Notes in Algebraic Geometry

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Chapter 1

Schemes and Varieties

1.0 Locally Ringed Space

1.0.1 Sheaves

Definition 1.0.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

For two open sets $V \subset U \subset X$, the morphism $\mathcal{F}(U) \to \mathcal{F}(V)$, often denoted by res^U_V, is called the *restriction map*.

Example 1.0.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 1.0.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U \to \mathbb{R}\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0, 1) \cup (2, 3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0, 1), (2, 3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0, 1))$ and $s_2 = 2 \in \mathcal{C}((2, 3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 1.0.4. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be presheaves on X with values in the same category (e.g., $\mathbf{Set}$, $\mathbf{Ab}$, $\mathbf{Ring}$, etc.). A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$. In other words, for every open set $U \subset X$, there is a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every inclusion of open sets $V \subset U$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}^U_V"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V). \end{tikzcd}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then φ is called a *morphism of sheaves.*

Fix a topological space X and a category $\mathbf{C}$. The sheaves (resp. presheaves) on X with values in $\mathbf{C}$ form a category, denoted by $\mathbf{Sh}(X, \mathbf{C})$ (resp. $\mathbf{PSh}(X, \mathbf{C})$), where the objects are sheaves (resp. presheaves) on X with values in $\mathbf{C}$ and the morphisms are morphisms of sheaves (resp. presheaves).

Definition 1.0.5. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in a category $\mathbf{C}$. For a point $x \in X$, the *stalk* of $\mathcal{F}$ at x, denoted by $\mathcal{F}_x$, is defined as the colimit

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the colimit is taken over all open neighborhoods U of x. An element of $\mathcal{F}_x$ is called a *germ* of sections of $\mathcal{F}$ at x.

More concretely, we have

$$\mathcal{F}_x = \{(U, s) : U \in \mathbf{Open}(X), U \ni x, s \in \mathcal{F}(U)\}/\sim,$$

where $(U, s) \sim (V, t)$ if there exists an open neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$.

Definition 1.0.6. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in $\mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.). A *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}^\dagger$ on X together with a morphism of presheaves $\eta : \mathcal{F} \to \mathcal{F}^+$ such that for every sheaf $\mathcal{G}$ on X and every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \eta$.
In other words, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & \mathcal{F}^\dagger \arrow[d, "\varphi^\dagger"] \\ & \mathcal{G}. \end{tikzcd}$$

To be checked.

The concrete describe of sheafification.

Definition 1.0.7. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The morphism φ is called *injective* (resp. *surjective*) if for every $x \in X$, the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).

Proposition 1.0.8. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is injective if and only if for every open set $U \subset X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. To be checked.

Remark 1.0.9. The surjectivity on stalks cannot imply the surjectivity on sections. A counterexample is given by the exponential map $\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^*$ defined by $\exp(f) = e^f$, where $\mathcal{O}_{\mathbb{C}}$ is the sheaf of holomorphic functions on $\mathbb{C}$ and $\mathcal{O}_{\mathbb{C}}^*$ is the sheaf of non-vanishing holomorphic functions on $\mathbb{C}$. The induced map on stalks $\exp_z : \mathcal{O}_{\mathbb{C},z} \to \mathcal{O}_{\mathbb{C},z}^*$ is surjective for every $z \in \mathbb{C}$ by the existence of logarithm locally. However, the map on global sections $\exp(\mathbb{C}) : \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}^*(\mathbb{C})$ is not surjective since there is no entire function f such that $e^{f(z)} = z$ for all $z \in \mathbb{C}^*$. To be continued. This is wrong, need to be revised.

Proposition 1.0.10. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is an isomorphism if and only if it is injective and surjective.

Now we consider sheaves with values in an abelian category.

Definition 1.0.11. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The *kernel* of φ, denoted by $\ker\varphi$, is the sheaf defined by

$$(\ker\varphi)(U) := \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$.
The *cokernel* of φ, denoted by $\operatorname{coker}\varphi$, is the sheafification of the presheaf defined by

$$(\operatorname{coker}\varphi)_{\text{pre}}(U) := \operatorname{coker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$. To be continued.

Theorem 1.0.12. Let X be a topological space and $\mathbf{C}$ be an abelian category (e.g., $\mathbf{Ab}$). Then the category of sheaves on X with values in $\mathbf{C}$ is an abelian category.

Proof. To be continued. □

To be checked and continuous.

1.0.2 Locally ringed space

Definition 1.0.13. Let $f : X \to Y$ be a continuous map between topological spaces. The *push-forward* functor $f_* : \mathbf{Sh}(X, \mathbf{C}) \to \mathbf{Sh}(Y, \mathbf{C})$ is defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

for every open set $V \subset Y$ and sheaf $\mathcal{F} \in \mathbf{Sh}(X, \mathbf{C})$.

Definition 1.0.14. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 1.0.15. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Construction 1.0.16 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Construction 1.0.17 (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$
\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}
$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i\in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i\in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^{\sharp}(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ_i' satisfying the same properties, then by gluing $\psi_i' \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

Definition 1.0.18. A morphism of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a *closed immersion* (resp. *open immersion*) if f induces a homeomorphism from X to a closed (resp. open) subset of Y and the map $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective (resp. an isomorphism). To be checked.

1.0.3 Manifolds as locally ringed spaces

1.0.4 Vector bundles and $\mathcal{O}_X$-modules

Let $(X, \mathcal{O}_X)$ be a manifold (real or complex) and $(\mathcal{E}, \pi, X)$ a vector bundle over X.

It can regard as a sheaf on X.

Definition 1.0.19. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F}$ of abelian groups on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\mathsf{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear, where the $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(V)$ is induced by the restriction map $\mathsf{res}_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

A *morphism of $\mathcal{O}_X$-modules* is a morphism of sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ such that for every open set $U \subseteq X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. To be checked...

Definition 1.0.20. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is said to be *locally free of rank r* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_U^r$, where $\mathcal{O}_U^r$ is the direct sum of r copies of $\mathcal{O}_U$. To be continued.

1.1 The First Properties of Schemes

If you learn the following content for the first time, it is recommended to skip all the proofs in this section and focus on the examples, remarks and the statements of propositions and theorems.

1.1.1 Schemes

Let R be a ring. Recall that the *spectrum* of R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set*.

Proposition 1.1.1. Let R be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

Proof. To be continued ☐

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 1.1.2. An *affine scheme* is a locally ringed space isomorphic to $(\mathsf{Spec}\, R, \mathcal{O}_{\mathsf{Spec}\, R})$ for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i\in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.
These data form a category, denoted by $\mathbf{Sch}$. If we fix a base scheme S, then an S*-scheme* is a scheme X together with a morphism $X \to S$. The category of S-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

Theorem 1.1.3. The functor $\mathsf{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. To be continued

Definition 1.1.4. A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if f induces an isomorphism of X onto an open (resp. closed) subscheme of Y. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. To be continued

Construction 1.1.5. Let R be a graded ring. The *projective scheme* $\mathsf{Proj}\, R$ is defined as the scheme associated to the sheaf of rings

$$\mathcal{O}_{\mathsf{Proj}\, R} = \bigoplus_{d\geq 0} R_d.$$

It can be covered by open affine subschemes of the form $\mathsf{Spec}\, R_f$ for homogeneous elements $f \in R$. To be checked.

Construction 1.1.6 (Glue open subschemes)**.** The construction in Construction 1.0.17 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Construction 1.0.17. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

Definition 1.1.7. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme Z of Y such that f factors through Z. More precisely, if $Y = \mathsf{Spec}\, A$ is affine, then the scheme theoretic image of f is $\mathsf{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover Y by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of f. To be checked.

1.1.2 Fiber product and base change

Definition 1.1.8. Let $\mathbf{C}$ be a category and $X, Y, S \in \mathsf{Obj}(\mathbf{C})$ with morphisms $f : X \to S$ and $g : Y \to S$. A *fiber product* of X and Y over S is an object $X \times_S Y \in \mathsf{Obj}(\mathbf{C})$ together with morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ such that $f \circ p = g \circ q$ and satisfies the universal property that for any object $W \in \mathsf{Obj}(\mathbf{C})$ with morphisms $u : W \to X$ and $v : W \to Y$ such that $f \circ u = g \circ v$, there exists a unique morphism $h = (u, v) : W \to X \times_S Y$ such that $p \circ h = u$ and $q \circ h = v$.

$$\begin{tikzcd} W \arrow[dr, "h"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & X \times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & S. \end{tikzcd}$$

To be checked.

Example 1.1.9. In the category of sets, the fiber product $X \times_S Y$ is given by

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\},$$

with the projections $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ being the restrictions of the natural projections. To be checked.

Remark 1.1.10. If one reverses the arrows in Definition 1.1.8, one gets the notion of *fiber coproduct*. It is also called the *pushout* or *amalgamated sum* in some literature. We denote the fiber coproduct of X and Y over S by $X \amalg_S Y$. Note that in the category of rings, the fiber coproduct $A \amalg_R B$ of R-algebras A and B over R is given by the tensor product $A \otimes_R B$. Dually, one can expect that fiber products of affine schemes correspond to tensor products of rings.

Theorem 1.1.11. The category of schemes admits fiber products. To be continued

Definition 1.1.12. Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a point. The *scheme theoretic fiber* of f over y is the fiber product $X_y = X \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y)$ is the residue field of the local ring $\mathcal{O}_{Y,y}$. To be checked.

Definition 1.1.13. Let X be a scheme and $Z_1, Z_2 \subset X$ be closed subschemes of X with inclusion morphisms $i_1 : Z_1 \to X$ and $i_2 : Z_2 \to X$. The *scheme theoretic intersection* of Z_1 and Z_2 is the fiber product $Z_1 \times_X Z_2$. To be checked.

1.1.3 Noetherian schemes and morphisms of finite type

Definition 1.1.14. A scheme X is *noetherian* if it admits a finite open cover $\{U_i\}_{i=1}^n$ such that each U_i is an affine scheme $\operatorname{Spec} A_i$ with A_i a noetherian ring. To be checked.

Proposition 1.1.15. A noetherian scheme is quasi-compact. To be checked.

Definition 1.1.16. Let $f : X \to S$ be a morphism of schemes. We say that f is *of finite type*, or X is *of finite type* over S, if there exists a finite affine cover $\{U_i\}_{i=1}^n$ of S such that for each i, $f^{-1}(U_i)$ can be covered by finitely many affine open subsets $\{V_{ij}\}_{j=1}^{m_i}$ with $f(V_{ij}) \subseteq U_i$ and the induced morphism $f|_{V_{ij}} : V_{ij} \to U_i$ corresponds to a finitely generated algebra over the ring of global sections of U_i. Given S, the category consisted of S-scheme of finite type over S, together with morphisms of S-schemes, is denoted by $\mathbf{sch}_S$. To be checked.

1.1.4 Integral, reduced and irreducible schemes

Definition 1.1.17. A topological space X is *irreducible* if it is non-empty and cannot be expressed as the union of two proper closed subsets. Equivalently, every non-empty open subset of X is dense in X. To be checked.

Proposition 1.1.18. Let X be a topological space satisfying the descending chain condition on closed subsets. Then X can be written as a finite union of irreducible closed subsets, called the *irreducible components* of X. Moreover, this decomposition is unique up to permutation of the components. To be checked.

Definition 1.1.19. A scheme X is *reduced* if its structure sheaf $\mathcal{O}_X$ has no nilpotent elements. To be checked.

Proposition 1.1.20. A scheme X is reduced if and only if for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a reduced ring. To be checked.

Proposition 1.1.21. Let X be a scheme. There exists a unique closed subscheme X of X such that X is reduced and has the same underlying topological space as X. Moreover, for any morphism of schemes $f : Y \to X$ with Y reduced, f factors uniquely through the inclusion $X \to X$. To be checked.

Definition 1.1.22. A scheme X is *integral* if it is both reduced and irreducible. To be checked.

Proposition 1.1.23. A scheme X is integral if and only if for every open affine subset $U = \operatorname{Spec} A \subset X$, the ring A is an integral domain. To be checked.

Corollary 1.1.24. Let $\Bbbk$ be an algebraically closed field and $n \geq 1$ be an integer. Then the polynomial $\det(x_{ij}) \in k[x_{ij} : 1 \leq i, j \leq n]$ is irreducible. To be checked.

1.1.5 Dimension

Definition 1.1.25. The *Krull dimension* of a topological space X, denoted by $\dim X$, is the supremum of the lengths n of chains of distinct irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X. If no such finite supremum exists, we say that X has infinite dimension. To be checked.

Definition 1.1.26. Let $\xi \in X$ be a point in a scheme X. The *local dimension* of X at ξ, denoted by $\dim_\xi X$, is defined as the infimum of the dimensions of all open neighborhoods U of ξ:

$$\dim_\xi X = \inf\{\dim U : U \text{ is an open neighborhood of } \xi\}.$$

To be checked.

1.1.6 Separated, proper and projective morphisms

Definition 1.1.27. A morphism of schemes $f : X \to Y$ is *separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. A scheme X is *separated* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. To be checked.

Proposition 1.1.28. Any affine scheme is separated. More generally, any morphism between affine schemes is separated. To be checked.

Proposition 1.1.29. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any scheme T and any pair of morphisms $g_1, g_2 : T \to X$ such that $f \circ g_1 = f \circ g_2$, the equalizer of g_1 and g_2 is a closed subscheme of T. To be checked.

Proposition 1.1.30. A scheme X is separated if and only if for any pair of affine open subschemes $U, V \subset X$, the intersection $U \cap V$ is also an affine open subscheme. To be checked.

Proposition 1.1.31. The composition of separated morphisms is separated. Moreover, separatedness is stable under base change, i.e., if $f : X \to Y$ is a separated morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also separated. To be checked.

Proposition 1.1.32. A morphism of schemes $f : X \to Y$ is separated if and only if for every commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[d] \arrow[dr] & \\
\operatorname{Spec} R \arrow[r, dashed] \arrow[dr] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists at most one morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. To be checked.

Definition 1.1.33. A morphism of schemes $f : X \to Y$ is *universally closed* if for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map. To be checked.

Definition 1.1.34. A morphism of schemes $f : X \to Y$ is *proper* if it is of finite type, separated, and universally closed. A scheme X is *proper* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is proper. To be checked.

Theorem 1.1.35. Any projective morphism is proper. In particular, any projective scheme is proper. To be checked.

Proposition 1.1.36. The composition of proper morphisms is proper. Moreover, properness is stable under base change, i.e., if $f : X \to Y$ is a proper morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also proper. To be checked.

Proposition 1.1.37. A morphism of schemes $f : X \to Y$ is proper if and only if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists a unique morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. To be checked.

1.1.7 Varieties

Definition 1.1.38. Let $\Bbbk$ be a algebraically closed field. A *variety over* $\Bbbk$ is an integral scheme of finite type over $\operatorname{Spec} \Bbbk$. The category of varieties over $\Bbbk$ is denoted by $\mathbf{Var}_{\Bbbk}$. To be checked.

Let X be a variety over $\Bbbk$. The closed points $X(\Bbbk)$ is a locally ringed subspace of X with the induced topology and structure sheaf. We denote the category of such locally ringed spaces by $\mathbf{ClaVar}_{\Bbbk}$, meaning the category of *classical varieties* over $\Bbbk$.

Theorem 1.1.39. Let X be a variety over $\Bbbk$. Then there is an equivalence of categories between $\mathbf{Var}_{\Bbbk}$ and $\mathbf{ClaVar}_{\Bbbk}$.

Slogan *Closed points determine varieties.*

Proof. To be continued. □

1.2 Category of sheaves of modules

Mostly results in this section fits into the context of ringed spaces.

1.2.1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 1.2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets.

Definition 1.2.2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that there exists a surjective morphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

Remark 1.2.3. There are many versions of "local" properties for sheaves of $\mathcal{O}_X$-modules. To be continued.

Definition 1.2.4. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it is finitely generated, and for every open set $U \subseteq X$ and every morphism of sheaves of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of φ is finitely generated.

Slogan

$$\mathbf{Sh}_X(\mathbf{Ab}) \supseteq \mathbf{Mod}_{\mathcal{O}_X} \supseteq \mathbf{QCoh}_X \supseteq \mathbf{Coh}_X.$$

Definition 1.2.5. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a coherent sheaf on X. The *support* of $\mathcal{F}$ is defined to be the set

$$\mathrm{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\},$$

where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x. To be checked.

1.2.2 As abelian categories

Theorem 1.2.6. Let $(X, \mathcal{O}_X)$ be a ringed space. All of $\mathbf{Sh}_X(\mathbf{Ab}), \mathbf{Mod}(\mathcal{O}_X), \mathbf{QCoh}_X, \mathbf{Coh}_X$ are abelian categories.

Theorem 1.2.7. Let $(X, \mathcal{O}_X)$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules has enough injectives. To be checked.

Remark 1.2.8. The category of sheaves of $\mathcal{O}_X$-modules generally does not have enough projectives. To be checked.

Theorem 1.2.9. Let X be a noetherian, integral, separated, regular scheme. Then every coherent sheaf on X admits a finite resolution by locally free sheaves.

Proof. To be continued. □

1.2.3 Relevant functors

Definition 1.2.10. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *sheaf Hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf of abelian groups defined as follows: for an open set $U \subseteq X$, we define

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of morphisms of sheaves of $\mathcal{O}_U$-modules from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. For an inclusion of open sets $V \subseteq U$, the restriction map

$$\mathrm{res}_{UV} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(V)$$

is defined by sending a morphism $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ to its restriction $\varphi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. To be continued.

Definition 1.2.11. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *dual sheaf* $\mathcal{F}^\vee$ is defined to be

$$\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$$

To be continued.

Definition 1.2.12. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules defined as follows: for an open set $U \subseteq X$, we define

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

where $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is the tensor product of $\mathcal{O}_X(U)$-modules. For an inclusion of open sets $V \subseteq U$, the restriction map
To be continued.

Definition 1.2.13. Let $f : X \to Y$ be a morphism of ringed spaces. The *pull-back functor* $f^* : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$ is defined as follows: for an $\mathcal{O}_Y$-module $\mathcal{F}$, we define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

where $f^{-1}\mathcal{F}$ is the inverse image sheaf of $\mathcal{F}$. For a morphism of $\mathcal{O}_Y$-modules $\varphi : \mathcal{F} \to \mathcal{G}$, we define

$$f^*\varphi : f^*\mathcal{F} \to f^*\mathcal{G}$$

to be the morphism induced by the morphism of sheaves of abelian groups $f^{-1}\varphi : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$.
To be continued.

Definition 1.2.14. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $Z \subseteq X$ be a closed subset. The *functor of sections with support in* Z is defined as follows: for an $\mathcal{O}_X$-module $\mathcal{F}$, we define

$$\Gamma_Z(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) \mid \mathrm{Supp}(s) \subseteq Z\},$$

where $\mathrm{Supp}(s)$ is the support of the section s. To be checked.

1.2.4 Cohomological theory

Definition 1.2.15. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *sheaf cohomology* $H^i(X, \mathcal{F})$ is defined as the i-th right derived functor of the global section functor $\Gamma(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ applied to $\mathcal{F}$, i.e.,

$$H^i(X, \mathcal{F}) := \mathrm{R}^i\Gamma(X, \mathcal{F}).$$

To be checked.

Definition 1.2.16. Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *i-th higher direct image* $\mathrm{R}^i f_*\mathcal{F}$ is defined as the i-th right derived functor of the direct image functor $f_* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ applied to $\mathcal{F}$, i.e.,

$$\mathsf{R}^i f_* \mathcal{F} := \mathsf{R}^i(f_* \mathcal{F}).$$

To be checked.

Definition 1.2.17. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *i-th sheaf Ext functor* $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is defined as the i-th right derived functor of the sheaf Hom functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$ applied to $\mathcal{G}$, i.e.,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \mathsf{R}^i\, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

To be checked.

Proposition 1.2.18. Let $f : X \to Y$ be a morphism of ringed spaces, and let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. Then there are long exact sequences of $\mathcal{O}_Y$-modules

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to \mathsf{R}^1 f_*\mathcal{F} \to \mathsf{R}^1 f_*\mathcal{G} \to \mathsf{R}^1 f_*\mathcal{H} \to \mathsf{R}^2 f_*\mathcal{F} \to \cdots$$

To be checked.

Theorem 1.2.19 (Affine criterion by Serre)**.** Let X be a scheme. Then X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i > 0$. To be checked.

Theorem 1.2.20 (Leray spectral sequence)**.** Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there exists a spectral sequence

$$E_2^{p,q} = H^p(Y, \mathsf{R}^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

To be checked.

1.3 Line bundles and divisors

1.3.1 Cartier divisors

Definition 1.3.1. Let X be a scheme. A *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of X. Equivalently, a Cartier divisor D can be represented by an open covering $\{U_i\}$ of X and a collection of rational functions $f_i \in \mathcal{K}_X^*(U_i)$ such that for any i, j, the function $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. We denote a Cartier divisor by $D = \{(U_i, f_i)\}$.

1.3.2 Line bundles and Picard group

Definition 1.3.2. Let X be a scheme. A *line bundle* on X is a locally free sheaf of $\mathcal{O}_X$-modules of rank 1.

Example 1.3.3. Let $X = \mathbb{P}_A^n = \operatorname{Proj} A[T_0, T_1, \dots, T_n] = \operatorname{Proj} B$ be the projective n-space over a ring A. For each integer $d \in \mathbb{Z}$, the sheaf $\mathcal{O}_X(d)$, defined by

$$\{f \neq 0\} \mapsto B(d)_{(f)},$$

is a line bundle on X, called the *twisted line bundle* of degree d. Recall that here $B(d)_{(f)}$ is the degree-zero part of the localization of the shifted graded ring $B(d)$ at the multiplicative set generated by f, and $B(d)$ is defined by $B(d)_m = B_{m+d}$ for all $m \in \mathbb{Z}$.

Let us verify this by direct computation. On the standard open subset $U_i = D_+(T_i) = \operatorname{Spec} B_i$, where $B_i = A\,[T_0/T_i, \dots, T_n/T_i]$, write $t_{j,i} = T_j/T_i$. We have

$$\mathcal{O}_X(d)(U_i) = B(d)^0_{(T_i)} = \left\{ \frac{f}{T_i^k} \middle| f \in B, \deg f = k + d \right\} = B_i \cdot T_i^d =: B_i \cdot e_i,$$

where we denote $e_i = T_i^d$. Hence $\mathcal{O}_X(d)(U_i)$ is a free B_i-module of rank 1 and thus $\mathcal{O}_X(d)$ is locally free of rank 1.

In the language of bundles, on $U_{ij} = U_i \cap U_j$, we have

$$e_i = t_{i,j}^d \cdot e_j.$$

Thus the transition functions of $\mathcal{O}_X(d)$ are given by $\{(U_{ij},\ t_{i,j}^d : U_{ij} \to \mathbb{G}_m)\}$.

Proposition 1.3.4. Let X be a scheme and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. Then

(a) the tensor product $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is also a line bundle on X;

(b) the dual $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also a line bundle on X;

(c) there is a natural isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$.

Proof. □

Definition 1.3.5. Let X be a scheme. The *Picard group* of X is defined to be the group of isomorphism classes of line bundles on X with the group operation given by the tensor product. It is denoted by $\operatorname{Pic}(X)$.

Definition 1.3.6. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\mathbb{k})$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X \times \{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X \times \{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\text{alg}} \mathcal{L}'$. To be checked.

1.3.3 Weil divisors and reflexive sheaves

To talk about Weil divisors, we need to work with normal schemes.

Definition 1.3.7. Let X be a normal integral scheme. A *Weil divisor* on X is a formal sum

$$D = \sum_{Z} n_Z Z,$$

where the sum runs over all prime divisors Z of X (i.e., integral closed subschemes of codimension 1) and $n_Z \in \mathbb{Z}$, such that for any affine open subset $U = \operatorname{Spec} A \subseteq X$, only finitely many Z intersecting U have nonzero coefficients n_Z. The group of Weil divisors on X is denoted by $\mathrm{WDiv}(X)$.

Definition 1.3.8. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. The sheaf $\mathcal{F}$ is called *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

Proposition 1.3.9. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. If $\mathcal{F}$ is reflexive, then it is determined by its restriction to any open subset $U \subseteq X$ whose complement has codimension at least 2, i.e., $\mathcal{F} \cong i_*(\mathcal{F}|_U)$, where $i : U \hookrightarrow X$ is the inclusion map. To be checked.

Proof. To be continued. □

Theorem 1.3.10. Let X be a normal integral scheme. There is a one-to-one correspondence between the set of isomorphism classes of reflexive sheaves of rank 1 on X and the Weil divisor class group $\mathrm{WDiv}(X)$ of X. Under this correspondence, a Weil divisor D corresponds to the reflexive sheaf $\mathcal{O}_X(D)$. To be checked.

