

Basepoint Free Theorem on Positive Characteristic

This section refers to [Kee99]. For site and algebraic space, we refer to [Knu71], [Art70], [Stacks] and [FGA05]. Throughout this section, all schemes (or algebraic space) are of finite type over a base scheme S with S noetherian.

1 Preliminaries

Theorem 1 (Serre vanishing in relative setting, ref. [Laz04, Theorem 1.7.6]). Let $f : X \rightarrow S$ be a proper morphism of schemes, \mathcal{L} a line bundle and \mathcal{F} a coherent sheaf on X . Suppose that \mathcal{L} is relatively ample. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the higher direct image sheaves $R^i f_* \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ are zero for all $i > 0$.

Theorem 2 (ref. [Laz04, Proposition 1.4.37]). Let X be a projective scheme over a field \mathbf{k} . Then there exists a scheme T of finite type over \mathbf{k} and a line bundle \mathcal{L} on $X \times T$ such that every numerically trivial line bundle on X arises as the restriction $\mathcal{L}|_{X \times \{t\}}$ for some $t \in T$.

Theorem 3 (Theorem on Formal Functions, ref. [Har77, Chapter III, Theorem 11.1]). Let $f : X \rightarrow Y$ be a projective morphism of noetherian schemes, let \mathcal{F} be a coherent sheaf on X , and let $y \in Y$. Then the natural map

$$(R^i f_* \mathcal{F})_y^\wedge \rightarrow \varprojlim H^i(X_n, \mathcal{F}_n)$$

is an isomorphism for all $i \geq 0$, where $X_n = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n$ and $\mathcal{F}_n = \mathcal{F}|_{X_n}$.

Definition 4. Let X be a proper variety and \mathcal{L} a nef line bundle on X . A closed subvariety $Z \subseteq X$ is called the *exceptional* for \mathcal{L} if $\mathcal{L}^{\dim Z} \cdot Z = 0$. The *exceptional locus* of \mathcal{L} , denoted by $\operatorname{Exc} \mathcal{L}$, is defined as the closure of the union of all exceptional subvarieties of \mathcal{L} .

If \mathcal{L} is semiample, then $\operatorname{Exc} \mathcal{L} = \operatorname{Exc} \varphi$ for the fibration $\varphi : X \rightarrow Y$ induced by \mathcal{L} .

Definition 5. Let X be a proper scheme and \mathcal{L} a nef line bundle on X . We say that \mathcal{L} is *endowed with a map (EWM)* if there is a proper morphism $\varphi : X \rightarrow Y$ to a proper algebraic space such that $\dim Z > \dim f(Z)$ if and only if Z is an exceptional subvariety of \mathcal{L} . If such a morphism is a fibration, then it is unique, called the *fibration associated to \mathcal{L}* .

Proposition 6. Let X be a proper variety and \mathcal{L} a nef line bundle on X endowed with a map. Let $\varphi : X \rightarrow Y$ be the associated fibration. Then TFAE:

- (a) \mathcal{L} is semiample;
- (b) $\mathcal{L}^{\otimes m}$ is pulled back from an ample line bundle on Y for some $m \in \mathbb{Z}_{>0}$;

(c) $\mathcal{L}^{\otimes m}$ is pulled back from a line bundle on Y for some $m \in \mathbb{Z}_{>0}$;

Proof. (a) \Leftrightarrow (b) \implies (c) is clear. Replacing \mathcal{L} by $\mathcal{L}^{\otimes m}$ for some $m \in \mathbb{Z}_{>0}$, suppose that $\mathcal{L} = \varphi^* \mathcal{L}_Y$ for some line bundle \mathcal{L}_Y on Y . We show that \mathcal{L}_Y is ample. Indeed, for all closed subvarieties $Z \subset Y$, we can find $Z' \subset X$ such that $Z' \twoheadrightarrow Z$ and $\dim Z' = \dim Z$. Then

$$\mathcal{L}_Y^{\dim Z} \cdot Z = d \mathcal{L}^{\dim Z'} \cdot Z' > 0$$

where $d = \deg(Z' \rightarrow Z)$. Hence \mathcal{L}_Y is ample. \square

Definition 7. A morphism $f : X \rightarrow Y$ of schemes is called a *universal homeomorphism* if for every Y -scheme Y' , the base change $X \times_Y Y' \rightarrow Y'$ is a homeomorphism between the underlying topological spaces.

Example 8. Let X be a scheme of finite type over \mathbf{k} . Then the natural morphism $X_{\text{red}} \rightarrow X$ is a universal homeomorphism.

Let X be a scheme over S of characteristic p . Then the absolute and relative Frobenius morphisms are universal homeomorphisms. **Yang: To be completed.**

The morphism $\text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{R}$ is not a universal homeomorphism.

