Notebook in Algebraic Geometry

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Setup and the first examples

1 Notations

All schemes are assumed to be separated. For a "scheme" which is not separated, we will use the term "prescheme".

Let A be a ring. We denote by $\operatorname{Spec} A$ the spectrum of A. For an ideal $I \subset A$, we use $V(I)$ to denote the closed subscheme of $\operatorname{Spec} A$ defined by I.

Let S be $\operatorname{Spec} \mathsf{k}$, $\operatorname{Spec} \mathcal{O}_K$ or an algebraic variety. An S-variety is an integral scheme X which is of finite type and flat over S. For an algebraic variety, we mean a k-variety.

We will use k, K to denote fields, and $\mathbf{k}, \mathbf{K}$ to denote their algebraically closure relatively.

Let X be an integral scheme. We denote by $\mathscr{K}(X)$ the function field of X. For a closed point $x \in X$, we denote by $\kappa(x)$ the residue field of x.

We denote the category of S-varieties by $\mathbf{Var}_S$. We denote by $X(T)$ the set of T-points of X, that is, the set of morphisms $T \to X$.

Let X be an algebraic variety over k. A geometrical point is referred a morphism $\operatorname{Spec} \mathbf{k} \to X$.

When refer a point (may not be closed) in a scheme, we will use the notation $\xi \in X$. We use Z_ξ to denote the Zariski closure of $\{\xi\}$ in X. When we talk about a closed point on an algebraic variety, we will use the notation $x \in X(\mathbf{k})$.

1.1 Separated and proper morphisms

2 Examples

Example 1. Let $\mathbf{k}$ be an algebraically closed field and A the localization of $\mathbf{k}[x]$ at (x). Let $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} A[y]$. There are three types of points in X:

(i) closed points with residue field $\mathbf{k}$, like $p = (x, y - a)$;

(ii) closed points with residue field $\mathbf{k}(y)$, like $P = (xy - 1)$;

(iii) non-closed points, like $\eta_1 = (x), \eta_2 = (y), \eta_3 = (x - y)$.

3 Preparation in commutative algebra

3.1 Nakayama's Lemma Yang: To be completed

Theorem 2 (Nakayama's Lemma)**.** Let A be a ring and $\mathfrak{M}$ be its Jacobi radical. Suppose M is a finitely generated A-module. If $\mathfrak{a}M = M$ for $\mathfrak{a} \subset \mathfrak{M}$, then $M = 0$.

Proof. Suppose M is generated by $x_1, \cdots, x_n$. Since $M = \mathfrak{a}M$, formally we have $(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(\mathfrak{a})$. Then $(\Phi - \mathrm{id})(x_1, \cdots, x_n)^T = 0$. Note that $\det(\Phi - \mathrm{id}) = 1 + a$ for $a \in \mathfrak{a} \subset \mathfrak{M}$. Then $\Phi - \mathrm{id}$ is invertible and then $M = 0$. □

Proposition 3 (Geometric form of Nakayama's Lemma)**.** Let $X = \operatorname{Spec} A$ be an affine scheme, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proof. Yang: To be completed. □

Corollary 4.

Proof. Yang: To be completed. □

3.2 Associated prime ideals

This part refers to [**Mat70**].

Definition 5 (Associated prime ideals)**.** Let A be a noetherian ring and M an A-module. The *associated prime ideals* of M are the prime ideals $\mathfrak{p}$ of form $\operatorname{Ann}(x)$ for some $x \in M$. The set of associated prime ideals of M is denoted by

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$\mathrm{Ass}(M)$.

Example 6. Let $A = \mathbf{k}[x, y]/(xy)$ and $M = A$. First we see that $(x) = \mathrm{Ann}\, y, (y) = \mathrm{Ann}\, x \in \mathrm{Ass}\, M$. Then we check other prime ideals. For (x, y), if $xf = yf = 0$, then $f \in (x) \cap (y) = (0)$. If $(x - a) = \mathrm{Ann}\, f$ for some f, note that $y \in (x - a)$ for $a \in \mathbf{k}^*$, then $f \in (x)$. Hence $f = 0$. Therefore $\mathrm{Ass}\, M = \{(x), (y)\}$.

Example 7. Let $A = \mathbf{k}[x, y]/(x^2, xy)$ and $M = A$. The underlying space of $\mathrm{Spec}\, A$ is the y-axis since $\sqrt{(x^2, xy)} = (x)$. First note that $(x) = \mathrm{Ann}\, y, (x, y) = \mathrm{Ann}\, x \in \mathrm{Ass}\, M$. For $(x, y - a)$ with $a \in \mathbf{k}^*$, easily see that $xf = (y - a)f = 0$ implies $f = 0$ since $A = \mathbf{k} \cdot x \oplus \mathbf{k}[y]$ as $\mathbf{k}$-vector space. Hence $\mathrm{Ass}\, M = \{(x), (x, y)\}$.

Let A be a noetherian ring and M an A-module. Note that $S^{-1}M = 0$ if and only if $S \cap \mathrm{Ann}\, M \neq \emptyset$. Then the set

$$\{\mathfrak{p} \in \mathrm{Spec}\, A \colon M_{\mathfrak{p}} \neq 0\}$$

is equal to $V(\mathrm{Ann}\, M)$.

Definition 8. Let A be a noetherian ring and M an A-module. The *support* of M is the closed subset $V(\mathrm{Ann}\, M)$ of $\mathrm{Spec}\, A$, denoted by $\mathrm{Supp}\, M$.

Lemma 9. Let A be a noetherian ring and M an A-module. Then the maximal element of the set

$$\{\mathrm{Ann}\, x \colon x \in M_{\mathfrak{p}}, x \neq 0\}$$

belongs to $\mathrm{Ass}\, M$.

Proof. We just need to show that such $\mathrm{Ann}\, x$ is prime. Otherwise, there exist $a, b \in A$ such that $ab \in \mathrm{Ann}\, x$ but $a, b \notin \mathrm{Ann}\, x$. It follows that $\mathrm{Ann}\, x \subsetneq \mathrm{Ann}\, ax$ since $b \in \mathrm{Ann}\, ax \setminus \mathrm{Ann}\, x$. This contradicts the maximality of $\mathrm{Ann}\, x$. □

An element $a \in A$ is called a zero divisor for M if $M \to aM, m \mapsto am$ is not injective.

Corollary 10. Let A be a noetherian ring and M an A-module. Then

$$\{\text{zero divisors for } M\} = \bigcup_{\mathfrak{p} \in \mathrm{Ass}\, M} \mathfrak{p}.$$

Lemma 11. Let A be a noetherian ring and M an A-module. Then $\mathfrak{p} \in \mathrm{Ass}_A\, M$ iff $\mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}\, M_{\mathfrak{p}}$.