Proof. To be continued. □

1.3.4 The first Chern class

Definition 1.3.11. Let X be a normal scheme and $\mathcal{L}$ a vector bundle on X. The *first Chern class* of $\mathcal{L}$, denoted by $c_1(\mathcal{L})$, is a Weil divisor class defined as follows:
To be completed.

Definition 1.3.12. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. On X_{reg}, the regular locus of X, $\mathcal{F}|_{X_{\mathrm{reg}}}$ admits a finite resolution by vector bundles

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F}|_{X_{\mathrm{reg}}} \to 0.$$

The *first Chern class* of $\mathcal{F}$, denoted by $c_1(\mathcal{F})$, is defined to be

$$c_1(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i c_1(\mathcal{E}_i).$$

To be revised.

Proposition 1.3.13. Let X be a normal scheme and $\mathcal{F}$ a torsion sheaf on X. Then

$$c_1(\mathcal{F}) = \sum_Z \operatorname{length}_{\mathcal{O}_{X,Z}}(\mathcal{F}_Z) \cdot Z,$$

where the sum runs over all prime divisors Z of X and $\mathcal{F}_Z$ is the stalk of $\mathcal{F}$ at the generic point of Z. To be checked.

1.4 Morphisms by line bundles and ampleness

1.4.1 Globally generated line bundles

Definition 1.4.1. Let X be a scheme over a ring A and $\mathcal{F}$ a quasi-coherent sheaf on X. We say that $\mathcal{F}$ is *globally generated* or *generated by global sections* if the natural map $\Gamma(X, \mathcal{F}) \otimes_A \mathcal{O}_X \to \mathcal{F}$ is surjective.

Proposition 1.4.2. Let X be a scheme over a ring A and $\mathcal{F}$, $\mathcal{G}$ quasi-coherent sheaves on X. Then we have the following:

(a) if $\mathcal{F}$ is globally generated, then for any morphism $f : Y \to X$ over A, the pullback $f^*\mathcal{F}$ is globally generated on Y;

(b) if both $\mathcal{F}$ and $\mathcal{G}$ are globally generated, then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

To be revised.

The story begins with the following theorem, which uses global sections of a globally generated line bundle to construct a morphism to projective space.

Theorem 1.4.3. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$. We need a more "functorial" expression.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Example 1.4.4. Let $X = \mathbb{P}_A^n$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}_A^N$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 1.4.5. Let $X = \mathbb{P}_A^m \times_A \mathbb{P}_A^n$ with A a ring and $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}_A^{(m+1)(n+1)-1}$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Proposition 1.4.6. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}_{\mathbf{k}}^n$ and $g : X \to \mathbb{P}_{\mathbf{k}}^m$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}_{\mathbf{k}}^n \dashrightarrow \mathbb{P}_{\mathbf{k}}^m$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. To be continued. □

1.4.2 Ample line bundles

Definition 1.4.7. Let X be a scheme over a field $\mathbf{k}$. A line bundle $\mathcal{L}$ on a X is called *very ample* if there exists a closed embedding $i : X \to \mathbb{P}_{\mathbf{k}}^n$ such that $\mathcal{L} \cong i^* \mathcal{O}(1)$.

The following lemma due to Serre gives a good description of very ample line bundles.

Lemma 1.4.8. Let X be a scheme over a ring A and $\mathcal{L}$ a very ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that for all $n \ge N$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated.

Proof. To be added. □

By Lemma 1.4.8, we have a more intrinsic definition.

Definition 1.4.9. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. To be continued.

Theorem 1.4.10. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

To be continued.

Proof. To be continued. □

Remark 1.4.11. By Theorem 1.4.10, a scheme X which is proper over a field $\mathbf{k}$ is projective if and only if it admits an ample line bundle. More intrinsically, we will use the definition that a *projective scheme* over a field $\mathbf{k}$ is a scheme proper over $\mathbf{k}$ which admits an ample line bundle. And the ample line bundle is often denoted by $\mathcal{O}_X(1)$. Once fix the ample line bundle $\mathcal{O}_X(1)$, for any coherent sheaf $\mathcal{F}$ on X, we denote $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n)$ for any integer n.

Proposition 1.4.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

To be continued.

Proof. To be continued. □

Theorem 1.4.13 (Serre Vanishing)**.** Let X be a projective scheme over a field k and $\mathcal{L}$ a very ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that for all $n \geq N$, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

Corollary 1.4.14. Let X be a projective variety over a field $\mathbf{k}$ and $\mathcal{L}$ an ample line bundle on X. Then for any non-zero global section $s \in \Gamma(X, \mathcal{L})$, the support of the effective Cartier divisor $\mathbf{div}(s)$ is connected.

Definition 1.4.15. Let $(X, \mathcal{O}_X(1))$ be a projective variety over a field $\mathbf{k}$ and $\mathcal{F}$ a coherent sheaf on X. The *Hilbert polynomial* of $\mathcal{F}$ with respect to $\mathcal{O}_X(1)$ is the polynomial

$$P_{\mathcal{F}}(n) = \chi(X, \mathcal{F}(n)) = \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathcal{F}(n)).$$

Let $Z \subseteq X$ be a closed subscheme with structure sheaf $\mathcal{O}_Z$. The *Hilbert polynomial* of Z with respect to $\mathcal{O}_X(1)$ is defined as $P_Z(n) = P_{\mathcal{O}_Z}(n)$. To be revised.

Note that the Euler characteristic $\chi(X, \mathcal{F}(n))$ is additive on short exact sequences of coherent sheaves. Fix an hypersurface $H \subseteq X$ defined by a global section of $\mathcal{O}_X(1)$. Then we have $\mathcal{O}_H \cong \mathcal{O}_X/\mathcal{O}_X(-1)$.

Thus by the exact sequence

$$0 \to \mathcal{F}(n-1) \to \mathcal{F}(n) \to \mathcal{F}(n)|_H \to 0,$$

we have

$$P_{\mathcal{F}}(n) - P_{\mathcal{F}}(n-1) = P_{\mathcal{F}|_H}(n).$$

Inductively, ...

By Theorem 1.4.13, we have

$$P_{\mathcal{F}}(n) = h^0(X, \mathcal{F} \otimes \mathcal{O}_X(n)), \quad \text{for } n \gg 0.$$

Example 1.4.16. Let $Z \subseteq \mathbb{P}^r_{\mathbf{k}}$ be a hypersurface of degree d. Note that $h^0(\mathbb{P}^r_{\mathbf{k}}, \mathcal{O}_{\mathbb{P}^r}(n)) = C_r^{n+r}$. Then the Hilbert polynomial of Z with respect to $\mathcal{O}_{\mathbb{P}^r}(1)$ is

$$P_Z(n) = P_{\mathcal{O}}(n) - P_{\mathcal{O}(-d)}(n) = \binom{n+r}{r} - \binom{n+r-d}{r} = \frac{d}{(r-1)!} n^{r-1} + \text{lower degree terms.}$$

To be checked.

1.4.3 Linear systems

In this subsection, when work over a field $\mathbf{k}$, we give a more geometric interpretation of previous subsections using the language of linear systems.

Definition 1.4.17. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

Definition 1.4.18. Let $\mathcal{L}$ be a line bundle on a scheme X. To be continued.

1.5 Finite morphisms and fibrations

Theorem 1.5.1 (Zariski's Main Theorem)**.** Let $f : Y \to X$ be a quasi-finite and separated morphism of schemes. Then there exists a factorization

Theorem 1.5.2 (Stein factorization)**.** Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then there exists a factorization

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where g is a proper morphism with connected fibers and h is a finite morphism. Moreover, this factorization is unique up to isomorphism. To be checked.

1.5.1 Finite morphisms

Theorem 1.5.3. Let $f : Y \to X$ be a finite morphism of schemes. If $\mathcal{L}$ is an ample line bundle on X, then $f^*\mathcal{L}$ is an ample line bundle on Y. If and only if.

1.5.2 Fibrations

Definition 1.5.4. Let $f : Y \to X$ be a proper morphism of noetherian schemes. We say that f is a fibration if for every point $x \in X$, the fiber $f^{-1}(x)$ is a geometrically connected scheme.

Proposition 1.5.5. Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then f is a fibration if and only if the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is an isomorphism. In particular, if X is an algebraically closed field and f is a fibration, then the fibers $f^{-1}(x)$ are also algebraically closed in the function field $K(X)$. To be revised

Definition 1.5.6. Let $f : Y \dashrightarrow X$ be a rational map of noetherian schemes. We say that f is a fibration if there exists an open subset $U \subseteq Y$ such that the restriction $f|_U : U \to X$ is a fibration.

1.6 Differentials and duality

Let S be a base noetherian scheme, $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over S and all varieties are assumed to be defined over $\Bbbk$.

1.6.1 The sheaves of differentials

Definition 1.6.1. Let $f : X \to S$ be an S-scheme. The *sheaf of differentials* of X over S, denoted by $\Omega_{X/S}$, is the $\mathcal{O}_X$-module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$.

Proposition 1.6.2. Let X and T be S-schemes and $X_T := X \times_S T$ be the base change of X along $T \to S$. Let $p : X_T \to X$ be the projection morphism. Then there is a natural isomorphism of $\mathcal{O}_{X_T}$-modules

$$\Omega_{X_T/T} \cong p^*\Omega_{X/S}.$$

Proof. Given by algebras, see ref. To be continued. □

Proposition 1.6.3. Let X be an S-scheme and $U \subseteq X$ be an open subscheme. Then there is a natural isomorphism of $\mathcal{O}_U$-modules

$$\Omega_{U/S} \cong \Omega_{X/S}|_U.$$

Furthermore, let $\xi \in X$, then there is a natural isomorphism of $\mathcal{O}_{X,\xi}$-modules

$$\Omega_{X/S,\xi} \cong \Omega_{\mathcal{O}_{X,\xi}/\mathcal{O}_{S,f(\xi)}}.$$

To be checked.

Proof. To be continued. □

Proposition 1.6.4. Let X be a regular variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a locally free sheaf of rank n.

Proof. To be continued. □

Proposition 1.6.5. Let X be a normal variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a reflexive sheaf of rank n.

Proof. To be continued. □

Definition 1.6.6. Let X be a normal variety over $\Bbbk$. The *canonical divisor* K_X of X is defined to be the Weil divisor class $c_1(\Omega_{X/\Bbbk})$.

Theorem 1.6.7 (Euler sequence for projective bundle)**.** Let X be a normal variety over $\Bbbk$ and $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on X. Let $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. Then there is an exact sequence of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$-modules

$$0 \to \Omega_{\mathbb{P}_X(\mathcal{E})/X} \xrightarrow{\phi} \pi^*\mathcal{E}(-1) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \to 0.$$

Here $\pi^*\mathcal{E}(-1)$ is twisted by the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-1)$.

Proof.

Step 1. First assume that $X = \operatorname{Spec} A$ is affine and $\mathcal{E}$ is free. Under this assumption, find expressions for ϕ and ψ.

Fix a basis $T_0, \dots, T_r$ of the free A-module $\mathcal{E}(X)$. On the standard open subset $U_i = \{T_i \neq 0\} = \operatorname{Spec} B_i \subseteq \mathbb{P}_X(\mathcal{E})$, we have coordinates $t_{j,i} := T_j/T_i$ for $j \neq i$. The exact sequence becomes

$$0 \to \bigoplus_{k \neq i} B_i \mathrm{d}t_{k,i} \xrightarrow{\phi} \bigoplus_{k=0}^{r} B_i e_i \cdot T_k \xrightarrow{\psi} B_i \to 0.$$

Here e_i is the local generator of $\mathcal{O}_{\mathbb{P}_A(\mathcal{E})}(-1)$ on U_i, symbolically satisfying $e_i T_i = 1$.

Recall that on the overlap $U_{ij} = U_i \cap U_j$, the coordinates are related by

$$t_{i,j} e_i = e_j, \quad \mathrm{d}t_{k,i} = t_{j,i} \mathrm{d}t_{k,j} - t_{k,i} t_{j,i} \mathrm{d}t_{i,j}.$$

Here we set $t_{l,l} := 1$ for convenience. Symbolically, we have

$$\text{“ } \mathrm{d}t_{k,i} = \frac{T_i \mathrm{d}T_k - T_k \mathrm{d}T_i}{T_i^2} = e_i \mathrm{d}T_k - t_{k,i} e_i \mathrm{d}T_i \text{ ”.}$$

On the overlap U_{ij}, it transitions as

$$\begin{aligned}
\text{“}\mathrm{d}t_{k,i} &= t_{j,i}\mathrm{d}t_{k,j} - t_{k,i}t_{j,i}\mathrm{d}t_{i,j} \\
&= t_{j,i}e_j\mathrm{d}T_k - t_{j,i}t_{k,j}e_j\mathrm{d}T_j - t_{k,i}t_{j,i}\left(e_j\mathrm{d}T_i - t_{i,j}e_j\mathrm{d}T_j\right) \\
&= e_i\mathrm{d}T_k - t_{k,i}e_i\mathrm{d}T_i\text{”}.
\end{aligned}$$

To make sense of the above symbolic expressions, we define ϕ and ψ locally on each U_i by

$$\phi(\mathrm{d}t_{k,i}) = e_iT_k - t_{k,i}e_iT_i, \quad \psi(e_iT_k) = t_{k,i}.$$

Step 2. Verify that ϕ and ψ are well-defined and the sequence is exact.

By computations in Step 1, ϕ is well-defined on the overlaps U_{ij}. For ψ, on the overlap U_{ij}, we have

$$\psi(e_jT_k) = \psi(t_{i,j}e_iT_k) = t_{i,j}t_{k,i} = t_{k,j}.$$

Thus ψ is also well-defined. It is clear that $\psi \circ \phi = 0$. Consider the matrix representation of ϕ with respect to the bases $\{\mathrm{d}t_{k,i}\}_{k\neq i}$ and $\{e_iT_k\}_{k=0}^r$:

$$\begin{pmatrix}
1 & & & & & & \\
 & 1 & & & & & \\
 & & \ddots & & & & \\
 & & & 1 & & & \\
-t_{0,i} & -t_{1,i} & \cdots & -t_{i-1,i} & -t_{i+1,i} & \cdots & -t_{r,i} \\
 & & & & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{pmatrix}.$$

It has rank r, ϕ is injective and $\ker\psi = \sqrt{-1}\phi$. Thus the sequence is exact.

Step 3. General case: glue the local exact sequences on affine open subsets of X.

In the local case, choose a different basis $S_0, \dots, S_r$ of $\mathcal{E}(X)$ given by the transition matrix $g \in \mathrm{GL}_{r+1}(A)$. For simplicity, we just look at on the open subset $U = \{T_0 \neq 0, S_0 \neq 0\}$. Set B_U be the localization of $B = A[T_0, \dots, T_r]$ at the multiplicative set generated by T_0 and S_0. It is still a graded algebra.

Note that ϕ is formally given by differentials in $A[T_0, \dots, T_r]$ and then sending the symbol $\mathrm{d}T_i$ to T_i and $1/T_0$ to e_0. The differentials are intrinsic and linear over A, and the assignment of $1/T_0$ to e_0 is just a change of notation. Thus ϕ is independent of the choice of basis. For ψ, it is indeed given by multiplying $B_U(-1)$ by the linear part of B and then taking the degree 0 part. It is also independent of the choice of basis.

Therefore, after changing basis, ϕ and ψ remain the same. This allows us to glue the local exact sequences on each affine open subset of X to obtain a global exact sequence. □

Corollary 1.6.8. Let $\mathbf{k}$ be a field. We have

$$\omega_{\mathbb{P}^n_{\mathbf{k}}/\mathbf{k}} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbf{k}}}(-(n+1)) \quad \text{and} \quad K_{\mathbb{P}^n_{\mathbf{k}}} \sim -(n+1)H,$$

where H is a hyperplane in $\mathbb{P}^n_{\mathbf{k}}$.

1.6.2 Fundamental sequences

Theorem 1.6.9 (The first fundamental sequence of differentials)**.** Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Proof. To be completed. □

Proposition 1.6.10. Let $f : X \to Y$ be a surjective and generically finite morphism of normal varieties over $\Bbbk$. Then the first fundamental sequence of differentials is exact on the left.

Proof. To be completed. □

Corollary 1.6.11 (Ramification formula)**.** Let $f : X \to Y$ be a finite morphism of normal varieties. Then

$$K_X = f^*K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} \big(\operatorname{Mult}_D f^*\big(f(D)\big) - 1\big) D$$

is the ramification divisor of f. To be checked. definition of ramification divisor needs to be checked.

Proof. To be completed. □

Theorem 1.6.12 (The second fundamental sequence of differentials)**.** Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}|_Z \to \Omega_{Z/S} \to 0.$$

Suppose further that $Z \to X$ is a regular immersion. Then the above sequence is also exact on the left.

Proof. To be completed. □

Corollary 1.6.13 (Adjunction formula)**.** Let X be a normal variety and $Z \subseteq X$ be a prime Cartier divisor which is normal as variety. Then

$$K_Z = (K_X + Z)|_Z.$$

Proof. Since both X and Z are normal, they are smooth in codimension 1. Removing the singular locus of X and Z, we may assume that both X and Z are smooth varieties. This is valid since the canonical divisor is determined by the smooth locus.

Since Z is Cartier, it is a local complete intersection in X. By Theorem 1.6.12, we have the exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/\Bbbk}|_Z \to \Omega_{Z/\Bbbk} \to 0.$$

Note that Z is of codimension 1 in X, so $\mathcal{I}_Z \cong \mathcal{O}_X(-Z)$ and thus $\mathcal{I}_Z/\mathcal{I}_Z^2 \cong \mathcal{O}_X(-Z)|_Z$. Taking c_1, we obtain

$$c_1(\Omega_X)|_Z = c_1(\Omega_Z) + c_1(\mathcal{O}_X(-Z))|_Z.$$

That is,
$$K_X|_Z = K_Z - Z|_Z.$$
Rearranging gives the desired result. To be revised. restriction of Weil divisors needs to be clarified. □

1.6.3 Serre duality

Definition 1.6.14 (Dualizing sheaf)**.** Let X be a proper scheme of dimension n over $\Bbbk$. A *dualizing sheaf* on X is a coherent sheaf ω_X° together with a trace map $\mathrm{tr}_X : H^n(X, \omega_X^\circ) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$ on X, the natural pairing
$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}_X} \Bbbk$$
induces an isomorphism
$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

Theorem 1.6.15. Let X be a projective scheme of dimension n over $\Bbbk$. Then there exists a dualizing sheaf ω_X° on X up to isomorphism. Moreover, if X is smooth, $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\Bbbk}$.

Proof. To be completed. □

Theorem 1.6.16 (Serre duality)**.** Let X be a projective, Cohen-Macaulay variety of dimension n over $\Bbbk$ with dualizing sheaf ω_X°. Then for every coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphism
$$\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. To be completed. □

When $\mathcal{F}$ is locally free, we have $\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^i(X, \omega_X^\circ \otimes \mathcal{F}^\vee)$.

Corollary 1.6.17. Let X be a projective, normal variety of dimension n over $\Bbbk$. Then for every integer m and $0 \le i \le n$, there is a natural isomorphism To be completed.

1.6.4 Logarithm version

1.7 Flat, smooth and étale morphisms

1.7.1 Flat families

Definition 1.7.1. Let $f : X \to Y$ be a morphism of schemes. For a point $\xi \in X$, we say that f is *flat at* ξ if the local ring $\mathcal{O}_{X,\xi}$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that f is *flat* if it is flat at every point $\xi \in X$.

The notation and terminology of flatness can be extended to sheaves of modules over schemes.

Definition 1.7.2. Let X be Y-scheme via a morphism $f : X \to Y$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *flat over Y at $\xi \in X$* if the stalk $\mathcal{F}_\xi$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that $\mathcal{F}$ is *flat over Y* if it is flat over Y at every point $\xi \in X$.

Proposition 1.7.3. We have the following fundamental properties of flat morphisms:

(a) open immersions are flat;

(b) the composition of flat morphisms is flat;

(c) flatness is preserved under base change;

(d) a coherent sheaf $\mathcal{F}$ on a noetherian scheme X is flat over X iff it is locally free.

Proof. To be added. □

Proposition 1.7.4. Let X be a regular integral scheme of dimension 1 and $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is flat over X iff it is torsion-free. To be checked.

Proposition 1.7.5. Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field $\mathbf{k}$. Then for every point $\xi \in X$, we have

$$\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}.$$

To be checked.

Theorem 1.7.6 (Miracle flatness)**.** Let $f : X \to Y$ be a morphism between noetherian schemes. Suppose that X is Cohen–Macaulay and that Y is regular. Then f is flat at $\xi \in X$ iff $\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}$. To be checked.

Theorem 1.7.7. Let X be a projective scheme with relatively ample line bundle $\mathcal{O}_X(1)$ over a noetherian scheme T. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\mathcal{F}$ is flat over T. Then the Hilbert polynomials $P_{X_t,\mathcal{F}_t}(m)$ are independent of $t \in T$. Conversely, suppose that T is reduced, the constant Hilbert polynomial $P_{X_t,\mathcal{F}_t}(m)$ implies that $\mathcal{F}$ is flat over T. To be checked.

Theorem 1.7.8. Let S be a integral noetherian scheme, $f : X \to S$ be a morphism of finite type and $\mathcal{F}$ be a coherent sheaf on X. Then there exists a non-empty open subset $U \subseteq S$ such that the restriction $\mathcal{F}|_{f^{-1}(U)}$ is flat over U.

Proof. To be added. □

To be added: deformation, algebraic families...

1.7.2 Base change and semicontinuity

Theorem 1.7.9 (Grauert's theorem)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the sheaf $R^i f_* \mathcal{F}$ is coherent on Y, and there exists an open subset $U \subseteq Y$ such that for every point $y \in U$, the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism. To be checked.

Theorem 1.7.10 (Cohomology and base change)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. For each integer $i \geq 0$, the following are equivalent:

(a) the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism for all points $y \in Y$;

(b) the sheaf $R^i f_* \mathcal{F}$ is locally free on Y.

To be checked.

Theorem 1.7.11 (Semicontinuity of cohomology)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the function

$$h^i : Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is upper semicontinuous on Y.

To be checked.

1.7.3 Smooth morphisms

Definition 1.7.12. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. For $\xi \in X$ with image $\zeta = f(\xi) \in Y$, set $\overline{\zeta} : \operatorname{Spec} \overline{\kappa(\zeta)} \to Y$ to be the geometric point over ζ and $X_{\overline{\zeta}}$ be the geometric fiber over ζ. We say that f is *smooth at* ξ if f is flat at ξ and the geometric fiber $X_{\overline{\zeta}}$ is regular over $\overline{\kappa(\zeta)}$ at every point lying over ξ. We say that f is *smooth* if it is smooth at every point $\xi \in X$.

To be checked.

1.7.4 Étale morphisms

Definition 1.7.13. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. We say that f is *étale at* ξ if f is smooth and finite at ξ. We say that f is *étale* if it is étale at every point $\xi \in X$.

To be checked.

Chapter 2

Surfaces

2.1 The first properties of surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

2.1.1 Basic concepts

Definition 2.1.1. A *surface* is a two-dimensional integral scheme of finite type over an algebraically closed field $\Bbbk$. A *projective surface* is a surface that is projective over $\Bbbk$. A *smooth surface* is a surface that is smooth over $\Bbbk$. To be checked.

2.1.2 Riemann-Roch Theorem for surfaces

2.1.3 Hodge Index Theorem

2.2 Birational geometry on surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

2.2.1 Birational morphisms on surfaces

Let X be a smooth projective surface, $0 \in X(\Bbbk)$ and $\pi : \tilde{X} = \mathrm{Bl}_0 X \to X$ the blow-up of X at 0. Denote by E the exceptional divisor of π.

Proposition 2.2.1. We have $E^2 = -1$.

Proof. To be continued $\square$

Proposition 2.2.2. We have $K_{\tilde{X}} = \pi^* K_X + E$.

Proof. We have the exact sequence

$$\Omega_{\tilde{X}} \to \pi^*\Omega_X \to \Omega_{\tilde{X}/X} \to 0.$$

Since both $\tilde{X}$ and X are smooth, $\Omega_{\tilde{X}}$ and Ω_X are locally free sheaves of rank 2. The kernel of the first map is of rank 0 and torsion, thus it is zero. Therefore, we have the short exact sequence

$$0 \to \Omega_{\tilde{X}} \to \pi^*\Omega_X \to \Omega_{\tilde{X}/X} \to 0.$$

By taking c_1, we only need to show that $c_1(\Omega_{\tilde{X}/X}) = E$.