Lemma 9. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms of schemes with g finite. Let \mathcal{F} be a coherent sheaf on X . Then the we have

$$R^i(g \circ f)_* \mathcal{F} = g_*(R^i f_* \mathcal{F}).$$

Proof. **Yang: This is a simple application of the Grothendieck spectral sequence. However, I do not know anything about it.** \square

2 Algebraic space

Definition 10. Let \mathbf{C} be a category. A *Grothendieck topology* on \mathbf{C} is a collection of sets of arrows $\{U_i \rightarrow U\}_{i \in I}$, called *covering*, for each object U in \mathbf{C} such that:

- (a) if $V \rightarrow U$ is an isomorphism, then $\{V \rightarrow U\}$ is a covering;
- (b) if $\{U_i \rightarrow U\}_{i \in I}$ is a covering and $V \rightarrow U$ is a arrow, then the fiber product $U_i \times_U V \rightarrow V$ exists and $\{U_i \times_U V \rightarrow V\}$ is a covering of V ;
- (c) if $\{U_i \rightarrow U\}_{i \in I}$ and $\{U_{ij} \rightarrow U_i\}_{j \in J_i}$ are coverings, then the collection of composition $\{U_{ij} \rightarrow U_i \rightarrow U\}_{i \in I, j \in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{J})$ where \mathbf{C} is a category and \mathcal{J} is a Grothendieck topology on \mathbf{C} .

Note that sheaf is indeed defined on a site.

Definition 11. Let $(\mathbf{C}, \mathcal{J})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{J})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \rightarrow \mathbf{Set}$ satisfying the following condition: for every object U in \mathbf{C} and every covering $\{U_i \rightarrow U\}_{i \in I}$ of U , if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every i, j , the pullback $s_i|_{U_i \times_U U_j}$ and $s_j|_{U_i \times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every i , the pullback $s|_{U_i} = s_i$.

Definition 12. Let X be a scheme. The *big étale site* of X , denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over X with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \rightarrow U\}_{i \in I}$ is a covering if and only if each U_i is étale over U and the union of their images is the whole U .

Let X be a scheme over S . By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \rightarrow \mathbf{Set}$ such that for any S -scheme T , $h_X(T) = \text{Hom}_{\mathbf{Sch}_S}(T, X)$. **Yang:** Easy to check that h_X is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

Definition 13. Let U be a scheme over a base scheme S . An *étale equivalence relation* on U is a morphism $R \rightarrow U \times_S U$ between schemes over S such that:

- (a) the projections in two factors $R \rightarrow U$ are étale and surjective;
- (b) for every S -scheme T , $h_R(T) \rightarrow h_U(T) \times h_U(T)$ gives an equivalence relation on $h_U(T)$ set-theoretically.

Definition 14. An *algebraic space* X over a base scheme S is an S -scheme U together with an étale equivalence relation $R \rightarrow U \times_S U$.

Let $X = (U, R)$ be an algebraic space over S . We explain X as a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. For any scheme T over S , $h_R(T)$ is an equivalence relation on $h_U(T)$. The rule sending T to the set of equivalence classes of $h_R(T)$ gives a presheaf on the site $(\mathbf{Sch}/S)_{\text{ét}}$. The sheafification of this presheaf is the sheaf associated to the algebraic space X . Explicitly, we have

$$X(T) := \left\{ f = (f_i) \left| \begin{array}{l} \{T_i \rightarrow T\} \text{ a covering, } f_i \in h_U(T_i) \text{ such} \\ \text{that } (f_i|_{T_i \times_T T_j}, f_j|_{T_i \times_T T_j}) \in h_R(T_i \times_T T_j) \end{array} \right. \right\} / \sim,$$

where

$$\alpha \sim \beta \quad \text{if } \exists \{S_i \rightarrow T\} \text{ such that } (\alpha|_{S_i}, \beta|_{S_i}) \in h_R(S_i).$$

Definition 15. An *algebraic space* over a base scheme S is a sheaf F on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$ such that

- (a) the diagonal morphism $F \rightarrow F \times_S F$ is representable;
- (b) there exists a scheme U over S and a map $h_U \rightarrow F$ which is surjective and étale.

The *morphism between algebraic spaces* F_1, F_2 is defined as a natural transformation of functors F_1, F_2 .

Remark 16. By Yoneda's Lemma, given a morphism $h_U \rightarrow F$ between sheaves is the same as giving an element of $F(U)$. We may abuse the notation.

Definition 17. Let \mathcal{P} be a property of morphisms of schemes satisfying the following conditions:

- (a) is preserved under any base change;
- (b) is étale local on the base. Yang: In [Stacks], this requires that “fppf local”.

Let $\alpha : F \rightarrow G$ be a representable morphism of sheaves on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. We say that α has property \mathcal{P} if for every $h_T \rightarrow G$, the base change $h_T \times_G F \rightarrow F$ has property \mathcal{P} .