Proof. Suppose $\mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}\, M_{\mathfrak{p}}$. Let $\mathfrak{p}A_{\mathfrak{p}} = \mathrm{Ann}\, y_0/c$ with $y_0 \in M$ and $c \in A \setminus \mathfrak{p}$. For $a \in \mathrm{Ann}\, y_0$, $ay_0 = 0$. Then $a/1 \in \mathfrak{p}A_{\mathfrak{p}}$. It follows that $a \in \mathfrak{p}$. Hence $\mathrm{Ann}\, y_0 \subset \mathfrak{p}$.
Inductively, if $\mathrm{Ann}\, y_n \subsetneq \mathfrak{p}$, then there exists $b_n \in A \setminus \mathfrak{p}$ such that $y_{n+1} := b_n y_n$, $\mathrm{Ann}\, y_{n+1} \subset \mathfrak{p}$ and $\mathrm{Ann}\, y_n \subsetneq \mathrm{Ann}\, y_{n+1}$. To see this, choose $a_n \in \mathfrak{p} \setminus \mathrm{Ann}\, y_n$. Then $(a_n/1)y_n = 0$ since $a_n/1 \in \mathfrak{p}A_{\mathfrak{p}}$. By definition, there exist $b_n \in A \setminus \mathfrak{p}$ such that $a_n b_n y_n = 0$. This process must terminate since A is noetherian. Thus $\mathrm{Ann}\, y_n = \mathfrak{p}$ for some n. Hence $\mathfrak{p} \in \mathrm{Ass}_A\, M$.
Conversely, suppose $\mathfrak{p} = \mathrm{Ann}\, x \in \mathrm{Ass}\, M$. If $(a/s)(x/1) = 0 \in M_{\mathfrak{p}}$, there exist $t \in A \setminus \mathfrak{p}$ such that $tax = 0$. It follows that $ta \in \mathfrak{p}$ and then $(a/s) \in \mathfrak{p}A_{\mathfrak{p}}$. Hence $\mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}\, M_{\mathfrak{p}}$. □

Proposition 12. We have $\mathrm{Ass}\, M \subset \mathrm{Supp}\, M$. Moreover, if $\mathfrak{p} \in \mathrm{Supp}\, M$ satisfies $V(\mathfrak{p})$ is an irreducible component of $\mathrm{Supp}\, M$, then $\mathfrak{p} \in \mathrm{Ass}\, M$.

Proof. For any $\mathfrak{p} = \mathrm{Ann}\, x \in \mathrm{Ass}\, M$, we have $A/\mathfrak{p} \cong A \cdot x \subset M$. Tensoring with $A_{\mathfrak{p}}$ gives $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$ since $A_{\mathfrak{p}}$ is flat. Hence $M_{\mathfrak{p}} \neq 0$ and $\mathfrak{p} \in \mathrm{Supp}\, M$.
Now suppose $\mathfrak{p} \in \mathrm{Supp}\, M$ and $V(\mathfrak{p})$ is an irreducible component of $\mathrm{Supp}\, M$. First we show that $\mathfrak{p} \in \mathrm{Ass}_{A_{\mathfrak{p}}}\, M_{\mathfrak{p}}$. Let $x \in M_{\mathfrak{p}}$ such that $\mathrm{Ann}\, x$ is maximal in the set

$$\{\mathrm{Ann}\, x \colon x \in M_{\mathfrak{p}}, x \neq 0\}.$$

Then we claim that $\mathrm{Ann}\, x = \mathfrak{p}A_{\mathfrak{p}}$. First, $\mathrm{Ann}\, x$ is prime by Lemma 9. If $\mathrm{Ann}\, x \neq \mathfrak{p}$, then $V(\mathrm{Ann}\, x) \supset V(\mathfrak{p})$. This implies that $\mathrm{Ann}\, x \notin \mathrm{Supp}\, M_{\mathfrak{p}}$ since $\mathrm{Supp}\, M_{\mathfrak{p}} = \mathrm{Supp}\, M \cap \mathrm{Spec}\, A_{\mathfrak{p}}$. This is a contradiction. Thus $\mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}\, M_{\mathfrak{p}}$.
By Lemma 11, we have $\mathfrak{p} \in \mathrm{Ass}\, M$. □

Remark 13. The existence of irreducible component is guaranteed by Zorn's Lemma.

Definition 14. A prime ideal $\mathfrak{p} \in \operatorname{Ass} M$ is called *embedded* if $V(\mathfrak{p})$ is not an irreducible component of $\operatorname{Supp} M$.

Example 15. For $M = A = \mathbf{k}[x, y]/(x^2, xy)$, the origin (x, y) is an embedded point.

Proposition 16. If we have exact sequence $0 \to M_1 \to M_2 \to M_3$, then $\operatorname{Ass} M_2 \subset \operatorname{Ass} M_1 \cup \operatorname{Ass} M_3$.

Proof. Let $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M_2 \setminus \operatorname{Ass} M_1$. Then the image $[x]$ of x in M_3 is not equal to 0. We have that $\operatorname{Ann} x \subset \operatorname{Ann}[x]$. If $a \in \operatorname{Ann}[x] \setminus \operatorname{Ann} x$, then $ax \in M_1$. Since $\operatorname{Ann} x \subsetneq \operatorname{Ann} ax$, there is $b \in \operatorname{Ann} ax \setminus \operatorname{Ann} x$. However, it implies $ba \in \operatorname{Ann} x$, and then $a \in \operatorname{Ann} x$ since $\operatorname{Ann} x$ is prime, which is a contradiction. □

Corollary 17. If M is finitely generated, then the set $\operatorname{Ass} M$ is finite.

Proof. For $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$, we know that the submodule M_1 generated by x is isomorphic to $A/\mathfrak{p}$. Inductively, we can choose M_n be the preimage of a submodule of M/M_{n-1} which is isomorphic to $A/\mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass} M/M_{n-1}$. We can take an ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$. Since M is finitely generated, this is a finite sequence. Then the conclusion follows by Proposition 16. □

Definition 18. An A-module is called *co-primary* if $\operatorname{Ass} M$ has a single element. Let M be an A-module and $N \subset M$ a submodule. Then N is called *primary* if M/N is co-primary. If $\operatorname{Ass} M/N = \{\mathfrak{p}\}$, then N is called $\mathfrak{p}$-primary.

