Quotient by algebraic group

Everything in this section is over an arbitrary field $\mathbf{k}$ unless otherwise specified.

1 Quotient

Definition 1. Let G be an algebraic group acting on a variety X. A *quotient* of X by G is a variety Y together with a morphism $\pi : X \to Y$ such that

(a) π is G-invariant, i.e., $\pi(g \cdot x) = \pi(x)$ for all $g \in G$ and $x \in X$.

(b) For any variety Z and any G-invariant morphism $f : X \to Z$, there exists a unique morphism $\overline{f} : Y \to Z$ such that $f = \overline{f} \circ \pi$.

In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "\overline{f}"] \\ & Z \end{tikzcd}$$

If a quotient exists, it is unique up to a unique isomorphism. Yang: To be continued...

Theorem 2. Let G be an affine algebraic group acting on a variety X. Then there exists a variety Y and a rational morphism $\pi : X \dashrightarrow Y$ with commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "\overline{f}"] \\ & Z \end{tikzcd}$$

satisfying the following universal property: If a quotient exists, it is unique up to a unique isomorphism.
Furthermore, if all orbits of G in X are closed, then π is a morphism (i.e., defined everywhere). Yang: To be continued... Yang: Ref?

2 Quotient of affine algebraic group by closed subgroup

Lemma 3. Let V be a finite-dimensional vector space over $\mathbf{k}$ and G an abstract group acting linearly on V. Let $W \subseteq V$ be a subspace of dimension m. Then $G.W = W$ if and only if $G. \wedge^m W = \wedge^m W$.

Proof. Yang: To be filled. □

Lemma 4. Let G be an affine algebraic group and H a closed subgroup. Then there exists a finite-dimensional linear representation V of G and a one-dimensional subspace $L \subseteq V$ such that H is the stabilizer of L.

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Proof. Yang: To be filled. □

Theorem 5. Let G be an affine algebraic group and H a closed subgroup. Then the quotient G/H exists as a quasi-projective variety.

Proof. Yang: To be filled. □