For $\eta \in \tilde{X}$ of codimension 1, if $\eta \notin E$, then $(\Omega_{\tilde{X}/X})_\eta = \Omega_{\mathcal{O}_{\tilde{X},\eta}/\mathcal{O}_{X,\pi(\eta)}} = 0$. Hence we only need to consider the case $\overline{\{\eta\}} = E$. To be continued ☐

Corollary 2.2.3. We have $K_{\tilde{X}}^2 = K_X^2 - 1$.

Proof. By Proposition 2.2.2, we have

$$K_{\tilde{X}}^2 = (\pi^*K_X + E)^2 = (\pi^*K_X)^2 + 2\pi^*K_X \cdot E + E^2 = K_X^2 + 0 - 1 = K_X^2 - 1.$$

☐

Theorem 2.2.4. Let $f : X \to Y$ be a birational morphism between two smooth projective surfaces. Then f can be decomposed as a finite sequence of blow-ups at points.

Proof. To be continued ☐

2.2.2 Castelnuovo's Theorem

Definition 2.2.5. A (-1)-*curve* on a smooth projective surface X is an irreducible curve $C \subseteq X$ such that $C \cong \mathbb{P}^1$ and $C^2 = -1$.

Remark 2.2.6. Let C be a (-1)-curve on a smooth projective surface X. Then its numerical class $[C] \in \mathrm{N}_1(X)$ spans an extremal ray of $\mathrm{Psef}_1(X)$ such that $K_X \cdot C < 0$. To be revised.

Theorem 2.2.7 (Castelnuovo's contractibility criterion)**.** Let X be a smooth projective surface and $C \subseteq X$ an irreducible curve. Then there exists a birational morphism $f : X \to Y$ contracting C to a smooth point if and only if C is a (-1)-curve.

Proof. To be continued ☐

Definition 2.2.8. A *minimal surface* is a smooth projective surface that does not contain any (-1)-curves. To be checked.

2.2.3 Resolution of singularities on surface

Definition 2.2.9. A *resolution of singularities* of a projective surface X is a smooth projective surface $\widetilde{X}$ together with a birational and proper morphism $\pi : \widetilde{X} \to X$ such that π is an isomorphism over the smooth locus of X. To be checked.

Theorem 2.2.10 (Resolution of singularities on surfaces)**.** Let X be a projective surface over an algebraically closed field $\Bbbk$. Then X admits a resolution of singularities. To be checked.

Definition 2.2.11. Let X be a projective surface. A *minimal resolution* of X is a resolution of singularities $\pi : \widetilde{X} \to X$ such that for any other resolution of singularities $\pi' : \widetilde{X}' \to X$, there exists a morphism $f : \widetilde{X}' \to \widetilde{X}$ such that π' factors as $\pi' = \pi \circ f$.

Proposition 2.2.12. Let X be a projective surface. Then X admits a unique minimal resolution of singularities.

Proof. To be continued □

2.3 Coarse classification of surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. We want to classify X up to birational equivalence. Let K_X be the canonical divisor of X.

Theorem 2.3.1. Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. Suppose that the Kodaira dimension $\kappa(X) \geq 0$. Then the linear system $|12K_X|$ is base point free. To be checked.

2.3.1 Classification

Theorem 2.3.2 (Enriques-Kodaira classification)**.** Let X be a smooth projective surface over $\Bbbk$. Then X is birational to a unique minimal model X', unless X is birational to a ruled surface. Moreover, the minimal model X' falls into one of the following classes:

(a) $\kappa(X') = -\infty$: $X' \cong \mathbb{P}^2$ or X' is a ruled surface;

(b) $\kappa(X') = 0$: X' is a K3 surface, an abelian surface or their quotients;

(c) $\kappa(X') = 1$: X' is an elliptic surface;

(d) $\kappa(X') = 2$: X' is a surface of general type.

To be checked.

2.4 Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$. This section is mainly based on [Har77, Chapter V.2].

2.4.1 Minimal Section and Classification

Definition 2.4.1 (Ruled surface)**.** A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all geometric fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 2.4.2. There exists a section of π.

Proof. To be continued... □

Proposition 2.4.3. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Proof. Let $\sigma : C \to X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π, $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in C$. By Miracle Flatness (??), f is flat. By Grauert's Theorem (Theorem 1.7.9), $\mathcal{E}$ is a vector bundle of rank 2 on C and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in C$.

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every $x \in X$, we have

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \to \mathcal{L}$ is surjective.

By ??, we have a morphism $\varphi : X \to \mathbb{P}_C(\mathcal{E})$ over C such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Since $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, $\varphi|_{X_t} : X_t \to \mathbb{P}_C(\mathcal{E})_t$ is an isomorphism for any $t \in C$. Hence φ is bijection on the underlying sets. Here is a serious gap. Why fiberwise isomorphism implies isomorphism? □

Lemma 2.4.4. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*. In particular, if $\mathcal{E}$ is normalized, then $e = -\deg c_1(\mathcal{E})$ is an invariant of the ruled surface X.

Proof. We can suppose that $\mathcal{E}$ is globally generated since we can always twist $\mathcal{E}$ by a sufficiently ample line bundle on C. Then for all line bundle $\mathcal{L}$ of degree sufficiently large, $\mathcal{L}$ is very ample and hence $H^0(C, \mathcal{E} \otimes \mathcal{L}) \neq 0$. By Lemma 2.4.2 and ??, $\mathcal{E}$ is an extension of line bundles. Then for all line bundle $\mathcal{L}$ of degree sufficiently negative, $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ since line bundles of negative degree have no global sections. Hence we can find a line bundle $\mathcal{M}$ on C of lowest degree such that $H^0(C, \mathcal{E} \otimes \mathcal{M}) \neq 0$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{M}$, we are done. □

Remark 2.4.5. The invariant e is unique but the normalization of $\mathcal{E}$ is not unique. For example, if $\mathcal{E}$ is normalized, then so is $\mathcal{E} \otimes \mathcal{L}$ for any line bundle $\mathcal{L}$ on C of degree 0. To be continued...

Suppose that $X \cong \mathbb{P}_C(\mathcal{E})$ where $\mathcal{E}$ is a normalized vector bundle of rank 2 on C. Since $H^0(C, \mathcal{E}) \neq 0$, choosing a non-zero section s, we get an exact sequence

$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{E} \to \mathcal{E}/\mathcal{O}_C \to 0.$$

We claim that $\mathcal{E}/\mathcal{O}_C$ is a line bundle on C. Since C is a curve, we only need to check that $\mathcal{E}/\mathcal{O}_C$ is torsion-free.

To be continued...

Definition 2.4.6. A section C_0 of π is called a *minimal section* if to be continued...

Lemma 2.4.7. Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface over a smooth curve C of genus g with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on C with $\deg \mathcal{L} = -e$.

(b) If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.

Proof. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable, we can assume that $H^0(C, \mathcal{L}_1) \neq 0$. If $\deg \mathcal{L}_1 > 0$, then $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$, contradicting the normalization of $\mathcal{E}$. Similarly $\deg \mathcal{L}_2 \leq 0$. Then $\mathcal{L}_1 \cong \mathcal{O}_C$. And hence $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$.

If $\mathcal{E}$ is indecomposable, we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

which is a non-trivial extension, with $\mathcal{L}$ a line bundle on C of degree $-e$. Hence by ??, we have $0 \neq \mathrm{Ext}^1_C(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$. By Serre duality, we have $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$. Hence $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$.

On the other hand, let $\mathcal{M}$ be a line bundle on C of degree -1. Twist the above exact sequence by $\mathcal{M}$ and take global sections, we have an equation

$$h^0(\mathcal{M}) - h^0(\mathcal{E} \otimes \mathcal{M}) + h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{M}) + h^1(\mathcal{E} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = 0.$$

Since $\deg \mathcal{M} < 0$ and $\mathcal{E}$ is normalized, we have $h^0(\mathcal{M}) = h^0(\mathcal{E} \otimes \mathcal{M}) = 0$. By Riemann-Roch, we have $h^1(\mathcal{M}) = g$ and $h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = -e - 1 + 1 - g$. Hence

$$h^1(\mathcal{E} \otimes \mathcal{M}) = e + 2g \geq 0.$$

This gives $e \geq -2g$. □

Theorem 2.4.8. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Proof. This is a direct consequence of Lemma 2.4.7. □

Example 2.4.9. Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$.

Let C be covered by two standard affine charts U_0, U_1 with coordinate u on U_0 and v on U_1 such that $u = 1/v$ on $U_0 \cap U_1$. On U_i, let $\mathcal{O}(-e)|_{U_i}$ be generated by s_i for $i = 0, 1$. We have $s_0 = u^e s_1$ on $U_0 \cap U_1$.

On $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$, let $[x_0 : x_1]$ and $[y_0 : y_1]$ be the homogeneous coordinates of $\mathbb{P}^1$ on X_0 and X_1 respectively. Then the transition function on $X_0 \cap X_1$ is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

Remark 2.4.10. The surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ is also called the *Hirzebruch surface*.

Theorem 2.4.11. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and for each e there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on E with $\deg \mathcal{L} = -e$.

Proof. Only the indecomposable case needs a proof. By Lemma 2.4.7, we have $-2 \leq e \leq 0$ and a non-trivial extension

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle on E of degree $-e$.

Case 1. $e = 0$.

In this case, $\mathcal{L}$ is of degree 0 and $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \neq 0$. Hence $\mathcal{L} \cong \mathcal{O}_E$. To be continued...

Case 2. $e = -1$.

In this case, $\mathcal{L}$ is of degree 1 and $H^1(E, \mathcal{L}) \cong H^0(E, \mathcal{L}^{-1}) = 0$. By Riemann-Roch, we have $h^0(E, \mathcal{L}) = 1$.

Case 3. $e = -2$.

To be continued...

□

Example 2.4.12. To be continued...

2.4.2 The Néron-Severi Group of Ruled Surfaces

Proposition 2.4.13. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and F a fiber of π. Then $\mathrm{Pic}(X) \cong \mathbb{Z}[C_0] \oplus \pi^* \mathrm{Pic}(C)$.

Proof. Let D be any divisor on X with $D.F = a \in \mathbb{Z}$. Then $D - aC_0$ is numerically trivial on the fibers of π. Let $\mathcal{L} = \mathcal{O}_X(D - aC_0)$. Then $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$ for any $t \in C$. By Grauert's Theorem (Theorem 1.7.9), $\pi_* \mathcal{L}$ is a line bundle on C and the natural map $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ is an isomorphism. □

Proposition 2.4.14. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let F be a fiber of π. Then $K_X \sim -2C_0 + \pi^*(K_C - c_1(\mathcal{E}))$. Numerically, we have $K_X \equiv -2C_0 + (2g - 2 - e)F$ where e is the invariant of X. Check this carefully.

Proof. To be continued. □

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

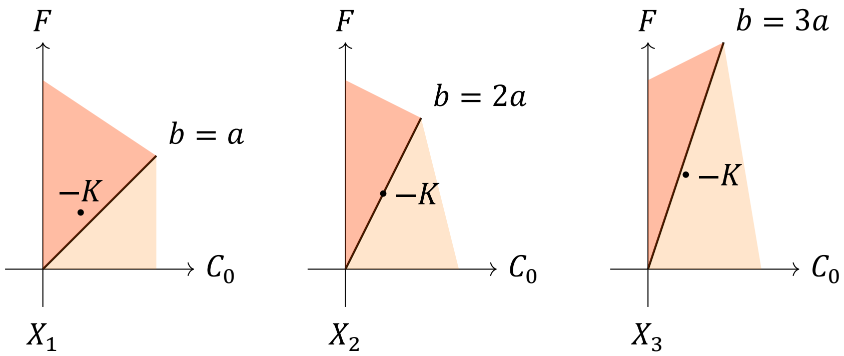
Theorem 2.4.15. Let $\pi : X \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with invariant e. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \sim aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff a, b \geq 0$;

(b) D is ample $\iff$ D is very ample $\iff a > 0$ and $b > ae$.

Proof. To be continued... □

Example 2.4.16. Here we draw the Néron-Severi group of the rational ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 1, 2, 3$.



We have $-K_{X_e} \equiv 2C_0 + (2+e)F$. For $e = 1$, $-K$ is ample and hence X_1 is a del Pezzo surface. For $e = 2$, $-K$ is nef and big but not ample. For $e \geq 3$, $-K$ is big but not nef.

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

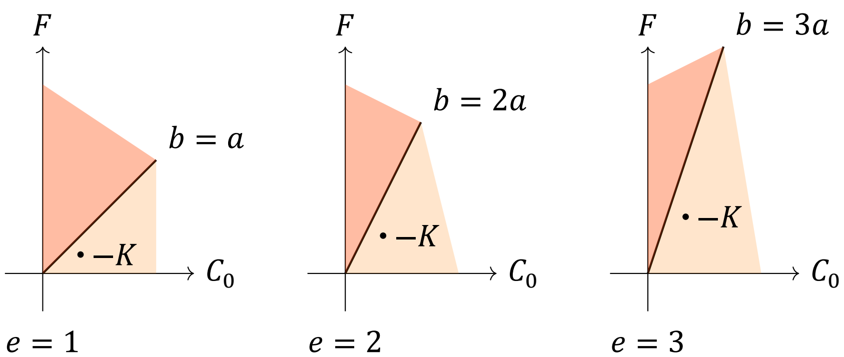
Theorem 2.4.17. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is decomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff a \geq 0$ and $b \geq ae$;

(b) D is ample $\iff$ D is very ample $\iff a > 0$ and $b > ae$.

Proof. To be continued... □

Example 2.4.18. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with decomposable normalized $\mathcal{E}$ for $e = 1, 2, 3$.



In this case, $-K \equiv 2C_0 + eF$ is always big but not nef.

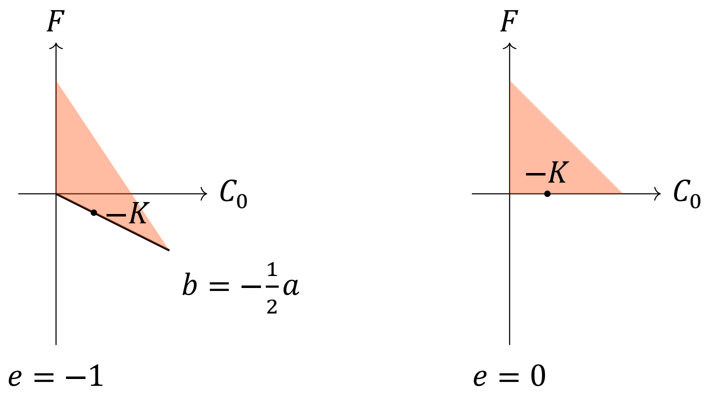
Theorem 2.4.19. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is indecomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq \frac{1}{2}ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > \frac{1}{2}ae$.

Proof. To be continued... □

Example 2.4.20. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with indecomposable normalized $\mathcal{E}$ for $e = -1, 0$.



In this case, $-K \equiv 2C_0 + eF$ is always nef but not big.

Proposition 2.4.21. Let $\pi : X \to C$ be a ruled surface over a smooth curve C. Then every nef divisor on X is semi-ample. Check this carefully.

2.5 K3 surface

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

2.5.1 The first properties

Definition 2.5.1. A *K3 surface* is a smooth, projective surface X with trivial canonical bundle $K_X \cong \mathcal{O}_X$ and irregularity $q(X) = h^1(X, \mathcal{O}_X) = 0$.

Example 2.5.2. A smooth quartic surface $X \subseteq \mathbb{P}^3$ is a K3 surface. Indeed, by the adjunction formula, we have

$$K_X = (K_{\mathbb{P}^3} + X)|_X = (-4H + 4H)|_X = 0,$$

where H is a hyperplane in $\mathbb{P}^3$. Moreover, by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0,$$

we have long exact sequence in cohomology

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(X, \mathcal{O}_X) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \to \cdots.$$

Since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$ and $H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$, we get $H^1(X, \mathcal{O}_X) = 0$.

2.5.2 Hodge Structure and Moduli of K3 surfaces

2.5.3 Neron-Severi group of K3 surfaces

2.6 Elliptic surfaces

2.6.1 The first properties

Definition 2.6.1. An *elliptic surface* is a smooth projective surface S together with a surjective morphism $\pi : S \to C$ to a smooth projective curve C such that the generic fiber of π is a smooth curve of genus 1, and π has a section $s : C \to S$. To be continued...

2.6.2 Classification of singular fibers

2.6.3 Mordell-Weil group and Neron-Severi group

2.7 Some Singular Surfaces

In this section, fix an algebraically closed field $\mathbb{k}$. Everything is over $\mathbb{k}$ unless otherwise specified.

2.7.1 Projective cone over smooth projective curve

Let $C \subset \mathbb{P}^n$ be a smooth projective curve. The *projective cone* over C is the projective variety $X \subset \mathbb{P}^{n+1}$ defined by the same homogeneous equations as C. The variety X is singular at the vertex of the cone, which corresponds to the point $[0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}$.

2.7.2 Du Val singularites

Du Val singularities (also known as rational double points, or ADE singularities) are a class of surface singularities that arise in algebraic geometry and complex surface theory. They are characterized by their resolution properties and can be classified according to the ADE classification of simply laced Dynkin diagrams.

A Du Val singularity can be locally described by one of the following equations in $\mathbb{C}^3$:

- A_n singularity: $x^2 + y^2 + z^{n+1} = 0$ for $n \geq 1$
- D_n singularity: $x^2 + y^{n-1} + yz^2 = 0$ for $n \geq 4$
- E_6 singularity: $x^2 + y^3 + z^4 = 0$
- E_7 singularity: $x^2 + y^3 + yz^3 = 0$

- E_8 singularity: $x^2 + y^3 + z^5 = 0$

These singularities are important in the study of algebraic surfaces, particularly in the context of minimal models and the classification of surfaces. They also appear in various areas of mathematics and theoretical physics, including string theory and mirror symmetry.

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2.7.3 Quotient singularities

Consider μ_n, the group of n-th roots of unity, acting on $\mathbb{A}^2$ by

$$\zeta \cdot (x, y) = (\zeta x, \zeta^m y)$$

for a fixed integer m with $\gcd(m, n) = 1$.

$(a_1, ..., a_n)/r$-singularity is the singularity obtained by taking the quotient of $\mathbb{A}^n$ by the action of μ_r defined by

$$\zeta \cdot (x_1, ..., x_n) = (\zeta^{a_1} x_1, ..., \zeta^{a_n} x_n)$$

where ζ is a primitive r-th root of unity.

A_n singularity is the quotient singularity of type $(1, -1)/(n + 1)$.

Its minimal resolution has exceptional locus consisting of a chain of n smooth rational curves, each with self-intersection -2. Looks like:

$$\bullet \text{ ——— } \bullet \text{ ——— } \bullet \text{ ——— } \cdots \text{ ——— } \bullet$$

Du Val singularities can be got by deforming A_n singularities.(general fiber A_n, special fiber Du Val).

Chapter 3

Birational Geometry

3.1 Technical Preparation

3.1.1 Resolution of singularities

Theorem 3.1.1. Let X be a normal variety over an algebraically closed field $\Bbbk$ of characteristic zero. Then there exists a log resolution of singularities $f : Y \to X$ such that Y is smooth and the exceptional divisor E is a simple normal crossing divisor.

3.1.2 Negativity Lemma

Theorem 3.1.2. Let $f : Y \to X$ be a proper birational morphism between normal varieties. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y such that $-D$ is f-nef. Then D is effective if and only if f_*D is.

Proof. To be completed. □

3.1.3 General adjunction formula

Theorem 3.1.3 (Adjuction formula)**.** Let X be a normal variety and S be a reduced divisor on X. Need to check the statement.

Proof. To be completed. □

3.1.4 Exceptional divisors

Proposition 3.1.4. Let $f : Y \to X$ be a proper birational morphism between normal varieties. Let E be an effective f-exceptional divisor on Y. Then we have $f_*\mathcal{O}_Y(E) \cong \mathcal{O}_X$.

3.2 Kodaira Vanishing Theorem

3.2.1 Preliminary

Theorem 3.2.1 (Serre Duality)**.** Let X be a Cohen-Macaulay projective variety of dimension n over $\mathbf{k}$ and D a divisor on X. Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \ldots, n.$$

Theorem 3.2.2 (Log Resolution of Singularities)**.** Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there is a smooth variety (or analytic space) Y and a projective morphism $f : Y \to X$ such that

(a) f is an isomorphism over $X - (\mathrm{Sing}(X) \cup \mathrm{Supp}\, Z)$,

(b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(c) $\mathrm{Exc}(f) \cup D$ is an snc divisor.

Theorem 3.2.3 (Lefschetz Hyperplane Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 3.2.4 (Hodge Decomposition)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Then for any k, there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 3.2.3 and Theorem 3.2.4, we have the following lemma.

Lemma 3.2.5. Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map $r_k : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \to H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \to H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n - 1$ and an injection for $p = n - 1$.

Theorem 3.2.6 (Leray spectral sequence)**.** Let $f : Y \to X$ be a morphism of varieties and $\mathcal{F}$ a coherent sheaf on Y. Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{F}) \implies H^{p+q}(Y, \mathcal{F}).$$

3.2.2 Kodaira Vanishing Theorem

Lemma 3.2.7. Let X be a smooth projective variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on X. Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \to X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D. It induces a homomorphism $\mathcal{L}^{-m} \to \mathcal{O}_X$. Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} \right) \Big/ (\mathcal{L}^{-m} \to \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then $\mathcal{A}$ is a finite $\mathcal{O}_X$-algebra. Let $Y \coloneqq \operatorname{Spec}_X \mathcal{A}$. Then Y is a finite $\mathcal{O}_X$-scheme and the natural morphism $f : Y \to X$ is finite and surjective.

For every $x \in X$, let $\mathcal{L}$ locally generated by t near x. Then $\mathcal{O}_Y$ locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y. Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. □

Remark 3.2.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D_i' = f^*(D_i)$. Then $D' + \sum_{i=1}^k D_i'$ is snc on Y by considering the local regular system of parameters.

Lemma 3.2.9. Let $f : Y \to X$ be a finite surjective morphism of projective varieties and $\mathcal{L}$ a line bundle on X. Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits by the trace map $(1/n)\operatorname{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. □

Theorem 3.2.10 (Kodaira Vanishing Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and A an ample divisor on X. Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 3.2.7 and 3.2.9, after taking a multiple of A, we can assume that A is effective.

Then we have an exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \to H^{i-1}(X, \mathcal{O}_A) \to H^i(X, \mathcal{O}_X(-A)) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 3.2.5 and Serre duality (Theorem 3.2.1). □

3.2.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

Theorem 3.2.11 (Kawamata-Viehweg Vanishing Theorem I)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{R}$-divisor on X. Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Theorem 3.2.12 (Kawamata-Viehweg Vanishing Theorem II)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{Q}$-divisor on X. Suppose that $\lceil D \rceil - D$ has snc support. Then

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall i > 0.$$

Theorem 3.2.13 (Kawamata-Viehweg Vanishing Theorem III)**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Let D be a nef $\mathbb{Q}$-divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

If we replace the assumption ”nef and big” of D by ”ample” in II and III, we denote them as “II(ample)” and “III(ample)” respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \implies \text{II(ample)} \implies \text{III(ample)} \implies \text{I} \implies \text{II} \implies \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

Proof of II (Theorem 3.2.12). Set $M := \lceil D \rceil$. Let

$$B := \sum_{i=1}^{k} b_i B_i := \lceil D \rceil - D = M - A, \quad b_i \in (0,1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k. When $k = 0$, the conclusion follows from Theorem 3.2.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 3.2.10.)) Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 3.2.15 and 3.2.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 3.2.7 on B_k, we can find a finite surjective morphism $f : X' \to X$ such that $f^*B_k = cB'_k, B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^k B'_i$ is an snc divisor on X'. Let $B' = \sum_{i=1}^{k-1} B'_i, A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$ vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^*\mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 3.2.9. □

Proof of III (Theorem 3.2.13). Let $f : \tilde{X} \to X$ be a resolution such that $\operatorname{Supp} f^*B \cup \operatorname{Exc} f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X},\tilde{B})} + f^*D,$$

where $\tilde{B} \in (0,1)$ has snc support and E is an effective exceptional divisor.

