Bend and Break

1 Preliminary

Definition 1 (Frobinius morphism)**.** Let X be a variety over a field $\mathbf{k}$ of characteristic $p > 0$. Denote the structure morphism by $\pi : X \to \operatorname{Spec}\mathbf{k}$. The *absolute Frobenius morphism* is the morphism given by $\mathcal{O}_X \to \mathcal{O}_X, f \mapsto f^p$, denoted by $\mathrm{Frob}_{X/\mathbb{F}_p}$. The *relative Frobenius morphism* is the morphism $\mathrm{Frob}_{X/\mathbf{k}}$ given by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{Frob}_{X/\mathbf{k}}"] \arrow[drr, bend left, "\mathrm{Frob}_{X/\mathbb{F}_p}"] \arrow[ddr, bend right, "\pi"'] & & \\
& X \times_{\mathbf{k}} \operatorname{Spec}\mathbf{k} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& \operatorname{Spec}\mathbf{k} \arrow[r, "\mathrm{Frob}_{\mathbf{k}/\mathbb{F}_p}"] & \operatorname{Spec}\mathbf{k}
\end{tikzcd}.$$

We usually denote $X \times_{\mathbf{k}} \operatorname{Spec}\mathbf{k}$ appearing above by $X^{(p)}$.

Proposition 2. Let X be a variety of dimension d over a field $\mathbf{k}$ of characteristic $p > 0$. Then the relative Frobenius morphism $\mathrm{Frob}_{X/\mathbf{k}} : X \to X^{(p)}$ is a finite morphism of degree p^d over $\mathbf{k}$.

2 Deformation of curves

Theorem 3 (ref. [Kol96, Chapter II, Theorem 1.2])**.** Let C be a projective smooth curve of genus g and X a smooth projective variety of dimension n. Let $f : C \to X$ be a non-constant morphism. Then every irreducible component of $\mathrm{Mor}(C, X)$ containing f has dimension at least

$$-K_Y \cdot f(C) + (1-g)n.$$

3 Find rational curves

Theorem 4. Let X be a smooth Fano variety. Then for any $x \in X(\mathbf{k})$, there is a rational curve C passing through x with

$$0 < -C \cdot K_X \leq \dim X + 1.$$

Theorem 5. Let X be a smooth projective variety such that $K_X \cdot C < 0$ for some irreducible curve $C \subset X$. Let H be an ample divisor on X. Then there exists a rational curve Γ such that

$$-(K_X \cdot C) \cdot \frac{H \cdot \Gamma}{H \cdot C} \leq -K_X \cdot \Gamma \leq \dim X + 1.$$

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References

[Kol96] János Kollár. *Rational Curves on Algebraic Varieties*. Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Berlin, Heidelberg: Springer-Verlag, 1996, p. 320. ISBN: 978-3-540-60168-5. DOI: 10.1007/978-3-662-03276-3. URL: https://doi.org/10.1007/978-3-662-03276-3.