to P_1 \xrightarrow{\mathrm{d}_1} P_0 \xrightarrow{\mathrm{d}_0} M \to 0.$$

Let $C := \operatorname{Im} \mathrm{d}_n$. Then we have

$$0 \to P_{n+1} \xrightarrow{\mathrm{d}_{n+1}} P_n \xrightarrow{\mathrm{d}_n} C \to 0.$$

Hence $\operatorname{Tor}^A_1(C, \mathsf{k}) \cong \operatorname{Tor}^A_{n+1}(M, \mathsf{k}) = 0$. Let $K = \operatorname{Ker} \mathrm{d}_n$. Then we have the short exact sequence

$$0 \to K \to P_n \to C \to 0.$$

Since $\operatorname{Tor}^A_1(C, \mathsf{k}) = 0$, there is an exact sequence

$$0 \to K \otimes_A \mathsf{k} \to P_n \otimes_A \mathsf{k} \to C \otimes_A \mathsf{k} \to 0.$$

Since $P_n \to C$ is minimal, we have $K \otimes_A \mathsf{k} = 0$. By the Nakayama's lemma, $K = 0$. This implies that $\operatorname{proj.dim} C \le 0$ and hence $\operatorname{proj.dim} M \le n$. □

Proposition 11. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring. Then $\operatorname{hl.dim} A = \operatorname{proj.dim} \mathsf{k}$ (finite or infinite).

Proof. The inequality $\operatorname{hl.dim} A \ge \operatorname{proj.dim} \mathsf{k}$ is by definition. Conversely, we can compute $\operatorname{Tor}^A_{n+1}(M, \mathsf{k})$ using minimal projective resolution of k for any finite A-module M. By Lemma 10, we have $\operatorname{proj.dim} M \le n$ if and only if $\operatorname{Tor}^A_{n+1}(M, \mathsf{k}) = 0$. This implies that $\operatorname{proj.dim} M \le n$ for all finite A-modules M if $\operatorname{proj.dim} \mathsf{k} = n$. By Remark 2, we have $\operatorname{hl.dim} A \le n$. □

Proposition 12. Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finite A-module. Let $a \in \mathfrak{m}$ be an M-regular element. Then $\operatorname{proj.dim} M/aM = \operatorname{proj.dim} M + 1$. Here we set $\infty + 1 = \infty$.

Proof. We have an exact sequence

$$0 \to M \xrightarrow{*a} M \to M/aM \to 0.$$

Take the long exact sequence with respect to $\mathrm{Tor}(-,\mathsf{k})$, we get

$$\cdots \to \mathrm{Tor}_{i+1}^A(M,\mathsf{k}) \to \mathrm{Tor}_{i+1}^A(M/aM,\mathsf{k}) \to \mathrm{Tor}_i^A(M,\mathsf{k}) \xrightarrow{*a} \mathrm{Tor}_i^A(M,\mathsf{k}) \to \cdots$$

Since the derived homomorphism of $*a$ is zero, we have $\mathrm{Tor}_{i+1}^A(M/aM,\mathsf{k}) = 0$ if and only if $\mathrm{Tor}_i^A(M,\mathsf{k}) = 0$. By Lemma 10, we have $\operatorname{proj.dim} M/aM = \operatorname{proj.dim} M + 1$. □

2 Depth and regularity by homological algebra

Proposition 13. Let $(A,\mathfrak{m},\mathsf{k})$ be a noetherian local ring and M a finite A-module. Then

$$\operatorname{depth} M := \inf\{i : \mathrm{Ext}_A^i(\mathsf{k},M) \neq 0\}.$$

Proof. Let $a \in \mathfrak{m}$ be M-regular and $N = M/aM$. Then we claim that

$$\inf\{i : \mathrm{Ext}_A^i(\mathsf{k},N) \neq 0\} = \inf\{i : \mathrm{Ext}_A^i(\mathsf{k},M) \neq 0\} - 1.$$

Indeed, we have an exact sequence

$$0 \to M \xrightarrow{a} M \to N \to 0.$$

It induces a long exact sequence

$$\cdots \to \mathrm{Ext}_A^{i-1}(\mathsf{k},M) \to \mathrm{Ext}_A^{i-1}(\mathsf{k},N) \to \mathrm{Ext}_A^i(\mathsf{k},M) \xrightarrow{\mathrm{Ext}_A^i(\mathsf{k},\mathrm{Mult}_a)} \mathrm{Ext}_A^i(\mathsf{k},M) \to \cdots.$$

