Varieties in more general settings

1 Geometric properties

Definition 1. Let $\mathbf{k}$ be a field and X a separated scheme of finite type over $\operatorname{Spec}\mathbf{k}$. We say that X has a *geometric property* $\mathcal{P}$ if $X_{\Bbbk}$ has the property $\mathcal{P}$ for the algebraic closure $\Bbbk$ of $\mathbf{k}$.

Definition 2. A *variety* over a field $\mathbf{k}$ is a geometrically integral scheme which is separated and of finite type over $\operatorname{Spec}\mathbf{k}$.

2 Points in varieties

Proposition 3. Let $\mathbf{k}$ be a field and $\mathbf{K}$ an extension of $\mathbf{k}$. Let X be a variety over $\mathbf{k}$. Then we have the following:

(a) there is a natural bijection between $X(\mathbf{K})$ and $X_{\mathbf{K}}(\mathbf{K})$;

(b) let $\mathbf{K}'/\mathbf{K}$ be an extension, then there is a natural inclusion $X(\mathbf{K}) \subseteq X(\mathbf{K}')$;

(c) suppose that $X = \operatorname{Spec}\mathbf{k}[T_1, \dots, T_n]/I$ is an affine variety, then there is a natural bijection between $X(\mathbf{K})$ and the set $\{(x_1, \dots, x_n) \in \mathbf{K}^n | f(x_1, \dots, x_n) = 0, \forall f \in I\}$.

Proof. Yang: To be continued □

Proposition 4. Let $\mathbf{k}$ be a field and X, Y varieties over $\mathbf{k}$. Let $f, g : X \to Y$ be two morphisms. If $f(x) = g(x)$ for all points $x \in X(\Bbbk)$, then $f = g$.

Proof. Yang: To be continued □

Example 5. Let $\mathbf{k} = \mathbb{F}_p$ and $X = \mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec}\mathbf{k}[T]$. Yang: To be continued

Date: October 1, 2025, **Author**: Tianle Yang, My Homepage