

# Stacks in category theory

## 1 Fibered categories and descent conditions

**Definition 1.** Let  $\mathbf{S}$  be a category and  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  a functor. A morphism  $f : b \rightarrow a$  in  $\mathbf{X}$  is called *strongly Cartesian* if for every object  $c \in \text{Obj}(\mathbf{X})$ , the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{X}}(c, b) & \xrightarrow{f \circ -} & \text{Hom}_{\mathbf{X}}(c, a) \\ \mathbf{p} \downarrow & & \downarrow \mathbf{p} \\ \text{Hom}_{\mathbf{S}}(w, v) & \xrightarrow{\mathbf{p}(f) \circ -} & \text{Hom}_{\mathbf{S}}(w, u) \end{array}$$

is a pullback of sets, where  $u = \mathbf{p}(a), v = \mathbf{p}(b), w = \mathbf{p}(c)$ .

The condition in [Definition 1](#) can be interpreted as follows: for any diagram as below black part with  $\mathbf{p}(g) = \mathbf{p}(f) \circ \alpha$ ,

$$\begin{array}{ccccc} c & & & & \\ & \searrow h & & \nearrow g & \\ & b & \xrightarrow{f} & a & \\ & \downarrow & & \downarrow & \\ w & \xrightarrow{\alpha} & v & \xrightarrow{\mathbf{p}(f)} & u \end{array}$$

there exists a unique gray morphism  $h : c \rightarrow a$  such that  $\mathbf{p}(h) = \alpha$  and  $f \circ h = g$ .

**Notation 2.** Let  $\mathbf{S}$  be a category and  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  a functor. For  $a, b \in \text{Obj}(\mathbf{X})$  and  $f \in \text{Hom}_{\mathbf{X}}(a, b)$ , we say that  $a$  is *over*  $\mathbf{p}(a)$  and  $f$  is *over*  $\mathbf{p}(f)$ . In a diagram, we have

$$\begin{array}{ccc} \mathbf{X} & & \\ \mathbf{p} \downarrow & & \\ \mathbf{S} & & \\ & a \xrightarrow{f} b & \\ & \downarrow \quad \downarrow & \\ & \mathbf{p}(a) \xrightarrow{\mathbf{p}(f)} \mathbf{p}(b) & \end{array}$$

**Definition 3.** Let  $\mathbf{S}$  be a category. A category  $\mathbf{X}$  over  $\mathbf{S}$  via  $\mathbf{p}$  is called a *category fibred* over the site  $\mathbf{S}$  if for every morphism  $\iota : v \rightarrow u$  in  $\mathbf{S}$  and every object  $a \in \text{Obj}(\mathbf{X})$  over  $u$ , there exists an object  $b \in \text{Obj}(\mathbf{X})$  over  $v$  and a strongly Cartesian morphism  $f : b \rightarrow a$  over  $\iota$ . Such an object  $b$  is called a *pullback* of  $a$  along  $\iota$ , and is often denoted by  $\iota^*a$ .

**Definition 4.** Let  $\mathbf{S}$  be a site and  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  a category fibred over  $\mathbf{S}$ . For every object  $u \in \text{Obj}(\mathbf{S})$ , the *fiber* of  $\mathbf{X}$  over  $u$  is the category  $\mathbf{X}_u$  given by

$$\text{Obj}(\mathbf{X}_u) = \{a \in \text{Obj}(\mathbf{X}) \mid \mathbf{p}(a) = u\}, \quad \text{Hom}_{\mathbf{X}_u}(a, b) = \{f \in \text{Hom}_{\mathbf{X}}(a, b) \mid \mathbf{p}(f) = \text{id}_u\}.$$

**Remark 5.** Note that in [Definition 3](#), the pullback  $r^*b$  of an object  $b$  along a morphism  $r$  is not necessarily unique. Yang: To be continued.

**Example 6.** Let  $\mathbf{S}$  be a category and  $\mathcal{F} : \mathbf{S}^{op} \rightarrow \mathbf{Set}$  be a presheaf on  $\mathbf{S}$  taking values in  $\mathbf{Set}$ . We can construct a category  $\mathbf{F}$  fibred over  $\mathbf{S}$  as follows:

- The objects of  $\mathbf{F}$  are pairs  $(U, x)$  where  $U \in \text{Obj}(\mathbf{S})$  and  $x \in \mathcal{F}(U)$ ;
- morphisms from  $(V, y)$  to  $(U, x)$  in  $\mathbf{F}$  are morphisms  $\iota : V \rightarrow U$  in  $\mathbf{S}$  such that  $\mathcal{F}(\iota)(x) = y$ , denoted by  $\text{res}_\iota$ .