Remark 19. This definition coincide with primary ideals in the case $M = A$. Recall an ideal $\mathfrak{q} \subset A$ is called *primary* if $\forall ab \in \mathfrak{p}$, $a \notin \mathfrak{q}$ implies $b^n \in \mathfrak{q}$ for some n.
Let $\mathfrak{q}$ be a $\mathfrak{q}$-primary ideal. Since $\operatorname{Supp} A/\mathfrak{q} = \{\mathfrak{p}\}$, $\mathfrak{p} \in \operatorname{Ass} A/\mathfrak{q}$. Suppose $\operatorname{Ann}[a] \in \operatorname{Ass} A/\mathfrak{q}$. Then $\mathfrak{p} \subset \operatorname{Ann}[a]$ since $V(\mathfrak{p}) = \operatorname{Supp} A/\mathfrak{q}$. If $b \in \operatorname{Ann}[a]$, then $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Hence $b^n \in \mathfrak{q}$, and then $b \in \mathfrak{p}$. This shows that $\operatorname{Ass} A/\mathfrak{q} = \{\mathfrak{p}\}$ and $\mathfrak{q}$ is $\mathfrak{p}$-primary as an A-submodule.
Let $\mathfrak{q} \subset A$ be a $\mathfrak{p}$-primary A-submodule. First we have $\mathfrak{p} = \sqrt{\mathfrak{q}}$ since $V(\mathfrak{p})$ is the unique irreducible component of $\operatorname{Supp} A/\mathfrak{q}$. Suppose $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Then $b \in \operatorname{Ann}[a] \subset \mathfrak{p}$ since $\mathfrak{p}$ is the unique maximal element in $\{\operatorname{Ann}[c] \colon c \in A \setminus \mathfrak{q}\}$. This implies that $b^n \in \mathfrak{q}$.

Definition 20. Let A be a noetherian ring, M an A-module and $N \subset M$ a submodule. A *minimal primary decomposition* of N in M is a finite set of primary submodules $\{Q_i\}_{i=1}^n$ such that

$$N = \bigcap_{i=1}^n Q_i,$$

no Q_i can be omitted and $\operatorname{Ass} M/Q_i$ are pairwise distinct. For $\operatorname{Ass} M/Q_i = \{\mathfrak{p}\}$, Q_i is called belonging to $\mathfrak{p}$.

Indeed, if $N \subset M$ admits a minimal primary decomposition $N = \bigcap Q_i$ with Q_i belonging to $\mathfrak{p}$, then $\operatorname{Ass}(M/N) = \{\mathfrak{p}_i\}$. For given i, consider $N_i := \bigcap_{j \neq i} Q_j$, then $N_i/N \cong (N_i + Q_i)/Q_i$. Since $N_i \neq N$, $\operatorname{Ass} N_i/N \neq \emptyset$. On the other hand, $\operatorname{Ass} N_i/N \subset \operatorname{Ass} M/Q_i = \{\mathfrak{p}\}$. It follows that $\operatorname{Ass} N_i/N = \{\mathfrak{p}_i\}$, whence $\mathfrak{p}_i \in \operatorname{Ass} M/N$. Conversely, we have an injection $M/N \hookrightarrow \bigoplus M/Q_i$, so $\operatorname{Ass} M/N \subset \bigcup \operatorname{Ass} M/Q_i$. Due to this, if Q_i belongs to $\mathfrak{p}$, we also say that Q_i is the $\mathfrak{p}$-component of N.

Proposition 21. Suppose $N \subset M$ has a minimal primary decomposition. If $\mathfrak{p} \in \operatorname{Ass} M/N$ is not embedded, then the $\mathfrak{p}$ component of N is unique. Explicitly, we have $Q = \nu^{-1}(N_\mathfrak{p})$, where $\nu : M \to M_\mathfrak{p}$.

Proof. First we show that $Q = \nu^{-1}(Q_\mathfrak{p})$. Clearly $Q \subset \nu^{-1}(Q_\mathfrak{p})$. Suppose $x \in \nu^{-1}(Q_\mathfrak{p})$. Then there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in Q$. That is, $[sx] = 0 \in M/Q$. If $[x] \neq 0$, we have $s \in \operatorname{Ann}[x] \subset \mathfrak{p}$. This contradiction enforces $Q = \nu^{-1}(Q_\mathfrak{p})$.
Then we show that $N_\mathfrak{p} = Q_\mathfrak{p}$. Just need to show that for $\mathfrak{p}' \neq \mathfrak{p}$ and the $\mathfrak{p}'$ component Q' of N, $Q'_\mathfrak{p} = M_\mathfrak{p}$. Since $\mathfrak{p}$ is not embedded, $\mathfrak{p}' \not\subset \mathfrak{p}$. Then $\mathfrak{p} \notin V(\mathfrak{p}) = \operatorname{Supp} M/Q'$. So $M_\mathfrak{p}/Q'_\mathfrak{p} = 0$. □

Example 22. If $\mathfrak{p}$ is embedded, then its components may not be unique. For example, let $M = A = \mathbf{k}[x, y]/(x^2, xy)$. Then for every $n \in \mathbb{Z}_{\geq 1}$, $(x) \cap (x^2, xy, y^n)$ is a minimal primary decomposition of $(0) \subset M$.

Let A be a noetherian ring and $\mathfrak{p} \subset A$ a prime ideal. We consider the $\mathfrak{p}$ component of $\mathfrak{p}^n$, which is called n-th symbolic power of $\mathfrak{p}$, denoted by $\mathfrak{p}^{(n)}$. We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_\mathfrak{p} \cap A$. In general, $\mathfrak{p}^{(n)}$ is not equal to $\mathfrak{p}^n$; see below example.

Example 23. Let $A = \mathsf{k}[x,y,z,w]/(y^2 - zx^2, yz - xw)$ and $\mathfrak{p} = (y,z,w)$. We have $z = y^2/x^2, w = yz/x \in \mathfrak{p}^2 A_\mathfrak{p}$, whence $\mathfrak{p}^2 A_\mathfrak{p} = (z,w) \neq \mathfrak{p}^2$.

Theorem 24. Let A be a noetherian ring and M an A-module. Then for every $\mathfrak{p} \in \operatorname{Ass} M$, there is a $\mathfrak{p}$-primary submodule $Q(\mathfrak{p})$ such that

$$(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M} Q(\mathfrak{p}).$$

Proof. Consider the set

$$\mathcal{N} := \{N \subset M : \mathfrak{p} \notin \operatorname{Ass} N\}.$$

Note that $\operatorname{Ass} \bigcup N_i = \bigcup \operatorname{Ass} N_i$ by definition of associated prime ideals. Then it is easy to check that $\mathcal{N}$ satisfies the conditions of Zorn's Lemma. Hence $\mathcal{N}$ has a maximal element $Q(\mathfrak{p})$. We claim that $Q(\mathfrak{p})$ is $\mathfrak{p}$-primary. If there is $\mathfrak{p}' \neq \mathfrak{p} \in \operatorname{Ass} M/Q(\mathfrak{p})$, then there is a submodule $N' \cong A/\mathfrak{p}$. Let N'' be the preimage of N' in M. We have $Q(\mathfrak{p}) \subsetneq N''$ and $N'' \in \mathcal{N}$. This is a contradiction. By the fact $\operatorname{Ass} \bigcap N_i = \bigcap \operatorname{Ass} N_i$, we get the conclusion. □

Corollary 25. Let A be a noetherian ring and M a finitely generated A-module. Then every submodule of M has a minimal primary decomposition.