Remark 18. The fiber product $F_1 \times_F F_2$ is just defined as $F_1 \times_F F_2(T) := F_1(T) \times_{F(T)} F_2(T)$ for any object $T \in \text{Obj}(\mathbf{Sch}_S)$. We say that a morphism $f : F_1 \rightarrow F_2$ of sheaves is *representable* if for every $T \in \text{Obj}(\mathbf{Sch}/S)$ and every $\xi \in F_2(T)$, the sheaf $F_1 \times_{F_2} h_T$ is representable as a functor. Here $h_T \rightarrow F_2$ is given by

$$h_T(U) \rightarrow F_2(U), \quad f \in \text{Hom}(U, T) \mapsto F_2(f)(\xi) \in F_2(U).$$

In our case, given an arbitrary $h_U \rightarrow F \times F$ is equivalent to giving morphisms $h_{U_i} \rightarrow F$ for $i = 1, 2$. And the fiber product $F \times_{F \times F} (h_{U_1} \times h_{U_2})$ is just the fiber product $h_{U_1} \times_F h_{U_2}$. Hence the first condition in Definition 15 is equivalent to that $h_{U_1} \times_F h_{U_2}$ is representable for any U_1, U_2 over F . This implies that $h_U \rightarrow F$ is representable, whence the second condition in Definition 15 makes sense.

Definition 19. Let X be an algebraic space over a base scheme S . Two morphisms from field $\text{Spec } k_i \rightarrow X$ is called equivalent if there is a common extension $K \supset k_1, k_2$ such that we have $\text{Spec } K \rightarrow \text{Spec } k_i \rightarrow X$ are the same for $i = 1, 2$. The *underlying point set* of X , denote by $|X|$, is defined as the set of equivalence classes of morphisms $\text{Spec } k \rightarrow X$ for all field k over the base field \mathbf{k} .

This definition coincides with the underlying set of a scheme. Let $\alpha : X \rightarrow Y$ be a morphism of algebraic spaces. It induces a map $|\alpha| : |X| \rightarrow |Y|$ by $x \mapsto \alpha \circ x$ (vertical composition).

Proposition 20 (ref. [Stacks, Lemma 66.4.6]). There is a unique topology on $|X|$ such that

- (a) if X is a scheme, then the topology coincides with the usual topology.
- (b) every morphism of algebraic spaces $f : X \rightarrow Y$ induces a continuous map $|f| : |X| \rightarrow |Y|$.
- (c) if U is a scheme and $U \rightarrow X$ is étale, then the induced map $|U| \rightarrow |X|$ is open.

This topology is called the *Zariski topology* on $|X|$.

Definition 21. Let X be an algebraic space over a base scheme S . All étale morphisms $U \rightarrow X$ with U scheme form a small site $X_{\text{ét}}$. All étale morphisms $U \rightarrow X$ with U algebraic space form a

small site $X_{\text{sp}, \text{ét}}$. The *structure sheaf* \mathcal{O}_X of X is given by $U \mapsto \Gamma(U, \mathcal{O}_U)$ for every étale morphism $U \rightarrow X$ from a scheme. It extends to a sheaf on the site $X_{\text{sp}, \text{ét}}$ uniquely.

Example 22. Let $U = \mathbb{A}_{\mathbb{C}}^1$ and $R \subset U \times U$ given by $y = x + n, n \in \mathbb{Z}$. Then R is a disjoint union of lines in $U \times U$. Write $R = \coprod_{n \in \mathbb{Z}} R_n$ with $R_n = \{(x, x + n) : x \in \mathbb{C}\}$. Then the projection is given by

$$\begin{aligned}\pi_1|_{R_n} : R_n &\rightarrow U, & (x, x + n) &\mapsto x, \\ \pi_2|_{R_n} : R_n &\rightarrow U, & (x, x + n) &\mapsto x + n.\end{aligned}$$

Easily see that the projection $\pi_i : R \rightarrow U$ is étale and surjective for $i = 1, 2$. Let $r_{ij} : R \times U \rightarrow U \times U \times U$ be the morphism which maps $((x, y), u)$ to (a_1, a_2, a_3) where $a_i = x$, $a_j = y$ and $a_k = u$ for $k \neq i, j$. Since $\Delta_U \rightarrow U \times U$ factors through R , $(\pi_1, \pi_2) = (\pi_2, \pi_1)$ and $r_{12} \times_{(U \times U \times U)} r_{23}$ factors through r_{13} , we have that $h_R(T)$ is an equivalence relation on $h_U(T)$ for all T over S . Then $X := (U, R)$ is an algebraic space.