The functor  $\mathbf{p} : \mathbf{F} \rightarrow \mathbf{S}$  is defined by  $\mathbf{p}(U, x) = U$  on objects and  $\mathbf{p}(\iota) = \iota$  on morphisms. If one has the diagram

$$\begin{array}{ccccc}
 (W, z) & & & \xrightarrow{\text{res}_\tau} & (U, x) \\
 \downarrow & & (V, y) & \xrightarrow{\text{res}_\iota} & \downarrow \\
 W & \xrightarrow{\sigma} & V & \xrightarrow{\iota} & U
 \end{array}$$

with  $\mathbf{p}(\text{res}_\tau) = \iota \circ \sigma$ . By definition, we have  $\tau = \iota \circ \sigma$  and  $\mathcal{F}(\tau)(x) = z, \mathcal{F}(\iota)(x) = y$ . Thus, we have  $\mathcal{F}(\sigma)(y) = z$ . This verifies that  $\text{res}_\sigma$  is a strongly Cartesian morphism. Note that the fiber of  $\mathbf{F}$  over an  $U \in \text{Obj}(\mathbf{S})$  is the discrete category associated to the set  $\mathcal{F}(U)$ . Therefore, presheaves of sets can be viewed as categories fibred in sets.

Conversely, given a category  $\mathbf{F}$  fibred in sets over  $\mathbf{S}$  via  $\mathbf{p} : \mathbf{F} \rightarrow \mathbf{S}$ , one can construct a presheaf of sets  $\mathcal{F} : \mathbf{S}^{op} \rightarrow \mathbf{Set}$  by defining  $\mathcal{F}(U) = \text{Obj}(\mathbf{F}_U)$  for each  $U \in \text{Obj}(\mathbf{S})$ , and for each morphism  $\iota : V \rightarrow U$  in  $\mathbf{S}$ , defining  $\mathcal{F}(\iota) : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$  by sending an object  $x \in \mathcal{F}(U)$  to its pullback  $\iota^*x \in \mathcal{F}(V)$  along  $\iota$ . This establishes an equivalence between presheaves of sets on  $\mathbf{S}$  and categories fibred in sets over  $\mathbf{S}$ .

**Example 7.** Yang: case  $\mathbf{S} = \text{set, group}$ . To be added.

**Slogan** *Presheaves of sets are categories fibered in sets.*

In following, we describe categories fibered in groupoids.

**Definition 8.** Let  $\mathbf{X}$  be a category fibred over a category  $\mathbf{S}$  via  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ . For every  $u \in \text{Obj}(\mathbf{S})$  and every pair of objects  $a, b$  over  $u$ , we define the *presheaf of morphisms*  $\text{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{op} \rightarrow \mathbf{Set}$  by

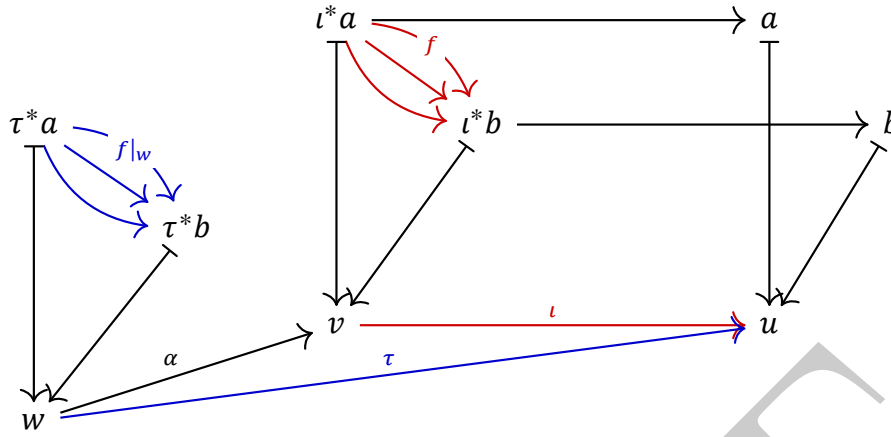
$$\text{Hom}_{\mathbf{X}}(a, b)(\iota : v \rightarrow u) = \text{Hom}_{\mathbf{X}_v}(\iota^*a, \iota^*b)$$

for every morphism  $\iota : v \rightarrow u$  in  $\mathbf{S}/u$ . For a morphism  $\alpha : w \rightarrow v$  in  $\mathbf{S}/u$ , the restriction map

$$\text{Hom}_{\mathbf{X}}(a, b)(\iota) \rightarrow \text{Hom}_{\mathbf{X}}(a, b)(\iota \circ \alpha)$$

is given by sending a morphism  $f : \iota^*a \rightarrow \iota^*b$  in  $\mathbf{X}_v$  to the pullback morphism Yang:  $\alpha^*f : (\iota \circ \alpha)^*a \rightarrow (\iota \circ \alpha)^*b$  need to conjugate with a natural transformation. in  $\mathbf{X}_w$ . Yang: To be checked.

