# Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$. This section is mainly based on [Har77, Chapter V.2].

## 1 Preliminaries

Let $S$ be a variety over $\Bbbk$ and $\mathcal{E}$ a vector bundle of rank $r+1$ on $S$.

**Proposition 1.** The $S$-varieties $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on $S$.

**Theorem 2.** Let $\pi : X = \mathbb{P}_S(\mathcal{E}) \to S$ be the projective bundle associated to a vector bundle $\mathcal{E}$ of rank $r+1$ on $S$. Then there is an exact sequence of vector bundles on $\mathbb{P}_S(\mathcal{E})$

$$0 \to \Omega_{\mathbb{P}_S(\mathcal{E})/S} \to \pi^*(\mathcal{E})(-1) \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \to 0.$$

In particular, $K_X \sim \pi^*(K_S + \det \mathcal{E}) - (r+1)\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. Yang: To be continued...

**Theorem 3** (Tsen's Theorem, [Stacks, Tag 03RD])**.** Let $C$ be a smooth curve over an algebraically closed field $\Bbbk$. Then $\mathbf{K} = \Bbbk(C)$ is a $C_1$ field, i.e., every degree $d$ hypersurface in $\mathbb{P}^n_{\mathbf{K}}$ has a $\mathbf{K}$-rational point provided $d \leq n$.

**Theorem 4** (Grauert's Theorem, [Har77, Corollary 12.9])**.** Let $f : X \to S$ be a projective morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf on $X$ which is flat over $S$. Suppose that $S$ is integral and the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$ is constant on $S$ for some $i \geq 0$. Then $\mathsf{R}^i f_*\mathcal{F}$ is locally free and the base change homomorphism

$$\varphi^i_s : \mathsf{R}^i f_*\mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s) \to H^i(X_s, \mathcal{F}_s)$$

is an isomorphism for all $s \in S$.

**Theorem 5** (Miracle Flatness, [Mat89, Theorem 23.1])**.** Let $f : X \to Y$ be a morphism of noetherian schemes. Assume that $Y$ is regular and $X$ is Cohen-Macaulay. If all fibers of $f$ have the same dimension $d = \dim X - \dim Y$, then $f$ is flat.

**Proposition 6** (Geometric form of Nakayama's Lemma)**.** Let $X$ be a variety, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on $X$. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

**Proposition 7.** Let $S$ be a noetherian scheme and $\mathcal{E}$ a vector bundle of rank $r+1$ on $S$. Denote by $\pi : \mathbb{P}_S(\mathcal{E}) \to S$ the projection. Let $X$ be an $S$-scheme via a morphism $g : X \to S$. Then there is a

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bijection

$$\left\{ \begin{array}{l} S\text{-morphisms} \\ X \to \mathbb{P}_S(\mathcal{E}) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \mathcal{L} \in \operatorname{Pic}(X) \text{ and surjective} \\ \text{homomorphisms } g^*\mathcal{E} \to \mathcal{L} \end{array} \right\}.$$

*Proof.* We have a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ by the definition of $\mathbb{P}_S(\mathcal{E})$. If we have a morphism $f : X \to \mathbb{P}_S(\mathcal{E})$ over $S$, then we have a surjective homomorphism $f^*\pi^*\mathcal{E} \to f^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$.

Suppose we have a surjective homomorphism $g^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ where $\mathcal{L}$ is a line bundle on $X$. Take an affine cover $\{U_i\}$ of $S$ such that $\mathcal{E}|_{U_i}$ is trivial. On $U_i$, choose a basis $e_0^{(i)}, \dots, e_r^{(i)}$ of $\mathcal{E}|_{U_i}$. Suppose $\mathbb{P}_S(\mathcal{E})$ is given by gluing $\mathbb{P}^r_{U_i}$ via $\varphi_{ij}$ induced by the transition functions of $\mathcal{E}$.