3.3 Length of modules

Definition 26. Let A be a ring and M an A module. *A simple module filtration of M* is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i-1} is a simple module, i.e. it has no submodule except 0 and itself. If M has a simple module filtration as above, we define the *length of M* as n and say that M has *finite length.*

The following proposition guarantees the length is well-defined.

Proposition 27. Suppose M has a simple module filtration $M = M_{0,0} \supsetneq M_{1,0} \supsetneq \cdots \supsetneq M_{n,0} = 0$. Then for any other filtration $M = M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,m} = 0$ with $m > n$, there exist $k < m$ such that $M_{0,k} = M_{0,k+1}$.

Proof. We claim that there are at least $0 \leq k_1 < \cdots < k_{m-n} < m$ satisfies that $M_{0,k_i} = M_{0,k_i+1}$. Let $M_{i,j} := M_{i,0} \cap M_{0,j}$. Inductively on n, we can assume that there exist $k_1, \cdots, k_{n-m+1}$ such that $M_{1,k} = M_{1,k+1}$. Consider the sequence

$$M_{0,0}/M_{1,0} \supset (M_{0,1} + M_{1,0})/M_{1,0} \supset \cdots \supset (M_{0,m} + M_{1,0})/M_{1,0} = 0$$

in $M_{0,0}/M_{1,0}$. Since $M_{0,0}/M_{1,0}$ is simple, there is at most one k_i with $M_{0,k_i} + M_{1,0} \neq M_{0,k_i+1} + M_{1,0}$. And note that if $M_{0,k_i} + M_{1,0} = M_{0,k_i+1} + M_{1,0}$ and $M_{0,k_i} \cap M_{1,0} = M_{0,k_i} \cap M_{1,0}$, then $M_{0,k_i} = M_{0,k_i+1}$ by the Five Lemma. □

Example 28. Let A be a ring and $\mathfrak{m} \in \operatorname{mSpec} A$. Then $A/\mathfrak{m}$ is a simple module.

Proposition 29. Let A be a ring and M an A-module. Then M is of finite length iff it satisfies both a.c.c and d.c.c.

Proof. Note that if M has either a strictly ascending chain or a strictly descending chain, M is of infinite length. Conversely, d.c.c guarantee M has a simple submodule and a.c.c guarantee the sequence terminates. □

Proposition 30. The length $l(-)$ is an additive function for modules of finite length. That is, if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ with M_i of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

Proof. The simple module filtrations of M_1 and M_3 will give a simple module filtration of M_2. □

Proposition 31. Let $(A, \mathfrak{m})$ be a local ring. Then A is artinian iff $\mathfrak{m}^n = 0$ for some $n \geq 0$.

Proof. Suppose A is artinian. Then the sequence $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ will stable. It follows that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n. By the Nakayama's Lemma 2, $\mathfrak{m}^n = 0$.

Conversely, we have

$$\mathfrak{m} \subset \mathfrak{N} \subset \bigcap_{\text{minimal prime ideal}} \mathfrak{p},$$

whence $\mathfrak{m}$ is minimal. □

Proposition 32. Let A be a ring. Then A is artinian iff A is of finite length.

Proof. First we show that A has only finite maximal ideal. Otherwise, consider the set $\{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k\}$. It has a minimal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ and for any maximal ideal $\mathfrak{m}$, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}$. It follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i. Let $\mathfrak{M} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ be the Jacobi radical of A. Consider the sequence $\mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots$ and by Nakayama's Lemma, we have $\mathfrak{M}^k = 0$ for some k. Consider the filtration

$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1^k \supset \mathfrak{m}_1^k \mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0).$$

We have $\mathfrak{m}_1^k \cdots \mathfrak{m}_i^j / \mathfrak{m}_1^k \cdots \mathfrak{m}_i^{j+1}$ is an $A/\mathfrak{m}_i$-vector space. It is artinian and then of finite length. Hence A is of finite length. □

Proposition 33. Let A be a ring. Then A is artinian iff A is noetherian and of dimension 0. For definition of dimension, see 37.

Proof. Suppose A is artinian. Then A is noetherian by Proposition 32. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A/\mathfrak{p}$ is an artinian integral domain. If there is $a \in A/\mathfrak{p}$ is not invertible, consider $(a) \supset (a^2) \supset \cdots$, we see $a = 0$. Hence $\mathfrak{p}$ is maximal and $\dim A = 0$.
Suppose that A is noetherian and of dimension 0. Then every maximal ideal is minimal. In particular, A has only finite maximal ideal $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$. Let $\mathfrak{q}_i$ be the $\mathfrak{p}_i$-component of (0). Then we have $A \hookrightarrow \bigoplus_i A/\mathfrak{q}_i$. We just need to show that $A/\mathfrak{q}_i$ is of finite length as A-module. If $\mathfrak{q}_i \subset \mathfrak{p}_j$, take radical we get $\mathfrak{p}_i \subset \mathfrak{q}_j$ and hence $i = j$. So $A/\mathfrak{q}_i$ is a local ring with maximal ideal $\mathfrak{p}_i A/\mathfrak{q}_i$. Then every element in $\mathfrak{p}_i A/\mathfrak{q}_i$ is nilpotent. Since $\mathfrak{p}_i$ is finitely generated, $(\mathfrak{p}_i A/\mathfrak{q}_i)^k = 0$ for some k. Then $A/\mathfrak{q}_i$ is artinian and then of finite length as $A/\mathfrak{q}_i$-module. Then the conclusion follows. □

3.4 Noether's Normalization Lemma and Hilbert's Nullstellensatz Yang: To be completed.

Theorem 34 (Noether's Normalization Lemma)**.** Let A be a k-algebra of finite type. Then there is an injection $\mathsf{k}[T_1, \cdots, T_d] \hookrightarrow A$ such that A is finite over $\mathsf{k}[T_1, \cdots, T_d]$.

Remark 35. Here A does not need to be integral. For example,

Theorem 36 (Hilbert's Nullstellensatz)**.** Let A be a

Normal, Cohen-Macaulay and regular schemes

1 Height, Depth and Dimension Yang: To be completed

Krull dimension and height of prime ideals Algebraically, we have the following definitions.

Definition 37. Let A be a noetherian ring. The *height of a prime ideal* $\mathfrak{p}$ in A is defined as the maximum length of chains of prime ideals contained in $\mathfrak{p}$, that is,

$$\operatorname{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as

$$\dim A := \max_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht}(\mathfrak{p}).$$

Geometrically, we have the corresponding definition.

