Regularity and Smoothness

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Smoothness

1 Modules of differentials and derivations

In this subsection, let R be a ring and A an R-algebra.

Definition 1 (Derivation)**.** A *derivation* of A over R is an R-linear map $\partial : A \to M$ with an A-module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module M, the set of all derivations of A over R into M forms an A-module, denoted by $\mathrm{Der}_R(A, M)$.

Given a module homomorphism $f : M \to N$ of A-modules and a derivation $\partial \in \mathrm{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of A over R into N.

Proposition 2. The functor $\mathrm{Der}_R(A, -)$ is representable. The representing object is denoted by $\Omega_{A/R}$, which is called the *module of differentials* of A over R.

Proof. First suppose A is a free R-algebra with a set of generators $a_\lambda, \lambda \in \Lambda$. Then an R-derivation $\partial \in \mathrm{Der}_R(A, M)$ is uniquely determined by its values on the generators a_λ. Let

$$\Omega_{A/R} \coloneqq \bigoplus_{\lambda \in \Lambda} A \cdot \mathrm{d}a_\lambda$$

and $\mathrm{d} : A \to \Omega_{A/R}$ be the R-derivation defined by $a_\lambda \mapsto \mathrm{d}a_\lambda$. For any R-derivation $\partial \in \mathrm{Der}_R(A, M)$, we can define a unique A-module homomorphism $\Phi_\partial : \Omega_{A/R} \to M$ by sending $\mathrm{d}a_\lambda$ to $\partial(a_\lambda)$ such that $\partial = \Phi_\partial \circ \mathrm{d}$. This gives a bijection

$$\mathrm{Der}_R(A, M) \cong \mathrm{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_\partial.$$

Now suppose $A = F/I$ is an arbitrary R-algebra, where F is a free R-algebra and I is an ideal of F. Then we can define the module of differentials

$$\Omega_{A/R} \coloneqq \big(\Omega_{F/R} \otimes_F A\big) \Big/ \sum_{f \in I} A \cdot \mathrm{d}f.$$

The R-linear map $\mathrm{d}_A : F \otimes_F A \xrightarrow{\mathrm{d}_F} \Omega_{F/R} \otimes_F A \to \Omega_{A/R}$ is a derivation of A over R.

For any R-derivation $\partial \in \mathrm{Der}_R(A, M)$, note that $F \to A \xrightarrow{\partial} M$ is an R-derivation of F over R into M. Then we get an F-module homomorphism $\Omega_F \to M$. It gives an A-module homomorphism $\Omega_F \otimes_F A \to M, \mathrm{d}f \otimes 1 \mapsto \partial f$. This map factors into $\Omega_F \otimes_F A \to \Omega_{A/R}$ and $\Phi_\partial : \Omega_{A/R} \to M$. Since Φ_∂ is A-linear and $\Omega_{A/R}$ is generated by $\mathrm{d}a_\lambda$ as A-module, such Φ_∂ is unique. ∎

Corollary 3. Suppose A is of finite type over R. Then the module of differentials $\Omega_{A/R}$ is a finitely generated A-module.

Remark 4. Let B be an A-algebra, M an A-module and N a B-module. If there is a homomorphism of A-modules $M \to N$, then we can extend it to a homomorphism of B-modules $M \otimes_A B \to N$ by sending $m \otimes b$ to $m \cdot b$. And such extension is unique in the sense of following commutative diagram:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N . \\ M \otimes_A B \arrow[ur, "\exists!"'] & \end{tikzcd}$$

Hence we get a natural bijection

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(M \otimes_A B, N).$$

Proposition 5. Let A, R' be R-algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

Date: June 26, 2025, **Author**: Tianle Yang, My Website

Proof. We check the universal property of $\Omega_{A/R}\otimes_A A'$. First, the map

$$\mathrm{d}_{A'} : A\otimes_R R' \to \Omega_{A/R}\otimes_R R' \cong \Omega_{A/R}\otimes_A A', \quad a\otimes r\mapsto \mathrm{d}a\otimes r$$

is an R'-derivation of A' into $\Omega_{A/R}\otimes_A A'$. For any R'-derivation $\partial' : A' \to M$ into an A'-module M, we can compose it with the homomorphism $A'\to A$ and get an R-derivation $\partial : A\to M$. By the universal property of $\Omega_{A/R}$, there is a unique A-module homomorphism $\Phi : \Omega_{A/R}\to M$ such that $\partial = \Phi\circ \mathrm{d}_A$. Then we can extend it to an A'-module homomorphism $\Phi' : \Omega_{A/R}\otimes_A A' \to M$ by Remark 4. By the construction, we have $\Phi'\circ \mathrm{d}_{A'} = \partial'$. □