Note that $a \in \mathfrak{m}$, then $\mathrm{Ext}_A^i(\mathsf{k},\mathrm{Mult}_a) = 0$. It follows that when $\mathrm{Ext}_A^{i-1}(\mathsf{k},M) = 0$, we have $\mathrm{Ext}_A^{i-1}(\mathsf{k},N) = 0$ iff $\mathrm{Ext}_A^i(\mathsf{k},M) = 0$, whence the claim.
Let $n = \inf\{i : \mathrm{Ext}_A^i(\mathsf{k},M) \neq 0\}$. Induct on n. Suppose first $n = 0$. Since k is a simple A-module, there is an injective homomorphism $\mathsf{k} \to M$. Then $\mathfrak{m} \in \operatorname{Ass} M$ and hence $\operatorname{depth} M = 0$.
Suppose $n > 0$., let $a_1,\cdots,a_m \in \mathfrak{m}$ be any M-regular sequence. Using the claim inductively on $M/(a_1,\cdots,a_m)M$, we have $n \geq$ depth. If M has no regular element, then $\mathfrak{m} \subset \bigcup_{\mathfrak{p}\in\operatorname{Ass} M} \mathfrak{p}$. Then $\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} M$. This show that we can find $x \neq 0 \in M$ such that $\mathfrak{p} = \operatorname{Ann} x$. It gives a homomorphism $\mathsf{k} = A/\mathfrak{m} \to M$. That is a contradiction and hence M has a regular element. Let a be M-regular and $N = M/aM$. Then $\operatorname{depth} N = n-1$ by the claim and induction hypothesis. Hence we have $\operatorname{depth} M \geq n$. □

Lemma 14. Let $(A,\mathfrak{m},\mathsf{k})$ be a noetherian local ring. Suppose we have exact sequences

$$0 \to A^{n_r} \xrightarrow{\mathrm{d}_r} A^{n_{r-1}} \xrightarrow{\mathrm{d}_{r-1}} \cdots \to A^{n_1} \xrightarrow{\mathrm{d}_1} A^{n_0},$$

such that $A^{n_i} \to \operatorname{Ker} \mathrm{d}_{i-1}$ is minimal for all i. Then $\operatorname{depth} A \geq r$.

Proof. Since d_r is injective and its image is contained in $\mathfrak{m}A^{n_{r-1}}$, we can choose $t \in \mathfrak{m}$ that is not a zero divisor. Denote the sequence by $C_\bullet$. Then we have a short exact sequence of complexes

$$0 \to C_\bullet \xrightarrow{*t} C_\bullet \to C_\bullet/tC_\bullet \to 0.$$

Consider the long exact sequence in homology

$$\cdots \to H_i(C_\bullet) \xrightarrow{*t} H_i(C_\bullet) \to H_i(C_\bullet/tC_\bullet) \to H_{i-1}(C_\bullet) \xrightarrow{*t} H_{i-1}(C_\bullet) \to \cdots.$$

Since $C_\bullet$ is exact, we have $H_i(C_\bullet) = 0$ for all i. In particular, $H_i(C_\bullet/tC_\bullet) = 0$ for all $i \geq 2$. Inductively, we can choose a regular sequence of length r in $\mathfrak{m}$. □

Lemma 15. Let $(A,\mathfrak{m},\mathsf{k})$ be a noetherian local ring and M a finite A-module. Suppose there is an injective homomorphism $\mathsf{k} \to M$. Then $\operatorname{proj.dim} M \geq \dim_{\mathsf{k}} T_{A,\mathfrak{m}}$.

Proof. Let $x_1,\cdots,x_n \subset \mathfrak{m} \setminus \mathfrak{m}^2$ such that their images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. Then we have a complex

$$K_\bullet := 0 \to \wedge^n A^{\oplus n} \xrightarrow{\mathrm{d}_n} \wedge^{n-1} A^{\oplus n} \xrightarrow{\mathrm{d}_{n-1}} \cdots \to \wedge^1 A^{\oplus n} \xrightarrow{\mathrm{d}_1} \wedge^0 A^{\oplus n} \xrightarrow{\mathrm{d}_0} \mathsf{k} \to 0,$$

where

$$\mathrm{d}_r : \wedge^r A^{\oplus n} \to \wedge^{r-1} A^{\oplus n}, \quad e_{i_1} \wedge \cdots \wedge e_{i_r} \mapsto \sum_{k=1}^r (-1)^k x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_r}.$$

Here $\widehat{e_{i_k}}$ means that we omit the k-th element. Let $P_\bullet \to M$ be the minimal projective resolution of M. Then we have a homomorphism of complexes

$$\varphi_\bullet : K_\bullet \to P_\bullet$$

induced by the injective homomorphism $\mathsf{k} \to M$.
We claim that φ_i is injective and splits P_i into a direct sum $K_i \oplus F_i$ with F_i free for all $i \geq 0$. Since K_i and P_i are free, we just need to show that $\varphi_i \otimes_A \mathrm{id}_\mathsf{k}$ is injective. Induct on i. For $i = 0$, note that $\mathsf{k} \to M \otimes_A \mathsf{k}$ is injective, by the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi_0 \otimes_A \mathrm{id}_\mathsf{k}"'] & \mathsf{k} \arrow[d] \\ P_0 \otimes_A \mathsf{k} \arrow[r, "\cong"] & M \otimes_A \mathsf{k} \end{tikzcd},$$

