Differentials and duality

Let $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes and varieties are assumed to be defined over $\Bbbk$.

1 The sheaves of differentials

Definition 1. Let X be a normal variety over $\Bbbk$ of dimension n. If X is smooth, then the *canonical divisor* K_X is defined to be $c_1(\omega_X)$. In general, let $U \subseteq X$ be the smooth locus of X and $i : U \hookrightarrow X$ be the inclusion map. Then the *canonical divisor* K_X is defined to be any Weil divisor on X such that $\mathcal{O}_X(K_X) \cong i_*\omega_U$. Note that U is big in X since X is normal, so such a Weil divisor always exists and is unique up to linear equivalence.

2 Fundamental sequence

Theorem 2. Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/\Bbbk} \to \Omega_{X/\Bbbk} \to \Omega_{X/Y} \to 0.$$

Yang: ... it will be exact.

Theorem 3 (Ramification formula)**.** Let $f : X \to Y$ be a morphism of schemes and let $Z \subseteq Y$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_Y$. Then there is a natural isomorphism

$$\Omega_{X/Y} \cong \mathcal{J}/\mathcal{J}^2,$$

where $\mathcal{J} = f^*\mathcal{I} \cdot \mathcal{O}_X \subseteq \mathcal{O}_X$ is the sheaf of ideals defining the preimage $W = f^{-1}(Z)$ in X. Yang: It is wrong.

Theorem 4. Let $Z \subseteq Y$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_Y$ and let $W = f^{-1}(Z)$ be the preimage of Z in X, defined by the sheaf of ideals $\mathcal{J} = f^*\mathcal{I} \cdot \mathcal{O}_X \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_W$-modules

$$\mathcal{J}/\mathcal{J}^2 \to \Omega_{X/\Bbbk}|_W \to \Omega_{W/\Bbbk} \to 0.$$

Yang: ... it will be exact.

Theorem 5 (Adjunction formula)**.** Let $f : X \to Y$ be a smooth morphism of schemes. Then there is a natural isomorphism

$$\Omega_{X/Y} \cong \Omega_{X/\Bbbk}|_W,$$

where $W = f^{-1}(Z)$ is the preimage of a closed subscheme $Z \subseteq Y$. Yang: It is wrong.

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3 Serre duality

Theorem 6 (Serre duality)**.** Let X be a proper variety over $\Bbbk$ and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a natural isomorphism

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee,$$

where ω_X is the canonical sheaf on X and $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual sheaf. Yang: there are some errors. Need to be revised