The surjection $g^*\mathcal{E}|_{U_i} \twoheadrightarrow \mathcal{L}|_{X_{U_i}}$ gives a unique morphism $f_i : X_{U_i} \to \mathbb{P}^r_{U_i}$ by ??. On $X_{U_i \cap U_j}$, $f_i$ and $f_j$ agree since we have

$$\begin{tikzcd}
X_{U_i\cap U_j} \arrow[r, "="] \arrow[d, "f_i"'] & X_{U_i\cap U_j} \arrow[d, "f_j"] \\
\mathbb{P}_{U_i\cap U_j}(\bigoplus \mathcal{O}_{U_i\cap U_j} e_k^{(i)}) \arrow[r, "\varphi_{ij}"] & \mathbb{P}_{U_i\cap U_j}(\bigoplus \mathcal{O}_{U_i\cap U_j} e_k^{(j)})
\end{tikzcd}$$

and the bottom arrow is identical to the identity map on $\mathbb{P}_S(\mathcal{E})_{U_i\cap U_j}$. Gluing $f_i$ gives a morphism $f : X \to \mathbb{P}_S(\mathcal{E})$ over $S$. In particular, we have $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. □

**Definition 8.** An *extension* of a coherent sheaf $\mathcal{F}$ by a coherent sheaf $\mathcal{G}$ on a scheme $X$ is an exact sequence of coherent sheaves

$$S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0).$$

Two extensions $S$ and $S'$ are *equivalent* if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{G}}"] & \mathcal{E} \arrow[r] \arrow[d, "\cong"] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{F}}"] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0.
\end{tikzcd}$$

**Proposition 9.** Let $X$ be a scheme and $\mathcal{F}, \mathcal{G}$ be coherent sheaves on $X$. Then there is a one-to-one correspondence between equivalence classes of extensions

$$S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0)$$

and elements of $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ given by

$$S \mapsto \delta(\mathrm{id}_{\mathcal{F}})$$

where $\delta : \operatorname{Hom}_X(\mathcal{F}, \mathcal{F}) \to \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ is the connecting homomorphism.

*Proof.* Take an exact sequence

$$0 \to \mathcal{G} \to \mathcal{I} \xrightarrow{\varphi} \mathcal{C} \to 0$$

with $\mathcal{I}$ injective. Applying $\operatorname{Hom}_X(\mathcal{F}, -)$ gives a long exact sequence

$$0 \to \operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}_X(\mathcal{F}, \mathcal{I}) \to \operatorname{Hom}_X(\mathcal{F}, \mathcal{C}) \xrightarrow{\delta} \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G}) \to 0.$$

For $a \in \operatorname{Ext}^1_X(\mathcal{F},\mathcal{G})$, choose a lifting $\alpha \in \operatorname{Hom}_X(\mathcal{F},\mathcal{C})$ of $a$. Let $\mathcal{E} := \operatorname{Ker}(\mathcal{I} \oplus \mathcal{F} \to \mathcal{C}, (i,f) \mapsto \varphi(i) - \alpha(f))$.

Let $\mathcal{E} \to \mathcal{F}$ be the projection to the second factor. It is surjective since $\varphi$ is surjective. Consider the inclusion $\mathcal{G} \to \mathcal{I} \to \mathcal{I} \oplus \mathcal{F}$, which factors through $\mathcal{E}$. On the other hand, if $e \in \mathcal{E}$ maps to $0$ in $\mathcal{F}$, then $e \in \mathcal{I}$ and $\varphi(e) = 0$, whence $e \in \mathcal{G}$. Hence we have an extension $S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0)$.

Yang: To be continued... □

## 2 Minimal Section and Classification

**Definition 10** (Ruled surface)**.** A *ruled surface* is a smooth projective surface $X$ together with a surjective morphism $\pi : X \to C$ to a smooth curve $C$ such that all geometric fibers of $\pi$ are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve $C$ of genus $g$.

**Lemma 11.** There exists a section of $\pi$.

*Proof.* Yang: To be continued... □

**Proposition 12.** Then there exists a vector bundle $\mathcal{E}$ of rank $2$ on $C$ such that $X \cong \mathbb{P}_C(\mathcal{E})$ over $C$.

*Proof.* Let $\sigma : C \to X$ be a section of $\pi$ and $D$ be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since $D$ is a section of $\pi$, $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in C$. By Miracle Flatness (Theorem 5), $f$ is flat. By Grauert's Theorem (Theorem 4), $\mathcal{E}$ is a vector bundle of rank $2$ on $C$ and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in C$.

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every $x \in X$, we have

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

Yang: The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \to \mathcal{L}$ is surjective.