We do not check the representability here but give an example. Let $U \rightarrow X$ be the natural morphism given by $\text{id}_U \in X(U)$. For any scheme T over \mathbb{C} , we have

$$(U \times_X U)(T) = \{(f, g) \in h_{U \times U}(T) : \exists \{T_i \rightarrow T\} \text{ s.t. } (f_i, g_i) \in h_R(T_i)\} = h_R(T).$$

Hence the fiber product $h_U \times_X h_U$ is represented by R .

We show that $X \not\cong \mathbb{C}^\times$ by computing the the global sections. Consider the covering $U \rightarrow X$, a section $s \in \mathcal{O}_X(X)$ is given by a section $s \in \Gamma(U, \mathcal{O}_U) = \mathbb{C}[t]$ such that $\pi_1^*s = \pi_2^*s$ in $\Gamma(R, \mathcal{O}_R)$. This means that $s(x + n) = s(x)$ for all $n \in \mathbb{Z}$. Hence s is a constant function. In particular, $\mathcal{O}_X(X) = \mathbb{C} \neq \mathbb{C}[t, t^{-1}]$.

The underlying set $|X|$ is union of the quotient set \mathbb{C}/\mathbb{Z} and a generic point. The Zariski topology on $|X|$ is the trivial topology.

In following, we will use the technique of *local construction* to construct many scheme-like objects on algebraic spaces. For local construction, see [Knu71]. Roughly speaking, for every étale morphism $U \rightarrow X$ with U a scheme, we construct a scheme-theoretic object on U which is compatible under base change. Then we glue these objects together to get a global object on X .

Definition 23. Let X be an algebraic space over a base scheme S . A *coherent sheaf* on X is a sheaf \mathcal{F} on $X_{\text{ét}}$ such that for every covering $\{U_i \rightarrow X\}$ with U_i schemes, the sheaf $\mathcal{F}|_{U_i}$ is coherent for every i . It extends to a sheaf on the site $X_{\text{sp}, \text{ét}}$ uniquely.

An *ideal sheaf* on X is a coherent sheaf $\mathcal{I} \subset \mathcal{O}_X$. It defines a closed subspace $V(\mathcal{I}) \subset X$ by **Yang: to be completed**. And every closed subspace $Y \subset X$ is defined by an ideal sheaf \mathcal{I}_Y such that $V(\mathcal{I}_Y) = Y$.

Definition 24. Let X be an algebraic space over a base scheme S . A *line bundle* on X is a coherent sheaf \mathcal{L} on X such that for every covering $\{U_i \rightarrow X\}$ with U_i schemes, the sheaf $\mathcal{L}|_{U_i}$ is a line bundle on U_i . It extends to a sheaf on the site $X_{\text{sp}, \text{ét}}$ uniquely.

Theorem 25 (ref. [Stacks, Theorem 76.36.4]). Let $f : X \rightarrow Y$ be a proper morphism of algebraic spaces over a base scheme S . Then there exists a factorization

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y,$$

where f_1 has geometrically connected fibers and $(f_1)_* \mathcal{O}_X = \mathcal{O}_Z$ and f_2 is finite.

Definition 26. Let X be an algebraic space over a base scheme S and Y a closed subset of $|X|$. The *formal completion* of X along Y , denoted by \mathfrak{X} , is

Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined as $\varprojlim_n \mathcal{O}_X / \mathcal{I}^n$ where \mathcal{I} is the ideal sheaf of Y in \mathcal{O}_X . **Yang: to be completed.**

Definition 27. Let X be an algebraic space and Y a closed subset of X . A *modification* of X along Y is a proper morphism $f : X' \rightarrow X$ and a closed subset $Y' \subset X'$ such that $X' \setminus Y' \rightarrow X \setminus Y$ is an isomorphism and $f^{-1}(Y) = Y'$.

Theorem 28 (ref. [Art70, Theorem 3.1]). Let Y' be a closed subset of an algebraic space X' of finite type over \mathbf{k} . Let \mathfrak{X}' be the formal completion of X' along Y' . Suppose that there is a formal modification $\mathfrak{f} : \mathfrak{X}' \rightarrow \mathfrak{X}$. Then there is a unique modification

$$f : X' \rightarrow X, \quad Y \subset X$$

such that the formal completion of X along Y is isomorphic to \mathfrak{X} and the induced morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ is isomorphic to \mathfrak{f} .

Theorem 29 (ref. [Art70, Theorem 6.2]). Let \mathfrak{X}' be a formal algebraic space and $Y' = V(\mathcal{I}')$ with \mathcal{I}' the defining ideal sheaf of \mathfrak{X}' . Let $f : Y' \rightarrow Y$ be a proper morphism. Suppose that

(a) for every coherent sheaf \mathcal{F} on \mathfrak{X}' , we have

$$R^1 f_* \mathcal{I}'^n \mathcal{F} / \mathcal{I}'^{n+1} \mathcal{F} = 0, \quad \forall n \gg 0;$$

(b) for every n , the homomorphism

$$f_*(\mathcal{O}_{\mathfrak{X}'} / \mathcal{I}'^n) \otimes_{f_* \mathcal{O}_{Y'}} \mathcal{O}_Y \rightarrow \mathcal{O}_Y$$

is surjective.

