

Stacks in category theory

1 Fibered categories and descent conditions

Definition 1. Let \mathbf{S} be a category and $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ a functor. A morphism $f : b \rightarrow a$ in \mathbf{X} is called *strongly Cartesian* if for every object $c \in \text{Obj}(\mathbf{X})$, the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{X}}(c, b) & \xrightarrow{f \circ -} & \text{Hom}_{\mathbf{X}}(c, a) \\ \mathbf{p} \downarrow & & \downarrow \mathbf{p} \\ \text{Hom}_{\mathbf{S}}(w, v) & \xrightarrow{\mathbf{p}(f) \circ -} & \text{Hom}_{\mathbf{S}}(w, u) \end{array}$$

is a pullback of sets, where $u = \mathbf{p}(a), v = \mathbf{p}(b), w = \mathbf{p}(c)$.

The condition in [Definition 1](#) can be interpreted as follows: for any diagram as below black part with $\mathbf{p}(g) = \mathbf{p}(f) \circ \alpha$,

$$\begin{array}{ccccc} c & \xrightarrow{h} & b & \xrightarrow{f} & a \\ \downarrow & \nearrow g & \downarrow & & \downarrow \\ w & \xrightarrow{\alpha} & v & \xrightarrow{\mathbf{p}(f)} & u \end{array}$$

there exists a unique gray morphism $h : c \rightarrow a$ such that $\mathbf{p}(h) = \alpha$ and $f \circ h = g$.

Notation 2. Let \mathbf{S} be a category and $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ a functor. For $a, b \in \text{Obj}(\mathbf{X})$ and $f \in \text{Hom}_{\mathbf{X}}(a, b)$, we say that a is *over* $\mathbf{p}(a)$ and f is *over* $\mathbf{p}(f)$. In a diagram, we have

$$\begin{array}{ccc} X & \xrightarrow{f} & b \\ \mathbf{p} \downarrow & & \downarrow \\ p(a) & \xrightarrow{\mathbf{p}(f)} & p(b) \end{array}$$

Definition 3. Let \mathbf{S} be a category. A category \mathbf{X} over \mathbf{S} via \mathbf{p} is called a *category fibred* over the site \mathbf{S} if for every morphism $\iota : v \rightarrow u$ in \mathbf{S} and every object $a \in \text{Obj}(\mathbf{X})$ over u , there exists an object $b \in \text{Obj}(\mathbf{X})$ over v and a strongly Cartesian morphism $f : b \rightarrow a$ over ι . Such an object b is called a *pullback* of a along ι , and is often denoted by ι^*a .

Definition 4. Let \mathbf{S} be a site and $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ a category fibred over \mathbf{S} . For every object $u \in \text{Obj}(\mathbf{S})$, the *fiber* of \mathbf{X} over u is the category \mathbf{X}_u given by

$$\text{Obj}(\mathbf{X}_u) = \{a \in \text{Obj}(\mathbf{X}) \mid \mathbf{p}(a) = u\}, \quad \text{Hom}_{\mathbf{X}_u}(a, b) = \{f \in \text{Hom}_{\mathbf{X}}(a, b) \mid \mathbf{p}(f) = \text{id}_u\}.$$

Remark 5. Note that in [Definition 3](#), the pullback r^*b of an object b along a morphism r is not necessarily unique. **Yang:** To be continued.

Example 6. Let \mathbf{S} be a category and $\mathcal{F} : \mathbf{S}^{op} \rightarrow \mathbf{Set}$ be a presheaf on \mathbf{S} taking values in \mathbf{Set} . We can construct a category \mathbf{F} fibred over \mathbf{S} as follows:

- The objects of \mathbf{F} are pairs (U, x) where $U \in \text{Obj}(\mathbf{S})$ and $x \in \mathcal{F}(U)$;
- morphisms from (V, y) to (U, x) in \mathbf{F} are morphisms $\iota : V \rightarrow U$ in \mathbf{S} such that $\mathcal{F}(\iota)(x) = y$, denoted by res_ι .