In a diagram, the presheaf of morphisms can be visualized as follows:



**Proposition 9.** Let  $\mathbf{S}$  be a category and  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  a category fibred over  $\mathbf{S}$ . Then  $\mathbf{X}$  is a category fibred in groupoids if and only if for every object  $u \in \text{Obj}(\mathbf{S})$  and every pair of objects  $a, b$  over  $u$ , the presheaf of morphisms  $\text{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\text{op}} \rightarrow \mathbf{Set}$  is a sheaf. Yang: To be checked.

**Definition 10.** Let  $\mathbf{S}$  be a category. A category  $\mathbf{X}$  fibred over  $\mathbf{S}$  via  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  is called a *category fibred in groupoids* over  $\mathbf{S}$  if for every object  $u \in \text{Obj}(\mathbf{S})$  and every pair of objects  $a, b$  over  $u$ , the presheaf of morphisms  $\text{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\text{op}} \rightarrow \mathbf{Set}$  is a sheaf. Yang: To be checked.

Now let us discuss how sheaves fit into the framework of fibered categories. Of course, we need assume the base category  $\mathbf{S}$  is a site. The glued condition for sheaves can be interpreted in terms of descent data in fibered categories.

**Definition 11.** Let  $\mathbf{S}$  be a site and  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  a fibered category over  $\mathbf{S}$ . Let  $U \in \text{Obj}(\mathbf{S})$  and  $\{U_i \rightarrow U\}$  be a covering in  $\mathbf{S}$ . A *descent datum* for objects of  $\mathbf{X}$  relative to the covering  $\{U_i \rightarrow U\}$  consists of

- a collection of objects  $a_i \in \text{Obj}(\mathbf{X}_{U_i})$  for each  $i$ ,
- a collection of isomorphisms  $\varphi_{ij} : a_j|_{U_{ij}} \rightarrow a_i|_{U_{ij}}$  in  $\mathbf{X}_{U_{ij}}$  for each pair  $(i, j)$ , where  $U_{ij} = U_i \times_U U_j$ ,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}}$$

holds for all triples  $(i, j, k)$ , where  $U_{ijk} = U_i \times_U U_j \times_U U_k$ . Yang: To be checked.

**Example 12.** Yang: To be added.

**Definition 13.** Let  $\mathbf{S}$  be a site and  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  a fibered category over  $\mathbf{S}$ . A descent datum  $(\{a_i\}, \{\varphi_{ij}\})$  for objects of  $\mathbf{X}$  relative to a covering  $\{U_i \rightarrow U\}$  in  $\mathbf{S}$  is called *effective* if there exists an object  $a \in \text{Obj}(\mathbf{X}_U)$  and isomorphisms  $\psi_i : a|_{U_i} \rightarrow a_i$  in  $\mathbf{X}_{U_i}$  such that for all pairs  $(i, j)$ , the diagram

$$\begin{array}{ccc} a|_{U_{ij}} & \xrightarrow{\psi_j|_{U_{ij}}} & a_j|_{U_{ij}} \\ \psi_i|_{U_{ij}} \downarrow & & \downarrow \varphi_{ij} \\ a_i|_{U_{ij}} & \xrightarrow{\varphi_{ij}} & a_j|_{U_{ij}} \end{array}$$

commutes. **Yang:** To be checked.

**Slogan** *Descent data are like gluing data for objects, and effectiveness means that the glued object exists.*

## 2 Prestacks and stacks

**Definition 14.** A *prestack* over the site  $\mathbf{S}$  is a category  $\mathbf{X}$  fibered in groupoids over  $\mathbf{S}$ .

