Flat, smooth and étale morphisms

1 Flat families

Definition 1. Let $f : X \to Y$ be a morphism of schemes. For a point $\xi \in X$, we say that f is *flat at* ξ if the local ring $\mathcal{O}_{X,\xi}$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that f is *flat* if it is flat at every point $\xi \in X$.

Definition 2. Let X be Y-scheme via a morphism $f : X \to Y$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *flat over* Y *at* $\xi \in X$ if the stalk $\mathcal{F}_\xi$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that $\mathcal{F}$ is *flat over* Y if it is flat over Y at every point $\xi \in X$.

Proposition 3. We have the following fundamental properties of flat morphisms:

(a) open immersions are flat;

(b) the composition of flat morphisms is flat;

(c) flatness is preserved under base change;

(d) a coherent sheaf $\mathcal{F}$ on a noetherian scheme X is flat over X iff it is locally free.

Yang: To be checked.

Proposition 4. Let X be a regular integral scheme of dimension 1 and $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is flat over X iff it is torsion-free, i.e., for every non-zero-divisor $s \in \mathcal{O}_{X,x}$, the multiplication map

$$s : \mathcal{F} \to \mathcal{F}$$

is injective. Yang: To be checked.

Proposition 5. Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field $\mathbf{k}$. Then for every point $\xi \in X$, we have

$$\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}.$$

Yang: To be checked.

Theorem 6 (Miracle flatness)**.** Let $f : X \to Y$ be a morphism between noetherian schemes. Suppose that X is Cohen–Macaulay and that Y is regular. Then f is flat at $\xi \in X$ iff $\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}$. Yang: To be checked.

Theorem 7. Let T be a integral noetherian scheme and $f : X \to T$ be a projective morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix a relatively ample line bundle H on X over T. Then $\mathcal{F}$ is flat over

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T iff the Hilbert polynomials

$$P(X_t, \mathcal{F}_t, H_t)(n) = \chi(X_t, \mathcal{F}_t \otimes H_t^{\otimes n})$$

are independent of $t \in T$. Yang: To be checked.

Yang: To be added: deformation, algebraic families...

2 Base change and semicontinuity

Theorem 8 (Grauert's theorem)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the sheaf $R^i f_* \mathcal{F}$ is coherent on Y, and there exists an open subset $U \subseteq Y$ such that for every point $y \in U$, the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism. Yang: To be checked.

Theorem 9 (Cohomology and base change)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. For each integer $i \geq 0$, the following are equivalent:

(a) the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism for all points $y \in Y$;

(b) the sheaf $R^i f_* \mathcal{F}$ is locally free on Y.

Yang: To be checked.

Theorem 10 (Semicontinuity of cohomology)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the function

$$h^i : Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is upper semicontinuous on Y.

Yang: To be checked.

3 Smooth morphisms

Definition 11. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. For $\xi \in X$ with image $\zeta = f(\xi) \in Y$, set $\overline{\zeta} : \operatorname{Spec} \overline{\kappa(\zeta)} \to Y$ to be the geometric point over ζ and $X_{\overline{\zeta}}$ be the geometric fiber over ζ. We say that f is *smooth at* ξ if f is flat at ξ and the geometric fiber $X_{\overline{\zeta}}$ is regular over $\overline{\kappa(\zeta)}$ at every point lying over ξ. We say that f is *smooth* if it is smooth at every point

$\xi \in X$.

Yang: To be checked.

4 Étale morphisms

Definition 12. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. We say that f is *étale at* ξ if f is smooth and finite at ξ. We say that f is *étale* if it is étale at every point $\xi \in X$.

Yang: To be checked.