Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$.

1 Preliminaries

Let S be a variety over $\Bbbk$ and $\mathcal{E}$ a vector bundle of rank $r+1$ on S.

Proposition 1. The S-varieties $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on S.

Theorem 2. Let $\pi : X = \mathbb{P}_S(\mathcal{E}) \to S$ be the projective bundle associated to a vector bundle $\mathcal{E}$ of rank $r+1$ on S. Then there is an exact sequence of vector bundles on $\mathbb{P}_S(\mathcal{E})$

$$0 \to \Omega_{\mathbb{P}_S(\mathcal{E})/S} \to \pi^*(\mathcal{E})(-1) \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \to 0.$$

In particular, $K_X \sim \pi^*(K_S + \det \mathcal{E}) - (r+1)\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. Yang: To be continued...

Theorem 3 (Tsen's Theorem, [Stacks, Tag 03RD])**.** Let C be a smooth curve over an algebraically closed field $\Bbbk$. Then $\mathbf{K} = \Bbbk(C)$ is a C_1 field, i.e., every degree d hypersurface in $\mathbb{P}^n_{\mathbf{K}}$ has a $\mathbf{K}$-rational point provided $d \leq n$.

Theorem 4 (Grauert's Theorem, [Har77, Corollary 12.9])**.** Let $f : X \to S$ be a projective morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf on X which is flat over S. Suppose that S is integral and the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$ is constant on S for some $i \geq 0$. Then $\mathrm{R}^i f_* \mathcal{F}$ is locally free and the base change homomorphism

$$\varphi^i_s : \mathrm{R}^i f_* \mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s) \to H^i(X_s, \mathcal{F}_s)$$

is an isomorphism for all $s \in S$.

Theorem 5 (Miracle Flatness, [Mat89, Theorem 23.1])**.** Let $f : X \to Y$ be a morphism of noetherian schemes. Assume that Y is regular and X is Cohen-Macaulay. If all fibers of f have the same dimension $d = \dim X - \dim Y$, then f is flat.

Proposition 6 (Geometric form of Nakayama's Lemma)**.** Let X be a variety, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proposition 7. Let S be a noetherian scheme and $\mathcal{E}$ a vector bundle of rank $r+1$ on S. Let X be

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a S-scheme via a morphism $g : X \to S$. Then there is a bijection

$$\{S\text{-morphisms } X \to \mathbb{P}_S(\mathcal{E})\} \leftrightarrow \left\{ \begin{array}{l} \text{surjective homomorphisms } g^*\mathcal{E} \to \mathcal{L} \\ \text{where } \mathcal{L} \text{ is a line bundle on } X \end{array} \right\}.$$

Yang: Need to check.

Proof. Take an affine cover $\{U_i\}$ of S such that $\mathcal{E}|_{U_i}$ is trivial. On U_i, the surjection $g^*\mathcal{E}|_{U_i} \twoheadrightarrow \mathcal{L}|_{X_{U_i}}$ gives a morphism $X_{U_i} \to \mathbb{P}_{U_i}(\mathcal{E}|_{U_i}) \cong \mathbb{P}_S(\mathcal{E})_{U_i}$ by Yang: ref. □

2 Minimal Section and Classification

Definition 8 (Ruled surface)**.** A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 9. There exists a section of π.

Proof. Yang: To be continued... □

Proposition 10. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Proof. Let $\sigma : C \to X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π, $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in C$. By Miracle Flatness (Theorem 5), f is flat. By Grauert's Theorem (Theorem 4), $\mathcal{E}$ is a vector bundle of rank 2 on C and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in C$.

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every $x \in X$, we have

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

Yang: The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \to \mathcal{L}$ is surjective.

Denote by $p : \mathbb{P}_C(\mathcal{E}) \to C$ the projection. Take an affine open cover $\{U_i\}$ of C such that $\mathcal{E}|_{U_i}$ is trivial. On U_i, the surjection $\pi^*\mathcal{E}|_{X_{U_i}} \twoheadrightarrow \mathcal{L}|_{X_{U_i}}$ gives a morphism $\varphi_i : X_{U_i} \to \mathbb{P}_{U_i}(\mathcal{E}|_{U_i}) \cong \mathbb{P}_C(\mathcal{E})_{U_i}$ by Yang: ref. Since φ_i and φ_j agree on $X_{U_i \cap U_j}$, they glue to give a morphism $\varphi : X \to \mathbb{P}_C(\mathcal{E})$ over C. Since $\varphi|_{X_t} : X_t \to \mathbb{P}_C(\mathcal{E})_t$ is an isomorphism for any $t \in C$, φ is □

Lemma 11. Fix a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$. There is a one-to-one correspondence between sections of π and quotient line bundles of $\mathcal{E}$ on $\mathcal{C}$.

Proof. Suppose we have a quotient $\mathcal{E} \to \mathcal{L} \to 0$ on C where $\mathcal{L}$ is a line bundle on C. By Proposition 7, we have a morphism $s : C \to \mathbb{P}_C(\mathcal{E})$ over C. Conversely, let $\sigma : C \to X$ be a section of π and D be its image. □

Lemma 12. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*.

Proof. □

Definition 13. A section C_0 of π is called a *minimal section* if Yang: to be continued...

Lemma 14. Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface over a smooth curve C of genus g with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on C with $\deg \mathcal{L} = -e$.

(b) If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.

Theorem 15. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Example 16. Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$. Yang: To be continued...

Theorem 17. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and for each e there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on E with $\deg \mathcal{L} = -e$.

Example 18. Yang: To be continued...

3 The Néron-Severi Group of Ruled Surfaces

Proposition 19. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let f be a fiber of π. Then $\operatorname{Pic}(X) \cong \mathbb{Z}C_0 \oplus \pi^* \operatorname{Pic}(C)$. Yang: Check this carefully.

Proof. Yang: To be continued... □

Proposition 20. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let f be a fiber of π. Then $K_X \sim -2C_0 + (K_C -)f$ where $e = -C_0^2$. Yang:

Check this carefully.

Proof. Yang: To be continued. ☐

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

Theorem 21. Let $\pi : X \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with invariant e. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \sim aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$;

(b) D is effective $\iff$ $a, b \geq 0$.

Proof. Yang: To be continued... ☐

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

Theorem 22. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is decomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$;

(b) D is effective $\iff$ $a \geq 0$ and $b \geq ae$.

Proof. Yang: To be continued... ☐

Theorem 23. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is indecomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > \frac{1}{2}ae$;

(b) D is effective $\iff$ $a \geq 0$ and $b \geq \frac{1}{2}ae$.

Proof. Yang: To be continued... ☐

References

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