By Lemma 3.2.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X},\tilde{B})} + f^*D) = H^i(X, f_*\mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 3.2.12 in either case relative to the assumption of D. □

Proof of I (Theorem 3.2.11). By Lemma 3.2.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor A. Then the theorem comes down to Theorem 3.2.13. □

Lemma 3.2.14. Let $f : Y \to X$ be a birational morphism of projective varieties with Y smooth and X has only rational singularities. Let E be an effective exceptional divisor on Y and D a divisor on X. Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

Proof. I am unable to proof this lemma. □

Lemma 3.2.15. Let X be a projective variety, $\mathcal{L}$ a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \to X$ and a line bundle $\mathcal{L}'$ on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X, then we can take f such that f^*D is an snc divisor on Y.

Proof. We can assume that $\mathcal{L}$ is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "g"] \\ X \arrow[r, "\varphi_{\mathcal{L}}"] & \mathbb{P}^N \end{tikzcd},$$

where $g : [x_0 : \ldots : x_N] \mapsto [x_0^m : \ldots : x_N^m]$. The morphism f is finite and surjective since so is g. Let $\mathcal{L}' := \psi^*\mathcal{L}$.

For smoothness, we can compose g with a general automorphism of $\mathbb{P}^N$. Then the conclusion follows from [Har77, Chapter III, Theorem 10.8]. □

Lemma 3.2.16 (ref. [KM98, Theorem 5.10, 5.22]). Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Lemma 3.2.17. Let X be a projective variety of dimension n and D a nef and big divisor on X. Then there exists an effective divisor B such that for every k, there is an ample divisor A_k such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k}B.$$

Proof. By definition of big divisor, there exists an ample divisor A_1 and effective divisor B such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k}B.$$

Since A is ample and D is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. □

3.3 Cone Theorem

3.3.1 Preliminary

Theorem 3.3.1 (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27])**.** Let X be a projective variety and $\mathcal{L}$ an semiample line bundle on X. Then there exists a fibration $\varphi : X \to Y$ of projective varieties such that for any $m \gg 0$ with $\mathcal{L}^m$ base point free, we have that the morphism $\varphi_{\mathcal{L}^m}$ induced by $\mathcal{L}^m$ is isomorphic to φ. Such a fibration is called the *Iitaka fibration* associated to $\mathcal{L}$.

Theorem 3.3.2 (Rigidity Lemma, ref. [Deb01, Lemma 1.15])**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Theorem 3.3.3. Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

Proposition 3.3.4. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

Schetch of proof. Take a resolution $f : Y \to X$, then f^*H is nef on Y and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor H_Y on Y closed enough to f^*H such that $K_Y \cdot H_Y^{n-1} < 0$. By [MM86, Theorem 5] and take limit for H_Y. □

Lemma 3.3.5 (ref. [Kaw91, Lemma])**.** Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Let E be an irreducible component of dimension d of the exceptional locus of f and $\nu : E^\nu \to X$ the normalization of E. Suppose that $f(E)$ is a point. Then for any ample divisor

H on X, we have
$$K_{E^\nu} \cdot \nu^* H^{d-1} \le K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

3.3.2 Non-vanishing Theorem

Theorem 3.3.6 (Non-vanishing Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have
$$H^0(X, mD) \neq 0.$$

Proof. To be completed. □

3.3.3 Base Point Free Theorem

Theorem 3.3.7 (Base Point Free Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, mD is base point free.

Proof. To be completed. □

Remark 3.3.8. In general, we say that a Cartier divisor D is *semiample* if there exists a positive integer m such that mD is base point free. The statement in Base Point Free Theorem (Theorem 3.3.7) is strictly stronger than the semiample condition. For example, let $\mathcal{L}$ be a torsion line bundle, then $\mathcal{L}$ is semiample, but there exists no positive integer M such that $m\mathcal{L}$ is base point free for all $m > M$.

3.3.4 Rationality Theorem

Lemma 3.3.9 (ref. [KM98, Theorem 1.36])**.** Let X be a proper variety of dimension n and $D_1, \ldots, D_m$ Cartier divisors on X. Then the Euler characteristic $\chi(n_1 D_1, \ldots, n_m D_m)$ is a polynomial in $(n_1, \cdots, n_m)$ of degree at most n.

Theorem 3.3.10 (Rationality Theorem)**.** Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let
$$t \coloneqq \inf\{s \ge 0 : K_{(X,B)} + sH \text{ is nef}\}$$
be the nef threshold of (X, B) with respect to H. Then $t = v/u \in \mathbb{Q}$ and
$$0 \le v \le a(X) \cdot (\dim X + 1).$$

Proof. For every $r \in \mathbb{R}_{>0}$, let
$$v(r) \coloneqq \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
We need to show that $v(t) \le a(\dim X + 1)$. For every $(p, q) \in \mathbb{Z}^2_{>0}$, set $D(p, q) \coloneqq paK_{(X,B)} + qH$. If $(p, q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, then we have $D(p, q)$ is not nef and $D(p, q) - K_{(X,B)}$ is ample.

Step 1. We show that a polynomial $P(x,y) \neq 0 \in \mathbb{Q}[x,y]$ of degree at most n is not identically zero on the set

$$\{(p,q) \in \mathbb{Z}^2 : p,q > M, 0 < atp - q < t\varepsilon\}, \quad \forall M > 0,$$

if $v(t)\varepsilon > a(n+1)$.

If $v(t) = \infty$, for any n, we show that we can find infinitely many lines L such that $\#L \cap \Lambda \geq n+1$. If so, Λ is Zariski dense in $\mathbb{Q}^2$. Since $1/at \in \mathbb{R} \setminus \mathbb{Q}$, there exist $p_0, q_0 > M$ such that

$$0 < \frac{p_0}{q_0} - \frac{1}{at} < \frac{\varepsilon}{(n+1)a} \cdot \frac{1}{q_0}, \text{ i.e. } 0 < atp_0 - q_0 < \frac{\varepsilon t}{n+1}.$$

Then $(ip_0, iq_0) \in \Lambda \cap \{p_0 y = q_0 x\}$ for $i = 1, \cdots, n+1$. Since M is arbitrary, there are infinitely many such lines L.

Suppose $v(t) = v < \infty$ and $t = v/u$. Then the inequality is equivalent to $0 < aup - vq < \varepsilon v$. Note that $\gcd(au, v)|a$, then $aup - vq = ai$ has integer solutions for $i = 1, \cdots, n+1$. Since $v(t)\varepsilon > a(n+1)$, there are at least $n+1$ lines which intersect Λ in infinitely many points. This enforce any polynomial which vanishes on Λ has degree at least $n+1$.

Step 2. There exists an index set $\Lambda \subset \mathbb{Z}^2$ such that Λ contains all sufficiently large (p,q) with $0 \leq atp - q \leq t$ and

$$Z := \mathrm{Bs}\,|D(p,q)| = \mathrm{Bs}\,|D(p',q')| \neq \emptyset, \quad \forall (p,q), (p',q') \in \Lambda.$$

For every $(p,q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, there exists $M > 0$ such that

$$D(\alpha, \beta) = \alpha a K_{(X,B)} + \beta H$$

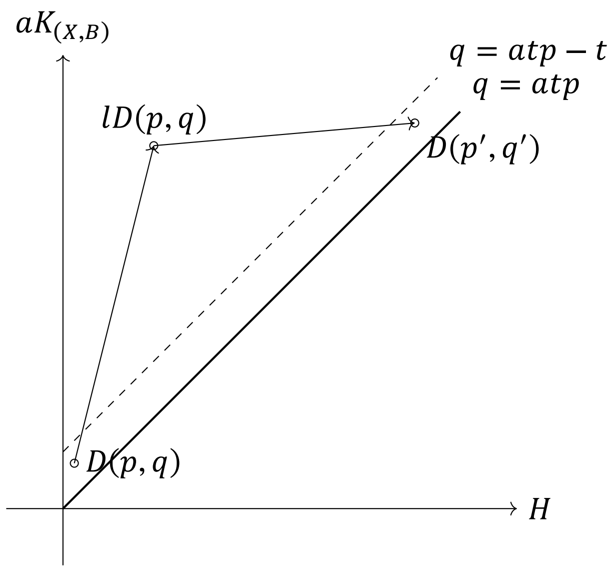
is base point free for all $\alpha = 0, \cdots, p$ and $\beta > M$. Choose M' large enough such that for all $(p', q') \in \mathbb{Z}^2_{>0}$ with $p', q' > M'$ and $0 < atp' - q' < t$, write

$$p' = lp + p_0, \quad q' = lq + q_0$$

for some $l \in \mathbb{Z}_{\geq 0}$ and $0 \leq p_0 < p$, we have $q_0 > M$. The existence of such M' follows from the estimate

$$q_0 = q' - lq = q' - \frac{p' - p_0}{p} q > q' - (p' - p_0)(at - \delta) > p'\delta,$$

where $\delta > 0$ is a small enough number such that $at - \delta > q/p$.



Then $D(p',q') - lD(p,q) = D(p_0,q_0)$ is base point free. It follows that $\mathrm{Bs}\,|D(p',q')| \subseteq \mathrm{Bs}\,|D(p,q)|$. By noetherian induction, there exists an index set Λ such that $\mathrm{Bs}\,|D(p,q)| = Z$ for all $(p,q) \in \Lambda$.

Step 3. Suppose the contradiction that $v(t) > a(\dim X + 1)$. Then we show that $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem (Theorem 3.3.7).

Let $P(x,y) := \chi(D(x,y))$ be the Hilbert polynomial of $D(x,y)$. Note that $P(0,n) = \chi(nH) \neq 0$ since H is ample. Then $P(x,y) \neq 0$ and $\deg P \leq \dim X$. By Step 1, P is not identically zero on Λ. Note that $D(p,q) - K_{(X,B)}$ is ample for all $(p,q) \in \Lambda$, then $h^i(X, D(p,q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem (Theorem 3.2.13). Then

$$P(p,q) = \chi(D(p,q)) = h^0(X, D(p,q)) \neq 0$$

for some $(p,q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$.

Step 4. We follow the same line of the proof of Base Point Free Theorem (Theorem 3.3.7) to show that there is a section which does not vanish on Z.

Fix $(p,q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n+1)/u$. Let $f : Y \to X$ be a resolution such that

(a) $K_{Y,B_Y} = f^*K_{(X,B)} + E_Y$ for some effective exceptional divisor E_Y, and Y, B_Y is a klt pair;

(b) $f^*|D(p,q)| = |L| + F$ for some effective divisor F and a base point free divisor L, and $f(\mathrm{Supp}\,F) = Z$;

(c) $f^*D(p,q) - f^*K_{(X,B)} - E_0$ is ample for some effective $\mathbb{Q}$-divisor $E_0 \in (0,1)$, and coefficients of E_0 are sufficiently small;

(d) $B_Y + E_Y + F + E_0$ has snc support.

Such resolution exists by [KM98].

Let $c := \inf\{\lfloor B_Y + E_0 + tF \rfloor \neq 0\}$. Adjust the coefficients of E_0 slightly such that $\lfloor B_Y + E_0 + cF \rfloor = F_0$ for unique prime divisor F_0 with $F_0 \subset \mathrm{Supp}\,F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then (Y, Δ_Y) is a klt pair.

Let

$$\begin{aligned} N(p',q') &:= f^*D(p',q') + E_Y - F_0 - K_{(Y,\Delta_Y)} \\ &= \Big(f^*D(p',q') - (1+c)f^*D(p,q)\Big) + \Big(f^*D(p,q) - f^*K_{(X,B)} - E_0\Big) + c\Big(f^*D(p,q) - F\Big). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p',q') \in \Lambda : 0 < atp' - q' < atp - q,\ p', q' > (1+c)\max\{p,q\}\},$$

the divisor $f^*D(p',q') - (1+c)f^*D(p,q) = f^*D(p' - (1+c)p, q' - (1+c)q)$ is ample, and hence $N(p',q')$ is ample.

By the exact sequence

$$0 \to \mathcal{O}_Y(f^*D(p',q') + E_Y - F_0) \to \mathcal{O}_Y(f^*D(p',q') + E_Y) \to \mathcal{O}_{F_0}((f^*D(p',q') + E_Y)|_{F_0}) \to 0$$

and Kawamata-Viehweg Vanishing Theorem (Theorem 3.2.13), we get a surjective map

$$H^0(Y, f^*D(p', q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}).$$

On F_0, consider the polynomial $\chi((f^*D(p', q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n - 1$ and by the construction of $(p, q), \Lambda_0$, similar to Step 3, we can show that $\chi((f^*D(p', q') + E_Y)|_{F_0})$ is not identically zero on Λ_0. By adjunction, we have $(f^*D(p', q') + E_Y)|_{F_0} = N(p', q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p', q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (Theorem 3.2.13) to get

$$h^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}) = \chi(F_0, (D(p', q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \mathrm{Bs}\,|D(p', q')|$. □

3.3.5 Cone Theorem and Contraction Theorem

Theorem 3.3.11 (Cone Theorem)**.** Let (X, B) be a projective klt pair. Then there exist countably many curves $C_i \subset X$ such that

(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. Let $F_D := \mathrm{Psef}_1(X) \cap D^\perp$ for a nef divisor D on X. If $\dim F_D = 1$, we also write $R_D := F_D$. Let $H_1, \cdots, H_{\rho-1}$ be ample divisors on X such that they together with $K_{(X,B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} := S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

Step 1. There exists an integer N such that for every $K_{(X,B)}$-negative extremal face F_D and for every ample divisor H, there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \neq F_{nD + rK_{(X,B)} + NH} \subset F_D$.

Let $N := (a(X)(\dim X + 1))!$, where $a(X)$ is the number in Theorem 3.3.10. For every n, $nD + H$ is an ample divisor and by Theorem 3.3.10, the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \geq 0 \,:\, K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since $K_{(X,B)} + (N/r_n)((n+1)D + H)$ is nef, we have $r_n \leq r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$ implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \to r \in \mathbb{Z}_{\geq 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors N_1, N_2, we have $F_{N_1 + N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists m large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

Step 2. Let $\Phi : N_1(X)_{K_{(X,B)}<0} \to \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left(\frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha}\right).$$

We show that the image of R_D under Φ lies in a $\mathbb{Z}$-lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class ξ. By Step 1, we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers n, r. Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of R_D under Φ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

Step 3. We show that every $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ is of the form R_D for some nef divisor D on X.

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$-negative exposed ray. Then R is of form $D^\perp \cap \mathrm{Psef}_1(X)$ for some nef $\mathbb{R}$-divisor D on X. We need to show that D can be choose as a nef $\mathbb{Q}$-divisor. There is a sequence of nef but not ample $\mathbb{Q}$-divisors D_m such that $D_m \to D$ as $m \to \infty$. We adjust D_m such that $\dim F_{D_m} = 1$ for all n.

By re-choosing H_i, we can assume that $D = a_1H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is D_m. Then F_{D_m} is $K_{(X,B)}$-negative. Note that $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by Step 1. If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{D_m}$. We can assume that$r_1K_{(X,B)} + NH_1$ is not identically zero on F_{D_m}. Then we can choose n large enough such that $\|r_1K_{(X,B)} + NH_1\|/n < 1/m$. Replace D_m by $D_m + (r_1K_{(X,B)} + NH_1)/n$. Inductively we construct D_m nef $\mathbb{Q}$-divisor with $D_m \to D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that ξ_m converges. Then $\xi_m \to \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, Φ is well-defined at ξ and the image of ξ_m under Φ is discrete. Hence $\xi = \xi_m$ for all m large enough. It follows that $R = R_{D_m}$ for a nef $\mathbb{Q}$-divisor D_m.

By Step 2, the $K_{(X,B)}$-negative extremal rays form a discrete set in $\{\alpha \in \mathrm{Psef}_1(X) : K_{(X,B)} \cdot \alpha < 0\}$. Hence every $K_{(X,B)}$-negative extremal ray is an exposed ray by Straszewicz's Theorem.

Step 4. Proof of the theorem.

Given an ample divisor H on X, note that εH has positive minimum δ on $\mathrm{Psef}_1(X) \cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \mathrm{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and Φ is well-defined on it. By Steps 2 and 3, there are only finitely many extremal rays on $\mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H \leq 0}$. Hence we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$\mathcal{C} := \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\} \subset \mathcal{c}$ such that $\alpha_n \to \alpha \in N_1(X)_{\mathbb{R}}$. Note that $\mathrm{Psef}_1(X)$ is closed, hence $\alpha \in \mathrm{Psef}_1(X)$. We only need to consider the case $\alpha \cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$. Then $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$ for all n large enough. Note that $\mathcal{c} \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$ is a polyhedral cone by Step 2 and hence is closed. Then $\alpha \in \mathcal{c}$ and the conclusion follows. □

Remark 3.3.12. Thanks for my friend Qin for pointing out that the extremal ray may not be exposed.

Theorem 3.3.13 (Contraction Theorem)**.** Let (X, B) be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;

(b) up to linearly equivalence, any Cartier divisor G with $F \subset G^\perp = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$ comes from a Cartier divisor on Y, i.e., there exists a Cartier divisor G_Y on Y such that $G \sim \varphi_F^* G_Y$.

Proof. We follow the following steps to prove the theorem.

Step 1. We show that there exists a nef divisor D on X such that $F = D^\perp \cap \mathrm{Psef}_1(X)$. In other words, F is defined on $N_1(X)_{\mathbb{Q}}$.

We can choose an ample divisor H and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on F since $F \cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in $N_1(X)_{\mathbb{R}}$. Then by Cone Theorem (Theorem 3.3.11), F is an extremal face of a rational polyhedral cone, namely $\mathrm{Psef}_1(X)_{K_{(X,B)}+(1/n)H\leq 0}$. It follows that $F^\perp \subset N^1(X)_{\mathbb{R}}$ is defined on $\mathbb{Q}$. Since F is extremal and $K_{(X,B)} + (1/n)H$-negative, the set $\{L \in F^\perp : L|_{\mathrm{Psef}_1(X)\setminus F} > 0\}$ has non-empty interior in $F^\perp$ by Theorems 3.3.3 and 3.3.11. Then there exists a Cartier divisor D such that $D \in F^\perp$ and $D|_{\mathrm{Psef}_1(X)\setminus F} > 0$. It follows that D is nef and $F = D^\perp \cap \mathrm{Psef}_1(X)$.

Step 2. Let $\varphi : X \to Y$ be the Iitaka fibration associated to D by Theorem 3.3.1. We show that φ is the desired fibration.

Note that $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$ is compact and D is strictly positive on it. Then there exist $a \geq 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since F is $K_{(X,B)}$-negative. Then by Base Point Free Theorem (Theorem 3.3.7), we know that mD is base point free for all $m \gg 0$. Hence we can apply Theorem 3.3.1 to get a fibration $\varphi_D : X \to Y$.

First we show that D comes from Y. Note that mD and $(m+1)D$ induces the same fibration φ_D for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m) D_{Y,m}$ and $D_{Y,m}$ is ample. Hence D_Y is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that C is contracted by φ_D if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let G be arbitrary Cartier divisor on X such that $F \subset G^\perp$. Since D is strictly positive on $\mathrm{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' := mD + G$, we have $D'^\perp \cap \mathrm{Psef}_1(X) = F$. Then by the same argument as above, we get an other fibration $\varphi_{D'} : X \to Y'$ such that a curve C is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 3.3.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on Y. In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor D'_Y on Y. Then $G = D' - mD$ also comes from Y. □

Remark 3.3.14. The Step 1 is amazing. If F is not $K_{(X,B)}$-negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E. By [Laz04, Lemma 1.5.4], we know that $\mathsf{Psef}_1(X)$ is a circular cone. The we see there indeed exist some irrational extremal faces of $\mathsf{Psef}_1(X)$.

Theorem 3.3.15 (Length of extremal rays)**.** Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathsf{Psef}_1(X)$. Then there exists a rational curve $C \subset X$ such that $[C] \in R$ and

$$0 < -K_{(X,B)} \cdot C \leq 2 \dim X.$$

Proof. By Theorem 3.3.13, let $\varphi_D : X \to Y$ be the contraction associated to R_D (note that we do not need the step to proof Theorem 3.3.13). If $\dim Y < \dim X$, let F be a general fiber of φ_D. By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that H is ample on F. By Proposition 3.3.4. In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By Lemma 3.3.5, we can use Proposition 3.3.4 to get the result. To be completed. □

Definition 3.3.16. Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathsf{Psef}_1(X)$ with contraction $\varphi_R : X \to Y$. There are three types of contractions:

(a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension one;

(b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension at least two;

(c) *Mori fiber space*: if $\dim X > \dim Y$.

Proposition 3.3.17. Let (X, B) be a $\mathbb{Q}$-factorial projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathsf{Psef}_1(X)$. Suppose that the contraction $\varphi : X \to Y$ associated to R is either divisorial or a Mori fiber space. Then Y is $\mathbb{Q}$-factorial.

Proof. Let D be a prime Weil divisor on Y and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\geq 0}[C]$ for an irreducible curve C contracted by φ. Set $D_X := \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$. Then D_X is a prime Weil divisor on X and hence is $\mathbb{Q}$-Cartier.

If φ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber F of φ. Then by Contraction Theorem (Theorem 3.3.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor D' on Y. We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence D is $\mathbb{Q}$-Cartier.

If φ is a divisorial contraction, let E be the exceptional divisor of φ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier $\mathbb{Q}$-divisor E_Y on Y). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 3.3.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor D' on Y. Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence D is $\mathbb{Q}$-Cartier. □

Remark 3.3.18. If φ is a small contraction, then Y is never $\mathbb{Q}$-factorial. Otherwise, let B_Y be the strict transform of B on Y. Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset U. Suppose $K_{(Y,B_Y)}$ is $\mathbb{Q}$-Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

Example 3.3.19. Let $X = E \times E \times \mathbb{P}^1$. To be completed.

Chapter 4

Abelian Varieties

4.1 The First Properties of Abelian Varieties

4.1.1 Definition and examples of Abelian Varieties

Definition 4.1.1. Let $\mathbf{k}$ be a field. An *abelian variety over* $\mathbf{k}$ is a proper variety A over $\mathbf{k}$ together with morphisms *identity* $e : \operatorname{Spec}\mathbf{k} \to A$, *multiplication* $m : A \times A \to A$ and *inversion* $i : A \to A$ such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd}
 & A \times A \times A \arrow[dl, "\mathrm{id}_A \times m"'] \arrow[dr, "m \times \mathrm{id}_A"] & \\
A \times A \arrow[dr, "m"'] & & A \times A \arrow[dl, "m"] \\
 & A &
\end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd}
A \times \operatorname{Spec}\mathbf{k} \arrow[r, "\mathrm{id}_A \times e"] \arrow[dr, "\cong"'] & A \times A \arrow[d, "m"] & \operatorname{Spec}\mathbf{k} \times A \arrow[l, "e \times \mathrm{id}_A"'] \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_A"] & \\
A \times A \arrow[dr, "m"'] & \operatorname{Spec}\mathbf{k} \arrow[d, "e"] & A \times A \arrow[dl, "m"] \\
 & A &
\end{tikzcd}.$$

In other words, an abelian variety is a group object in the category of proper varieties over $\mathbf{k}$.

Example 4.1.2. Let E be an elliptic curve over a field $\mathbf{k}$. Then E is an abelian variety of dimension 1. To be completed.

In the following, we will always assume that A is an abelian variety over a field $\mathbf{k}$ of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The *left translation* by $a \in A(\mathbf{k})$ is defined as

$$l_a : A \xrightarrow{\cong} \operatorname{Spec}\mathbf{k} \times A \xrightarrow{a\times \mathrm{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a.

Proposition 4.1.3. Let A be an abelian variety. Then A is smooth.

Proof. By base changing to the algebraic closure of $\mathbf{k}$, we may assume that $\mathbf{k}$ is algebraically closed. Note that there is a non-empty open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a. □

Proposition 4.1.4. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. But how to show it is a morphism of varieties? To be completed. □

Theorem 4.1.5. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism, i.e., for every $\mathbf{k}$-scheme T, the induced map $f_T : A(T) \to B(T)$ is a group homomorphism.