By Proposition 7, we have a morphism $\varphi : X \to \mathbb{P}_C(\mathcal{E})$ over $C$ such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Since $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, $\varphi|_{X_t} : X_t \to \mathbb{P}_C(\mathcal{E})_t$ is an isomorphism for any $t \in C$. Hence $\varphi$ is bijection on the underlying sets. Yang: Here is a serious gap. Why fiberwise isomorphism implies isomorphism? □

**Lemma 13.** It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C,\mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on $C$ with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*. In particular, if $\mathcal{E}$ is normalized, then $e = -\deg c_1(\mathcal{E})$ is an invariant of the ruled surface $X$.

*Proof.* We can suppose that $\mathcal{E}$ is globally generated since we can always twist $\mathcal{E}$ by a sufficiently ample line bundle on $C$. Then for all line bundle $\mathcal{L}$ of degree sufficiently large, $\mathcal{L}$ is very ample and hence $H^0(C, \mathcal{E} \otimes \mathcal{L}) \neq 0$. By Lemma 11 and Proposition 7, $\mathcal{E}$ is an extension of line bundles. Then for all line bundle $\mathcal{L}$ of degree sufficiently negative, $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ since line bundles of negative degree have no global sections. Hence we can find a line bundle $\mathcal{M}$ on $C$ of lowest degree such that $H^0(C, \mathcal{E} \otimes \mathcal{M}) \neq 0$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{M}$, we are done. □

**Remark 14.** The invariant $e$ is unique but the normalization of $\mathcal{E}$ is not unique. For example, if $\mathcal{E}$ is normalized, then so is $\mathcal{E} \otimes \mathcal{L}$ for any line bundle $\mathcal{L}$ on $C$ of degree $0$. Yang: To be continued...

Suppose that $X \cong \mathbb{P}_C(\mathcal{E})$ where $\mathcal{E}$ is a normalized vector bundle of rank $2$ on $C$. Since $H^0(C, \mathcal{E}) \neq 0$, choosing a non-zero section $s$, we get an exact sequence

$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{E} \to \mathcal{E}/\mathcal{O}_C \to 0.$$

We claim that $\mathcal{E}/\mathcal{O}_C$ is a line bundle on $C$. Since $C$ is a curve, we only need to check that $\mathcal{E}/\mathcal{O}_C$ is torsion-free.

Yang: To be continued...

**Definition 15.** A section $C_0$ of $\pi$ is called a *minimal section* if Yang: to be continued...

**Lemma 16.** Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface over a smooth curve $C$ of genus $g$ with invariant $e$ and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on $C$ with $\deg \mathcal{L} = -e$.

(b) If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.

*Proof.* If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable, we can assume that $H^0(C, \mathcal{L}_1) \neq 0$. If $\deg \mathcal{L}_1 > 0$, then $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$, contradicting the normalization of $\mathcal{E}$. Similarly $\deg \mathcal{L}_2 \leq 0$. Then $\mathcal{L}_1 \cong \mathcal{O}_C$. And hence $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$.

If $\mathcal{E}$ is indecomposable, we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

which is a non-trivial extension, with $\mathcal{L}$ a line bundle on $C$ of degree $-e$. Hence by Proposition 9, we have $0 \neq \operatorname{Ext}^1_C(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$. By Serre duality, we have $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$. Hence $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$.

On the other hand, let $\mathcal{M}$ be a line bundle on $C$ of degree $-1$. Twist the above exact sequence by $\mathcal{M}$ and take global sections, we have an equation

$$h^0(\mathcal{M}) - h^0(\mathcal{E} \otimes \mathcal{M}) + h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{M}) + h^1(\mathcal{E} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = 0.$$

Since $\deg \mathcal{M} < 0$ and $\mathcal{E}$ is normalized, we have $h^0(\mathcal{M}) = h^0(\mathcal{E} \otimes \mathcal{M}) = 0$. By Riemann-Roch, we have $h^1(\mathcal{M}) = g$ and $h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = -e - 1 + 1 - g$. Hence

$$h^1(\mathcal{E} \otimes \mathcal{M}) = e + 2g \geq 0.$$

This gives $e \geq -2g$. □

**Theorem 17.** Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant $e$. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

*Proof.* This is a direct consequence of Lemma 16. □

**Example 18.** Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$.

Let $C$ be covered by two standard affine charts $U_0, U_1$ with coordinate $u$ on $U_0$ and $v$ on $U_1$ such that $u = 1/v$ on $U_0 \cap U_1$. On $U_i$, let $\mathcal{O}(-e)|_{U_i}$ be generated by $s_i$ for $i = 0, 1$. We have $s_0 = u^e s_1$ on $U_0 \cap U_1$.