Then there exists a modification $\mathfrak{f} : \mathfrak{X}' \rightarrow \mathfrak{X}$ and a defining ideal sheaf \mathcal{I} of \mathfrak{X} such that $V(\mathcal{I}) = Y$ and \mathfrak{f} induces f on Y .

Theorem 30 (ref. [Art70, Theorem 6.1]). Let Y' be a closed algebraic subspace of an algebraic space X' and $f_0 : Y' \rightarrow Y$ a finite morphism. Then there exists a modification $f : X' \rightarrow X$ whose

restriction to Y' is f_0 . It is the amalgamated sum $X = X' \amalg_{Y'} Y$ in the category of algebraic spaces **AlgSp**.

Example 31. Let $X = \mathbb{A}^2 = \text{Spec } \mathbf{k}[x, y]$ and $Y = V(y)$ be the x -axis. Let $f_0 : Y' = \mathbb{A}^1 \rightarrow Y, x \mapsto x^2$. Then there exists a modification $f : X' \rightarrow X$ such that the restriction $f|_{Y'} : Y' \rightarrow Y$ is f_0 .

Yang: To be completed.

3 A sufficient and necessary condition for EWM

In this and next subsection, we assume that all schemes (algebraic spaces) are of finite type over a field \mathbf{k} with characteristic $p > 0$.

Lemma 32. Let $f : X \rightarrow Y$ be a finite morphism of algebraic space which is of finite type over \mathbf{k} . Suppose that f is a universal homeomorphism. Then there exists $q = p^n$ such that the relative Frobenius morphism $\text{Frob}_{X/\mathbf{k}}^n$ factors as

$$\text{Frob}_{X/\mathbf{k}}^n : X \xrightarrow{f} Y \rightarrow X^{(q)}.$$

Proof. Yang: I can only prove this for schemes. Suppose that X, Y are affine. Factor it as $A \twoheadrightarrow B \hookrightarrow C$ with A, B, C \mathbf{k} -algebras.

For $A \twoheadrightarrow B$, let I be the kernel of the surjection. Since $\text{Spec } B \rightarrow \text{Spec } A$ is finite universal homeomorphism, we have that I is a nilpotent ideal. Hence there exists q such that $I^q = 0$. Let $a, a' \in A$ with the same image b in B . Then we have $a^q - a'^q \in I^q = 0$. Hence $a^q = a'^q$ in A . This gives a map $B^q \rightarrow A, b^q \mapsto a^q$.

For $B \hookrightarrow C$, we induct on the dimension. If C is artinian, then $0 = C^q \subset B \subset C$. In general case, this shows that $B \cdot C^{q_1} \subset C$ is an isomorphism at generic points. Let $I := \text{Ann}(B \cdot C^q/B) \subset B$. This is the conductor of extension $B \cdot C^{q_1} \subset C$, whence also an ideal of $B \cdot C^{q_1}$. To see this, for every $x \in B \cdot C^{q_1}$, $b \in I$, we have $xbB \cdot C^{q_1} = bB \cdot C^{q_1} \subset B$. By induction hypothesis, we have $(BC^{q_1}/I)^{q_2} \subset B/I$. For $x \in BC^{q_1}$, there exists $b \in B$ and $\delta \in I \subset B$ such that $x^{q_2} = b + \delta \in B$. Hence we have $(BC^{q_1})^{q_2} \subset B$. In particular, we have $C^{q_1 q_2} \subset (B \cdot C^{q_1})^{q_2} \subset B$.

In general case, we have

$$\begin{array}{ccccc} C^{q_1 q_2} & \longrightarrow & A' & \twoheadrightarrow & C^{q_1} \\ & & \downarrow & & \downarrow \\ & & A & \twoheadrightarrow & B \hookrightarrow C \end{array},$$

where A' is the preimage of C^{q_1} in A . One we have $C^q \rightarrow A \rightarrow C$, note that $A \rightarrow C$ is over \mathbf{k} , then it gives

$$C^q \rightarrow C^{(q)} \rightarrow A \rightarrow C.$$

□

Corollary 33. Let $Z \rightarrow X$ be a finite universal homeomorphism of algebraic spaces and $Z \rightarrow Y$ any finite morphism of algebraic spaces. Suppose that X, Y, Z are all of finite type over \mathbf{k} . Then the amalgamated sum $X \amalg_Z Y$ exists in the category of algebraic spaces. Moreover, $Y \rightarrow X \amalg_Z Y$ is a finite universal homeomorphism.

