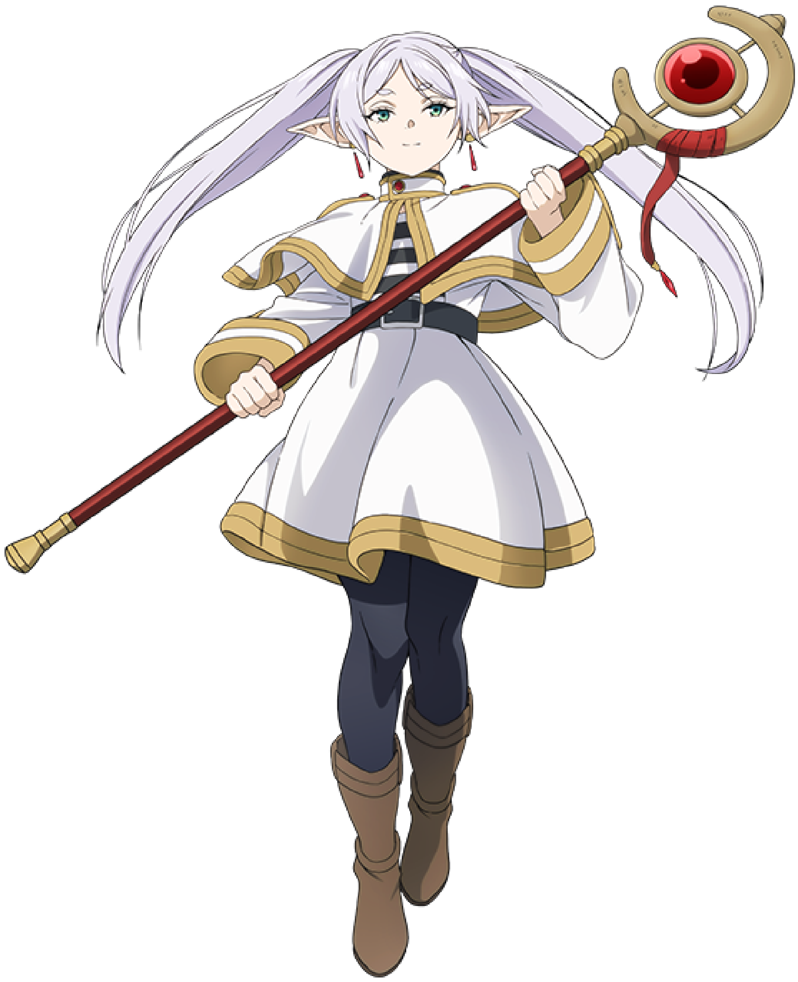
Normal and Cohen-Macaulay schemes



如果是勇者辛美尔，他一定会这么做的！

Normality and Cohen-Macaulay schemes

1 Height, Depth and Dimension

There are three numbers measuring the "size" of a local ring $(A, \mathfrak{m})$:

- $\dim A$: the Krull dimension of A.
- $\operatorname{depth} A$: the depth of A.
- $\dim_{\mathsf{k}} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a k-vector space.

Definition 1. The *height of a prime ideal* $\mathfrak{p}$ in A is defined as the maximum length of chains of prime ideals contained in $\mathfrak{p}$, that is,

$$\operatorname{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as $\dim A := \max_{\mathfrak{m}} \operatorname{ht}(\mathfrak{m})$.

Definition 2. Let $(A, \mathfrak{m})$ be a noetherian local ring with residue field k and M a finitely generated A-module. A sequence $t_1, \cdots, t_n \in \mathfrak{m}$ is called a *regular sequence for* M if t_i is not a zero divisor on $M/(t_1, \cdots, t_{i-1})$, that is, $M/(t_1, \cdots, t_{i-1}) \to t_i M/(t_1, \cdots, t_{i-1})$ is injective. The *depth* of M is defined as the maximum length of regular sequences for M.

These three numbers are related by the following inequalities.

Proposition 3. Let $(A, \mathfrak{m})$ be a local noetherian ring with residue field k. Then the following inequalities hold:

$$\operatorname{depth} A \leq \dim A \leq \dim_{\mathsf{k}} T_{A,\mathfrak{m}}.$$

To see these, we need the following well-known theorem.

Theorem 4 (Krull's Principal Ideal Theorem)**.**

Theorem 5 (Nakayama's Lemma)**.** thhhhh

Definition 6 (Cohen-Macaulay)**.** A local noetherian ring $(A, \mathfrak{m})$ is called *Cohen-Macaulay* if $\dim A = \operatorname{depth} A$.

Example 7 (Non Cohen-Macaulay rings)**.**

Proposition 8.

Theorem 9 (Serre's criterion for normality)**.** Let X be a

2 Normal schemes

Definition 10. A ring A is called *normal* if it is an integral domain and integrally closed in its field of fractions $\operatorname{Frac}(A)$.

Proposition 11. Normality is a local property. That is, TFAE:

(a) A is normal.

(b) For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is normal.

(c) For any maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$, the localization $A_{\mathfrak{m}}$ is normal.

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Proof. □

Proposition 12. Let A be a normal ring. Then $A[X]$ and $A[[X]]$ are normal rings.

Definition 13. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,x}$ is normal for any point $x \in X$.

Example 14.

Definition 15. Let X be a scheme. The *normalization* of X is a X-scheme X^ν with the following universal property: for any normal X-scheme Y, its structure morphism $Y \to X$ factors through X^ν.

Proposition 16. Let X be an integral scheme. Then the normalization X^ν of X exists. Moreover, X^ν/X is birational.

Theorem 17. Let X be a normal noetherian scheme. Let $F \subset X$ be a closed subset of codimension ≥ 2. Then the restriction $\mathcal{O}_X(X) \to \mathcal{O}_X(X \setminus F)$ is an isomorphism.