Differentials and duality

Let S be a base noetherian scheme, $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over S and all varieties are assumed to be defined over $\Bbbk$.

1 The sheaves of differentials

Definition 1. Let $f : X \to S$ be an S-scheme. The *sheaf of differentials* of X over S, denoted by $\Omega_{X/S}$, is the $\mathcal{O}_X$-module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$.

Example 2.

Proposition 3. Let X be a regular variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a locally free sheaf of rank n.

Proof. Yang: To be continued. □

Proposition 4. Let X be a normal variety over $\Bbbk$. Then $\Omega_{X/\Bbbk}$ is a reflexive sheaf.

Proof. Yang: To be continued. □

Theorem 5 (Euler sequence for projective bundle)**.** Let X be a normal variety over $\Bbbk$ and $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on X. Let $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. Then there is a natural exact sequence of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$-modules

$$0 \to \Omega_{\mathbb{P}_X(\mathcal{E})/X} \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \to 0.$$

Yang: To be checked.

Proof. □

Definition 6. Let X be a normal variety over $\Bbbk$ of dimension n. If X is smooth, then the *canonical divisor* K_X is defined to be $c_1(\omega_X)$. In general, let $U \subseteq X$ be the smooth locus of X and $i : U \hookrightarrow X$ be the inclusion map. Then the *canonical divisor* K_X is defined to be any Weil divisor on X such that $\mathcal{O}_X(K_X) \cong i_*\omega_U$. Note that U is big in X since X is normal, so such a Weil divisor always exists and is unique up to linear equivalence.

Example 7. Let $\mathbb{P}^n_{\Bbbk}$ be the projective space of dimension n over $\Bbbk$. Then the canonical divisor $K_{\mathbb{P}^n_{\Bbbk}} \sim -(n+1)H$, where H is a hyperplane in $\mathbb{P}^n_{\Bbbk}$. Yang: To be checked.

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2 Fundamental sequences

Theorem 8 (The first fundamental sequence of differentials)**.** Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Proof. Yang: To be completed. □

Proposition 9. Let $f : X \to Y$ be a surjective and generically finite morphism of normal varieties over $\Bbbk$. Then the first fundamental sequence of differentials is exact on the left.

Proof. Yang: To be completed. □

Corollary 10 (Ramification formula)**.** Let $f : X \to Y$ be a finite morphism of normal varieties. Then

$$K_X = f^*K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} \left(\mathrm{Mult}_D\, f^*\big(f(D)\big) - 1\right) D$$

is the ramification divisor of f. Yang: To be checked. definition of ramification divisor needs to be checked.

Proof. Yang: To be completed. □

Theorem 11 (The second fundamental sequence of differentials)**.** Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{J}/\mathcal{J}^2 \to \Omega_{X/S}|_Z \to \Omega_{Z/S} \to 0.$$

Suppose further that $Z \to X$ is a regular immersion. Then the above sequence is also exact on the left.

Proof. Yang: To be completed. □

Corollary 12 (Adjunction formula)**.** Let X be a normal variety and $Z \subseteq X$ be a prime Cartier divisor which is normal as variety. Then

$$K_Z = (K_X + Z)|_Z.$$

Proof. Since both X and Z are normal, they are smooth in codimension 1. Removing the singular locus of X and Z, we may assume that both X and Z are smooth varieties. This is valid since the canonical divisor is determined by the smooth locus.

Since Z is Cartier, it is a locally complete intersection in X. By Theorem 11, we have the exact sequence

$$0 \to \mathcal{J}_Z/\mathcal{J}_Z^2 \to \Omega_{X/\Bbbk}|_Z \to \Omega_{Z/\Bbbk} \to 0.$$

Note that Z is of codimension 1 in X, so $\mathcal{J}_Z \cong \mathcal{O}_X(-Z)$ and thus $\mathcal{J}_Z/\mathcal{J}_Z^2 \cong \mathcal{O}_X(-Z)|_Z$. Yang: Taking the top exterior power, we obtain

$$c_1(\Omega_X)|_Z = c_1(\Omega_Z) + c_1(\mathcal{O}_X(-Z))|_Z.$$

That is,

$$K_X|_Z = K_Z - Z|_Z.$$

Rearranging gives the desired result. Yang: To be revised. □

3 Serre duality

Definition 13 (Dualizing sheaf)**.** Let X be a proper scheme of dimension n over $\Bbbk$. A *dualizing sheaf* on X is a coherent sheaf ω_X° together with a trace map $\mathrm{tr}_X : H^n(X, \omega_X^\circ) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}_X} \Bbbk$$

induces an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

Theorem 14. Let X be a projective scheme of dimension n over $\Bbbk$. Then there exists a dualizing sheaf ω_X° on X up to isomorphism. Moreover, if X is smooth, $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\Bbbk}$.

Proof. Yang: To be completed. □

Theorem 15 (Serre duality)**.** Let X be a projective, Cohen-Macaulay variety of dimension n over $\Bbbk$ with dualizing sheaf ω_X°. Then for every coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. Yang: To be completed. □

Yang: When $\mathcal{F}$ is locally free, we have $\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^i(X, \omega_X^\circ \otimes \mathcal{F}^\vee)$.

Corollary 16. Let X be a projective, normal variety of dimension n over $\Bbbk$. Then for every integer m and $0 \le i \le n$, there is a natural isomorphism Yang: To be completed.

4 Logarithm version