Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$. This section is mainly based on [Har77, Chapter V.2].

1 Preliminaries

Let S be a variety over $\Bbbk$ and $\mathcal{E}$ a vector bundle of rank $r+1$ on S.

Proposition 1. The S-varieties $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on S.

Theorem 2. Let $\pi : X = \mathbb{P}_S(\mathcal{E}) \to S$ be the projective bundle associated to a vector bundle $\mathcal{E}$ of rank $r+1$ on S. Then there is an exact sequence of vector bundles on $\mathbb{P}_S(\mathcal{E})$

$$0 \to \Omega_{\mathbb{P}_S(\mathcal{E})/S} \to \pi^*(\mathcal{E})(-1) \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \to 0.$$

In particular, $K_X \sim \pi^*(K_S + \det \mathcal{E}) - (r+1)\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. Yang: To be continued...

Theorem 3 (Tsen's Theorem, [Stacks, Tag 03RD])**.** Let C be a smooth curve over an algebraically closed field $\Bbbk$. Then $\mathbf{K} = \Bbbk(C)$ is a C_1 field, i.e., every degree d hypersurface in $\mathbb{P}^n_{\mathbf{K}}$ has a $\mathbf{K}$-rational point provided $d \leq n$.

Theorem 4 (Grauert's Theorem, [Har77, Corollary 12.9])**.** Let $f : X \to S$ be a projective morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf on X which is flat over S. Suppose that S is integral and the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$ is constant on S for some $i \geq 0$. Then $\mathrm{R}^i f_* \mathcal{F}$ is locally free and the base change homomorphism

$$\varphi^i_s : \mathrm{R}^i f_* \mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s) \to H^i(X_s, \mathcal{F}_s)$$

is an isomorphism for all $s \in S$.

Theorem 5 (Miracle Flatness, [Mat89, Theorem 23.1])**.** Let $f : X \to Y$ be a morphism of noetherian schemes. Assume that Y is regular and X is Cohen-Macaulay. If all fibers of f have the same dimension $d = \dim X - \dim Y$, then f is flat.

Proposition 6 (Geometric form of Nakayama's Lemma)**.** Let X be a variety, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proposition 7. Let S be a noetherian scheme and $\mathcal{E}$ a vector bundle of rank $r+1$ on S. Denote by $\pi : \mathbb{P}_S(\mathcal{E}) \to S$ the projection. Let X be an S-scheme via a morphism $g : X \to S$. Then there is a

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bijection

$$\left\{\begin{array}{l} S\text{-morphisms} \\ X \to \mathbb{P}_S(\mathcal{E}) \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \mathcal{L} \in \operatorname{Pic}(X) \text{ and surjective} \\ \text{homomorphisms } g^*\mathcal{E} \to \mathcal{L} \end{array}\right\}.$$

Proof. We have a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ by the definition of $\mathbb{P}_S(\mathcal{E})$. If we have a morphism $f : X \to \mathbb{P}_S(\mathcal{E})$ over S, then we have a surjective homomorphism $f^*\pi^*\mathcal{E} \to f^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$.

Suppose we have a surjective homomorphism $g^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ where $\mathcal{L}$ is a line bundle on X. Take an affine cover $\{U_i\}$ of S such that $\mathcal{E}|_{U_i}$ is trivial. On U_i, choose a basis $e_0^{(i)}, \dots, e_r^{(i)}$ of $\mathcal{E}|_{U_i}$. Suppose $\mathbb{P}_S(\mathcal{E})$ is given by gluing $\mathbb{P}^r_{U_i}$ via φ_{ij} induced by the transition functions of $\mathcal{E}$.