Definition 38. Let X be a noetherian scheme. The *codimension of an irreducible subscheme* Y in X is defined as the length of the longest chain of irreducible closed subsets containing Y, that is,

$$\operatorname{codim}_X(Y) := \sup\{n \mid \exists \text{ a chain of irreducible closed subsets } Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n\}.$$

The *dimension* of X is defined as
$$\dim X := \max_{\xi \in X} \operatorname{codim}_X Z_\xi.$$

For an affine scheme $X = \operatorname{Spec} A$, above two definitions coincide by the correspondence of prime ideals and irreducible closed subsets.

Proposition 39. Let A be a noetherian ring and $\mathfrak{p} \in \operatorname{Spec} A$. Then
$$\operatorname{ht}(\mathfrak{p}) = \operatorname{codim}_{\operatorname{Spec} A} V(\mathfrak{p}) = \dim A_{\mathfrak{p}}.$$

Lemma 40. Let $A \subset B$ be noetherian rings such that B is finite over A. Then the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Proof. For $\mathfrak{p} \in \operatorname{Spec} A$, let $S := A - \mathfrak{p}$ and denote $S^{-1}B$ by $B_{\mathfrak{p}}$. Then we have $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$ is finite over $A_{\mathfrak{p}}$. Let $\mathfrak{P}B_{\mathfrak{p}}$ be a maximal ideal of $B_{\mathfrak{p}}$. We claim that $\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$ is maximal. Indeed, consider $A_{\mathfrak{p}}/(\mathfrak{P} \cap A_{\mathfrak{p}}) \hookrightarrow B_{\mathfrak{p}}/\mathfrak{P}B_{\mathfrak{p}}$, the latter is finite over the former. This enforces $A_{\mathfrak{p}}/(\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}})$ be a field. Hence $\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, and then $\mathfrak{P} \cap A = \mathfrak{p}$. □

Proposition 41. Let $A \subset B$ be noetherian rings such that B is finite over A. Then $\dim A = \dim B$.

Proof. If we have a sequence $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ of prime ideals in B, then there exists $f \in \mathfrak{P}_2 \setminus \mathfrak{P}_1$. Since B is finite over A, there exist $a_1, \cdots, a_n \in A$ such that
$$f^n + a_1 f^{n-1} + \cdots + a_n = 0.$$
Then $a_n \in \mathfrak{P}_2 \cap A$. If $a_n \in \mathfrak{P}_1$, $f^{n-1} + \cdots + a_{n_1} \in \mathfrak{P}_1$ since $f \notin \mathfrak{P}_1$. Then $a_{n-1} \in \mathfrak{P}_2$. Repeat the process, it will terminate, whence $\mathfrak{P}_1 \cap A \subsetneq \mathfrak{P}_2 \cap A$. Otherwise, we have $f^n \in a_1 B + \cdots + a_n B \subset \mathfrak{P}_1$.
Conversely, suppose we have $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} A$ with $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Choose $\mathfrak{P}_1 \in \operatorname{Spec} B$ such that $\mathfrak{P}_1 \cap A = \mathfrak{p}_1$, then we have $A/\mathfrak{p}_1 \subset B/\mathfrak{P}_1$. Let $\mathfrak{P}_2$ be the preimage of the prime ideal in $B/\mathfrak{P}_1$ which is over image of $\mathfrak{p}_2$ in $A/\mathfrak{p}_1$. Proposition 40 guarantees that such $\mathfrak{P}_2$ exists. Then we get $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$. Repeat this progress, we get $\dim B \geq \dim A$. □

Theorem 42 (Krull's Principal Ideal Theorem)**.** Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let $\mathfrak{p}$ be a minimal prime ideal among those containing f. Then $\operatorname{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_{\mathfrak{p}}$, we may assume A is local with maximal ideal $\mathfrak{p}$. Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.
Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is $\mathfrak{q}$-primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1} A_{\mathfrak{q}}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_{\mathfrak{q}} = 0$. It follows that $\mathfrak{q}A_{\mathfrak{q}}$ is minimal, whence $A_{\mathfrak{q}}$ is artinian. Therefore, $\mathfrak{q}$ is minimal in A. □

Corollary 43. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n. Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.
Choose $\mathfrak{q}$ be a minimal prime ideal among those containing $(\mathfrak{p}_{k-1}, f)$ and contained in $\mathfrak{p}_{k+1}$. Then by Krull's Principal Ideal Theorem 42, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace $\mathfrak{p}_k$ by $\mathfrak{q}_k$, we have $f \in \mathfrak{q}_k \setminus \mathfrak{p}_{k-1}$
Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.
Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$ has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A. It follows that $\dim A/(f) + 1 \leq \dim A$. □

For varieties, the Krull dimension behaves well by follows.

Lemma 44. Let X be an algebraic variety over k. Then for every closed point $x \in X(\mathbf{k})$, we have
$$\dim X = \dim \mathcal{O}_{X,x} = \operatorname{trdeg}(\mathscr{K}(X)/\mathsf{k}).$$

Proof. Since X is irreducible, we may assume that $X = \operatorname{Spec} A$ is affine. Let $d = \operatorname{trdeg}(\mathscr{K}(X)/\mathsf{k})$.
By Noether's Normalization Lemma 34, there is an injective and finite homomorphism $A_0 = \mathsf{k}[T_1, \cdots, T_d] \hookrightarrow A$. Let $\mathfrak{M}$ be the corresponding maximal ideal of x in A and $\mathfrak{m} = \mathfrak{M} \cap \mathsf{k}[T_1, \cdots, T_d]$. Denote the image of T_i in $\mathsf{l} := A_0/\mathfrak{m}$ by t_i. The extension l/k is finite by Nullstellensatz 36. Let $f_i \in \mathsf{k}[T]$ be the minimal polynomial of t_i and $g_i := f_i(T_i) \in A_0$. Then $g_i \in \mathfrak{m}$ and $\mathfrak{m} = g_1 A_0 + \cdots, g_d A_0$. In particular, $g_1, \cdots, g_d \in \mathfrak{M}$.
We have $A/g_1 A + \cdots + g_d A$ is finite over $A_0/\mathfrak{m}$, whence it is artinian. This implies that $A_{\mathfrak{M}}/g_1 A_{\mathfrak{M}} + \cdots + g_d A_{\mathfrak{M}}$ is also artinian. Since g_{k+1} is not a zero divisor in $A_0/g_1 A_0 + \cdots + g_k A_0$, g_{k+1} is not contained in any minimal prime ideal of $A_0/g_1 A_0 + \cdots + g_k A_0$. Then g_{k+1} is also not contained in any minimal prime ideal of $A/g_1 A + \cdots + g_k A$. By Corollary 43, $\dim A_{\mathfrak{M}} = \dim(A_{\mathfrak{M}}/g_1 A_{\mathfrak{M}} + \cdots + g_d A_{\mathfrak{M}}) + d = d$. □

Theorem 45. Let S be spectrum of a field k or an algebraic integer ring $\mathcal{O}_K$ and X an integral S-variety. Then we have the follows:

(i) For every point $\xi \in X$, $\dim X = \dim \mathcal{O}_{X,\xi} + \operatorname{codim} Z_\xi$.

