Definition and First Properties of Schemes

If you learn the following content for the first time, it is recommended to skip all the proofs in this section and focus on the examples, remarks and the statements of propositions and theorems.

1 Schemes

Let R be a ring. Recall that the *spectrum* of R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set*.

Proposition 1. Let R be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

Proof. Yang: To be continued □

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 2. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i \in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.
These data form a category, denoted by $\mathbf{Sch}$. If we fix a base scheme S, then an S*-scheme* is a scheme X together with a morphism $X \to S$. The category of S-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

Theorem 3. The functor $\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. Yang: To be continued

Definition 4. A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if f induces an isomorphism of X onto an open (resp. closed) subscheme of Y. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. Yang: To be continued

Example 5. Let R be a graded ring. The *projective scheme* $\operatorname{Proj} R$ is defined as the scheme associated to the sheaf of rings

$$\mathcal{O}_{\operatorname{Proj} R} = \bigoplus_{d \geq 0} R_d.$$

It can be covered by open affine subschemes of the form $\operatorname{Spec} R_f$ for homogeneous elements $f \in R$. Yang: To be checked.

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Example 6 (Glue open subschemes)**.** The construction in ?? allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in ??. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

Definition 7. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme Z of Y such that f factors through Z. More precisely, if $Y = \operatorname{Spec} A$ is affine, then the scheme theoretic image of f is $\operatorname{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover Y by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of f. Yang: To be checked.

2 Fiber product

Definition 8. Let $\mathcal{C}$ be a category and $X, Y, S \in \operatorname{Obj}(\mathcal{C})$ with morphisms $f : X \to S$ and $g : Y \to S$. A *fiber product* of X and Y over S is an object $Z \in \operatorname{Obj}(\mathcal{C})$ together with morphisms $p : Z \to X$ and $q : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

and satisfies the universal property that for any object $W \in \operatorname{Obj}(\mathcal{C})$ with morphisms $u : W \to X$ and $v : W \to Y$ such that $f \circ u = g \circ v$, there exists a unique morphism $h : W \to Z$ such that $p \circ h = u$ and $q \circ h = v$.

If a fiber product exists, it is unique up to a unique isomorphism. We denote the fiber product by $X \times_S Y$. Yang: To be checked.

Example 9. In the category of sets, the fiber product $X \times_S Y$ is given by

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\},$$

with the projections $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ being the restrictions of the natural projections. Yang: To be checked.

Remark 10. If one reverses the arrows in Definition 8, one gets the notion of *fiber coproduct*. It is also called the *pushout* or *amalgamated sum* in some literature. We denote the fiber coproduct of X and Y over S by $X \amalg_S Y$. Note that in the category of rings, the fiber coproduct $A \amalg_R B$ of R-algebras A and B over R is given by the tensor product $A \otimes_R B$. Dually, one can expect that fiber products of affine schemes correspond to tensor products of rings.

Theorem 11. The category of schemes admits fiber products. More precisely, given morphisms of schemes $f : X \to S$ and $g : Y \to S$, there exists a scheme Z together with morphisms $p : Z \to X$

and $q : Z \to Y$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

commutes and satisfies the universal property of the fiber product. We denote this scheme by $X \times_S Y$.
Yang: To be continued

Definition 12. Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a point. The *scheme theoretic fiber* of f over y is the fiber product $X_y = X \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y)$ is the residue field of the local ring $\mathcal{O}_{Y,y}$. Yang: To be checked.

Definition 13. Let X be a scheme and $Z_1, Z_2 \subset X$ be closed subschemes defined by quasi-coherent sheaves of ideals $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{O}_X$, respectively. The *scheme theoretic intersection* of Z_1 and Z_2 is the closed subscheme $Z_1 \cap Z_2$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}_1 + \mathcal{I}_2$. Yang: To be checked.

3 Noetherian and finite type

Definition 14. A scheme X is *Noetherian* if it admits a finite open cover $\{U_i\}_{i=1}^n$ such that each U_i is an affine scheme $\operatorname{Spec} A_i$ with A_i a Noetherian ring. Yang: To be checked.

