Stacks in category theory

1 Prestacks

Definition 1. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. A morphism $f : a \to b$ in $\mathbf{X}$ is called *strongly Cartesian* if for every object $c \in \mathrm{Obj}(\mathbf{X})$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{X}}(c,a) \arrow[r, "f\circ-"] \arrow[d, "\mathbf{p}"'] & \mathrm{Hom}_{\mathbf{X}}(c,b) \arrow[d, "\mathbf{p}"] \\
\mathrm{Hom}_{\mathbf{S}}(\mathbf{p}(c),\mathbf{p}(a)) \arrow[r, "\mathbf{p}(f)\circ-"] & \mathrm{Hom}_{\mathbf{S}}(\mathbf{p}(c),\mathbf{p}(b))
\end{tikzcd}$$

is a pullback of sets.

Notation 2. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. For $a, b \in \mathrm{Obj}(\mathbf{X})$ and $f \in \mathrm{Hom}_{\mathbf{X}}(a,b)$, we say that a is *over* $\mathbf{p}(a)$ and f is *over* $\mathbf{p}(f)$. In a diagram, we have

$$\begin{tikzcd}
\mathbf{X} \arrow[d, "\mathbf{p}"'] & a \arrow[r, "f"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\
\mathbf{S} & \mathbf{p}(a) \arrow[r, "\mathbf{p}(f)"] & \mathbf{p}(b)
\end{tikzcd}$$

Definition 3. Let $\mathbf{S}$ be a site. A category $\mathbf{X}$ over $\mathbf{S}$ via $\mathbf{p}$ is called a *category fibred* over the site $\mathbf{S}$ if for every morphism $r : u \to v$ in $\mathbf{S}$ and every object $b \in \mathrm{Obj}(\mathbf{X})$ over v, there exists an object $a \in \mathrm{Obj}(\mathbf{X})$ over u and a strongly Cartesian morphism $f : a \to b$ over r. Such an object a is called a *pullback* of b along r, and is often denoted by r^*b.

Definition 4. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a category fibred over $\mathbf{S}$. For every object $u \in \mathrm{Obj}(\mathbf{S})$, the *fiber* of $\mathbf{X}$ over u is the category $\mathbf{X}_u$ given by

$$\mathrm{Obj}(\mathbf{X}_u) = \{a \in \mathrm{Obj}(\mathbf{X}) \mid \mathbf{p}(a) = u\}, \quad \mathrm{Hom}_{\mathbf{X}_u}(a,b) = \{f \in \mathrm{Hom}_{\mathbf{X}}(a,b) \mid \mathbf{p}(f) = \mathrm{id}_u\}.$$

Remark 5. Note that in Definition 3, the pullback r^*b of an object b along a morphism r is not necessarily unique. Yang: To be continued.

Yang: Why do we need the Cartesian morphisms exists?

Remark 6. Yang: presheaves as category fibered in set, right?

Slogan *Presheaf is a category fibered in sets.*

Definition 7. A *prestack* over the site $\mathbf{S}$ is a category $\mathbf{X}$ fibered in groupoids over $\mathbf{S}$ via $\mathbf{p} : \mathbf{X} \to \mathbf{S}$. Yang: To be revised.

Remark 8. Let $\mathbf{S}$ be a site. A presheaf of sets on $\mathbf{S}$ can be viewed as a functor $\mathbf{S}^{op} \to \mathbf{Set}$. A prestack over $\mathbf{S}$ can be viewed as a functor $\mathbf{S}^{op} \to \mathbf{Grpd}$ by associating to each object $u \in \mathrm{Obj}(\mathbf{S})$ the fiber category $\mathbf{X}_u$, which is a groupoid, and to each morphism $u \to v$ in $\mathbf{S}$ the pullback functor

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$\mathbf{X}_v \to \mathbf{X}_u$. Thus, prestacks can be seen as a generalization of presheaves of sets, where the values are groupoids instead of sets. Yang: To be checked.

