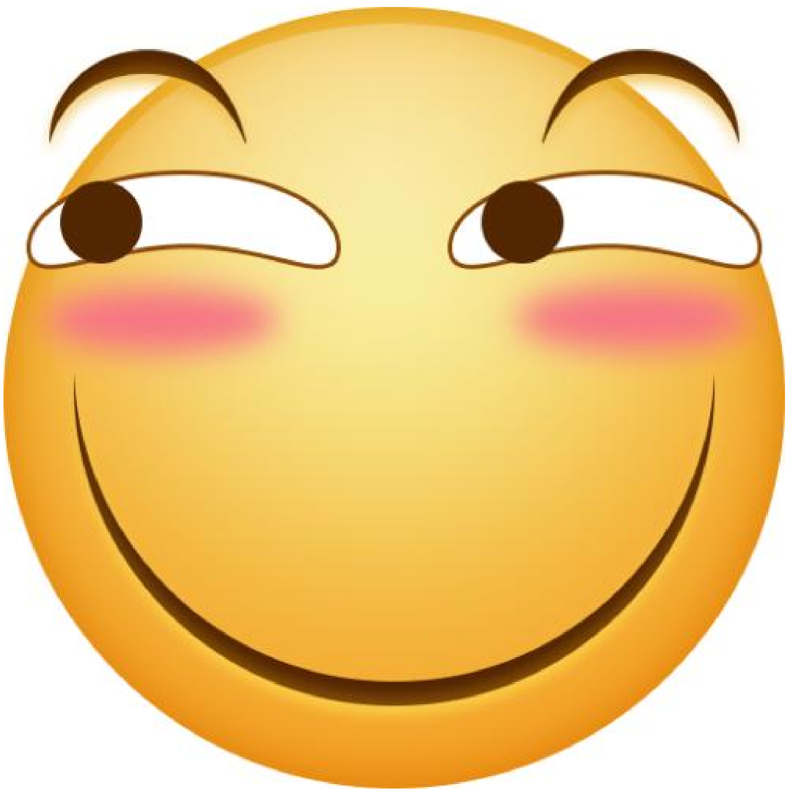
Notes in Algebraic Geometry



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Notes in Algebraic Geometry

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Contents

Chapter 1

The First Properties

1.1 Setup and the first examples

1.1.1 Notations

All schemes are assumed to be separated. For a "scheme" which is not separated, we will use the term "prescheme".

Let A be a ring. We denote by $\operatorname{Spec} A$ the spectrum of A. For an ideal $I \subset A$, we use $V(I)$ to denote the closed subscheme of $\operatorname{Spec} A$ defined by I.

Let S be $\operatorname{Spec} \mathbf{k}$, $\operatorname{Spec} \mathscr{O}_K$ or an algebraic variety. An S-variety is an integral scheme X which is of finite type and flat over S. For an algebraic variety, we mean a $\mathbf{k}$-variety.

We will use $\mathbf{k}, \mathbf{K}$ to denote fields, and $\Bbbk, \mathbb{K}$ to denote their algebraically closure relatively.

Let X be an integral scheme. We denote by $\mathscr{K}(X)$ the function field of X. For a closed point $x \in X$, we denote by $\kappa(x)$ the residue field of x.

We denote the category of S-varieties by $\mathbf{Var}_S$. We denote by $X(T)$ the set of T-points of X, that is, the set of morphisms $T \to X$.

Let X be an algebraic variety over $\mathbf{k}$. A geometrical point is referred a morphism $\operatorname{Spec} \Bbbk \to X$.

When refer a point (may not be closed) in a scheme, we will use the notation $\xi \in X$. We use Z_ξ to denote the Zariski closure of $\{\xi\}$ in X. When we talk about a closed point on an algebraic variety, we will use the notation $x \in X(\Bbbk)$.

Separated and proper morphisms

1.1.2 Examples

Chapter 2

Birational Geometry

2.1 Bend and Break

2.1.1 Preliminary

Definition 2.1.1 (Frobinius morphism)**.** Let X be a variety over a field $\mathbb{k}$ of characteristic $p > 0$. Denote the structure morphism by $\pi : X \to \operatorname{Spec}\mathbb{k}$. The *absolute Frobenius morphism* is the morphism given by $\mathcal{O}_X \to \mathcal{O}_X, f \mapsto f^p$, denoted by $\mathrm{Frob}_{X/\mathbb{F}_p}$. The *relative Frobenius morphism* is the morphism $\mathrm{Frob}_{X/\mathbb{k}}$ given by the following commutative diagram:

$$
\begin{tikzcd}
X \arrow[dr, "\mathrm{Frob}_{X/\mathbb{k}}"] \arrow[drr, bend left, "\mathrm{Frob}_{X/\mathbb{F}_p}"] \arrow[ddr, bend right, "\pi"'] & & \\
& X \times_{\mathbb{k}} \operatorname{Spec}\mathbb{k} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& \operatorname{Spec}\mathbb{k} \arrow[r, "\mathrm{Frob}_{\mathbb{k}/\mathbb{F}_p}"] & \operatorname{Spec}\mathbb{k}
\end{tikzcd}.
$$

We usually denote $X \times_{\mathbb{k}} \operatorname{Spec}\mathbb{k}$ appearing above by $X^{(p)}$.

Proposition 2.1.2. Let X be a variety of dimension d over a field $\mathbb{k}$ of characteristic $p > 0$. Then the relative Frobenius morphism $\mathrm{Frob}_{X/\mathbb{k}} : X \to X^{(p)}$ is a finite morphism of degree p^d over $\mathbb{k}$.

2.1.2 Deformation of curves

Theorem 2.1.3 (ref. [Kol96, Chapter II, Theorem 1.2])**.** Let C be a smooth projective curve of genus g and X a smooth projective variety of dimension n. Let $f : C \to X$ be a non-constant morphism. Then every irreducible component of $\mathrm{Mor}(C, X)$ containing f has dimension at least

$$-K_Y \cdot f(C) + (1 - g)n.$$

Proposition 2.1.4. Let X be a projective variety and $f : C \to X$ a non-constant morphism from a pointed smooth projective curve $p_0 \in C$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t : C \to X$ for $t \in T$ such that $f_0 = f$ and $f_t(p_0) = x_0$ for some point $x_0 \in X$ and all t. Then there exist some rational curves $\Gamma_1, \dots, \Gamma_m \subset X$ such that

(a) $x_0 \in \bigcup_{i=1}^m \Gamma_i$;

(b) there is a morphism $g : C \to X$ such that $f(C) \equiv_{alg} g(C) + \sum_{i=1}^m a_i\Gamma_i$ with $a_i > 0$ for all i.

Proposition 2.1.5. Let X be a projective variety and $f : \mathbb{P}^1 \to X$ a non-constant morphism with $f(0) = x_0, f(\infty) = x_\infty$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t : \mathbb{P}^1 \to X$ for $t \in T$ such that $f_0 = f$ and $f_t(0) = x_0, f_t(\infty) = x_\infty$ for all t. Then there exists a curve $C \subset X$ such that $f(\mathbb{P}^1) \equiv_{alg} aC$ with $a > 1$.

2.1.3 Find rational curves

Theorem 2.1.6. Let X be a smooth Fano variety. Then for any $x \in X(\mathbb{k})$, there is a rational curve C passing through x with

$$0 < -C \cdot K_X \leq \dim X + 1.$$

Proof. Yang: To be completed. ☐

Theorem 2.1.7. Let X be a smooth projective variety such that $K_X \cdot C < 0$ for some irreducible curve $C \subset X$. Let H be an ample divisor on X. Then there exists a rational curve Γ such that

$$-(K_X \cdot C) \cdot \frac{H \cdot \Gamma}{H \cdot C} \leq -K_X \cdot \Gamma \leq \dim X + 1.$$

Proof. Yang: To be completed. ☐

Theorem 2.1.8. Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Suppose that $K_{(X,B)}$ is f-ample. Then the exceptional locus of f is covered by rational curves Γ with

$$0 < -K_{(X,B)} \cdot \Gamma \leq 2 \dim X.$$

Theorem 2.1.9. Let X be a smooth projective variety of dimension n and $H, H_1, \cdots, H_{n-1}$ ample divisors on X. Suppose that $K_X \cdot H_1 \cdots H_{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H \cdot H_1 \cdots H_{n-1}}{K_X \cdot H_1 \cdots H_{n-1}}.$$

Proposition 2.1.10. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

2.2 Kodaira Vanishing Theorem

2.2.1 Preliminary

Theorem 2.2.1 (Serre Duality)**.** Let X be a Cohen-Macaulay projective variety of dimension n over $\mathbf{k}$ and D a divisor on X. Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \ldots, n.$$

Theorem 2.2.2 (Log Resolution of Singularities)**.** Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there is a smooth variety (or analytic space) Y and a projective morphism $f : Y \to X$ such that

(a) f is an isomorphism over $X - (\mathrm{Sing}(X) \cup \mathrm{Supp}\, Z)$,

(b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(c) $\mathrm{Exc}(f) \cup D$ is an snc divisor.

Theorem 2.2.3 (Lefschetz Hyperplane Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 2.2.4 (Hodge Decomposition)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Then for any k, there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 2.2.3 and Theorem 2.2.4, we have the following lemma.

Lemma 2.2.5. Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map $r_k : H^k(X,\mathbb{C}) \to H^k(Y,\mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X,\Omega_X^q) \to H^p(Y,\Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p+q<n-1$ and an injection for $p+q=n-1$. In particular,

$$H^p(X,\mathcal{O}_X) \to H^p(Y,\mathcal{O}_Y)$$

is an isomorphism for $p<n-1$ and an injection for $p=n-1$.

Theorem 2.2.6 (Leray spectral sequence)**.** Let $f : Y \to X$ be a morphism of varieties and $\mathcal{F}$ a coherent sheaf on Y. Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_*\mathcal{F}) \implies H^{p+q}(Y,\mathcal{F}).$$

2.2.2 Kodaira Vanishing Theorem

Lemma 2.2.7. Let X be a smooth projective variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on X. Suppose there is an integer m and a smooth divisor $D \in H^0(X,\mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \to X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D. It induces a homomorphism $\mathcal{L}^{-m} \to \mathcal{O}_X$. Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}\right) / (\mathcal{L}^{-m} \to \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then $\mathcal{A}$ is a finite $\mathcal{O}_X$-algebra. Let $Y := \operatorname{Spec}_X \mathcal{A}$. Then Y is a finite $\mathcal{O}_X$-scheme and the natural morphism $f : Y \to X$ is finite and surjective.

For every $x \in X$, let $\mathcal{L}$ locally generated by t near x. Then $\mathcal{O}_Y$ locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t=0$ on Y. Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. □

Remark 2.2.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D_i' = f^*(D_i)$. Then $D' + \sum_{i=1}^k D_i'$ is snc on Y by considering the local regular system of parameters.

Lemma 2.2.9. Let $f : Y \to X$ be a finite surjective morphism of projective varieties and $\mathcal{L}$ a line bundle on X. Suppose that X is normal. Then for any $i \geq 0$, $H^i(X,\mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion

$\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits by the trace map $(1/n)\,\mathrm{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. □

Theorem 2.2.10 (Kodaira Vanishing Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and A an ample divisor on X. Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 2.2.7 and 2.2.9, after taking a multiple of A, we can assume that A is effective. Then we have an exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \to H^{i-1}(X, \mathcal{O}_A) \to H^i(X, \mathcal{O}_X(-A)) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 2.2.5 and Serre duality (Theorem 2.2.1). □

2.2.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

Theorem 2.2.11 (Kawamata-Viehweg Vanishing Theorem I)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{R}$-divisor on X. Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Theorem 2.2.12 (Kawamata-Viehweg Vanishing Theorem II)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{Q}$-divisor on X. Suppose that $\lceil D \rceil - D$ has snc support. Then

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall i > 0.$$

Theorem 2.2.13 (Kawamata-Viehweg Vanishing Theorem III)**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Let D be a nef $\mathbb{Q}$-divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

If we replace the assumption ”nef and big” of D by ”ample” in II and III, we denote them as

"II(ample)" and "III(ample)" respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \implies \text{II(ample)} \implies \text{III(ample)} \implies \text{I} \implies \text{II} \implies \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

Proof of II (Theorem 2.2.12). Set $M := \lceil D \rceil$. Let

$$B := \sum_{i=1}^{k} b_i B_i := \lceil D \rceil - D = M - A, \quad b_i \in (0,1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k. When $k = 0$, the conclusion follows from Theorem 2.2.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 2.2.10.)) Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 2.2.15 and 2.2.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 2.2.7 on B_k, we can find a finite surjective morphism $f : X' \to X$ such that $f^*B_k = cB'_k, B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^{k} B'_i$ is an snc divisor on X'. Let $B' = \sum_{i=1}^{k-1} B'_i, A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$ vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^*\mathcal{O}_X(-M + (a - i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 2.2.9. □

Proof of III (Theorem 2.2.13). Let $f : \tilde{X} \to X$ be a resolution such that $\operatorname{Supp} f^*B \cup \operatorname{Exc} f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X},\tilde{B})} + f^*D,$$

where $\tilde{B} \in (0,1)$ has snc support and E is an effective exceptional divisor.

By Lemma 2.2.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X},\tilde{B})} + f^*D) = H^i(X, f_*\mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 2.2.12 in either case relative to the assumption of D. □

Proof of I (Theorem 2.2.11). By Lemma 2.2.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor A. Then the theorem comes down to Theorem 2.2.13. □

Lemma 2.2.14. Let $f : Y \to X$ be a birational morphism of projective varieties with Y smooth and X has only rational singularities. Let E be an effective exceptional divisor on Y and D a divisor on X. Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

Proof. Yang: I am unable to proof this lemma. □

Lemma 2.2.15. Let X be a projective variety, $\mathcal{L}$ a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \to X$ and a line bundle $\mathcal{L}'$ on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X, then we can take f such that f^*D is an snc divisor on Y.

Proof. We can assume that $\mathcal{L}$ is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "g"] \\ X \arrow[r, "\varphi_{\mathcal{L}}"] & \mathbb{P}^N \end{tikzcd},$$

where $g : [x_0 : \ldots : x_N] \mapsto [x_0^m : \ldots : x_N^m]$. The morphism f is finite and surjective since so is g. Let $\mathcal{L}' := \psi^*\mathcal{L}$.

For smoothness, we can compose g with a general automorphism of $\mathbb{P}^N$. Then the conclusion follows from [Har77, Chapter III, Theorem 10.8]. □

Lemma 2.2.16 (ref. [KM98, Theorem 5.10, 5.22])**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Lemma 2.2.17. Let X be a projective variety of dimension n and D a nef and big divisor on X. Then there exists an effective divisor B such that for every k, there is an ample divisor A_k such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k}B.$$

Proof. By Yang: definition of big divisor, there exists an ample divisor A_1 and effective divisor B such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k}B.$$

Since A is ample and D is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. □

2.3 Cone Theorem

2.3.1 Preliminary

Theorem 2.3.1 (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27])**.** Let X be a projective variety and ℓ an semiample line bundle on X. Then there exists a fibration $\varphi : X \to Y$ of projective varieties such that for any $m \gg 0$ with ℓ^m base point free, we have that the morphism

φ_{ℓ^m} induced by ℓ^m is isomorphic to φ. Such a fibration is called the *Iitaka fibration* associated to ℓ.

Theorem 2.3.2 (Rigidity Lemma, ref. [Deb01, Lemma 1.15]). Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Theorem 2.3.3. Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

Proposition 2.3.4. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

Schetch of proof. Take a resolution $f : Y \to X$, then f^*H is nef on Y and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor H_Y on Y closed enough to f^*H such that $K_Y \cdot H_Y^{n-1} < 0$. By [MM86, Theorem 5] and take limit for H_Y. □

Lemma 2.3.5 (ref. [Kaw91, Lemma]). Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Let E be an irreducible component of dimension d of the exceptional locus of f and $\nu : E^\nu \to X$ the normalization of E. Suppose that $f(E)$ is a point. Then for any ample divisor H on X, we have

$$K_{E^\nu} \cdot \nu^* H^{d-1} \leq K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

2.3.2 Non-vanishing Theorem

Theorem 2.3.6 (Non-vanishing Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

2.3.3 Base Point Free Theorem

Theorem 2.3.7 (Base Point Free Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, mD is base point free.

Remark 2.3.8. In general, we say that a Cartier divisor D is *semiample* if there exists a positive integer m such that mD is base point free. The statement in Base Point Free Theorem (Theorem 2.3.7) is strictly stronger than the semiample condition. For example, let ℓ be a torsion line bundle, then ℓ is semiample but there exists no positive integer M such that $m\ell$ is base point free for all $m > M$.

2.3.4 Rationality Theorem

Lemma 2.3.9 (ref. [KM98, Theorem 1.36]). Let X be a proper variety of dimension n and $D_1, \dots, D_m$ Cartier divisors on X. Then the Euler characteristic $\chi(n_1 D_1, \dots, n_m D_m)$ is a polynomial in $(n_1, \cdots, n_m)$ of degree at most n.

Theorem 2.3.10 (Rationality Theorem). Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let

$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H. Then $t = v/u \in \mathbb{Q}$ and

$$0 \leq v \leq a(X) \cdot (\dim X + 1).$$

Proof. For every $r \in \mathbb{R}_{>0}$, let

$$v(r) := \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We need to show that $v(t) \leq a(\dim X + 1)$. For every $(p, q) \in \mathbb{Z}^2_{>0}$, set $D(p, q) := paK_{(X,B)} + qH$. If $(p, q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, then we have $D(p, q)$ is not nef and $D(p, q) - K_{(X,B)}$ is ample.

Step 1. We show that a polynomial $P(x, y) \neq 0 \in \mathbb{Q}[x, y]$ of degree at most n is not identically zero on the set

$$\{(p, q) \in \mathbb{Z}^2 : p, q > M, 0 < atp - q < t\varepsilon\}, \quad \forall M > 0,$$

if $v(t)\varepsilon > a(n + 1)$.

If $v(t) = \infty$, for any n, we show that we can find infinitely many lines L such that $\#L \cap \Lambda \geq n+1$. If so, Λ is Zariski dense in $\mathbb{Q}^2$. Since $1/at \in \mathbb{R} \setminus \mathbb{Q}$, there exist $p_0, q_0 > M$ such that

$$0 < \frac{p_0}{q_0} - \frac{1}{at} < \frac{\varepsilon}{(n + 1)a} \cdot \frac{1}{q_0}, \text{ i.e. } 0 < atp_0 - q_0 < \frac{\varepsilon t}{n + 1}.$$

Then $(ip_0, iq_0) \in \Lambda \cap \{p_0 y = q_0 x\}$ for $i = 1, \cdots, n + 1$. Since M is arbitrary, there are infinitely many such lines L.

Suppose $v(t) = v < \infty$ and $t = v/u$. Then the inequality is equivalent to $0 < aup - vq < \varepsilon v$. Note that $\gcd(au, v)|a$, then $aup - vq = ai$ has integer solutions for $i = 1, \cdots, n + 1$. Since $v(t)\varepsilon > a(n + 1)$, there are at least $n + 1$ lines which intersect Λ in infinitely many points. This enforce any polynomial which vanishes on Λ has degree at least $n + 1$.

Step 2. There exists an index set $\Lambda \subset \mathbb{Z}^2$ such that Λ contains all sufficiently large (p, q) with $0 \leq atp - q \leq t$ and

$$Z := \operatorname{Bs}|D(p, q)| = \operatorname{Bs}|D(p', q')| \neq \emptyset, \quad \forall (p, q), (p', q') \in \Lambda.$$

For every $(p,q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, choose $k \in \mathbb{Z}_{>0}$ such that $k(atp-q) > t$. Then for all $p', q' > kp$ with $0 < atp' - q' < t$, we have

$$p' - kp \geq 0, \quad q' - kp > t(p' - kp).$$

It follows that

Yang: To be completed.

Step 3. Suppose the contradiction that $v(t) > a(\dim X + 1)$. Then we show that $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem (Theorem 2.3.7).

Let $P(x,y) := \chi(D(x,y))$ be the Hilbert polynomial of $D(x,y)$. Note that $P(0,n) = \chi(nH) \neq 0$ since H is ample. Then $P(x,y) \neq 0$ and $\deg P \leq \dim X$. By Step 1, P is not identically zero on Λ. Note that $D(p,q) - K_{(X,B)}$ is ample for all $(p,q) \in \Lambda$, then $h^i(X, D(p,q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem (Theorem 2.2.13). Then

$$P(p,q) = \chi(D(p,q)) = h^0(X, D(p,q)) \neq 0$$

for some $(p,q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$.

Step 4. We follow the same line of the proof of Base Point Free Theorem (Theorem 2.3.7) to show that there is a section which does not vanish on Z.

Fix $(p,q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n+1)/u$. Let $f : Y \to X$ be a resolution such that

(a) $K_{Y,B_Y} = f^*K_{(X,B)} + E_Y$ for some effective exceptional divisor E_Y, and Y, B_Y is a klt pair;

(b) $f^*|D(p,q)| = |L| + F$ for some effective divisor F and a base point free divisor L, and $f(\operatorname{Supp} F) = Z$;

(c) $f^*D(p,q) - f^*K_{(X,B)} - E_0$ is ample for some effective $\mathbb{Q}$-divisor $E_0 \in (0,1)$, and coefficients of E_0 are sufficiently small;

(d) $B_Y + E_Y + F + E_0$ has snc support.

Yang: Such resolution exists by [KM98].

Let $c := \inf\{\lfloor B_Y + E_0 + tF \rfloor \neq 0\}$. Adjust the coefficients of E_0 slightly such that $\lfloor B_Y + E_0 + cF \rfloor = F_0$ for unique prime divisor F_0 with $F_0 \subset \operatorname{Supp} F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then (Y, Δ_Y) is a klt pair.

Let

$$\begin{aligned} N(p',q') &:= f^*D(p',q') + E_Y - F_0 - K_{(Y,\Delta_Y)} \\ &= \Big(f^*D(p',q') - (1+c)f^*D(p,q)\Big) + \Big(f^*D(p,q) - f^*K_{(X,B)} - E_0\Big) + c\Big(f^*D(p,q) - F\Big). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p',q') \in \Lambda : 0 < atp' - q' < atp - q,\ p', q' > (1+c)\max\{p,q\}\},$$

the divisor $f^*D(p',q') - (1+c)f^*D(p,q) = f^*D(p'-(1+c)p, q'-(1+c)q)$ is ample, and hence $N(p',q')$ is ample.

By the exact sequence

$$0 \to \mathcal{O}_Y(f^*D(p',q') + E_Y - F_0) \to \mathcal{O}_Y(f^*D(p',q') + E_Y) \to \mathcal{O}_{F_0}((f^*D(p',q') + E_Y)|_{F_0}) \to 0$$

and Kawamata-Viehweg Vanishing Theorem (Theorem 2.2.13), we get a surjective map

$$H^0(Y, f^*D(p',q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p',q') + E_Y)|_{F_0}).$$

On F_0, consider the polynomial $\chi((f^*D(p',q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n-1$ and by the construction of $(p,q), \Lambda_0$, similar to Step 3, we can show that $\chi((f^*D(p',q') + E_Y)|_{F_0})$ is not identically zero on Λ_0. By adjunction, we have $(f^*D(p',q') + E_Y)|_{F_0} = N(p',q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p',q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (Theorem 2.2.13) to get

$$h^0(F_0, (f^*D(p',q') + E_Y)|_{F_0}) = \chi(F_0, (D(p',q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \mathrm{Bs}\,|D(p',q')|$. □

2.3.5 Cone Theorem and Contraction Theorem

Theorem 2.3.11 (Cone Theorem)**.** Let (X,B) be a projective klt pair. Then there exist countably many rational curves $C_i \subset X$ with

$$0 < -K_{(X,B)} \cdot C_i \leq 2\dim X$$

such that

(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. Let $F_D := \mathrm{Psef}_1(X) \cap D^\perp$ for a nef divisor D on X. If $\dim F_D = 1$, we also write $R_D := F_D$. Let $H_1, \cdots, H_{\rho-1}$ be ample divisors on X such that they together with $K_{(X,B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} := S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

Step 1. There exists an integer N such that for every $K_{(X,B)}$-negative extremal face F_D and for every ample divisor H, there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \neq F_{nD+rK_{(X,B)}+NH} \subset F_D$.

Let $N := (a(X)(\dim X + 1))!$, where $a(X)$ is the number in Theorem 2.3.10. For every n, $nD+H$ is an ample divisor and by Theorem 2.3.10, the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \geq 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since $K_{(X,B)} + (N/r_n)((n + 1)D + H)$ is nef, we have $r_n \leq r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$ implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \to r \in \mathbb{Z}_{\geq 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors N_1, N_2, we have $F_{N_1+N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists m large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

Step 2. Let $\Phi : N_1(X)_{K_{(X,B)}<0} \to \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left(\frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha}\right).$$

We show that the image of R_D under Φ lies in a $\mathbb{Z}$-lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class ξ. By Step 1, we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers n, r. Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of R_D under Φ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

Step 3. We show that every $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ is of the form R_D for some nef divisor D on X.

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$-negative extremal ray. Yang: Then R is of form $D^\perp \cap \mathrm{Psef}_1(X)$ for some nef $\mathbb{R}$-divisor D on X by Theorem 2.3.3. We need to show that D can be choose as a nef $\mathbb{Q}$-divisor. There is a sequence of nef but not ample $\mathbb{Q}$-divisors D_m such that $D_m \to D$ as $m \to \infty$. We adjust D_m such that $\dim F_{D_m} = 1$ for all n.

