Stacks in category theory

1 Prestacks

Notation 1. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. For $a, b \in \mathrm{Obj}(\mathbf{X})$ and $f \in \mathrm{Hom}_{\mathbf{X}}(a, b)$, we say that a is *over* $\mathbf{p}(a)$ and f is *over* $\mathbf{p}(f)$. In a diagram, we have

$$\begin{tikzcd} \mathbf{X} \arrow[d, "\mathbf{p}"'] \\ \mathbf{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[d] & b \arrow[d] \\ \mathbf{p}(a) \arrow[r, "\mathbf{p}(f)"] & \mathbf{p}(b) \end{tikzcd}$$

Definition 2. A functor $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *prestack* over the site $\mathbf{S}$ if for every object $U \in \mathrm{Obj}(\mathbf{S})$ and every pair of objects $a, b \in \mathrm{Obj}(\mathbf{X})$ over U (i.e., $\mathbf{p}_{\mathbf{X}}(a) = U = \mathbf{p}_{\mathbf{X}}(b)$), the presheaf

$$\underline{\mathrm{Hom}}_U(a, b) : (\mathbf{S}/U)^{op} \to \mathbf{Set}$$

defined by

$$(V \xrightarrow{g} U) \mapsto \mathrm{Hom}_{\mathbf{X}_V}(g^*a, g^*b)$$

is a sheaf on the site $\mathbf{S}/U$. Yang: To be revised.

Proposition 3. A functor $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is a prestack over the site $\mathbf{S}$ if and only if $\mathbf{X}$ is a category fibred in groupoids over $\mathbf{S}$ and for every object $U \in \mathrm{Obj}(\mathbf{S})$ and every pair of objects $a, b \in \mathrm{Obj}(\mathbf{X})$ over U, the presheaf

$$\underline{\mathrm{Hom}}_U(a, b) : (\mathbf{S}/U)^{op} \to \mathbf{Set}$$

defined by

$$(V \xrightarrow{g} U) \mapsto \mathrm{Hom}_{\mathbf{X}_V}(g^*a, g^*b)$$

is a sheaf on the site $\mathbf{S}/U$. Yang: To be checked.

Slogan *Prestacks are "presheaf remembering automorphisms".*

Theorem 4 (Yoneda 2-Lemma)**.** Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ and $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Then the functor

$$\mathrm{Fun}_{\mathbf{S}}(\mathbf{X}, \mathbf{Y}) \to (\mathbf{p}_*, \mathbf{q}_*)$$

given by $\Phi \mapsto \Phi_*$ is an equivalence of categories. Yang: To be revised.

Theorem 5. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$, $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$, and $\mathbf{r} : \mathbf{Z} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Let $\Phi : \mathbf{X} \to \mathbf{Z}$ and $\Psi : \mathbf{Y} \to \mathbf{Z}$ be morphisms of prestacks over $\mathbf{S}$. Then the fiber product $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ exists in the category of prestacks over $\mathbf{S}$. Yang: To be checked.

Date: January 14, 2026, **Author**: Tianle Yang, My Homepage

2 Stacks

Definition 6. Let $\mathbf{S}$ be a site. A prestack $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *stack* over the site $\mathbf{S}$ if for every object $U \in \mathsf{Obj}(\mathbf{S})$ and every covering $\{U_i \to U\}$ in $\mathbf{S}$, the descent data for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ are effective. Yang: To be revised.

Appendix