Proof. Consider the diagram

$$\begin{tikzcd} A \times A \arrow[d, "p_1"'] \arrow[dr, "\varphi"] & \\ A & B \end{tikzcd}$$

with φ be given by

$$\begin{aligned} &A \times A \xrightarrow{\Delta\times\Delta} A \times A \times A \times A \xrightarrow{\cong} A \times A \times A \times A \xrightarrow{(f\circ m_A)\times(i_B\circ f)\times(i_B\circ f)} B \times B \times B \xrightarrow{m_B} B,\\ &(x,y) \mapsto (x,x,y,y) \mapsto (x,y,y,x) \mapsto (f(xy), f(y)^{-1}, f(x)^{-1}) \mapsto f(xy)f(y)^{-1}f(x)^{-1}. \end{aligned}$$

We have $\varphi(p_1^{-1}(e_A)) = \varphi(\{e_A\} \times A) = \{e_B\}$. Then by Rigidity Lemma (??), there exists a unique rational map $\psi : A \dashrightarrow B$ such that $\varphi = \psi \circ p_1$. Note that $A \to A \times \{e_A\} \to A \times A$ gives a section of p_1. On this section, we have that φ is constant equal to e_B. Thus ψ is well-defined and $\psi(A) = e_B$. It follows that φ factors through the constant map $A \times A \to \{e_B\} \to B$. Then for every $(x, y) \in A(\Bbbk) \times A(\Bbbk)$, we have

$$f(xy) = f(x)f(y).$$

Since $A(\Bbbk)$ is dense in A, the conclusion follows. □

Proposition 4.1.6. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 4.1.5. □

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 4.1.7. Let A be an abelian variety over $\Bbbk$ and n a positive integer not divisible by $\operatorname{char} \Bbbk$. Then the multiplication by n morphism $[n]_A : A \to A$ is finite surjective and étale.

Proof. To be completed. □

4.1.2 Complex abelian varieties

Theorem 4.1.8. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if there exists a positive definite Hermitian form H on $\mathbb{C}^n$ such that $\Im(H)(\Lambda, \Lambda) \subset \mathbb{Z}$. To be completed.

4.2 Picard Groups of Abelian Varieties

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. Let A be an abelian variety over $\mathbf{k}$.

4.2.1 Pullback along group operations

Theorem 4.2.1 (Theorem of the cube)**.** Let X, Y, Z be proper varieties over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\}\times Y\times Z}$, $\mathcal{L}|_{X\times\{y\}\times Z}$ and $\mathcal{L}|_{X\times Y\times\{z\}}$ are trivial. Then $\mathcal{L}$ is trivial.

Proof. To be completed. □

Remark 4.2.2. If we assume the existence of the Picard scheme, then the Theorem 4.2.1 can be deduced from the Rigidity Lemma. Consider the morphism

$$\varphi : X \times Y \to \mathrm{Pic}(Z), \quad (x, y) \mapsto \mathcal{L}|_{\{x\}\times\{y\}\times Z}.$$

Since $\varphi(x, y) = \mathcal{O}_Z$, φ factors through $\mathrm{Pic}^0(Z)$. Then the assumption implies that φ contracts $\{x\}\times Y$, $X \times \{y\}$ and hence it maps $X \times Y$ to a point. Thus $\varphi(x', y') = \mathcal{O}_Z$ for every $(x', y') \in X \times Y$. Then by Grauert's theorem, we have $\mathcal{L} \cong p^* p_* \mathcal{L}$ where $p : X \times Y \times Z \to X \times Y$ is the projection. Note that $p_*\mathcal{L} \cong \mathcal{L}|_{X\times Y\times\{z\}} \cong \mathcal{O}_{X\times Y}$. Hence $\mathcal{L}$ is trivial.

Lemma 4.2.3. Let A be an abelian variety over $\mathbf{k}$, $f, g, h : X \to A$ morphisms from a variety X to A and $\mathcal{L}$ a line bundle on A. Then we have

$$(f + g + h)^*\mathcal{L} \cong (f + g)^*\mathcal{L} \otimes (f + h)^*\mathcal{L} \otimes (g + h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof. First consider $X = A\times A\times A$, $p : X \to A, (x_1, x_2, x_3) \mapsto x_1+x_2+x_3$, $p_{ij} : X \to A, (x_1, x_2, x_3) \mapsto x_i + x_j$ for $1 \le i < j \le 3$ and $p_i : X \to A, (x_1, x_2, x_3) \mapsto x_i$ for $1 \le i \le 3$. Then the conclusion follows from the theorem of the cube by taking $\mathcal{L}' = p^*\mathcal{L}^{-1}\otimes p_{12}^*\mathcal{L}\otimes p_{13}^*\mathcal{L}\otimes p_{23}^*\mathcal{L}\otimes p_1^*\mathcal{L}^{-1}\otimes p_2^*\mathcal{L}^{-1}\otimes p_3^*\mathcal{L}^{-1}$ and considering the restriction to $\{0\} \times A \times A$, $A \times \{0\} \times A$ and $A \times A \times \{0\}$.

In general, consider the morphism $\varphi = (f, g, h) : X \to A \times A \times A$ and pull back the above isomorphism along φ. □

Proposition 4.2.4. Let A be an abelian variety over $\mathbf{k}$, $n \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle on A. Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes\frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes\frac{1}{2}(n^2-n)}.$$

Proof. For $n = 0, 1$, the conclusion is trivial. For $n \ge 2$, we can use the previous lemma on $[n-2]_A, [1]_A, [1]_A$ and induct on n. Hence we have

$$[n]_A^*\mathcal{L} \cong [n-1]_A^*\mathcal{L} \otimes [n-1]_A^*\mathcal{L} \otimes [2]_A^*\mathcal{L} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [n-2]_A^*\mathcal{L}^{-1}.$$

Then the conclusion follows from induction. To be completed. □

Definition 4.2.5. Let A be an abelian variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on A. We say that $\mathcal{L}$ is *symmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}$ and *antisymmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}^{-1}$.

Theorem 4.2.6 (Theorem of the square)**.** Let A be an abelian variety over $\mathbf{k}$, $x, y \in A(\mathbf{k})$ two points and $\mathcal{L}$ a line bundle on A. Then

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Proof. To be completed. □

Example 4.2.7. Let E be an elliptic curve over $\Bbbk$ with origin 0. For $x \in E(\Bbbk)$, let P_x be the corresponding prime divisor on E. Denote $\mathsf{Pic}^0(E)$ the subgroup of $\mathsf{Pic}(E)$ consisting of line bundles of degree zero. Given $x, y \in E(\Bbbk)$, by the theorem of the square, we have

$$t_{-x-y}^*\mathcal{O}_E(P_0) \otimes \mathcal{O}_E(P_0) \cong t_{-x}^*\mathcal{O}_E(P_0) \otimes t_{-y}^*\mathcal{O}_E(P_0).$$

Note that $t_{-x}^*\mathcal{O}_E(P_0) \cong \mathcal{O}_E(P_x)$. Hence

$$\mathcal{O}_E(P_{x+y} - P_0) \cong \mathcal{O}_E(P_x - P_0 + P_y - P_0).$$

This shows that the map $E(\Bbbk) \to \mathsf{Pic}^0(E), x \mapsto \mathcal{O}_E(P_x - P_0)$ is a group homomorphism. It is injective since if $\mathcal{O}_E(P_x - P_0) \cong \mathcal{O}_E$, then $P_x \sim P_0$ and hence $x = 0$. It is also surjective since for any $\mathcal{L} \in \mathsf{Pic}^0(E)$, we can write $\mathcal{L} \cong \mathcal{O}_E(\sum a_i P_{x_i})$ with $\sum a_i = 0$ and then

$$\mathcal{L} \cong \mathcal{O}_E\left(\sum a_i(P_{x_i} - P_0)\right) \cong \mathcal{O}_E\left(P_{\sum a_i x_i} - P_0\right).$$

Hence we have an isomorphism of groups $E(\Bbbk) \cong \mathsf{Pic}^0(E)$.

Remark 4.2.8. We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \to \mathsf{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary $\mathcal{L}$, the map

$$\Phi_{\square} : \mathsf{Pic}(A) \to \mathsf{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \mathsf{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is also a group homomorphism. For any $x \in A(\mathbf{k})$, we have

$$\Phi_{t_x^*\mathcal{L}} = \Phi_{\mathcal{L}}$$

by Theorem 4.2.6. In the other words,

$$\Phi_{\mathcal{L}}(x) \in \mathsf{Ker}\,\Phi_{\square}, \quad \forall \mathcal{L} \in \mathsf{Pic}(A), x \in A(\mathbf{k}).$$

If we assume the scheme structure on $\mathsf{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\mathsf{Pic}^0(A)$. Let $K(\mathcal{L}) := \mathsf{Ker}\,\Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A. We give another description of $K(\mathcal{L})$. From this point, when $K(\mathcal{L})$ is finite, we can recover the dual abelian variety $A^\vee = \mathsf{Pic}^0_{A/\mathbf{k}}$ as the quotient $A/K(\mathcal{L})$.

Example 4.2.9. Let E be an elliptic curve over $\Bbbk$ with origin 0. We have $\mathsf{Pic}^0(E) \cong E(\Bbbk)$ by sending $x \in E(\Bbbk)$ to $\mathcal{O}_E(P_x - P_0)$ where P_x is the point on E corresponding to x.

Then

$$\Phi_{n\mathcal{O}(P_x)}(y) = t_y^*\mathcal{O}_E(nP_x) \otimes \mathcal{O}_E(-nP_x) \cong \mathcal{O}_E(nP_{x-y} - nP_0 + nP_0 - nP_x) \leftrightarrow -ny$$

Hence

$$\Phi_{n\mathcal{O}(P_x)} : E(\Bbbk) \to E(\Bbbk), \quad y \mapsto -ny.$$

Hence $K(\mathcal{L}) = \{x \in E(\Bbbk) : n \cdot x \sim n \cdot 0\} = E[n](\Bbbk)$ is the subgroup of n-torsion points of E.

4.2.2 Projectivity

In this subsection, we work over the algebraically closed field $\Bbbk$.

Proposition 4.2.10. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. Then $|2D|$ is base point free.

Proof. To be completed. □

Theorem 4.2.11. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. TFAE:

(a) the stabilizer $\mathsf{Stab}(D)$ of D is finite;

(b) the morphism $\phi_{|2D|}$ induced by the complete linear system $|2D|$ is finite;

(c) D is ample;

(d) $K(\mathcal{O}_A(D))$ is finite.

Proof. To be completed. □

Theorem 4.2.12. Let A be an abelian variety over $\Bbbk$. Then A is projective.

Proof. To be completed. □

Corollary 4.2.13. Let A be an abelian variety over $\Bbbk$ and D a divisor on A. Then D is pseudo-effective if and only if it is nef, i.e. $\mathrm{Psef}^1(A) = \mathrm{Nef}^1(A)$.

Proof. To be completed. ☐

4.2.3 Dual abelian varieties

In this subsection, we work over the algebraically closed field $\Bbbk$.

Definition 4.2.14. Let A be an abelian variety over $\Bbbk$. We define the *dual abelian variety* of A to be $A/K(\mathcal{L})$ for some ample line bundle $\mathcal{L}$ on A. We denote it by $A^\vee$.

We have a natural map $A^\vee(\Bbbk) \to \mathrm{Pic}^0(A)$ by sending $x + K(\mathcal{L}) \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$. We will show that this map is an isomorphism.

Lemma 4.2.15. There exists a unique line bundle $\mathcal{P}$ on $A \times A^\vee$ such that for every $y = \mathcal{L} \in A^\vee = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{y\}} \cong \mathcal{L}$.

Proof. To be completed. ☐

Lemma 4.2.16. Let A be an abelian variety over $\Bbbk$ and B a group variety over $\Bbbk$. Then there is a natural bijection between the morphisms $f : B \to A^\vee$ and the line bundles $\mathcal{L}$ on $A \times B$ such that for every $b \in B(\Bbbk)$, we have $\mathcal{L}|_{A\times\{b\}} \in \mathrm{Pic}^0(A)$. The bijection is given by $f \mapsto (1_A \times f)^*\mathcal{P}$ where $\mathcal{P}$ is the Poincaré line bundle on $A \times A^\vee$. To be completed.

Proof. To be completed. ☐

Theorem 4.2.17. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ and the Poincaré line bundle $\mathcal{P}$ on $A \times A^\vee$ do not depend on the choice of the ample line bundle $\mathcal{L}$. Moreover, there is a natural bijection $A^\vee(\mathbf{k}) \to \mathrm{Pic}^0(A)$ of groups. Under this bijection, for every $x = \mathcal{L} \in A^\vee(\mathbf{k}) = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{x\}} \cong \mathcal{L}$.

Proof. To be completed. ☐

Proposition 4.2.18. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ is also an abelian variety and the natural morphism $A \to A^{\vee\vee}$ is an isomorphism.

Proof. To be completed. ☐

4.2.4 The Néron-Severi group

Theorem 4.2.19. Let A be an abelian variety over $\Bbbk$. The we have an inclusion $\mathrm{NS}(A) \hookrightarrow \mathrm{Hom}_{\mathbf{Av}}(A, A^\vee)$ of groups given by

$$\mathcal{L} \mapsto \left(\Phi_{\mathcal{L}} : A \to A^\vee, \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}\right).$$

To be completed.

Example 4.2.20. Let E be an elliptic curve over $\Bbbk$ without complex multiplication and $A = E^n$ for some $n \geq 1$. Set $D_i = E^{i-1} \times \{0\} \times E^{n-i}$ for $1 \leq i \leq n$ and $D_{ij} = \Delta_{ij} - D_i - D_j$ for $1 \leq i < j \leq n$ where Δ_{ij} is the pullback of the diagonal divisor $\Delta_E \subseteq E \times E$ along the projection $A \to E \times E$ to the i-th and j-th factors. Then $\mathrm{NS}(A)$ is generated by the classes of D_i's and D_{ij}'s.why?

The homomorphism $\Phi : \mathrm{NS}(A) \to \mathrm{Hom}_{\mathbf{Av}}(A, A^\vee)$ can be described as follows. Note that $A^\vee \cong (E^\vee)^n \cong E^n = A$. For D_i, $\Phi_{D_i} : A \to A^\vee$ is given by

$$\Phi_{D_i}(x_1, \dots, x_n) = t^*_{(x_1,\dots,x_n)}\mathcal{O}_A(D_i) \otimes \mathcal{O}_A(D_i)^{-1} \cong (0, \dots, 0, x_i, 0, \dots, 0).$$

For D_{ij}, $\Phi_{D_{ij}} : A \to A^\vee$ is given by

$$\Phi_{D_{ij}}(x_1, \dots, x_n) = t^*_{(x_1,\dots,x_n)}\mathcal{O}_A(D_{ij}) \otimes \mathcal{O}_A(D_{ij})^{-1} \cong (0, \dots, 0, x_j, 0, \dots, 0, x_i, 0, \dots, 0).$$

Hence under the identification $\mathrm{Hom}_{\mathbf{Av}}(A, A^\vee) \cong M_n(\mathbb{Z})$, the map $\Phi : \mathrm{NS}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A, A^\vee)$ is given by

$$D_i \mapsto E_{ii}, \quad D_{ij} \mapsto E_{ij} + E_{ji}$$

where E_{ij} is the matrix with 1 at the (i, j)-th entry and 0 elsewhere.

Chapter 5

Algebraic Groups

5.1 First properties of algebraic groups

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. Everything are defined over $\mathbf{k}$ unless otherwise specified.

5.1.1 Basic concepts

Definition 5.1.1. A *group scheme* over S is an S-scheme G together with morphisms *multiplication* $\mu : G \times G \to G$, *identity* $\varepsilon : S \to G$ and *inversion* $\iota : G \to G$ over S such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd} & G \times G \times G \arrow[dl, "\mathrm{id}_G \times \mu"'] \arrow[dr, "\mu \times \mathrm{id}_G"] & \\ G \times G \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd} G \times S \arrow[r, "\mathrm{id}_G \times \varepsilon"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] & S \times G \arrow[l, "\varepsilon \times \mathrm{id}_G"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd} & G \arrow[dl, "\mathrm{id}_G \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_G"] & \\ G \times G \arrow[dr, "\mu"'] & S \arrow[d, "\varepsilon"] & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd}.$$

In other words, an S-group scheme is a group object in the category $\mathbf{Sch}_S$.

Definition 5.1.2. An *algebraic group* is a $\mathbf{k}$-group scheme G which is reduced, separated and of finite type over a field $\mathbf{k}$.

Remark 5.1.3. We often identify an algebraic group G with its set of closed points $G(\Bbbk)$ when there is no confusion. Why can we do this?

Remark 5.1.4. Even if we work over $\Bbbk$ and just consider the closed points $G(\Bbbk)$ of an algebraic group G, $G(\Bbbk)$ is not a topological group with respect to the Zariski topology in general. The reason is that the topology on $G(\Bbbk) \times G(\Bbbk)$ is not the product topology of the topologies on $G(\Bbbk)$.

Definition 5.1.5. Let G be an algebraic group and $x \in G(\mathbf{k})$ a $\mathbf{k}$-point. The *left translation* by x is the morphism

$$l_x : G \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{\mu} G.$$

Similar definition applies to the right translation r_x.

Remark 5.1.6. In the context of algebraic groups, we often use multiplicative notation for the group law. That is, for $g, h \in G(\mathbf{k})$, we write gh instead of $\mu(g, h)$ and g^{-1} instead of $\iota(g)$.

Sometimes we also abuse the notation by $\mu : G \times \cdots \times G \to G$ to denote the multiplication of multiple elements, i.e. $\mu(g_1, \ldots, g_n) = g_1 \cdots g_n$ for $g_1, \ldots, g_n \in G(\mathbf{k})$.

Proposition 5.1.7. Let G be an algebraic group. Then G is smooth over $\mathbf{k}$.

Proof. Since G is reduced and of finite type over a field, it is generically regular. Let $g \in G(\Bbbk)$ be a regular point. Then the left translation $l_{gh^{-1}} : G \to G$ is an isomorphism, hence G is regular at $h \in G(\Bbbk)$. It follows that G is regular at every $\Bbbk$-point, hence G is smooth over $\mathbf{k}$. □

Remark 5.1.8. Let G be an algebraic group. Then the irreducible components of G coincide with the connected components of G. We will use the term "connected" to refer to both concepts since "irreducible" has other meanings in the theory of representations.

Example 5.1.9. The *additive group* $\mathbb{G}_a$ is defined to be the affine line $\mathbb{A}^1$ with the group law given by addition. Concretely, we can write $\mathbb{G}_a = \operatorname{Spec} \mathbf{k}[T]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a, &\quad (x, y) \mapsto x + y, \\ \iota : \mathbb{G}_a \to \mathbb{G}_a, &\quad x \mapsto -x, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_a, &\quad * \mapsto 0. \end{aligned}$$

Example 5.1.10. The *multiplicative group* $\mathbb{G}_m$ is defined to be the affine variety $\mathbb{A}^1 \setminus \{0\}$ with the group law given by multiplication. Concretely, we can write $\mathbb{G}_m = \operatorname{Spec} \mathbf{k}[T, T^{-1}]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m, &\quad (x, y) \mapsto xy, \\ \iota : \mathbb{G}_m \to \mathbb{G}_m, &\quad x \mapsto x^{-1}, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_m, &\quad * \mapsto 1. \end{aligned}$$

Example 5.1.11. The *general linear group* GL_n is defined to be the open subvariety of $\mathbb{A}^{n^2}$ consisting of invertible matrices, with the group law given by matrix multiplication. Concretely, we can write

$\mathrm{GL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}, \det(T_{ij})^{-1}]$ where $1 \leq i, j \leq n$ and the group law is given by the morphism

$$\begin{aligned} \mu &: \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, \quad (A, B) \mapsto AB, \\ \iota &: \mathrm{GL}_n \to \mathrm{GL}_n, \quad A \mapsto A^{-1}, \\ \varepsilon &: \operatorname{Spec} \mathbf{k} \to \mathrm{GL}_n, \quad * \mapsto I_n. \end{aligned}$$

Example 5.1.12. An abelian variety is an algebraic group that is also a proper variety.

Example 5.1.13. Let G and H be algebraic groups. The *product* $G \times H$ is an algebraic group with the group law defined by

$$\begin{aligned} \mu_{G \times H} = \mu_G \times \mu_H &: (G \times H) \times (G \times H) \cong (G \times G) \times (H \times H) \to G \times H, \\ \varepsilon_{G \times H} = \varepsilon_G \times \varepsilon_H &: \operatorname{Spec} \mathbf{k} \cong \operatorname{Spec} \mathbf{k} \times \operatorname{Spec} \mathbf{k} \to G \times H, \\ \iota_{G \times H} = \iota_G \times \iota_H &: G \times H \to G \times H. \end{aligned}$$

Example 5.1.14. Let G be an algebraic group over $\mathbf{k}$ and $\mathbf{K}/\mathbf{k}$ a field extension. The base change $G_{\mathbf{K}} = G \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} \mathbf{K}$ is an algebraic group over $\mathbf{K}$ with the group law defined by the base change of the original group law of G to $\mathbf{K}$.

Definition 5.1.15. A *homomorphism* of algebraic groups is a morphism of schemes that is also a group homomorphism. Explicitly, a morphism $f : G \to H$ between algebraic groups G and H is a homomorphism if the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f \times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

where μ_G and μ_H are the group laws of G and H, respectively.

Definition 5.1.16. An *algebraic subgroup* of an algebraic group G is a closed subscheme $H \subseteq G$ that is also a subgroup of G. More precisely, H is an algebraic subgroup and the inclusion morphism $H \hookrightarrow G$ is compatible with the group laws.

An algebraic subgroup H of G is called *normal* if for any $\mathbf{k}$-scheme S, the subgroup $H(S)$ is a normal subgroup of the abstract group $G(S)$.

Remark 5.1.17. To check $H < G$ whether H is a normal subgroup of G, it suffices to check that $H(\Bbbk)$ is normal in $G(\Bbbk)$. To be continued.

Example 5.1.18. The *special linear group* SL_n is defined to be the closed subvariety of GL_n defined by the equation $\det = 1$. It is an algebraic subgroup of GL_n.

Proposition 5.1.19. Let G be an algebraic group and S is a closed subgroup of $G(\Bbbk)$. Then there exists a unique algebraic subgroup H of G such that $H(\Bbbk) = S$.

Proof. To be continued... □

Remark 5.1.20. By Proposition 5.1.19, we often identify an algebraic group G with its set of closed points $G(\Bbbk)$ when there is no confusion.

Remark 5.1.21. If one replaces $\Bbbk$ by $\mathbf{k}$ in Proposition 5.1.19, the statement may not hold. For example, let $\mathbf{k} = \mathbb{Q}$ and G be the elliptic curve defined by $X^3 + Y^3 = Z^3$ in $\mathbb{P}^2$. It is well-known that $\#G(\mathbb{Q}) = 3$. Let S be the disjoint union of the three $\mathbb{Q}$-points of G endowed with the reduced subscheme structure and the group structure induced from G. Then S is a proper closed subgroup of G and we have $S(\mathbb{Q}) = G(\mathbb{Q})$. This contradicts the uniqueness in Proposition 5.1.19.

Indeed, in this chapter, despite working over an arbitrary field $\mathbf{k}$, we mostly consider the closed points of algebraic groups over $\Bbbk$.

Definition 5.1.22. Let G be an algebraic group. The *neutral component* G^0 is the connected component of G containing the identity element ε.

Proposition 5.1.23. The neutral component G^0 is a closed, normal algebraic subgroup of G.

Proof. To be continued... □

Proposition 5.1.24. Let G be an algebraic group and $H \subseteq G(\Bbbk)$ a subgroup (not necessarily closed). Then the Zariski closure $\overline{H}$ of H in G is an algebraic subgroup of G. If $H \subset G(\Bbbk)$ is constructible, then $H = \overline{H}(\Bbbk)$.

Proof. To be continued... □

Remark 5.1.25. In general, we can only expect the image of a morphism of varieties to be a constructible subset. This is not sufficient to guarantee that the image is closed, even if the original variety is closed. However, the group structure provides additional constraints that ensure the constructible subgroup is indeed closed. Example 5.1.26 provides an example where the product of two closed algebraic subgroups is not closed, illustrating that the importance of the subgroup condition.

To be continued...

Example 5.1.26. Let $G = \mathrm{SL}_2$ over $\Bbbk$, $T = \{\mathrm{diag}(t, t^{-1}) | t \in \Bbbk^\times\}$ and $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Set $S = gTg^{-1}$. Then both T and S are closed algebraic subgroups of $G(\Bbbk)$, but the product TS is not closed in $G(\Bbbk)$. By direct computation, we have

$$S = \left\{ \begin{pmatrix} s & s^{-1} - s \\ 0 & s^{-1} \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

Then

$$TS = \left\{ \begin{pmatrix} ts & t(s^{-1} - s) \\ 0 & (ts)^{-1} \end{pmatrix} \middle| t, s \in \Bbbk^\times \right\}.$$

We have

$$TS \cap \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \Bbbk \right\} = \left\{ \begin{pmatrix} 1 & s^{-2} - 1 \\ 0 & 1 \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

The right hand side is not closed in $\mathrm{SL}_2(\Bbbk)$ since it does not contain the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Hence TS is not closed in $G(\Bbbk)$.