By re-choosing H_i, we can assume that $D = a_1H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is D_m. Then F_{D_m} is $K_{(X,B)}$-negative. Note that $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by Step 1. If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{D_m}$. We can assume that$r_1K_{(X,B)}+NH_1$ is not identically zero on F_{D_m}. Then we can choose n large enough such that $\|r_1K_{(X,B)} + NH_1\|/n < 1/m$. Replace

D_m by $D_m + (r_1 K_{(X,B)} + NH_1)/n$. Inductively we construct D_m nef $\mathbb{Q}$-divisor with $D_m \to D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that ξ_m converges. Then $\xi_m \to \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, Φ is well-defined at ξ and the image of ξ_m under Φ is discrete. Hence $\xi = \xi_m$ for all m large enough. It follows that $R = R_{D_m}$ for a nef $\mathbb{Q}$-divisor D_m.

Step 4. We show that any $K_{(X,B)}$-negative extremal ray R_D contains the class of a rational curve C with $0 < -K_{(X,B)} \cdot C \leq 2\dim X$.

By Theorem 2.3.13, let $\varphi_D : X \to Y$ be the contraction associated to R_D (note that we do not need the step to proof Theorem 2.3.13). If $\dim Y < \dim X$, let F be a general fiber of φ_D. Yang: By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that H is ample on F. By Proposition 2.3.4. Yang: In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By Lemma 2.3.5, we can use Proposition 2.3.4 to get the result.

Yang: To be completed.

Step 5. Proof of the theorem.

Given an ample divisor H on X, note that εH has positive minimum δ on $\operatorname{Psef}_1(X) \cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \operatorname{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and Φ is well-defined on it. By Steps 2 and 3, there are only finitely many extremal rays on $\operatorname{Psef}_1(X)_{K_{(X,B)}+\varepsilon H \leq 0}$. By Step 4, we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$\mathcal{C} := \operatorname{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\} \subset \mathcal{C}$ such that $\alpha_n \to \alpha \in N_1(X)_{\mathbb{R}}$. Note that $\operatorname{Psef}_1(X)$ is closed, hence $\alpha \in \operatorname{Psef}_1(X)$. We only need to consider the case $\alpha \cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$. Then $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$ for all n large enough. Note that $\mathcal{C} \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$ is a polyhedral cone by Step 2 and hence is closed. Then $\alpha \in \mathcal{C}$ and the conclusion follows. □

Remark 2.3.12. Yang: Thanks for my friend Qin for pointing out that the extremal ray in Theorem 2.3.11 may not be exposed.

Theorem 2.3.13 (Contraction Theorem). Let (X, B) be a projective klt pair and $F \subset \operatorname{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\operatorname{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;

(b) up to linearly equivalence, any Cartier divisor G with $F \subset G^\perp = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$ comes from a Cartier divisor on Y, i.e., there exists a Cartier divisor G_Y on Y such that $G \sim \varphi_F^* G_Y$.

Proof. We follow the following steps to prove the theorem.

Step 1. We show that there exists a nef divisor D on X such that $F = D^{\perp} \cap \mathrm{Psef}_1(X)$. In other words, F is defined on $N_1(X)_{\mathbb{Q}}$.

We can choose an ample divisor H and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on F since $F \cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in $N_1(X)_{\mathbb{R}}$. Then by Cone Theorem (Theorem 2.3.11), F is an extremal face of a rational polyhedral cone, namely $\mathrm{Psef}_1(X)_{K_{(X,B)}+(1/n)H\le 0}$. It follows that $F^{\perp} \subset N^1(X)_{\mathbb{R}}$ is defined on $\mathbb{Q}$. Since F is extremal and $K_{(X,B)} + (1/n)H$-negative, the set $\{L \in F^{\perp} : L|_{\mathrm{Psef}_1(X)\setminus F} > 0\}$ has non-empty interior in $F^{\perp}$ by Theorems 2.3.3 and 2.3.11. Then there exists a Cartier divisor D such that $D \in F^{\perp}$ and $D|_{\mathrm{Psef}_1(X)\setminus F} > 0$. It follows that D is nef and $F = D^{\perp} \cap \mathrm{Psef}_1(X)$.

Step 2. Let $\varphi : X \to Y$ be the Iitaka fibration associated to D by Theorem 2.3.1. We show that φ is the desired fibration.

Note that $\mathrm{Psef}_1(X)_{K_{(X,B)}\ge 0} \cap S^{\rho-1}$ is compact and D is strictly positive on it. Then there exist $a \ge 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\mathrm{Psef}_1(X)_{K_{(X,B)}\ge 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since F is $K_{(X,B)}$-negative. Then by Base Point Free Theorem (Theorem 2.3.7), we know that mD is base point free for all $m \gg 0$. Hence we can apply Theorem 2.3.1 to get a fibration $\varphi_D : X \to Y$.

First we show that D comes from Y. Note that mD and $(m+1)D$ induces the same fibration φ_D for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m)D_{Y,m}$ and $D_{Y,m}$ is ample. Hence D_Y is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that C is contracted by φ_D if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let G be arbitrary Cartier divisor on X such that $F \subset G^{\perp}$. Since D is strictly positive on $\mathrm{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' := mD + G$, we have $D'^{\perp} \cap \mathrm{Psef}_1(X) = F$. Then by the same argument as above, we get an other fibration $\varphi_{D'} : X \to Y'$ such that a curve C is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 2.3.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on Y. In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor D'_Y on Y. Then $G = D' - mD$ also comes from Y. □

Remark 2.3.14. The Step 1 is amazing. If F is not $K_{(X,B)}$-negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E. By [Laz04, Lemma 1.5.4], we know that $\mathrm{Psef}_1(X)$ is a circular cone. The we see there indeed exist some irrational extremal faces of $\mathrm{Psef}_1(X)$.

Definition 2.3.15. Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ with contraction $\varphi_R : X \to Y$. There are three types of contractions:

(a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension one;

(b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension at least two;

(c) *Mori fiber space*: if $\dim X > \dim Y$.

Proposition 2.3.16. Let (X, B) be a $\mathbb{Q}$-factorial projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Suppose that the contraction $\varphi : X \to Y$ associated to R is either divisorial or a Mori fiber space. Then Y is $\mathbb{Q}$-factorial.

Proof. Let D be a prime Weil divisor on Y and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\geq 0}[C]$ for an irreducible curve C contracted by φ. Set $D_X := \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$. Then D_X is a prime Weil divisor on X and hence is $\mathbb{Q}$-Cartier.

If φ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber F of φ. Then by Contraction Theorem (Theorem 2.3.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor D' on Y. We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence D is $\mathbb{Q}$-Cartier.

If φ is a divisorial contraction, let E be the exceptional divisor of φ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier $\mathbb{Q}$-divisor E_Y on Y). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 2.3.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor D' on Y. Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence D is $\mathbb{Q}$-Cartier. □

Remark 2.3.17. If φ is a small contraction, then Y is never $\mathbb{Q}$-factorial. Otherwise, let B_Y be the strict transform of B on Y. Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset U. Suppose $K_{(Y,B_Y)}$ is $\mathbb{Q}$-Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

2.4 Basepoint Free Theorem on Positive Characteristic

This section refers to [Kee99]. For site and algebraic space, we refer to [Knu71], [Art70], [Stacks] and [FGA05]. Throughout this section, all schemes (or algebraic space) are of finite type over a base scheme S with S noetherian.

2.4.1 Preliminaries

Theorem 2.4.1 (Serre vanishing in relative setting, ref. [Laz04, Theorem 1.7.6])**.** Let $f : X \to S$ be a proper morphism of schemes, $\mathscr{L}$ a line bundle and $\mathscr{F}$ a coherent sheaf on X. Suppose that $\mathscr{L}$ is relatively ample. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the higher direct image sheaves $R^i f_* \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ are zero for all $i > 0$.

Theorem 2.4.2 (ref. [Laz04, Proposition 1.4.37])**.** Let X be a projective scheme over a field $\Bbbk$. Then there exists a scheme T of finite type over $\Bbbk$ and a line bundle $\mathscr{L}$ on $X \times T$ such that every numerically trivial line bundle on X arises as the restriction $\mathscr{L}|_{X \times \{t\}}$ for some $t \in T$.

Theorem 2.4.3 (Theorem on Formal Functions, ref. [Har77, Chapter III, Theorem 11.1])**.** Let $f : X \to Y$ be a projective morphism of noetherian schemes, let $\mathscr{F}$ be a coherent sheaf on X, and let $y \in Y$. Then the natural map

$$(R^i f_* \mathscr{F})^\wedge_y \to \varprojlim H^i(X_n, \mathscr{F}_n)$$

is an isomorphism for all $i \geq 0$, where $X_n = X \times_Y \operatorname{Spec} \mathscr{O}_{Y,y}/\mathfrak{m}_y^n$ and $\mathscr{F}_n = \mathscr{F}|_{X_n}$.

Definition 2.4.4. Let X be a proper variety and $\mathscr{L}$ a nef line bundle on X. A closed subvariety $Z \subseteq X$ is called the *exceptional* for $\mathscr{L}$ if $\mathscr{L}^{\dim Z} \cdot Z = 0$. The *exceptional locus* of $\mathscr{L}$, denoted by $\operatorname{Exc} \mathscr{L}$, is defined as the closure of the union of all exceptional subvarieties of $\mathscr{L}$.

If $\mathscr{L}$ is semiample, then $\operatorname{Exc} \mathscr{L} = \operatorname{Exc} \varphi$ for the fibration $\varphi : X \to Y$ induced by $\mathscr{L}$.

Definition 2.4.5. Let X be a proper scheme and $\mathscr{L}$ a nef line bundle on X. We say that $\mathscr{L}$ is *endowed with a map (EWM)* if there is a proper morphism $\varphi : X \to Y$ to a proper algebraic space such that $\dim Z > \dim f(Z)$ if and only if Z is an exceptional subvariety of $\mathscr{L}$. If such a morphism is a fibration, then it is unique, called the *fibration associated to* $\mathscr{L}$.

Proposition 2.4.6. Let X be a proper variety and $\mathscr{L}$ a nef line bundle on X endowed with a map. Let $\varphi : X \to Y$ be the associated fibration. Then TFAE:

(a) $\mathscr{L}$ is semiample;

(b) $\mathscr{L}^{\otimes m}$ is pulled back from an ample line bundle on Y for some $m \in \mathbb{Z}_{>0}$;

(c) $\mathscr{L}^{\otimes m}$ is pulled back from a line bundle on Y for some $m \in \mathbb{Z}_{>0}$;

Proof. (a) $\Leftrightarrow$ (b) $\implies$ (c) is clear. Replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes m}$ for some $m \in \mathbb{Z}_{>0}$, suppose that $\mathscr{L} = \varphi^* \mathscr{L}_Y$ for some line bundle $\mathscr{L}_Y$ on Y. We show that $\mathscr{L}_Y$ is ample. Indeed, for all closed subvarieties $Z \subset Y$, we can find $Z' \subset X$ such that $Z' \twoheadrightarrow Z$ and $\dim Z' = \dim Z$. Then

$$\mathscr{L}_Y^{\dim Z} \cdot Z = d \mathscr{L}^{\dim Z'} \cdot Z' > 0$$

where $d = \deg(Z' \to Z)$. Hence $\mathscr{L}_Y$ is ample. □

Definition 2.4.7. A morphism $f : X \to Y$ of schemes is called a *universal homeomorphism* if for every Y-scheme Y', the base change $X \times_Y Y' \to Y'$ is a homeomorphism between the underlying topological spaces.

Example 2.4.8. Let X be a scheme of finite type over $\mathbb{k}$. Then the natural morphism $X_{\text{red}} \to X$ is a universal homeomorphism.

Let X be a scheme over S of characteristic p. Then the absolute and relative Frobenius morphisms are universal homeomorphisms. Yang: To be completed.

The morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ is not a universal homeomorphism.

Lemma 2.4.9. Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes with g finite. Let $\mathcal{F}$ be a coherent sheaf on X. Then the we have

$$R^i(g \circ f)_*\mathcal{F} = g_*(R^i f_*\mathcal{F}).$$

Proof. Yang: This is a simple application of the Grothendieck spectral sequence. However, I do not know anything about it. □

2.4.2 Algebraic space

Definition 2.4.10. Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of sets of arrows $\{U_i \to U\}_{i\in I}$, called *covering*, for each object U in $\mathbf{C}$ such that:

(a) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering;

(b) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a arrow, then the fiber product $U_i \times_U V \to V$ exists and $\{U_i \times_U V \to V\}$ is a covering of V;

(c) if $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ are coverings, then the collection of composition $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{J})$ where $\mathbf{C}$ is a category and $\mathcal{J}$ is a Grothendieck topology on $\mathbf{C}$.

Note that sheaf is indeed defined on a site.

Definition 2.4.11. Let $(\mathbf{C}, \mathcal{J})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{J})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ satisfying the following condition: for every object U in $\mathbf{C}$ and every covering $\{U_i \to U\}_{i\in I}$ of U, if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every i, j, the pullback $s_i|_{U_i \times_U U_j}$ and $s_j|_{U_i \times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every i, the pullback $s|_{U_i} = s_i$.

Definition 2.4.12. Let X be a scheme. The *big étale site* of X, denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over X with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if and only if each U_i is étale over U and the union of their images is the whole U.

Let X be a scheme over S. By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \to \mathbf{Set}$ such that for any S-scheme T, $h_X(T) = \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$. Yang: Easy to check that h_X is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

Definition 2.4.13. Let U be a scheme over a base scheme S. An *étale equivalence relation* on U is a morphism $R \to U \times_S U$ between schemes over S such that:

(a) the projections in two factors $R \to U$ are étale and surjective;

(b) for every S-scheme T, $h_R(T) \to h_U(T) \times h_U(T)$ gives an equivalence relation on $h_U(T)$ set-theoretically.

Definition 2.4.14. An *algebraic space* X over a base scheme S is an S-scheme U together with an étale equivalence relation $R \to U \times_S U$.

Let $X = (U, R)$ be an algebraic space over S. We explain X as a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. For any scheme T over S, $h_R(T)$ is an equivalence relation on $h_U(T)$. The rule sending T to the set of equivalence classes of $h_R(T)$ gives a presheaf on the site $(\mathbf{Sch}/S)_{\text{ét}}$. The sheafification of this presheaf is the sheaf associated to the algebraic space X. Explicitly, we have

$$X(T) := \left\{ f = (f_i) \,\middle|\, \begin{array}{c} \{T_i \to T\} \text{ a covering, } f_i \in h_U(T_i) \text{ such} \\ \text{that } (f_i|_{T_i \times_T T_j}, f_j|_{T_i \times_T T_j}) \in h_R(T_i \times_T T_j) \end{array} \right\} / \sim,$$

where

$$\alpha \sim \beta \quad \text{if } \exists \{S_i \to T\} \text{ such that } (\alpha|_{S_i}, \beta|_{S_i}) \in h_R(S_i).$$

Definition 2.4.15. An *algebraic space* over a base scheme S is a sheaf F on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$ such that

(a) the diagonal morphism $F \to F \times_S F$ is representable;

(b) there exists a scheme U over S and a map $h_U \to F$ which is surjective and étale.

The *morphism between algebraic spaces* F_1, F_2 is defined as a natural transformation of functors F_1, F_2.

Remark 2.4.16. By Yoneda's Lemma, given a morphism $h_U \to F$ between sheaves is the same as giving an element of $F(U)$. We may abuse the notation.

Definition 2.4.17. Let $\mathcal{P}$ be a property of morphisms of schemes satisfying the following conditions:

(a) is preserved under any base change;

(b) is étale local on the base. Yang: In [Stacks], this requires that "fppf local".

Let $\alpha : F \to G$ be a representable morphism of sheaves on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. We say that α has property $\mathcal{P}$ if for every $h_T \to G$, the base change $h_T \times_G F \to F$ has property $\mathcal{P}$.

Remark 2.4.18. The fiber product $F_1 \times_F F_2$ is just defined as $F_1 \times_F F_2(T) := F_1(T) \times_{F(T)} F_2(T)$ for any object $T \in \mathrm{Obj}(\mathbf{Sch}_S)$. We say that a morphism $f : F_1 \to F_2$ of sheaves is *representable* if for every $T \in \mathrm{Obj}(\mathbf{Sch}/S)$ and every $\xi \in F_2(T)$, the sheaf $F_1 \times_{F_2} h_T$ is representable as a functor. Here $h_T \to F_2$ is given by

$$h_T(U) \to F_2(U), \quad f \in \mathrm{Hom}(U,T) \mapsto F_2(f)(\xi) \in F_2(U).$$

In our case, given an arbitrary $h_U \to F \times F$ is equivalent to giving morphisms $h_{U_i} \to F$ for $i = 1, 2$. And the fiber product $F \times_{F\times F} (h_{U_1} \times h_{U_2})$ is just the fiber product $h_{U_1} \times_F h_{U_2}$. Hence the first condition in Definition 2.4.15 is equivalent to that $h_{U_1} \times_F h_{U_2}$ is representable for any U_1, U_2 over F. This implies that $h_U \to F$ is representable, whence the second condition in Definition 2.4.15 makes sense.

Definition 2.4.19. Let X be an algebraic space over a base scheme S. Two two morphisms form field $\operatorname{Spec} k_i \to X$ is called equivalent if there is a common extension $K \supset k_1, k_2$ such that we have $\operatorname{Spec} K \to \operatorname{Spec} k_i \to X$ are the same for $i = 1, 2$. The *underlying point set* of X, denote by $|X|$, is defined as the set of equivalence classes of morphisms $\operatorname{Spec} k \to X$ for all field k over the base field $\mathbb{k}$.

This definition coincides with the underlying set of a scheme. Let $\alpha : X \to Y$ be a morphism of algebraic spaces. It induces a map $|\alpha| : |X| \to |Y|$ by $x \mapsto \alpha \circ x$ (vertical composition).

Proposition 2.4.20 (ref. [Stacks, Lemma 66.4.6])**.** There is a unique topology on $|X|$ such that

(a) if X is a scheme, then the topology coincides with the usual topology.

(b) every morphism of algebraic spaces $f : X \to Y$ induces a continuous map $|f| : |X| \to |Y|$.

(c) if U is a scheme and $U \to X$ is étale, then the induced map $|U| \to |X|$ is open.

This topology is called the *Zariski topology* on $|X|$.

Definition 2.4.21. Let X be an algebraic space over a base scheme S. All étale morphisms $U \to X$ with U scheme form a small site $X_{\text{ét}}$. All étale morphisms $U \to X$ with U algebraic space form a small site $X_{\text{sp, ét}}$. The *structure sheaf* $\mathscr{O}_X$ of X is given by $U \mapsto \Gamma(U, \mathscr{O}_U)$ for every étale morphism $U \to X$ from a scheme. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

Example 2.4.22. Let $U = \mathbb{A}^1_{\mathbb{C}}$ and $R \subset U \times U$ given by $y = x + n, n \in \mathbb{Z}$. Then R is a disjoint union of lines in $U \times U$. Write $R = \coprod_{n\in\mathbb{Z}} R_n$ with $R_n = \{(x, x + n) : x \in \mathbb{C}\}$. Then the projection is given by

$$\pi_1|_{R_n} : R_n \to U, \quad (x, x + n) \mapsto x,$$
$$\pi_2|_{R_n} : R_n \to U, \quad (x, x + n) \mapsto x + n.$$

Easily see that the projection $\pi_i : R \to U$ is étale and surjective for $i = 1, 2$. Let $r_{ij} : R\times U \to U\times U\times U$ be the morphism which maps $((x, y), u)$ to (a_1, a_2, a_3) where $a_i = x$, $a_j = y$ and $a_k = u$ for $k \neq i, j$. Since $\Delta_U \to U \times U$ factors through R, $(\pi_1, \pi_2) = (\pi_2, \pi_1)$ and $r_{12} \times_{(U\times U\times U)} r_{23}$ factors through r_{13}, we have that $h_R(T)$ is an equivalence relation on $h_U(T)$ for all T over S. Then $X := (U, R)$ is an algebraic space.

We do not check the representability here but give an example. Let $U \to X$ be the natural morphism given by $\mathrm{id}_U \in X(U)$. For any scheme T over $\mathbb{C}$, we have

$$(U \times_X U)(T) = \{(f, g) \in h_{U\times U}(T) : \exists \{T_i \to T\} \text{ s.t. } (f_i, g_i) \in h_R(T_i)\} = h_R(T).$$

Hence the fiber product $h_U \times_X h_U$ is represented by R.

We show that $X \ncong \mathbb{C}^\times$ by computing the the global sections. Consider the covering $U \to X$, a section $s \in \mathcal{O}_X(X)$ is given by a section $s \in \Gamma(U, \mathcal{O}_U) = \mathbb{C}[t]$ such that $\pi_1^* s = \pi_2^* s$ in $\Gamma(R, \mathcal{O}_R)$. This means that $s(x + n) = s(x)$ for all $n \in \mathbb{Z}$. Hence s is a constant function. In particular, $\mathcal{O}_X(X) = \mathbb{C} \neq \mathbb{C}[t, t^{-1}]$.

The underlying set $|X|$ is union of the quotient set $\mathbb{C}/\mathbb{Z}$ and a generic point. The Zariski topology on $|X|$ is the trivial topology.

In following, we will use the technique of *local construction* to construct many scheme-like objects on algebraic spaces. For local construction, see [Knu71]. Roughly speaking, for every étale morphism $U \to X$ with U a scheme, we construct a scheme-theoretic object on U which is compatible under base change. Then we glue these objects together to get a global object on X.

Definition 2.4.23. Let X be an algebraic space over a base scheme S. A *coherent sheaf* on X is a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ such that for every covering $\{U_i \to X\}$ with U_i schemes, the sheaf $\mathcal{F}|_{U_i}$ is coherent for every i. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.
An *ideal sheaf* on X is a coherent sheaf $\mathcal{I} \subset \mathcal{O}_X$. It defines a closed subspace $V(\mathcal{I}) \subset X$ by Yang: to be completed. And every closed subspace $Y \subset X$ is defined by an ideal sheaf $\mathcal{I}_Y$ such that $V(\mathcal{I}_Y) = Y$.

Definition 2.4.24. Let X be an algebraic space over a base scheme S. A *line bundle* on X is a coherent sheaf $\mathcal{L}$ on X such that for every covering $\{U_i \to X\}$ with U_i schemes, the sheaf $\mathcal{L}|_{U_i}$ is a line bundle on U_i. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

Theorem 2.4.25 (ref. [Stacks, Theorem 76.36.4])**.** Let $f : X \to Y$ be a proper morphism of algebraic spaces over a base scheme S. Then there exists a factorization

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y,$$

where f_1 has geometrically connected fibers and $(f_1)_* \mathcal{O}_X = \mathcal{O}_Z$ and f_2 is finite.

Definition 2.4.26. Let X be an algebraic space over a base scheme S and Y a closed subset of $|X|$. The *formal completion* of X along Y, denoted by $\mathfrak{X}$, is
Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined as $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$ where $\mathcal{I}$ is the ideal sheaf of Y in $\mathcal{O}_X$. Yang: to be completed.

Definition 2.4.27. Let X be an algebraic space and Y a closed subset of X. A *modification* of X along Y is a proper morphism $f : X' \to X$ and a closed subset $Y' \subset X'$ such that $X' \setminus Y' \to X \setminus Y$ is an isomorphism and $f^{-1}(Y) = Y'$.

Theorem 2.4.28 (ref. [Art70, Theorem 3.1])**.** Let Y' be a closed subset of an algebraic space X' of finite type over $\Bbbk$. Let $\mathfrak{X}'$ be the formal completion of X' along Y'. Suppose that there is a formal modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$. Then there is a unique modification

$$f : X' \to X, \quad Y \subset X$$

such that the formal completion of X along Y is isomorphic to $\mathfrak{X}$ and the induced morphism $\mathfrak{X}' \to \mathfrak{X}$ is isomorphic to $\mathfrak{f}$.

Theorem 2.4.29 (ref. [Art70, Theorem 6.2])**.** Let $\mathfrak{X}'$ be a formal algebraic space and $Y' = V(\mathscr{i}')$ with $\mathscr{i}'$ the defining ideal sheaf of $\mathfrak{X}'$. Let $f : Y' \to Y$ be a proper morphism. Suppose that

(a) for every coherent sheaf $\mathscr{f}$ on $\mathfrak{X}'$, we have

$$R^1 f_* \mathscr{i}'^n \mathscr{f} / \mathscr{i}'^{n+1} \mathscr{f} = 0, \quad \forall n \gg 0;$$

(b) for every n, the homomorphism

$$f_*(\mathscr{O}_{\mathfrak{X}'}/\mathscr{i}'^n) \otimes_{f_* \mathscr{O}_{Y'}} \mathscr{O}_Y \to \mathscr{O}_Y$$

is surjective.

Then there exists a modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ and a defining ideal sheaf $\mathscr{i}$ of $\mathfrak{X}$ such that $V(\mathscr{i}) = Y$ and $\mathfrak{f}$ induces f on Y.

Theorem 2.4.30 (ref. [Art70, Theorem 6.1])**.** Let Y' be a closed algebraic subspace of an algebraic space X' and $f_0 : Y' \to Y$ a finite morphism. Then there exists a modification $f : X' \to X$ whose restriction to Y' is f_0. It is the amalgamated sum $X = X' \amalg_{Y'} Y$ in the category of algebraic spaces **AlgSp**.

