Schemes and Varieties

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Date: August 31, 2025, **Author**: Tianle Yang, My Homepage

1 Definition and First Properties of Schemes

1.1 Locally Ringed Space

1.2 Schemes

Example 1.1 (Glue open subschemes)**.** We construct a scheme by gluing open subschemes. Let X_i be schemes for $i \in I$ and $U_{ij} \subseteq X_i$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Yang:

1.3 Integral, reduced and irreducible

1.4 Fiber product

1.5 Dimension

1.6 Noetherian and finite type

1.7 Separated and proper

2 Category of sheaves of modules

2.1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 2.1. Let X be a ringed space with structure sheaf $\mathcal{O}_X$. A **sheaf of (left) $\mathcal{O}_X$-modules** is a sheaf $\mathcal{F}$ on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the restriction map

$\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ in the sense that for every $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$, we have

$$\rho_{UV}(s \cdot m) = \rho_{UV}(s) \cdot \rho_{UV}(m).$$

Yang: To be continued...

Example 2.2. Let X be a scheme. The structure sheaf $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-modules. More generally, any quasi-coherent sheaf (to be defined later) is a sheaf of $\mathcal{O}_X$-modules. In particular, if $X = \operatorname{Spec} A$ is an affine scheme, then for any A-module M, the associated sheaf $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules. Yang: To be continued...

Definition 2.3. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **quasi-coherent** if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets. Yang: To be continued...

Definition 2.4. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **coherent** if it is quasi-coherent and for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of finite free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0,$$

where m, n are finite integers. Yang: To be continued...

2.2 As abelian categories

Theorem 2.5. Let X be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules is an abelian category. Yang: To be continued...

Theorem 2.6. Let X be a scheme. The category of quasi-coherent sheaves on X is an abelian category. Yang: To be continued...

Theorem 2.7. Let X be a noetherian scheme. The category of coherent sheaves on X is an abelian category. Yang: To be continued...

2.3 Relevant functors

Theorem 2.8. Let X be a ringed space. The global sections functor

$$\Gamma(X,-) : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\mathcal{O}_X(X)\text{-modules})$$

is left exact. Yang: To be continued...

Theorem 2.9. Let $f : X \to Y$ be a morphism of ringed spaces. The direct image functor

$$f_* : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_Y\text{-modules})$$

is left exact. Yang: To be continued...

Theorem 2.10. Let $f : X \to Y$ be a morphism of ringed spaces. The inverse image functor

$$f^* : (\text{Sheaves of } \mathcal{O}_Y\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_X\text{-modules})$$

is right exact. Yang: To be continued...

2.4 Locally free sheaves and vector bundles

Definition 2.11. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **locally free** if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to a finite free $\mathcal{O}_U$-module, i.e., there exists an isomorphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{F}|_U \cong \mathcal{O}_U^n,$$

where n is a finite integer called the **rank** of $\mathcal{F}$ at x. Yang: To be continued...

Example 2.12. A **line bundle** on a scheme X is a locally free sheaf of rank 1. The sheaf of differentials $\Omega_{X/k}$ on a smooth variety X over a field k is a locally free sheaf of rank equal to the dimension of X. Yang: To be continued...

Theorem 2.13. Let X be a scheme. There is an equivalence of categories between the category of locally free sheaves of finite rank on X and the category of vector bundles on X. Yang: To be continued...

2.5 Cohomological theory

Theorem 2.14. Let X be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then the cohomology groups $H^i(X,\mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

Theorem 2.15. Let X be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

Theorem 2.16. Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

3 Normal, Cohen-Macaulay, and regular schemes

4 Line Bundles and Divisors

4.1 Cartier Divisors

4.2 Line Bundles and Picard Group

4.3 Weil Divisors and Reflexive Sheaves

5 Line bundles induce morphisms

5.1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 5.1. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Let $U_i \coloneqq \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. $\square$

Proposition 5.2. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. Yang: To be continued. □

Example 5.3. Let $X = \mathbb{P}^n_{\mathbf{k}}$ with $\mathbf{k}$ a field and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\ldots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_{\mathbf{k}}$ where $N = \binom{n+d}{d} - 1$. On $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\ldots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \ldots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 5.4. Let $X = \mathbb{P}^m_{\mathbf{k}} \times \mathbb{P}^n_{\mathbf{k}}$ with $\mathbf{k}$ a field and $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \ldots, T_m$ and $S_0, \ldots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}^{(m+1)(n+1)-1}_{\mathbf{k}}$. On $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\ldots : x_i y_j : \ldots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Example 5.5. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ over $\mathbb{P}^1$. Yang: To be continued.