Proposition 6. Let A be an R-algebra and S a multiplicative set of A. Then we have an isomorphism

$$\Omega_{S^{-1}A/R}\cong S^{-1}\Omega_{A/R}.$$

Proof. Let

$$\mathrm{d}_{S^{-1}A} : S^{-1}A \to S^{-1}\Omega_{A/R}, \quad \frac{a}{s}\mapsto \frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}.$$

By direct computation, $\mathrm{d}_{S^{-1}A}$ is an R-derivation of $S^{-1}A$ over R into $S^{-1}\Omega_{A/R}$. For any R-derivation $\partial : S^{-1}A\to M$ into an $S^{-1}A$-module M, we can get an $S^{-1}A$-module homomorphism $\Phi' : S^{-1}\Omega_{A/R}\to M$ as proof of Proposition 5. We have

$$\partial(s\cdot \frac{a}{s}) = s\partial(\frac{a}{s}) + \frac{a}{s}\partial s.$$

It follows that

$$\partial(\frac{a}{s}) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(\mathrm{d}a) - a\Phi'(\mathrm{d}s)}{s^2} = \Phi'(\frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}).$$

Thus, $\Phi'\circ \mathrm{d}_{S^{-1}A} = \partial$. □

Theorem 7. Let A be an R-algebra and B an A-algebra. Then there is a natural short exact sequence

$$\Omega_{A/R}\otimes_A B\to \Omega_{B/R}\to \Omega_{B/A}\to 0$$

of B-modules.

Proof. Let $\mathrm{d}_{A/R} : A\to \Omega_{A/R}$ be the R-derivation of A over R. The map $A\to B\xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R}$ induces a B-linear map

$$u : \Omega_{A/R}\otimes_A B\to \Omega_{B/R}, \quad \mathrm{d}_{A/R}(a)\otimes b\mapsto b\mathrm{d}_{B/R}(a).$$

The map $\mathrm{d}_{B/A}$ is an A-derivation and hence R-derivation. Then it induces a B-linear map

$$v : \Omega_{B/R}\to \Omega_{B/A}, \quad \mathrm{d}_{B/R}(b)\mapsto \mathrm{d}_{B/A}(b).$$

Since $\Omega_{B/A}$ is generated by elements of the form $\mathrm{d}_{B/A}(b)$ for $b\in B$, the map v is surjective. And clearly $\mathrm{d}_{B/A}(a) = a\mathrm{d}_{B/A}(1) = 0$ for $a\in A$.
Consider the composition $B\xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R}\to \Omega_{B/R}/\operatorname{Im} u$. For every $a\in A, b\in B$, we have

$$[\mathrm{d}_{B/R}(ab)] = [b\mathrm{d}_{B/R}(a) + a\mathrm{d}_{B/R}(b)] = [b\mathrm{d}_{B/R}(a)] + [a\mathrm{d}_{B/A}(b)] = [a\mathrm{d}_{B/A}(b)].$$

Hence it is indeed an A-derivation of B. Then it induces a B-linear map

$$\varphi : \Omega_{B/A}\to \Omega_{B/R}/\operatorname{Im} u, \quad \mathrm{d}_{B/A}(b)\mapsto [\mathrm{d}_{B/R}(b)].$$

The map φ is surjective since $\Omega_{B/R}$ is generated by elements of the form $\mathrm{d}_{B/R}(b)$ for $b\in B$. Note that the composition

$$\Omega_{B/A}\xrightarrow{\varphi} \Omega_{B/R}/\operatorname{Im} u\to \Omega_{B/A}/\operatorname{Ker} v$$

is the identity map. Thus, φ is injective and hence an isomorphism. In particular, we have $\operatorname{Ker} v = \operatorname{Im} u$. □

Remark 8. The exact sequence in Theorem 7 is left exact if and only if every R-derivation of A into B-module extends to an R-derivation of B into B-module.
Yang: To be completed.