Example 2.4.31. Let $X = \mathbb{a}^2 = \operatorname{Spec} \Bbbk[x, y]$ and $Y = V(y)$ be the x-axis. Let $f_0 : Y' = \mathbb{a}^1 \to Y, x \mapsto x^2$. Then there exists a modification $f : X' \to X$ such that the restriction $f|_{Y'} : Y' \to Y$ is f_0. Yang: To be completed.

2.4.3 A sufficient and necessary condition for EWM

In this and next subsection, we assume that all schemes (algebraic spaces) are of finite type over a field $\Bbbk$ with characteristic $p > 0$.

Lemma 2.4.32. Let $f : X \to Y$ be a finite morphism of algebraic space which is of finite type over $\Bbbk$. Suppose that f is a universal homeomorphism. Then there exists $q = p^n$ such that the relative Frobinius morphism $\mathrm{Frob}^n_{X/\Bbbk}$ factors as

$$\mathrm{Frob}^n_{X/\Bbbk} : X \xrightarrow{f} Y \to X^{(q)}.$$

Proof. Yang: I can only prove this for schemes. Suppose that X, Y are affine. Factor it as $A \twoheadrightarrow B \hookrightarrow C$ with A, B, C $\Bbbk$-algebras.

For $A \twoheadrightarrow B$, let I be the kernel of the surjection. Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is finite universal homeomorphism, we have that I is a nilpotent ideal. Hence there exists q such that $I^q = 0$. Let $a, a' \in A$ with the same image b in B. Then we have $a^q - a'^q \in I^q = 0$. Hence $a^q = a'^q$ in A. This gives a map $B^q \to A, b^q \mapsto a^q$.

For $B \hookrightarrow C$, we induct on the dimension. If C is artinian, then $0 = C^q \subset B \subset C$. In general case, this shows that $B \cdot C^{q_1} \subset C$ is an isomorphism at generic points. Let $I := \operatorname{Ann}(B \cdot C^q / B) \subset B$. This is the conductor of extension $B \cdot C^{q_1} \subset C$, whence also an ideal of $B \cdot C^{q_1}$. To see this, for every $x \in B \cdot C^{q_1}$, $b \in I$, we have $xbB \cdot C^{q_1} = bB \cdot C^{q_1} \subset B$. By induction hypothesis, we have $(BC^{q_1}/I)^{q_2} \subset B/I$. For $x \in BC^{q_1}$, there exists $b \in B$ and $\delta \in I \subset B$ such that $x^{q_2} = b + \delta \in B$. Hence we have $(BC^{q_1})^{q_2} \subset B$. In particular, we have $C^{q_1 q_2} \subset (B \cdot C^{q_1})^{q_2} \subset B$.

In general case, we have

$$\begin{tikzcd} C^{q_1 q_2} \arrow[r] & A' \arrow[r, two heads] \arrow[d] & C^{q_1} \arrow[d] & \\ & A \arrow[r, two heads] & B \arrow[r, hook] & C \end{tikzcd},$$

where A' is the preimage of C^{q_1} in A. One we have $C^q \to A \to C$, note that $A \to C$ is over $\Bbbk$, then it gives

$$C^q \to C^{(q)} \to A \to C.$$

□

Corollary 2.4.33. Let $Z \to X$ be a finite universal homeomorphism of algebraic spaces and $Z \to Y$ any finite morphism of algebraic spaces. Suppose that X, Y, Z are all of finite type over $\Bbbk$. Then the amalgamated sum $X \amalg_Z Y$ exists in the category of algebraic spaces. Moreover, $Y \to X \amalg_Z Y$ is a finite universal homeomorphism.

Proof. By Lemma 2.4.32, we have a diagram

$$\begin{tikzcd} Y^{(q)} & Y \arrow[l] \\ Z^{(q)} \arrow[u] & \\ X \arrow[u] & Z \arrow[l, "f"] \arrow[uu, "g"] \end{tikzcd}.$$

Denote $X \to Y^{(q)}$ by f. Let

$$\mathscr{a} := \mathrm{Ker}(\mathscr{O}_X \times \mathscr{O}_Y \to \mathscr{O}_Z, \quad (s,t) \mapsto f^*s - g^*t).$$

Then $\mathscr{a}$ is an $\mathscr{O}_{Y^{(q)}}$-algebra. Set $W := \mathrm{Spec}_{Y^{(q)}} \mathscr{a}$. Then $W = X \amalg_Z Y$ is the amalgamated sum in the category of algebraic spaces. Yang: The most important point is that $Z \to W$ is finite. Yang: At least in the cat of schemes. ☐

Proposition 2.4.34. Let $g : X' \to X$ be a proper, finite universal homeomorphism between algebraic spaces. Then a line bundle ℓ on X is endowed with a map if and only if $g^*\ell$ is endowed with a map.

Proof. Let $f : X' \to Z$ be the map endowed on $g^*\ell$. By Lemma 2.4.32, we have a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[dd, "f"] & X \arrow[d] \\
& X'^{(q)} \arrow[d] \\
Z \arrow[r] & Z^{(q)}
\end{tikzcd}.$$

Easy to check that $X \to Z^{(q)}$ is a map associated to ℓ. ☐

Proposition 2.4.35. Let X be a projective scheme and ℓ a nef line bundle on X. Assume that $X = X_1 \cup X_2$ for closed subsets X_1 and X_2. Suppose that $\ell|_{X_i}$ is endowed with a fibration $g_i : X_i \to Z_i$ for $i = 1, 2$. Then ℓ is endowed with a map $g : X \to Z$.

Proof. Let $X_{12} := X_1 \cap X_2$. Let $X_{12} \to Z_{12}$ be the Stein factorization of the map $g_1|_{X_{12}}$. Then by Yang: Rigidity Lemma, it is also the Stein factorization of the map $g_2|_{X_{12}}$. Denote Y_i be the image of Z_{12} in Z_i for $i = 1, 2$. Then we have a commutative diagram

$$\begin{tikzcd}
& & Z_1 & \\
& X & X_1 \arrow[l] \arrow[u] & Y_1 \arrow[ul, "h'"'] \\
Z_2 & X_2 \arrow[l] \arrow[u] & X_{12} \arrow[l] \arrow[u] \arrow[dr] & \\
& Y_2 \arrow[ul] & & Z_{12} \arrow[ll] \arrow[uu, "h"'] \arrow[ulll, "f"]
\end{tikzcd}.$$

Consider the sub-diagram

$$\begin{tikzcd}
& Z_1 \\
& Y_1 \arrow[u, "h'"] \\
Z_2 & Z_{12} \arrow[l, "f"] \arrow[u, "h"]
\end{tikzcd}.$$

Here f is finite, h is finite universal homeomorphism and h' is a closed immersion. By Corollary 2.4.33, we have the amalgamated sum $Z' := Y_1 \amalg_{Z_{12}} Z_2$ exists in the category of algebraic spaces. Since f is finite, so is the induced morphism $Y_1 \to Z'$. Then by Theorem 2.4.30, the amalgamated sum $Z := Z' \amalg_{Y_1} Z_1$ exists in the category of algebraic spaces.

Then we have a commutative diagram

$$\begin{tikzcd}
Z & & Z_1 \arrow[ll, hook'] \\
& X \arrow[ul, "g"'] & X_1 \arrow[l, hook'] \arrow[u, hook] \\
Z_2 \arrow[uu, hook] & X_2 \arrow[l, hook'] \arrow[u, hook] & X_{12} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}.$$

Directly check shows that g is a map associated to ℓ. ☐

Proposition 2.4.36. Let X be a projective scheme and D a nef and big divisor on X. Then we can write $D = A + E$ where A is an ample divisor and E is an effective divisor. Then D is endowed with a map iff $D|_{E_{red}}$ is endowed with a map.

Proof. By Proposition 2.4.34, we may assume that $D|_E$ is endowed with a map $f : E \to Z$. Let $\ell = \mathcal{O}_X(-E)$ be the ideal sheaf of E. note that $-E = A - D$ and D is f-numerically trivial. Hence $\ell|_E$ is f-ample. By Serre's vanishing, for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 \in \mathbb{m}$ such that for all $n \geq n_0$, we have

$$R^i f_* \mathcal{F}|_E \otimes \ell|_E^{\otimes n} = 0$$

for all $i > 0$. In particular, let $n \in \mathbb{z}$ such that $R^i f_* \mathcal{O}_X/\ell \otimes \ell^{\otimes m} = 0$ for all $i > 0, m \geq n$. Set $i := \ell^{\otimes n}$. Then by the exact sequence

$$0 \to \ell^{n-1} \otimes \mathcal{O}_X/\ell \to \mathcal{O}_X/\ell^n \to \mathcal{O}_X/\ell \to 0,$$

we have that $R^i f_*(\mathcal{O}_X/i \otimes i^t) = 0$ for all $i > 0, t \geq 1$. This implies that $f_*\mathcal{O}_X/i^t \to f_*\mathcal{O}_X/i$ is surjective for all $t \geq 1$.

Let

$$a := \mathcal{O}_X \oplus iT \oplus i^2T^2 \oplus \cdots,$$
$$m := \mathcal{F} \oplus i\mathcal{F}T \oplus i^2\mathcal{F}T^2 \oplus \cdots,$$

where T is a formal variable to denote the grading. Then a is a graded $\mathcal{O}_X$-algebra of finite type and m is a finite graded a-module. We have an exact sequence of graded a-modules

$$0 \to k \to m \otimes_a iT \to m \to 0,$$

where $k = \bigoplus k_r T^r$ is a finite graded a-module. Hence for $r \gg 1$, we have that $iT \cdot k_r T^r = k_{r+1}T^{r+1}$. It implies that the image of $k_{r+1}T^{r+1} \to m_r T^r \otimes_a iT$ is contained in im_r for all $r \gg 1$. Tensor with $a \otimes_{\mathcal{O}_X} \mathcal{O}_X/i$, we have that

$$k_{r+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X/i \to 0 \to m_r \otimes_{\mathcal{O}_X} i \otimes_{\mathcal{O}_X} \mathcal{O}_X/i \to m_{r+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X/i \to 0.$$

That is, $i^r\mathcal{f}/i^{r+1}\mathcal{f}\otimes_{\mathcal{O}_X/i} i/i^2 \cong i^{r+1}\mathcal{f}/i^{r+2}\mathcal{f}$ for all $r \gg 1$. Hence we have that

$$R^i f_*(i^{r-1}\mathcal{f}/i^r\mathcal{f}) = 0$$

for all $i > 0, r \gg 1$.

Let $E' := V(i)$, we have that $D|_{E'}$ is endowed with a map $f' : E' \to Z'$ by Proposition 2.4.34. Moreover, we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & Z \arrow[d, "g"] \\ E' \arrow[r, "f'"] & Z' \end{tikzcd}$$

with g finite. Then by Grothendieck Spectral Sequence, we have that

$$R^i f'_*(i^{r-1}\mathcal{f}/i^r\mathcal{f}) = 0$$

for all $i > 0, r \gg 1$.

Then we can apply Theorems 2.4.28 and 2.4.29 to get a modification $X \to Y$. Note that $\operatorname{Exc} D \subset \operatorname{Supp} E$. It follows that $X \to Y$ is a map associated to D. □

Theorem 2.4.37. Let X be a proper variety and ℓ a nef line bundle on X. Then ℓ is endowed with a map if and only if $\ell|_{\operatorname{Exc}\ell}$ is endowed with a map.

Proof. By Proposition 2.4.35, we can assume that ℓ is big. Then the result follows from Proposition 2.4.36 and induction on dimension. □

2.4.4 For semiample

Lemma 2.4.38. Let X be a projective scheme over $\Bbbk = \overline{\mathbb{f}_p}$. Then ℓ is numerically trivial if and only if ℓ is torsion in $\operatorname{Pic}(X)$.

Proof. Let T be the scheme in Theorem 2.4.2. Then ℓ corresponds to a $\mathbb{f}_q$-point of T. Note that there are only finitely many $\mathbb{f}_q$-points in T. Hence ℓ is torsion in $\operatorname{Pic}(X)$. □

Proposition 2.4.39. Let $f : X \to Y$ be a finite universal homeomorphism between algebraic spaces of finite type over $\Bbbk$ and ℓ a line bundle on Y. Then there exists $q = p^n$ such that

(a) for every section $s \in H^0(X, f^*\ell)$, we have $s^q \in \Im(H^0(Y, \ell^{\otimes q}) \to H^0(X, f^*\ell^{\otimes q}))$;

(b) ℓ is semiample if and only if $f^*\ell$ is semiample;

(c) the map

$$f^* : \operatorname{Pic}(Y) \otimes \mathbb{z}[1/q] \to \operatorname{Pic}(X) \otimes \mathbb{z}[1/q]$$

is an isomorphism;

(d) if $f^*s_1 = f^*s_2$ for two sections $s_1, s_2 \in H^0(Y, \ell)$, then $s_1^q = s_2^q$ in $H^0(X, \ell^{\otimes q})$.

Proof. Note that $\mathrm{Frob}^* \ell \cong \ell^{\otimes p}$. Then all the properties follows from Lemma 2.4.32. □

Proposition 2.4.40. Let X be a projective scheme and ℓ a nef line bundle on X. Assume that $X = X_1 \cup X_2$ for closed subsets X_1 and X_2. Suppose that $\ell|_{X_i}$ is semiample for $i = 1, 2$. Then ℓ is semiample.

Proof. Yang: To be learned. □

Lemma 2.4.41. Let $f : X \to Y$ be a proper map between algebraic spaces with $f_*\mathcal{O}_X = \mathcal{O}_Y$ and ℓ a line bundle on X. Let $D = V(i) \subset X$ be a closed subspace defined by an ideal sheaf i, $Z = f(D)$ and $D_k := V(i^k)$. Suppose that f is a modification with respect to D, Z and $R^1 f_* i^k/i^{k+1} = 0$ for all $k \gg 0$. Suppose for every k, there exists $r > 0$ such that $\ell^{\otimes r}|_{D_k}$ is pulled back from $f(D_k)$. Then $\ell^{\otimes r}$ is pulled back from Y for some $r > 0$.

Proof. Replace D by D_k and ℓ by $\ell^{\otimes r}$ for some $k, r > 0$, we can assume that $R^1 f_* i^k/i^{k+1} = 0$ for all k and $\ell|_D$ is pulled back from $f(D)$. Then we show that $f_*\ell$ is a line bundle and $f^* f_*\ell \cong \ell$. Both of them are local, so we can assume that $X = \operatorname{Spec} B, Z = \operatorname{Spec} A$ are spectrum of local rings. Hence $\ell|_{D_k}$ is trivial for all k. By vanishing of $R^1 f_* i^k/i^{k+1}$, we have a surjection $H^0(D_{k+1}, \ell|_{D_{k+1}}) \twoheadrightarrow H^0(D_k, \ell|_{D_k})$ for all k. This allow us to choose a section $s_k \in H^0(D_k, \ell|_{D_k})$ such that $s_k = s_{k+1}|_{D_k}$ for all k. Then we have a section $s \in H^0(D, \ell|_D)$ such that $s|_{D_k} = s_k$ for all k. By Nakayama's Lemma, we can assume that s_k is nowhere vanishing. Yang: To be completed. □

Proposition 2.4.42. Let X be a projective scheme and D a nef and big divisor on X. Then we can write $D = A + E$ where A is an ample divisor and E is an effective divisor. Then D is semiample iff $D|_{E_{red}}$ is semiample.

Proof. Yang: To be completed. □

Theorem 2.4.43. Let X be a proper variety and ℓ a nef line bundle on X. Then ℓ is semiample if and only if $\ell|_{\mathrm{Exc}\,\ell}$ is semiample.

Proof. Yang: To be completed. □

2.4.5 Basepoint free theorem on positive characteristic

Proposition 2.4.44 (ref. Yang:). Let $T \subset X$ be a reduced Weil divisor on a normal variety X. Let $T^\nu \to T$ be the normalization, $C \subset T^\nu$ the effective Weil divisor defined by the conductor and $p : T^\nu \to T \hookrightarrow X$ the composition. Suppose that $K_X + T$ is $\mathbb{Q}$-Cartier. Then there exists an effective $\mathbb{Q}$-Weil divisor D on T^ν such that

$$K_{T^\nu} + C + D = p^*(K_X + T).$$

Theorem 2.4.45. Let X be a normal projective $\mathbb{Q}$-factorial threefold and $B \in (0, 1)$ a $\mathbb{Q}$-divisor. Let ℓ be a nef and big line bundle on X such that $\ell - K_{(X,B)}$ is nef and big. Then ℓ is endowed with a map. Moreover, if $\Bbbk = \overline{\mathbb{F}_p}$, ℓ is semiample.

Proof. Let $\ell = \mathcal{O}_X(A+E)$ with A an ample divisor and E an effective divisor. Write $E = E_0+E_1+E_2$ such that the restriction of ℓ to every irreducible component of E_i is of numerical dimension i. Let $S := \operatorname{Supp} E_1$ and $S = \sum S_i$ with S_i irreducible components. Let $S^\nu \to S$ and $S_i^\nu \to S_i$ be the normalizations.

Step 1. Reduce to show that $\ell|_S$ is endowed with a map (semiample).

Yang: To be completed.

Step 2. Reduce to show that $\ell|_{S_i^\nu}$ is endowed with a map (semiample).

Yang: To be completed.

Step 3. Show that $\ell|_{S_i^\nu}$ is endowed with a map (semiample).

Yang: To be completed. □

2.5 F-singularities

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let X be a projective variety over $\Bbbk$. Let F denote the relative Frobenius morphism on X.

Definition 2.5.1. We say that X is *F-finite* if $F : X \to X^{(p)}$ is finite.

Definition 2.5.2. We say that X is *globally F-split* if $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ splits as $\mathcal{O}_X$-modules for some $e \geq 0$. This is equivalent to for every $e \in \mathbb{Z}_{>0}$, $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ splits as $\mathcal{O}_X$-modules.

Definition 2.5.3. Fix $\phi : F^e_* L \to \mathcal{O}_X$ a splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X$. Define $\phi^n : F^{ne}_* L^{1+p^e+\cdots+p^{(n-1)e}} \to \mathcal{O}_X$ by induction:

$$\phi^n := \phi \circ F^e_*(\phi^{n-1}).$$

Theorem 2.5.4. Above ϕ^n will be stable. That is, $\Im\phi^n = \Im\phi^{n+1}$ for all $n \gg 0$.

Definition 2.5.5. Let $\sigma(X,\phi) := \Im\phi^n$. We say that (X,ϕ) is *F-pure* if $\sigma(X,\phi) = \mathcal{O}_X$.

Proposition 2.5.6. There is a bijection between

$$\{\text{effective } \mathbb{Q}\text{-divisor } \Delta \text{ such that } (p^e-1)(K_X+\Delta) \text{ is Cartier}\}/\sim$$

and

$$\{\text{line bundles } \ell \text{ and } \phi : F^e_*\ell \to \mathcal{O}_X\}.$$

Proof. We have

$$F^e_X\mathcal{O}_X((1-p^e)K_X) \to \mathcal{O}_X$$

given by $F^e\mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X)$ and reflexivity of $\mathcal{O}_X(K_X)$. Since Δ is effective, we have

$$F^e(\mathcal{O}_X((1-p^e)(K_X+\Delta))) \to F^e\mathcal{O}_X((1-p^e)(K_X)) \to \mathcal{O}_X.$$

The another direction is by Grothendieck's duality

$$\mathcal{H}om_{\mathcal{O}_X}(F^e\ell, \mathcal{O}_X) \cong F_*^e(\ell^{-1} \otimes \mathcal{O}_X((1-p^e)K_X)).$$

□

Definition 2.5.7. Let $\phi_{e,\Delta} : F_*^e(\mathcal{O}_X((1-p^e)(K_X+\Delta))) \to \mathcal{O}_X$ be the morphism corresponding to the effective $\mathbb{Q}$-divisor Δ.
We say that (X, Δ) is *F-pure* if $(X, \phi_{e,\Delta})$ is F-pure.
We say that (X, Δ) is *globally F-split* if for every Weil divisor $D \geq 0$, $\mathcal{O}_X \to F_*^e(\mathcal{O}_X(\lceil(p^e-1)\Delta\rceil + D))$ admits a splitting for some $e \geq 0$.
We say that (X, Δ) is *strongly F-split* if for every Weil divisor $D \geq 0$, $\mathcal{O}_X \to F_*^e(\mathcal{O}_X(\lceil(p^e-1)\Delta\rceil + D))$ admits a local splitting for some $e \geq 0$.

Definition 2.5.8.

Definition 2.5.9. $S^0(X, \sigma(X, \Delta) \otimes m)$

Proposition 2.5.10. Let X be a globally F-split projective variety. Then we have

(a) suppose that $H^i(X, \ell^n) = 0$ for all $i > 0$ and all $n \gg 0$, then $H^i(X, \ell) = 0$ for all $i > 0$;

(b) for every ample divisor A on X, we have $H^i(X, \mathcal{O}_X(A)) = 0$ for all $i > 0$;

(c) suppose that X is Cohen-Macaulay and A-ample, then $H^i(X, \mathcal{O}_X(-A)) = 0$ for all $i < \dim X$;

(d) suppose that X is normal and A-ample, then $H^i(X, \omega_X(A)) = 0$ for all $i > 0$.

Appendix A

Commutative Algebra

A.1 Elementary Results

Yang: To be completed

A.1.1 Rings and modules

In the appendix and all the note, the "ring" is always commutative and with identity. We denote by $\operatorname{Spec} A$ the set of prime ideals of a ring A. We denote by $\operatorname{mSpec} A$ the set of maximal ideals of A. Let $I \subset A$ be an ideal of A. We define

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} A : I \subset \mathfrak{p}\}.$$

Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A. We define

$$(\mathfrak{a} : \mathfrak{b}) := \{a \in A : a\mathfrak{b} \subset \mathfrak{a}\}.$$

This is an ideal of A.

Let $\operatorname{rad}(A)$ be the Jacobian radical of A, i.e., the intersection of all maximal ideals of A. Let $\operatorname{nil}(A)$ be the nilradical of A, i.e., the ideal of A consisting of all nilpotent elements.

Proposition A.1.1. Let A be a ring. Then we have

$$\operatorname{nil}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}.$$

Proof. Yang: To be completed. □

Proposition A.1.2. Let A be a ring, $\mathfrak{p}, \mathfrak{p}_i$ prime ideals of A and $\mathfrak{a}, \mathfrak{a}_i$ ideals of A.

(a) Suppose $\mathfrak{a} \subset \bigcup_{i=1}^n \mathfrak{p}_i$. Then there exists i such that $\mathfrak{a} \subset \mathfrak{p}_i$.

(b) Suppose $\bigcap_{i=1}^n \mathfrak{a}_i \subset \mathfrak{p}$. Then there exists i such that $\mathfrak{a}_i \subset \mathfrak{p}$.

Proof. Yang: To be completed. □

Let M be an A-module. We say that M is *finite* if there exists an exact sequence

$$A^n \to M \to 0.$$

We say that M is *finite presented* if there exists an exact sequence

$$A^m \to A^n \to M \to 0.$$

If A is a noetherian ring, then every finite A-module is finite presented.

Definition A.1.3. Let A be a ring and M an A-module. The *support* of M is defined as

$$\operatorname{Supp} M := \{\mathfrak{p} \in \operatorname{Spec} A \colon M_{\mathfrak{p}} \neq 0\}.$$

The *annihilator* of M is defined as

$$\operatorname{Ann} M := \{a \in A \colon aM = 0\}.$$

This is an ideal of A.

Proposition A.1.4. Let A be a ring and M a finite A-module. Then $\operatorname{Supp} M = V(\operatorname{Ann} M)$. In particular, $\operatorname{Supp} M$ is a closed subset of $\operatorname{Spec} A$.

Proof. Yang: To be completed. □

A.1.2 Localization

Definition A.1.5. Let A be a ring and $S \subset A$ a multiplicative subset, i.e., $1 \in S$ and $s_1, s_2 \in S$ implies $s_1 s_2 \in S$. The *localization* of A at S is defined as

$$S^{-1}A := A \times S/\sim,$$

where $(a,s) \sim (b,t)$ if there exists $u \in S$ such that $u(at - bs) = 0$. Yang: To be completed.

Proposition A.1.6.

A.1.3 Chain conditions

A.1.4 Nakayama's Lemma

Theorem A.1.7 (Nakayama's Lemma)**.** Let A be a ring and $\mathfrak{M}$ be its Jacobi radical. Suppose M is a finitely generated A-module. If $\mathfrak{a}M = M$ for $\mathfrak{a} \subset \mathfrak{M}$, then $M = 0$.

Proof. Suppose M is generated by $x_1, \cdots, x_n$. Since $M = \mathfrak{a}M$, formally we have $(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(\mathfrak{a})$. Then $(\Phi - \mathrm{id})(x_1, \cdots, x_n)^T = 0$. Note that $\det(\Phi - \mathrm{id}) = 1 + a$ for $a \in \mathfrak{a} \subset \mathfrak{M}$. Then $\Phi - \mathrm{id}$ is invertible and then $M = 0$. □

Remark A.1.8. The finiteness of M is crucial in Nakayama's Lemma. For example, let $\overline{\mathbb{Z}}$ be the ring of algebraic integers in $\overline{\mathbb{Q}}$. Choose a non-zero prime ideal $\mathfrak{p}$ of $\overline{\mathbb{Z}}$. Then we have that $\mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}} = \mathfrak{p}^2\overline{\mathbb{Z}}_{\mathfrak{p}}$. Indeed, if $a \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$, let $b = \sqrt{a} \in \overline{\mathbb{Z}}_{\mathfrak{p}}$. Then $b^2 = a \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$ and whence $b \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$ since $\mathfrak{p}$ is prime. It follows that $a = b^2 \in \mathfrak{p}^2\overline{\mathbb{Z}}_{\mathfrak{p}}$.

