Cone Theorem

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Cone Theorem

1 Preliminary

Theorem 1 (Iitaka fibration)**.** Let X be a projective variety and $\mathcal{L}$ a line bundle on X. Let $\varphi_n : X \dashrightarrow Y_n$ be the dominant rational map associated to $\mathcal{L}^n$. Then for $n \gg 0$, the rational maps φ_n stable to a fibration $\varphi_\infty : X \dashrightarrow Y_\infty$ up to birational equivalence.

2 Non-vanishing Theorem

Theorem 2 (Non-vanishing Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

3 Base Point Free Theorem

Theorem 3 (Base Point Free Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then D is semiample.

4 Rationality Theorem

Theorem 4 (Rationality Theorem)**.** Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let

$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H. Then $t = u/v \in \mathbb{Q}$ and

$$0 \leq u \leq a(X) \cdot (\dim X + 1).$$

5 Cone Theorem and Contraction Theorem

Theorem 5 (Cone Theorem)**.** Let (X, B) be a projective klt pair. Then there exist countably many rational curves $C_i \subset X$ with

$$0 < -K_{(X,B)} \cdot C_i \leq 2 \dim X$$

such that

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(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. We only need to prove (b) and (a) follows from (b) by taking $\varepsilon = 1/n$.

Yang: To be completed. □

Theorem 6 (Contraction Theorem)**.** Let (X, B) be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;

(b) any line bundle $\mathcal{L}$ with $F \subset \mathcal{L}^\perp = \{\alpha \in N_1(X) : \alpha \cdot \mathcal{L} = 0\}$ comes from a line bundle on Y, i.e., there exists a line bundle $\mathcal{L}_Y$ on Y such that $\mathcal{L} \cong \varphi_F^* \mathcal{L}_Y$.