Proposition 5.1.27. Let G be an algebraic group, X_i varieties over $\mathbf{k}$ and $f_i : X_i \to G$ morphisms for $i = 1, \dots, n$ with images $Y_i = f_i(X_i)$. Suppose that Y_i pass through the identity element of G. Let H be the closed subgroup of G generated by $Y_1, \dots, Y_n$, i.e. the smallest closed subgroup of G containing $Y_1, \dots, Y_n$. Then H is connected and $H = Y_{a_1}^{e_1} \cdots Y_{a_m}^{e_m}$ for some $a_1, \dots, a_m \in \{1, \dots, n\}$ and $e_1, \dots, e_m \in \{\pm 1\}$.

Proof. To be continued... □

Remark 5.1.28. We can take $m \leq 2 \dim G$ in Proposition 5.1.27.

5.1.2 Action and representations

Definition 5.1.29. An *action* of an algebraic group G on a variety X is a morphism

$$\sigma : G \times X \to X$$

such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu\times\mathrm{id}_X"] \arrow[d, "\mathrm{id}_G\times\sigma"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}\mathbf{k} \times X \arrow[r, "\varepsilon\times\mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

where μ is the group law of G and ε is the identity element of G. In other words, for any $\mathbf{k}$-scheme S, the induced map $G(S) \times X(S) \to X(S)$ defines a group action of the abstract group $G(S)$ on the set $X(S)$.

For simplicity, we often write $g.x$ instead of $\sigma(g, x)$ for $g \in G(\mathbf{k})$ and $x \in X(\mathbf{k})$.

Example 5.1.30. There are three natural actions of an algebraic group G on itself:

(a) Left translation: $g.h = l_g(h) = gh$;

(b) Right translation: $g.h = r_g(h) = hg^{-1}$;

(c) Conjugation: $g.h = \mathrm{Ad}_g(h) = ghg^{-1}$.

All of them are morphisms of varieties since they are defined by the group law and inversion of G.

Example 5.1.31. The general linear group GL_n acts on the affine space $\mathbb{A}^n$ by matrix multiplication. It is given by polynomials, hence is a morphism of varieties.

Example 5.1.32. The general linear group GL_{n+1} acts on the projective space $\mathbb{P}^n$ by

$$A \cdot [x_0 : \cdots : x_n] = [y_0 : \cdots : y_n], \quad \text{where } (y_0, \dots, y_n)^T = A(x_0, \dots, x_n)^T.$$

Let U_i be the standard affine open subset of $\mathbb{P}^n$ defined by $x_i \neq 0$. The map is given by polynomials on the principal open subset of $\mathrm{GL}_{n+1} \times U_i$ defined by $y_j \neq 0$ for any j. Hence it is a morphism of varieties.

Definition 5.1.33. A *linear representation* of an algebraic group G on a finite-dimensional vector space V over $\Bbbk$ is an abstract group representation $\rho : G(\Bbbk) \to \mathrm{GL}(V)$ such that if we identify V

with $\mathbb{A}^n$ for some n, then the map $G(\Bbbk) \times \mathbb{A}^n(\Bbbk) \to \mathbb{A}^n(\Bbbk)$ is is a morphism of varieties.

Definition 5.1.34. Let G be an algebraic group acting on a variety X. For any $x \in X(\mathbf{k})$, the *orbit* of x is the locally closed subvariety $G \cdot x = \sigma(G \times \{x\})$ of X.

Proposition 5.1.35. Let G be an algebraic group acting on a variety X. Then for any $x \in X(\mathbf{k})$, the orbit $G \cdot x$ is a locally closed subvariety of X, and $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of strictly smaller dimension.

Proof. To be continued... □

Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For $x \in G(\mathbf{k})$, we have the left translation of functions $\tau_x : A \to A$ defined by $\tau_x(f)(y) = f(x^{-1}y)$ for $y \in X(\mathbf{k})$.

Lemma 5.1.36. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For any finite-dimensional subspace $V \subseteq A$, there exists a finite-dimensional G-invariant subspace $W \subseteq A$ containing V.

Proof. To be continued... □

Theorem 5.1.37. Any affine algebraic group is linear, i.e. is isomorphic to a closed algebraic subgroup of some GL_n.

Proof. To be continued... □

5.1.3 Lie algebra of an algebraic group

Let G be an algebraic group. The *Lie algebra* of G is defined to be the tangent space of G at the identity element ε:

$$\operatorname{Lie}(G) = T_\varepsilon G.$$

It is a finite-dimensional vector space over $\mathbf{k}$.

Proposition 5.1.38. The group law $\mu : G \times G \to G$ induces the plus map on $\operatorname{Lie}(G)$:

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)} : T_{(\varepsilon,\varepsilon)}(G \times G) \cong T_\varepsilon G \oplus T_\varepsilon G \to T_\varepsilon G, \quad (v, w) \mapsto v + w.$$

Proof. We have

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v, w) = \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v, 0) + \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(0, w) = (\mathrm{d}\mu \circ (\mathrm{id}_G \times \varepsilon))_\varepsilon(v) + (\mathrm{d}\mu \circ (\varepsilon \times \mathrm{id}_G))_\varepsilon(w) = v + w.$$

□

Proposition 5.1.39. Let G be an algebraic group and n a positive integer which is not divisible by $\operatorname{char} \mathbf{k}$. Then the power map $p_n : G \to G$ is generically finite.

Proof. To be added. □

Corollary 5.1.40. Let G be a connected algebraic group and H a closed subgroup of $G(\Bbbk)$ with finite index. Then $H = G(\Bbbk)$.

Proof. To be added. □

Corollary 5.1.41. Let G be an algebraic group and H a closed subgroup of $G(\Bbbk)$. Suppose that there exists a positive integer n which is not divisible by $\operatorname{char}\mathbf{k}$ such that $h^n = e$ for all $h \in H$. Then H is finite. To be completed.

Remark 5.1.42. Thanks for my mathematical brother Zelong Chen for telling me this. To be revised

Remark 5.1.43. The classical Burnside theorem states that a finite exponent subgroup of $\mathrm{GL}_n(\mathbb{C})$ is finite. Corollary 5.1.41 can be viewed as a generalization of the classical Burnside theorem to arbitrary algebraic groups over arbitrary fields.

5.2 Quotient by algebraic group

Everything in this section is over an arbitrary field $\mathbf{k}$ unless otherwise specified.

5.2.1 Quotient

Definition 5.2.1. Let G be an algebraic group acting on a variety X. A *quotient* of X by G is a variety Y together with a morphism $\pi : X \to Y$ such that

(a) π is G-invariant, i.e., $\pi(g \cdot x) = \pi(x)$ for all $g \in G$ and $x \in X$.

(b) For any variety Z and any G-invariant morphism $f : X \to Z$, there exists a unique morphism $\overline{f} : Y \to Z$ such that $f = \overline{f} \circ \pi$.

In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\overline{f}"] \\ & Z \end{tikzcd}$$

If a quotient exists, it is unique up to a unique isomorphism. To be continued...

Such a quotient does not always exist.

Theorem 5.2.2. Let G be an affine algebraic group acting on a variety X. Then there exists a variety Y and a rational morphism $\pi : X \dashrightarrow Y$ with commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "\overline{f}"] \\ & Z \end{tikzcd}$$

satisfying the following universal property: If a quotient exists, it is unique up to a unique isomorphism.
Furthermore, if all orbits of G in X are closed, then π is a morphism (i.e., defined everywhere). To be continued... Ref?

5.2.2 Quotient of affine algebraic group by closed subgroup

Lemma 5.2.3. Let V be a finite-dimensional vector space over $\mathbf{k}$ and G an abstract group acting linearly on V. Let $W \subseteq V$ be a subspace of dimension m. Then $G.W = W$ if and only if $G. \wedge^m W = \wedge^m W$.

Proof. To be filled. □

Lemma 5.2.4. Let G be an affine algebraic group and H a closed subgroup. Then there exists a finite-dimensional linear representation V of G and a one-dimensional subspace $L \subseteq V$ such that H is the stabilizer of L.

Proof. To be filled. □

Theorem 5.2.5. Let G be an affine algebraic group and H a closed subgroup. Then the quotient G/H exists as a quasi-projective variety.

Proof. To be filled. □

5.3 Decomposition of algebraic groups

Theorem 5.3.1 (Chavellaye Decomposition)**.** Let G be an algebraic group. Then there exists a unique maximal connected affine normal algebraic subgroup G_{aff} of G such that the quotient G/G_{aff} is an abelian variety. This subgroup is called the *affine part* of G. To be continued...

Theorem 5.3.2 (Rosenlicht Decomposition)**.** Let G be an algebraic group. Then there exists a smallest normal connected algebraic subgroup G_{ant} of G such that the quotient G/G_{ant} is affine. This subgroup is called the *anti-affine part* of G. Moreover, G_{ant} is contained in the center of G^0. To be continued...

5.3.1

5.4 Structure of linear algebraic groups I: commutative and solvable groups

In this section, everything is defined on an algebraically closed field $\Bbbk$.

5.4.1 Commutative algebraic groups and character groups

Definition 5.4.1. Let G be a linear algebraic group over a field $\Bbbk$. The *character group* of G, denoted by $\chi(G)$, is defined to be the group of all homomorphisms of linear algebraic groups from G to the multiplicative group $\mathbb{G}_m$:

$$\chi(G) = \mathrm{Hom}_{\mathbf{AlgGrp}_{\Bbbk}}(G, \mathbb{G}_m).$$

The group operation on $\chi(G)$ is given by pointwise multiplication of characters.

Example 5.4.2. Let us compute the character group of the multiplicative group $\mathbb{G}_m$. Let $\varphi : \mathbb{G}_m \to \mathbb{G}_m$ be a character of $\mathbb{G}_m$. Since φ is a morphism of algebraic varieties, it induces a homomorphism of coordinate rings $\varphi^\sharp : \Bbbk[T, T^{-1}] \to \Bbbk[T, T^{-1}]$. Note that $(\Bbbk[T, T^{-1}])^\times = \{cT^n \mid c \in \Bbbk^\times, n \in \mathbb{Z}\}$. Thus, we have $\varphi^\sharp(T) = cT^n$ for some $c \in \Bbbk^\times$ and $n \in \mathbb{Z}$. That is, $\varphi(a) = ca^n$ for all $a \in \mathbb{G}_m(\Bbbk) = \Bbbk^\times$. However, since φ is a group homomorphism, we must have $\varphi(1) = c = 1$. Therefore, $\varphi(a) = a^n$ for some integer n. Conversely, for each integer n, the map $\chi_n : \mathbb{G}_m \to \mathbb{G}_m$ defined by $\chi_n(a) = a^n$ is indeed a character of $\mathbb{G}_m$. Hence, we have shown that $\chi(\mathbb{G}_m) \cong \mathbb{Z}$.

Example 5.4.3. The character group of the additive group $\mathbb{G}_a$ is trivial, i.e., $\chi(\mathbb{G}_a) = \{0\}$. Indeed, every morphism from $\mathbb{A}^1$ to $\mathbb{A}^1 \setminus \{0\}$ is constant since every regular function on $\mathbb{A}^1$ is a polynomial, and there is no non-constant polynomial function without zeros.

Definition 5.4.4. A linear algebraic group T over a field $\Bbbk$ is called a *torus* if T is isomorphic to a finite product of copies of the multiplicative group $\mathbb{G}_m$, i.e., $T \cong \mathbb{G}_m^n$ for some non-negative integer n.

Lemma 5.4.5. Let T be a torus and H be a connected closed algebraic subgroup of T. Then H is also a torus.

Proof. To be continued. □

Recall that for a linear operator $T : V \to V$ of finite-dimensional $\Bbbk$-vector space V is called *semisimple* if it is diagonalizable, and *unipotent* if $T - \mathrm{id}_V$ is nilpotent.

Proposition 5.4.6. Let $T \subset \mathrm{GL}_n(\Bbbk)$ be a torus. Then every element of $T(\Bbbk)$ is semisimple. Conversely, if $g \in \mathrm{GL}_n(\Bbbk)$ is semisimple and of infinite order, then the neutral component of the algebraic subgroup generated by g is a torus.

Proposition 5.4.7. Let $g \subset \mathrm{GL}_n(\Bbbk)$. Then g is unipotent if and only if the algebraic subgroup generated by g is isomorphic to $\mathbb{G}_a$. To be revised.

Theorem 5.4.8. Let G be a connected commutative linear algebraic group over an algebraically closed field $\Bbbk$ of characteristic zero. Then G is isomorphic to $\mathbb{G}_m^r \times \mathbb{G}_a^s$ for some non-negative integers r and s.

Proof. To be continued. □

5.4.2 Jordan-Chevalley Decomposition of elements

Definition 5.4.9. Let G be a linear algebraic group and $g \in G(\Bbbk)$. We say that g is *semisimple* (resp. *unipotent*) if its image under some (equivalently, any) faithful linear representation of G is a semisimple (resp. unipotent) linear operator.

Lemma 5.4.10. The notion of semisimple and unipotent elements in Definition 5.4.9 does not depend on the choice of faithful linear representation.

Proof. To be added. ☐

Theorem 5.4.11 (Jordan-Chevalley Decomposition)**.** Let G be a linear algebraic group and $g \in G(\Bbbk)$. Then there exist unique commuting elements $g_s, g_u \in G(\Bbbk)$ such that $g = g_s g_u$, where g_s is semisimple and g_u is unipotent.
Moreover, this decomposition is functorial in the sense that for any homomorphism of linear algebraic groups $\varphi : G \to H$, we have $\varphi(g)_s = \varphi(g_s)$ and $\varphi(g)_u = \varphi(g_u)$. To be checked

Proof. To be continued. ☐

5.4.3 Solvable groups and Borel subgroups

Definition 5.4.12. A group G is said to be *solvable* if there exists a finite sequence of algebraic subgroups

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = \{e\}$$

such that each G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is commutative for all $0 \leq i < n$. to be checked.

Theorem 5.4.13. Let G be a solvable linear algebraic group acting on a proper variety X. Then there exists a fixed point $x \in X(\Bbbk)$ such that $g \cdot x = x$ for all $g \in G(\Bbbk)$.

Corollary 5.4.14 (Lie-Kolchin Theorem)**.** Let $G < \mathrm{GL}_n(\Bbbk)$ be a solvable linear algebraic group over an algebraically closed field $\Bbbk$. Then there exists a basis of $\Bbbk^n$ such that G is contained in the group of upper triangular matrices with respect to this basis.

Theorem 5.4.15. Let G be a linear algebraic group of dimension 1 over an algebraically closed field $\Bbbk$. Then G is isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$.

5.4.4 Decomposition of linear algebraic groups

Definition 5.4.16. Let G be a linear algebraic group over a field $\Bbbk$. The *radical* of G, denoted by $\mathrm{rad}(G)$, is defined to be the unique maximal connected normal solvable subgroup of G.

Well-defined?

Definition 5.4.17. Let G be a linear algebraic group. The *unipotent radical* of G, denoted by $\mathrm{rad}_u(G)$, is defined to be the subgroup of $\mathrm{rad}(G)$ consisting of all unipotent elements.

Why a group?

Definition 5.4.18. Let G be a linear algebraic group over a field $\Bbbk$. We say that G is *semisimple* if $\mathrm{rad}(G)$ is trivial.

Definition 5.4.19. Let G be a linear algebraic group over a field $\Bbbk$. We say that G is *reductive* if the unipotent radical of G is trivial.

Slogan

$$\begin{array}{ccc} \text{“\textit{unipotent radical}”} & \to\!\leftarrow & \text{“\textit{reductive}”} \\ \Downarrow & & \Uparrow \\ \text{“\textit{solvable radical}”} & \to\!\leftarrow & \text{“\textit{semisimple}”} \end{array}$$

Theorem 5.4.20 (Levi Decomposition)**.** Let G be a linear algebraic group over an algebraically closed field $\Bbbk$. Then there exists a reductive subgroup H of G such that the multiplication map $\mathrm{rad}_u(G) \rtimes H \to G$ is an isomorphism of algebraic groups. Such a subgroup H is called a *Levi subgroup* of G. To be checked.

Proof. To be continued. □

5.5 Application: birational group of varieties of general type

In this section, we apply the results from the previous sections to study the birational automorphism groups of varieties of general type.

Theorem 5.5.1. Let X be a projective variety of general type over an algebraically closed field $\Bbbk$ of characteristic zero. Then the group of birational automorphisms $\mathrm{Bir}(X)$ is finite.

Proof. We will prove this theorem in several steps. By replacing X with its resolution of singularities, we may assume that X is smooth.

Step 1. For every $m \geq 1$, $\mathrm{Bir}(X)$ linearly acts on $H^0(X, mK_X)$ via pull-back of functions (as abstract group).

Let $\mathscr{K}(X)$ be the function field of X. Then for every $g \in \mathrm{Bir}(X)$, g induces an automorphism of $\mathscr{K}(X)$ over $\Bbbk$, which we denote by g^*. In particular we know that g^* is injective and $\Bbbk$-linear. By definition, $H^0(X, mK_X) = \{s \in \mathscr{K}(X) \mid \mathrm{div}(s) + mK_X \geq 0\}$. We only need to show that for every $s \in H^0(X, mK_X)$, $g^*(s) \in H^0(X, mK_X)$ since $\dim_{\Bbbk} H^0(X, mK_X) < \infty$. Consider the commutative diagram

$$\begin{tikzcd} \Gamma \arrow[d, "p"'] \arrow[dr, "q"] & \\ X \arrow[r, dashed, "g"] & X \end{tikzcd}$$

with Γ smooth and p, q birational morphisms. Then we have

$$K_\Gamma = p^*K_X + E_p = q^*K_X + E_q,$$

where E_p and E_q are p- and q-exceptional divisors respectively. Moreover, E_p and E_q are effective since X is smooth. For every $s \in H^0(X, mK_X)$, we have

$$\mathrm{div}(q^*s) + mK_\Gamma = q^*(\mathrm{div}(s) + mK_X) + mE_q \geq 0.$$

Then

$$\begin{aligned}\operatorname{div}(g^*s) + mK_X &= p_*p^*(\operatorname{div}(g^*s) + mK_X) \\ &= p_*\left(\operatorname{div}(q^*s) + mK_\Gamma - mE_p\right) \\ &= p_*\left(\operatorname{div}(q^*s) + mK_\Gamma\right) \geq 0.\end{aligned}$$

It follows that $g^*(s) \in H^0(X, mK_X)$.

Note this action $g \mapsto g^*$ is contravariant, i.e., for every $g_1, g_2 \in \operatorname{Bir}(X)$, we have $(g_1 \circ g_2)^* = g_2^* \circ g_1^*$.

Step 2. The group $\operatorname{Bir}(X)$ is a linear algebraic group by identifying it with a closed subgroup of $\operatorname{Aut}(\mathbb{P}(V))$ for some finite-dimensional $\Bbbk$-vector space V (subspace of $H^0(X, mK_X)$ for some $m > 0$). Moreover, its rational action on X is algebraic.

By ??, there exists an integer $m > 0$ such that the map $\psi : X \dashrightarrow \mathbb{P}(H^0(X, mK_X))$ is birational onto its image Y. Let V be the subspace of $H^0(X, mK_X)$ spanned by the affine cone over Y. Since $\operatorname{Bir}(X)$ linearly acts on $H^0(X, mK_X)$ by Step 1, it also linearly acts on V. we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, dashed, "\psi"] & X \arrow[d, dashed, "\psi"] \\ Y \arrow[r, "\varphi_g|_Y"] \arrow[d, hook] & Y \arrow[d, hook] \\ \mathbb{P}(V) \arrow[r, "\varphi_g"] & \mathbb{P}(V) \end{tikzcd}$$

for every $g \in \operatorname{Bir}(X)$, where φ_g is the induced automorphism of $\mathbb{P}(V)$.

Since ψ is birational, the map $g \mapsto \varphi_g$ defines an injective group homomorphism from $\operatorname{Bir}(X)$ to $\operatorname{Aut}(\mathbb{P}(V))$. Consider the natural algebraic group structure on $\operatorname{Aut}(\mathbb{P}(V))$ and let G be the Zariski closure of the image of $\operatorname{Bir}(X)$ in $\operatorname{Aut}(\mathbb{P}(V))$. Note that $\operatorname{Bir}(X)$ fixes Y. Thus G also fixes Y. Since the affine cone over Y spans V, we conclude that any element $g \in G$ is uniquely determined by its restriction to Y. In particular, we have $G = \operatorname{Bir}(X)$. Note that $\operatorname{Aut}(\mathbb{P}(V))$ is a linear algebraic group and so is its closed subgroup $\operatorname{Bir}(X)$.

Step 3. If $\dim \operatorname{Bir}(X) > 0$, then it contains $\mathbb{G}_a$ or $\mathbb{G}_m$ as a subgroup. We show that the action of $\mathbb{G}_a$ or $\mathbb{G}_m$ on X leads to X being uniruled, which contradicts the assumption that X is of general type.

By Lemma 5.5.5 and Theorem 5.4.15, if $\dim \operatorname{Bir}(X) > 0$, then $\operatorname{Bir}(X)$ contains either $\mathbb{G}_a$ or $\mathbb{G}_m$ as a subgroup. Note that both $\mathbb{G}_a$ and $\mathbb{G}_m$ are rational varieties, without loss of generality, we may assume that $\operatorname{Bir}(X)$ contains $\mathbb{G}_m$ as a subgroup. Then we have a rational map

$$\Phi : \mathbb{G}_m \times X \dashrightarrow X.$$

Fix $x \in X$ such that $\Phi|_{\mathbb{G}_m \times \{x\}} : \mathbb{G}_m \to X$ is not constant. Choose $Z \subset X$ a closed subvariety of codimension 1 passing through x such that $\mathbb{G}_m.x \not\subseteq Z$. Then the closure of $\Phi(\mathbb{G}_m \times Z)$ in X has dimension at least $\dim Z + 1 = \dim X$. Hence we have a dominant rational map

$$\Phi : \mathbb{P}^1 \times Z \dashrightarrow X.$$

This contradicts ?? and the assumption that X is of general type. Therefore, we must have $\dim \operatorname{Bir}(X) = 0$, i.e., $\operatorname{Bir}(X)$ is finite. □

Remark 5.5.2. In the proof of Theorem 5.5.1, by $\mathbb{P}(V)$ we mean the projective space associated to the vector space V in the sense of Grothendieck, i.e., $\mathbb{P}(V) = \mathrm{Proj}(\bigoplus_{k\geq 0} \mathrm{Sym}^k V)$. Hence if one have a linear map $f : V \to W$ between two finite-dimensional $\Bbbk$-vector spaces, then it induces a morphism $\mathbb{P}(W) \to \mathbb{P}(V)$ (not $\mathbb{P}(V) \to \mathbb{P}(W)$).

Corollary 5.5.3. Let X be a projective variety of general type over an algebraically closed field $\Bbbk$ of characteristic zero. Then there exists a projective variety Y birational to X such that $\mathrm{Bir}(Y) = \mathrm{Aut}(Y)$.

Corollary 5.5.4. Let X be a smooth projective Fano variety over an algebraically closed field $\Bbbk$ of characteristic zero. Then the group of automorphisms $\mathrm{Aut}(X)$ is a linear algebraic group.

Proof. Note that for every $g \in \mathrm{Aut}(X)$, g induces an automorphism of $H^0(X, -mK_X)$ for every integer $m \geq 1$ via pull-back of functions. Then the same argument as in Step 2 shows that $\mathrm{Aut}(X)$ is a linear algebraic group. □

Lemma 5.5.5. Let G be a linear algebraic group over an algebraically closed field $\Bbbk$. Then G has a one-dimensional algebraic subgroup.

Chapter 6

Sites, algebraic space and stacks

6.1 Preliminaries in Category Theory

6.1.1 Sites

Definition 6.1.1. Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of sets of arrows $\{U_i \to U\}_{i\in I}$, called *covering*, for each object U in $\mathbf{C}$ such that:

(a) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering;

(b) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a arrow, then the fiber product $U_i \times_U V \to V$ exists and $\{U_i \times_U V \to V\}$ is a covering of V;

(c) if $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ are coverings, then the collection of composition $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{J})$ where $\mathbf{C}$ is a category and $\mathcal{J}$ is a Grothendieck topology on $\mathbf{C}$.

Note that sheaf is indeed defined on a site.

Definition 6.1.2. Let $(\mathbf{C}, \mathcal{J})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{J})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ satisfying the following condition: for every object U in $\mathbf{C}$ and every covering $\{U_i \to U\}_{i\in I}$ of U, if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every i, j, the pullback $s_i|_{U_i\times_U U_j}$ and $s_j|_{U_i\times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every i, the pullback $s|_{U_i} = s_i$.

Definition 6.1.3. Let X be a scheme. The *big étale site* of X, denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over X with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if and only if each U_i is étale over U and the union of their images is the whole U.

