Picard Groups of Abelian Varieties

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. Let A be an abelian variety over $\mathbf{k}$.

1 Pullback along group operations

Theorem 1 (Theorem of the cube)**.** Let X, Y, Z be proper varieties over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\}\times Y\times Z}$, $\mathcal{L}|_{X\times\{y\}\times Z}$ and $\mathcal{L}|_{X\times Y\times\{z\}}$ are trivial. Then $\mathcal{L}$ is trivial.

Proof. Yang: To be completed. □

Remark 2. If we assume the existence of the Picard scheme, then the Theorem 1 can be deduced from the Rigidity Lemma. Consider the morphism

$$\varphi : X \times Y \to \mathrm{Pic}(Z), \quad (x, y) \mapsto \mathcal{L}|_{\{x\}\times\{y\}\times Z}.$$

Since $\varphi(x, y) = \mathcal{O}_Z$, φ factors through $\mathrm{Pic}^0(Z)$. Then the assumption implies that φ contracts $\{x\}\times Y$, $X \times \{y\}$ and hence it maps $X \times Y$ to a point. Thus $\varphi(x', y') = \mathcal{O}_Z$ for every $(x', y') \in X \times Y$. Then by Grauert's theorem, we have $\mathcal{L} \cong p^* p_* \mathcal{L}$ where $p : X \times Y \times Z \to X \times Y$ is the projection. Note that $p_*\mathcal{L} \cong \mathcal{L}|_{X\times Y\times\{z\}} \cong \mathcal{O}_{X\times Y}$. Hence $\mathcal{L}$ is trivial.

Lemma 3. Let A be an abelian variety over $\mathbf{k}$, $f, g, h : X \to A$ morphisms from a variety X to A and $\mathcal{L}$ a line bundle on A. Then we have

$$(f + g + h)^*\mathcal{L} \cong (f + g)^*\mathcal{L} \otimes (f + h)^*\mathcal{L} \otimes (g + h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof. First consider $X = A\times A\times A$, $p : X \to A, (x_1, x_2, x_3) \mapsto x_1+x_2+x_3$, $p_{ij} : X \to A, (x_1, x_2, x_3) \mapsto x_i + x_j$ for $1 \le i < j \le 3$ and $p_i : X \to A, (x_1, x_2, x_3) \mapsto x_i$ for $1 \le i \le 3$. Then the conclusion follows from the theorem of the cube by taking $\mathcal{L}' = p^*\mathcal{L}^{-1} \otimes p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \otimes p_3^*\mathcal{L}^{-1}$ and considering the restriction to $\{0\} \times A \times A$, $A \times \{0\} \times A$ and $A \times A \times \{0\}$.

In general, consider the morphism $\varphi = (f, g, h) : X \to A \times A \times A$ and pull back the above isomorphism along φ. □

Proposition 4. Let A be an abelian variety over $\mathbf{k}$, $n \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle on A. Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes\frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes\frac{1}{2}(n^2-n)}.$$

Proof. For $n = 0, 1$, the conclusion is trivial. For $n \ge 2$, we can use the previous lemma on $[n-2]_A, [1]_A, [1]_A$ and induct on n. Hence we have

$$[n]_A^*\mathcal{L} \cong [n-1]_A^*\mathcal{L} \otimes [n-1]_A^*\mathcal{L} \otimes [2]_A^*\mathcal{L} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [n-2]_A^*\mathcal{L}^{-1}.$$

Then the conclusion follows from induction. Yang: To be completed. □

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Definition 5. Let A be an abelian variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on A. We say that $\mathcal{L}$ is *symmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}$ and *antisymmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}^{-1}$.

Theorem 6 (Theorem of the square)**.** Let A be an abelian variety over $\mathbf{k}$, $x, y \in A(\mathbf{k})$ two points and $\mathcal{L}$ a line bundle on A. Then
$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Proof. Yang: To be completed. □

Remark 7. We can define a map
$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \to \mathrm{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$
Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary $\mathcal{L}$, the map
$$\Phi_{\square} : \mathrm{Pic}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \mathrm{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$
is also a group homomorphism.

If we assume the scheme structure on $\mathrm{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\mathrm{Pic}^0(A)$. Let $K(\mathcal{L}) := \mathrm{Ker}\,\Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A. We give another description of $K(\mathcal{L})$. From this point, when $K(\mathcal{L})$ is finite, we can recover the dual abelian variety $A^\vee = \mathrm{Pic}^0_{A/\mathbf{k}}$ as the quotient $A/K(\mathcal{L})$.

2 Projectivity

In this subsection, we work over the algebraically closed field $\Bbbk$.

Proposition 8. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. Then $|2D|$ is base point free.

Proof. Yang: To be completed. □

Theorem 9. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. TFAE:

(a) the stabilizer $\mathrm{Stab}(D)$ of D is finite;

(b) the morphism $\phi_{|2D|}$ induced by the complete linear system $|2D|$ is finite;

(c) D is ample;

(d) $K(\mathcal{O}_A(D))$ is finite.

Proof. Yang: To be completed. □

Theorem 10. Let A be an abelian variety over $\Bbbk$. Then A is projective.

Proof. Yang: To be completed. □

Corollary 11. Let A be an abelian variety over $\Bbbk$ and D a divisor on A. Then D is pseudo-effective if and only if it is nef, i.e. $\mathrm{Psef}^1(A) = \mathrm{Nef}^1(A)$.

Proof. Yang: To be completed. □

3 Dual abelian varieties

In this subsection, we work over the algebraically closed field $\Bbbk$.

Proposition 12. Let A be an abelian variety over $\mathbf{k}$ and $\mathcal{L}$ an ample line bundle on A. Then the homomorphism $\Phi_{\mathcal{L}} : A(\Bbbk) \to \mathrm{Pic}(A)$ factors through $\mathrm{Pic}^0(A)$ and $A(\Bbbk) \to \mathrm{Pic}^0(A)$ is surjective.

Proof. Yang: To be completed. □

Definition 13. Let A be an abelian variety over $\Bbbk$. We define the *dual abelian variety* of A to be $A/K(\mathcal{L})$ for some ample line bundle $\mathcal{L}$ on A. We denote it by $A^\vee$.

Theorem 14. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ does not depend on the choice of the ample line bundle $\mathcal{L}$. Moreover, there is a natural bijection $A^\vee(\mathbf{k}) \to \mathrm{Pic}^0(A)$.

Proof. Yang: To be completed. □

Proposition 15. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ is also an abelian variety and the natural map $A \to A^{\vee\vee}$ is an isomorphism.

Proof. Yang: To be completed. □

Proposition 16. There exists a unique line bundle $\mathcal{P}$ on $A \times A^\vee$ such that for every $y = \mathcal{L} \in A^\vee = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{y\}} \cong \mathcal{L}$.

Proof. Yang: To be completed. □

4 The Néron-Severi group

Theorem 17. Let A be an abelian variety over $\Bbbk$. The we have an inclusion $\mathrm{NS}(A) \hookrightarrow \mathrm{Hom}_{\mathbf{Grp}}(A, A^\vee)$ given by Yang: To be completed.