Theorem 9. Let A be an R-algebra and I an ideal of A. Set $B \coloneqq A/I$. Then there is a natural short exact sequence

$$I/I^2 \to \Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to 0$$

of B-modules.

Proof. Suppose $A = F/\mathfrak{b}$ for some free R-algebra F and an ideal $\mathfrak{b}$ of F. Let $\mathfrak{a}$ be the preimage of I in F. Let $\mathrm{d}\mathfrak{b}$ (resp. $\mathrm{d}\mathfrak{a}$) denote the image of $\mathfrak{b}$ (resp. $\mathfrak{a}$) in $\Omega_{F/R}$. Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \to (\mathrm{d}\mathfrak{a} \otimes_F B)/(\mathrm{d}\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows. □

Definition 10. Let k be a field and A an integral k-algebra of finite type of dimension n. We say A is *smooth at* $\mathfrak{p} \in \operatorname{Spec} A$ if the module of differentials $\Omega_{A,\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank n.

Example 11. Let K/k be a finite generated field extension and k' be the algebraic closure of k in K. Then

$$\dim_{\mathsf{K}} \Omega_{\mathsf{K}/\mathsf{k}} = \operatorname{trdeg}(\mathsf{K}/\mathsf{k}) + \dim_{\mathsf{k}'} \Omega_{\mathsf{k}'/\mathsf{k}},$$

and $\dim_{\mathsf{k}'} \Omega_{\mathsf{k}'/\mathsf{k}} = 0$ if and only if k' is separable over k.
First suppose $\mathsf{K} = \mathsf{k}'$ is algebraic over k. Suppose k'/k is separable. For every $\alpha \in \mathsf{k}'$, suppose $f(\alpha) = 0$ for $f \in \mathsf{k}[T]$. Then $\mathrm{d}f(\alpha) = f'(\alpha)\mathrm{d}\alpha = 0$. By the separability of k'/k, we have $f'(\alpha) \neq 0$. It follows that $\mathrm{d}\alpha = 0$. Conversely, let $\alpha \in \mathsf{k}'$ be a inseparable element over k. Since $\mathsf{k}[\alpha] \to \mathsf{k}[\alpha], \alpha^n \mapsto n\alpha^{n-1}$ is a non-zero R-derivation, we have $\Omega_{\mathsf{k}[\alpha]/\mathsf{k}} \neq 0$. By induction on number of generated elements, choosing a middle field $\mathsf{k} \subset \mathsf{k}'' \subset \mathsf{k}'$, at least one of $\Omega_{\mathsf{k}''/\mathsf{k}}$ and $\Omega_{\mathsf{k}'/\mathsf{k}''}$ is non-zero. Then $\Omega_{\mathsf{K}/\mathsf{k}} \neq 0$ by Theorem 7.
Then suppose $\mathsf{k}' = \mathsf{k}$. By the Noether's Normalization Lemma, we can find a finite set of elements $T_1, \cdots, T_n \in \mathsf{K}$ such that K is algebraic over $\mathsf{k}'(T_1, \dots, T_n)$. Note that we can choose T_i such that $\mathsf{K}/\mathsf{k}'(T_1, \cdots, T_n)$ is separable. To see this, if $\alpha \in \mathsf{K}$ is an inseparable element over $\mathsf{k}'(T_1, \cdots, T_n)$, then by replacing a suitable T_i with α, we reduce the inseparable degree of $\mathsf{K}/\mathsf{k}'(T_1, \cdots, T_n)$.
Since $\mathsf{K}/\mathsf{k}'(T_1, \cdots, T_n)$ is finite, every k-derivation of $\mathsf{k}'(T_1, \cdots, T_n)$ into K-module extends to a k-derivation of K into K-module. Then by Remark 8, we have

$$0 \to \Omega_{\mathsf{k}'(T_1,\cdots,T_n)/\mathsf{k}} \otimes_{\mathsf{k}'(T_1,\cdots,T_n)} \mathsf{K} \to \Omega_{\mathsf{K}/\mathsf{k}} \to \Omega_{\mathsf{K}/\mathsf{k}'(T_1,\cdots,T_n)} \to 0.$$