Let X be a scheme over S. By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \to \mathbf{Set}$ such that for any S-scheme T, $h_X(T) = \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$. Easy to check that h_X is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

6.1.2 Fibered categories and descent conditions

Definition 6.1.4. Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. A morphism $f : b \to a$ in $\mathbf{X}$ is called *strongly Cartesian* if for every object $c \in \mathrm{Obj}(\mathbf{X})$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{X}}(c,b) \arrow[r, "f\circ-"] \arrow[d, "\mathbf{p}"'] & \mathrm{Hom}_{\mathbf{X}}(c,a) \arrow[d, "\mathbf{p}"] \\ \mathrm{Hom}_{\mathbf{S}}(w,v) \arrow[r, "\mathbf{p}(f)\circ-"] & \mathrm{Hom}_{\mathbf{S}}(w,u) \end{tikzcd}$$

is a pullback of sets, where $u = \mathbf{p}(a), v = \mathbf{p}(b), w = \mathbf{p}(c)$.

The condition in Definition 6.1.4 can be interpreted as follows: for any diagram as below black part with $\mathbf{p}(g) = \mathbf{p}(f) \circ \alpha$,

$$\begin{tikzcd} c \arrow[rrd, bend left, "g"] \arrow[rd, "h"] \arrow[dd, mapsto] & & \\ & b \arrow[r, "f"] \arrow[d, mapsto] & a \arrow[d, mapsto] \\ w \arrow[r, "\alpha"] & v \arrow[r, "\mathbf{p}(f)"] & u \end{tikzcd}$$

there exists a unique gray morphism $h : c \to a$ such that $\mathbf{p}(h) = \alpha$ and $f \circ h = g$.

Notation 6.1.5. Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. For $a, b \in \mathrm{Obj}(\mathbf{X})$ and $f \in \mathrm{Hom}_{\mathbf{X}}(a, b)$, we say that a is *over* $\mathbf{p}(a)$ and f is *over* $\mathbf{p}(f)$. In a diagram, we have

$$\begin{tikzcd} \mathbf{X} \arrow[d, "\mathbf{p}"'] & a \arrow[r, "f"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ \mathbf{S} & \mathbf{p}(a) \arrow[r, "\mathbf{p}(f)"] & \mathbf{p}(b) \end{tikzcd}$$

Definition 6.1.6. Let $\mathbf{S}$ be a category. A category $\mathbf{X}$ over $\mathbf{S}$ via $\mathbf{p}$ is called a *category fibred* over the site $\mathbf{S}$ if for every morphism $\iota : v \to u$ in $\mathbf{S}$ and every object $a \in \mathrm{Obj}(\mathbf{X})$ over u, there exists an object $b \in \mathrm{Obj}(\mathbf{X})$ over v and a strongly Cartesian morphism $f : b \to a$ over ι. Such an object b is called a *pullback* of a along ι, and is often denoted by $\iota^* a$.

Definition 6.1.7. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a category fibred over $\mathbf{S}$. For every object $u \in \mathrm{Obj}(\mathbf{S})$, the *fiber* of $\mathbf{X}$ over u is the category $\mathbf{X}_u$ given by

$$\mathrm{Obj}(\mathbf{X}_u) = \{a \in \mathrm{Obj}(\mathbf{X}) \mid \mathbf{p}(a) = u\}, \quad \mathrm{Hom}_{\mathbf{X}_u}(a, b) = \{f \in \mathrm{Hom}_{\mathbf{X}}(a, b) \mid \mathbf{p}(f) = \mathrm{id}_u\}.$$

Remark 6.1.8. Note that in Definition 6.1.6, the pullback $r^* b$ of an object b along a morphism r is not necessarily unique. To be continued.

Example 6.1.9. Let $\mathbf{S}$ be a category and $\mathcal{F} : \mathbf{S}^{op} \to \mathbf{Set}$ be a presheaf on $\mathbf{S}$ taking values in $\mathbf{Set}$. We can construct a category $\mathbf{F}$ fibred over $\mathbf{S}$ as follows:

- The objects of $\mathbf{F}$ are pairs (U, x) where $U \in \mathrm{Obj}(\mathbf{S})$ and $x \in \mathcal{F}(U)$;
- morphisms from (V, y) to (U, x) in $\mathbf{F}$ are morphisms $\iota : V \to U$ in $\mathbf{S}$ such that $\mathcal{F}(\iota)(x) = y$, denoted by res_ι.

The functor $\mathbf{p} : \mathbf{F} \to \mathbf{S}$ is defined by $\mathbf{p}(U, x) = U$ on objects and $\mathbf{p}(\iota) = \iota$ on morphisms. If one has the diagram

$$\begin{tikzcd}
(W,z) \arrow[rrd, bend left, "\mathrm{res}_\tau"] \arrow[dd, mapsto] & & \\
& (V,y) \arrow[r, "\mathrm{res}_\iota"] \arrow[d, mapsto] & (U,x) \arrow[d, mapsto] \\
W \arrow[r, "\sigma"] & V \arrow[r, "\iota"] & U
\end{tikzcd}$$

with $\mathbf{p}(\mathrm{res}_\tau) = \iota \circ \sigma$. By definition, we have $\tau = \iota \circ \sigma$ and $\mathcal{F}(\tau)(x) = z, \mathcal{F}(\iota)(x) = y$. Thus, we have $\mathcal{F}(\sigma)(y) = z$. This verifies that res_σ is a strongly Cartesian morphism. Note that the fiber of $\mathbf{F}$ over an $U \in \mathrm{Obj}(\mathbf{S})$ is the discrete category associated to the set $\mathcal{F}(U)$. Therefore, presheaves of sets can be viewed as categories fibred in sets.

Conversely, given a category $\mathbf{F}$ fibred in sets over $\mathbf{S}$ via $\mathbf{p} : \mathbf{F} \to \mathbf{S}$, one can construct a presheaf of sets $\mathcal{F} : \mathbf{S}^{op} \to \mathbf{Set}$ by defining $\mathcal{F}(U) = \mathrm{Obj}(\mathbf{F}_U)$ for each $U \in \mathrm{Obj}(\mathbf{S})$, and for each morphism $\iota : V \to U$ in $\mathbf{S}$, defining $\mathcal{F}(\iota) : \mathcal{F}(U) \to \mathcal{F}(V)$ by sending an object $x \in \mathcal{F}(U)$ to its pullback $\iota^* x \in \mathcal{F}(V)$ along ι. This establishes an equivalence between presheaves of sets on $\mathbf{S}$ and categories fibred in sets over $\mathbf{S}$.

Example 6.1.10. case $\mathbf{S} = set, group$. To be added.

Slogan *Presheaves of sets are categories fibered in sets.*

In following, we describe categories fibered in groupoids.

Definition 6.1.11. Let $\mathbf{X}$ be a category fibred over a category $\mathbf{S}$ via $\mathbf{p} : \mathbf{X} \to \mathbf{S}$. For every $u \in \mathrm{Obj}(\mathbf{S})$ and every pair of objects a, b over u, we define the *presheaf of morphisms* $\mathrm{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ by

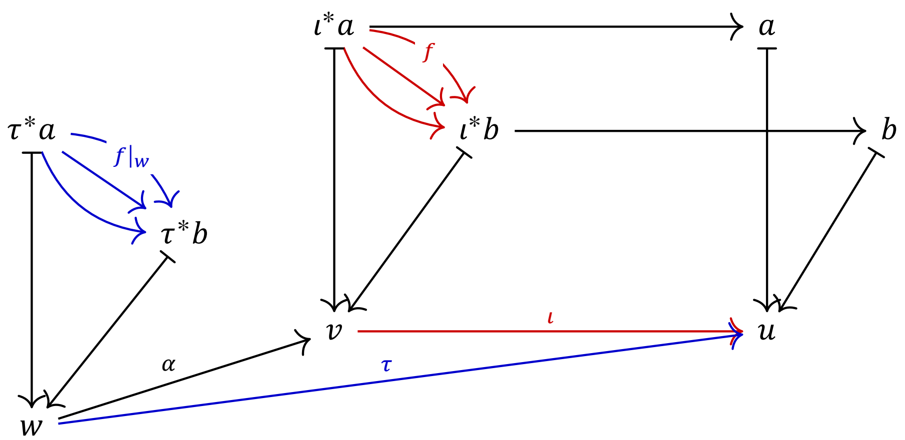
$$\mathrm{Hom}_{\mathbf{X}}(a, b)(\iota : v \to u) = \mathrm{Hom}_{\mathbf{X}_v}(\iota^* a, \iota^* b)$$

for every morphism $\iota : v \to u$ in $\mathbf{S}/u$. For a morphism $\alpha : w \to v$ in $\mathbf{S}/u$, the restriction map

$$\mathrm{Hom}_{\mathbf{X}}(a, b)(\iota) \to \mathrm{Hom}_{\mathbf{X}}(a, b)(\iota \circ \alpha)$$

is given by sending a morphism $f : \iota^* a \to \iota^* b$ in $\mathbf{X}_v$ to the pullback morphism $\alpha^* f : (\iota \circ \alpha)^* a \to (\iota \circ \alpha)^* b$ need to conjugate with a natural transformation. in $\mathbf{X}_w$. To be checked.

In a diagram, the presheaf of morphisms can be visualized as follows:



Proposition 6.1.12. Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a category fibred over $\mathbf{S}$. Then $\mathbf{X}$ is a category fibred in groupoids if and only if for every object $u \in \mathsf{Obj}(\mathbf{S})$ and every pair of objects a, b over u, the presheaf of morphisms $\mathsf{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf. To be checked.

Definition 6.1.13. Let $\mathbf{S}$ be a category. A category $\mathbf{X}$ fibred over $\mathbf{S}$ via $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *category fibred in groupoids* over $\mathbf{S}$ if for every object $u \in \mathsf{Obj}(\mathbf{S})$ and every pair of objects a, b over u, the presheaf of morphisms $\mathsf{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf. To be checked.

Now let us discuss how sheaves fit into the framework of fibered categories. Of course, we need assume the base category $\mathbf{S}$ is a site. The glued condition for sheaves can be interpreted in terms of descent data in fibered categories.

Definition 6.1.14. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. Let $U \in \mathsf{Obj}(\mathbf{S})$ and $\{U_i \to U\}$ be a covering in $\mathbf{S}$. A *descent datum* for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ consists of

- a collection of objects $a_i \in \mathsf{Obj}(\mathbf{X}_{U_i})$ for each i,
- a collection of isomorphisms $\varphi_{ij} : a_j|_{U_{ij}} \to a_i|_{U_{ij}}$ in $\mathbf{X}_{U_{ij}}$ for each pair (i, j), where $U_{ij} = U_i \times_U U_j$,

such that the cocycle condition
$$\varphi_{ik}|_{U_{ijk}} = \varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}}$$
holds for all triples (i, j, k), where $U_{ijk} = U_i \times_U U_j \times_U U_k$. To be checked.

Example 6.1.15. To be added.

Definition 6.1.16. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. A descent datum $(\{a_i\}, \{\varphi_{ij}\})$ for objects of $\mathbf{X}$ relative to a covering $\{U_i \to U\}$ in $\mathbf{S}$ is called *effective* if there exists an object $a \in \mathsf{Obj}(\mathbf{X}_U)$ and isomorphisms $\psi_i : a|_{U_i} \to a_i$ in $\mathbf{X}_{U_i}$ such that for all pairs (i, j), the diagram

$$\begin{tikzcd}
a|_{U_{ij}} \arrow[r, "\psi_j|_{U_{ij}}"] \arrow[d, "\psi_i|_{U_{ij}}"'] & a_j|_{U_{ij}} \arrow[d, "\varphi_{ij}"] \\
a_i|_{U_{ij}} \arrow[r, "\varphi_{ij}"'] & a_j|_{U_{ij}}
\end{tikzcd}$$

commutes. To be checked.

Slogan *Descent data are like gluing data for objects, and effectiveness means that the glued object exists.*

6.1.3 Prestacks and stacks

Definition 6.1.17. A *prestack* over the site $\mathbf{S}$ is a category $\mathbf{X}$ fibred in groupoids over $\mathbf{S}$.

Slogan *Prestacks are "presheaf remembering automorphisms".*

Example 6.1.18. presheaf is a prestack. To be added.

Example 6.1.19. The moduli problem of classifying algebraic curves of a fixed genus g can be formulated as a prestack over the site of schemes. Consider the category $\mathbf{M}_g$ whose objects are families of smooth projective curves of genus g over schemes, and whose morphisms are isomorphisms of such families. The functor $\mathbf{p} : \mathbf{M}_g \to \mathbf{Sch}$ sending a family of curves to its base scheme makes $\mathbf{M}_g$ a category fibred in groupoids over $\mathbf{Sch}$. For each scheme S, the fiber category $\mathbf{M}_{g,S}$ consists of families of smooth projective curves of genus g over S and their isomorphisms. The descent data for objects in $\mathbf{M}_g$ relative to a covering of schemes correspond to gluing families of curves along isomorphisms on overlaps, which is effective due to the nature of algebraic curves. Thus, $\mathbf{M}_g$ is a prestack over the site of schemes. To be revised.

Proposition 6.1.20. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$, $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$, and $\mathbf{r} : \mathbf{Z} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Let $\Phi : \mathbf{X} \to \mathbf{Z}$ and $\Psi : \mathbf{Y} \to \mathbf{Z}$ be morphisms of prestacks over $\mathbf{S}$. Then the fiber product $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ exists in the category of prestacks over $\mathbf{S}$. To be checked.

Definition 6.1.21. Let $\mathbf{S}$ be a site. A prestack $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *stack* over the site $\mathbf{S}$ if for every object $U \in \mathrm{Obj}(\mathbf{S})$ and every covering $\{U_i \to U\}$ in $\mathbf{S}$, the descent data for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ are effective. To be revised.

Definition 6.1.22. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ and $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$ be stacks over $\mathbf{S}$. A *morphism of stacks* $F : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$ is a functor $F : \mathbf{X} \to \mathbf{Y}$ such that $\mathbf{q} \circ F = \mathbf{p}$. To be checked.

Slogan *Stacks are to prestacks as sheaves are to presheaves.*

Example 6.1.23. Let X be a scheme over a base noetherian scheme S. The functor of points $h_X : (\mathbf{Sch}/S)_{\text{ét}}^{\text{op}} \to \mathbf{Set}$ is a sheaf, and thus a stack.

Construction 6.1.24. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ be a prestack over $\mathbf{S}$. There exists a stack $\mathbf{p}^+ : \mathbf{X}^+ \to \mathbf{S}$ over $\mathbf{S}$ together with a morphism of prestacks $F : \mathbf{X} \to \mathbf{X}^+$ over $\mathbf{S}$ satisfying the following universal property: for every stack $\mathbf{p}' : \mathbf{Y} \to \mathbf{S}$ over $\mathbf{S}$ and every morphism of prestacks $G : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$, there exists a unique morphism of stacks $G^+ : \mathbf{X}^+ \to \mathbf{Y}$ over $\mathbf{S}$ such that $G = G^+ \circ F$. The stack $\mathbf{X}^+$ is called the *stackification* of the prestack $\mathbf{X}$. To be checked.

Notation 6.1.25. As Example 6.1.9, we can associate a prestack $\mathbf{X}$ over a $\mathbf{S}$ to a functor $\mathcal{X} : \mathbf{S}^{\text{op}} \to \mathbf{Grpd}$ by setting $\mathbf{X}_u = \mathcal{X}(u)$ for each $u \in \mathrm{Obj}(\mathbf{S})$ and defining the pullback functors accordingly. In particular, we can talk about representability of such prestacks. To be revised. Why do not we just talk about sheaves of groupoid?

Definition 6.1.26. Let $\mathbf{S}$ be a site, and let $\mathbf{X}, \mathbf{Y}$ be prestacks over $\mathbf{S}$. A morphism of prestacks $F : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$ is called *representable* if for every $\mathbf{Z} \to \mathbf{Y}$ over $\mathbf{S}$ with $\mathbf{Z}$ representable in $\mathbf{S}$, the fiber product $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}$ is representable in $\mathbf{S}$.

6.2 Algebraic spaces

Definition 6.2.1. Let U be a scheme over a base scheme S. An *étale equivalence relation* on U is a morphism $R \to U \times_S U$ between schemes over S such that:

(a) the projections in two factors $R \to U$ are étale and surjective;

(b) for every S-scheme T, $h_R(T) \to h_U(T) \times h_U(T)$ gives an equivalence relation on $h_U(T)$ set-theoretically.

Definition 6.2.2. An *algebraic space* X over a base scheme S is an S-scheme U together with an étale equivalence relation $R \to U \times_S U$.

Let $X = (U, R)$ be an algebraic space over S. We explain X as a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. For any scheme T over S, $h_R(T)$ is an equivalence relation on $h_U(T)$. The rule sending T to the set of equivalence classes of $h_R(T)$ gives a presheaf on the site $(\mathbf{Sch}/S)_{\text{ét}}$. The sheafification of this presheaf is the sheaf associated to the algebraic space X. Explicitly, we have

$$X(T) := \left\{ f = (f_i) \,\middle|\, \begin{array}{c} \{T_i \to T\} \text{ a covering, } f_i \in h_U(T_i) \text{ such} \\ \text{that } (f_i|_{T_i \times_T T_j}, f_j|_{T_i \times_T T_j}) \in h_R(T_i \times_T T_j) \end{array} \right\} \Big/ \sim,$$

where

$$\alpha \sim \beta \quad \text{if } \exists \{S_i \to T\} \text{ such that } (\alpha|_{S_i}, \beta|_{S_i}) \in h_R(S_i).$$

Definition 6.2.3. An *algebraic space* over a base scheme S is a sheaf F on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$ such that

(a) the diagonal morphism $F \to F \times_S F$ is representable;

(b) there exists a scheme U over S and a map $h_U \to F$ which is surjective and étale.

The *morphism between algebraic spaces* F_1, F_2 is defined as a natural transformation of functors F_1, F_2.

Remark 6.2.4. By Yoneda's Lemma, given a morphism $h_U \to F$ between sheaves is the same as giving an element of $F(U)$. We may abuse the notation.

Definition 6.2.5. Let $\mathcal{p}$ be a property of morphisms of schemes satisfying the following conditions:

(a) is preserved under any base change;

(b) is étale local on the base.In [Stacks], this requires that "fppf local".

Let $\alpha : F \to G$ be a representable morphism of sheaves on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. We say that α has property $\mathcal{p}$ if for every $h_T \to G$, the base change $h_T \times_G F \to F$ has property $\mathcal{p}$.

Remark 6.2.6. The fiber product $F_1 \times_F F_2$ is just defined as $F_1 \times_F F_2(T) := F_1(T) \times_{F(T)} F_2(T)$ for any object $T \in \mathrm{Obj}(\mathbf{Sch}_S)$. We say that a morphism $f : F_1 \to F_2$ of sheaves is *representable* if for every $T \in \mathrm{Obj}(\mathbf{Sch}/S)$ and every $\xi \in F_2(T)$, the sheaf $F_1 \times_{F_2} h_T$ is representable as a functor. Here

$h_T \to F_2$ is given by

$$h_T(U) \to F_2(U), \quad f \in \mathrm{Hom}(U, T) \mapsto F_2(f)(\xi) \in F_2(U).$$

In our case, given an arbitrary $h_U \to F \times F$ is equivalent to giving morphisms $h_{U_i} \to F$ for $i = 1, 2$. And the fiber product $F \times_{F \times F} (h_{U_1} \times h_{U_2})$ is just the fiber product $h_{U_1} \times_F h_{U_2}$. Hence the first condition in Definition 6.2.3 is equivalent to that $h_{U_1} \times_F h_{U_2}$ is representable for any U_1, U_2 over F. This implies that $h_U \to F$ is representable, whence the second condition in Definition 6.2.3 makes sense.

Definition 6.2.7. Let X be an algebraic space over a base scheme S. Two two morphisms form field $\mathsf{Spec}\, k_i \to X$ is called equivalent if there is a common extension $K \supset k_1, k_2$ such that we have $\mathsf{Spec}\, K \to \mathsf{Spec}\, k_i \to X$ are the same for $i = 1, 2$. The *underlying point set* of X, denote by $|X|$, is defined as the set of equivalence classes of morphisms $\mathsf{Spec}\, k \to X$ for all field k over the base field $\Bbbk$.

This definition coincides with the underlying set of a scheme. Let $\alpha : X \to Y$ be a morphism of algebraic spaces. It induces a map $|\alpha| : |X| \to |Y|$ by $x \mapsto \alpha \circ x$ (vertical composition).

Proposition 6.2.8 (ref. [Stacks, Lemma 66.4.6])**.** There is a unique topology on $|X|$ such that

(a) if X is a scheme, then the topology coincides with the usual topology.

(b) every morphism of algebraic spaces $f : X \to Y$ induces a continuous map $|f| : |X| \to |Y|$.

(c) if U is a scheme and $U \to X$ is étale, then the induced map $|U| \to |X|$ is open.

This topology is called the *Zariski topology* on $|X|$.

Definition 6.2.9. Let X be an algebraic space over a base scheme S. All étale morphisms $U \to X$ with U scheme form a small site $X_{\text{ét}}$. All étale morphisms $U \to X$ with U algebraic space form a small site $X_{\text{sp, ét}}$. The *structure sheaf* $\mathcal{O}_X$ of X is given by $U \mapsto \Gamma(U, \mathcal{O}_U)$ for every étale morphism $U \to X$ from a scheme. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

Example 6.2.10. Let $U = \mathbb{A}^1_{\mathbb{C}}$ and $R \subset U \times U$ given by $y = x + n, n \in \mathbb{Z}$. Then R is a disjoint union of lines in $U \times U$. Write $R = \coprod_{n \in \mathbb{Z}} R_n$ with $R_n = \{(x, x+n) : x \in \mathbb{C}\}$. Then the projection is given by

$$\begin{aligned} \pi_1|_{R_n} &: R_n \to U, \quad (x, x+n) \mapsto x, \\ \pi_2|_{R_n} &: R_n \to U, \quad (x, x+n) \mapsto x+n. \end{aligned}$$

Easily see that the projection $\pi_i : R \to U$ is étale and surjective for $i = 1, 2$. Let $r_{ij} : R \times U \to U \times U \times U$ be the morphism which maps $((x, y), u)$ to (a_1, a_2, a_3) where $a_i = x$, $a_j = y$ and $a_k = u$ for $k \neq i, j$. Since $\Delta_U \to U \times U$ factors through R, $(\pi_1, \pi_2) = (\pi_2, \pi_1)$ and $r_{12} \times_{(U \times U \times U)} r_{23}$ factors through r_{13}, we have that $h_R(T)$ is an equivalence relation on $h_U(T)$ for all T over S. Then $X := (U, R)$ is an algebraic space.

We do not check the representability here but give an example. Let $U \to X$ be the natural morphism given by $\mathrm{id}_U \in X(U)$. For any scheme T over $\mathbb{C}$, we have

$$(U \times_X U)(T) = \{(f, g) \in h_{U \times U}(T) : \exists \{T_i \to T\} \text{ s.t. } (f_i, g_i) \in h_R(T_i)\} = h_R(T).$$

Hence the fiber product $h_U \times_X h_U$ is represented by R.

We show that $X \not\cong \mathbb{C}^\times$ by computing the the global sections. Consider the covering $U \to X$, a section $s \in \mathcal{O}_X(X)$ is given by a section $s \in \Gamma(U, \mathcal{O}_U) = \mathbb{C}[t]$ such that $\pi_1^* s = \pi_2^* s$ in $\Gamma(R, \mathcal{O}_R)$. This means that $s(x+n) = s(x)$ for all $n \in \mathbb{Z}$. Hence s is a constant function. In particular, $\mathcal{O}_X(X) = \mathbb{C} \neq \mathbb{C}[t, t^{-1}]$.

The underlying set $|X|$ is union of the quotient set $\mathbb{C}/\mathbb{Z}$ and a generic point. The Zariski topology on $|X|$ is the trivial topology.

In following, we will use the technique of *local construction* to construct many scheme-like objects on algebraic spaces. For local construction, see [**Knu71**]. Roughly speaking, for every étale morphism $U \to X$ with U a scheme, we construct a scheme-theoretic object on U which is compatible under base change. Then we glue these objects together to get a global object on X.

Definition 6.2.11. Let X be an algebraic space over a base scheme S. A *coherent sheaf* on X is a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ such that for every covering $\{U_i \to X\}$ with U_i schemes, the sheaf $\mathcal{F}|_{U_i}$ is coherent for every i. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.
An *ideal sheaf* on X is a coherent sheaf $\mathcal{I} \subset \mathcal{O}_X$. It defines a closed subspace $V(\mathcal{I}) \subset X$ by to be completed. And every closed subspace $Y \subset X$ is defined by an ideal sheaf $\mathcal{I}_Y$ such that $V(\mathcal{I}_Y) = Y$.

