# Category of sheaves of modules

## 1 Sheaves of modules, quasi-coherent and coherent sheaves

**Definition 1.** Let $X$ be a ringed space with structure sheaf $\mathcal{O}_X$. A **sheaf of (left) $\mathcal{O}_X$-modules** is a sheaf $\mathcal{F}$ on $X$ such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ in the sense that for every $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$, we have

$$\rho_{UV}(s \cdot m) = \rho_{UV}(s) \cdot \rho_{UV}(m).$$

Yang: To be continued...

**Example 2.** Let $X$ be a scheme. The structure sheaf $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-modules. More generally, any quasi-coherent sheaf (to be defined later) is a sheaf of $\mathcal{O}_X$-modules. In particular, if $X = \operatorname{Spec} A$ is an affine scheme, then for any $A$-module $M$, the associated sheaf $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules. Yang: To be continued...

**Definition 3.** Let $X$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **quasi-coherent** if for every point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where $I, J$ are (possibly infinite) index sets. Yang: To be continued...

**Definition 4.** Let $X$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **coherent** if it is quasi-coherent and for every point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of finite free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0,$$

where $m, n$ are finite integers. Yang: To be continued...

## 2 As abelian categories

**Theorem 5.** Let $X$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules is an abelian category. Yang: To be continued...

**Theorem 6.** Let $X$ be a scheme. The category of quasi-coherent sheaves on $X$ is an abelian category. Yang: To be continued...

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**Theorem 7.** Let $X$ be a noetherian scheme. The category of coherent sheaves on $X$ is an abelian category. Yang: To be continued...

## 3 Relevant functors

**Theorem 8.** Let $X$ be a ringed space. The global sections functor

$$\Gamma(X, -) : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\mathcal{O}_X(X)\text{-modules})$$

is left exact. Yang: To be continued...

**Theorem 9.** Let $f : X \to Y$ be a morphism of ringed spaces. The direct image functor

$$f_* : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_Y\text{-modules})$$

is left exact. Yang: To be continued...

**Theorem 10.** Let $f : X \to Y$ be a morphism of ringed spaces. The inverse image functor

$$f^* : (\text{Sheaves of } \mathcal{O}_Y\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_X\text{-modules})$$

is right exact. Yang: To be continued...

## 4 Cohomological theory

**Theorem 11.** Let $X$ be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

**Theorem 12.** Let $X$ be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on $X$. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

**Theorem 13.** Let $X$ be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on $X$. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...