On $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$, let $[x_0 : x_1]$ and $[y_0 : y_1]$ be the homogeneous coordinates of $\mathbb{P}^1$ on $X_0$ and $X_1$ respectively. Then the transition function on $X_0 \cap X_1$ is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

**Remark 19.** The surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ is also called the *Hirzebruch surface*.

**Theorem 20.** Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve $E$ with invariant $e$ and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or $-1$, and for each $e$ there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on $E$ with $\deg \mathcal{L} = -e$.

*Proof.* Only the indecomposable case needs a proof. By Lemma 16, we have $-2 \leq e \leq 0$ and a non-trivial extension

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle on $E$ of degree $-e$.

**Case 1.** $e = 0$.

In this case, $\mathcal{L}$ is of degree $0$ and $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \neq 0$. Hence $\mathcal{L} \cong \mathcal{O}_E$. Yang: To be continued...

**Case 2.** $e = -1$.

In this case, $\mathcal{L}$ is of degree $1$ and $H^1(E, \mathcal{L}) \cong H^0(E, \mathcal{L}^{-1}) = 0$. By Riemann-Roch, we have $h^0(E, \mathcal{L}) = 1$.

**Case 3.** $e = -2$.

Yang: To be continued...

□

**Example 21.** Yang: To be continued...

# 3 The Néron-Severi Group of Ruled Surfaces

**Proposition 22.** Let $\pi : X \to C$ be a ruled surface over a smooth curve $C$ of genus $g$. Let $C_0$ be a minimal section of $\pi$ and $F$ a fiber of $\pi$. Then $\operatorname{Pic}(X) \cong \mathbb{Z}[C_0] \oplus \pi^* \operatorname{Pic}(C)$.

*Proof.* Let $D$ be any divisor on $X$ with $D.F = a \in \mathbb{Z}$. Then $D - aC_0$ is numerically trivial on the fibers of $\pi$. Let $\mathcal{L} = \mathcal{O}_X(D - aC_0)$. Then $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$ for any $t \in C$. By Grauert's Theorem (Theorem 4), $\pi_*\mathcal{L}$ is a line bundle on $C$ Yang: and the natural map $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. □

**Proposition 23.** Let $\pi : X \to C$ be a ruled surface over a smooth curve $C$ of genus $g$. Let $C_0$ be a minimal section of $\pi$ and let $F$ be a fiber of $\pi$. Then $K_X \sim -2C_0 + \pi^*(K_C - c_1(\mathcal{E}))$. Numerically, we have $K_X \equiv -2C_0 + (2g - 2 - e)F$ where $e$ is the invariant of $X$. Yang: Check this carefully.

*Proof.* Yang: To be continued. □

**Rational case.** Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

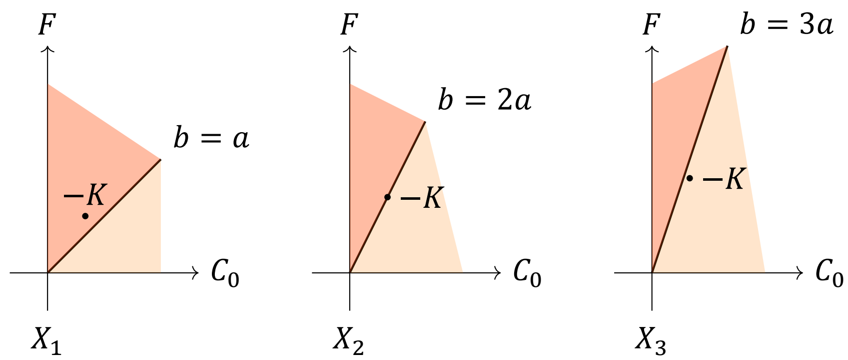
**Theorem 24.** Let $\pi : X \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with invariant $e$. Let $C_0$ be a minimal section of $\pi$ and let $F$ be a fiber of $\pi$. Let $D \sim aC_0 + bF$ be a divisor on $X$ with $a, b \in \mathbb{Z}$.

(a) $D$ is ample $\iff$ $D$ is very ample $\iff$ $a > 0$ and $b > ae$;

(b) $D$ is effective $\iff$ $a, b \geq 0$.

*Proof.* Yang: To be continued... □

**Example 25.** Here we draw the Néron-Severi group of the rational ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 1, 2, 3$.