Definition 5.6. A *linear system* on a scheme X is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on X and $V \subseteq \Gamma(X, \mathcal{L})$ is a subspace. The dimension of the linear system is $\dim V - 1$. A linear system is *base-point free* if V is base-point free. A linear system is *complete* if $V = \Gamma(X, \mathcal{L})$. Yang: To be continued.

Definition 5.7. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

Definition 5.8. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^* \mathcal{O}(1)$. Yang: To be continued.

Definition 5.9. Let $\mathcal{L}$ be a line bundle on a scheme X and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace. The *base locus* of V is the closed subset

$$\mathrm{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If $\mathrm{Bs}(V) = \emptyset$, we say that V is *base-point free*. Yang: To be continued.

Definition 5.10. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. Yang: To be continued.

Definition 5.11. Let $\mathcal{L}$ be a line bundle on a scheme X. Yang: To be continued.

Theorem 5.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

Proposition 5.13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

Yang: To be continued.

Proof. Yang: To be continued. □

Proposition 5.14. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then $\mathcal{L}$ is very ample if and only if the following two conditions hold:

(a) (separate points) for any two distinct points $x, y \in X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $s(y) \neq 0$;

(b) (separate tangent vectors) for any point $x \in X$ and non-zero tangent vector $v \in T_x X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $v(s) \neq 0$.

Yang: To be continued.

5.2 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 5.15. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X,\mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X,\mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X,\mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X,\mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X,\mathcal{L}))$.

5.3 Asymptotic behavior

Definition 5.16. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X,\mathcal{L}) = \bigoplus_{n\geq 0} \Gamma(X,\mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

Definition 5.17. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. Yang: To be continued.

Theorem 5.18. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \mathrm{Proj}\, R(X,\mathcal{L})$ and f is induced by the natural map $R(X,\mathcal{L}) \to \Gamma(X,\mathcal{L}^{\otimes n})$. Yang: To be continued.

Definition 5.19. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X,\mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X,\mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large n. Yang: To be continued.

Example 5.20. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. Yang: To be continued.

6 Differentials and duality

7 Flat, smooth and étale morphisms

8 Relative objects

8.1 Relative schemes

8.2 Relative ampleness and relative morphisms

9 Finite morphisms and fibrations

10 Varieties in more general settings

10.1 Varieties

Definition 10.1. A *variety* over an algebraically closed field $\mathbb{k}$ is an integral separated scheme of finite type over $\operatorname{Spec} \mathbf{k}$.

Yang: Suppose that $\mathbf{k}$ is not algebraically closed, let $\mathbf{k}'$ be an algebraic extension of $\mathbf{k}$. What is the relation between X, $X_{\mathbf{k}'}$, $X(\mathbf{k}')$ and $X_{\mathbf{k}'}(\mathbf{k}')$?

10.2 Geometric properties

10.3 Points in varieties

Proposition 10.2. Let $\mathscr{k}$ be a field and ℓ an extension of $\mathscr{k}$. Let X be a variety over $\mathscr{k}$. Then we have the following:

(a) there is a natural bijection between $X(\ell)$ and $X_\ell(\ell)$;

(b) let m/ℓ be an extension, then there is a natural inclusion $X(\ell) \subseteq X(m)$;

(c) suppose that $X = \operatorname{Spec} \mathscr{k}[T_1, \dots, T_n]/I$ is an affine variety, then there is a natural bijection between $X(\ell)$ and the set $\{(x_1, \dots, x_n) \in \ell^n | f(x_1, \dots, x_n) = 0, \forall f \in I\}$.