Category of sheaves of modules

1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 1. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F}$ of abelian groups on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\mathrm{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear, where the $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(V)$ is induced by the restriction map $\mathrm{res}_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.
A *morphism of $\mathcal{O}_X$-modules* is a morphism of sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ such that for every open set $U \subseteq X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear.

Definition 2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets. Yang: To be checked...

Definition 3. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that there exists a surjective morphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

Definition 4. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it is finitely generated, and for every open set $U \subseteq X$ and every morphism of sheaves of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of φ is finitely generated. Yang: To be checked...

2 As abelian categories

Theorem 5. Let X be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules is an abelian category. Yang: To be continued...

Theorem 6. Let X be a scheme. The category of quasi-coherent sheaves on X is an abelian category. Yang: To be continued...

Theorem 7. Let X be a noetherian scheme. The category of coherent sheaves on X is an abelian category. Yang: To be continued...

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3 Relevant functors

Theorem 8. Let X be a ringed space. The global sections functor

$$\Gamma(X,-) : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\mathcal{O}_X(X)\text{-modules})$$

is left exact. Yang: To be continued...

Theorem 9. Let $f : X \to Y$ be a morphism of ringed spaces. The direct image functor

$$f_* : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_Y\text{-modules})$$

is left exact. Yang: To be continued...

Theorem 10. Let $f : X \to Y$ be a morphism of ringed spaces. The inverse image functor

$$f^* : (\text{Sheaves of } \mathcal{O}_Y\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_X\text{-modules})$$

is right exact. Yang: To be continued...

4 Locally free sheaves and vector bundles

Definition 11. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **locally free** if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to a finite free $\mathcal{O}_U$-module, i.e., there exists an isomorphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{F}|_U \cong \mathcal{O}_U^n,$$

where n is a finite integer called the **rank** of $\mathcal{F}$ at x. Yang: To be continued...

Example 12. A **line bundle** on a scheme X is a locally free sheaf of rank 1. The sheaf of differentials $\Omega_{X/k}$ on a smooth variety X over a field k is a locally free sheaf of rank equal to the dimension of X. Yang: To be continued...

Theorem 13. Let X be a scheme. There is an equivalence of categories between the category of locally free sheaves of finite rank on X and the category of vector bundles on X. Yang: To be continued...

5 Cohomological theory

Theorem 14. Let X be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then the cohomology groups $H^i(X,\mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

Theorem 15. Let X be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. Then the cohomology groups $H^i(X,\mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

Theorem 16. Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...