Structure of linear algebraic groups

1 Jordan-Chevalley Decomposition of elements

Recall that for a linear operator $T : V \to V$ of finite-dimensional $\Bbbk$-vector space V is called *semisimple* if it is diagonalizable, and *unipotent* if $T - \mathrm{id}_V$ is nilpotent.

Definition 1. Let G be a linear algebraic group and $g \in G(\Bbbk)$. We say that g is *semisimple* (resp. *unipotent*) if its image under some (equivalently, any) faithful linear representation of G is a semisimple (resp. unipotent) linear operator.

Lemma 2. The notion of semisimple and unipotent elements in Definition 1 does not depend on the choice of faithful linear representation.

Proof. Yang: To be added. □

Theorem 3 (Jordan-Chevalley Decomposition)**.** Let G be a linear algebraic group and $g \in G(\Bbbk)$. Then there exist unique commuting elements $g_s, g_u \in G(\Bbbk)$ such that $g = g_s g_u$, where g_s is semisimple and g_u is unipotent.
Moreover, this decomposition is functorial in the sense that for any homomorphism of linear algebraic groups $\varphi : G \to H$, we have $\varphi(g)_s = \varphi(g_s)$ and $\varphi(g)_u = \varphi(g_u)$. Yang: To be checked

2 Decomposition of linear algebraic groups

Definition 4. Let G be a linear algebraic group over a field $\Bbbk$. The *radical* of G, denoted by $\mathrm{rad}(G)$, is defined to be the unique maximal connected normal solvable subgroup of G.

Definition 5. Let G be a linear algebraic group. The *unipotent radical* of G, denoted by $\mathrm{rad}_u(G)$, is defined to be the subgroup of $\mathrm{rad}(G)$ consisting of all unipotent elements.

Definition 6. Let G be a linear algebraic group over a field $\Bbbk$. We say that G is *semisimple* if $\mathrm{rad}(G)$ is trivial.

Definition 7. Let G be a linear algebraic group over a field $\Bbbk$. We say that G is *reductive* if the unipotent radical of G is trivial.

Slogan

$$\begin{array}{ccc} \text{“\textit{unipotent radical}”} & \rightarrow\!\leftarrow & \text{“\textit{reductive}”} \\ \Downarrow & & \Uparrow \\ \text{“\textit{solvable radical}”} & \rightarrow\!\leftarrow & \text{“\textit{semisimple}”} \end{array}$$

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Theorem 8 (Levi Decomposition)**.** Let G be a linear algebraic group over an algebraically closed field $\Bbbk$. Then there exists a reductive subgroup H of G such that the multiplication map $\mathrm{rad}_u(G) \rtimes H \to G$ is an isomorphism of algebraic groups. Such a subgroup H is called a *Levi subgroup* of G. Yang: To be checked.

3 Solvable groups and Borel subgroups

Definition 9. A group G is said to be *solvable* if there exists a finite sequence of algebraic subgroups

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = \{e\}$$

such that each G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is commutative for all $0 \leq i < n$. Yang: to be checked.

Theorem 10. Let G be a solvable linear algebraic group acting on a proper variety X. Then there exists a fixed point $x \in X(\Bbbk)$ such that $g \cdot x = x$ for all $g \in G(\Bbbk)$.

Corollary 11 (Lie-Kolchin Theorem)**.** Let $G < \mathrm{GL}_n(\Bbbk)$ be a solvable linear algebraic group over an algebraically closed field $\Bbbk$. Then there exists a basis of $\Bbbk^n$ such that G is contained in the group of upper triangular matrices with respect to this basis.

Theorem 12. Let G be a linear algebraic group of dimension 1 over an algebraically closed field $\Bbbk$. Then G is isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$.

4 Semisimple and reductive algebraic groups