Category of sheaves of modules

Mostly results in this section fits into the context of ringed spaces.

1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets.

Definition 2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that there exists a surjective morphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

Remark 3. There are many versions of "local" properties for sheaves of $\mathcal{O}_X$-modules. Yang: To be continued.

Definition 4. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it is finitely generated, and for every open set $U \subseteq X$ and every morphism of sheaves of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of φ is finitely generated.

Slogan

$$\mathbf{Sh}_X(\mathbf{Ab}) \supseteq \mathbf{Mod}_{\mathcal{O}_X} \supseteq \mathbf{QCoh}_X \supseteq \mathbf{Coh}_X.$$

2 As abelian categories

Theorem 5. Let $(X, \mathcal{O}_X)$ be a ringed space. All of $\mathbf{Sh}_X(\mathbf{Ab}), \mathbf{Mod}(\mathcal{O}_X), \mathbf{QCoh}_X, \mathbf{Coh}_X$ are abelian categories.

Theorem 6. Let $(X, \mathcal{O}_X)$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules has enough injectives. Yang: To be checked.

Remark 7. The category of sheaves of $\mathcal{O}_X$-modules generally does not have enough projectives. Yang: To be checked.

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3 Relevant functors

Definition 8. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *sheaf Hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf of abelian groups defined as follows: for an open set $U \subseteq X$, we define

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of morphisms of sheaves of $\mathcal{O}_U$-modules from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. For an inclusion of open sets $V \subseteq U$, the restriction map

$$\mathrm{res}_{UV} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(V)$$

is defined by sending a morphism $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ to its restriction $\varphi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. Yang: To be continued.

Definition 9. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules defined as follows: for an open set $U \subseteq X$, we define

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

where $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is the tensor product of $\mathcal{O}_X(U)$-modules. For an inclusion of open sets $V \subseteq U$, the restriction map
Yang: To be continued.

Definition 10. Let $f : X \to Y$ be a morphism of ringed spaces. The *pull-back functor* $f^* : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$ is defined as follows: for an $\mathcal{O}_Y$-module $\mathcal{F}$, we define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

where $f^{-1}\mathcal{F}$ is the inverse image sheaf of $\mathcal{F}$. For a morphism of $\mathcal{O}_Y$-modules $\varphi : \mathcal{F} \to \mathcal{G}$, we define

$$f^*\varphi : f^*\mathcal{F} \to f^*\mathcal{G}$$

to be the morphism induced by the morphism of sheaves of abelian groups $f^{-1}\varphi : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$.
Yang: To be continued.

4 Cohomological theory

Definition 11. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *sheaf cohomology* $H^i(X, \mathcal{F})$ is defined as the i-th right derived functor of the global section functor $\Gamma(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ applied to $\mathcal{F}$, i.e.,

$$H^i(X, \mathcal{F}) := \mathrm{R}^i\Gamma(X, \mathcal{F}).$$

Yang: To be checked.

Definition 12. Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *i-th higher direct image* $\mathsf{R}^i f_* \mathcal{F}$ is defined as the i-th right derived functor of the direct image functor $f_* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ applied to $\mathcal{F}$, i.e.,

$$\mathsf{R}^i f_* \mathcal{F} := \mathsf{R}^i (f_* \mathcal{F}).$$

Yang: To be checked.

Definition 13. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *i-th sheaf Ext functor* $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is defined as the i-th right derived functor of the sheaf Hom functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$ applied to $\mathcal{G}$, i.e.,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \mathsf{R}^i\, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Yang: To be checked.

Proposition 14. Let $f : X \to Y$ be a morphism of ringed spaces, and let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. Then there are long exact sequences of $\mathcal{O}_Y$-modules

$$0 \to f_* \mathcal{F} \to f_* \mathcal{G} \to f_* \mathcal{H} \to \mathsf{R}^1 f_* \mathcal{F} \to \mathsf{R}^1 f_* \mathcal{G} \to \mathsf{R}^1 f_* \mathcal{H} \to \mathsf{R}^2 f_* \mathcal{F} \to \cdots$$

Yang: To be checked.

Theorem 15 (Affine criterion by Serre)**.** Let X be a scheme. Then X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i > 0$. Yang: To be checked.

Theorem 16 (Leray spectral sequence)**.** Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there exists a spectral sequence

$$E_2^{p,q} = H^p(Y, \mathsf{R}^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Yang: To be checked.