The surjection $g^*\mathcal{E}|_{U_i} \twoheadrightarrow \mathcal{L}|_{X_{U_i}}$ gives a unique morphism $f_i : X_{U_i} \to \mathbb{P}^r_{U_i}$ by ??. On $X_{U_i \cap U_j}$, f_i and f_j agree since we have

$$\begin{tikzcd} X_{U_i\cap U_j} \arrow[r, "="] \arrow[d, "f_i"'] & X_{U_i\cap U_j} \arrow[d, "f_j"] \\ \mathbb{P}_{U_i\cap U_j}(\bigoplus \mathcal{O}_{U_i\cap U_j} e_k^{(i)}) \arrow[r, "\varphi_{ij}"] & \mathbb{P}_{U_i\cap U_j}(\bigoplus \mathcal{O}_{U_i\cap U_j} e_k^{(j)}) \end{tikzcd}$$

and the bottom arrow is identical to the identity map on $\mathbb{P}_S(\mathcal{E})_{U_i\cap U_j}$. Gluing f_i gives a morphism $f : X \to \mathbb{P}_S(\mathcal{E})$ over S. In particular, we have $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. □

Definition 8. An *extension* of a coherent sheaf $\mathcal{F}$ by a coherent sheaf $\mathcal{G}$ on a scheme X is an exact sequence of coherent sheaves

$$S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0).$$

Two extensions S and S' are *equivalent* if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{G}}"] & \mathcal{E} \arrow[r] \arrow[d, "\cong"] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{F}}"] & 0 \\ 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0. \end{tikzcd}$$

Proposition 9. Let X be a scheme and $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. Then there is a one-to-one correspondence between equivalence classes of extensions

$$S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0)$$

and elements of $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ given by

$$S \mapsto \delta(\mathrm{id}_{\mathcal{F}})$$

where $\delta : \operatorname{Hom}_X(\mathcal{F}, \mathcal{F}) \to \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ is the connecting homomorphism.

Proof. Take an exact sequence

$$0 \to \mathcal{G} \to \mathcal{I} \xrightarrow{\varphi} \mathcal{C} \to 0$$

with $\mathcal{I}$ injective. Applying $\operatorname{Hom}_X(\mathcal{F}, -)$ gives a long exact sequence

$$0 \to \operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}_X(\mathcal{F}, \mathcal{I}) \to \operatorname{Hom}_X(\mathcal{F}, \mathcal{C}) \xrightarrow{\delta} \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G}) \to 0.$$

For $a \in \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})$, choose a lifting $\alpha \in \mathrm{Hom}_X(\mathcal{F}, \mathcal{C})$ of a. Let $\mathcal{E} := \mathrm{Ker}(\mathcal{I} \oplus \mathcal{F} \to \mathcal{C}, (i, f) \mapsto \varphi(i) - \alpha(f))$.

Let $\mathcal{E} \to \mathcal{F}$ be the projection to the second factor. It is surjective since φ is surjective. Consider the inclusion $\mathcal{G} \to \mathcal{I} \to \mathcal{I} \oplus \mathcal{F}$, which factors through $\mathcal{E}$. On the other hand, if $e \in \mathcal{E}$ maps to 0 in $\mathcal{F}$, then $e \in \mathcal{I}$ and $\varphi(e) = 0$, whence $e \in \mathcal{G}$. Hence we have an extension $S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0)$.

Yang: To be continued... □

2 Minimal Section and Classification

Definition 10 (Ruled surface)**.** A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all geometric fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 11. There exists a section of π.

Proof. Yang: To be continued... □

Proposition 12. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Proof. Let $\sigma : C \to X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π, $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in C$. By Miracle Flatness (Theorem 5), f is flat. By Grauert's Theorem (Theorem 4), $\mathcal{E}$ is a vector bundle of rank 2 on C and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in C$.

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every $x \in X$, we have

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

Yang: The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \to \mathcal{L}$ is surjective.

By Proposition 7, we have a morphism $\varphi : X \to \mathbb{P}_C(\mathcal{E})$ over C such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Since $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, $\varphi|_{X_t} : X_t \to \mathbb{P}_C(\mathcal{E})_t$ is an isomorphism for any $t \in C$. Hence φ is bijection on the underlying sets. By Miracle Flatness (Theorem 5), φ is flat. Yang: $\mathcal{O}_{\mathbb{P}_C(\mathcal{E}), \varphi(x)} \to \mathcal{O}_{X,x}$ is finite. □

Lemma 13. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle.*

Proof. □

Yang: To be continued...