(ii) For every non-empty open subset $U \subset X$, $\dim U = \dim X$.

(iii) $\dim X = \operatorname{trdeg}(\mathscr{K}(X)/\mathscr{K}(S)) + \dim S$.

Proof. Yang: To be continued. □

Example 46. For general noetherian schemes, Theorem 45 may not hold. Let $A = \mathsf{k}[t]$, $\mathfrak{m} = (t)$, $B = A_{\mathfrak{m}}[x]$ and $X = \operatorname{Spec} B$. Then we have $\dim X = 2$ since Yang: To be added.

Depth For a noetherian local ring $(A, \mathfrak{m})$, we can define the depth of an A-module M. Somehow the Krull dimension is "homological" and the depth is "cohomological".

Definition 47. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A-module. A sequence $t_1, \cdots, t_n \in \mathfrak{m}$ is called an *M-regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \cdots, t_{i-1})M$ for all i.

Example 48. Let $A = \mathsf{k}[x, y]/(x^2, xy)$ and $I = (x, y)$. Then $\operatorname{depth}_I A = 0$.

Definition 49. The *I-depth* of M is defined as the maximum length of M-regular sequences in I, denoted by $\operatorname{depth}_I M$. When A is a local ring with maximal ideal $\mathfrak{m}$, we write $\operatorname{depth} M$ for $\operatorname{depth}_{\mathfrak{m}} M$.

Regular and Serre's conditions Up to now, there are three numbers measuring the "size" of a local ring $(A, \mathfrak{m})$:

- $\dim A$: the Krull dimension of A.
- $\operatorname{depth} A$: the depth of A.
- $\dim_{\kappa(\mathfrak{m})} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$-vector space.

These three numbers are related by the following inequalities.

Proposition 50. Let $(A, \mathfrak{m})$ be a local noetherian ring with residue field k. Then the following inequalities hold:

$$\operatorname{depth} A \leq \dim A \leq \dim_{\mathsf{k}} T_{A,\mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary 43.
Let $t_1, \cdots, t_n$ be a $\kappa(\mathfrak{m})$-basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \cdots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \cdots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \cdots, t_n))$. It follows that $\mathfrak{m} = (t_1, \cdots, t_n)$ by Nakayama's Lemma. By Corollary 43,

$$n + \dim A/(t_1, \cdots, t_n) \geq n - 1 + \dim A/(t_1, \cdots, t_{n-1}) \geq \cdots \geq 1 + \dim A/(t_1) \geq \dim A.$$

We conclude the result. □

Definition 51. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property* (R_k) or *is regular in codimension* k if $\forall \xi \in X$ with $\operatorname{codim} Z_\xi \leq k$,

$$\dim_{\kappa(\xi)} T_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

We say that X *verifies property* (S_k) if $\forall \xi \in X$ with $\operatorname{depth} \mathcal{O}_{X,\xi} < k$,

$$\operatorname{depth} \mathcal{O}_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

Lemma 52. Let A be a ring and $\mathfrak{a} \subset \bigcup_i \mathfrak{p}_i$. Then $\mathfrak{a} \subset \mathfrak{p}_i$ for some i.

Proof. Yang: To be completed. □

Example 53. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.
Suppose A verifies (S_1). If $\mathfrak{p} \in AssA$, every element in $\mathfrak{p}$ is a zero divisor. Then $\operatorname{depth} A_{\mathfrak{p}} = 0$. It follows that $\dim A_{\mathfrak{p}} = 0$ and then $\mathfrak{p}$ is minimal.
Suppose A has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_{\mathfrak{p}} = 0$. This means every element in $\mathfrak{p}A_{\mathfrak{p}}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Lemma 52, $\mathfrak{p} = \mathfrak{q}$ for some minimal $\mathfrak{q}$, whence $\dim A_{\mathfrak{p}} = 0$.

Example 54. Let A be a noetherian ring verifies (S_1). Then A verifies (S_2) iff for any nonzero divisor $f \in A$, $\operatorname{Ass}_A A/fA$ has no embedded point.
Suppose A verifies (S_2). Let $f \in A$ be a nonzero divisor and $\mathfrak{p} \in \operatorname{Ass}_A A/fA$. There exist $g \in A \setminus fA$ such that $\mathfrak{p} = (f : g)$. For any $t_1, t_2 \in \mathfrak{p}$, there exist s_1, s_2 with $s_i \notin (t_i)$ and $t_i g = f s_i$. Then $t_1 t_2 g = f s_1 t_2 = f s_2 t_1$. Since f is not a zero divisor, $s_1 t_2 = s_2 t_1$. Then t_2 is a zero divisor in $A_{\mathfrak{p}}/t_1 A_{\mathfrak{p}}$ since $s_1 \notin (t_1)$. Since $f \in \mathfrak{p}$, $\operatorname{depth} A_{\mathfrak{p}} = 1$ and then $\operatorname{ht} \mathfrak{p} = 1$. This show that $\mathfrak{p}$ is not embedded in $\operatorname{Ass}_A A/fA$.
Conversely, suppose $\operatorname{Ass}_A A/fA$ has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_{\mathfrak{p}} = 1$. Then there exists $f \in A_{\mathfrak{p}}$ which is not a zero divisor. We have $\operatorname{depth} A_{\mathfrak{p}}/fA_{\mathfrak{p}} = 0$ and $\operatorname{Ass}_A A/fA$ has no embedded point, whence $\mathfrak{p}$ is minimal in A/fA. Then $\operatorname{ht} \mathfrak{p} = 1$ by Krull's Principal Ideal Theorem 42 and the fact f is not a zero divisor.

Example 55. Let X be a locally noetherian scheme. Then X is reduced iff it verifies (R_0) and (S_1).
The properties are local, whence we can assume $X = \operatorname{Spec} A$. Suppose A is reduced. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be all minimal prime ideals of A. We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where $\mathfrak{N}$ is the nilradical of A. Hence A has no embedded point. Since $A_{\mathfrak{p}}$ is artinian, local and reduced, $A_{\mathfrak{p}}$ is a field and hence regular.
Conversely, let $\operatorname{Ass} A$ be equal to $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Then every $\mathfrak{p}_i$ is minimal by (S_1). Let f be in $\mathfrak{N}$. Then the image of f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0), $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where $\mathfrak{q}_i$ is the $\mathfrak{p}_i$ component of (0) in A. Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

2 Normal schemes Yang: To be completed

Definition 56. An integral domain A is called *normal* if it is integrally closed in its field of fractions $\operatorname{Frac}(A)$.