The functor $\mathbf{p} : \mathbf{F} \rightarrow \mathbf{S}$ is defined by $\mathbf{p}(U, x) = U$ on objects and $\mathbf{p}(\iota) = \iota$ on morphisms. If one has the diagram

$$\begin{array}{ccccc}
 & (W, z) & & & \\
 \downarrow & & \nearrow \text{res}_\tau & & \\
 & (V, y) & \xrightarrow{\text{res}_\iota} & (U, x) & \\
 \downarrow & & & \downarrow & \\
 W & \xrightarrow{\sigma} & V & \xrightarrow{\iota} & U
 \end{array}$$

with $\mathbf{p}(\text{res}_\tau) = \iota \circ \sigma$. By definition, we have $\tau = \iota \circ \sigma$ and $\mathcal{F}(\tau)(x) = z, \mathcal{F}(\iota)(x) = y$. Thus, we have $\mathcal{F}(\sigma)(y) = z$. This verifies that res_σ is a strongly Cartesian morphism. Note that the fiber of \mathbf{F} over an $U \in \text{Obj}(\mathbf{S})$ is the discrete category associated to the set $\mathcal{F}(U)$. Therefore, presheaves of sets can be viewed as categories fibred in sets.

Conversely, given a category \mathbf{F} fibred in sets over \mathbf{S} via $\mathbf{p} : \mathbf{F} \rightarrow \mathbf{S}$, one can construct a presheaf of sets $\mathcal{F} : \mathbf{S}^{op} \rightarrow \mathbf{Set}$ by defining $\mathcal{F}(U) = \text{Obj}(\mathbf{F}_U)$ for each $U \in \text{Obj}(\mathbf{S})$, and for each morphism $\iota : V \rightarrow U$ in \mathbf{S} , defining $\mathcal{F}(\iota) : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ by sending an object $x \in \mathcal{F}(U)$ to its pullback $\iota^*x \in \mathcal{F}(V)$ along ι . This establishes an equivalence between presheaves of sets on \mathbf{S} and categories fibred in sets over \mathbf{S} .

Example 7. Yang: case $\mathbf{S} = \text{set, group}$. To be added.

Slogan Presheaves of sets are categories fibered in sets.

In following, we describe categories fibered in groupoids.

Definition 8. Let \mathbf{X} be a category fibred over a category \mathbf{S} via $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$. For every $u \in \text{Obj}(\mathbf{S})$ and every pair of objects a, b over u , we define the *presheaf of morphisms* $\text{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{op} \rightarrow \mathbf{Set}$ by

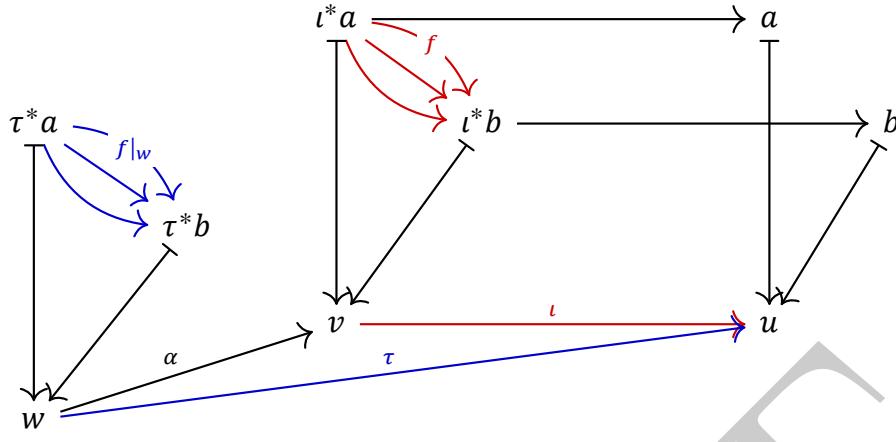
$$\text{Hom}_{\mathbf{X}}(a, b)(\iota : v \rightarrow u) = \text{Hom}_{\mathbf{X}_v}(\iota^*a, \iota^*b)$$

for every morphism $\iota : v \rightarrow u$ in \mathbf{S}/u . For a morphism $\alpha : w \rightarrow v$ in \mathbf{S}/u , the restriction map

$$\text{Hom}_{\mathbf{X}}(a, b)(\iota) \rightarrow \text{Hom}_{\mathbf{X}}(a, b)(\iota \circ \alpha)$$

is given by sending a morphism $f : \iota^*a \rightarrow \iota^*b$ in \mathbf{X}_v to the pullback morphism Yang: $\alpha^*f : (\iota \circ \alpha)^*a \rightarrow (\iota \circ \alpha)^*b$ need to conjugate with a natural transformation. in \mathbf{X}_w . Yang: To be checked.