Definition 14. A section C_0 of π is called a *minimal section* if Yang: to be continued...

Lemma 15. Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface over a smooth curve C of genus g with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on C with $\deg \mathcal{L} = -e$.

(b) If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.

Proof. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable, we can assume that $H^0(C, \mathcal{L}_1) \neq 0$. If $\deg \mathcal{L}_1 > 0$, then $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$, contradicting the normalization of $\mathcal{E}$. Similarly $\deg \mathcal{L}_2 \leq 0$. Then $\mathcal{L}_1 \cong \mathcal{O}_C$. And hence $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$.

If $\mathcal{E}$ is indecomposable, we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

which is a non-trivial extension, with $\mathcal{L}$ a line bundle on C of degree $-e$. Hence by Proposition 9, we have $0 \neq \operatorname{Ext}^1_C(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$. By Serre duality, we have $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$. Hence $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$.

Yang: To be continued... □

Theorem 16. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Proof. This is a direct consequence of Lemma 15. □

Example 17. Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$.

Let C be covered by two standard affine charts U_0, U_1 with coordinate u on U_0 and v on U_1 such that $u = 1/v$ on $U_0 \cap U_1$. On U_i, let $\mathcal{O}(-e)|_{U_i}$ be generated by s_i for $i = 0, 1$. We have $s_0 = u^e s_1$ on $U_0 \cap U_1$.

On $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$, let $[x_0 : x_1]$ and $[y_0 : y_1]$ be the homogeneous coordinates of $\mathbb{P}^1$ on X_0 and X_1 respectively. Then the transition function on $X_0 \cap X_1$ is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

Remark 18. The surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ is also called the *Hirzebruch surface*.

Theorem 19. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and for each e there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on E with

$\deg \mathcal{L} = -e$.

Proof. Only the indecomposable case needs a proof. By Lemma 15, we have $-2 \le e \le 0$ and a non-trivial extension

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle on E of degree $-e$.

Case 1. $e = 0$.

In this case, $\mathcal{L}$ is of degree 0 and $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \ne 0$. Hence $\mathcal{L} \cong \mathcal{O}_E$.

Yang: To be continued...

Case 2. $e = -1$.

In this case, $\mathcal{L}$ is of degree 1 and $H^1(E, \mathcal{L}) \cong H^0(E, \mathcal{L}^{-1}) = 0$. By Riemann-Roch, we have $h^0(E, \mathcal{L}) = 1$.

Case 3. $e = -2$.

Yang: To be continued...

□

Example 20. Yang: To be continued...

3 The Néron-Severi Group of Ruled Surfaces

Proposition 21. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and F a fiber of π. Then $\operatorname{Pic}(X) \cong \mathbb{Z}[C_0] \bigoplus \pi^* \operatorname{Pic}(C)$.

Proof. Let D be any divisor on X with $D.F = a \in \mathbb{Z}$. Then $D - aC_0$ is numerically trivial on the fibers of π. Let $\mathcal{L} = \mathcal{O}_X(D - aC_0)$. Then $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$ for any $t \in C$. By Grauert's Theorem (Theorem 4), $\pi_* \mathcal{L}$ is a line bundle on C Yang: and the natural map $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ is an isomorphism. □

Proposition 22. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let F be a fiber of π. Then $K_X \sim -2C_0 + \pi^*(K_C - c_1(\mathcal{E}))$. Numerically, we have $K_X \equiv -2C_0 + (2g - 2 - e)F$ where e is the invariant of X. Yang: Check this carefully.

Proof. Yang: To be continued. □

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \bigoplus \mathcal{O}(-e)$ for some $e \ge 0$.