Finally, note that every k-derivation ∂ of k' into K-module can be extended to $\mathsf{k}'[T_1, \cdots, T_n]$ by setting $\partial T_i = 0$. Thus, we have

$$0 \to \Omega_{\mathsf{k}'/\mathsf{k}} \otimes_{\mathsf{k}'} \mathsf{k}'[T_1, \cdots, T_n] \to \Omega_{\mathsf{k}'[T_1,\cdots,T_n]/\mathsf{k}} \to \Omega_{\mathsf{k}'[T_1,\cdots,T_n]/\mathsf{k}'} \to 0.$$

This follows that

$$\dim_{\mathsf{K}} \Omega_{\mathsf{K}/\mathsf{k}} = \dim_{\mathsf{K}} \Omega_{\mathsf{K}/\mathsf{k}'} + \dim_{\mathsf{k}'} \Omega_{\mathsf{k}'/\mathsf{k}}.$$

2 Applications to affine varieties

Let k be arbitrary field, $A = \mathsf{k}[T_1, \dots, T_n]$ and $\mathfrak{m}$ a maximal ideal of A such that $\kappa(\mathfrak{m})$ is separable over k. We try to give an explanation of Zariski's tangent space at $\mathfrak{m}$ using the language of derivation. We know that $\Omega_{A/\mathsf{k}} = \bigoplus_{i=1}^n A\mathrm{d}T_i$, thus $\Omega_{A_\mathfrak{m}/\mathsf{k}} \cong \bigoplus_{i=1}^n A_\mathfrak{m}\mathrm{d}T_i$. Then

$$\operatorname{Der}_{\mathsf{k}}(A_\mathfrak{m}, A_\mathfrak{m}) \cong \operatorname{Hom}_{\mathsf{k}}(\Omega_{A_\mathfrak{m}/\mathsf{k}}, A_\mathfrak{m}) \cong \bigoplus_{i=1}^n A_\mathfrak{m}\partial_i,$$

where $\partial_i \in \operatorname{Der}_{\mathsf{k}}(A_\mathfrak{m}, A_\mathfrak{m})$ is the derivation defined by $\mathrm{d}T_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. It coincides with the usual derivation $f \mapsto \partial f/\partial T_i$. Consider the restriction of ∂_i to $\mathfrak{m}$ and take values in the residue field $\kappa(\mathfrak{m})$, we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1,\cdots,\partial_n)^T} A_\mathfrak{m}^n \to \kappa(\mathfrak{m})^n.$$

Since $\kappa(\mathfrak{m})$ is separable over k, we claim that $\operatorname{Ker}\Phi = \mathfrak{m}^2$. Indeed, by Remark 12, we can write every $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ as $\sum_i a_i g_i$. Then

$$\frac{\partial f}{\partial T_i} = a_i \frac{\partial g_i}{\partial T_i} + g_i \frac{\partial a_i}{\partial T_i}.$$

Since g_i is separable, the image of $\partial g_i/\partial T_i$ in $\kappa(\mathfrak{m})$ is not zero. Hence $\Phi(f) \neq 0$. By the claim, Φ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$ of $\kappa(\mathfrak{m})$-vector spaces. Then we get

$$T_{A,\mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where $x \in \mathbb{A}^n_{\mathsf{k}}$ is the point corresponding to $\mathfrak{m}$. This coincides with the usual tangent space at x in language of differential geometry.

Remark 12. Let k be arbitrary field, $A = \mathsf{k}[T_1, \cdots, T_n]$ and g_i irreducible polynomials in one variable T_i over k. Then for every $f \in A$, we can write

$$f = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{Z}^n_{\geq 0}} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \leq \deg g_i.$$

This is called the *Taylor expansion of f with respect to $g_1, \cdots, g_n$*.
When $n = 1$, it follows from division algorithm. For $n > 1$, we can use induction on n. Let $\mathsf{K} = \mathsf{k}(T_1, \cdots, T_{n-1})$. Then we can write f as

$$f = \sum_{i=0}^r a_i g_n^i, \quad a_i \in \mathsf{K}[T_n], \quad \deg a_i < \deg g_n.$$

. Comparing the coefficients of two sides from the highest degree of T_n to the lowest degree, we see that

$$a_i \in \mathsf{k}[T_1, \cdots, T_{n-1}].$$

By induction hypothesis, the conclusion follows.