Lemma 57. Let $A \subset C$ be rings and B the integral closure of A in C, S a multiplicatively closed subset of A. Then the integral closure of $S^{-1}A$ in $S^{-1}C$ is $S^{-1}B$.

Proof. For every $b \in B$ and $\forall s \in S$, there exists $a_i \in A$ s.t.

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then

$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s^1}\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0.$$

Hence b/s is integral over $S^{-1}A$, $S^{-1}B$ is integral over $S^{-1}A$.
If $c/s \in S^{-1}C$ is integral over $S^{-1}A$, then $\exists a_i \in S^{-1}A$ s.t.

$$\left(\frac{c}{s}\right)^n + a_1\left(\frac{c}{s}\right)^{n-1} + \cdots + a_n = 0.$$

Then

$$c^n + a_1 s c^{n-1} + \cdots + a_n s^n = 0 \in S^{-1}C$$

Then $\exists t \in S$ s.t.
$$t(c^n + a_1 sc^{n-1} + \cdots + a_n s^n) = 0 \in C.$$
Then
$$(ct)^n + a_1 st(ct)^{n-1} + \cdots + a_n s^n t^n = t^n(c^n + a_1 sc^{n-1} + \cdots + a_n s^n) = 0.$$
Hence ct is integral over A, then $ct \in B$. Then $c/s = (ct)/(st) \in S^{-1}B$. This completes the proof. □

Proposition 58. Normality is a local property. That is, for an integral domain A, TFAE:

(i) A is normal.

(ii) For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_\mathfrak{p}$ is normal.

(iii) For any maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$, the localization $A_\mathfrak{m}$ is normal.

Proof. When A is normal, $A_\mathfrak{p}$ is normal by Lemma 57.
Assume that $A_\mathfrak{m}$ is normal for every $\mathfrak{m} \in \operatorname{mSpec} A$. If A is not normal, let $\tilde{A}$ be the integral closure of A in $\operatorname{Frac} A$, $\tilde{A}/A$ is a nonzero A-module. Suppose $\mathfrak{p} \in \operatorname{Supp} \tilde{A}/A$ and $\mathfrak{p} \subset \mathfrak{m}$. We have $\tilde{A}_\mathfrak{m}/A_\mathfrak{m} = 0$ and $\tilde{A}_\mathfrak{p}/A_\mathfrak{p} = (\tilde{A}_\mathfrak{m}/A_\mathfrak{m})_\mathfrak{p} \neq 0$. This is a contradiction. □

Definition 59. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,\xi}$ is normal for any point $\xi \in X$. A ring A is called *normal* if $\operatorname{Spec} A$ is normal.

Remark 60. For a general ring A, let $S := A \setminus (\bigcup_{\mathfrak{p} \in \operatorname{Ass} A} \mathfrak{p}) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} A} A \setminus \mathfrak{p}$. Then S is a multiplicative set. The localization $S^{-1}A$ is called *the total ring of fractions* of A.
Suppose A is reduced and $\operatorname{Ass} A = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Denote its total ring of fractions by Q. Note that elements in Q are either unit or zero divisor. Hence any maximal ideal $\mathfrak{m}$ is contained in $\bigcup \mathfrak{p}_i Q$, whence contained in some $\mathfrak{p}_i Q$. Thus $\mathfrak{p}_i Q$ are maximal ideals. And we have $\bigcap \mathfrak{p}_i Q = 0$. By the Chinese Remainder Theorem, we have $Q = \prod Q/\mathfrak{p}_i Q = \prod A_{\mathfrak{p}_i}$.
Let A be a reduced ring with total ring of fractions Q. Then A is normal iff A is integral closed in Q. If A is normal, then for every $\mathfrak{p} \in \operatorname{Spec} A$, $A_\mathfrak{p}$ is integral. Then there is unique minimal prime ideal $\mathfrak{p}_i \subset \mathfrak{p}$. In particular, any two minimal prime ideal are relatively prime. By the Chinese Remainder Theorem, $A = \prod A/\mathfrak{p}_i$. Just need to check $A/\mathfrak{p}_i$ is integral closed in $A_{\mathfrak{p}_i}$. This is clear by check pointwise.
Conversely, suppose A is integral closed in Q. Let e_i be the unit element of $A_{\mathfrak{p}_i}$. It belongs to A since $e_i^2 - e_i = 0$. Since $1 = e_1 + \cdots + e_n$ and $e_i e_j = \delta_{ij}$, we have $A = \prod Ae_i$. Since Ae_i is integral closed in $A_{\mathfrak{p}_i}$, it is normal. Hence A is normal.

Definition 61. Let X be a scheme. The *normalization* of X is an X-scheme X^ν with the following universal property: for any normal X-scheme Y with dominant structure morphism, its structure morphism $Y \to X$ factors through X^ν.

Proposition 62. Let X be an integral scheme. Then the normalization X^ν of X exists. Moreover, $X^\nu \to X$ is birational.

Proof. Suppose there is a dominant morphism $Y \to X$ with Y normal. Since Y is normal, it is reduced. Then it factors through X_{red}. Hence we can assume that X is reduced by replacing X by X_{red}.
Suppose $X = \operatorname{Spec} A$ is affine. Let A^ν be the integral closure of A in it total ring of fractions and $X^\nu := \operatorname{Spec} A^\nu$. It gives a homomorphism $A \to \mathcal{O}_Y(Y)$. We claim that it is injective. Otherwise, it factors through $A \to A/I$ and then $Y \to \operatorname{Spec} A$ factors through $\operatorname{Spec} A/I \to \operatorname{Spec} A$. It contradicts that $Y \to X$ is dominant. Since Y is normal, $\mathcal{O}_Y(Y)$ is integral closed in its total ring of fraction. Then $\mathcal{O}_Y(Y)$ contains A^ν. This shows that X^ν is the normalization of X.
In general case, take an affine cover $\{U_i\}$ of X and clue these U_i^ν by universal property. □

Lemma 63. Let A be a normal ring. Then A verifies (R_1) and (S_2).

