The First Properties of Abelian Varieties

1 Preliminaries

Proposition 1. Let $f : X \to Y$ be a morphism of varieties over a field $\mathbf{k}$. Then the function $y \mapsto \dim f^{-1}(y)$ is upper semicontinuous, i.e., for every integer m, the set $\{y \in Y : \dim f^{-1}(y) \geq m\}$ is closed in Y. Yang: To be check.

2 Definition and examples of Abelian Varieties

Theorem 2 (Rigidity Lemma)**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Definition 3. Let S be a scheme. An *abelian scheme over* S is a group object in the category $\mathbf{Sch}_S$ such that the structure morphism is proper, smooth and a fibration. If $S = \operatorname{Spec} \mathbf{k}$ for some field $\mathbf{k}$, then it is called an *abelian variety over* $\mathbf{k}$.

Definition 4. Let $\mathbf{k}$ be an algebraically closed field. An *abelian variety over* $\mathbb{k}$ is a proper variety A over $\mathbf{k}$ together with morphisms *identity* $e : \operatorname{Spec} \mathbf{k} \to A$, *multiplication* $m : A \times A \to A$ and *inversion* $i : A \to A$ such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd}
 & A \times A \times A \arrow[dl, "\mathrm{id}_A \times m"'] \arrow[dr, "m \times \mathrm{id}_A"] & \\
A \times A \arrow[dr, "m"'] & & A \times A \arrow[dl, "m"] \\
 & A &
\end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd}
A \times \operatorname{Spec} \mathbf{k} \arrow[r, "\mathrm{id}_A \times e"] \arrow[dr, "\cong"'] & A \times A \arrow[d, "m"] & \operatorname{Spec} \mathbf{k} \times A \arrow[l, "e \times \mathrm{id}_A"'] \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_A"] & \\
A \times A \arrow[dr, "m"'] & \operatorname{Spec} \mathbf{k} \arrow[d, "e"] & A \times A \arrow[dl, "m"] \\
 & A &
\end{tikzcd}.$$

Yang: Can we just say that $A(\mathbf{k})$ is a group with e, m, i satisfying the axioms?

Example 5. Let E be an elliptic curve over a field $\mathbf{k}$. Then E is an abelian variety of dimension 1.

Example 6.

Example 7.

In the following, we will always assume that A is an abelian variety over a field $\mathbf{k}$ of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The left translation by $a \in A(\mathbf{k})$ is defined as

$$l_a : A \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times A \xrightarrow{a \times \mathrm{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a.

Proposition 8. Let A be an abelian variety. Then A is smooth.

Proof. Note that there is an open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a. □

Proposition 9. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. Yang: But how to show it is a morphism of varieties? Yang: To be completed. □

Lemma 10. Let X, Y, Z be proper varieties over a field $\mathbf{k}$ and $g : X \times Y \to Z$ a morphism over $\mathbf{k}$. Suppose that g contracts $X \times y_0$ for some point $y_0 \in X(\mathbf{k})$. Then there exists a unique morphism $f : Y \to Z$ such that $g = f \circ p_Y$, where $p_Y : X \times Y \to Y$ is the projection to the second factor.

Proof. Yang: To be completed. □

Theorem 11. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism.

Proof. Yang: We have $(m_A)_*(\mathcal{O}_{A\times A}) \cong \mathcal{O}_A$. Then consider the diagram

$$\begin{tikzcd}
A \times A \arrow[r, "f\times f"] \arrow[d, "m_A"'] & B \times B \arrow[d, "m_B"] \\
A & B.
\end{tikzcd}$$

For every closed point $a \in A$, the fiber $m_A^{-1}(a)$ is isomorphic to $m_{A_{\kappa(a)}}^{-1}(a)$ and Yang: To be completed. □

Proposition 12. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 11. □

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 13. Let A be an abelian variety over $\Bbbk$ and n a positive integer. Then the multiplication by n morphism $[n]_A : A \to A$ is finite surjective and étale.

Proof. Yang: To be completed. □

3 Complex abelian varieties

Theorem 14. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if Λ Yang: To be completed.