Proposition A.1.9 (Geometric form of Nakayama's Lemma)**.** Let $X = \operatorname{Spec} A$ be an affine scheme, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proof. Yang: To be completed. □

Corollary A.1.10. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. Then the function $x \mapsto \dim_{\kappa(x)} \mathcal{F}|_x$ is upper semicontinuous.

Proof. Yang: To be completed. □

A.1.5 Nullstellensatz

Theorem A.1.11 (Noether's Normalization Lemma)**.** Let A be a $\mathbf{k}$-algebra of finite type. Then there is an injection $\mathbf{k}[T_1, \cdots, T_d] \hookrightarrow A$ such that A is finite over $\mathbf{k}[T_1, \cdots, T_d]$.

Remark A.1.12. Here A does not need to be integral. For example,

Theorem A.1.13 (Hilbert's Nullstellensatz)**.** Let A be a

A.2 Associated prime ideals

A.2.1 Associated prime ideals

Definition A.2.1 (Associated prime ideals)**.** Let A be a noetherian ring and M an A-module. The *associated prime ideals* of M are the prime ideals $\mathfrak{p}$ of form $\operatorname{Ann}(x)$ for some $x \in M$. The set of associated prime ideals of M is denoted by $\operatorname{Ass}(M)$.

Example A.2.2. Let $A = \Bbbk[x, y]/(xy)$ and $M = A$. First we see that $(x) = \operatorname{Ann} y, (y) = \operatorname{Ann} x \in \operatorname{Ass} M$. Then we check other prime ideals. For (x, y), if $xf = yf = 0$, then $f \in (x) \cap (y) = (0)$. If $(x - a) = \operatorname{Ann} f$ for some f, note that $y \in (x - a)$ for $a \in \Bbbk^*$, then $f \in (x)$. Hence $f = 0$. Therefore $\operatorname{Ass} M = \{(x), (y)\}$.

Example A.2.3. Let $A = \Bbbk[x, y]/(x^2, xy)$ and $M = A$. The underlying space of $\operatorname{Spec} A$ is the y-axis since $\sqrt{(x^2, xy)} = (x)$. First note that $(x) = \operatorname{Ann} y, (x, y) = \operatorname{Ann} x \in \operatorname{Ass} M$. For $(x, y - a)$ with $a \in \Bbbk^*$, easily see that $xf = (y - a)f = 0$ implies $f = 0$ since $A = \Bbbk \cdot x \oplus \Bbbk[y]$ as $\Bbbk$-vector space. Hence $\operatorname{Ass} M = \{(x), (x, y)\}$.

Lemma A.2.4. Let A be a noetherian ring and M an A-module. Then the maximal element of the set

$$\{\operatorname{Ann} x : x \in M_{\mathfrak{p}}, x \neq 0\}$$

belongs to $\operatorname{Ass} M$.

Proof. We just need to show that such $\operatorname{Ann} x$ is prime. Otherwise, there exist $a, b \in A$ such that $ab \in \operatorname{Ann} x$ but $a, b \notin \operatorname{Ann} x$. It follows that $\operatorname{Ann} x \subsetneq \operatorname{Ann} ax$ since $b \in \operatorname{Ann} ax \setminus \operatorname{Ann} x$. This contradicts the maximality of $\operatorname{Ann} x$. □

An element $a \in A$ is called a zero divisor for M if $M \to aM, m \mapsto am$ is not injective.

Corollary A.2.5. Let A be a noetherian ring and M an A-module. Then

$$\{\text{zero divisors for } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}.$$

Lemma A.2.6. Let A be a noetherian ring and M an A-module. Then $\mathfrak{p} \in \operatorname{Ass}_A M$ iff $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Proof. Suppose $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Ann} y_0/c$ with $y_0 \in M$ and $c \in A \setminus \mathfrak{p}$. For $a \in \operatorname{Ann} y_0$, $ay_0 = 0$. Then $a/1 \in \mathfrak{p}A_{\mathfrak{p}}$. It follows that $a \in \mathfrak{p}$. Hence $\operatorname{Ann} y_0 \subset \mathfrak{p}$.

Inductively, if $\operatorname{Ann} y_n \subsetneq \mathfrak{p}$, then there exists $b_n \in A \setminus \mathfrak{p}$ such that $y_{n+1} := b_n y_n$, $\operatorname{Ann} y_{n+1} \subset \mathfrak{p}$ and $\operatorname{Ann} y_n \subsetneq \operatorname{Ann} y_{n+1}$. To see this, choose $a_n \in \mathfrak{p} \setminus \operatorname{Ann} y_n$. Then $(a_n/1)y_n = 0$ since $a_n/1 \in \mathfrak{p}A_{\mathfrak{p}}$. By definition, there exist $b_n \in A \setminus \mathfrak{p}$ such that $a_n b_n y_n = 0$. This process must terminate since A is noetherian. Thus $\operatorname{Ann} y_n = \mathfrak{p}$ for some n. Hence $\mathfrak{p} \in \operatorname{Ass}_A M$.

Conversely, suppose $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$. If $(a/s)(x/1) = 0 \in M_{\mathfrak{p}}$, there exist $t \in A \setminus \mathfrak{p}$ such that $tax = 0$. It follows that $ta \in \mathfrak{p}$ and then $(a/s) \in \mathfrak{p}A_{\mathfrak{p}}$. Hence $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. □

Proposition A.2.7. We have $\operatorname{Ass} M \subset \operatorname{Supp} M$. Moreover, if $\mathfrak{p} \in \operatorname{Supp} M$ satisfies $V(\mathfrak{p})$ is an irreducible component of $\operatorname{Supp} M$, then $\mathfrak{p} \in \operatorname{Ass} M$.

Proof. For any $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$, we have $A/\mathfrak{p} \cong A \cdot x \subset M$. Tensoring with $A_{\mathfrak{p}}$ gives $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$ since $A_{\mathfrak{p}}$ is flat. Hence $M_{\mathfrak{p}} \neq 0$ and $\mathfrak{p} \in \operatorname{Supp} M$.

Now suppose $\mathfrak{p} \in \operatorname{Supp} M$ and $V(\mathfrak{p})$ is an irreducible component of $\operatorname{Supp} M$. First we show that $\mathfrak{p} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $x \in M_{\mathfrak{p}}$ such that $\operatorname{Ann} x$ is maximal in the set

$$\{\operatorname{Ann} x : x \in M_{\mathfrak{p}}, x \neq 0\}.$$

Then we claim that $\operatorname{Ann} x = \mathfrak{p}A_{\mathfrak{p}}$. First, $\operatorname{Ann} x$ is prime by Lemma A.2.4. If $\operatorname{Ann} x \neq \mathfrak{p}$, then $V(\operatorname{Ann} x) \supset V(\mathfrak{p})$. This implies that $\operatorname{Ann} x \notin \operatorname{Supp} M_{\mathfrak{p}}$ since $\operatorname{Supp} M_{\mathfrak{p}} = \operatorname{Supp} M \cap \operatorname{Spec} A_{\mathfrak{p}}$. This is a contradiction. Thus $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. By Lemma A.2.6, we have $\mathfrak{p} \in \operatorname{Ass} M$. □

Remark A.2.8. The existence of irreducible component is guaranteed by Zorn's Lemma.

Definition A.2.9. A prime ideal $\mathfrak{p} \in \operatorname{Ass} M$ is called *embedded* if $V(\mathfrak{p})$ is not an irreducible component of $\operatorname{Supp} M$.

Example A.2.10. For $M = A = \Bbbk[x, y]/(x^2, xy)$, the origin (x, y) is an embedded point.

Proposition A.2.11. If we have exact sequence $0 \to M_1 \to M_2 \to M_3$, then $\operatorname{Ass} M_2 \subset \operatorname{Ass} M_1 \cup \operatorname{Ass} M_3$.

Proof. Let $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M_2 \setminus \operatorname{Ass} M_1$. Then the image $[x]$ of x in M_3 is not equal to 0. We have that $\operatorname{Ann} x \subset \operatorname{Ann}[x]$. If $a \in \operatorname{Ann}[x] \setminus \operatorname{Ann} x$, then $ax \in M_1$. Since $\operatorname{Ann} x \subsetneq \operatorname{Ann} ax$, there is $b \in \operatorname{Ann} ax \setminus \operatorname{Ann} x$. However, it implies $ba \in \operatorname{Ann} x$, and then $a \in \operatorname{Ann} x$ since $\operatorname{Ann} x$ is prime, which is a contradiction. □

Corollary A.2.12. If M is finitely generated, then the set $\operatorname{Ass} M$ is finite.

Proof. For $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$, we know that the submodule M_1 generated by x is isomorphic to $A/\mathfrak{p}$. Inductively, we can choose M_n be the preimage of a submodule of M/M_{n-1} which is isomorphic to $A/\mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass} M/M_{n-1}$. We can take an ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$. Since M is finitely generated, this is a finite sequence. Then the conclusion follows by Proposition A.2.11. □

A.2.2 Primary decomposition

Definition A.2.13. An A-module is called *co-primary* if $\operatorname{Ass} M$ has a single element. Let M be an A-module and $N \subset M$ a submodule. Then N is called *primary* if M/N is co-primary. If $\operatorname{Ass} M/N = \{\mathfrak{p}\}$, then N is called $\mathfrak{p}$-primary.

Remark A.2.14. This definition coincide with primary ideals in the case $M = A$. Recall an ideal $\mathfrak{q} \subset A$ is called *primary* if $\forall ab \in \mathfrak{p}$, $a \notin \mathfrak{q}$ implies $b^n \in \mathfrak{q}$ for some n.

Let $\mathfrak{q}$ be a $\mathfrak{q}$-primary ideal. Since $\operatorname{Supp} A/\mathfrak{q} = \{\mathfrak{p}\}$, $\mathfrak{p} \in \operatorname{Ass} A/\mathfrak{q}$. Suppose $\operatorname{Ann}[a] \in \operatorname{Ass} A/\mathfrak{q}$. Then $\mathfrak{p} \subset \operatorname{Ann}[a]$ since $V(\mathfrak{p}) = \operatorname{Supp} A/\mathfrak{q}$. If $b \in \operatorname{Ann}[a]$, then $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Hence $b^n \in \mathfrak{q}$, and then $b \in \mathfrak{p}$. This shows that $\operatorname{Ass} A/\mathfrak{q} = \{\mathfrak{p}\}$ and $\mathfrak{q}$ is $\mathfrak{p}$-primary as an A-submodule.

Let $\mathfrak{q} \subset A$ be a $\mathfrak{p}$-primary A-submodule. First we have $\mathfrak{p} = \sqrt{\mathfrak{q}}$ since $V(\mathfrak{p})$ is the unique irreducible component of $\operatorname{Supp} A/\mathfrak{q}$. Suppose $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Then $b \in \operatorname{Ann}[a] \subset \mathfrak{p}$ since $\mathfrak{p}$ is the unique maximal element in $\{\operatorname{Ann}[c] : c \in A \setminus \mathfrak{q}\}$. This implies that $b^n \in \mathfrak{q}$.

Definition A.2.15. Let A be a noetherian ring, M an A-module and $N \subset M$ a submodule. A *minimal primary decomposition* of N in M is a finite set of primary submodules $\{Q_i\}_{i=1}^n$ such that

$$N = \bigcap_{i=1}^n Q_i,$$

no Q_i can be omitted and $\operatorname{Ass} M/Q_i$ are pairwise distinct. For $\operatorname{Ass} M/Q_i = \{\mathfrak{p}\}$, Q_i is called belonging to $\mathfrak{p}$.

Indeed, if $N \subset M$ admits a minimal primary decomposition $N = \bigcap Q_i$ with Q_i belonging to $\mathfrak{p}$, then $\operatorname{Ass}(M/N) = \{\mathfrak{p}_i\}$. For given i, consider $N_i := \bigcap_{j \neq i} Q_j$, then $N_i/N \cong (N_i + Q_i)/Q_i$. Since $N_i \neq N$, $\operatorname{Ass} N_i/N \neq \emptyset$. On the other hand, $\operatorname{Ass} N_i/N \subset \operatorname{Ass} M/Q_i = \{\mathfrak{p}\}$. It follows that $\operatorname{Ass} N_i/N = \{\mathfrak{p}_i\}$, whence $\mathfrak{p}_i \in \operatorname{Ass} M/N$. Conversely, we have an injection $M/N \hookrightarrow \bigoplus M/Q_i$, so $\operatorname{Ass} M/N \subset \bigcup \operatorname{Ass} M/Q_i$. Due to this, if Q_i belongs to $\mathfrak{p}$, we also say that Q_i is the $\mathfrak{p}$-component of N.

Proposition A.2.16. Suppose $N \subset M$ has a minimal primary decomposition. If $\mathfrak{p} \in \operatorname{Ass} M/N$ is not embedded, then the $\mathfrak{p}$ component of N is unique. Explicitly, we have $Q = \nu^{-1}(N_{\mathfrak{p}})$, where $\nu : M \to M_{\mathfrak{p}}$.

Proof. First we show that $Q = \nu^{-1}(Q_{\mathfrak{p}})$. Clearly $Q \subset \nu^{-1}(Q_{\mathfrak{p}})$. Suppose $x \in \nu^{-1}(Q_{\mathfrak{p}})$. Then there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in Q$. That is, $[sx] = 0 \in M/Q$. If $[x] \neq 0$, we have $s \in \operatorname{Ann}[x] \subset \mathfrak{p}$. This contradiction enforces $Q = \nu^{-1}(Q_{\mathfrak{p}})$.

Then we show that $N_{\mathfrak{p}} = Q_{\mathfrak{p}}$. Just need to show that for $\mathfrak{p}' \neq \mathfrak{p}$ and the $\mathfrak{p}'$ component Q' of N, $Q'_{\mathfrak{p}} = M_{\mathfrak{p}}$. Since $\mathfrak{p}$ is not embedded, $\mathfrak{p}' \not\subset \mathfrak{p}$. Then $\mathfrak{p} \notin V(\mathfrak{p}) = \operatorname{Supp} M/Q'$. So $M_{\mathfrak{p}}/Q'_{\mathfrak{p}} = 0$. □

Example A.2.17. If $\mathfrak{p}$ is embedded, then its components may not be unique. For example, let $M = A = \Bbbk[x,y]/(x^2, xy)$. Then for every $n \in \mathbb{Z}_{\geq 1}$, $(x) \cap (x^2, xy, y^n)$ is a minimal primary decomposition of $(0) \subset M$.

Let A be a noetherian ring and $\mathfrak{p} \subset A$ a prime ideal. We consider the $\mathfrak{p}$ component of $\mathfrak{p}^n$, which is called n-th symbolic power of $\mathfrak{p}$, denoted by $\mathfrak{p}^{(n)}$. We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A$. In general, $\mathfrak{p}^{(n)}$ is not equal to $\mathfrak{p}^n$; see below example.

Example A.2.18. Let $A = \mathbf{k}[x,y,z,w]/(y^2 - zx^2, yz - xw)$ and $\mathfrak{p} = (y,z,w)$. We have $z = y^2/x^2, w = yz/x \in \mathfrak{p}^2 A_{\mathfrak{p}}$, whence $\mathfrak{p}^2 A_{\mathfrak{p}} = (z,w) \neq \mathfrak{p}^2$.

Theorem A.2.19. Let A be a noetherian ring and M an A-module. Then for every $\mathfrak{p} \in \operatorname{Ass} M$, there is a $\mathfrak{p}$-primary submodule $Q(\mathfrak{p})$ such that

$$(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M} Q(\mathfrak{p}).$$

Proof. Consider the set

$$\mathfrak{n} := \{N \subset M : \mathfrak{p} \notin \operatorname{Ass} N\}.$$

Note that $\operatorname{Ass} \bigcup N_i = \bigcup \operatorname{Ass} N_i$ by definition of associated prime ideals. Then it is easy to check that $\mathfrak{n}$ satisfies the conditions of Zorn's Lemma. Hence $\mathfrak{n}$ has a maximal element $Q(\mathfrak{p})$. We claim that $Q(\mathfrak{p})$ is $\mathfrak{p}$-primary. If there is $\mathfrak{p}' \neq \mathfrak{p} \in \operatorname{Ass} M/Q(\mathfrak{p})$, then there is a submodule $N' \cong A/\mathfrak{p}$. Let N'' be the preimage of N' in M. We have $Q(\mathfrak{p}) \subsetneq N''$ and $N'' \in \mathfrak{n}$. This is a contradiction. By the fact $\operatorname{Ass} \bigcap N_i = \bigcap \operatorname{Ass} N_i$, we get the conclusion. □

Corollary A.2.20. Let A be a noetherian ring and M a finite A-module. Then every submodule of M has a minimal primary decomposition.

A.3 Dimension and Depth

There are three numbers measuring the "size" of a local ring $(A, \mathfrak{m})$:

- $\dim A$: the Krull dimension of A.
- $\operatorname{depth} A$: the depth of A.
- $\dim_{\kappa(\mathfrak{m})} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$-vector space.

Somehow the Krull dimension is "homological" and the depth is "cohomological".

Definition A.3.1. Let A be a noetherian ring. The *height of a prime ideal* $\mathfrak{p}$ in A is defined as the maximum length of chains of prime ideals contained in $\mathfrak{p}$, that is,

$$\operatorname{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as

$$\dim A := \max_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht}(\mathfrak{p}).$$

Example A.3.2. Let A be a PID. For every two non-zero prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$, if $\mathfrak{p}_1 = t_1 A \subset \mathfrak{p}_2 = t_2 A$, then $t_2 \mid t_1$ and hence $\mathfrak{p}_1 = \mathfrak{p}_2$. It follows that $\dim A = 1$. Consequently, the ring of integers $\mathbb{Z}$ and the polynomial ring $\mathbf{k}[T]$ in one variable over a field have Krull dimension 1.

Definition A.3.3. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A-module. A sequence $t_1, \cdots, t_n \in I$ is called an *M-regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \cdots, t_{i-1})M$ for all i.

Example A.3.4. Let $A = \mathbf{k}[x, y]/(x^2, xy)$ and $I = (x, y)$. Then $\operatorname{depth}_I A = 0$.

Definition A.3.5. Let A be a noetherian ring. For every $\mathfrak{p} \in \operatorname{Spec} A$, $\mathfrak{p}/\mathfrak{p}^2$ is a vector space over $\kappa(\mathfrak{p})$. The *Zariski's tangent space* $T_{A,\mathfrak{p}}$ of A at $\mathfrak{p}$ is defined as $(\mathfrak{p}/\mathfrak{p}^2)^\vee$, the dual $\kappa(\mathfrak{p})$-vector space of $\mathfrak{p}/\mathfrak{p}^2$.

A.3.1 Artinian Rings and Length of Modules

Definition A.3.6. Let A be a ring and M an A module. *A simple module filtration of M* is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i-1} is a simple module, i.e. it has no submodule except 0 and itself. If M has a simple module filtration as above, we define the *length of M* as n and say that M has *finite length*.

The following proposition guarantees the length is well-defined.

Proposition A.3.7. Suppose M has a simple module filtration $M = M_{0,0} \supsetneq M_{1,0} \supsetneq \cdots \supsetneq M_{n,0} = 0$. Then for any other filtration $M = M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,m} = 0$ with $m > n$, there exist $k < m$ such that $M_{0,k} = M_{0,k+1}$.

Proof. We claim that there are at least $0 \le k_1 < \cdots < k_{m-n} < m$ satisfies that $M_{0,k_i} = M_{0,k_i+1}$. Let $M_{i,j} := M_{i,0} \cap M_{0,j}$. Inductively on n, we can assume that there exist $k_1, \cdots, k_{n-m+1}$ such that $M_{1,k} = M_{1,k+1}$. Consider the sequence

$$M_{0,0}/M_{1,0} \supset (M_{0,1} + M_{1,0})/M_{1,0} \supset \cdots \supset (M_{0,m} + M_{1,0})/M_{1,0} = 0$$

in $M_{0,0}/M_{1,0}$. Since $M_{0,0}/M_{1,0}$ is simple, there is at most one k_i with $M_{0,k_i} + M_{1,0} \neq M_{0,k_i+1} + M_{1,0}$. And note that if $M_{0,k_i} + M_{1,0} = M_{0,k_i+1} + M_{1,0}$ and $M_{0,k_i} \cap M_{1,0} = M_{0,k_i} \cap M_{1,0}$, then $M_{0,k_i} = M_{0,k_i+1}$ by the Five Lemma. □

Example A.3.8. Let A be a ring and $\mathfrak{m} \in \operatorname{mSpec} A$. Then $A/\mathfrak{m}$ is a simple module. Yang: To be completed.

Proposition A.3.9. Let A be a ring and M an A-module. Then M is of finite length iff it satisfies both a.c.c and d.c.c.

Proof. Note that if M has either a strictly ascending chain or a strictly descending chain, M is of infinite length. Conversely, d.c.c guarantee M has a simple submodule and a.c.c guarantee the sequence terminates. □

Proposition A.3.10. The length $l(-)$ is an additive function for modules of finite length. That is, if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ with M_i of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

Proof. The simple module filtrations of M_1 and M_3 will give a simple module filtration of M_2. □

Proposition A.3.11. Let $(A, \mathfrak{m})$ be a local ring. Then A is artinian iff $\mathfrak{m}^n = 0$ for some $n \ge 0$.

Proof. Suppose A is artinian. Then the sequence $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ is stable. It follows that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n. By the Nakayama's Lemma A.1.7, $\mathfrak{m}^n = 0$.

Conversely, we have

$$\mathfrak{m} \subset \mathfrak{N} \subset \bigcap_{\text{minimal prime ideal}} \mathfrak{p},$$

whence $\mathfrak{m}$ is minimal. □

Proposition A.3.12. Let A be a ring. Then A is artinian iff A is of finite length.

Proof. First we show that A has only finite maximal ideal. Otherwise, consider the set $\{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k\}$. It has a minimal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ and for any maximal ideal $\mathfrak{m}$, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}$. It follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i. Let $\mathfrak{M} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ be the Jacobi radical of A. Consider the sequence $\mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots$ and by Nakayama's Lemma, we have $\mathfrak{M}^k = 0$ for some k. Consider the

filtration
$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1^k \supset \mathfrak{m}_1^k\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0).$$
We have $\mathfrak{m}_1^k \cdots \mathfrak{m}_i^j / \mathfrak{m}_1^k \cdots \mathfrak{m}_i^{j+1}$ is an $A/\mathfrak{m}_i$-vector space. It is artinian and then of finite length. Hence A is of finite length. □

Theorem A.3.13. Let A be a ring. Then A is artinian iff A is noetherian and of dimension 0.

Proof. Suppose A is artinian. Then A is noetherian by Proposition A.3.12. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A/\mathfrak{p}$ is an artinian integral domain. If there is $a \in A/\mathfrak{p}$ is not invertible, consider $(a) \supset (a^2) \supset \cdots$, we see $a = 0$. Hence $\mathfrak{p}$ is maximal and $\dim A = 0$.

Suppose that A is noetherian and of dimension 0. Then every maximal ideal is minimal. In particular, A has only finite maximal ideal $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$. Let $\mathfrak{q}_i$ be the $\mathfrak{p}_i$-component of (0). Then we have $A \hookrightarrow \bigoplus_i A/\mathfrak{q}_i$. We just need to show that $A/\mathfrak{q}_i$ is of finite length as A-module. If $\mathfrak{q}_i \subset \mathfrak{p}_j$, take radical we get $\mathfrak{p}_i \subset \mathfrak{q}_j$ and hence $i = j$. So $A/\mathfrak{q}_i$ is a local ring with maximal ideal $\mathfrak{p}_i A/\mathfrak{q}_i$. Then every element in $\mathfrak{p}_i A/\mathfrak{q}_i$ is nilpotent. Since $\mathfrak{p}_i$ is finitely generated, $(\mathfrak{p}_i A/\mathfrak{q}_i)^k = 0$ for some k. Then $A/\mathfrak{q}_i$ is artinian and then of finite length as $A/\mathfrak{q}_i$-module. Then the conclusion follows. □

A.3.2 Dedekind Domains

Yang: To be completed

A.3.3 Krull's Principal Ideal Theorem

Theorem A.3.14 (Krull's Principal Ideal Theorem)**.** Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let $\mathfrak{p}$ be a minimal prime ideal among those containing f. Then $\operatorname{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_{\mathfrak{p}}$, we may assume A is local with maximal ideal $\mathfrak{p}$. Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.

Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is $\mathfrak{q}$-primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1} A_{\mathfrak{q}}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_{\mathfrak{q}} = 0$. It follows that $\mathfrak{q} A_{\mathfrak{q}}$ is minimal, whence $A_{\mathfrak{q}}$ is artinian. Therefore, $\mathfrak{q}$ is minimal in A. □

Corollary A.3.15. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n. Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.

Choose $\mathfrak{q}$ be a minimal prime ideal among those containing $(\mathfrak{p}_{k-1}, f)$ and contained in $\mathfrak{p}_{k+1}$.

Then by Krull's Principal Ideal Theorem A.3.14, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace $\mathfrak{p}_k$ by $\mathfrak{q}_k$, we have $f \in \mathfrak{q}_k \backslash \mathfrak{p}_{k-1}$

Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.

Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$ has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A. It follows that $\dim A/(f) + 1 \leq \dim A$. □

Proposition A.3.16. Let $(A, \mathfrak{m})$ be a local noetherian ring with residue field $\mathbf{k}$. Then the following inequalities hold:

$$\operatorname{depth} A \leq \dim A \leq \dim_{\mathbf{k}} T_{A,\mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary A.3.15.

Let $t_1, \cdots, t_n$ be a $\kappa(\mathfrak{m})$-basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \cdots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \cdots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \cdots, t_n))$. It follows that $\mathfrak{m} = (t_1, \cdots, t_n)$ by Nakayama's Lemma. By Corollary A.3.15,

$$n + \dim A/(t_1, \cdots, t_n) \geq n - 1 + \dim A/(t_1, \cdots, t_{n-1}) \geq \cdots \geq 1 + \dim A/(t_1) \geq \dim A.$$

We conclude the result. □

Definition A.3.17. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property* (R_k) or *is regular in codimension* k if $\forall \xi \in X$ with $\operatorname{codim} Z_\xi \leq k$,

$$\dim_{\kappa(\xi)} T_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

We say that X *verifies property* (S_k) if $\forall \xi \in X$ with $\operatorname{depth} \mathcal{O}_{X,\xi} < k$,

$$\operatorname{depth} \mathcal{O}_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

Example A.3.18. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.

Suppose A verifies (S_1). If $\mathfrak{p} \in \operatorname{Ass} A$, every element in $\mathfrak{p}$ is a zero divisor. Then $\operatorname{depth} A_{\mathfrak{p}} = 0$. It follows that $\dim A_{\mathfrak{p}} = 0$ and then $\mathfrak{p}$ is minimal.

Suppose A has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_{\mathfrak{p}} = 0$. This means every element in $\mathfrak{p}A_{\mathfrak{p}}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Proposition A.1.2, $\mathfrak{p} = \mathfrak{q}$ for some minimal $\mathfrak{q}$, whence $\dim A_{\mathfrak{p}} = 0$.

Example A.3.19. Let A be a noetherian ring. Then A is reduced iff it verifies (R_0) and (S_1).

Suppose A is reduced. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be all minimal prime ideals of A. We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where $\mathfrak{N}$ is the nilradical of A. Hence A has no embedded point. Since $A_{\mathfrak{p}}$ is artinian, local and reduced, $A_{\mathfrak{p}}$ is a field and hence regular.

Conversely, let $\operatorname{Ass} A$ be equal to $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Then every $\mathfrak{p}_i$ is minimal by (S_1). Let f be in $\mathfrak{N}$.

Then the image of f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0), $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where $\mathfrak{q}_i$ is the $\mathfrak{p}_i$ component of (0) in A. Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

A.3.4 Cohen-Macaulay rings

Definition A.3.20 (Cohen-Macaulay)**.** A noetherian local ring $(A, \mathfrak{m})$ is called *Cohen-Macaulay* if $\dim A = \operatorname{depth} A$. A noetherian ring A is called *Cohen-Macaulay* if for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is Cohen-Macaulay. This is equivalent to that A verifies (S_k) for all $k \geq 0$.

Example A.3.21 (Non Cohen-Macaulay rings)**.** Yang: To be completed.

Corollary A.3.22. Let A be a noetherian ring, M a finite A-module and $a \in A$ an M-regular element. Then $\operatorname{depth} M = \operatorname{depth} M/aM + 1$.

Corollary A.3.23. Let A be a noetherian ring $a \in A$ a nonzero divisor. Then A verifies (S_d) iff A/aA verifies (S_{d-1}).

Definition A.3.24. An ideal I of a noetherian ring A is called *unmixed* if

$$\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \operatorname{Ass}(A/I).$$

Here $\operatorname{ht}(I)$ is defined as

$$\operatorname{ht}(I) := \inf\{\operatorname{ht}(\mathfrak{p}) : I \subset \mathfrak{p}\}.$$

We say that *the unmixedness theorem holds for a noetherian ring* A if any ideal $I \subset A$ generated by $\operatorname{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme* X if $\mathscr{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Theorem A.3.25. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Proof. We can assume that $X = \operatorname{Spec} A$ is affine.

Suppose X is Cohen-Macaulay. Let $I \subset A$ be an ideal generated by $a_1, \cdots, a_r$ with $r = \operatorname{ht}(I)$. We claim that $a_1, \cdots, a_r$ is an A-regular sequence. If so, we get that the unmixedness theorem holds for A by applying Example A.3.18 on A/I. Since $\operatorname{ht}(a_1, \cdots, a_{r-1}) \leq r - 1$ by Krull's Principal Ideal Theorem A.3.14 and $\operatorname{ht}(a_1, \cdots, a_r) = r \leq \operatorname{ht}(a_1, \cdots, a_{r-1}) + 1$, we have $\operatorname{ht}(a_1, \cdots, a_{r-1}) = r - 1$. By induction on r, we can assume that $a_1, \cdots, a_{r-1}$ is an A-regular sequence. Hence any prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1, \cdots, a_{r-1})$ has height $r - 1$. Now suppose a_r is a zero divisor in $A/(a_1, \cdots, a_{r-1})$. Then there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1, \cdots, a_{r-1})$ such that $a_r \in \mathfrak{p}$. Then $I \subset \mathfrak{p}$ and $\operatorname{ht}(I) \leq r - 1$. This contradicts that $\operatorname{ht}(I) = r$.

Suppose the unmixedness theorem holds for A. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal with $\operatorname{ht}(\mathfrak{p}) = r$. Then $\mathfrak{p} \in \operatorname{Ass} A$ if and only if $\operatorname{ht}(\mathfrak{p}) = 0$. If $r > 0$, there is a nonzero divisor $a \in \mathfrak{p}$. By Krull's Principal Ideal Theorem A.3.14, $\operatorname{ht}(\mathfrak{p}A/aA) = r - 1$. Inductively, we can find a regular sequence

$a_1, \cdots, a_r$ in $\mathfrak{p}$. Then $\operatorname{depth} A_{\mathfrak{p}} = r$. □

Theorem A.3.26. Let X be a locally noetherian scheme. Suppose that X is Cohen-Macaulay. Let $F \subset X$ be a closed subset of codimension $\geq k$. Then the restriction $H^i(X, \mathscr{O}_X) \to H^i(X \setminus F, \mathscr{O}_X)$ is an isomorphism.

Proof. Yang: To be completed. □

A.3.5 Regular rings

Definition A.3.27. A noetherian ring A is said to be *regular at* $\mathfrak{p} \in \operatorname{Spec} A$ if we have

$$\dim_{\kappa(\mathfrak{p})} T_{A,\mathfrak{p}} = \dim A_{\mathfrak{p}},$$

where $\dim A_{\mathfrak{p}}$ is the Krull dimension of the local ring $A_{\mathfrak{p}}$.
A noetherian ring A is said to be *regular* if it is regular at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. This is equivalent to the condition that A verifies (R_k) for all $k \geq 0$.

Remark A.3.28. A noetherian ring A is regular if and only if it is regular at every maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$. The proof uses homological tools; see Theorem B.3.17 and Corollary B.3.18.

Definition A.3.29. Let A be a noetherian ring that is regular at $\mathfrak{p} \in \operatorname{Spec} A$. A sequence $t_1, \cdots, t_n \in \mathfrak{p}$ is called a *regular system of parameters* at $\mathfrak{p}$ if their images form a basis of the $\kappa(\mathfrak{p})$-vector space $\mathfrak{p}/\mathfrak{p}^2$.

Proposition A.3.30. Let $(A, \mathfrak{m})$ be a noetherian local ring that is regular at $\mathfrak{m}$. Let $t_1, \cdots, t_n$ be a regular system of parameters at $\mathfrak{m}$, $\mathfrak{p}_i = (t_1, \cdots, t_i)$ and $\mathfrak{p}_0 = (0)$. Then $\mathfrak{p}_i$ is a prime ideal of height i, and $A/\mathfrak{p}_i$ is a regular local ring for all i. In particular, regular local ring is integral, and the regular system of parameters $t_1, \cdots, t_n$ is a regular sequence in A.

Proof. By the Krull's Principal Ideal Theorem A.3.14, we have

$$n - 1 = \dim A - 1 \leq \dim A/(t_1) \leq \dim_{\kappa(\mathfrak{m}/(t_1))} T_{A/(t_1), \mathfrak{m}/(t_1)} \leq n - 1.$$

Hence $\dim A/(t_1) = n - 1$ and $\operatorname{ht}(t_1) = 1$. Since $t_2, \cdots, t_n$ generate $\mathfrak{m}/(t_1)$, we have that $A/(t_1)$ is regular at $\mathfrak{m}/(t_1)$ and the images of $t_2, \cdots, t_n$ form a regular system of parameters.

For integrality, we induct on the dimension of A. If $\dim A = 0$, then A is a field and hence integral. Suppose $\dim A > 0$, let $\mathfrak{q}$ be a minimal prime ideal of A. Then $t_1 \notin \mathfrak{q}$. We have

$$n - 1 = \dim A - 1 \leq \dim A/(\mathfrak{q} + t_1 A) \leq \dim_{\kappa(\mathfrak{q}/(t_1))} T_{A/(\mathfrak{q}+t_1 A), \mathfrak{q}/(t_1)} \leq n - 1.$$

By similar arguments, we have $A/(\mathfrak{q} + t_1 A)$ is regular at $\mathfrak{m}/(\mathfrak{q} + t_1 A)$. By induction hypothesis, both of $A/t_1 A$ and $A/(\mathfrak{q} + t_1 A)$ are integral and of dimension $n - 1$. Hence $t_1 A = t_1 A + \mathfrak{q}$, i.e. $\mathfrak{q} \subset t_1 A$. For every $a = bt_1 \in \mathfrak{q}$, we have $b \in \mathfrak{q}$ since $t_1 \notin \mathfrak{q}$. Then $\mathfrak{q} \subset t_1 \mathfrak{q} \subset \mathfrak{m}\mathfrak{q}$. By Nakayama's Lemma, $\mathfrak{q} = 0$, whence A is integral. □

Corollary A.3.31. A regular noetherian ring is Cohen-Macaulay.

Corollary A.3.32. A regular noetherian ring is normal.

Remark A.3.33. Indeed we can show a stronger result: a noetherian regular local ring is a UFD; see Yang: ref.

A.4 Finite Algebra and Normality

Let R be a ring and A be an R-algebra. We say that A is *of finite type* over R if there exists a surjective R-algebra homomorphism $R[T_1, \cdots, T_n] \to A$ for some $n \geq 0$. We say that A is finite over R if it is finite as an R-module.

A.4.1 Finite algebra

Let A be a ring and B a finite A-algebra.

Example A.4.1. Let K be a number field. Then O_K is a finite $\mathbb{Z}$-algebra. Yang: To be completed.

Lemma A.4.2. Let $A \subset B$ be noetherian rings such that B is finite over A. Then the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Proof. For $\mathfrak{p} \in \operatorname{Spec} A$, let $S := A - \mathfrak{p}$ and denote $S^{-1}B$ by $B_\mathfrak{p}$. Then we have $A_\mathfrak{p} \hookrightarrow B_\mathfrak{p}$ and $B_\mathfrak{p}$ is finite over $A_\mathfrak{p}$. Let $\mathfrak{P}B_\mathfrak{p}$ be a maximal ideal of $B_\mathfrak{p}$. We claim that $\mathfrak{P}B_\mathfrak{p} \cap A_\mathfrak{p}$ is maximal. Indeed, consider $A_\mathfrak{p}/(\mathfrak{P} \cap A_\mathfrak{p}) \hookrightarrow B_\mathfrak{p}/\mathfrak{P}B_\mathfrak{p}$, the latter is finite over the former. This enforces $A_\mathfrak{p}/(\mathfrak{P}B_\mathfrak{p} \cap A_\mathfrak{p})$ be a field. Hence $\mathfrak{P}B_\mathfrak{p} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, and then $\mathfrak{P} \cap A = \mathfrak{p}$. □

Proposition A.4.3. Let $A \subset B$ be noetherian rings such that B is finite over A. Then $\dim A = \dim B$.

Proof. If we have a sequence $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ of prime ideals in B, then there exists $f \in \mathfrak{P}_2 \setminus \mathfrak{P}_1$. Since B is finite over A, there exist $a_1, \cdots, a_n \in A$ such that

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0.$$

Then $a_n \in \mathfrak{P}_2 \cap A$. If $a_n \in \mathfrak{P}_1$, $f^{n-1} + \cdots + a_{n_1} \in \mathfrak{P}_1$ since $f \notin \mathfrak{P}_1$. Then $a_{n-1} \in \mathfrak{P}_2$. Repeat the process, it will terminate, whence $\mathfrak{P}_1 \cap A \subsetneq \mathfrak{P}_2 \cap A$. Otherwise, we have $f^n \in a_1 B + \cdots + a_n B \subset \mathfrak{P}_1$.

Conversely, suppose we have $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} A$ with $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Choose $\mathfrak{P}_1 \in \operatorname{Spec} B$ such that $\mathfrak{P}_1 \cap A = \mathfrak{p}_1$, then we have $A/\mathfrak{p}_1 \subset B/\mathfrak{P}_1$. Let $\mathfrak{P}_2$ be the preimage of the prime ideal in $B/\mathfrak{P}_1$ which is over image of $\mathfrak{p}_2$ in $A/\mathfrak{p}_1$. Proposition A.4.2 guarantees that such $\mathfrak{P}_2$ exists. Then we get $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$. Repeat this progress, we get $\dim B \geq \dim A$. □

Yang: To be completed

Definition A.4.4. An integral domain A is called *normal* if it is integrally closed in its field of fractions $\mathrm{Frac}(A)$.

Lemma A.4.5. Let $A \subset C$ be rings and B the integral closure of A in C, S a multiplicatively closed subset of A. Then the integral closure of $S^{-1}A$ in $S^{-1}C$ is $S^{-1}B$.

Proof. For every $b \in B$ and $\forall s \in S$, there exists $a_i \in A$ s.t.

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then

$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s^1}\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0.$$

Hence b/s is integral over $S^{-1}A$, $S^{-1}B$ is integral over $S^{-1}A$.

If $c/s \in S^{-1}C$ is integral over $S^{-1}A$, then $\exists a_i \in S^{-1}A$ s.t.

$$\left(\frac{c}{s}\right)^n + a_1\left(\frac{c}{s}\right)^{n-1} + \cdots + a_n = 0.$$

Then

$$c^n + a_1 s c^{n-1} + \cdots + a_n s^n = 0 \in S^{-1}C$$

Then $\exists t \in S$ s.t.

$$t(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0 \in C.$$

Then

$$(ct)^n + a_1 st(ct)^{n-1} + \cdots + a_n s^n t^n = t^n(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0.$$

Hence ct is integral over A, then $ct \in B$. Then $c/s = (ct)/(st) \in S^{-1}B$. This completes the proof. □

Proposition A.4.6. Normality is a local property. That is, for an integral domain A, TFAE:

(i) A is normal.

(ii) For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is normal.

(iii) For any maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$, the localization $A_{\mathfrak{m}}$ is normal.

Proof. When A is normal, $A_{\mathfrak{p}}$ is normal by Lemma A.4.5.

Assume that $A_{\mathfrak{m}}$ is normal for every $\mathfrak{m} \in \operatorname{mSpec} A$. If A is not normal, let $\tilde{A}$ be the integral closure of A in $\operatorname{Frac} A$, $\tilde{A}/A$ is a nonzero A-module. Suppose $\mathfrak{p} \in \operatorname{Supp} \tilde{A}/A$ and $\mathfrak{p} \subset \mathfrak{m}$. We have $\tilde{A}_{\mathfrak{m}}/A_{\mathfrak{m}} = 0$ and $\tilde{A}_{\mathfrak{p}}/A_{\mathfrak{p}} = (\tilde{A}_{\mathfrak{m}}/A_{\mathfrak{m}})_{\mathfrak{p}} \neq 0$. This is a contradiction. □

Proposition A.4.7. Let A be a normal ring. Then $A[X]$ is also normal.

Definition A.4.8. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,\xi}$ is normal for any point $\xi \in X$. A ring A is called *normal* if $\operatorname{Spec} A$ is normal.

Remark A.4.9. For a general ring A, let $S := A \setminus (\bigcup_{\mathfrak{p}\in\operatorname{Ass} A} \mathfrak{p}) = \bigcap_{\mathfrak{p}\in\operatorname{Ass} A} A \setminus \mathfrak{p}$. Then S is a multiplicative set. The localization $S^{-1}A$ is called *the total ring of fractions* of A.

Suppose A is reduced and $\operatorname{Ass} A = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Denote its total ring of fractions by Q. Note that elements in Q are either unit or zero divisor. Hence any maximal ideal $\mathfrak{m}$ is contained in $\bigcup \mathfrak{p}_i Q$, whence contained in some $\mathfrak{p}_i Q$. Thus $\mathfrak{p}_i Q$ are maximal ideals. And we have $\bigcap \mathfrak{p}_i Q = 0$. By the Chinese Remainder Theorem, we have $Q = \prod Q/\mathfrak{p}_i Q = \prod A_{\mathfrak{p}_i}$.

Let A be a reduced ring with total ring of fractions Q. Then A is normal iff A is integral closed in Q. If A is normal, then for every $\mathfrak{p} \in \operatorname{Spec} A$, $A_{\mathfrak{p}}$ is integral. Then there is unique minimal prime ideal $\mathfrak{p}_i \subset \mathfrak{p}$. In particular, any two minimal prime ideal are relatively prime. By the Chinese Remainder Theorem, $A = \prod A/\mathfrak{p}_i$. Just need to check $A/\mathfrak{p}_i$ is integral closed in $A_{\mathfrak{p}_i}$. This is clear by check pointwise.

Conversely, suppose A is integral closed in Q. Let e_i be the unit element of $A_{\mathfrak{p}_i}$. It belongs to A since $e_i^2 - e_i = 0$. Since $1 = e_1 + \cdots + e_n$ and $e_i e_j = \delta_{ij}$, we have $A = \prod Ae_i$. Since Ae_i is integral closed in $A_{\mathfrak{p}_i}$, it is normal. Hence A is normal.

Lemma A.4.10. Let A be a normal ring. Then A verifies (R_1) and (S_2).

Proof. Since all properties are local, we can assume A is integral and local.

For (S_2), by Example ??, we only need to show that $\operatorname{Ass}_A A/f$ has no embedded point. Let $\mathfrak{p} = (f : g) =\in \operatorname{Ass}_A A/fA$ and $t := f/g \in \operatorname{Frac} A$. After Replacing A by $A_{\mathfrak{p}}$, we can assume that $\mathfrak{p}$ is maximal. By definition, $t^{-1}\mathfrak{p} \subset A$. If $t^{-1}\mathfrak{p} \subset \mathfrak{p}$, suppose $\mathfrak{p}$ is generated by $(x_1, \cdots, x_n)$ and $t^{-1}(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(A)$. There is a monic polynomial $\chi(T) \in A[T]$ vanishing Φ. Then $\chi(t^{-1}) = 0$ and $t^{-1} \in A$. This is impossible by definition of t. Then $t^{-1}\mathfrak{p} = A$, and $\mathfrak{p} = (t)$ is principal. By Krull's Principal Ideal Theorem A.3.14, $\operatorname{ht}(\mathfrak{p}) = 1$.

Now we show that A verifies (R_1). Suppose $(A, \mathfrak{m})$ is local of dimension 1. Choosing $a \in \mathfrak{m}$, A/a is of dimension 0. Then by A.3.11, $\mathfrak{m}^n \subset aA$ for some $n \geq 1$. Suppose $\mathfrak{m}^{n-1} \not\subset aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$ and let $t = a/b$. By construction, $t^{-1} \notin A$ and $t^{-1}\mathfrak{m} \subset A$. After similar argument, we see that $\mathfrak{m} = tA$, whence A is regular. □

Lemma A.4.11. Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension 1. Then A is normal iff A is regular.

Proof. By lemma A.4.10, we just need to show that regularity implies normality.

Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since A is regular, $\mathfrak{m} = (t)$. Let $I \subset \mathfrak{m}$ be an ideal. If $I \subset \bigcap_n \mathfrak{m}^n$, then for every $a \in I$, there exists a_n such that $a = a_n t^n$. Then we get an ascending chain of ideals $(a_1) \subset (a_2) \subset \cdots$. Hence $a = 0$ by Nakayama's Lemma. Suppose I is not zero. Then there is some n such that $I \subset \mathfrak{m}^n$ and $I \not\subset \mathfrak{m}^{n+1}$. For every $at^n \in I \setminus \mathfrak{m}^{n+1}$, $a \notin \mathfrak{m}$, whence a is a unit in A. Then $I = (t^n)$. Hence A is PID and hence normal. □

Proposition A.4.12. Let A be a noetherian integral domain of dimension ≥ 1 verifying (S_2). Then

$$A = \bigcap_{\mathfrak{p}\in \operatorname{Spec} A, \operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

Proof. Clearly $A \subset \bigcap A_{\mathfrak{p}}$. Let $t = f/g \in \bigcap A_{\mathfrak{p}}$. Since $f \in gA_{\mathfrak{p}}$ and we have $gA = \bigcap(gA_{\mathfrak{p}} \cap A)$, $f \in gA$. It follows that $t \in A$. □

Theorem A.4.13 (Serre's criterion for normality)**.** Let X be a locally noetherian scheme. Then X is normal if and only if it verifies (R_1) and (S_2).

Proof. One direction has been proved in Lemma A.4.10. Suppose X verifies (R_1) and (S_2). Again we can assume $X = \operatorname{Spec} A$ is affine and A is local. By Remark A.4.9, we just need to show that A is integral closed in its total ring of fractions Q. Suppose we have

$$\left(\frac{a}{b}\right)^n + c_1\left(\frac{a}{b}\right)^{n-1} + \cdots + c_n = 0 \in Q.$$

Since A verifies (S_2), $bA = \bigcap \nu_{\mathfrak{p}}^{-1}(b_{\mathfrak{p}}A_{\mathfrak{p}})$. So it is sufficient to show that $a_{\mathfrak{p}} \in b_{\mathfrak{p}}A_{\mathfrak{p}}$ with $\operatorname{ht}(\mathfrak{p}) = 1$. Note that $A_{\mathfrak{p}}$ is regular and hence normal by Lemma A.4.11. Then above equation gives us desired result. □

A.5 Smoothness

A.5.1 Modules of differentials and derivations

In this subsection, let R be a ring and A an R-algebra.

Definition A.5.1 (Derivation)**.** A *derivation* of A over R is an R-linear map $\partial : A \to M$ with an A-module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module M, the set of all derivations of A over R into M forms an A-module, denoted by $\operatorname{Der}_R(A, M)$.

Given a module homomorphism $f : M \to N$ of A-modules and a derivation $\partial \in \operatorname{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of A over R into N.

Proposition A.5.2. The functor $\operatorname{Der}_R(A, -)$ is representable. The representing object is denoted by $\Omega_{A/R}$, which is called the *module of differentials* of A over R.

Proof. First suppose A is a free R-algebra with a set of generators $a_\lambda, \lambda \in \Lambda$. Then an R-derivation

$\partial \in \mathrm{Der}_R(A, M)$ is uniquely determined by its values on the generators a_λ. Let

$$\Omega_{A/R} := \bigoplus_{\lambda \in \Lambda} A \cdot \mathrm{d}a_\lambda$$

and $\mathrm{d} : A \to \Omega_{A/R}$ be the R-derivation defined by $a_\lambda \mapsto \mathrm{d}a_\lambda$. For any R-derivation $\partial \in \mathrm{Der}_R(A, M)$, we can define a unique A-module homomorphism $\Phi_\partial : \Omega_{A/R} \to M$ by sending $\mathrm{d}a_\lambda$ to $\partial(a_\lambda)$ such that $\partial = \Phi_\partial \circ \mathrm{d}$. This gives a bijection

$$\mathrm{Der}_R(A, M) \cong \mathrm{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_\partial.$$

Now suppose $A = F/I$ is an arbitrary R-algebra, where F is a free R-algebra and I is an ideal of F. Then we can define the module of differentials

$$\Omega_{A/R} := \left(\Omega_{F/R} \otimes_F A\right) / \sum_{f \in I} A \cdot \mathrm{d}f.$$

The R-linear map $\mathrm{d}_A : F \otimes_F A \xrightarrow{\mathrm{d}_F} \Omega_{F/R} \otimes_F A \to \Omega_{A/R}$ is a derivation of A over R.

For any R-derivation $\partial \in \mathrm{Der}_R(A, M)$, note that $F \to A \xrightarrow{\partial} M$ is an R-derivation of F over R into M. Then we get an F-module homomorphism $\Omega_F \to M$. It gives an A-module homomorphism $\Omega_F \otimes_F A \to M, \mathrm{d}f \otimes 1 \mapsto \partial f$. This map factors into $\Omega_F \otimes_F A \to \Omega_{A/R}$ and $\Phi_\partial : \Omega_{A/R} \to M$. Since Φ_∂ is A-linear and $\Omega_{A/R}$ is generated by $\mathrm{d}a_\lambda$ as A-module, such Φ_∂ is unique. □

Corollary A.5.3. Suppose A is of finite type over R. Then the module of differentials $\Omega_{A/R}$ is a finitely generated A-module.

