Formal Completion

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1 Formal completion of rings and modules

Definition 1. Let A be a ring and $\mathcal{T}$ a topology on A. We say that $(A, \mathcal{T})$ is a *topological ring* if the operations of addition and multiplication are continuous with respect to the topology $\mathcal{T}$.
Given a topological ring A. A *topological A-module* is a pair $(M, \mathcal{T}_M)$ where M is an A-module and $\mathcal{T}_M$ is a topology on M such that the addition and scalar multiplication is continuous. The morphisms of topological A-modules are the continuous A-linear maps. They form a category denoted by $\mathbf{TopMod}_A$.

Definition 2. Let A be a ring and I an ideal of A. The *I-adic topology* on A is the topology defined by the basis of open sets $a + I^n$ for all $n \geq 0$.
A sequence $\{a_n\}$ in A is said to *converge to* $a \in A$ if for every n, there exists N such that for all $m \geq N$, we have $a_m - a \in I^n$.
A sequence $\{a_n\}$ in A is said to be *Cauchy* if for every n, there exists N such that for all $m, k \geq N$, we have $a_m - a_k \in I^n$.

Definition 3 (Formal Completion)**.** Let A be a ring and I an ideal of A. The *formal completion* of A with respect to I, denoted by $\widehat{A}$, is defined as

$$\widehat{A} := \varprojlim(\cdots \to A/I^n \to A/I^{n-1} \to \cdots \to A/I),$$

where the maps are the natural projections $A/I^n \to A/I^{n-1}$.
Let M be a A-module. The *formal completion* of M with respect to I, denoted by $\widehat{M}$, is defined as

$$\widehat{M} := \varprojlim(\cdots \to M/I^nM \to M/I^{n-1}M \to \cdots \to M/IM),$$

where the maps are the natural projections $M/I^nM \to M/I^{n-1}M$.

By the universal property of the inverse limit, we get a covariant functor from the category of A-modules to the category of $\widehat{A}$-modules, which sends an A-module M to $\widehat{M}$ and a morphism $f : M \to N$ to the induced morphism $\widehat{f} : \widehat{M} \to \widehat{N}$.

Lemma 4. The functor of completion with respect to an ideal is exact. Yang: finite.

Proof. Yang: To be completed. □

Proposition 5. The formal completion $\widehat{A}$ of a ring A with respect to an ideal I is a complete topological ring with respect to the I-adic topology. That is, every Cauchy sequence in $\widehat{A}$ converges to an element in $\widehat{A}$.

Yang: To be completed.

Lemma 6. Let $\widehat{A}$ be the formal completion of a noetherian ring A with respect to an ideal I. Suppose that I is generated by $a_1, ..., a_n$. Then we have an isomorphism of topological rings

$$\widehat{A} \cong A[[X_1, \dots, X_n]]/(X_1 - a_1, \cdots, X_n - a_n).$$

Proof. Yang: To be completed. □

Proposition 7. Let A be a noetherian ring and I an ideal of A. Then the formal completion $\widehat{A}$ of A with respect to I is a noetherian ring.

Proof. Yang: To be completed. □

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Proposition 8. Let A be a noetherian ring and I an ideal of A. Then the formal completion $\widehat{A}$ of A with respect to I is a flat A-module.

Proof. Yang: To be completed. □

Proposition 9. Let $\widehat{A}$ be completion of a noetherian ring A with respect to an ideal I and M a finite A-module. Then the natural map $M \otimes_A \widehat{A} \to \widehat{M}$ is an isomorphism.

Proof. Yang: To be completed. □

Theorem 10 (Artin-Rees Lemma)**.** Let A be a noetherian ring, I an ideal of A, M a finite A-module and N a submodule of M. Then there exists an integer N such that for all $n \geq 0$, we have

$$(I^{N+n}M) \cap N = I^n(I^N M \cap N).$$

Proof. Yang: To be completed. □

Proposition 11. Let A be a noetherian ring and $\mathfrak{m}$ a maximal ideal of A. Then the formal completion $\widehat{A}$ of A with respect to $\mathfrak{m}$ is a local ring with maximal ideal $\mathfrak{m}\widehat{A}$.

Proof. Yang: To be completed. □

2 Complete local rings

Definition 12 (Coefficient rings)**.**

Theorem 13 (Weierstrass Preparation Theorem)**.** Let $(A, \mathfrak{m})$ be a noetherian complete local ring, $f = \sum_{n=0}^{\infty} a_n X^n \in A[[X]]$ a power series with $f \not\equiv 0 \mod \mathfrak{m}$. Then there exists a unique factorization of the form $f = ug$, where u is a unit in $A[[X]]$ and g is a polynomial of the form

$$g = X^d + b_{d-1}X^{d-1} + \cdots + b_0,$$

where $b_i \in \mathfrak{m}$ for all i.

Theorem 14 (Hensel's Lemma)**.** Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian complete local ring,

Theorem 15 (Cohen Structure Theorem)**.** Let $A, \mathfrak{m}, \mathsf{k}$ be a noetherian complete local ring with coefficient field k. Then

(a) If A is regular of dimension d, then $A \cong \mathsf{k}[[X_1, \ldots, X_d]]$.

3 Unique factorization of regular local rings