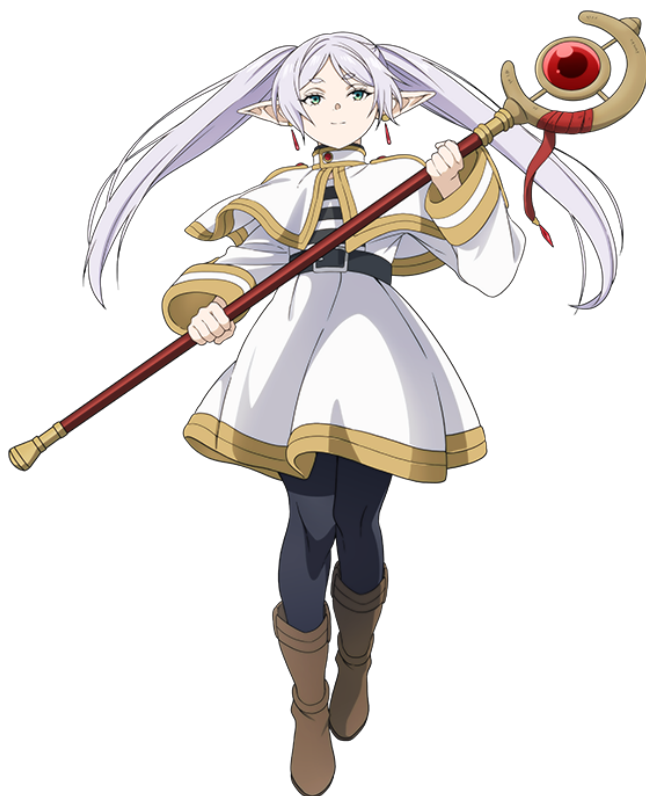


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# *Abelian Varieties*



“如果是勇者辛美尔，他一定会这么做的！”

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## 1 The First Properties of Abelian Varieties

### 1.1 Definition and examples of Abelian Varieties

**Theorem 1.1** (Rigidity Lemma). Let  $\pi_i : X \rightarrow Y_i$  be proper morphisms of varieties over a field  $\mathbf{k}$  for  $i = 1, 2$ . Suppose that  $\pi_1$  is a fibration and  $\pi_2$  contracts  $\pi_1^{-1}(y_0)$ . Then there exists a rational map  $\varphi : Y_1 \dashrightarrow Y_2$  such that  $\pi_2 \circ \varphi = \pi_1$  and  $\varphi$  is well-defined near  $Y_1 \setminus \{y_0\}$ .

**Definition 1.2.** Let  $S$  be a scheme. An *abelian scheme over  $S$*  is a group object in the category  $\mathbf{Sch}_S$  such that the structure morphism is proper, smooth and a fibration. If  $S = \text{Spec } \mathbf{k}$  for some field  $\mathbf{k}$ , then it is called an *abelian variety over  $\mathbf{k}$* .

**Example 1.3.**

**Example 1.4.**

**Example 1.5.**

In the following, we will always assume that  $A$  is an abelian variety over a field  $\mathbf{k}$  of dimension  $d$ .

Temporarily, we will use the notation  $e_A, m_A, i_A$  to denote the identity section, multiplication morphism and inversion morphism of an abelian variety  $A$ .

**Proposition 1.6.** Let  $A$  be an abelian variety. Then  $A$  is smooth.

*Proof.* Note that there is an open subset  $U \subset A$  which is smooth. Then apply the left translation morphism  $l_a$ . □

**Proposition 1.7.** Let  $A$  be an abelian variety. Then the cotangent bundle  $\Omega_A$  is trivial, i.e.,  $\Omega_A \cong \mathcal{O}_A^{\oplus d}$  where  $d = \dim A$ .

*Proof.* **Yang:** To be completed. □

**Lemma 1.8.** Let  $p : X \times Y \rightarrow Z$  be a proper morphism of varieties over  $\mathbf{k}$  such that  $p$  contracts  $\{x_0\} \times Y$  for some point  $x_0 \in X$ . Then there exists a unique morphism  $f : Y \rightarrow Z$  such that  $p = f \circ p_Y$ .

*Proof.* Yang: To be completed. □

**Theorem 1.9.** Let  $A$  and  $B$  be abelian varieties. Then any morphism  $f : A \rightarrow B$  with  $f(e_A) = e_B$  is a group homomorphism.

*Proof.* Yang: To be completed. □

**Proposition 1.10.** Let  $A$  be an abelian variety. Then  $A$  is an abelian group.

*Proof.* Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 1.9. □

From now on, we will use the notation  $0, +, [-1]_A, t_a$  to denote the identity section, addition morphism, inversion morphism and translation by  $a$  of an abelian variety  $A$ . For every  $n \in \mathbb{Z}_{>0}$ , the homomorphism of multiplication by  $n$  is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \text{id}_A} A \times A \xrightarrow{+} A,$$

where  $\Delta$  is the diagonal morphism.

**Proposition 1.11.** Let  $A$  be an abelian variety over  $\mathbb{k}$  and  $n$  a positive integer. Then the multiplication by  $n$  morphism  $[n]_A : A \rightarrow A$  is finite surjective and étale.

*Proof.* Yang: To be completed. □

## 1.2 Complex abelian varieties

**Theorem 1.12.** Let  $A$  be a complex abelian variety. Then  $A$  is a complex torus, i.e., there exists a lattice  $\Lambda \subset \mathbb{C}^d$  such that  $A \cong \mathbb{C}^d / \Lambda$ . Conversely, let  $A = \mathbb{C}^n / \Lambda$  be a complex torus for some lattice  $\Lambda$ . Then  $A$  is a complex abelian variety if and only if  $\Lambda$  Yang: To be completed.