Let $B = A/I$ be a k of finite type, $I = (F_1, \ldots, F_m) \subset \mathfrak{m}$ and $\mathfrak{n}$ the image of $\mathfrak{m}$ in B. We have an exact sequence of $\kappa(\mathfrak{m})$-vector spaces

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} \colon \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of $F_1, \ldots, F_m$ is the $m \times n$ matrix

$$J(F_1, \ldots, F_m) \coloneqq \left(\frac{\partial F_i}{\partial T_j}\right)_{1\leq i\leq m, 1\leq j\leq n}$$

with entries in B.

Theorem 13. Setting as above. Then B is regular at $\mathfrak{n}$ if and only if the Jacobian matrix J has maximal rank $n - \dim B_\mathfrak{n}$ after taking values in the residue field $\kappa(\mathfrak{m})$.

Proof. We have an exact sequence

$$0 \to T_{B,\mathfrak{n}} \to T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa^m \to 0,$$

where Ψ sends $\partial \in T_{A,\mathfrak{m}}$ to $(\partial(F_1), \ldots, \partial(F_m))^T$. Note that the matrix of Ψ is just J^T, the transpose of the Jacobian matrix. Hence

$$\operatorname{rank} J = n - \dim_\kappa T_{B,\mathfrak{n}} \leq n - \dim B_\mathfrak{n}$$

and the equality holds if and only if B is regular at $\mathfrak{n}$. □

Remark 14. If $\kappa(\mathfrak{m})$ is not separable over k, then we still have the inequality

$$\operatorname{rank} J \leq n - \dim B_\mathfrak{n}.$$

Indeed, in any case, we have an exact sequence

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Hence $\dim_\kappa I/(I \cap \mathfrak{m}^2) = n - \dim B_\mathfrak{n}$. There is a $\kappa(\mathfrak{m})$-linear map

$$I/(I \cap \mathfrak{m}^2) \to \kappa(\mathfrak{m})^n, \quad [f] \mapsto (\partial_1(f), \ldots, \partial_n(f))^T,$$

and every row of the Jacobian matrix J is in the image of this map. Thus, the rank of J is at most $n - \dim B_\mathfrak{n}$. Hence if $\operatorname{rank} J = n - \dim B_\mathfrak{n}$, we can still see that B is regular at $\mathfrak{n}$. However, the converse does not hold in general.

Proposition 15. Let k be a field, $\mathbf{k}$ the algebraic closure of k, A a k-algebra of finite type and $A_{\mathbf{k}} := A \otimes_{\mathsf{k}} \mathbf{k}$. Yang: Suppose $A_{\mathbf{k}}$ is integral. Let $\mathfrak{m} \in \operatorname{mSpec} A$ and $\mathfrak{m}'$ be a maximal ideal of $A_{\mathbf{k}}$ lying over $\mathfrak{m}$. Then

(a) If $A_{\mathbf{k}}$ is regular at $\mathfrak{m}'$, then A is regular at $\mathfrak{m}$;

(b) suppose $\kappa(\mathfrak{m})$ is separable over k, the converse holds.

Proof. Regarding $J_{\mathfrak{m}}$ and $J_{\mathfrak{m}'}$ as matrices with entries in $\mathbf{k}$, they are the same and hence have the same rank. If $A_{\mathbf{k}}$ is regular at $\mathfrak{m}'$, since $\kappa(\mathfrak{m}) = \mathbf{k}$, then $\operatorname{rank} J_{\mathfrak{m}'} = n - \dim A_{\mathbf{k},\mathfrak{m}'}$. Note that $\dim A_{\mathbf{k},\mathfrak{m}'} = \operatorname{trdeg}(\mathscr{K}(A_{\mathbf{k}})/\mathbf{k}) = \operatorname{trdeg}(\mathscr{K}(A)/\mathsf{k}) = \dim A_{\mathfrak{m}}$, we have $\operatorname{rank} J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence A is regular at $\mathfrak{m}$.
Conversely, suppose A is regular at $\mathfrak{m}$ and $\kappa(\mathfrak{m})$ is separable over k. Then $\operatorname{rank} J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence $A_{\mathbf{k}}$ is regular at $\mathfrak{m}'$. □