**Slogan** *Prestacks are “presheaf remembering automorphisms”.*

**Example 15.** presheaf is a prestack. **Yang:** To be added.

**Example 16.** The moduli problem of classifying algebraic curves of a fixed genus  $g$  can be formulated as a prestack over the site of schemes. Consider the category  $\mathbf{M}_g$  whose objects are families of smooth projective curves of genus  $g$  over schemes, and whose morphisms are isomorphisms of such families. The functor  $\mathbf{p} : \mathbf{M}_g \rightarrow \mathbf{Sch}$  sending a family of curves to its base scheme makes  $\mathbf{M}_g$  a category fibered in groupoids over  $\mathbf{Sch}$ . For each scheme  $S$ , the fiber category  $\mathbf{M}_{g,S}$  consists of families of smooth projective curves of genus  $g$  over  $S$  and their isomorphisms. The descent data for objects in  $\mathbf{M}_g$  relative to a covering of schemes correspond to gluing families of curves along isomorphisms on overlaps, which is effective due to the nature of algebraic curves. Thus,  $\mathbf{M}_g$  is a prestack over the site of schemes. **Yang:** To be revised.

**Proposition 17.** Let  $\mathbf{S}$  be a site, and let  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ ,  $\mathbf{q} : \mathbf{Y} \rightarrow \mathbf{S}$ , and  $\mathbf{r} : \mathbf{Z} \rightarrow \mathbf{S}$  be prestacks over  $\mathbf{S}$ . Let  $\Phi : \mathbf{X} \rightarrow \mathbf{Z}$  and  $\Psi : \mathbf{Y} \rightarrow \mathbf{Z}$  be morphisms of prestacks over  $\mathbf{S}$ . Then the fiber product  $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$  exists in the category of prestacks over  $\mathbf{S}$ . **Yang:** To be checked.

**Definition 18.** Let  $\mathbf{S}$  be a site. A prestack  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  is called a *stack* over the site  $\mathbf{S}$  if for every object  $U \in \text{Obj}(\mathbf{S})$  and every covering  $\{U_i \rightarrow U\}$  in  $\mathbf{S}$ , the descent data for objects of  $\mathbf{X}$  relative to the covering  $\{U_i \rightarrow U\}$  are effective. **Yang:** To be revised.

**Slogan** *Stacks are to prestacks as sheaves are to presheaves.*

**Example 19.** Let  $X$  be a scheme over a base noetherian scheme  $S$ . The functor of points  $h_X : (\mathbf{Sch}/S)_{\text{et}}^{\text{op}} \rightarrow \mathbf{Set}$  is a sheaf, and thus a stack.

**Construction 20.** Let  $\mathbf{S}$  be a site, and let  $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$  be a prestack over  $\mathbf{S}$ . There exists a stack  $\mathbf{p}^+ : \mathbf{X}^+ \rightarrow \mathbf{S}$  over  $\mathbf{S}$  together with a morphism of prestacks  $F : \mathbf{X} \rightarrow \mathbf{X}^+$  over  $\mathbf{S}$  satisfying the following universal property: for every stack  $\mathbf{p}' : \mathbf{Y} \rightarrow \mathbf{S}$  over  $\mathbf{S}$  and every morphism of prestacks  $G : \mathbf{X} \rightarrow \mathbf{Y}$  over  $\mathbf{S}$ , there exists a unique morphism of stacks  $G^+ : \mathbf{X}^+ \rightarrow \mathbf{Y}$  over  $\mathbf{S}$  such that  $G = G^+ \circ F$ . The stack  $\mathbf{X}^+$  is called the *stackification* of the prestack  $\mathbf{X}$ . **Yang:** To be checked.

**Example 21.** Let  $S$  be a noetherian scheme, and let  $G$  be a group scheme over  $S$  acting on a scheme  $X$  over  $S$  via a morphism  $\sigma : G \times_S X \rightarrow X$ . The *quotient stack*  $[X/G]$  is defined as following:

- For each scheme  $U$  over  $S$ , the objects of  $[X/G](U)$  are pairs  $(P, f)$  where  $P$  is a  $G$ -torsor over  $U$  and  $f : P \rightarrow X$  is a  $G$ -equivariant morphism over  $S$ .

- Morphisms between two objects  $(P, f)$  and  $(P', f')$  in  $[X/G](U)$  are given by  $G$ -equivariant morphisms  $\varphi : P \rightarrow P'$  over  $U$  such that  $f' \circ \varphi = f$ .

The assignment  $U \mapsto [X/G](U)$  defines a stack over the site  $(\mathbf{Sch}/S)_{\text{ét}}$ . This stack captures the quotient of  $X$  by the action of  $G$  in a way that respects the group action and the torsor structure.

Yang: To be added.

## Appendix

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