the image of $\varphi_0 \otimes_A \mathrm{id}_\mathsf{k}$ is not zero in $P_0 \otimes_A \mathsf{k}$.
For $i > 0$, since K_{i-1} and P_{i-1} are free, we have a natural isomorphism between

$$\mathfrak{m}K_{i-1} \otimes_A \mathsf{k} \to \mathfrak{m}P_{i-1} \otimes_A \mathsf{k}$$

and

$$K_{i-1} \otimes_A \mathfrak{m}/\mathfrak{m}^2 \to P_{i-1} \otimes_A \mathfrak{m}/\mathfrak{m}^2.$$

We have a commutative diagram

$$\begin{tikzcd} K_i \otimes_A \mathsf{k} \arrow[r] \arrow[d] & \mathfrak{m}K_{i-1} \otimes_A \mathsf{k} \arrow[d] \\ P_i \otimes_A \mathsf{k} \arrow[r] & \mathfrak{m}P_{i-1} \otimes_A \mathsf{k} \end{tikzcd}. \quad (1)$$

Since $P_{i-1}/K_{i-1} \cong F_{i-1}$ is free, the right vertical map in (1) is injective. By construction of $K_\bullet$, $K_i \otimes_A \mathsf{k} \to \mathfrak{m}K_{i-1} \otimes_A \mathsf{k}$ is injective. Hence the left vertical map in (1) is injective. This completes the proof of the claim.
By the claim, $P_i \neq 0$ for all $i \leq n$ and the conclusion follows. □

Proposition 16 (Auslander-Buchsbaum formula)**.** Let A be a noetherian local ring and M a finite A-module. Suppose $\operatorname{proj.dim} M < \infty$. Then $\operatorname{proj.dim} M = \operatorname{depth} A - \operatorname{depth} M$.

Proof. We have a minimal projective resolution

$$0 \to A^{n_r} \to A^{n_{r-1}} \to \cdots \to A^{n_1} \to A^{n_0} \to M \to 0.$$

By Lemma 14, we have $\operatorname{depth} A \geq \operatorname{proj.dim} M$.
Induct on $\operatorname{depth} M$. Suppose $\operatorname{depth} M = 0$. Then by Proposition 13, we have $\operatorname{Hom}_A(\mathsf{k}, M) \neq 0$, whence there is an injective homomorphism $\mathsf{k} \to M$. By Lemma 15, we have

$$\operatorname{depth} A \geq \operatorname{proj.dim} M \geq \dim_\mathsf{k} T_{A,\mathfrak{m}} \geq \operatorname{depth} A.$$

If $\operatorname{depth} M > 0$, choose a regular element $a \in \mathfrak{m}$ that is M-regular. Then by Propostion 12, we have

$$\operatorname{depth} M + \operatorname{proj.dim} M = \operatorname{depth}(M/aM) - 1 + \operatorname{proj.dim}(M/aM) + 1 = \operatorname{depth} A.$$

□

Theorem 17. Let $(A, \mathfrak{m})$ be a noetherian local ring. Then A is regular at $\mathfrak{m}$ if and only if $\operatorname{hl.dim} A < \infty$.

Proof. Suppose A is regular at $\mathfrak{m}$. Let $x_1, \cdots, x_n$ be a minimal generating set of $\mathfrak{m}$. Then $x_1, \cdots, x_n$ is an A-regular sequence since A is regular at $\mathfrak{m}$. By Proposition 12, we have $\operatorname{proj.dim} \mathsf{k} = \operatorname{proj.dim} A/(x_1, \cdots, x_n)A = n + \operatorname{proj.dim} A = n$.
Conversely, suppose $\operatorname{hl.dim} A < \infty$. Then by Proposition 11, we have $\operatorname{proj.dim} \mathsf{k} < \infty$. We have

$$\dim_\mathsf{k} T_{A,\mathfrak{m}} \leq \operatorname{proj.dim} \mathsf{k} \leq \operatorname{depth} A \leq \dim_\mathsf{k} T_{A,\mathfrak{m}}.$$

The first "$\leq$" follows from Lemma 15. The second "$\leq$" follows from Proposition 16. Hence we see that A is regular at $\mathfrak{m}$. □

Lemma 18. Let A be a noetherian integral domain. Then A is a UFD if and only if every height 1 prime ideal of A is principal.

Proof. Yang: To be completed. □

Lemma 19. Let A be a noetherian integral domain and $(x) \subset A$ a non-zero prime ideal. Then A is a UFD if and only if $A[1/x]$ is a UFD.

Proof. Yang: To be completed. □

Theorem 20. Let $A, \mathfrak{m}$ be a regular noetherian local ring. Then A is UFD.

Proof. Yang: To be completed. □