Proof. By Lemma 32, we have a diagram

$$\begin{array}{ccc} Y^{(q)} & \longleftarrow & Y \\ \uparrow & & \uparrow g \\ Z^{(q)} & & \\ \uparrow & & \\ X & \xleftarrow{f} & Z \end{array} \quad .$$

Denote $X \rightarrow Y^{(q)}$ by f . Let

$$\mathcal{A} := \text{Ker}(\mathcal{O}_X \times \mathcal{O}_Y \rightarrow \mathcal{O}_Z, \quad (s, t) \mapsto f^*s - g^*t).$$

Then \mathcal{A} is an $\mathcal{O}_{Y^{(q)}}$ -algebra. Set $W := \text{Spec}_{Y^{(q)}} \mathcal{A}$. Then $W = X \amalg_Z Y$ is the amalgamated sum in the category of algebraic spaces. **Yang: The most important point is that $Z \rightarrow W$ is finite. Yang: At least in the cat of schemes.** \square

Proposition 34. Let $g : X' \rightarrow X$ be a proper, finite universal homeomorphism between algebraic spaces. Then a line bundle \mathcal{L} on X is endowed with a map if and only if $g^*\mathcal{L}$ is endowed with a map.

Proof. Let $f : X' \rightarrow Z$ be the map endowed on $g^*\mathcal{L}$. By Lemma 32, we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow f & & \downarrow \\ & X'^{(q)} & \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z^{(q)} \end{array} \quad .$$

Easy to check that $X \rightarrow Z^{(q)}$ is a map associated to \mathcal{L} . \square

Proposition 35. Let X be a projective scheme and \mathcal{L} a nef line bundle on X . Assume that $X = X_1 \cup X_2$ for closed subsets X_1 and X_2 . Suppose that $\mathcal{L}|_{X_i}$ is endowed with a fibration $g_i : X_i \rightarrow Z_i$ for $i = 1, 2$. Then \mathcal{L} is endowed with a map $g : X \rightarrow Z$.

Proof. Let $X_{12} := X_1 \cap X_2$. Let $X_{12} \rightarrow Z_{12}$ be the Stein factorization of the map $g_1|_{X_{12}}$. Then by **Yang: Rigidity Lemma**, it is also the Stein factorization of the map $g_2|_{X_{12}}$. Denote Y_i be the image

of Z_{12} in Z_i for $i = 1, 2$. Then we have a commutative diagram

$$\begin{array}{ccccc}
 & & Z_1 & & \\
 & & \uparrow & \swarrow h' & \\
 & X & \longleftarrow X_1 & & Y_1 \\
 & \uparrow & \uparrow & & \uparrow h \\
 Z_2 & \longleftarrow X_2 & \longleftarrow X_{12} & & Z_{12} \\
 & \searrow f & \searrow & & \\
 & Y_2 & \longleftarrow & &
 \end{array}$$

Consider the sub-diagram

$$\begin{array}{ccc}
 & Z_1 & \\
 & \uparrow h' & \\
 & Y_1 & \\
 & \uparrow h & \\
 Z_2 & \xleftarrow{f} & Z_{12}
 \end{array}$$

Here f is finite, h is finite universal homeomorphism and h' is a closed immersion. By [Corollary 33](#), we have the amalgamated sum $Z' := Y_1 \amalg_{Z_{12}} Z_2$ exists in the category of algebraic spaces. Since f is finite, so is the induced morphism $Y_1 \rightarrow Z'$. Then by [Theorem 30](#), the amalgamated sum $Z := Z' \amalg_{Y_1} Z_1$ exists in the category of algebraic spaces.

Then we have a commutative diagram

$$\begin{array}{ccccc}
 Z & \longleftarrow & & Z_1 & \\
 \uparrow & \swarrow g & & \uparrow & \\
 & X & \longleftarrow & X_1 & \\
 & \uparrow & & \uparrow & \\
 Z_2 & \longleftarrow & X_2 & \longleftarrow & X_{12}
 \end{array}$$

Directly check shows that g is a map associated to \mathcal{L} . □

Proposition 36. Let X be a projective scheme and D a nef and big divisor on X . Then we can write $D = A + E$ where A is an ample divisor and E is an effective divisor. Then D is endowed with a map iff $D|_{E_{red}}$ is endowed with a map.