Remark A.5.4. Let B be an A-algebra, M an A-module and N a B-module. If there is a homomorphism of A-modules $M \to N$, then we can extend it to a homomorphism of B-modules $M \otimes_A B \to N$ by sending $m \otimes b$ to $m \cdot b$. And such extension is unique in the sense of following commutative diagram:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N. \\ M \otimes_A B \arrow[ur, "\exists!"'] & \end{tikzcd}$$

Hence we get a natural bijection

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(M \otimes_A B, N).$$

Proposition A.5.5. Let A, R' be R-algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

Proof. We check the universal property of $\Omega_{A/R} \otimes_A A'$. First, the map

$$\mathrm{d}_{A'} : A \otimes_R R' \to \Omega_{A/R} \otimes_R R' \cong \Omega_{A/R} \otimes_A A', \quad a \otimes r \mapsto \mathrm{d}a \otimes r$$

is an R'-derivation of A' into $\Omega_{A/R} \otimes_A A'$. For any R'-derivation $\partial' : A' \to M$ into an A'-module M, we can compose it with the homomorphism $A' \to A$ and get an R-derivation $\partial : A \to M$. By the universal property of $\Omega_{A/R}$, there is a unique A-module homomorphism $\Phi : \Omega_{A/R} \to M$ such that $\partial = \Phi \circ \mathrm{d}_A$. Then we can extend it to an A'-module homomorphism $\Phi' : \Omega_{A/R} \otimes_A A' \to M$ by Remark A.5.4. By the construction, we have $\Phi' \circ \mathrm{d}_{A'} = \partial'$. □

Proposition A.5.6. Let A be an R-algebra and S a multiplicative set of A. Then we have an isomorphism

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

Proof. Let

$$\mathrm{d}_{S^{-1}A} : S^{-1}A \to S^{-1}\Omega_{A/R}, \quad \frac{a}{s} \mapsto \frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}.$$

By direct computation, $\mathrm{d}_{S^{-1}A}$ is an R-derivation of $S^{-1}A$ over R into $S^{-1}\Omega_{A/R}$. For any R-derivation $\partial : S^{-1}A \to M$ into an $S^{-1}A$-module M, we can get an $S^{-1}A$-module homomorphism $\Phi' : S^{-1}\Omega_{A/R} \to M$ as proof of Proposition A.5.5. We have

$$\partial(s \cdot \frac{a}{s}) = s\partial(\frac{a}{s}) + \frac{a}{s}\partial s.$$

It follows that

$$\partial(\frac{a}{s}) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(\mathrm{d}a) - a\Phi'(\mathrm{d}s)}{s^2} = \Phi'(\frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}).$$

Thus, $\Phi' \circ \mathrm{d}_{S^{-1}A} = \partial$. □

Theorem A.5.7. Let A be an R-algebra and B an A-algebra. Then there is a natural short exact sequence

$$\Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to \Omega_{B/A} \to 0$$

of B-modules.

Proof. Let $\mathrm{d}_{A/R} : A \to \Omega_{A/R}$ be the R-derivation of A over R. The map $A \to B \xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R}$ induces a B-linear map

$$u : \Omega_{A/R} \otimes_A B \to \Omega_{B/R}, \quad \mathrm{d}_{A/R}(a) \otimes b \mapsto b\mathrm{d}_{B/R}(a).$$

The map $\mathrm{d}_{B/A}$ is an A-derivation and hence R-derivation. Then it induces a B-linear map

$$v : \Omega_{B/R} \to \Omega_{B/A}, \quad \mathrm{d}_{B/R}(b) \mapsto \mathrm{d}_{B/A}(b).$$

Since $\Omega_{B/A}$ is generated by elements of the form $\mathrm{d}_{B/A}(b)$ for $b \in B$, the map v is surjective. And clearly $\mathrm{d}_{B/A}(a) = a\mathrm{d}_{B/A}(1) = 0$ for $a \in A$.

Consider the composition $B \xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R} \to \Omega_{B/R}/\Im u$. For every $a \in A, b \in B$, we have

$$[\mathrm{d}_{B/R}(ab)] = [b\mathrm{d}_{B/R}(a) + a\mathrm{d}_{B/R}(b)] = [b\mathrm{d}_{B/R}(a)] + [a\mathrm{d}_{B/A}(b)] = [a\mathrm{d}_{B/A}(b)].$$

Hence it is indeed an A-derivation of B. Then it induces a B-linear map

$$\varphi : \Omega_{B/A} \to \Omega_{B/R}/\Im u, \quad \mathrm{d}_{B/A}(b) \mapsto [\mathrm{d}_{B/R}(b)].$$

The map φ is surjective since $\Omega_{B/R}$ is generated by elements of the form $\mathrm{d}_{B/R}(b)$ for $b \in B$. Note that the composition

$$\Omega_{B/A} \xrightarrow{\varphi} \Omega_{B/R}/\mathfrak{J}u \to \Omega_{B/A}/\operatorname{Ker} v$$

is the identity map. Thus, φ is injective and hence an isomorphism. In particular, we have $\operatorname{Ker} v = \mathfrak{J}u$. □

Remark A.5.8. The exact sequence in Theorem A.5.7 is left exact if and only if every R-derivation of A into B-module extends to an R-derivation of B into B-module.

Yang: To be completed.

Theorem A.5.9. Let A be an R-algebra and I an ideal of A. Set $B := A/I$. Then there is a natural short exact sequence

$$I/I^2 \to \Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to 0$$

of B-modules.

Proof. Suppose $A = F/\mathfrak{b}$ for some free R-algebra F and an ideal $\mathfrak{b}$ of F. Let $\mathfrak{a}$ be the preimage of I in F. Let $\mathrm{d}\mathfrak{b}$ (resp. $\mathrm{d}\mathfrak{a}$) denote the image of $\mathfrak{b}$ (resp. $\mathfrak{a}$) in $\Omega_{F/R}$. Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \to (\mathrm{d}\mathfrak{a} \otimes_F B)/(\mathrm{d}\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows. □

Definition A.5.10. Let $\mathbf{k}$ be a field and A an integral $\mathbf{k}$-algebra of finite type of dimension n. We say A is *smooth at* $\mathfrak{p} \in \operatorname{Spec} A$ if the module of differentials $\Omega_{A,\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank n.

Example A.5.11. Let $\mathbf{K}/\mathbf{k}$ be a finite generated field extension and $\mathbf{k}'$ be the algebraic closure of $\mathbf{k}$ in $\mathbf{K}$. Then

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \operatorname{trdeg}(\mathbf{K}/\mathbf{k}) + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}},$$

and $\dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}} = 0$ if and only if $\mathbf{k}'$ is separable over $\mathbf{k}$.

First suppose $\mathbf{K} = \mathbf{k}'$ is algebraic over $\mathbf{k}$. Suppose $\mathbf{k}'/\mathbf{k}$ is separable. For every $\alpha \in \mathbf{k}'$, suppose $f(\alpha) = 0$ for $f \in \mathbf{k}[T]$. Then $\mathrm{d}f(\alpha) = f'(\alpha)\mathrm{d}\alpha = 0$. By the separability of $\mathbf{k}'/\mathbf{k}$, we have $f'(\alpha) \neq 0$. It follows that $\mathrm{d}\alpha = 0$. Conversely, let $\alpha \in \mathbf{k}'$ be a inseparable element over $\mathbf{k}$. Since $\mathbf{k}[\alpha] \to \mathbf{k}[\alpha], \alpha^n \mapsto n\alpha^{n-1}$ is a non-zero R-derivation, we have $\Omega_{\mathbf{k}[\alpha]/\mathbf{k}} \neq 0$. By induction on number of generated elements, choosing a middle field $\mathbf{k} \subset \mathbf{k}'' \subset \mathbf{k}'$, at least one of $\Omega_{\mathbf{k}''/\mathbf{k}}$ and $\Omega_{\mathbf{k}'/\mathbf{k}''}$ is non-zero. Then $\Omega_{\mathbf{K}/\mathbf{k}} \neq 0$ by Theorem A.5.7.

Then suppose $\mathbf{k}' = \mathbf{k}$. By the Noether's Normalization Lemma, we can find a finite set of elements $T_1, \cdots, T_n \in \mathbf{K}$ such that $\mathbf{K}$ is algebraic over $\mathbf{k}'(T_1, \ldots, T_n)$. Note that we can choose T_i such that $\mathbf{K}/\mathbf{k}'(T_1, \cdots, T_n)$ is separable. To see this, if $\alpha \in \mathbf{K}$ is an inseparable element over $\mathbf{k}'(T_1, \cdots, T_n)$, then by replacing a suitable T_i with α, we reduce the inseparable degree of $\mathbf{K}/\mathbf{k}'(T_1, \cdots, T_n)$.

Since $\mathbf{K}/\mathbf{k}'(T_1, \cdots, T_n)$ is finite, every $\mathbf{k}$-derivation of $\mathbf{k}'(T_1, \cdots, T_n)$ into $\mathbf{K}$-module extends to a

$\mathbf{k}$-derivation of $\mathbf{K}$ into $\mathbf{K}$-module. Then by Remark A.5.8, we have

$$0 \to \Omega_{\mathbf{k}'(T_1,\cdots,T_n)/\mathbf{k}} \otimes_{\mathbf{k}'(T_1,\cdots,T_n)} \mathbf{K} \to \Omega_{\mathbf{K}/\mathbf{k}} \to \Omega_{\mathbf{K}/\mathbf{k}'(T_1,\cdots,T_n)} \to 0.$$

Finally, note that every $\mathbf{k}$-derivation ∂ of $\mathbf{k}'$ into $\mathbf{K}$-module can be extended to $\mathbf{k}'[T_1,\cdots,T_n]$ by setting $\partial T_i = 0$. Thus, we have

$$0 \to \Omega_{\mathbf{k}'/\mathbf{k}} \otimes_{\mathbf{k}'} \mathbf{k}'[T_1,\cdots,T_n] \to \Omega_{\mathbf{k}'[T_1,\cdots,T_n]/\mathbf{k}} \to \Omega_{\mathbf{k}'[T_1,\cdots,T_n]/\mathbf{k}'} \to 0.$$

This follows that

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}'} + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}}.$$

A.5.2 Applications to affine varieties

Let $\mathbf{k}$ be arbitrary field, $A = \mathbf{k}[T_1,\dots,T_n]$ and $\mathfrak{m}$ a maximal ideal of A such that $\kappa(\mathfrak{m})$ is separable over $\mathbf{k}$. We try to give an explanation of Zariski's tangent space at $\mathfrak{m}$ using the language of derivation. We know that $\Omega_{A/\mathbf{k}} = \bigoplus_{i=1}^n A\mathrm{d}T_i$, thus $\Omega_{A_\mathfrak{m}/\mathbf{k}} \cong \bigoplus_{i=1}^n A_\mathfrak{m}\mathrm{d}T_i$. Then

$$\mathrm{Der}_{\mathbf{k}}(A_\mathfrak{m}, A_\mathfrak{m}) \cong \mathrm{Hom}_{\mathbf{k}}(\Omega_{A_\mathfrak{m}/\mathbf{k}}, A_\mathfrak{m}) \cong \bigoplus_{i=1}^n A_\mathfrak{m}\partial_i,$$

where $\partial_i \in \mathrm{Der}_{\mathbf{k}}(A_\mathfrak{m}, A_\mathfrak{m})$ is the derivation defined by $\mathrm{d}T_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. It coincides with the usual derivation $f \mapsto \partial f/\partial T_i$. Consider the restriction of ∂_i to $\mathfrak{m}$ and take values in the residue field $\kappa(\mathfrak{m})$, we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1,\cdots,\partial_n)^T} A_\mathfrak{m}^n \to \kappa(\mathfrak{m})^n.$$

Since $\kappa(\mathfrak{m})$ is separable over $\mathbf{k}$, we claim that $\mathrm{Ker}\,\Phi = \mathfrak{m}^2$. Indeed, by Remark A.5.12, we can write every $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ as $\sum_i a_i g_i$. Then

$$\frac{\partial f}{\partial T_i} = a_i \frac{\partial g_i}{\partial T_i} + g_i \frac{\partial a_i}{\partial T_i}.$$

Since g_i is separable, the image of $\partial g_i/\partial T_i$ in $\kappa(\mathfrak{m})$ is not zero. Hence $\Phi(f) \neq 0$. By the claim, Φ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$ of $\kappa(\mathfrak{m})$-vector spaces. Then we get

$$T_{A,\mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where $x \in \mathbb{a}_{\mathbf{k}}^n$ is the point corresponding to $\mathfrak{m}$. This coincides with the usual tangent space at x in language of differential geometry.

Remark A.5.12. Let $\mathbf{k}$ be arbitrary field, $A = \mathbf{k}[T_1,\cdots,T_n]$ and g_i irreducible polynomials in one variable T_i over $\mathbf{k}$. Then for every $f \in A$, we can write

$$f = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{Z}_{\geq 0}^n} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \leq \deg g_i.$$

This is called the *Taylor expansion of f with respect to $g_1,\cdots,g_n$*.

When $n = 1$, it follows from division algorithm. For $n > 1$, we can use induction on n. Let

$\mathbf{K} = \mathbf{k}(T_1, \cdots, T_{n-1})$. Then we can write f as

$$f = \sum_{i=0}^{r} a_i g_n^i, \quad a_i \in \mathbf{K}[T_n], \quad \deg a_i < \deg g_n.$$

Comparing the coefficients of two sides from the highest degree of T_n to the lowest degree, we see that

$$a_i \in \mathbf{k}[T_1, \cdots, T_{n-1}].$$

By induction hypothesis, the conclusion follows.

Let $B = A/I$ be a $\mathbf{k}$ of finite type, $I = (F_1, \ldots, F_m) \subset \mathfrak{m}$ and $\mathfrak{n}$ the image of $\mathfrak{m}$ in B. We have an exact sequence of $\kappa(\mathfrak{m})$-vector spaces

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} : \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of $F_1, \ldots, F_m$ is the $m \times n$ matrix

$$J(F_1, \ldots, F_m) := \left(\frac{\partial F_i}{\partial T_j}\right)_{1 \le i \le m, 1 \le j \le n}$$

with entries in B.

Theorem A.5.13. Setting as above. Then B is regular at $\mathfrak{n}$ if and only if the Jacobian matrix J has maximal rank $n - \dim B_{\mathfrak{n}}$ after taking values in the residue field $\kappa(\mathfrak{m})$.

Proof. We have an exact sequence

$$0 \to T_{B,\mathfrak{n}} \to T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa^m \to 0,$$

where Ψ sends $\partial \in T_{A,\mathfrak{m}}$ to $(\partial(F_1), \ldots, \partial(F_m))^T$. Note that the matrix of Ψ is just J^T, the transpose of the Jacobian matrix. Hence

$$\operatorname{rank} J = n - \dim_\kappa T_{B,\mathfrak{n}} \le n - \dim B_{\mathfrak{n}}$$

and the equality holds if and only if B is regular at $\mathfrak{n}$. □

Remark A.5.14. If $\kappa(\mathfrak{m})$ is not separable over $\mathbf{k}$, then we still have the inequality

$$\operatorname{rank} J \le n - \dim B_{\mathfrak{n}}.$$

Indeed, in any case, we have an exact sequence

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Hence $\dim_\kappa I/(I \cap \mathfrak{m}^2) = n - \dim B_{\mathfrak{n}}$. There is a $\kappa(\mathfrak{m})$-linear map

$$I/(I \cap \mathfrak{m}^2) \to \kappa(\mathfrak{m})^n, \quad [f] \mapsto (\partial_1(f), \ldots, \partial_n(f))^T,$$

and every row of the Jacobian matrix J is in the image of this map. Thus, the rank of J is at most $n - \dim B_{\mathfrak{n}}$.

Hence if $\operatorname{rank} J = n - \dim B_{\mathfrak{n}}$, we can still see that B is regular at $\mathfrak{n}$. However, the converse does not hold in general.

Proposition A.5.15. Let $\mathbf{k}$ be a field, $\Bbbk$ the algebraic closure of $\mathbf{k}$, A a $\mathbf{k}$-algebra of finite type and $A_{\Bbbk} := A \otimes_{\mathbf{k}} \Bbbk$. Yang: Suppose $A_{\Bbbk}$ is integral. Let $\mathfrak{m} \in \operatorname{mSpec} A$ and $\mathfrak{m}'$ be a maximal ideal of $A_{\Bbbk}$ lying over $\mathfrak{m}$. Then

(a) If $A_{\Bbbk}$ is regular at $\mathfrak{m}'$, then A is regular at $\mathfrak{m}$;

(b) suppose $\kappa(\mathfrak{m})$ is separable over $\mathbf{k}$, the converse holds.

Proof. Regarding $J_{\mathfrak{m}}$ and $J_{\mathfrak{m}'}$ as matrices with entries in $\Bbbk$, they are the same and hence have the same rank. If $A_{\Bbbk}$ is regular at $\mathfrak{m}'$, since $\kappa(\mathfrak{m}) = \Bbbk$, then $\operatorname{rank} J_{\mathfrak{m}'} = n - \dim A_{\Bbbk,\mathfrak{m}'}$. Note that $\dim A_{\Bbbk,\mathfrak{m}'} = \operatorname{trdeg}(\mathscr{K}(A_{\Bbbk})/\Bbbk) = \operatorname{trdeg}(\mathscr{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}}$, we have $\operatorname{rank} J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence A is regular at $\mathfrak{m}$.

Conversely, suppose A is regular at $\mathfrak{m}$ and $\kappa(\mathfrak{m})$ is separable over $\mathbf{k}$. Then $\operatorname{rank} J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence $A_{\Bbbk}$ is regular at $\mathfrak{m}'$. Yang: To be modified. ☐

Proposition A.5.16. Let $\mathbf{k}$ be a field and A an integral $\mathbf{k}$-algebra of finite type and of dimension n. Let $\Bbbk$ be the algebraic closure of $\mathbf{k}$ and $A_{\Bbbk} := A \otimes_{\mathbf{k}} \Bbbk$. Then A is smooth at $\mathfrak{p} \in \operatorname{Spec} A$ if and only if $A_{\Bbbk}$ is regular at every $\mathfrak{m}'$ over $\mathfrak{m}$.

Proof. Since $\Omega_{A_{\Bbbk}/\Bbbk} \cong \Omega_{A/\mathbf{k}} \otimes_A A_{\Bbbk}$ is free of rank n if and only if $\Omega_{A/\mathbf{k}}$ is free of rank n, we can assume that $\mathbf{k} = \Bbbk$. If A is smooth at $\mathfrak{p}$, then $\Omega_{A_{\mathfrak{p}}/\mathbf{k}} \cong \bigoplus A_{\mathfrak{p}} \mathrm{d}f_i$ is free of rank n. Let $\mathfrak{P}_i \in \operatorname{Der}_{\Bbbk}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ be the derivation defined by $\mathrm{d}f_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. Then we have $\partial_i f_j = \delta_{ij}$ for $1 \leq i, j \leq n$. Then similar to above argument, we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{(\partial_1, \ldots, \partial_n)^T} \Bbbk^n.$$

This shows that $A_{\Bbbk}$ is regular at $\mathfrak{m}$.

Conversely, suppose $A_{\Bbbk}$ is regular at $\mathfrak{m}$. Note that $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A,\Bbbk} \otimes_A \Bbbk$ is surjective since $\Omega_{A_{\Bbbk}/\Bbbk} = 0$. Then by Nakayama's lemma, $\Omega_{A_{\mathfrak{m}}/\Bbbk}$ is generated by n elements as an $A_{\mathfrak{m}}$-module.

Note that $\dim_{\mathscr{K}(A)} \Omega_{\mathscr{K}(A)/\mathbf{k}} = \operatorname{trdeg}(\mathscr{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}} = n$. Yang: By induction on transcendental degree.

Yang: By Nakayama's Lemma, $\Omega_{A_{\mathfrak{m}}/\mathbf{k}}$ is free of rank n as an $A_{\mathfrak{m}}$-module.

Yang: To be completed. ☐

Example A.5.17. Let $\mathbf{k}$ be an imperfect field of characteristic $p > 2$. Suppose $\alpha = \beta^p \in \mathbf{k}$ and β is not in $\mathbf{k}$. Let $A = \mathbf{k}[x, y]/(x^2 - y^p - \alpha)$ and $\mathfrak{m} = (x, y^p - \alpha) = (x)$. Note that $\mathfrak{m}$ is principal, so A is regular at $\mathfrak{m}$. However,

$$J_{\mathfrak{m}} = \left(\frac{\partial}{\partial x}(x^2 - y^p - \alpha), \frac{\partial}{\partial y}(y^p - \alpha) \right) = (2x, 0) = (0, 0) \in M_{1\times 2}(\kappa(\mathfrak{m})).$$

Thus, A is not smooth at $\mathfrak{m}$. From the view of differentials, we have

$$\Omega_{A_\mathfrak{m}/\mathrm{k}} = A_\mathfrak{m}\mathrm{d}x \oplus A_\mathfrak{m}\mathrm{d}y / A_\mathfrak{m} \cdot x\mathrm{d}x = \kappa(\mathfrak{m})\mathrm{d}x \oplus A_\mathfrak{m}\mathrm{d}y,$$

which is not free as an $A_\mathfrak{m}$-module.

A.6 Formal Completion

A.6.1 Formal completion of rings and modules

Definition A.6.1. Let A be a ring and $\mathcal{T}$ a topology on A. We say that $(A, \mathcal{T})$ is a *topological ring* if the operations of addition and multiplication are continuous with respect to the topology $\mathcal{T}$. Given a topological ring A. A *topological A-module* is a pair $(M, \mathcal{T}_M)$ where M is an A-module and $\mathcal{T}_M$ is a topology on M such that the addition and scalar multiplication is continuous. The morphisms of topological A-modules are the continuous A-linear maps. They form a category denoted by $\mathbf{TopMod}_A$.

Definition A.6.2. Let A be a ring, I an ideal of A and M an A-module. The *I-adic topology* on M is the topology defined by the basis of open sets $x + I^kM$ for all $x \in M, k \geq 0$.

Example A.6.3. Let $A = \mathbb{Z}$ be the ring of integers and p a prime number. The p-adic topology on $\mathbb{Z}$ is defined by the metric

$$d(x, y) := \|x - y\|_p := p^{-v(x-y)},$$

where v is the valuation defined by the ideal $p\mathbb{Z}$.

Note that for I-adic topology, any homomorphism $f : M \to N$ of A-modules is continuous since $f(x + I^kN) \subset f(x) + I^kM$ for all $x \in M$ and $k \geq 0$. Hence the forgotten functor $\mathbf{TopMod}_A \to \mathbf{Mod}_A$ gives an equivalence of categories.

Let M be an A-module equipped with the I-adic topology. Note that M is Hausdorff as a topological space if and only if $\bigcap_{n\geq 0} I^nM = \{0\}$. In this case, we say that M is *I-adically separated.*

When M is I-adically separated, we can see that M is indeed a metric space. Fix $r \in (0, 1)$. For every $x \neq y \in M$, there is a unique $k \geq 0$ such that $x - y \in I^kM$ but $x - y \notin I^{k+1}M$. We can define a metric on M by

$$d(x, y) := r^k.$$

This metric induces the I-adic topology on M.

To analyze the I-adic separation property of M, the following Artin-Rees Lemma is particularly useful.

Theorem A.6.4 (Artin-Rees Lemma)**.** Let A be a noetherian ring, I an ideal of A, M a finite

A-module and N a submodule of M. Then there exists an integer r such that for all $n \geq 0$, we have

$$(I^{r+n}M) \cap N = I^n(I^r M \cap N).$$

Proof. Let

$$A' := A \oplus IX \oplus I^2X^2 \oplus \cdots \subset A[X]$$

be a graded A-algebra. Note that if $I = (a_1, \dots, a_k)$, then $A' = A[a_1X, \dots, a_kX]$. Hence A' is a noetherian ring. Let

$$M' := M \oplus IMX \oplus I^2MX^2 \oplus \cdots$$

be a graded A'-module. Then M' is a finite A'-module since it is generated by M and M is finite over A. Let

$$N' := N \oplus (IM \cap N)X \oplus (I^2M \cap N)X^2 \oplus \cdots$$

be a graded submodule of M'. Then N' is finite over A'. Suppose $N' = \sum A'x_i$ with $x_i \in I^{d_i}M \cap N$. Choose $r \geq d_i$ for all i. Then the degree $n+r$ part of N' is equal to degree n part of A' timing the degree r part of N'. That is, for all $n \geq 0$, $I^{n+r}M \cap N = I^n(I^rM \cap N)$. □

Corollary A.6.5. Let A be a noetherian ring, I an ideal of A, M a finite A-module and N a submodule of M. Then the subspace topology on N induced by $N \subset M$ coincides with the I-adic topology on N.

Proof. This is a direct consequence of the Artin-Rees Lemma. □

Corollary A.6.6. Let A be a noetherian ring, I an ideal of A, and M a finite A-module. Let $N = \bigcap_{n\geq 0} I^nM$. Then $IN = N$. In particular, if $I \subset \mathrm{rad}(A)$, then M is I-adically separated.

Proof. We have that

$$N = I^{n+r}M \cap N = I^n(I^rM \cap N) = I^nN \subset IN \subset N.$$

The latter conclusion follows from the Nakayama's Lemma. □

Definition A.6.7. Let A be a ring, I an ideal of A and M an A-module. We say that M is *complete (with respect to I-adic topology)* if M is I-adically separated and complete as a metric space with respect to the metric induced by the I-adic topology.

Lemma A.6.8. Let A be a ring, I an ideal of A and M an A-module. Then the inverse limit

$$\widehat{M} := \varprojlim(\cdots \to M/I^nM \to M/I^{n-1}M \to \cdots \to M/IM)$$

exists in the category of A-modules. Moreover, $\widehat{A}$ is an A-algebra and $\widehat{M}$ is an $\widehat{A}$-module.