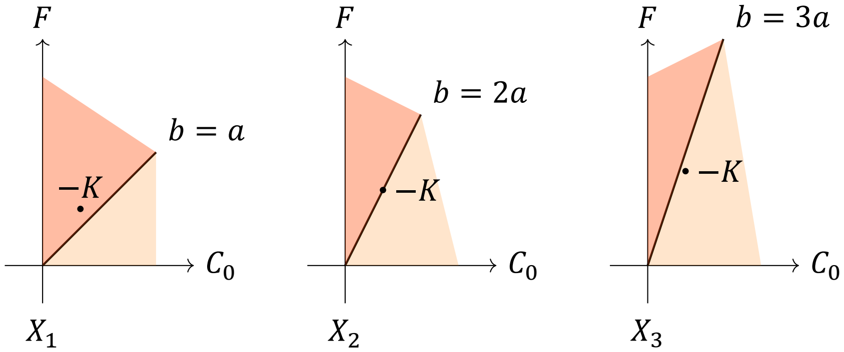
Theorem 23. Let $\pi : X \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with invariant e. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \sim aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$;

(b) D is effective $\iff$ $a, b \ge 0$.

Proof. Yang: To be continued... □

Example 24. Here we draw the Néron-Severi group of the rational ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 1, 2, 3$.



We have $-K_{X_e} \equiv 2C_0 + (2+e)F$. For $e = 1$, $-K$ is ample and hence X_1 is a del Pezzo surface. For $e = 2$, $-K$ is nef and big but not ample. For $e \geq 3$, $-K$ is big but not nef.

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

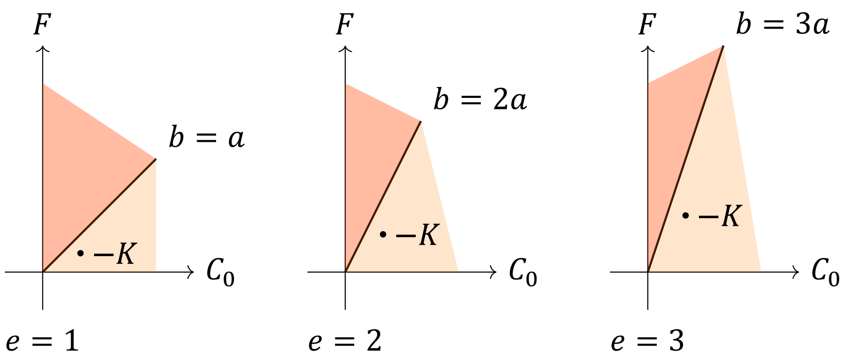
Theorem 25. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is decomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$;

(b) D is effective $\iff$ $a \geq 0$ and $b \geq ae$.

Proof. Yang: To be continued... □

Example 26. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with decomposable normalized $\mathcal{E}$ for $e = 1, 2, 3$.



In this case, $-K \equiv 2C_0 + eF$ is always big but not nef.

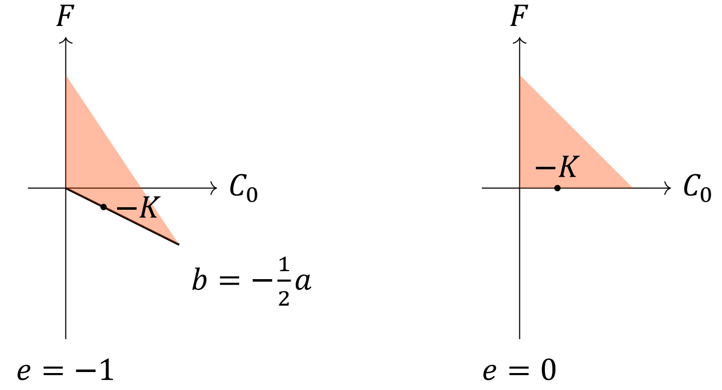
Theorem 27. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is indecomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > \frac{1}{2}ae$;

(b) D is effective $\iff$ $a \geq 0$ and $b \geq \frac{1}{2}ae$.

Proof. Yang: To be continued... □

Example 28. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with indecomposable normalized $\mathcal{E}$ for $e = -1, 0$.



In this case, $-K \equiv 2C_0 + eF$ is always nef but not big.

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