Proof. By [Proposition 34](#), we may assume that $D|_E$ is endowed with a map $f : E \rightarrow Z$. Let $\mathcal{L} = \mathcal{O}_X(-E)$ be the ideal sheaf of E . note that $-E = A - D$ and D is f -numerically trivial. Hence $\mathcal{L}|_E$ is f -ample. By Serre's vanishing, for every coherent sheaf \mathcal{F} on X , there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$R^i f_* \mathcal{F}|_E \otimes \mathcal{L}|_E^{\otimes n} = 0$$

for all $i > 0$. In particular, let $n \in \mathbb{Z}$ such that $R^i f_* \mathcal{O}_X / \mathcal{L} \otimes \mathcal{L}^{\otimes m} = 0$ for all $i > 0, m \geq n$. Set

$\mathcal{I} := \mathcal{L}^{\otimes n}$. Then by the exact sequence

$$0 \rightarrow \mathcal{L}^{n-1} \otimes \mathcal{O}_X / \mathcal{L} \rightarrow \mathcal{O}_X / \mathcal{L}^n \rightarrow \mathcal{O}_X / \mathcal{L} \rightarrow 0,$$

we have that $R^i f_*(\mathcal{O}_X / \mathcal{I} \otimes \mathcal{I}^t) = 0$ for all $i > 0, t \geq 1$. This implies that $f_* \mathcal{O}_X / \mathcal{I}^t \rightarrow f_* \mathcal{O}_X / \mathcal{I}$ is surjective for all $t \geq 1$.

Let

$$\begin{aligned} \mathcal{A} &:= \mathcal{O}_X \oplus \mathcal{I}T \oplus \mathcal{I}^2 T^2 \oplus \cdots, \\ \mathcal{M} &:= \mathcal{F} \oplus \mathcal{I}FT \oplus \mathcal{I}^2 \mathcal{F}T^2 \oplus \cdots, \end{aligned}$$

where T is a formal variable to denote the grading. Then \mathcal{A} is a graded \mathcal{O}_X -algebra of finite type and \mathcal{M} is a finite graded \mathcal{A} -module. We have an exact sequence of graded \mathcal{A} -modules

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{M} \otimes_{\mathcal{A}} \mathcal{I}T \rightarrow \mathcal{M} \rightarrow 0,$$

where $\mathcal{K} = \bigoplus \mathcal{K}_r T^r$ is a finite graded \mathcal{A} -module. Hence for $r \gg 1$, we have that $\mathcal{I}T \cdot \mathcal{K}_r T^r = \mathcal{K}_{r+1} T^{r+1}$. It implies that the image of $\mathcal{K}_{r+1} T^{r+1} \rightarrow \mathcal{M}_r T^r \otimes_{\mathcal{A}} \mathcal{I}T$ is contained in $\mathcal{I} \mathcal{M}_r$ for all $r \gg 1$. Tensor with $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I}$, we have that

$$\mathcal{K}_{r+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I} \rightarrow 0 \rightarrow \mathcal{M}_r \otimes_{\mathcal{O}_X} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I} \rightarrow \mathcal{M}_{r+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I} \rightarrow 0.$$

That is, $\mathcal{I}^r \mathcal{F} / \mathcal{I}^{r+1} \mathcal{F} \otimes_{\mathcal{O}_X / \mathcal{I}} \mathcal{I} / \mathcal{I}^2 \cong \mathcal{I}^{r+1} \mathcal{F} / \mathcal{I}^{r+2} \mathcal{F}$ for all $r \gg 1$. Hence we have that

$$R^i f_*(\mathcal{I}^{r-1} \mathcal{F} / \mathcal{I}^r \mathcal{F}) = 0$$

for all $i > 0, r \gg 1$.

Let $E' := V(\mathcal{I})$, we have that $D|_{E'}$ is endowed with a map $f' : E' \rightarrow Z'$ by [Proposition 34](#). Moreover, we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & Z \\ \downarrow & & \downarrow g \\ E' & \xrightarrow{f'} & Z' \end{array}$$

with g finite. Then by Grothendieck Spectral Sequence, we have that

$$R^i f'_*(\mathcal{I}^{r-1} \mathcal{F} / \mathcal{I}^r \mathcal{F}) = 0$$

for all $i > 0, r \gg 1$.

Then we can apply [Theorems 28](#) and [29](#) to get a modification $X \rightarrow Y$. Note that $\text{Exc } D \subset \text{Supp } E$. It follows that $X \rightarrow Y$ is a map associated to D . \square

Theorem 37. Let X be a proper variety and \mathcal{L} a nef line bundle on X . Then \mathcal{L} is endowed with a map if and only if $\mathcal{L}|_{\text{Exc } \mathcal{L}}$ is endowed with a map.

Proof. By [Proposition 35](#), we can assume that \mathcal{L} is big. Then the result follows from [Proposition 36](#) and induction on dimension. \square

4 For semiample

Lemma 38. Let X be a projective scheme over $\mathbf{k} = \overline{\mathbb{F}_p}$. Then \mathcal{L} is numerically trivial if and only if \mathcal{L} is torsion in $\text{Pic}(X)$.

