Schemes as functors

1 The functor of points

Let X be a scheme over a base scheme S. The *functor of points* of X is the functor $h_X(-) : (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ defined by $T \mapsto h_X(T) = \mathrm{Hom}_S(T, X)$.

When we say that $f(x) = y$ for $x \in X(T)$ and $y \in Y(T)$, we mean that the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "x"] \arrow[dr, "y"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

2 What is a scheme?

For a scheme X over S, we will often identify X with its functor of points h_X. In this way, we can think of a scheme as a functor from $(\mathbf{Sch}/S)^{\mathrm{op}}$ to $\mathbf{Set}$.

The underlying topological space of X can be recovered from the functor of points h_X as follows: The points of X correspond to the morphisms from the spectrum of a field to X.

The structure sheaf of X can also be recovered from the functor of points h_X.

Date: October 1, 2025, **Author**: Tianle Yang, My Homepage