Bend and Break

1 Preliminary

Definition 1 (Frobinius morphism)**.** Let X be a variety over a field $\mathbf{k}$ of characteristic $p > 0$. Denote the structure morphism by $\pi : X \to \operatorname{Spec} \mathbf{k}$. The *absolute Frobenius morphism* is the morphism given by $\mathcal{O}_X \to \mathcal{O}_X, f \mapsto f^p$, denoted by $\mathrm{Frob}_{X/\mathbb{F}_p}$. The *relative Frobenius morphism* is the morphism $\mathrm{Frob}_{X/\mathbf{k}}$ given by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{Frob}_{X/\mathbf{k}}"] \arrow[drr, bend left, "\mathrm{Frob}_{X/\mathbb{F}_p}"] \arrow[ddr, bend right, "\pi"'] & & \\
& X \times_{\mathbf{k}} \operatorname{Spec} \mathbf{k} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& \operatorname{Spec} \mathbf{k} \arrow[r, "\mathrm{Frob}_{\mathbf{k}/\mathbb{F}_p}"] & \operatorname{Spec} \mathbf{k}
\end{tikzcd}.$$

We usually denote $X \times_{\mathbf{k}} \operatorname{Spec} \mathbf{k}$ appearing above by $X^{(p)}$.

Proposition 2. Let X be a variety of dimension d over a field $\mathbf{k}$ of characteristic $p > 0$. Then the relative Frobenius morphism $\mathrm{Frob}_{X/\mathbf{k}} : X \to X^{(p)}$ is a finite morphism of degree p^d over $\mathbf{k}$.

2 Deformation of curves

Theorem 3 (ref. [Kol96, Chapter II, Theorem 1.2])**.** Let C be a smooth projective curve of genus g and X a smooth projective variety of dimension n. Let $f : C \to X$ be a non-constant morphism. Then every irreducible component of $\mathrm{Mor}(C, X)$ containing f has dimension at least

$$-K_Y \cdot f(C) + (1 - g)n.$$

Proposition 4. Let X be a projective variety and $f : C \to X$ a non-constant morphism from a pointed smooth projective curve $p_0 \in C$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t : C \to X$ for $t \in T$ such that $f_0 = f$ and $f_t(p_0) = x_0$ for some point $x_0 \in X$ and all t. Then there exist some rational curves $\Gamma_1, \ldots, \Gamma_m \subset X$ such that

(a) $x_0 \in \bigcup_{i=1}^m \Gamma_i$;

(b) there is a morphism $g : C \to X$ such that $f(C) \equiv_{alg} g(C) + \sum_{i=1}^m a_i \Gamma_i$ with $a_i > 0$ for all i.

Date: July 21, 2025, **Author**: Tianle Yang, My Website

Proposition 5. Let X be a projective variety and $f : \mathbb{P}^1 \to X$ a non-constant morphism with $f(0) = x_0, f(\infty) = x_\infty$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t : \mathbb{P}^1 \to X$ for $t \in T$ such that $f_0 = f$ and $f_t(0) = x_0, f_t(\infty) = x_\infty$ for all t. Then there exists a curve $C \subset X$ such that $f(\mathbb{P}^1) \equiv_{alg} aC$ with $a > 1$.

3 Find rational curves

Theorem 6. Let X be a smooth Fano variety. Then for any $x \in X(\mathbf{k})$, there is a rational curve C passing through x with

$$0 < -C \cdot K_X \leq \dim X + 1.$$

Proof. Yang: To be completed. □

Theorem 7. Let X be a smooth projective variety such that $K_X \cdot C < 0$ for some irreducible curve $C \subset X$. Let H be an ample divisor on X. Then there exists a rational curve Γ such that

$$-(K_X \cdot C) \cdot \frac{H \cdot \Gamma}{H \cdot C} \leq -K_X \cdot \Gamma \leq \dim X + 1.$$

Proof. Yang: To be completed. □

Theorem 8. Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Suppose that $K_{(X,B)}$ is f-ample. Then the exceptional locus of f is covered by rational curves Γ with

$$0 < -K_{(X,B)} \cdot \Gamma \leq 2 \dim X.$$

Theorem 9. Let X be a smooth projective variety of dimension n and $H, H_1, \cdots, H_{n-1}$ ample divisors on X. Suppose that $K_X \cdot H_1 \cdots H_{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H \cdot H_1 \cdots H_{n-1}}{K_X \cdot H_1 \cdots H_{n-1}}.$$

Proposition 10. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

References

[Kol96] János Kollár. *Rational Curves on Algebraic Varieties*. Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Berlin, Heidelberg:

Springer-Verlag, 1996, p. 320. ISBN: 978-3-540-60168-5. DOI: 10.1007/978-3-662-03276-3. URL: https://doi.org/10.1007/978-3-662-03276-3.