Proof. Let T be the scheme in [Theorem 2](#). Then \mathcal{L} corresponds to a \mathbb{F}_q -point of T . Note that there are only finitely many \mathbb{F}_q -points in T . Hence \mathcal{L} is torsion in $\text{Pic}(X)$. \square

Proposition 39. Let $f : X \rightarrow Y$ be a finite universal homeomorphism between algebraic spaces of finite type over \mathbf{k} and \mathcal{L} a line bundle on Y . Then there exists $q = p^n$ such that

- (a) for every section $s \in H^0(X, f^*\mathcal{L})$, we have $s^q \in \text{Im}(H^0(Y, \mathcal{L}^{\otimes q}) \rightarrow H^0(X, f^*\mathcal{L}^{\otimes q}))$;
- (b) \mathcal{L} is semiample if and only if $f^*\mathcal{L}$ is semiample;
- (c) the map

$$f^* : \text{Pic}(Y) \otimes \mathbb{Z}[1/q] \rightarrow \text{Pic}(X) \otimes \mathbb{Z}[1/q]$$

is an isomorphism;

- (d) if $f^*s_1 = f^*s_2$ for two sections $s_1, s_2 \in H^0(Y, \mathcal{L})$, then $s_1^q = s_2^q$ in $H^0(X, \mathcal{L}^{\otimes q})$.

Proof. Note that $\text{Frob}^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$. Then all the properties follows from [Lemma 32](#). \square

Proposition 40. Let X be a projective scheme and \mathcal{L} a nef line bundle on X . Assume that $X = X_1 \cup X_2$ for closed subsets X_1 and X_2 . Suppose that $\mathcal{L}|_{X_i}$ is semiample for $i = 1, 2$. Then \mathcal{L} is semiample.

Proof. **Yang:** To be learned. \square

Lemma 41. Let $f : X \rightarrow Y$ be a proper map between algebraic spaces with $f_*\mathcal{O}_X = \mathcal{O}_Y$ and \mathcal{L} a line bundle on X . Let $D = V(\mathcal{I}) \subset X$ be a closed subspace defined by an ideal sheaf \mathcal{I} , $Z = f(D)$ and $D_k := V(\mathcal{I}^k)$. Suppose that f is a modification with respect to D, Z and $R^1f_*\mathcal{I}^k/\mathcal{I}^{k+1} = 0$ for all $k \gg 0$. Suppose for every k , there exists $r > 0$ such that $\mathcal{L}^{\otimes r}|_{D_k}$ is pulled back from $f(D_k)$. Then $\mathcal{L}^{\otimes r}$ is pulled back from Y for some $r > 0$.

Proof. **Yang:** To be completed. \square

Proposition 42. Let X be a projective scheme and D a nef and big divisor on X . Then we can write $D = A + E$ where A is an ample divisor and E is an effective divisor. Then D is semiample iff $D|_{E_{\text{red}}}$ is semiample.

Proof. **Yang:** To be completed. \square

Theorem 43. Let X be a proper variety and \mathcal{L} a nef line bundle on X . Then \mathcal{L} is semiample if and only if $\mathcal{L}|_{\text{Exc } \mathcal{L}}$ is semiample.

Proof. Yang: To be completed. □

5 Basepoint free theorem on positive characteristic

Proposition 44 (ref. Yang:). Let $T \subset X$ be a reduced Weil divisor on a normal variety X . Let $T^\nu \rightarrow T$ be the normalization, $C \subset T^\nu$ the effective Weil divisor defined by the conductor and $p : T^\nu \rightarrow T \hookrightarrow X$ the composition. Suppose that $K_X + T$ is \mathbb{Q} -Cartier. Then there exists an effective \mathbb{Q} -Weil divisor D on T^ν such that

$$K_{T^\nu} + C + D = p^*(K_X + T).$$

Theorem 45. Let X be a normal projective \mathbb{Q} -factorial threefold and $B \in (0, 1)$ a \mathbb{Q} -divisor. Let \mathcal{L} be a nef and big line bundle on X such that $\mathcal{L} - K_{(X,B)}$ is nef and big. Then \mathcal{L} is endowed with a map. Moreover, if $\mathbf{k} = \overline{\mathbb{F}_p}$, \mathcal{L} is semiample.

Proof. Let $\mathcal{L} = \mathcal{O}_X(A + E)$ with A an ample divisor and E an effective divisor. Write $E = E_0 + E_1 + E_2$ such that the restriction of \mathcal{L} to every irreducible component of E_i is of numerical dimension i . Let $S := \text{Supp } E_1$ and $S = \sum S_i$ with S_i irreducible components. Let $S^\nu \rightarrow S$ and $S_i^\nu \rightarrow S_i$ be the normalizations.

Step 1. Reduce to show that $\mathcal{L}|_S$ is endowed with a map (semiample).

Yang: To be completed.

Step 2. Reduce to show that $\mathcal{L}|_{S_i^\nu}$ is endowed with a map (semiample).

Yang: To be completed.

Step 3. Show that $\mathcal{L}|_{S_i^\nu}$ is endowed with a map (semiample).

Yang: To be completed. □

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