The First Properties of Abelian Varieties

1 Preliminaries

Proposition 1. Let $f: X \to Y$ be a morphism of varieties over a field **k**. Then the function $y \mapsto \dim f^{-1}(y)$ is upper semicontinuous, i.e., for every integer m, the set $\{y \in Y : \dim f^{-1}(y) \ge m\}$ is closed in Y. Yang: To be check.

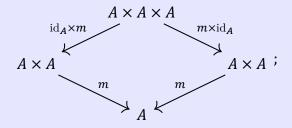
2 Definition and examples of Abelian Varieties

Theorem 2 (Rigidity Lemma). Let $\pi_i: X \to Y_i$ be proper morphisms of varieties over a field **k** for i=1,2. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi: Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Definition 3. Let S be a scheme. An *abelian scheme over* S is a group object in the category \mathbf{Sch}_{S} such that the structure morphism is proper, smooth and a fibration. If $S = \operatorname{Spec} \mathbf{k}$ for some field \mathbf{k} , then it is called an *abelian variety over* \mathbf{k} .

Definition 4. Let **k** be an algebraically closed field. An *abelian variety over* \mathbb{k} is a proper variety A over **k** together with morphisms *identity* $e : \operatorname{Spec} \mathbf{k} \to A$, *multiplication* $m : A \times A \to A$ and *inversion* $i : A \to A$ such that the following diagrams commute:

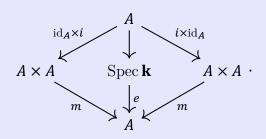
(a) (Associativity)



(b) (Identity)

$$A \times \operatorname{Spec} \mathbf{k} \xrightarrow{\operatorname{id}_A \times e} A \times A \xleftarrow{e \times \operatorname{id}_A} \operatorname{Spec} \mathbf{k} \times A$$

(c) (Inversion)



Date: August 30, 2025, Author: Tianle Yang, My Website

Yang: Can we just say that $A(\mathbf{k})$ is a group with e, m, i satisfying the axioms?

- **Example 5.** Let E be an elliptic curve over a field \mathbf{k} . Then E is an abelian variety of dimension 1.
- Example 6.
- Example 7.

In the following, we will always assume that A is an abelian variety over a field \mathbf{k} of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The left translation by $a \in A(\mathbf{k})$ is defined as

$$l_a: A \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times A \xrightarrow{a \times \operatorname{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a .

Proposition 8. Let A be an abelian variety. Then A is smooth.

Proof. Note that there is an open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a .

Proposition 9. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. Yang: But how to show it is a morphism of varieties? Yang: To be completed.

Lemma 10. Let X,Y,Z be proper varieties over a field \mathbf{k} and $g:X\times Y\to Z$ a morphism over \mathbf{k} . Suppose that g contracts $X\times y_0$ for some point $y_0\in X(\mathbf{k})$. Then there exists a unique morphism $f:Y\to Z$ such that $g=f\circ p_Y$, where $p_Y:X\times Y\to Y$ is the projection to the second factor.

Proof. Yang: To be completed.

Theorem 11. Let A and B be abelian varieties. Then any morphism $f:A\to B$ with $f(e_A)=e_B$ is a group homomorphism.

Proof. Yang: We have $(m_A)_*(\mathcal{O}_{A\times A})\cong\mathcal{O}_A$. Then consider the diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ \downarrow^{m_A} & & \downarrow^{m_B} \\ A & & B. \end{array}$$

For every closed point $a \in A$, the fiber $m_A^{-1}(a)$ is isomorphic to $m_{A_{\kappa(a)}}^{-1}(a)$ and Yang: To be completed.

Proposition 12. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 11.

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition mor-

$$[n]_A: A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 13. Let A be an abelian variety over \mathbbm{k} and n a positive integer. Then the multiplication by n morphism $[n]_A:A\to A$ is finite surjective and étale.

Proof. Yang: To be completed.

3 Complex abelian varieties

Theorem 14. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ . Then A is a complex abelian variety if and only if Λ Yang: To be completed.