In a diagram, the presheaf of morphisms can be visualized as follows:



Proposition 9. Let \mathbf{S} be a category and $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ a category fibred over \mathbf{S} . Then \mathbf{X} is a category fibred in groupoids if and only if for every object $u \in \text{Obj}(\mathbf{S})$ and every pair of objects a, b over u , the presheaf of morphisms $\text{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\text{op}} \rightarrow \mathbf{Set}$ is a sheaf. **Yang:** To be checked.

Definition 10. Let \mathbf{S} be a category. A category \mathbf{X} fibred over \mathbf{S} via $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ is called a *category fibred in groupoids* over \mathbf{S} if for every object $u \in \text{Obj}(\mathbf{S})$ and every pair of objects a, b over u , the presheaf of morphisms $\text{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\text{op}} \rightarrow \mathbf{Set}$ is a sheaf. **Yang:** To be checked.

Now let us discuss how sheaves fit into the framework of fibered categories. Of course, we need assume the base category \mathbf{S} is a site. The glued condition for sheaves can be interpreted in terms of descent data in fibered categories.

Definition 11. Let \mathbf{S} be a site and $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ a fibered category over \mathbf{S} . Let $U \in \text{Obj}(\mathbf{S})$ and $\{U_i \rightarrow U\}$ be a covering in \mathbf{S} . A *descent datum* for objects of \mathbf{X} relative to the covering $\{U_i \rightarrow U\}$ consists of

- a collection of objects $a_i \in \text{Obj}(\mathbf{X}_{U_i})$ for each i ,
- a collection of isomorphisms $\varphi_{ij} : a_j|_{U_{ij}} \rightarrow a_i|_{U_{ij}}$ in $\mathbf{X}_{U_{ij}}$ for each pair (i, j) , where $U_{ij} = U_i \times_U U_j$,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}}$$

holds for all triples (i, j, k) , where $U_{ijk} = U_i \times_U U_j \times_U U_k$. **Yang:** To be checked.

Example 12. **Yang:** To be added.

Definition 13. Let \mathbf{S} be a site and $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ a fibered category over \mathbf{S} . A descent datum $(\{a_i\}, \{\varphi_{ij}\})$ for objects of \mathbf{X} relative to a covering $\{U_i \rightarrow U\}$ in \mathbf{S} is called *effective* if there exists an object $a \in \text{Obj}(\mathbf{X}_U)$ and isomorphisms $\psi_i : a|_{U_i} \rightarrow a_i$ in \mathbf{X}_{U_i} such that for all pairs (i, j) , the diagram

$$\begin{array}{ccc} a|_{U_{ij}} & \xrightarrow{\psi_j|_{U_{ij}}} & a_j|_{U_{ij}} \\ \psi_i|_{U_{ij}} \downarrow & & \downarrow \varphi_{ij} \\ a_i|_{U_{ij}} & \xrightarrow{\varphi_{ij}} & a_j|_{U_{ij}} \end{array}$$

commutes. Yang: To be checked.

Slogan Descent data are like gluing data for objects, and effectiveness means that the glued object exists.

2 Prestacks and stacks

Definition 14. A prestack over the site \mathbf{S} is a category \mathbf{X} fibered in groupoids over \mathbf{S} .

Slogan Prestacks are “presheaf remembering automorphisms”.

Example 15. presheaf is a prestack. Yang: To be added.

Example 16. The moduli problem of classifying algebraic curves of a fixed genus g can be formulated as a prestack over the site of schemes. Consider the category \mathbf{M}_g whose objects are families of smooth projective curves of genus g over schemes, and whose morphisms are isomorphisms of such families. The functor $\mathbf{p} : \mathbf{M}_g \rightarrow \mathbf{Sch}$ sending a family of curves to its base scheme makes \mathbf{M}_g a category fibred in groupoids over \mathbf{Sch} . For each scheme S , the fiber category $\mathbf{M}_{g,S}$ consists of families of smooth projective curves of genus g over S and their isomorphisms. The descent data for objects in \mathbf{M}_g relative to a covering of schemes correspond to gluing families of curves along isomorphisms on overlaps, which is effective due to the nature of algebraic curves. Thus, \mathbf{M}_g is a prestack over the site of schemes. Yang: To be revised.