Proof. Let

$$\widehat{M} := \left\{ (x_n) \in \prod_{n\geq 0} M/I^nM \,\middle|\, x_{n+1} \mapsto x_n \right\}.$$

We claim that $\widehat{M}$ is that we desired. Yang: To be completed. □

Definition A.6.9 (Formal Completion)**.** Let A be a ring, I an ideal of A and M an A-module. The *formal completion* of M with respect to I, denoted by $\widehat{M}$, is defined as

$$\widehat{M} := \varprojlim(\cdots \to M/I^nM \to M/I^{n-1}M \to \cdots \to M/IM),$$

where the maps are the natural projections $M/I^nM \to M/I^{n-1}M$.

Example A.6.10. Let $A = \mathbb{Z}$ be the ring of integers and $I = p\mathbb{Z}$. The formal completion of $\mathbb{Z}$ with respect to $p\mathbb{Z}$ is the ring of p-adic integers, denoted by $\mathbb{Z}_p$. The elements of $\mathbb{Z}_p$ can be represented as infinite series of the form

$$a_0 + a_1p + a_2p^2 + \cdots,$$

where $a_i \in \{0, 1, \dots, p-1\}$.

Example A.6.11. Let R be a ring, $A = R[X_1, \dots, X_n]$ and $I = (X_1, \dots, X_n)$. The formal completion of A with respect to I is the ring of formal power series $R[[X_1, \dots, X_n]]$. The elements of $R[[X_1, \dots, X_n]]$ can be represented as infinite series of the form

$$\sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} X_1^{i_1} \cdots X_n^{i_n},$$

where $a_{i_1,\dots,i_n} \in R$ and the multi-index $(i_1, \dots, i_n)$ runs over all non-negative integers.

Proposition A.6.12. The formal completion $\widehat{M}$ of a A-module M is complete, and image of M is dense in $\widehat{M}$. Moreover, $\widehat{M}$ is uniquely characterized by above properties.

Proof. Yang: To be completed. □

By the universal property of the inverse limit, we get a covariant functor from the category of A-modules to the category of topological $\widehat{A}$-modules, which sends an A-module M to $\widehat{M}$ and a morphism $f : M \to N$ to the induced morphism $\widehat{f} : \widehat{M} \to \widehat{N}$.

Lemma A.6.13. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of finite A-modules. Then the sequence of $\widehat{A}$-modules

$$0 \to \widehat{M_1} \to \widehat{M_2} \to \widehat{M_3} \to 0$$

is still exact.

Proof. Yang: To be completed. □

Proposition A.6.14. Let $\widehat{A}$ be completion of a noetherian ring A with respect to an ideal I and M a finite A-module. Then the natural map $M \otimes_A \widehat{A} \to \widehat{M}$ is an isomorphism.

Proof. Since A is noetherian and M is finite, we have an exact sequence

$$A^m \to A^n \to M \to 0.$$

By Lemma A.6.13, we have an exact sequence

$$\widehat{A^m} \to \widehat{A^n} \to \widehat{M} \to 0.$$

On the other hand, we have

$$A^m \otimes_A \widehat{A} \to A^n \otimes_A \widehat{A} \to M \otimes_A \widehat{A} \to 0$$

by right exactness of the tensor product. Since the inverse limit commutes with finite direct sums, we complete the proof by the Five Lemma. □

Proposition A.6.15. Let A be a noetherian ring and I an ideal of A. Then the formal completion $\widehat{A}$ of A with respect to I is a flat A-module.

Proof. This is a direct consequence of Lemma A.6.13 and Proposition A.6.14. □

Lemma A.6.16. Let $\widehat{A}$ be the formal completion of a noetherian ring A with respect to an ideal I. Suppose that I is generated by $a_1, ..., a_n$. Then we have an isomorphism of topological rings

$$\widehat{A} \cong A[[X_1, \dots, X_n]]/(X_1 - a_1, \cdots, X_n - a_n).$$

Proof. Yang: To be completed. □

Proposition A.6.17. Let A be a noetherian ring and I an ideal of A. Then the formal completion $\widehat{A}$ of A with respect to I is a noetherian ring.

Proof. Note that $A[[X_1, \dots, X_n]]$ is noetherian by Hilbert's Basis Theorem. Then the conclusion follows from Lemma A.6.16. □

Proposition A.6.18. Let A be a noetherian ring and $\mathfrak{m}$ a maximal ideal of A. Then the formal completion $\widehat{A}$ of A with respect to $\mathfrak{m}$ is a local ring with maximal ideal $\mathfrak{m}\widehat{A}$.

Proof. Yang: To be completed. □

A.6.2 Complete local rings

Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring with respect to the $\mathfrak{m}$-adic topology. We say that A is *of equal characteristic* if $\operatorname{char} A = \operatorname{char} \mathbf{k}$, and *of mixed characteristic* if $\operatorname{char} A \neq \operatorname{char} \mathbf{k}$. In latter case, $\operatorname{char} \mathbf{k} = p$ and $\operatorname{char} A = 0$ or $\operatorname{char} A = p^k$.

The goal of this subsection is the following structure theorem for noetherian complete local rings due to Cohen.

Theorem A.6.19 (Cohen Structure Theorem)**.** Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring of dimension d. Then

(a) A is a quotient of a noetherian regular complete local ring;

(b) if A is regular and of equal characteristic, then $A \cong \mathbf{k}[[X_1, \dots, X_d]]$;

(c) if A is regular, of mixed characteristic $(0,p)$ and $p \notin \mathfrak{m}^2$, then $A \cong D[[X_1,\dots,X_{d-1}]]$, where $(D,p,\mathbf{k})$ is a complete DVR;

(d) if A is regular, of mixed characteristic $(0,p)$ and $p \in \mathfrak{m}^2$, then $A \cong D[[X_1,\dots,X_d]]/(f)$, where $(D,p,\mathbf{k})$ is a complete DVR and f a regular parameter.

To prove the Cohen Structure Theorem, we first list some preliminary results on complete local rings. They are independently important and can be used in other contexts.

Theorem A.6.20 (Hensel's Lemma)**.** Let $(A,\mathfrak{m},\mathbf{k})$ be a complete local ring, $f \in A[X]$ a monic polynomial and $\overline{f} \in \mathbf{k}[X]$ its reduction modulo $\mathfrak{m}$. Suppose that $\overline{f} = \overline{g}\cdot\overline{h}$ for some monic polynomials $\overline{g},\overline{h} \in \mathbf{k}[X]$ such that $\gcd(\overline{g},\overline{h}) = 1$. Then the factorization lifts to a unique factorization $f = g\cdot h$ in $A[X]$ such that g and h are monic polynomials.

Proof. Lift $\overline{g}$ and $\overline{h}$ to monic polynomials $g_1, h_1 \in A[X]$. We inductively construct a sequence of monic polynomials $g_n, h_n \in A[X]$ such that $\Delta_n = f - g_n h_n \in \mathfrak{m}^n[X]$ and $g_n - g_{n+1}, h_n - h_{n+1} \in \mathfrak{m}^n[X]$ for all $n \geq 1$. Suppose that g_n and h_n are constructed. Let $g_{n+1} = g_n + \varepsilon_n$ and $h_{n+1} = h_n + \eta_n$ for $\varepsilon_n, \eta_n \in \mathfrak{m}^n[X]$. Then we have

$$f - g_{n+1}h_{n+1} = \Delta_n - (\varepsilon_n h_n + \eta_n g_n) + \varepsilon_n\eta_n.$$

Hence we just need to choose ε_n and η_n such that

$$\varepsilon_n h_n + \eta_n g_n \equiv \Delta_n \mod \mathfrak{m}^{n+1}, \quad \deg \varepsilon_n < \deg g_n, \quad \deg \eta_n < \deg h_n.$$

Since $\gcd(\overline{g},\overline{h}) = 1$, there exist $\overline{u},\overline{v} \in \mathbf{k}[X]$ such that $\overline{ug} + \overline{vh} = 1$ and $\deg\overline{u} < \deg\overline{g}$, $\deg\overline{v} < \deg\overline{h}$. Lift $\overline{u}$ and $\overline{v}$ to $u, v \in A[X]$ preserving the degrees. Then we have $ug_n + vh_n \equiv 1 \mod \mathfrak{m}$. Let $\varepsilon_n = u\Delta_n$ and $\eta_n = v\Delta_n$. Then we get the desired equation. ☐

Proposition A.6.21. Let $(A,\mathfrak{m},\mathbf{k})$ be a noetherian complete local ring and M an A-module that is $\mathfrak{m}$-adically separated. Suppose $\dim_{\mathbf{k}} M/\mathfrak{m}M < \infty$. Then the basis of $M \otimes_A \mathbf{k}$ as $\mathbf{k}$-vector space can be lifted to a generating set of M as an A-module.

Proof. Let $t_1,\dots,t_n \in M$ such that their images in $M/\mathfrak{m}M$ form a basis of $M/\mathfrak{m}M$ as a $\mathbf{k}$-vector space. Then $M = t_1A + \cdots + t_nA + \mathfrak{m}M$. For every $x \in M$, we can write

$$x = a_{0,1}t_1 + \cdots + a_{0,n}t_n + m_1$$

for some $a_{0,i} \in A$ and $m_1 \in \mathfrak{m}M$. Inductively, we have $\mathfrak{m}^kM = t_1\mathfrak{m}^k + \cdots + t_n\mathfrak{m}^k + \mathfrak{m}^{k+1}M$. Suppose that we have constructed $m_k \in \mathfrak{m}^kM$. Then we can write

$$m_k = a_{k,1}t_1 + \cdots + a_{k,n}t_n + m_{k+1}.$$

Note that $\sum_{k\geq 0} a_{k,i}$ converges in A, denote its limit by a_i. Then we have

$$x - a_1t_1 + \cdots + a_nt_n = \sum_{i=1}^{n}\sum_{r\geq k} a_{r,i}t_i + m_k \in \mathfrak{m}^kM$$

for all k. Since M is $\mathfrak{m}$-adically separated, $x = a_1t_1 + \cdots + a_nt_n$. It follows that $M = \sum At_i$. □

The key to prove the Cohen Structure Theorem is the existence of coefficient rings.

Definition A.6.22 (Coefficient rings)**.** Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring.
When A is equal-characteristic, the coefficient ring (or coefficient field) is a homomorphism of rings $\mathbf{k} \to A$ such that $\mathbf{k} \to A \to A/\mathfrak{m}$ is an isomorphism.
When A is mixed-characteristic, the coefficient ring is a complete local ring $(R, pR, \mathbf{k})$ with a local homomorphism of rings $R \hookrightarrow A$ such that the induced homomorphism $R/pR \to A/\mathfrak{m}$ is an isomorphism.

Remark A.6.23. Recall that a homomorphism of local rings $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is said to be local if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Theorem A.6.24. Every noetherian complete local ring $(A, \mathfrak{m}, \mathbf{k})$ has a coefficient ring.

Assume the existence of coefficient rings, we can prove the Cohen Structure Theorem.

Proof of Cohen Structure Theorem. Let R be a coefficient ring of A and $\mathfrak{m} = (f_1, \dots, f_d)$ a minimal generating set of $\mathfrak{m}$. Then we have a homomorphism of complete local rings

$$\Phi : R[[X_1, \dots, X_d]] \to A, \quad X_i \mapsto f_i.$$

Let $\mathfrak{n}$ be the maximal ideal of $R[[X_1, \dots, X_d]]$. Then $\mathfrak{n}A = \mathfrak{m}$. By Proposition A.6.21, A is generated by 1 as an $R[[X_1, \dots, X_d]]$-module. This implies that Φ is surjective and (a) follows.

If A is regular of equal characteristic, then $\mathfrak{m}$ is generated by a regular sequence. By consider the dimension of $R[[X_1, \dots, X_d]]$ and A, we have that Φ is an isomorphism. This proves (b).

Note that if A is regular of mixed characteristic $(0, p)$ and $p \notin \mathfrak{m}^2$, then $\mathfrak{m}$ is generated by $p, f_1, \dots, f_{d-1}$. Then consider the homomorphism of complete local rings

$$R[[X_1, \dots, X_{d-1}]] \to A, \quad X_i \mapsto f_i.$$

By the same argument as above, we have that it is an isomorphism. This proves (c).

For (d), we have that $\ker \Phi$ is of height 1 by the dimension argument. Since regular local rings are UFDs, we can write $\ker \Phi = (f)$ for some $f \in R[[X_1, \dots, X_d]]$. Then we finish. □

Existence of coefficient rings

Proof of Theorem A.6.24 in characteristic 0. Note that for any $n \in \mathbb{Z}$, $n \notin \mathfrak{m}$. Hence $\mathbb{Q} \subset A$. Let $\Sigma := \{\text{subfield in } A\}$ and K a maximal element in Σ with respect to the inclusion. The set Σ is non-empty since $\mathbb{Q} \in \Sigma$. By Zorn's Lemma, K exists. Then K is a subfield of $\mathbf{k}$ by $K \hookrightarrow A \twoheadrightarrow A/\mathfrak{m} \cong \mathbf{k}$. We claim that K is a coefficient field of A.

Suppose there is $\bar{t} \in \mathbf{k} \setminus K$. If $\bar{t}$ is transcendent over K, lift $\bar{t}$ to an element $t \in A$. Then for any polynomial $f \neq 0 \in K[T]$, we have $f(\bar{t}) \neq 0 \in \mathbf{k}$. Hence $f(t) \notin \mathfrak{m}$. This implies that $1/f(t) \in A$, whence $K(t) \subset A$. This contradicts the maximality of K. If $\bar{t}$ is algebraic over K, let $f \in K[T]$ be the minimal polynomial of $\bar{t}$. Then f is irreducible in $K[T]$ and $f(\bar{t}) = 0$. Regard f as a polynomial in $A[T]$ by $K \hookrightarrow A$. Note that $\operatorname{char} A = 0$ implies that f is separable. By Hensel's Lemma (Theorem A.6.20), we can lift the root $\bar{t}$ to an element $t \in A$ such that $f(t) = 0$. Then $K(t)$ is a field extension of K and $K(t) \subset A$. This contradicts the maximality of K again. ☐

The same strategy does not work when $\operatorname{char} \mathbf{k} = p > 0$ since there might be inseparable extensions. To fix this, we need to introduce the notion of p-basis.

Definition A.6.25. Let $\mathbf{k}$ be a field of characteristic p. A finite set $\{t_1, \dots, t_n\} \subset \mathbf{k} \setminus \mathbf{k}^p$ is called *p-independent* if $[\mathbf{k}(t_1, \dots, t_n) : \mathbf{k}] = p^n$. A set $\Theta \subset \mathbf{k} \setminus \mathbf{k}^p$ is called a *p-independent* if its any finite subset is p-independent. A *p-basis* for $\mathbf{k}$ is a maximal p-independent set $\Theta \subset \mathbf{k} \setminus \mathbf{k}^p$.

By definition, we have that $\mathbf{k} = \mathbf{k}^p[\Theta]$ for any p-basis Θ of $\mathbf{k}$. For any $a \in \mathbf{k}$ and $\theta \in \Theta$, we can write a as a polynomial in Θ with coefficients in $\mathbf{k}^p$. The degree of θ in such polynomial representation is at most $p - 1$. Such polynomial representation is unique by definition of p-independence.

Applying the Frobenius map n times, we have that $\mathbf{k}^{p^n} = \mathbf{k}^{p^{n+1}}[\Theta^{p^n}]$. This follows that $\mathbf{k} = \mathbf{k}^{p^n}[\Theta]$ for all n. Moreover, for any $a \in \mathbf{k}$ and $\theta \in \Theta$, we can write a as a polynomial in Θ with coefficients in $\mathbf{k}^{p^n}$ and the degree of θ is at most $p^n - 1$. Such polynomial representation is unique.

Let $\mathbf{k}$ be a perfect field of characteristic p. If there is $a \in \mathbf{k} \setminus \mathbf{k}^p$, then $\mathbf{k}(a^{1/p})/\mathbf{k}$ is an inseparable extension. This contradicts the perfectness of $\mathbf{k}$. Hence $\mathbf{k} = \mathbf{k}^p$ and $\mathbf{k}$ has no nonempty p-basis.

Example A.6.26. Let $\mathbf{k} = \mathbb{F}_p(t_1, \dots, t_n)$. Then $\mathbf{k}^p = \mathbb{F}_p(t_1^p, \dots, t_n^p)$. The set $\{t_1, \dots, t_n\}$ is a p-basis for $\mathbf{k}$.

Proof of Theorem A.6.24 in characteristic p. Choose $\Theta \subset A$ such that its image in $A/\mathfrak{m}$ is a p-basis for $\mathbf{k}$. Let $A_n := A^{p^n} = \{a^{p^n} : a \in A\}$ and $K := \bigcap_{n \geq 0}(A_n[\Theta])$. Then we claim that K is a coefficient field of A.

First we show that $A_n[\Theta] \cap \mathfrak{m} \subset \mathfrak{m}^{p^n}$. For every $a \in A_n[\Theta]$, if the degree of θ in the polynomial representation of a is more than $p^n - 1$, we can write $\theta^k = \theta^{ap^n} \cdot \theta^b$ for some $b < p^n$. Regard $\theta^{ap^n} \in A^{p^n}$ as coefficients. Now assume that $a \in A_n[\Theta] \cap \mathfrak{m}$. Then consider the image of a in $A/\mathfrak{m}$. The image of a equals 0 implies every coefficient of a is in $\mathfrak{m}$. Such coefficients are of form b^{p^n} for some $b \in A$, whence $b \in \mathfrak{m}$. Hence $a \in \mathfrak{m}^{p^n}$. This implies that $K \cap \mathfrak{m} = \bigcap_{n \geq 0}(A_n[\Theta] \cap \mathfrak{m}) \subset \bigcap_{n \geq 0} \mathfrak{m}^{p^n} = \{0\}$. Then K is a field and hence a subfield of $\mathbf{k}$.

For any $\bar{a} \in \mathbf{k}$, note that $\mathbf{k} = \mathbf{k}^p[\overline{\Theta}] = \mathbf{k}^{p^2}[\overline{\Theta}] = \cdots = \mathbf{k}^{p^n}[\overline{\Theta}] = \cdots$. For every n, write

$$\bar{a} = \sum_{\mu_n} \bar{c}_{\mu_n}^{p^n} \mu_n =: P_{\bar{a},n}(\bar{c}_{\mu_n}),$$

where μ_n runs over all monomials in $\overline{\Theta}$ with degree at most $p^n - 1$ and $\bar{c}_{\mu_n} \in \mathbf{k}$. We call this representation the p^n-development of $\bar{a}$ with respect to $\overline{\Theta}$. Plug the p^m-development of c_{μ_n} into $P_{\bar{a},n}$, we get the p^{n+m}-development of $\bar{a}$. In formula, that is,

$$P_{\bar{a},n}(P_{\bar{a},m}(\bar{c}_{\mu_{n+m}})) = P_{\bar{a},n+m}(\bar{c}_{\mu_{n+m}}).$$

Lift $\bar{c}_{\mu_n}$ to $c_{\mu_n} \in A$ for all μ_n. Let $a_n := P_{\bar{a},n}(c_{\mu_n}) = \sum_{\mu_n} c_{\mu_n}^{p^n} \mu_n \in A_n[\Theta]$. For $m \geq n$, we have $a_n - a_m \in A_n[\Theta] \cap \mathfrak{m} \subset \mathfrak{m}^{p^n}$. Hence a_n converges to an element $a \in A$. Now we show that $a \in K$.

For every μ_k, let $b_{\mu_k,n} \in A$ be the element getting by plugging $c_{\mu_{n+k}}$ into the $P_{\overline{c}_{\mu_k},n}$. Then $b_{\mu_k,n}$ converges to an element $b_{\mu_k} \in A$. By construction, we have

$$a = \lim_{n\to\infty} P_{\overline{a},n+k}(c_{\mu_{n+k}}) = \lim_{n\to\infty} P_{\overline{a},k}(b_{\mu_k,n}) = P_{\overline{a}}(b_{\mu_k}) = \sum_{\mu_k} b_{\mu_k}^{p^k}\mu_k \in A_k[\Theta], \quad \forall k.$$

It follows that $a \in K$. □

Lemma A.6.27. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring of mixed characteristic. Suppose that $\mathfrak{m}^n = 0$ for some $n \geq 1$. Then there exists a complete local ring $(R, pR, \mathbf{k})$ with $R \subset A$.

Proof. Fix a p-basis of $\mathbf{k}$ and lift it to $\Theta \subset R$. Let $q = p^{n-1}$ and

$$m := \left\{\theta_1^{k_1} \cdots \theta_d^{k_d} \mid \theta_i \in \Theta, k_i \leq q-1\right\}, \quad S := \left\{ \sum_{\mu\in m,\text{ finite}} a_\mu \mu \,\middle|\, a_\mu \in R^q \right\}.$$

For any $a, b \in A$, we claim that $a \equiv b \mod \mathfrak{m}$ if and only if $a^q \equiv b^q \mod \mathfrak{m}^n$. If $a \equiv b \mod \mathfrak{m}$, write $a = b + m$ for some $m \in \mathfrak{m}$. Then $a^p = b^p + pb^{q-1}m + \cdots + m^q$. Hence $a^p \equiv b^p \mod \mathfrak{m}^2$. Inductively, we have $a^q \equiv b^q \mod \mathfrak{m}^n$. Conversely, if $a^q \equiv b^q \mod \mathfrak{m}^n$, then $a^q - b^q \in \mathfrak{m}^n \subset \mathfrak{m}$. Note that the Frobenius map $x \mapsto x^q$ is injective on $A/\mathfrak{m}$. It follows that $a \equiv b \mod \mathfrak{m}$. By the claim, S maps to $\mathbf{k}^q[\Theta] = \mathbf{k}$ bijectively.

Let

$$R := S + pS + p^2 S + \cdots + p^{n-1}S.$$

We claim that R is a subring of A. If so, $R/pR \cong \mathbf{k}$ and we get a complete local ring $(R, pR, \mathbf{k})$.

Take $a, b \in A$. We have

$$a^q + b^q = (a+b)^q + pc \in A^q + pA.$$

Inductively, we have

$$a^q + b^q \in A^q + pA^q + \cdots + p^{n-1}A^q.$$

This implies that R is closed under addition. Note that $\theta^a = \theta^{aq} \cdot \theta^b$ with $b < q$. Then for any $\mu, \nu \in m$, we have $\mu\nu \in S$. Hence R is closed under multiplication. □

Lemma A.6.28. Let $\mathbf{k}$ be a field of characteristic p. Then there exists a DVR $(D, p, \mathbf{k})$ of mixed characteristic $(0, p)$.

Proof. Fix a well order $\leq$ on $\mathbf{k}$ and for any $a \in \mathbf{k}$, set $\mathbf{k}_a$ be the subfield of $\mathbf{k}$ generated by all elements $b \in \mathbf{k}$ such that $b \leq a$. Then $\mathbf{k} = \bigcup_{a\in\mathbf{k}} \mathbf{k}_a$. We construct DVRs D_a with residue field $\mathbf{k}_a$ such that $D_a \subset D_b$ for $a \leq b$. Begin from $\mathbf{k}_0 = \mathbb{F}_p$ and let $D_0 = \mathbb{Z}_{(p)}$. Suppose that D_a is constructed for all $a < b$. If $\mathbf{k}_b/\mathbf{k}_a$ is transcendental, then let D_b be the localization of $D_a[b]$ at the prime ideal generated by p.

If $\mathbf{k}_b/\mathbf{k}_a$ is algebraic, then let $\overline{f} \in \mathbf{k}_a[T]$ be the monic minimal polynomial of b. Let $\mathbf{K}_a = \mathrm{Frac}(D_a)$ and $K_b = \mathbf{K}_a[T]/(f)$, where f is a monic lift of $\overline{f}$ to $D_a[T]$. Note that f is irreducible since

$\overline{f}$ is irreducible. Let D_b be the integral closure of D_a in K_b. In general, D_b is a Dedekind domain. Consider the prime factorization $pD_b = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_k^{e_k}$ in D_b. For every i, $D_b/\mathfrak{p}_i$ is a field extension of $\mathbf{k}_a$ and $\overline{f}$ has a root in $D_b/\mathfrak{p}_i$. Suppose $\deg \overline{f} = \deg f = d$. It follows that $[(D_b/\mathfrak{p}_i) : \mathbf{k}_a] = d$. Note that we have $\sum_{i=1}^k e_i f_i = [\mathbf{K}_b : \mathbf{K}_a] = d$. Hence $k = 1$ and $e_1 = 1$. It follows that pD_b is prime and D_b is a DVR with residue field $\mathbf{k}_b$.

Let $D = \bigcup_{a \in \mathbf{k}} D_a$. Then $(D, pD, \mathbf{k})$ is the desired DVR. □

Example A.6.29. Let $\mathbf{k} = \mathbb{f}_p(t)$. Then $D = \mathbb{z}[t]_{(p)}$ is a DVR satisfying the condition in Lemma A.6.28.

Let $\mathbf{k} = \overline{\mathbb{f}_p}$. For any $n \geq 1$, let $K_n = K_{n-1}(\zeta_{p^n-1})$ and $K_0 = \mathbb{q}$. Let $D_n := \mathscr{O}_{K_n, \mathfrak{p}_n}$ be the localization of the ring of integers of K_n at the prime $\mathfrak{p}_n$ lying above $\mathfrak{p}_{n-1}$. Then $D := \bigcup_n D_n$ is a DVR with residue field $\mathbf{k}$.