Proposition 16. Let k be a field and A an integral k-algebra of finite type and of dimension n. Let $\mathbf{k}$ be the algebraic closure of k and $A_{\mathbf{k}} := A \otimes_{\mathsf{k}} \mathbf{k}$. Then A is smooth at $\mathfrak{p} \in \operatorname{Spec} A$ if and only if $A_{\mathbf{k}}$ is regular at every $\mathfrak{m}'$ over $\mathfrak{m}$.

Proof. Since $\Omega_{A_{\mathbf{k}}/\mathbf{k}} \cong \Omega_{A/\mathsf{k}} \otimes_A A_{\mathbf{k}}$ is free of rank n if and only if $\Omega_{A/\mathsf{k}}$ is free of rank n, we can assume that $\mathsf{k} = \mathbf{k}$. If A is smooth at $\mathfrak{p}$, then $\Omega_{A_{\mathfrak{p}}/\mathsf{k}} \cong \bigoplus A_{\mathfrak{p}} \mathrm{d}f_i$ is free of rank n. Let $\mathfrak{P}_i \in \operatorname{Der}_{\mathbf{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ be the derivation defined by $\mathrm{d}f_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. Then we have $\partial_i f_j = \delta_{ij}$ for $1 \leq i, j \leq n$. Then similar to above argument, we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{(\partial_1,\ldots,\partial_n)^T} \mathbf{k}^n.$$

This shows that $A_{\mathbf{k}}$ is regular at $\mathfrak{m}$.
Conversely, suppose $A_{\mathbf{k}}$ is regular at $\mathfrak{m}$. Note that $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A,\mathbf{k}} \otimes_A \mathbf{k}$ is surjective since $\Omega_{A_{\mathbf{k}}/\mathbf{k}} = 0$. Then by Nakayama's lemma, $\Omega_{A_{\mathfrak{m}}/\mathbf{k}}$ is generated by n elements as an $A_{\mathfrak{m}}$-module.
Note that $\dim_{\mathscr{K}(A)} \Omega_{\mathscr{K}(A)/\mathsf{k}} = \operatorname{trdeg}(\mathscr{K}(A)/\mathsf{k}) = \dim A_{\mathfrak{m}} = n$. Yang: By induction on transcendental degree.
Yang: By Nakayama's Lemma, $\Omega_{A_{\mathfrak{m}}/\mathsf{k}}$ is free of rank n as an $A_{\mathfrak{m}}$-module.
Yang: To be completed. □

Example 17. Let k be an imperfect field of characteristic $p > 2$. Suppose $\alpha = \beta^p \in \mathsf{k}$ and β is not in k. Let $A = \mathsf{k}[x,y]/(x^2 - y^p - \alpha)$ and $\mathfrak{m} = (x, y^p - \alpha) = (x)$. Note that $\mathfrak{m}$ is principal, so A is regular at $\mathfrak{m}$. However,

$$J_{\mathfrak{m}} = \left(\frac{\partial}{\partial x}(x^2 - y^p - \alpha), \frac{\partial}{\partial y}(y^p - \alpha) \right) = (2x, 0) = (0,0) \in M_{1\times 2}(\kappa(\mathfrak{m})).$$

Thus, A is not smooth at $\mathfrak{m}$. From the view of differentials, we have

$$\Omega_{A_{\mathfrak{m}}/\mathsf{k}} = A_{\mathfrak{m}}\mathrm{d}x \oplus A_{\mathfrak{m}}\mathrm{d}y / A_{\mathfrak{m}} \cdot x\mathrm{d}x = \kappa(\mathfrak{m})\mathrm{d}x \oplus A_{\mathfrak{m}}\mathrm{d}y,$$

which is not free as an $A_{\mathfrak{m}}$-module.