Differentials and duality

Let S be a base noetherian scheme, $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over S and all varieties are assumed to be defined over $\Bbbk$.

1 The sheaves of differentials

Definition 1. Let $f : X \to S$ be an S-scheme. The *sheaf of differentials* of X over S, denoted by $\Omega_{X/S}$, is the $\mathcal{O}_X$-module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$.

Proposition 2. Let X and T be S-schemes and $X_T := X \times_S T$ be the base change of X along $T \to S$. Let $p : X_T \to X$ be the projection morphism. Then there is a natural isomorphism of $\mathcal{O}_{X_T}$-modules

$$\Omega_{X_T/T} \cong p^*\Omega_{X/S}.$$

Proof. Given by algebras, see Yang: ref. Yang: To be continued. □

Proposition 3. Let X be an S-scheme and $U \subseteq X$ be an open subscheme. Then there is a natural isomorphism of $\mathcal{O}_U$-modules

$$\Omega_{U/S} \cong \Omega_{X/S}|_U.$$

Furthermore, let $\xi \in X$, then there is a natural isomorphism of $\mathcal{O}_{X,\xi}$-modules

$$\Omega_{X/S,\xi} \cong \Omega_{\mathcal{O}_{X,\xi}/\mathcal{O}_{S,f(\xi)}}.$$

Yang: To be checked.

Proof. Yang: To be continued. □

Proposition 4. Let X be a regular variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a locally free sheaf of rank n.

Proof. Yang: To be continued. □

Proposition 5. Let X be a normal variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a reflexive sheaf of rank n.

Proof. Yang: To be continued. □

Definition 6. Let X be a normal variety over $\Bbbk$. The *canonical divisor* K_X of X is defined to be the Weil divisor class $c_1(\Omega_{X/\Bbbk})$.

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Theorem 7 (Euler sequence for projective bundle)**.** Let X be a normal variety over $\Bbbk$ and $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on X. Let $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. Then there is an exact sequence of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$-modules

$$0 \to \Omega_{\mathbb{P}_X(\mathcal{E})/X} \xrightarrow{\phi} \pi^*\mathcal{E}(-1) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \to 0.$$

Here $\pi^*\mathcal{E}(-1)$ is twisted by the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-1)$.

Proof.

Step 1. First assume that $X = \operatorname{Spec} A$ is affine and $\mathcal{E}$ is free. Under this assumption, find expressions for ϕ and ψ.

Fix a basis $T_0, \dots, T_r$ of the free A-module $\mathcal{E}(X)$. On the standard open subset $U_i = \{T_i \neq 0\} = \operatorname{Spec} B_i \subseteq \mathbb{P}_X(\mathcal{E})$, we have coordinates $t_{j,i} := T_j/T_i$ for $j \neq i$. The exact sequence becomes

$$0 \to \bigoplus_{k \neq i} B_i \mathrm{d}t_{k,i} \xrightarrow{\phi} \bigoplus_{k=0}^{r} B_i e_i \cdot T_k \xrightarrow{\psi} B_i \to 0.$$

Here e_i is the local generator of $\mathcal{O}_{\mathbb{P}_A(\mathcal{E})}(-1)$ on U_i, symbolically satisfying $e_i T_i = 1$.

Recall that on the overlap $U_{ij} = U_i \cap U_j$, the coordinates are related by

$$t_{i,j} e_i = e_j, \quad \mathrm{d}t_{k,i} = t_{j,i} \mathrm{d}t_{k,j} - t_{k,i} t_{j,i} \mathrm{d}t_{i,j}.$$

Here we set $t_{l,l} := 1$ for convenience. Symbolically, we have

$$\text{“ } \mathrm{d}t_{k,i} = \frac{T_i \mathrm{d}T_k - T_k \mathrm{d}T_i}{T_i^2} = e_i \mathrm{d}T_k - t_{k,i} e_i \mathrm{d}T_i \text{ ”.}$$

On the overlap U_{ij}, it transitions as

$$\begin{aligned} \text{“}\mathrm{d}t_{k,i} &= t_{j,i} \mathrm{d}t_{k,j} - t_{k,i} t_{j,i} \mathrm{d}t_{i,j} \\ &= t_{j,i} e_j \mathrm{d}T_k - t_{j,i} t_{k,j} e_j \mathrm{d}T_j - t_{k,i} t_{j,i} \left(e_j \mathrm{d}T_i - t_{i,j} e_j \mathrm{d}T_j \right) \\ &= e_i \mathrm{d}T_k - t_{k,i} e_i \mathrm{d}T_i\text{”.} \end{aligned}$$

To make sense of the above symbolic expressions, we define ϕ and ψ locally on each U_i by

$$\phi(\mathrm{d}t_{k,i}) = e_i T_k - t_{k,i} e_i T_i, \quad \psi(e_i T_k) = t_{k,i}.$$

Step 2. Verify that ϕ and ψ are well-defined and the sequence is exact.

By computations in Step 1, ϕ is well-defined on the overlaps U_{ij}. For ψ, on the overlap U_{ij}, we have

$$\psi(e_j T_k) = \psi(t_{i,j} e_i T_k) = t_{i,j} t_{k,i} = t_{k,j}.$$

Thus ψ is also well-defined. It is clear that $\psi \circ \phi = 0$. Consider the matrix representation of ϕ with respect to the bases $\{\mathrm{d}t_{k,i}\}_{k\neq i}$ and $\{e_iT_k\}_{k=0}^r$:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ -t_{0,i} & -t_{1,i} & \cdots & -t_{i-1,i} & -t_{i+1,i} & \cdots & -t_{r,i} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

It has rank r, ϕ is injective and $\ker\psi = \sqrt{-1}\phi$. Thus the sequence is exact.