Definition 6.2.12. Let X be an algebraic space over a base scheme S. A *line bundle* on X is a coherent sheaf $\mathcal{L}$ on X such that for every covering $\{U_i \to X\}$ with U_i schemes, the sheaf $\mathcal{L}|_{U_i}$ is a line bundle on U_i. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

Theorem 6.2.13 (ref. [Stacks, Theorem 76.36.4])**.** Let $f : X \to Y$ be a proper morphism of algebraic spaces over a base scheme S. Then there exists a factorization

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y,$$

where f_1 has geometrically connected fibers and $(f_1)_* \mathcal{O}_X = \mathcal{O}_Z$ and f_2 is finite.

Definition 6.2.14. Let X be an algebraic space over a base scheme S and Y a closed subset of $|X|$. The *formal completion* of X along Y, denoted by $\mathfrak{X}$, is
Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined as $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$ where $\mathcal{I}$ is the ideal sheaf of Y in $\mathcal{O}_X$. to be completed.

Definition 6.2.15. Let X be an algebraic space and Y a closed subset of X. A *modification* of X along Y is a proper morphism $f : X' \to X$ and a closed subset $Y' \subset X'$ such that $X' \setminus Y' \to X \setminus Y$ is an isomorphism and $f^{-1}(Y) = Y'$.

Theorem 6.2.16 (ref. [**Art70**])**.** Let Y' be a closed subset of an algebraic space X' of finite type over $\Bbbk$. Let $\mathfrak{X}'$ be the formal completion of X' along Y'. Suppose that there is a formal modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$. Then there is a unique modification

$$f : X' \to X, \quad Y \subset X$$

such that the formal completion of X along Y is isomorphic to $\mathfrak{X}$ and the induced morphism $\mathfrak{X}' \to \mathfrak{X}$ is isomorphic to $\mathfrak{f}$.

Theorem 6.2.17 (ref. [**Art70**]). Let $\mathfrak{X}'$ be a formal algebraic space and $Y' = V(\mathcal{I}')$ with $\mathcal{I}'$ the defining ideal sheaf of $\mathfrak{X}'$. Let $f : Y' \to Y$ be a proper morphism. Suppose that

(a) for every coherent sheaf $\mathcal{F}$ on $\mathfrak{X}'$, we have

$$R^1 f_* \mathcal{I}'^n \mathcal{F}/\mathcal{I}'^{n+1}\mathcal{F} = 0, \quad \forall n \gg 0;$$

(b) for every n, the homomorphism

$$f_*(\mathcal{O}_{\mathfrak{X}'}/\mathcal{I}'^n) \otimes_{f_*\mathcal{O}_{Y'}} \mathcal{O}_Y \to \mathcal{O}_Y$$

is surjective.

Then there exists a modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ and a defining ideal sheaf $\mathcal{I}$ of $\mathfrak{X}$ such that $V(\mathcal{I}) = Y$ and $\mathfrak{f}$ induces f on Y.

Theorem 6.2.18 (ref. [**Art70**]). Let Y' be a closed algebraic subspace of an algebraic space X' and $f_0 : Y' \to Y$ a finite morphism. Then there exists a modification $f : X' \to X$ whose restriction to Y' is f_0. It is the amalgamated sum $X = X' \amalg_{Y'} Y$ in the category of algebraic spaces **AlgSp**.

Example 6.2.19. Let $X = \mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y]$ and $Y = V(y)$ be the x-axis. Let $f_0 : Y' = \mathbb{A}^1 \to Y, x \mapsto x^2$. Then there exists a modification $f : X' \to X$ such that the restriction $f|_{Y'} : Y' \to Y$ is f_0. To be completed.

6.3 Algebraic stacks

6.3.1 Definitions

Conventions Throughout this section, we fix a base noetherian scheme S. All schemes are viewed as its associated functor of points over S. In other words, we work in the category $\mathrm{Fun}((\mathbf{Sch}/S)^{\mathrm{op}}, \mathbf{Grpd})$. On the base category $\mathbf{Sch}/S$, we consider the étale topology unless otherwise specified.

Definition 6.3.1. A morphism $f : X \to Y$ of stacks is said to be *representable (by schemes)* if for every morphism of schemes $U \to Y$, the fiber product $X \times_Y U$ is a scheme.

Definition 6.3.2. Let P be a property of morphisms of schemes which is stable under base change, for example, being étale, smooth, flat, surjective, etc. A representable morphism of stacks $f : X \to Y$ is said to *satisfy property* P if for every morphism of schemes $U \to Y$, the projection morphism $X \times_Y U \to U$ satisfies property P.

Definition 6.3.3. A *Deligne-Mumford stack* over S is a stack $\mathcal{X}$ over S such that

(a) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, and

(b) there exists a scheme U over S and an étale surjective morphism $U \to \mathcal{X}$.

Definition 6.3.4. An *algebraic stack* over S is a stack $\mathcal{X}$ over S such that

(a) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, and

(b) there exists a scheme U over S and a smooth surjective morphism $U \to \mathcal{X}$.

Construction 6.3.5. Let G be a group scheme over S acting on a scheme X over S via a morphism $\sigma : G \times_S X \to X$. The *quotient stack* $[X/G]$ is defined as following:

- For each scheme U over S, the objects of $[X/G](U)$ are pairs (P, f) where P is a G-torsor over U and $f : P \to X$ is a G-equivariant morphism over S.

- Morphisms between two objects (P, f) and (P', f') in $[X/G](U)$ are given by G-equivariant morphisms $\varphi : P \to P'$ over U such that $f' \circ \varphi = f$.

The assignment $U \mapsto [X/G](U)$ defines a stack over the site $(\mathbf{Sch}/S)_{\text{ét}}$. This stack captures the quotient of X by the action of G in a way that respects the group action and the torsor structure. To be added.

Example 6.3.6. Let $\mathbb{k}$ be a field. Consider the projective plane $\mathbb{P}^2_{\mathbb{k}}$ over $\mathbb{k}$ and all cubic curve $C \subseteq \mathbb{P}^2_{\mathbb{k}}$. Its moduli stack $\mathcal{M}$ of cubic curves is an algebraic stack. To be revised.

Chapter 7

Moduli Spaces

7.1 Introduction to moduli problems

7.1.1 Moduli problem by representable functors

Moduli space is a geometric space whose points represent isomorphism classes of certain geometric objects. For example, fix a field $\mathbb{k}$, all elliptic curves over $\mathbb{k}$ can be classified by the j-invariant, which gives a bijection between isomorphism classes of elliptic curves and elements of $\mathbb{k}$. However, this classification is just a set-theoretic one. We would like to have a geometric "parameter space" such that we can "deform" the objects continuously. This is the initial motivation for moduli spaces.

In algebraic geometry, "deforming objects continuously" is usually described by flat families. Hence the most perfect object to represent this moduli problem is such a flat family $\mathcal{X} \to M$, where M is a variety parameterizing all elliptic curves, and the fiber over each point $m \in M(\mathbb{k})$ is the elliptic curve corresponding to m. Furthermore, we hope that this family is universal, i.e., any other flat family $\mathcal{Y} \to B$ of elliptic curves is obtained by pulling back $\mathcal{X} \to M$ along a unique morphism $B \to M$. (Despite in this example, such a perfect family does not exist; we will discuss this later.) It can be described by the language of functors. Consider the functor

$$\mathcal{M} : \mathbf{Var}_{\mathbb{k}}^{\mathrm{op}} \to \mathbf{Set}, \quad X \mapsto \{\text{flat families of elliptic curves over } X\}/\sim,$$

where $\sim$ is the equivalence relation given by isomorphisms of families over X. If the family $\mathcal{X} \to M$ above exists, then it represents the functor $\mathcal{M}$. In this case, we say that the moduli problem $\mathcal{M}$ is *representable*, M is called a *fine moduli space* and $\mathcal{X} \to M$ is called a *universal object*.

Hence, to study a moduli problem, we follow the steps:

1. Define the moduli functor $\mathcal{M}$.

2. Check whether $\mathcal{M}$ is representable.

Slogan

Moduli problem $\longleftrightarrow$ *Find a representable functor describing "deformations".*

Definition 7.1.1. Let S be a noetherian scheme. A *moduli functor* over S is a functor $\mathcal{M} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$.

Remark 7.1.2. Let S be a noetherian scheme and $\mathcal{M} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ be a functor. In other words, $\mathcal{M}$ is a presheaf on the category $\mathbf{Sch}_S$ with values in $\mathbf{Set}$. By Example 6.1.9, we can view $\mathcal{M}$ as a category fibred in sets over $\mathbf{Sch}_S$.

For every $X \in \mathbf{Sch}_S$, the fiber category $\mathcal{M}(X)$ is the set of "deformation families over X". If $x \in \mathbf{Sch}_S$ is a point, then the fiber category $\mathcal{M}(x)$ is the isomorphism classes of objects represented by the point x.

Definition 7.1.3. Let S be a noetherian scheme and $\mathcal{M} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ be a functor.
If $\mathcal{M}$ is representable by a scheme M of finite type over S, then we say that M is a *fine moduli space* for the moduli problem $\mathcal{M}$. The object $\mathcal{U} \in \mathcal{M}(M)$ corresponding to the identity morphism $\mathrm{id}_M \in \mathrm{Hom}_{\mathbf{Sch}_S}(M, M)$ is called a *universal object* over M.

We give a simple example of a fine moduli space.

Example 7.1.4. Consider the moduli problem of lines in the projective plane $\mathbb{P}^2_{\Bbbk}$ over a field $\Bbbk$. Define the moduli functor

$$\mathcal{G} : \mathbf{Sch}_{\Bbbk}^{\mathrm{op}} \to \mathbf{Set}, \quad X \mapsto \{L \subset \mathbb{P}^2 \times X | L \text{ is flat over } X, L_x \text{ is a line in } \mathbb{P}^2 \text{ for all } x \in X(\Bbbk)\}.$$

We claim that the dual projective plane $G = \mathbb{P}^2_{\Bbbk}$ is a fine moduli space for the moduli problem $\mathcal{G}$. The universal object is given by

$$U = \{([x : y : z], [a : b : c]) \in \mathbb{P}^2_{\Bbbk} \times G | ax + by + cz = 0\} \subset \mathbb{P}^2_{\Bbbk} \times G.$$

For every $X \in \mathbf{Sch}_{\Bbbk}$ and $f \in G(X)$, we can give a family of lines $L = U \times_G X \in \mathcal{G}(X)$ by pulling back the universal object U along $\mathrm{id}_{\mathbb{P}^2_{\Bbbk}} \times f : \mathbb{P}^2_{\Bbbk} \times X \to \mathbb{P}^2_{\Bbbk} \times G$. The difficult part is to construct the inverse map, i.e., given a family of lines $L \in \mathcal{G}(X)$, we need to construct a morphism $f : X \to G$ such that L is obtained by pulling back U along $\mathrm{id}_{\mathbb{P}^2_{\Bbbk}} \times f$.

We need a more "functorial" way to describe the dual projective plane G. Set $V = H^0(\mathcal{O}_{\mathbb{P}^2_{\Bbbk}}(1))$, let $G = \mathrm{Proj}(\mathrm{Sym}^{\bullet} V^{\vee})$ be the dual projective plane. To give a morphism $f : X \to G$, it is equivalent to giving a line bundle $\mathcal{L}$ on X and a surjective morphism $\mathcal{O}_X \otimes_{\Bbbk} V^{\vee} \twoheadrightarrow \mathcal{L}$ by To be added..

Let $\mathcal{I}_L$ be the ideal sheaf of L in $\mathbb{P}^2_X = \mathbb{P}^2 \times X$. Consider the short exact sequence on $\mathbb{P}^2_X$:

$$0 \to \mathcal{I}_L(1) \to \mathcal{O}_{\mathbb{P}^2_X}(1) \to \mathcal{O}_L(1) \to 0.$$

Since L is a family of lines, we have $\mathcal{I}_L(1)|_{\mathbb{P}^2 \times \{x\}} \cong \mathcal{O}_{\mathbb{P}^2}$ for all $x \in X(\Bbbk)$. By Theorem of Cohomology and Base Change (To be added.), we have $\mathcal{L}^{\vee} \coloneqq (\mathrm{pr}_X)_*(\mathcal{I}_L(1))$ is a line bundle on X and $\mathrm{R}^1(\mathrm{pr}_X)_*\mathcal{I}_L(1) = 0$. Then, pushing forward the above short exact sequence along the projection $\mathrm{pr}_X : \mathbb{P}^2_X \to X$ gives a short exact sequence

$$0 \to \mathcal{L}^{\vee} \to \mathcal{O}_X \otimes_{\Bbbk} V \to (\mathrm{pr}_X)_*\mathcal{O}_L(1) \to 0.$$

Dualizing it gives a surjective morphism $\mathcal{O}_X \otimes_{\Bbbk} V^{\vee} \twoheadrightarrow \mathcal{L}$. In particular, if $X = G$ and $L = U$ is the universal family, we get the line bundle $\mathcal{U} = \mathcal{O}_G(1)$ on G and the surjective morphism $\mathcal{O}_G \otimes_{\Bbbk} V^{\vee} \twoheadrightarrow \mathcal{O}_G(1)$ corresponding to the identity morphism id_G.

Let $f : X \to G$ be the morphism induced by the surjective morphism $\mathcal{O}_X \otimes_{\Bbbk} V^\vee \twoheadrightarrow \mathcal{L}$. Then we have the following commutative diagram:(Why?)

$$\begin{tikzcd}
f^*(\mathcal{O}_G \otimes_{\Bbbk} V^\vee) \arrow[r] \arrow[d, "\cong"] & f^*\mathcal{U} \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathcal{O}_X \otimes_{\Bbbk} V^\vee \arrow[r] & \mathcal{L} \arrow[r] & 0.
\end{tikzcd}$$

Take duals and pull back along $\mathrm{pr}_X : \mathbb{P}^2_X \to X$, we get a commutative diagram: (To be added.)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_X^* f^*\mathcal{U}^\vee \arrow[r] \arrow[d, "\cong"] & \mathrm{pr}_X^* f^*\mathcal{O}_G \otimes_{\Bbbk} V \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{pr}_X^* \mathcal{L}^\vee \arrow[r] & \mathrm{pr}_X^* \mathcal{O}_X \otimes_{\Bbbk} V.
\end{tikzcd}$$

Note that $\mathcal{L}^\vee = (\mathrm{pr}_X)_*(\mathcal{I}_L(1))$ and $\mathcal{U}^\vee = (\mathrm{pr}_G)_*(\mathcal{I}_U(1))$. Hence, by cohomology commuting with flat base change To be added., we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_X^*(\mathrm{pr}_X)_*\Big((\mathrm{id}_{\mathbb{P}^2} \times f)^*\mathcal{I}_U(1)\Big) \arrow[r] \arrow[d, "\cong"] & \mathrm{pr}_X^*\Big(f^*\mathcal{O}_G \otimes_{\Bbbk} V\Big) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{pr}_X^*(\mathrm{pr}_X)_*\Big(\mathcal{I}_L(1)\Big) \arrow[r] & \mathrm{pr}_X^*\Big(\mathcal{O}_X \otimes_{\Bbbk} V\Big).
\end{tikzcd}$$

Note that $(\mathrm{pr}_X)_*\mathcal{I}_L(1)$ is a line bundle and hence we have $\mathrm{pr}_X^*(\mathrm{pr}_X)_*(\mathcal{I}_L(1)) \cong \mathcal{I}_L(1)$. Together with the natural surjective homomorphism $\mathrm{pr}_X^*(\mathcal{O}_X \otimes_{\Bbbk} V) \to \mathcal{O}_{\mathbb{P}^2_X}(1)$, we have

$$\begin{tikzcd}
 & & \mathrm{pr}_X^*\Big(f^*\mathcal{O}_G \otimes_{\Bbbk} V\Big) \arrow[dr, two heads] & \\
0 \arrow[r] & (\mathrm{id}_{\mathbb{P}^2} \times f)^*\mathcal{I}_U(1) \arrow[ur] \arrow[rr] \arrow[d, "\cong"] & & \mathcal{O}_{\mathbb{P}^2_X}(1) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathcal{I}_L(1) \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^2_X}(1). \\
 & & \mathrm{pr}_X^*\Big(\mathcal{O}_X \otimes_{\Bbbk} V\Big) \arrow[ur, two heads] &
\end{tikzcd}$$

Why this homomorphism injective and why it is the natural inclusion? After identifying the last vertical isomorphism, we have the equality of sheaf ideals $(\mathrm{id}_{\mathbb{P}^2} \times f)^*\mathcal{I}_U = \mathcal{I}_L$. It follows that $L = U \times_G X$ and we are done.

7.1.2 Coarse moduli space

7.2 The Quot functor

7.2.1 Definitions and examples

Definition 7.2.1. Let S be a noetherian scheme, $(X, \mathcal{O}(1))$ a projective scheme over S, and $\mathcal{E}$ a vector bundle on X. For a polynomial $P \in \mathbb{Q}[\lambda]$, The *Quot functor* $\mathfrak{Quot}_{\mathcal{E}/X/S,P} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ is defined as

$$T \mapsto \left\{ \mathcal{E}_{X_T} \to \mathcal{Q} \to 0 \text{ on } X_T \;\middle|\; \mathcal{Q} \text{ is flat over } T, \text{ and } P_{\mathcal{Q}|_{X_\xi}} = P, \ \forall \xi \in T \right\} / \sim,$$

where two quotients $\mathcal{E}_{X_T} \to \mathcal{Q} \to 0$ and $\mathcal{E}_{X_T} \to \mathcal{Q}' \to 0$ are equivalent if there is an isomorphism $\mathcal{Q} \cong \mathcal{Q}'$ making the diagram

$$\begin{tikzcd} \mathcal{E}_{X_T} \arrow[r] \arrow[dr] & \mathcal{Q} \arrow[d, "\cong"] \\ & \mathcal{Q}' \end{tikzcd}$$

commute.

The main goal of this section is to prove the following representability theorem.

Theorem 7.2.2. The Quot functor $\mathfrak{Quot}_{\mathcal{E}/X/S,P}$ is representable by a projective S-scheme $\mathrm{Quot}_{\mathcal{E}/X/S,P}$ and a universal quotient $p_X^* \mathcal{E} \to \mathcal{Q} \to 0$ on $X \times_S \mathrm{Quot}_{\mathcal{E}/X/S,P}$. To be checked.

Many important moduli spaces can be realized as special cases of the Quot scheme.

Grassmannian scheme The first example is the Grassmannian scheme.

Definition 7.2.3. Let S be a noetherian scheme and $\mathcal{E}$ a vector bundle of rank n on S. The *Grassmannian functor* $\mathfrak{Grass}_{\mathcal{E},r} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ is defined as

$$T \mapsto \{ \mathcal{E}_T \to \mathcal{Q} \to 0 \mid \mathcal{Q} \text{ locally free of rank } r \text{ on } T \} / \sim,$$

where two quotients $\mathcal{E}_T \to \mathcal{Q} \to 0$ and $\mathcal{E}_T \to \mathcal{Q}' \to 0$ are equivalent if there is an isomorphism $\mathcal{Q} \cong \mathcal{Q}'$ making the diagram

$$\begin{tikzcd} \mathcal{E}_{T} \arrow[r] \arrow[dr] & \mathcal{Q} \arrow[d, "\cong"] \\ & \mathcal{Q}' \end{tikzcd}$$

commute.

Let $i : \xi \to S$ be a point of S. Then the fiber $\mathcal{Q}|_\xi = i^* \mathcal{Q}$ is a vector space over the residue field $\kappa(\xi)$ of dimension r. By taking $(X, \mathcal{O}(1)) = (S, \mathcal{O}_S)$ and $P(\lambda) = r$, the Grassmannian functor $\mathfrak{Grass}_{\mathcal{E},r}$ is a special case of the Quot functor $\mathfrak{Quot}_{\mathcal{E}/X/S,P}$.

Let us further specialize to the case where $S = \operatorname{Spec} \mathbf{k}$ for a field $\mathbf{k}$ and $\mathcal{E} = V = \mathbf{k}^{\oplus n}$ is a finite-dimensional $\mathbf{k}$-vector space. Note that $V \to W \to 0$ is equivalent to $V \to W' \to 0$ if and only if $\ker(V \to W) = \ker(V \to W')$. Hence in this case, the Grassmannian functor $\mathfrak{Grass}_{\mathcal{E},r}$ becomes the classical Grassmannian variety $\mathrm{Gr}(n-r,V)$ parameterizing $n-r$-dimensional subspaces of $\mathbf{k}^{\oplus n}$.

More specially, when $r=1$ or $r=n-1$, the Grassmannian variety $\mathrm{Gr}(n-r,V)$ is the projective space $\mathbb{P}^{n-1}_{\mathbf{k}}$. However, although the space is the same, the universal object is different. When $r=1$, i.e., $\mathbb{P}^{n-1}_{\mathbf{k}}$ parameterizes quotients $V\to W\to 0$ where W is a one-dimensional vector space, the universal object is

$$\bigoplus_{i=1}^{n}\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}}\cdot e_i\to\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}}(1)\to 0,\quad e_i\mapsto x_i,$$

where $x_1,\dots,x_n$ are the homogeneous coordinates on $\mathbb{P}^{n-1}_{\mathbf{k}}$. When $r=n-1$, i.e., $\mathbb{P}^{n-1}_{\mathbf{k}}$ parameterizes quotients $V\to W\to 0$ where W is an $(n-1)$-dimensional vector space, the universal object is

$$0\to\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}}(-1)\xrightarrow{\varphi}\bigoplus_{i=1}^{n}\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}}\cdot e_i\to\mathcal{Q}\to 0,\quad \varphi(1/x_i)=e_i,$$

where we view $\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}}(-1)$ locally generated by $1/x_i$ on the chart $x_i\neq 0$.

When we say $\mathbb{P}(V)$, we usually mean the case $r=1$. This is also called the projectivization of the vector space V of hyperplanes in V or in the sense of Grothendieck. Under this convention, the functor $\mathbb{P}:\mathbf{vect}_{\mathbf{k}}\to\mathbf{Sch}_{\mathbf{k}}$ sending a finite-dimensional vector space V to the projective space $\mathbb{P}(V)$ is contravariant. Hence one should view V as the space of linear functions rather than points.

To be continued, describe $\mathbb{P}^n_S$ for general S.

Hilbert scheme Another important example is the Hilbert scheme.

Definition 7.2.4. Let S be a noetherian scheme and X a projective scheme over S. For a polynomial $P\in\mathbb{Q}[\lambda]$, the *Hilbert functor* $\mathfrak{Hilb}_{X/S,P}:\mathbf{Sch}_S^{\mathrm{op}}\to\mathbf{Set}$ is defined as

$$T\mapsto\left\{Y\subseteq X_T\;\middle|\;Y\to T\text{ is flat and for all }\xi\in T,\ P_{\mathcal{O}_{Y_\xi}}=P\right\},$$

where Y_ξ is the fiber of Y over the point ξ and $P_{\mathcal{O}_{Y_\xi}}$ is the Hilbert polynomial of $\mathcal{O}_{Y_\xi}$ with respect to $\mathcal{O}_X(1)|_{X_\xi}$.

By taking $\mathcal{E}=\mathcal{O}_X$ and noting that a quotient $\mathcal{O}_{X_T}\to\mathcal{Q}\to 0$ corresponds to a closed subscheme $Y=\mathcal{S}pec_{X_T}\mathcal{Q}\subseteq X_T$, we see that the Hilbert functor $\mathfrak{Hilb}_{X/S,P}$ is a special case of the Quot functor $\mathfrak{Quot}_{\mathcal{E}/X/S,P}$.

To be continued

Morphisms space Yet another example is the morphisms space.

Definition 7.2.5. Let S be a noetherian scheme, X,Y projective schemes over S, and $f:X\to Y$ a morphism over S. The *functor of morphism space through* f $\mathfrak{Mor}_{(X,Y)/S,f}:\mathbf{Sch}_S^{\mathrm{op}}\to\mathbf{Set}$ is defined as

$$T\mapsto\left\{g_T:X_T\to Y_T\text{ over }T\middle|\Gamma_g\text{ is flat over }T\text{ and }P_{\Gamma_{g_\xi}}=P_{\Gamma_{f_\xi}}\text{ for all }\xi\in T\right\},$$

where $X_T=X\times_S T$ and $Y_T=Y\times_S T$. To be revised.

7.2.2 Castelnuovo-Mumford regularity

7.2.3 Construction of Quot scheme

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