# 2 Picard Groups of Abelian Varieties

## 2.1 Pullback along group operations

**Theorem 2.1** (Seesaw Theorem). Let  $A$  be an abelian variety over  $\mathbb{k}$ .

**Theorem 2.2** (Theorem of the cube). Let  $X, Y, Z$  be completed varieties over  $\mathbb{k}$  and  $\mathcal{L}$  a line bundle on  $X \times Y \times Z$ . Suppose that there exist  $x \in X(\mathbb{k}), y \in Y(\mathbb{k}), z \in Z(\mathbb{k})$  such that the restriction  $\mathcal{L}|_{\{x\} \times Y \times Z}, \mathcal{L}|_{X \times \{y\} \times Z}$  and  $\mathcal{L}|_{X \times Y \times \{z\}}$  are trivial. Then  $\mathcal{L}$  is trivial.

*Proof.* Yang: To be completed. □

**Remark 2.3.** If we assume the existence of the Picard scheme, then the theorem of the cube can be deduced from the Rigidity Lemma. **Yang: To be completed.**

**Proposition 2.4.** Let  $A$  be an abelian variety over  $\mathbb{k}$ ,  $f, g, h : X \rightarrow A$  morphisms from a variety  $X$  to  $A$  and  $\mathcal{L}$  a line bundle on  $A$ . Then

$$(f + g + h)^*\mathcal{L} \cong (f + g)^*\mathcal{L} \otimes (f + h)^*\mathcal{L} \otimes (g + h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

*Proof.* **Yang: To be completed.** □

**Proposition 2.5.** Let  $A$  be an abelian variety over  $\mathbb{k}$ ,  $n \in \mathbb{Z}$  and  $\mathcal{L}$  a line bundle on  $A$ . Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes \frac{1}{2}(n^2-n)}.$$

*Proof.* **Yang: To be completed.** □

**Theorem 2.6** (Theorem of the square). Let  $A$  be an abelian variety over  $\mathbb{k}$ ,  $x, y \in A(\mathbb{k})$  two points and  $\mathcal{L}$  a line bundle on  $A$ . Then

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

**Remark 2.7.** We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbb{k}) \rightarrow \text{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that  $\Phi_{\mathcal{L}}$  is a homomorphism of groups. When we vary  $\mathcal{L}$ , the map

$$\Phi_{\square} : \text{Pic}(A) \rightarrow \text{Hom}_{\mathbf{Grp}}(A(\mathbb{k}), \text{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is a group homomorphism. For any  $x \in A(\mathbb{k})$ , we have

$$\Phi_{t_x^*\mathcal{L}} = \Phi_{\mathcal{L}}.$$

In the other words,

$$\Phi_{\mathcal{L}}(x) \in \text{Ker } \Phi_{\square}, \quad \forall \mathcal{L} \in \text{Pic}(A), x \in A(\mathbb{k}).$$

**Yang: To be completed.**

If we assume the scheme structure on  $\text{Pic}(A)$ , then  $\Phi_{\mathcal{L}}$  is a morphism of scheme and factors through  $\text{Pic}^0(A)$ . Let  $K(\mathcal{L}) := \text{Ker } \Phi_{\mathcal{L}}$ , then  $K(\mathcal{L})$  is a subgroup scheme of  $A$ . We give another description of  $K(\mathcal{L})$ . From this point, we can recover the dual abelian variety  $A^\vee = \text{Pic}^0(A)$  as the quotient  $A/K(\mathcal{L})$ .

**Yang: To be completed.**

## 2.2 Projectivity

**Proposition 2.8.** Let  $A$  be an abelian variety over  $\mathbb{k}$  and  $D$  an effective divisor on  $A$ . Then  $|2D|$  is base point free.

**Theorem 2.9.** Let  $A$  be an abelian variety over  $\mathbb{k}$  and  $D$  an effective divisor on  $A$ . TFAE:

- (a) the stabilizer  $\text{Stab}(D)$  of  $D$  is finite;
- (b) the morphism  $\Phi_{|2D|}$  induced by the complete linear system  $|2D|$  is finite;
- (c)  $D$  is ample;
- (d)  $K(\mathcal{O}_A(D))$  is finite.

**Theorem 2.10.** Let  $A$  be an abelian variety over  $\mathbf{k}$ . Then  $A$  is projective.

*Proof.* Yang: To be completed. □

## 2.3 Isogenies and finite subgroups

**Theorem 2.11.** Let  $A$  be an abelian variety of dimension  $d$  over  $\mathbb{k}$ . Then the subgroup  $A[n]$  of  $n$  torsion points is finite and we have

- (a) if  $n$  is coprime to  $\text{char}(\mathbf{k})$ , then  $A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2d}$ ;
- (b) if  $n = p^k$  for  $p = \text{char}(\mathbf{k}) > 0$

*Proof.* Yang: To be completed. □

**Theorem 2.12.** Let  $A$  be an abelian variety over  $\mathbb{k}$ . There is a bijection between the isogenies from  $A$  over  $\mathbb{k}$  and the finite subgroup schemes of  $A$ .

## 2.4 Dual abelian varieties

**Theorem 2.13.** Let  $A$  be an abelian variety over  $\mathbf{k}$ . Then  $\text{Pic}^0(A)$  has a natural structure of an abelian variety, called the *dual abelian variety* of  $A$ , denoted by  $A^\vee$ .

**Proposition 2.14.** There exists a unique line bundle  $\mathcal{P}$  on  $A \times A^\vee$  such that for every  $y = \mathcal{L} \in A^\vee = \text{Pic}^0(A)$ , we have  $\mathcal{P}|_{A \times \{y\}} \cong \mathcal{L}$ .