Algebraic stacks

1 Definitions

Conventions Throughout this section, we fix a base noetherian scheme S. All schemes are viewed as its associated functor of points over S. In other words, we work in the category $\mathrm{Fun}((\mathbf{Sch}/S)^{\mathrm{op}}, \mathbf{Grpd})$. On the base category $\mathbf{Sch}/S$, we consider the étale topology unless otherwise specified.

Definition 1. A morphism $f : X \to Y$ of stacks is said to be *representable (by schemes)* if for every morphism of schemes $U \to Y$, the fiber product $X \times_Y U$ is a scheme.

Definition 2. Let P be a property of morphisms of schemes which is stable under base change, for example, being étale, smooth, flat, surjective, etc. A representable morphism of stacks $f : X \to Y$ is said to *satisfy property* P if for every morphism of schemes $U \to Y$, the projection morphism $X \times_Y U \to U$ satisfies property P.

Definition 3. A *Deligne-Mumford stack* over S is a stack $\mathcal{X}$ over S such that

(a) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, and

(b) there exists a scheme U over S and an étale surjective morphism $U \to \mathcal{X}$.

Definition 4. An *algebraic stack* over S is a stack $\mathcal{X}$ over S such that

(a) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, and

(b) there exists a scheme U over S and a smooth surjective morphism $U \to \mathcal{X}$.

Construction 5. Let G be a group scheme over S acting on a scheme X over S via a morphism $\sigma : G \times_S X \to X$. The *quotient stack* $[X/G]$ is defined as following:

- For each scheme U over S, the objects of $[X/G](U)$ are pairs (P, f) where P is a G-torsor over U and $f : P \to X$ is a G-equivariant morphism over S.
- Morphisms between two objects (P, f) and (P', f') in $[X/G](U)$ are given by G-equivariant morphisms $\varphi : P \to P'$ over U such that $f' \circ \varphi = f$.

The assignment $U \mapsto [X/G](U)$ defines a stack over the site $(\mathbf{Sch}/S)_{\text{ét}}$. This stack captures the quotient of X by the action of G in a way that respects the group action and the torsor structure. Yang: To be added.

Example 6. Let $\Bbbk$ be a field. Consider the projective plane $\mathbb{P}^2_{\Bbbk}$ over $\Bbbk$ and all cubic curve $C \subseteq \mathbb{P}^2_{\Bbbk}$. Its moduli stack $\mathcal{M}$ of cubic curves is an algebraic stack. Yang: To be revised.

Appendix

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