Proof. Since all properties are local, we can assume A is integral and local.
For (S_2), by Example 54, we only need to show that $\operatorname{Ass}_A A/f$ has no embedded point. Let $\mathfrak{p} = (f : g) =\in \operatorname{Ass}_A A/fA$ and $t := f/g \in \operatorname{Frac} A$. After Replacing A by $A_\mathfrak{p}$, we can assume that $\mathfrak{p}$ is maximal. By definition, $t^{-1}\mathfrak{p} \subset A$. If $t^{-1}\mathfrak{p} \subset \mathfrak{p}$, suppose $\mathfrak{p}$ is generated by $(x_1, \cdots, x_n)$ and $t^{-1}(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(A)$. There is a monic polynomial $\chi(T) \in A[T]$ vanishing Φ. Then $\chi(t^{-1}) = 0$ and $t^{-1} \in A$. This is impossible by definition of t. Then $t^{-1}\mathfrak{p} = A$, and $\mathfrak{p} = (t)$ is principal. By Krull's Principal Ideal Theorem 42, $\operatorname{ht}(\mathfrak{p}) = 1$.
Now we show that A verifies (R_1). Suppose $(A, \mathfrak{m})$ is local of dimension 1. Choosing $a \in \mathfrak{m}$, A/a is of dimension 0. Then by 31, $\mathfrak{m}^n \subset aA$ for some $n \geq 1$. Suppose $\mathfrak{m}^{n-1} \not\subset aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$ and let $t = a/b$. By construction,

$t^{-1} \notin A$ and $t^{-1}\mathfrak{m} \subset A$. After similar argument, we see that $\mathfrak{m} = tA$, whence A is regular. □

Lemma 64. Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension 1. Then A is normal iff A is regular.

Proof. By lemma 63, we just need to show that regularity implies normality.
Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since A is regular, $\mathfrak{m} = (t)$. First we show that A is an integral domain.
Let $I \subset A$ be a nonzero ideal of A. Then $\mathfrak{m}^n \subset I$ for some $n \geq 1$. Yang: To be completed. □

Proposition 65. Let A be a noetherian integral domain of dimension ≥ 1 verifying (S_2). Then

$$A = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A, \operatorname{ht}(\mathfrak{p}) = 1} A_{\mathfrak{p}}.$$

Proof. Clearly $A \subset \bigcap A_{\mathfrak{p}}$. Let $t = f/g \in \bigcap A_{\mathfrak{p}}$. Since $f \in gA_{\mathfrak{p}}$ and we have $gA = \bigcap (gA_{\mathfrak{p}} \cap A)$, $f \in gA$. It follows that $t \in A$. □

Theorem 66 (Serre's criterion for normality)**.** Let X be a locally noetherian scheme. Then X is normal if and only if it verifies (R_1) and (S_2).

Proof. One direction has been proved in Lemma 63. Suppose X verifies (R_1) and (S_2). Again we can assume $X = \operatorname{Spec} A$ is affine and A is local. By Remark 60, we just need to show that A is integral closed in its total ring of fractions Q. Suppose we have

$$\left(\frac{a}{b}\right)^n + c_1 \left(\frac{a}{b}\right)^{n-1} + \cdots + c_n = 0 \in Q.$$

Since A verifies (S_2), $bA = \bigcap \nu_{\mathfrak{p}}^{-1}(b_{\mathfrak{p}} A_{\mathfrak{p}})$. So it is sufficient to show that $a_{\mathfrak{p}} \in b_{\mathfrak{p}} A_{\mathfrak{p}}$ with $\operatorname{ht}(\mathfrak{p}) = 1$. Note that $A_{\mathfrak{p}}$ is regular and hence normal by Lemma 64. Then above equation gives us desired result. □

Theorem 67. Let X be a normal and locally noetherian scheme. Let $F \subset X$ be a closed subset of codimension ≥ 2. Then the restriction $H^0(X, \mathcal{O}_X) \to H^0(X \setminus F, \mathcal{O}_X)$ is an isomorphism.

Proof. By the exact sequences

$$0 \to \mathcal{F}(X) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j),$$

where $\{U_i\}$ is an affine open cover of X, we can reduce to the case that X is affine. Then $X = \operatorname{Spec} A$ for some normal noetherian ring A. For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{ht}(\mathfrak{p}) = 1$, we have $\mathfrak{p} \in X \setminus F$. By Proposition 65, the conclusion follows. □

Theorem 68 (Valuation criterion for properness)**.** Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. Then f is proper iff for any valuation ring A, $\mathsf{K} = \operatorname{Frac} A$ and commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathsf{K} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd},$$

the morphism $\operatorname{Spec} A \to Y$ factors through f.

Proposition 69. Yang: To be completed. Let X, Y be noetherian S-schemes with S noetherian. Suppose X is normal and of finite type over S, and Y is proper over S. Let $\xi \in X$ be a generic point and $\eta \in Y$ a point such that there exists a morphism $\operatorname{Spec} \mathscr{K}(\xi) \to Y$. Then there exists an open subset $U \subset X$ containing ξ such that the morphism extends to a morphism $U \to Y$.

Theorem 70. Let X, Y be noetherian S-schemes with S noetherian. Suppose X is normal and of finite type over S, and Y is proper over S. Let $f : X \dashrightarrow Y$ be a rational map. Then f is well-defined on an open subset $U \subset X$ whose complement has codimension ≥ 2.

Proof. Yang: To be completed. □

Remark 71. Theorem 67 and Theorem 70 are very similar. However, they are base on different properties. Theorem 67 is based on (S_2), while Theorem 70 is based on (R_1). Philosophically, the (S_k) conditions are used to control the "bad part of codimension larger than k". The (R_k) conditions are used to control the "bad part of codimension smaller than or equal to k". We will see more examples in the next section.Yang: To be completed.

3 Cohen-Macaulay schemes

Definition 72 (Cohen-Macaulay)**.** A noetherian local ring $(A, \mathfrak{m})$ is called *Cohen-Macaulay* if $\dim A = \operatorname{depth} A$. A locally noetherian scheme X is called *Cohen-Macaulay* if $\mathcal{O}_{X,\xi}$ is Cohen-Macaulay for any point $\xi \in X$.

By definition, it is easy to see that X is Cohen-Macaulay if and only if it verifies (S_k) for all $k \geq 0$.

Example 73 (Non Cohen-Macaulay rings)**.**

Definition 74. An ideal I of a noetherian ring A is called *unmixed* if

$$\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \operatorname{Ass}(A/I).$$

We say that *the unmixedness theorem holds for a noetherian ring* A if any ideal $I \subset A$ generated by $\operatorname{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme* X if $\mathcal{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Proposition 75. Yang: To be completed.

Theorem 76. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Theorem 77. Let X be a locally noetherian scheme. Suppose that X is Cohen-Macaulay. Let $F \subset X$ be a closed subset of codimension $\geq k$. Then the restriction $H^i(X, \mathcal{O}_X) \to H^i(X \setminus F, \mathcal{O}_X)$ induced by the is an isomorphism.

4 Regular schemes

Proposition 78. Let $(A, \mathfrak{m})$ be a regular local ring. Then A is integral.

Proposition 79. If X verifies (R_k), then $\operatorname{codim}_X X_{\text{sing}} \geq k + 1$.

Proposition 80. A regular scheme is Cohen-Macaulay.

Corollary 81. A regular scheme is normal.