Differentials and duality

Let S be a base noetherian scheme, $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over S and all varieties are assumed to be defined over $\Bbbk$.

1 The sheaves of differentials

Definition 1. Let $f : X \to S$ be an S-scheme. The *sheaf of differentials* of X over S, denoted by $\Omega_{X/S}$, is the $\mathcal{O}_X$-module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$.

Proposition 2. Let X and T be S-schemes and $X_T := X \times_S T$ be the base change of X along $T \to S$. Let $p : X_T \to X$ be the projection morphism. Then there is a natural isomorphism of $\mathcal{O}_{X_T}$-modules

$$\Omega_{X_T/T} \cong p^* \Omega_{X/S}.$$

Proof. Yang: To be continued. □

Proposition 3. Let X be an S-scheme and $U \subseteq X$ be an open subscheme. Then there is a natural isomorphism of $\mathcal{O}_U$-modules

$$\Omega_{U/S} \cong \Omega_{X/S}|_U.$$

Furthermore, let $\xi \in X$, then there is a natural isomorphism of $\mathcal{O}_{X,\xi}$-modules

$$\Omega_{X/S,\xi} \cong \Omega_{\mathcal{O}_{X,\xi}/\mathcal{O}_{S,f(\xi)}}.$$

Yang: To be checked.

Proof. Yang: To be continued. □

Proposition 4. Let X be a regular variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a locally free sheaf of rank n.

Proof. Yang: To be continued. □

Proposition 5. Let X be a normal variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a reflexive sheaf of rank n.

Proof. Yang: To be continued. □

Theorem 6 (Euler sequence for projective bundle)**.** Let X be a normal variety over $\Bbbk$ and $\mathcal{E}$ be a locally free sheaf of rank $r + 1$ on X. Let $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$.

Then there is an exact sequence of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$-modules

$$0 \to \Omega_{\mathbb{P}_X(\mathcal{E})/X} \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \to 0.$$

Yang: To be checked.

Proof. Yang: To be continued. □

Definition 7. Let X be a normal variety over $\Bbbk$. The *canonical divisor* K_X of X is defined to be the Weil divisor class $c_1(\Omega_{X/\Bbbk})$.

Example 8. Let $\mathbb{P}^n_{\Bbbk}$ be the projective space of dimension n over $\Bbbk$. Then the canonical divisor $K_{\mathbb{P}^n_{\Bbbk}} \sim -(n+1)H$, where H is a hyperplane in $\mathbb{P}^n_{\Bbbk}$. Yang: To be checked.

2   Fundamental sequences

Theorem 9 (The first fundamental sequence of differentials). Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Proof. Yang: To be completed. □

Proposition 10. Let $f : X \to Y$ be a surjective and generically finite morphism of normal varieties over $\Bbbk$. Then the first fundamental sequence of differentials is exact on the left.

Proof. Yang: To be completed. □

Corollary 11 (Ramification formula). Let $f : X \to Y$ be a finite morphism of normal varieties. Then

$$K_X = f^*K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} \left(\text{Mult}_D f^*(f(D)) - 1 \right) D$$

is the ramification divisor of f. Yang: To be checked. definition of ramification divisor needs to be checked.

Proof. Yang: To be completed. □

Theorem 12 (The second fundamental sequence of differentials). Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}|_Z \to \Omega_{Z/S} \to 0.$$

Suppose further that $Z \to X$ is a regular immersion. Then the above sequence is also exact on the left.

Proof. Yang: To be completed. □

Corollary 13 (Adjunction formula)**.** Let X be a normal variety and $Z \subseteq X$ be a prime Cartier divisor which is normal as variety. Then

$$K_Z = (K_X + Z)|_Z.$$

Proof. Since both X and Z are normal, they are smooth in codimension 1. Removing the singular locus of X and Z, we may assume that both X and Z are smooth varieties. This is valid since the canonical divisor is determined by the smooth locus.

Since Z is Cartier, it is a local complete intersection in X. By Theorem 12, we have the exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/\Bbbk}|_Z \to \Omega_{Z/\Bbbk} \to 0.$$

Note that Z is of codimension 1 in X, so $\mathcal{I}_Z \cong \mathcal{O}_X(-Z)$ and thus $\mathcal{I}_Z/\mathcal{I}_Z^2 \cong \mathcal{O}_X(-Z)|_Z$. Taking c_1, we obtain

$$c_1(\Omega_X)|_Z = c_1(\Omega_Z) + c_1(\mathcal{O}_X(-Z))|_Z.$$

That is,

$$K_X|_Z = K_Z - Z|_Z.$$

Rearranging gives the desired result. Yang: To be revised. restriction of Weil divisors needs to be clarified. □

3 Serre duality

Definition 14 (Dualizing sheaf)**.** Let X be a proper scheme of dimension n over $\Bbbk$. A *dualizing sheaf* on X is a coherent sheaf ω_X° together with a trace map $\mathrm{tr}_X : H^n(X, \omega_X^\circ) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}_X} \Bbbk$$

induces an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

Theorem 15. Let X be a projective scheme of dimension n over $\Bbbk$. Then there exists a dualizing sheaf ω_X° on X up to isomorphism. Moreover, if X is smooth, $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\Bbbk}$.

Proof. Yang: To be completed. □

Theorem 16 (Serre duality)**.** Let X be a projective, Cohen-Macaulay variety of dimension n over $\Bbbk$ with dualizing sheaf ω_X°. Then for every coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. Yang: To be completed. □

Yang: When $\mathcal{F}$ is locally free, we have $\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^i(X, \omega_X^\circ \otimes \mathcal{F}^\vee)$.

Corollary 17. Let X be a projective, normal variety of dimension n over $\Bbbk$. Then for every integer m and $0 \leq i \leq n$, there is a natural isomorphism Yang: To be completed.

4 Logarithm version