Proposition 17. Let \mathbf{S} be a site, and let $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$, $\mathbf{q} : \mathbf{Y} \rightarrow \mathbf{S}$, and $\mathbf{r} : \mathbf{Z} \rightarrow \mathbf{S}$ be prestacks over \mathbf{S} . Let $\Phi : \mathbf{X} \rightarrow \mathbf{Z}$ and $\Psi : \mathbf{Y} \rightarrow \mathbf{Z}$ be morphisms of prestacks over \mathbf{S} . Then the fiber product $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ exists in the category of prestacks over \mathbf{S} . Yang: To be checked.

Definition 18. Let \mathbf{S} be a site. A prestack $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ is called a *stack* over the site \mathbf{S} if for every object $U \in \text{Obj}(\mathbf{S})$ and every covering $\{U_i \rightarrow U\}$ in \mathbf{S} , the descent data for objects of \mathbf{X} relative to the covering $\{U_i \rightarrow U\}$ are effective. Yang: To be revised.

Definition 19. Let \mathbf{S} be a site, and let $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ and $\mathbf{q} : \mathbf{Y} \rightarrow \mathbf{S}$ be stacks over \mathbf{S} . A *morphism of stacks* $F : \mathbf{X} \rightarrow \mathbf{Y}$ over \mathbf{S} is a functor $F : \mathbf{X} \rightarrow \mathbf{Y}$ such that $\mathbf{q} \circ F = \mathbf{p}$. Yang: To be checked.

Slogan Stacks are to prestacks as sheaves are to presheaves.

Example 20. Let X be a scheme over a base noetherian scheme S . The functor of points $h_X : (\mathbf{Sch}/S)_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Set}$ is a sheaf, and thus a stack.

Construction 21. Let \mathbf{S} be a site, and let $\mathbf{p} : \mathbf{X} \rightarrow \mathbf{S}$ be a prestack over \mathbf{S} . There exists a stack $\mathbf{p}^+ : \mathbf{X}^+ \rightarrow \mathbf{S}$ over \mathbf{S} together with a morphism of prestacks $F : \mathbf{X} \rightarrow \mathbf{X}^+$ over \mathbf{S} satisfying the following universal property: for every stack $\mathbf{p}' : \mathbf{Y} \rightarrow \mathbf{S}$ over \mathbf{S} and every morphism of prestacks $G : \mathbf{X} \rightarrow \mathbf{Y}$ over \mathbf{S} , there exists a unique morphism of stacks $G^+ : \mathbf{X}^+ \rightarrow \mathbf{Y}$ over \mathbf{S} such that $G = G^+ \circ F$. The stack \mathbf{X}^+ is called the *stackification* of the prestack \mathbf{X} . Yang: To be checked.

Example 22. Let S be a noetherian scheme, and let G be a group scheme over S acting on a scheme X over S via a morphism $\sigma : G \times_S X \rightarrow X$. The *quotient stack* $[X/G]$ is defined as following:

- For each scheme U over S , the objects of $[X/G](U)$ are pairs (P, f) where P is a G -torsor over U and $f : P \rightarrow X$ is a G -equivariant morphism over S .
- Morphisms between two objects (P, f) and (P', f') in $[X/G](U)$ are given by G -equivariant morphisms $\varphi : P \rightarrow P'$ over U such that $f' \circ \varphi = f$.

The assignment $U \mapsto [X/G](U)$ defines a stack over the site $(\mathbf{Sch}/S)_{\text{ét}}$. This stack captures the quotient of X by the action of G in a way that respects the group action and the torsor structure.
Yang: To be added.

Notation 23. As Example 6, we can associate a prestack \mathbf{X} over a \mathbf{S} to a functor $\mathcal{X} : \mathbf{S}^{\text{op}} \rightarrow \mathbf{Grpd}$ by setting $\mathbf{X}_u = \mathcal{X}(u)$ for each $u \in \text{Obj}(\mathbf{S})$ and defining the pullback functors accordingly. In particular, we can talk about representability of such prestacks. Yang: To be revised. Yang: Why do not we just talk about sheaves of groupoid?

Definition 24. Let \mathbf{S} be a site, and let \mathbf{X}, \mathbf{Y} be prestacks over \mathbf{S} . A morphism of prestacks $F : \mathbf{X} \rightarrow \mathbf{Y}$ over \mathbf{S} is called *representable* if for every $\mathbf{Z} \rightarrow \mathbf{Y}$ over \mathbf{S} with \mathbf{Z} representable in \mathbf{S} , the fiber product $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}$ is representable in \mathbf{S} .

Appendix