
Kodaira Vanishing Theorem



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1 Preliminary

Theorem 1 (Serre Duality). Let X be a Cohen-Macaulay projective variety of dimension n over k and D a divisor on X . Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \dots, n.$$

Theorem 2 (Log Resolution of Singularities). Let X be an irreducible reduced algebraic variety over \mathbb{C} (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z . Then there is a smooth variety (or analytic space) Y and a projective morphism $f : Y \rightarrow X$ such that

- (a) f is an isomorphism over $X - (\text{Sing}(X) \cup \text{Supp } Z)$,
- (b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and
- (c) $\text{Exc}(f) \cup D$ is an snc divisor.

Theorem 3 (Lefschetz Hyperplane Theorem). Let X be a smooth projective variety of dimension n over \mathbb{C} and Y a hyperplane section of X . Then the restriction map

$$H^k(X, \mathbb{C}) \rightarrow H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 4 (Hodge Decomposition). Let X be a smooth projective variety of dimension n over \mathbb{C} . Then for any k , there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 3 and Theorem 4, we have the following lemma.

Lemma 5. Let X be a smooth projective variety of dimension n over \mathbb{C} and Y a hyperplane section of X . Then the restriction map $r_k : H^k(X, \mathbb{C}) \rightarrow H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \rightarrow H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \rightarrow H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n - 1$ and an injection for $p = n - 1$.

Theorem 6 (Leray spectral sequence). Let $f : Y \rightarrow X$ be a morphism of varieties and \mathcal{F} a coherent sheaf on Y . Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{F}) \implies H^{p+q}(Y, \mathcal{F}).$$

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Lemma 7. Let X be a smooth projective variety over k and \mathcal{L} a line bundle on X . Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \rightarrow X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D . It induces a homomorphism $\mathcal{L}^{-m} \rightarrow \mathcal{O}_X$. Consider the \mathcal{O}_X -algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} \right) / (\mathcal{L}^{-m} \rightarrow \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then \mathcal{A} is a finite \mathcal{O}_X -algebra. Let $Y := \text{Spec}_X \mathcal{A}$. Then Y is a finite \mathcal{O}_X -scheme and the natural morphism $f : Y \rightarrow X$ is finite and surjective.

For every $x \in X$, let \mathcal{L} locally generated by t near x . Then \mathcal{O}_Y locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y . Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. \square

Remark 8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D'_i = f^*(D_i)$. Then $D' + \sum_{i=1}^k D'_i$ is snc on Y by considering the local regular system of parameters.

Lemma 9. Let $f : Y \rightarrow X$ be a finite surjective morphism of projective varieties and \mathcal{L} a line bundle on X . Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^* \mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^* \mathcal{L}) \cong H^i(X, f_* \mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$ splits by the trace map $(1/n) \text{Tr}_{Y/X}$. Thus we have $f_* \mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_* \mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. \square

Theorem 10 (Kodaira Vanishing Theorem). Let X be a smooth projective variety of dimension n over k of characteristic 0 and A an ample divisor on X . Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 7 and 9, after taking a multiple of A , we can assume that A is effective. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(-A) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_A \rightarrow 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \rightarrow H^{i-1}(X, \mathcal{O}_A) \rightarrow H^i(X, \mathcal{O}_X(-A)) \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 5 and Serre duality (Theorem 1). \square

3 Vanishing theorem for nef and big divisors

Lemma 11. Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0, A an ample divisor and E an snc divisor on X . Then

$$H^i(X, K_X + A + E) = 0, \quad \forall i > 0.$$

Proof. Let $E = \sum_{i=1}^k E_i$. We induct on k . Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-A - \sum_{i=1}^k E_i) \rightarrow \mathcal{O}_X(-A - \sum_{i=1}^{k-1} E_i) \rightarrow \mathcal{O}_{E_k}(-A - \sum_{i=1}^{k-1} E_i) \rightarrow 0.$$

Yang: To be completed. \square

Theorem 12 (Kawamata-Viehweg Vanishing Theorem for nef and big divisors). Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0 and D a nef and big \mathbb{R} -divisor on X . Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Proof. Yang: To be completed. \square

4 Kawamata-Viehweg Vanishing Theorem for klt pairs

Lemma 13. Let X be a projective variety, \mathcal{L} a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \rightarrow X$ and a line bundle \mathcal{L}' on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X , then we can take f such that f^*D is an snc divisor on Y .

Proof. We can assume that \mathcal{L} is very ample by writing it as a difference of two very ample line

bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & \mathbb{P}^N \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{\varphi_{\mathcal{L}}} & \mathbb{P}^N \end{array}$$

where $g : [x_0 : \dots : x_N] \mapsto [x_0^m : \dots : x_N^m]$. The morphism f is finite and surjective since so is g . Let $\mathcal{L}' := \psi^* \mathcal{L} \mathcal{O}(1)$.

For smoothness, we can compose g with a general automorphism of \mathbb{P}^N . Then the conclusion follows from [Har13, Chapter III, Theorem 10.8]. \square

Theorem 14. Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0 and D a nef \mathbb{R} -divisor on X . Suppose that $\lceil D \rceil - D$ has snc support. Then

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall i > 0.$$

Proof. By the Bertini, we can assume that $A := D$ is ample and a \mathbb{Q} -divisor by adding a sufficiently small ample divisor and adjusting the coefficients slightly. Set $M := \lceil D \rceil$. Let

$$B := \sum_{i=1}^k b_i B_i := \lceil D \rceil - D = M - A, \quad b_i \in (0, 1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k . Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 13 and 9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 7 on B_k , we can find a finite surjective morphism $f : X' \rightarrow X$ such that $f^*B_k = cB'_k$, $B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^k B'_i$ is an snc divisor on X' . Let $B' = \sum_{i=1}^{k-1} B'_i$, $A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$ vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^* \mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 9. \square

Lemma 15 (ref. [KM98, Theorem 5.10, 5.22]). Let (X, B) be a klt pair over \mathbf{k} of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Theorem 16 (Kawamata-Viehweg Vanishing Theorem for klt pairs). Let (X, B) be a klt pair over \mathbf{k} of characteristic 0. Let D be a nef \mathbb{R} -divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

Proof. Let $f : \tilde{X} \rightarrow X$ be a resolution such that $\text{Supp } f^*B \cup \text{Exc } f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X}, \tilde{B})} + f^*D,$$

where $\tilde{B} \in (0, 1)$ has snc support and E is an effective exceptional divisor.

Claim 17. The higher direct image sheaves $R^i f_*(\mathcal{O}_{\tilde{X}}(f^*(K_{(X,B)} + D) + E))$ vanish for $i > 0$ and $f_*(\mathcal{O}_{\tilde{X}}(f^*(K_{(X,B)} + D) + E)) \cong \mathcal{O}_X(K_{(X,B)} + D)$.

By the Claim, we have

$$H^i(\tilde{X}, K_{(\tilde{X}, \tilde{B})} + f^*D) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 14. □

Proof of Claim 17. Let $\mathcal{F} := \mathcal{O}_{\tilde{X}}(f^*(K_{(X,B)} + D) + E)$. **Yang: To be completed.** □

References

- [Har13] Robin Hartshorne. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 2013.
- [KM98] János Kollár and Shigefumi Mori. “Birational geometry of algebraic varieties”. In: *(No Title)* 134 (1998).