We have $-K_{X_e} \equiv 2C_0 + (2 + e)F$. For $e = 1$, $-K$ is ample and hence $X_1$ is a del Pezzo surface. For $e = 2$, $-K$ is nef and big but not ample. For $e \geq 3$, $-K$ is big but not nef.

**Elliptic case.** Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve $E$ with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

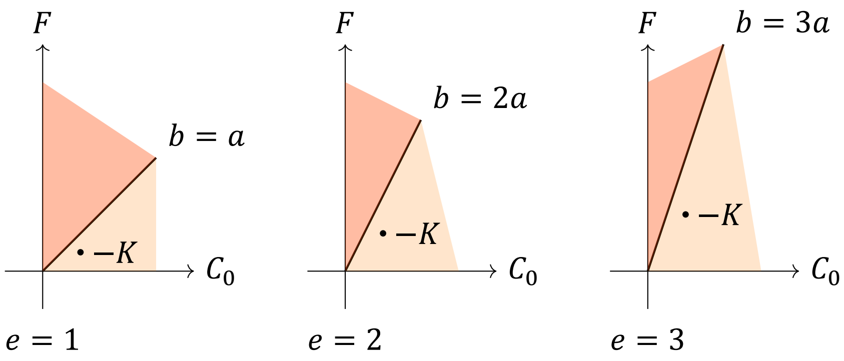
**Theorem 26.** Let $\pi : X \to E$ be a ruled surface over an elliptic curve $E$ with invariant $e$. Assume that $\mathcal{E}$ is decomposable. Let $C_0$ be a minimal section of $\pi$ and let $F$ be a fiber of $\pi$. Let $D \equiv aC_0 + bF$ be a divisor on $X$ with $a, b \in \mathbb{Z}$.

(a) $D$ is ample $\iff$ $D$ is very ample $\iff$ $a > 0$ and $b > ae$;

(b) $D$ is effective $\iff$ $a \geq 0$ and $b \geq ae$.

*Proof.* Yang: To be continued... □

**Example 27.** Here we draw the Néron-Severi group of the ruled surface $X$ over an elliptic curve $E$ with decomposable normalized $\mathcal{E}$ for $e = 1, 2, 3$.




In this case, $-K \equiv 2C_0 + eF$ is always big but not nef.

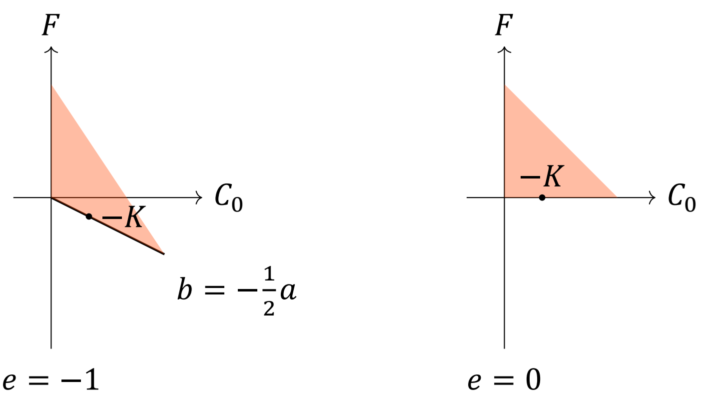
**Theorem 28.** Let $\pi : X \to E$ be a ruled surface over an elliptic curve $E$ with invariant $e$. Assume that $\mathcal{E}$ is indecomposable. Let $C_0$ be a minimal section of $\pi$ and let $F$ be a fiber of $\pi$. Let $D \equiv aC_0 + bF$ be a divisor on $X$ with $a, b \in \mathbb{Z}$.

(a) $D$ is ample $\iff$ $D$ is very ample $\iff$ $a > 0$ and $b > \frac{1}{2}ae$;

(b) $D$ is effective $\iff$ $a \geq 0$ and $b \geq \frac{1}{2}ae$.

*Proof.* Yang: To be continued... □

**Example 29.** Here we draw the Néron-Severi group of the ruled surface $X$ over an elliptic curve $E$ with indecomposable normalized $\mathcal{E}$ for $e = -1, 0$.




In this case, $-K \equiv 2C_0 + eF$ is always nef but not big.

**Proposition 30.** Let $\pi : X \to C$ be a ruled surface over a smooth curve $C$. Then every nef divisor on $X$ is semi-ample. Yang: Check this carefully.

# References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