Step 3. General case: glue the local exact sequences on affine open subsets of X.

In the local case, choose a different basis $S_0, \dots, S_r$ of $\mathcal{E}(X)$ given by the transition matrix $g \in \mathrm{GL}_{r+1}(A)$. For simplicity, we just look at on the open subset $U = \{T_0 \neq 0, S_0 \neq 0\}$. Set B_U be the localization of $B = A[T_0, \dots, T_r]$ at the multiplicative set generated by T_0 and S_0. It is still a graded algebra.

Note that ϕ is formally given by differentials in $A[T_0, \dots, T_r]$ and then sending the symbol $\mathrm{d}T_i$ to T_i and $1/T_0$ to e_0. The differentials are intrinsic and linear over A, and the assignment of $1/T_0$ to e_0 is just a change of notation. Thus ϕ is independent of the choice of basis. For ψ, it is indeed given by multiplying $B_U(-1)$ by the linear part of B and then taking the degree 0 part. It is also independent of the choice of basis.

Therefore, after changing basis, ϕ and ψ remain the same. This allows us to glue the local exact sequences on each affine open subset of X to obtain a global exact sequence. □

Corollary 8. Let $\mathbf{k}$ be a field. We have

$$\omega_{\mathbb{P}^n_{\mathbf{k}}/\mathbf{k}} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbf{k}}}(-(n+1)) \quad \text{and} \quad K_{\mathbb{P}^n_{\mathbf{k}}} \sim -(n+1)H,$$

where H is a hyperplane in $\mathbb{P}^n_{\mathbf{k}}$.

2 Fundamental sequences

Theorem 9 (The first fundamental sequence of differentials)**.** Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Proof. Yang: To be completed. □

Proposition 10. Let $f : X \to Y$ be a surjective and generically finite morphism of normal varieties over $\mathbb{k}$. Then the first fundamental sequence of differentials is exact on the left.

Proof. Yang: To be completed. □

Corollary 11 (Ramification formula)**.** Let $f : X \to Y$ be a finite morphism of normal varieties. Then

$$K_X = f^*K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} \big(\mathrm{Mult}_D\, f^*\big(f(D)\big) - 1\big)\, D$$

is the ramification divisor of f. Yang: To be checked. definition of ramification divisor needs to be checked.

Proof. Yang: To be completed. ☐

Theorem 12 (The second fundamental sequence of differentials)**.** Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}|_Z \to \Omega_{Z/S} \to 0.$$

Suppose further that $Z \to X$ is a regular immersion. Then the above sequence is also exact on the left.

Proof. Yang: To be completed. ☐

Corollary 13 (Adjunction formula)**.** Let X be a normal variety and $Z \subseteq X$ be a prime Cartier divisor which is normal as variety. Then

$$K_Z = (K_X + Z)|_Z.$$

Proof. Since both X and Z are normal, they are smooth in codimension 1. Removing the singular locus of X and Z, we may assume that both X and Z are smooth varieties. This is valid since the canonical divisor is determined by the smooth locus.

Since Z is Cartier, it is a local complete intersection in X. By Theorem 13, we have the exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/\Bbbk}|_Z \to \Omega_{Z/\Bbbk} \to 0.$$

Note that Z is of codimension 1 in X, so $\mathcal{I}_Z \cong \mathcal{O}_X(-Z)$ and thus $\mathcal{I}_Z/\mathcal{I}_Z^2 \cong \mathcal{O}_X(-Z)|_Z$. Taking c_1, we obtain

$$c_1(\Omega_X)|_Z = c_1(\Omega_Z) + c_1(\mathcal{O}_X(-Z))|_Z.$$

That is,

$$K_X|_Z = K_Z - Z|_Z.$$

Rearranging gives the desired result. Yang: To be revised. restriction of Weil divisors needs to be clarified. ☐

3 Serre duality

Definition 14 (Dualizing sheaf)**.** Let X be a proper scheme of dimension n over $\Bbbk$. A *dualizing sheaf* on X is a coherent sheaf ω_X° together with a trace map $\mathrm{tr}_X : H^n(X, \omega_X^\circ) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}_X} \Bbbk$$

induces an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

Theorem 15. Let X be a projective scheme of dimension n over $\Bbbk$. Then there exists a dualizing sheaf ω_X° on X up to isomorphism. Moreover, if X is smooth, $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\Bbbk}$.

Proof. Yang: To be completed. □

Theorem 16 (Serre duality)**.** Let X be a projective, Cohen-Macaulay variety of dimension n over $\Bbbk$ with dualizing sheaf ω_X°. Then for every coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. Yang: To be completed. □

Yang: When $\mathcal{F}$ is locally free, we have $\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^i(X, \omega_X^\circ \otimes \mathcal{F}^\vee)$.

Corollary 17. Let X be a projective, normal variety of dimension n over $\Bbbk$. Then for every integer m and $0 \le i \le n$, there is a natural isomorphism Yang: To be completed.

4 Logarithm version