Generic principle and general varieties

The main references for this section are [Har77; Laz04; FOV99].

1 Generic principle

Let X be a scheme of finite type over an excellent integral scheme S and $\mathcal{F}$ be a coherent sheaf on X. Consider the following properties of $\mathcal{F}$:

(a) $\mathcal{F}$ is locally free;

(b) $\mathcal{F}$ is flat over S;

(c) $\mathcal{F}$ satisfies the condition S_k;

Consider the following properties of X:

(a) X is smooth over S;

(b) X satisfies the condition R_k;

(c) X is Gorenstein;

Theorem 1. Let X be a scheme of finite type over an excellent integral scheme S and $\mathcal{F}$ be a coherent sheaf on X. Then for a general point $\sigma \in S$, the fiber X_σ satisfies the same properties as X and the restriction $\mathcal{F}|_{X_\sigma}$ satisfies the same properties as $\mathcal{F}$. Yang:

Theorem 2.

Corollary 3. Let X be a smooth variety over an algebraically closed field k and let $\mathcal{L}$ be a very ample line bundle on X. Then for a general member $D \in |\mathcal{L}|$, the divisor D is smooth. Yang:

2 Fibration

Example 4. Consider the morphism $f : \operatorname{Spec} k[x,y]/(xy) \to \operatorname{Spec} k[t]$ given by $t \mapsto xy$. Then the generic fiber of f is not integral. Yang:

Here we want to find out a relative version of the notion of "integral".

Definition 5. Let S be an integral excellent scheme and X be an integral scheme over S of finite type via the structure morphism $f : X \to S$. Suppose that f is dominant. We say that f is a *fibration* if $\mathscr{K}(S)$ is algebraically closed in $\mathscr{K}(X)$.

Theorem 6. Let $f : X \to S$ be a fibration. Then for a general point $\sigma \in S$, the fiber X_σ is integral.

Proof. Yang: □

Slogan *Fibration is the relatively version of "integral".*

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Proposition 7. Let S be an integral excellent scheme and X be an integral scheme over S of finite type via the structure morphism $f : X \to S$. Suppose that f is proper surjective and that S is normal. Then f is a fibration if and only if $f_*\mathcal{O}_X = \mathcal{O}_S$.

Proof. Yang: □

Proposition 8. Let $f : X \to Y$ be a fibration with Y normal. Then the fibers of f are connected.

Remark 9. If we drop the normality assumption in Proposition 7, then the condition $f_*\mathcal{O}_X = \mathcal{O}_Y$ is still sufficient to guarantee that f is a fibration. However, the converse may fail.

Yang: To be added.

Theorem 10 (Zariski's Main Theorem)**.** Let $f : Y \to X$ be a birational finite type morphism of excellent integral schemes. Suppose that X is normal. Then the fiber of f are connected.

Theorem 11 (Stein factorization)**.** Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then there exists a factorization

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where g is a proper morphism with connected fibers and h is a finite morphism. Moreover, this factorization is unique up to isomorphism. Yang: To be checked.

Theorem 12 (Rigidity Lemma)**.** Let $f : Y \to X$ be a fibration of noetherian schemes. Let $g : Y \to Z$ be a morphism such that the restriction $g|_{f^{-1}(x)} : f^{-1}(x) \to Z$ is constant for every point $x \in X$. Then there exists a unique morphism $h : X \to Z$ such that $g = h \circ f$. Yang:

3 Varieties in general setting

Definition 13. Let S be an integral excellent scheme. An S*-variety* is a scheme X of finite type over S such that the structure morphism $f : X \to S$ is a fibration. Yang:

Theorem 14. Let X be a variety over a field $\mathbf{k}$. Then there exists an integral excellent scheme S of finite type over $\operatorname{Spec}\mathbb{Z}$ and an S-variety X_S such that the generic fiber of the structure morphism $X_S \to S$ is isomorphic to X. Yang:

Appendix

References

[FOV99] H Flenner, Liam O'Carroll, and W Vogel. *Joins and Intersections*. Springer Science & Business Media, 1999 (cit. on p. 1).

[Har77] Robin Hartshorne. *Algebraic geometry*. Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9 (cit. on p. 1).

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: 10.1007/978-3-642-18808-4. URL: https://doi.org/10.1007/978-3-642-18808-4 (cit. on p. 1).