Proposition 15. A Noetherian scheme is quasi-compact. Yang: To be checked.

Definition 16. Let S be a scheme. A scheme X is of *finite type* over S if there exists a finite open cover $\{U_i\}_{i=1}^n$ of S such that for each i, $f^{-1}(U_i)$ can be covered by finitely many affine open subsets $\{V_{ij}\}_{j=1}^{m_i}$ with $f(V_{ij}) \subseteq U_i$ and the induced morphism $f|_{V_{ij}} : V_{ij} \to U_i$ corresponds to a finitely generated algebra over the ring of global sections of U_i.
A scheme is called *Noetherian* if it is of finite type over $\operatorname{Spec} \mathbb{Z}$. Yang: To be checked.

4 Integral, reduced and irreducible

Definition 17. A topological space X is *irreducible* if it is non-empty and cannot be expressed as the union of two proper closed subsets. Equivalently, every non-empty open subset of X is dense in X. Yang: To be checked.

Proposition 18. Let X be a topological space satisfying the descending chain condition on closed subsets. Then X can be written as a finite union of irreducible closed subsets, called the *irreducible components* of X. Moreover, this decomposition is unique up to permutation of the components. Yang: To be checked.

Definition 19. A scheme X is *reduced* if its structure sheaf $\mathcal{O}_X$ has no nilpotent elements. Yang: To be checked.

Proposition 20. A scheme X is reduced if and only if for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a reduced ring. Yang: To be checked.

Proposition 21. Let X be a scheme. There exists a unique closed subscheme X of X such that X is reduced and has the same underlying topological space as X. Moreover, for any morphism of schemes $f : Y \to X$ with Y reduced, f factors uniquely through the inclusion $X \to X$. Yang: To be checked.

Definition 22. A scheme X is *integral* if it is both reduced and irreducible. Yang: To be checked.

Proposition 23. A scheme X is integral if and only if for every open affine subset $U = \operatorname{Spec} A \subset X$, the ring A is an integral domain. Yang: To be checked.

5 Dimension

Definition 24. The *Krull dimension* of a topological space X, denoted by $\dim X$, is the supremum of the lengths n of chains of distinct irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X. If no such finite supremum exists, we say that X has infinite dimension. Yang: To be checked.

6 Separated and proper

Definition 25. A morphism of schemes $f : X \to Y$ is *separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. A scheme X is *separated* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. Yang: To be checked.

Proposition 26. Any affine scheme is separated. More generally, any morphism between affine schemes is separated. Yang: To be checked.

Proposition 27. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any scheme T and any pair of morphisms $g_1, g_2 : T \to X$ such that $f \circ g_1 = f \circ g_2$, the equalizer of g_1 and g_2 is a closed subscheme of T. Yang: To be checked.

Proposition 28. A scheme X is separated if and only if for any pair of affine open subschemes $U, V \subset X$, the intersection $U \cap V$ is also an affine open subscheme. Yang: To be checked.

Proposition 29. The composition of separated morphisms is separated. Moreover, separatedness is stable under base change, i.e., if $f : X \to Y$ is a separated morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also separated. Yang: To be checked.

Proposition 30. A morphism of schemes $f : X \to Y$ is separated if and only if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists at most one morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. Yang: To be checked.

Definition 31. A morphism of schemes $f : X \to Y$ is *universally closed* if for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map. Yang: To be checked.

Definition 32. A morphism of schemes $f : X \to Y$ is *proper* if it is of finite type, separated, and universally closed (i.e., for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map). A scheme X is *proper* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is proper. Yang: To be checked.

Theorem 33. Any projective morphism is proper. In particular, any projective scheme is proper. Yang: To be checked.

Proposition 34. The composition of proper morphisms is proper. Moreover, properness is stable under base change, i.e., if $f : X \to Y$ is a proper morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also proper. Yang: To be checked.

Proposition 35. A morphism of schemes $f : X \to Y$ is proper if and only if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists a unique morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. Yang: To be checked.