Lemma A.6.30. Given $\mathbf{k}$ a field of characteristic p, there exists a unique complete local ring $(R, pR, \mathbf{k})$ of mixed characteristic (p^n, p).

Proof. The existence follows from Lemma A.6.28. To show the uniqueness, suppose that $(R', pR', \mathbf{k})$ is another complete local ring of mixed characteristic (p^n, p). Fix a p-basis of $\mathbf{k}$ and lift it to $\Theta \subset R$ and $\Theta' \subset R'$ relatively. Let $q = p^{n-1}$ and

$$m := \left\{ \theta_1^{k_1} \cdots \theta_d^{k_d} \mid \theta_i \in \Theta, k_i \leq q-1 \right\}, \quad S := \left\{ \sum_{\mu \in m, \text{ finite}} a_\mu \mu \middle| a_\mu \in R^q \right\}.$$

Define m', S' similarly with Θ' and R'. Since $S \to R \to \mathbf{k}$ and $S' \to R' \to \mathbf{k}$ are bijections, we can define a bijective map $\Phi : S \to S'$.

Note that any element in S can be written as $s + pr$ with $s \in S$ and $r \in R$ uniquely since $S \to \mathbf{k}$ is bijective. Inductively, we can write any element in R as

$$r = s + ps_1 + p^2 s_2 + \cdots + p^{n-1} s_{n-1},$$

where $s_i \in S$. The similarly for R'. Extend Φ to R and we get a bijection between R and R'. Note that by construction, Φ preserves addition and multiplication. Hence we get a ring isomorphism $\Phi : R \to R'$. □

Proof of Theorem A.6.24 in mixed characteristic. Since A is complete, we have $A = \varprojlim_n A/\mathfrak{m}^n$. By Lemma A.6.27, there is a complete local ring $(R_n, pR_n, \mathbf{k})$ with $R_n \subset A/\mathfrak{m}^n$. By Lemma A.6.30, such R_n is unique up to isomorphism. It follows that $R_n \cong R_m/p^{k_n}$ for $m \geq n$. We get an inverse system

$$\cdots \to R_n \to R_{n-1} \to \cdots \to R_1 \cong \mathbf{k}.$$

Let $R := \varprojlim_n R_n$. Then $(R, pR, \mathbf{k})$ is a complete local ring. The homomorphisms $R_n \hookrightarrow A/\mathfrak{m}^n$ induce a homomorphism of complete local rings $R \hookrightarrow A$. This concludes the proof. □

Appendix B

Homological Algebra

B.1 Complexes and Homology

Definition B.1.1. Let $A_\bullet$ and $B_\bullet$ be two complexes in $\mathcal{A}$ and $\varphi_\bullet, \psi_\bullet : A_\bullet \to B_\bullet$ be two morphisms of complexes. A *homotopy* between $\varphi_\bullet$ and $\psi_\bullet$ is a collection of morphisms $h_n : A_n \to B_{n-1}$ such that

$$\varphi_n - \psi_n = \mathrm{d}_{B_{n+1}} \circ h_n + h_{n-1} \circ \mathrm{d}_{A_n}.$$

In diagram, we have

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r, "\mathrm{d}_{A_n}"] \arrow[dl, "h_n"'] \arrow[d, shift right, "\psi_n"'] \arrow[d, shift left, "\varphi_n"] & A_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] & \cdots . \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\mathrm{d}_{B_{n+1}}"] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

B.2 Derived Functors

In this section, fix an abelian category $\mathcal{A}$.

B.2.1 Resolution

Definition B.2.1 (Resolution)**.** Let $A \in \mathcal{A}$. A *projective resolution* (resp. *flat resolution*, *free resolution*) of A is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0,$$

where each P_i is a projective (resp. flat, free) object in $\mathcal{A}$.
An *injective resolution* of A is an exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots \to I^n \to \cdots,$$

where each I^i is an injective object in $\mathcal{A}$.

Proposition B.2.2. Let $P_\bullet : \cdots \to P_1 \to P_0 \to A \to 0$ and $Q_\bullet : \cdots \to Q_1 \to Q_0 \to B \to 0$ be complexes in $\mathcal{A}$ such that P_i is projective and $Q_\bullet$ is exact. Given a morphism $f : A \to B$, there exists a morphism of complexes $f_\bullet : P_\bullet \to Q_\bullet$ such that $f_0 = f$. In particular, any two such morphism of complexes are homotopic.
Dually, let $I^\bullet : 0 \to A \to I^0 \to I^1 \to \cdots$ and $J^\bullet : 0 \to B \to J^0 \to J^1 \to \cdots$ be complexes in $\mathcal{A}$ such that J^i is injective and $I^\bullet$ is exact. Given a morphism $f : A \to B$, there exists a morphism of complexes $f^\bullet : I^\bullet \to J^\bullet$ such that $f^0 = f$. In particular, any two such morphism of complexes are homotopic.

Proof. Yang: To be completed. □

Definition B.2.3. For an object $A \in \mathcal{A}$, the *projective dimension* of A, denoted $\operatorname{proj.dim} A$, is the smallest integer n such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

of A of length n. If no such n exists, we set $\operatorname{proj.dim} A = \infty$.
Dually, the *injective dimension* of A, denoted $\operatorname{inj.dim} A$, is the smallest integer n such that there exists an injective resolution

$$0 \to A \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to 0$$

of A of length n. If no such n exists, we set $\operatorname{inj.dim} A = \infty$.

B.3 Applications to Commutative Algebra

B.3.1 Homological dimension

Lemma B.3.1. Let A be a ring and M an A-module. Then

$$\sup_M \operatorname{proj.dim} M = \sup_N \operatorname{inj.dim} N.$$

Proof. Note that
$$\operatorname{proj.dim} M \leq n$$
if and only if
$$\operatorname{Ext}^A_{n+1}(M, N) = 0, \quad \forall N.$$
And this is equivalent to
$$\operatorname{inj.dim} N \leq n.$$
□

Remark B.3.2. In fact, for fix N, we have
$$\operatorname{inj.dim} N \leq n$$
if and only if
$$\operatorname{Ext}^A_{n+1}(A/I, N) = 0, \quad \forall I$$
By Lemma Yang: ?. Hence we have
$$\sup_{M \text{ finite}} \operatorname{proj.dim} M = \sup_M \operatorname{proj.dim} M = \sup_N \operatorname{inj.dim} N.$$

Definition B.3.3. Let A be a ring. The *homological dimension* of A, denoted $\operatorname{hl.dim} A$, is defined as
$$\operatorname{hl.dim} A := \sup_M \operatorname{proj.dim} M = \sup_M \operatorname{inj.dim} M.$$

Lemma B.3.4. Let A be a noetherian ring, B a flat A-algebra and M a finite A-module. Then we have
$$\operatorname{Ext}^i_A(M, N) \otimes B \cong \operatorname{Ext}^i_B(M \otimes B, N \otimes M), \quad \forall N.$$

Proof. Yang: To be completed. □

Proposition B.3.5. Let A be a noetherian ring. Then
$$\operatorname{hl.dim} A = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{hl.dim} A_{\mathfrak{p}}.$$

Proof. Compute homological dimension of A using $\operatorname{Ext}^i_A(M, N)$ for finite M. The conclusion follows from Propostion B.3.4. □

Definition B.3.6. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring. We say that a homomorphism of A-modules $f : M \to N$ is *minimal* if the induced map $M \otimes \mathbf{k} \to N \otimes \mathbf{k}$ is an isomorphism. Equivalently, f is minimal if and only if f is surjective and $\operatorname{Ker} f \subset \mathfrak{m}M$.

Definition B.3.7. Let A be a noetherian local ring and M a finite A-module. A *minimal projective* *resolution* of M is a projective resolution

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

such that each homomorphism $P_i \to \operatorname{Ker} d_{i-1}$ is minimal.

Proposition B.3.8. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring and M a finite A-module. Then M has a minimal projective resolution. Moreover, any two minimal projective resolutions of M are isomorphic.

Proof. Suppose $M \otimes_A \mathbf{k} = \bigoplus \mathbf{k} \cdot \overline{x_i}$. Lift x_i to elements of M. Then we have a minimal homomorphism $\mathrm{d}_0 : \bigoplus A \cdot x_i \to M$. Similarly choose minimal homomorphisms $\mathrm{d}_k : A^{n_i} \to \operatorname{Ker} \mathrm{d}_{i-1}$ for $i = 1, 2, \cdots$. This gives a minimal projective resolution.

Suppose we have two minimal homomorphism $f, g : A^n \to M$. After tensoring with $\mathbf{k}$, we have isomorphisms between $f \otimes \mathbf{k}$ and $g \otimes \mathbf{k}$. Lifting to A, we get an homomorphism $\varphi : f \to g$. Here homomorphism between f, g means a homomorphism $A^n \to A^n$ such that $f = g \circ \varphi$. The homomorphism φ is represented by a matrix T. We have $\det T \notin \mathfrak{m}$, whence φ is an isomorphism. □

Proposition B.3.9. Let $L_\bullet \to M$ be a minimal projective resolution and $P_\bullet$ be an arbitrary projective resolution of M. Then we have $P_\bullet \cong L_\bullet \oplus P'_\bullet$ for some exact complexes $P'_\bullet$.

Proof. By Propostion B.2.2, we have homomorphism

$$L_\bullet \xrightarrow{\varphi_\bullet} P_\bullet \xrightarrow{\psi_\bullet} L_\bullet.$$

between complexes. By Propostion B.2.2 again, $T_\bullet := \psi_\bullet \circ \varphi_\bullet$ is homotopic to the identity by $h_\bullet$. Suppose $T_\bullet$ is represented by a matrix. Since $L_\bullet$ is minimal, we have

$$(T - \mathrm{id})(L_n) = (\mathrm{d}_{n+1} \circ h_n + h_{n-1} \circ \mathrm{d}_n)(L_n) \subset \mathfrak{m} L_n.$$

Then $\det T \notin \mathfrak{m}$ and hence $T_\bullet$ is an isomorphism. It follows that $\psi_\bullet$ is surjective, whence it splits $P_\bullet$ into a direct sum $L \oplus P'_\bullet$ since $L_\bullet$ is projective. By the Five Lemma, we see that $P'_\bullet$ is exact. □

Lemma B.3.10. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring and M a finite A-module. Then $\operatorname{proj.dim} M \leq n$ if and only if $\operatorname{Tor}^A_{n+1}(M, \mathbf{k}) = 0$.

Proof. The necessity is clear. For the sufficiency, we have a minimal projective resolution

$$\cdots \to P_{n+1} \xrightarrow{\mathrm{d}_{n+1}} P_n \xrightarrow{\mathrm{d}_n} P_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \cdots \to P_1 \xrightarrow{\mathrm{d}_1} P_0 \xrightarrow{\mathrm{d}_0} M \to 0.$$

Let $C := \Im \mathrm{d}_n$. Then we have

$$0 \to P_{n+1} \xrightarrow{\mathrm{d}_{n+1}} P_n \xrightarrow{\mathrm{d}_n} C \to 0.$$

Hence $\operatorname{Tor}^A_1(C, \mathbf{k}) \cong \operatorname{Tor}^A_{n+1}(M, \mathbf{k}) = 0$. Let $K = \operatorname{Ker} \mathrm{d}_n$. Then we have the short exact sequence

$$0 \to K \to P_n \to C \to 0.$$

Since $\mathrm{Tor}_1^A(C, \mathbf{k}) = 0$, there is an exact sequence

$$0 \to K \otimes_A \mathbf{k} \to P_n \otimes_A \mathbf{k} \to C \otimes_A \mathbf{k} \to 0.$$

Since $P_n \to C$ is minimal, we have $K \otimes_A \mathbf{k} = 0$. By the Nakayama's lemma, $K = 0$. This implies that $\mathrm{proj.dim}\, C \leq 0$ and hence $\mathrm{proj.dim}\, M \leq n$. □

Proposition B.3.11. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring. Then $\mathrm{hl.dim}\, A = \mathrm{proj.dim}\, \mathbf{k}$ (finite or infinite).

Proof. The inequality $\mathrm{hl.dim}\, A \geq \mathrm{proj.dim}\, \mathbf{k}$ is by definition. Conversely, we can compute $\mathrm{Tor}_{n+1}^A(M, \mathbf{k})$ using minimal projective resolution of $\mathbf{k}$ for any finite A-module M. By Lemma B.3.10, we have $\mathrm{proj.dim}\, M \leq n$ if and only if $\mathrm{Tor}_{n+1}^A(M, \mathbf{k}) = 0$. This implies that $\mathrm{proj.dim}\, M \leq n$ for all finite A-modules M if $\mathrm{proj.dim}\, \mathbf{k} = n$. By Remark B.3.2, we have $\mathrm{hl.dim}\, A \leq n$. □

Proposition B.3.12. Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finite A-module. Let $a \in \mathfrak{m}$ be an M-regular element. Then $\mathrm{proj.dim}\, M/aM = \mathrm{proj.dim}\, M + 1$. Here we set $\infty + 1 = \infty$.

Proof. We have an exact sequence

$$0 \to M \xrightarrow{*a} M \to M/aM \to 0.$$

Take the long exact sequence with respect to $\mathrm{Tor}(-, \mathbf{k})$, we get

$$\cdots \to \mathrm{Tor}_{i+1}^A(M, \mathbf{k}) \to \mathrm{Tor}_{i+1}^A(M/aM, \mathbf{k}) \to \mathrm{Tor}_i^A(M, \mathbf{k}) \xrightarrow{*a} \mathrm{Tor}_i^A(M, \mathbf{k}) \to \cdots$$

Since the derived homomorphism of $*a$ is zero, we have $\mathrm{Tor}_{i+1}^A(M/aM, \mathbf{k}) = 0$ if and only if $\mathrm{Tor}_i^A(M, \mathbf{k}) = 0$. By Lemma B.3.10, we have $\mathrm{proj.dim}\, M/aM = \mathrm{proj.dim}\, M + 1$. □

B.3.2 Depth and regularity by homological algebra

Proposition B.3.13. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring and M a finite A-module. Then

$$\mathrm{depth}\, M := \inf\{i : \mathrm{Ext}_A^i(\mathbf{k}, M) \neq 0\}.$$

Proof. Let $a \in \mathfrak{m}$ be M-regular and $N = M/aM$. Then we claim that

$$\inf\{i : \mathrm{Ext}_A^i(\mathbf{k}, N) \neq 0\} = \inf\{i : \mathrm{Ext}_A^i(\mathbf{k}, M) \neq 0\} - 1.$$

Indeed, we have an exact sequence

$$0 \to M \xrightarrow{a} M \to N \to 0.$$

It induces a long exact sequence

$$\cdots \to \mathrm{Ext}_A^{i-1}(\mathbf{k}, M) \to \mathrm{Ext}_A^{i-1}(\mathbf{k}, N) \to \mathrm{Ext}_A^i(\mathbf{k}, M) \xrightarrow{\mathrm{Ext}_A^i(\mathbf{k}, \mathrm{Mult}_a)} \mathrm{Ext}_A^i(\mathbf{k}, M) \to \cdots.$$

Note that $a \in \mathfrak{m}$, then $\mathrm{Ext}_A^i(\mathbf{k}, \mathrm{Mult}_a) = 0$. It follows that when $\mathrm{Ext}_A^{i-1}(\mathbf{k}, M) = 0$, we have $\mathrm{Ext}_A^{i-1}(\mathbf{k}, N) = 0$ iff $\mathrm{Ext}_A^i(\mathbf{k}, M) = 0$, whence the claim.

Let $n = \inf\{i : \mathrm{Ext}_A^i(\mathbf{k}, M) \neq 0\}$. Induct on n. Suppose first $n = 0$. Since $\mathbf{k}$ is a simple A-module, there is an injective homomorphism $\mathbf{k} \to M$. Then $\mathfrak{m} \in \mathrm{Ass}\, M$ and hence $\operatorname{depth} M = 0$.

Suppose $n > 0$., let $a_1, \cdots, a_m \in \mathfrak{m}$ be any M-regular sequence. Using the claim inductively on $M/(a_1, \cdots, a_m)M$, we have $n \geq$ depth. If M has no regular element, then $\mathfrak{m} \subset \bigcup_{\mathfrak{p} \in \mathrm{Ass}\, M} \mathfrak{p}$. Then $\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}\, M$. This show that we can find $x \neq 0 \in M$ such that $\mathfrak{p} = \mathrm{Ann}\, x$. It gives a homomorphism $\mathbf{k} = A/\mathfrak{m} \to M$. That is a contradiction and hence M has a regular element. Let a be M-regular and $N = M/aM$. Then $\operatorname{depth} N = n - 1$ by the claim and induction hypothesis. Hence we have $\operatorname{depth} M \geq n$. □

Lemma B.3.14. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring. Suppose we have exact sequences

$$0 \to A^{n_r} \xrightarrow{\mathrm{d}_r} A^{n_{r-1}} \xrightarrow{\mathrm{d}_{r-1}} \cdots \to A^{n_1} \xrightarrow{\mathrm{d}_1} A^{n_0},$$

such that $A^{n_i} \to \mathrm{Ker}\, \mathrm{d}_{i-1}$ is minimal for all i. Then $\operatorname{depth} A \geq r$.

Proof. Since d_r is injective and its image is contained in $\mathfrak{m}A^{n_{r-1}}$, we can choose $t \in \mathfrak{m}$ that is not a zero divisor. Denote the sequence by $C_\bullet$. Then we have a short exact sequence of complexes

$$0 \to C_\bullet \xrightarrow{*t} C_\bullet \to C_\bullet/tC_\bullet \to 0.$$

Consider the long exact sequence in homology

$$\cdots \to H_i(C_\bullet) \xrightarrow{*t} H_i(C_\bullet) \to H_i(C_\bullet/tC_\bullet) \to H_{i-1}(C_\bullet) \xrightarrow{*t} H_{i-1}(C_\bullet) \to \cdots.$$

Since $C_\bullet$ is exact, we have $H_i(C_\bullet) = 0$ for all i. In particular, $H_i(C_\bullet/tC_\bullet) = 0$ for all $i \geq 2$. Inductively, we can choose a regular sequence of length r in $\mathfrak{m}$. □

Lemma B.3.15. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring and M a finite A-module. Suppose there is an injective homomorphism $\mathbf{k} \to M$. Then $\operatorname{proj.dim} M \geq \dim_{\mathbf{k}} T_{A,\mathfrak{m}}$.

Proof. Let $x_1, \cdots, x_n \subset \mathfrak{m} \backslash \mathfrak{m}^2$ such that their images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. Then we have a complex

$$K_\bullet := 0 \to \wedge^n A^{\oplus n} \xrightarrow{\mathrm{d}_n} \wedge^{n-1} A^{\oplus n} \xrightarrow{\mathrm{d}_{n-1}} \cdots \to \wedge^1 A^{\oplus n} \xrightarrow{\mathrm{d}_1} \wedge^0 A^{\oplus n} \xrightarrow{\mathrm{d}_0} \mathbf{k} \to 0,$$

where

$$\mathrm{d}_r : \wedge^r A^{\oplus n} \to \wedge^{r-1} A^{\oplus n}, \quad e_{i_1} \wedge \cdots \wedge e_{i_r} \mapsto \sum_{k=1}^{r} (-1)^k x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_r}.$$

Here $\widehat{e_{i_k}}$ means that we omit the k-th element. Let $P_\bullet \to M$ be the minimal projective resolution of M. Then we have a homomorphism of complexes

$$\varphi_\bullet : K_\bullet \to P_\bullet$$

induced by the injective homomorphism $\mathbf{k} \to M$.

We claim that φ_i is injective and splits P_i into a direct sum $K_i \oplus F_i$ with F_i free for all $i \geq 0$. Since K_i and P_i are free, we just need to show that $\varphi_i \otimes_A \mathrm{id}_{\mathbf{k}}$ is injective. Induct on i. For $i = 0$, note that $\mathbf{k} \to M \otimes_A \mathbf{k}$ is injective, by the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi_0 \otimes_A \mathrm{id}_{\mathbf{k}}"'] & \mathbf{k} \arrow[d] \\ P_0 \otimes_A \mathbf{k} \arrow[r, "\cong"] & M \otimes_A \mathbf{k} \end{tikzcd},$$

the image of $\varphi_0 \otimes_A \mathrm{id}_{\mathbf{k}}$ is not zero in $P_0 \otimes_A \mathbf{k}$.

For $i > 0$, since K_{i-1} and P_{i-1} are free, we have a natural isomorphism between

$$\mathfrak{m}K_{i-1} \otimes_A \mathbf{k} \to \mathfrak{m}P_{i-1} \otimes_A \mathbf{k}$$

and

$$K_{i-1} \otimes_A \mathfrak{m}/\mathfrak{m}^2 \to P_{i-1} \otimes_A \mathfrak{m}/\mathfrak{m}^2.$$

We have a commutative diagram

$$\begin{tikzcd} K_i \otimes_A \mathbf{k} \arrow[r] \arrow[d] & \mathfrak{m}K_{i-1} \otimes_A \mathbf{k} \arrow[d] \\ P_i \otimes_A \mathbf{k} \arrow[r] & \mathfrak{m}P_{i-1} \otimes_A \mathbf{k} \end{tikzcd}. \tag{B.1}$$

Since $P_{i-1}/K_{i-1} \cong F_{i-1}$ is free, the right vertical map in (B.1) is injective. By construction of $K_\bullet$, $K_i \otimes_A \mathbf{k} \to \mathfrak{m}K_{i-1} \otimes_A \mathbf{k}$ is injective. Hence the left vertical map in (B.1) is injective. This completes the proof of the claim.

By the claim, $P_i \neq 0$ for all $i \leq n$ and the conclusion follows. □

Proposition B.3.16 (Auslander-Buchsbaum formula)**.** Let A be a noetherian local ring and M a finite A-module. Suppose $\operatorname{proj.dim} M < \infty$. Then $\operatorname{proj.dim} M = \operatorname{depth} A - \operatorname{depth} M$.

Proof. We have a minimal projective resolution

$$0 \to A^{n_r} \to A^{n_{r-1}} \to \cdots \to A^{n_1} \to A^{n_0} \to M \to 0.$$

By Lemma B.3.14, we have $\operatorname{depth} A \geq \operatorname{proj.dim} M$.

Induct on $\operatorname{depth} M$. Suppose $\operatorname{depth} M = 0$. Then by Proposition B.3.13, we have $\operatorname{Hom}_A(\mathbf{k}, M) \neq 0$, whence there is an injective homomorphism $\mathbf{k} \to M$. By Lemma B.3.15, we have

$$\operatorname{depth} A \geq \operatorname{proj.dim} M \geq \dim_{\mathbf{k}} T_{A,\mathfrak{m}} \geq \operatorname{depth} A.$$

If $\operatorname{depth} M > 0$, choose a regular element $a \in \mathfrak{m}$ that is M-regular. Then by Propostion B.3.12, we have

$$\operatorname{depth} M + \operatorname{proj.dim} M = \operatorname{depth}(M/aM) - 1 + \operatorname{proj.dim}(M/aM) + 1 = \operatorname{depth} A.$$

□

Theorem B.3.17. Let $(A, \mathfrak{m})$ be a noetherian local ring. Then A is regular at $\mathfrak{m}$ if and only if $\operatorname{hl.dim} A < \infty$.

Proof. Suppose A is regular at $\mathfrak{m}$. Let $x_1, \cdots, x_n$ be a minimal generating set of $\mathfrak{m}$. Then $x_1, \cdots, x_n$ is an A-regular sequence since A is regular at $\mathfrak{m}$. By Proposition B.3.12, we have $\operatorname{proj.dim} \mathbf{k} = \operatorname{proj.dim} A/(x_1, \cdots, x_n)A = n + \operatorname{proj.dim} A = n$.

Conversely, suppose $\operatorname{hl.dim} A < \infty$. Then by Proposition B.3.11, we have $\operatorname{proj.dim} \mathbf{k} < \infty$. We have

$$\dim_{\mathbf{k}} T_{A,\mathfrak{m}} \leq \operatorname{proj.dim} \mathbf{k} \leq \operatorname{depth} A \leq \dim_{\mathbf{k}} T_{A,\mathfrak{m}}.$$

The first "≤" follows from Lemma B.3.15. The second "≤" follows from Proposition B.3.16. Hence we see that A is regular at $\mathfrak{m}$. □

Corollary B.3.18. Let $(A, \mathfrak{m})$ be a noetherian local ring. Then A is regular if and only if it is regular at $\mathfrak{m}$.

Proof. The sufficiency is trivial. For the necessity, note that if A is regular, then $\operatorname{hl.dim} A < \infty$ by Theorem B.3.17. For any $\mathfrak{p} \in \operatorname{Spec} A$, we have a finite projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A/\mathfrak{p} \to 0.$$

Tensoring with $A_\mathfrak{p}$, we have a finite projective resolution of $\kappa(\mathfrak{p})$. By Theorem B.3.17 again, we see that $A_\mathfrak{p}$ is regular at $\mathfrak{p}$. □

Lemma B.3.19. Let A be a noetherian integral domain. Then A is a UFD if and only if every height 1 prime ideal of A is principal.

Proof. Yang: To be completed. □

Lemma B.3.20. Let A be a noetherian integral domain and $(x) \subset A$ a non-zero prime ideal. Then A is a UFD if and only if $A[1/x]$ is a UFD.

Proof. Yang: To be completed. □

Theorem B.3.21. Let $A, \mathfrak{m}$ be a regular noetherian local ring. Then A is UFD.

Proof. Yang: To be completed. □

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