Slogan *Prestacks are "presheaf remembering automorphisms".*

Yang: Where is the 2-category?

Theorem 9 (Yoneda 2-Lemma)**.** Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ and $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Then the functor

$$\mathsf{Fun}_{\mathbf{S}}(\mathbf{X}, \mathbf{Y}) \to (\mathbf{p}_*, \mathbf{q}_*)$$

given by $\Phi \mapsto \Phi_*$ is an equivalence of categories. Yang: To be revised.

Theorem 10. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$, $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$, and $\mathbf{r} : \mathbf{Z} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Let $\Phi : \mathbf{X} \to \mathbf{Z}$ and $\Psi : \mathbf{Y} \to \mathbf{Z}$ be morphisms of prestacks over $\mathbf{S}$. Then the fiber product $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ exists in the category of prestacks over $\mathbf{S}$. Yang: To be checked.

2 Descent conditions

Definition 11. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. Let $U \in \mathsf{Obj}(\mathbf{S})$ and $\{U_i \to U\}$ be a covering in $\mathbf{S}$. A *descent datum* for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ consists of

- a collection of objects $a_i \in \mathsf{Obj}(\mathbf{X}_{U_i})$ for each i,
- a collection of isomorphisms $\varphi_{ij} : a_j|_{U_{ij}} \to a_i|_{U_{ij}}$ in $\mathbf{X}_{U_{ij}}$ for each pair (i, j), where $U_{ij} = U_i \times_U U_j$,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}}$$

holds for all triples (i, j, k), where $U_{ijk} = U_i \times_U U_j \times_U U_k$. Yang: To be checked.

Definition 12. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. A descent datum $(\{a_i\}, \{\varphi_{ij}\})$ for objects of $\mathbf{X}$ relative to a covering $\{U_i \to U\}$ in $\mathbf{S}$ is called *effective* if there exists an object $a \in \mathsf{Obj}(\mathbf{X}_U)$ and isomorphisms $\psi_i : a|_{U_i} \to a_i$ in $\mathbf{X}_{U_i}$ such that for all pairs (i, j), the diagram

$$\begin{tikzcd} a|_{U_{ij}} \arrow[r, "\psi_j|_{U_{ij}}"] \arrow[d, "\psi_i|_{U_{ij}}"'] & a_j|_{U_{ij}} \arrow[d, "\varphi_{ij}"] \\ a_i|_{U_{ij}} \arrow[r, "\varphi_{ij}"'] & a_j|_{U_{ij}} \end{tikzcd}$$

commutes. Yang: To be checked.

Slogan *Descent data are like gluing data for objects, and effectiveness means that the glued object exists.*

3 Stacks

Definition 13. Let $\mathbf{S}$ be a site. A prestack $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *stack* over the site $\mathbf{S}$ if for every object $U \in \mathsf{Obj}(\mathbf{S})$ and every covering $\{U_i \to U\}$ in $\mathbf{S}$, the descent data for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ are effective. Yang: To be revised.

Slogan *Stacks to prestacks are like sheaves to presheaves.*

Definition 14. Let $\mathbf{S}$ be a site, and let G be a group object in $\mathbf{S}$ acting on an object $X \in \mathsf{Obj}(\mathbf{S})$. The *quotient stack* $[X/G]$ is the stack over $\mathbf{S}$ defined as follows:

- For each object $U \in \mathsf{Obj}(\mathbf{S})$, the groupoid $[X/G](U)$ has as objects the pairs (P, f), where P is a G-torsor over U and $f : P \to X$ is a G-equivariant morphism.
- Morphisms between two objects (P, f) and (P', f') in $[X/G](U)$ are given by G-equivariant morphisms $\varphi : P \to P'$ such that $f' \circ \varphi = f$.

The assignment $U \mapsto [X/G](U)$ defines a stack over $\mathbf{S}$. Yang: To be checked.

Appendix