Structure of linear algebraic groups

Theorem 1. Let G be a linear algebraic group of dimension 1 over an algebraically closed field $\Bbbk$. Then G is isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$.

Lemma 2. Let G be a linear algebraic group over an algebraically closed field $\Bbbk$. Then G has a one-dimensional algebraic subgroup.

1 Jordan-Chevalley Decomposition

Theorem 3. Let $G \subset \mathrm{GL}_n(\Bbbk)$ be a linear algebraic group over a field $\Bbbk$. Then for every element $g \in G(\Bbbk)$, there exist unique commuting elements $g_s, g_u \in G(\Bbbk)$ such that $g = g_s g_u$, where g_s is semisimple and g_u is unipotent. Moreover, this decomposition is functorial in the sense that for any morphism of linear algebraic groups $\varphi : G \to H$, we have $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$. Yang: To be checked

2 Solvable part

Definition 4. A group G is said to be *solvable* if there exists a finite sequence of subgroups

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = \{e\}$$

such that each G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is abelian for all $0 \leq i < n$. Yang: to be checked.

Definition 5. Let G be a linear algebraic group over a field $\Bbbk$. The *radical* of G, denoted by $\mathrm{rad}(G)$, is defined to be the unique maximal connected normal solvable subgroup of G.

Theorem 6. Let $G \subset \mathrm{GL}_n(\Bbbk)$ be a solvable linear algebraic group over an algebraically closed field $\Bbbk$. Then there exists a basis of $\Bbbk^n$ such that G is contained in the group of upper triangular matrices with respect to this basis.

3 Semisimple

Definition 7. Let G be a linear algebraic group over a field $\Bbbk$.

(a) We say that G is *simple* if G is non-abelian and has no non-trivial proper connected normal algebraic subgroups.

(b) We say that G is *semisimple* if $\mathrm{rad}(G)$ is trivial.

Yang: To be checked.

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Definition 8. Let G be a linear algebraic group over a field $\Bbbk$. We say that G is *reductive* if the unipotent radical of G is trivial. Yang: To be checked.

Appendix