The First Properties of Abelian Varieties

1 Definition and examples of Abelian Varieties

Definition 1. Let S be a scheme. An *abelian scheme over* S is a group object in the category $\mathbf{Sch}_S$ such that the structure morphism is proper, smooth and a fibration. If $S = \operatorname{Spec}\mathbf{k}$ for some field $\mathbf{k}$, then it is called an *abelian variety over* $\mathbf{k}$.

Definition 2. Let $\mathbf{k}$ be a field. An *abelian variety over* $\mathbf{k}$ is a proper variety A over $\mathbf{k}$ together with morphisms *identity* $e : \operatorname{Spec}\mathbf{k} \to A$, *multiplication* $m : A \times A \to A$ and *inversion* $i : A \to A$ such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd} & A\times A\times A \arrow[dl, "\mathrm{id}_A\times m"'] \arrow[dr, "m\times\mathrm{id}_A"] & \\ A\times A \arrow[dr, "m"] & & A\times A \arrow[dl, "m"'] \\ & A & \end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd} A\times\operatorname{Spec}\mathbf{k} \arrow[r, "\mathrm{id}_A\times e"] \arrow[dr, "\cong"'] & A\times A \arrow[d, "m"] & \operatorname{Spec}\mathbf{k}\times A \arrow[l, "e\times\mathrm{id}_A"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A\times i"'] \arrow[d] \arrow[dr, "i\times\mathrm{id}_A"] & \\ A\times A \arrow[dr, "m"'] & \operatorname{Spec}\mathbf{k} \arrow[d, "e"] & A\times A \arrow[dl, "m"] \\ & A & \end{tikzcd}.$$

Yang: Can we just say that $A(\mathbf{k})$ is a group with e, m, i satisfying the axioms?

Example 3. Let E be an elliptic curve over a field $\mathbf{k}$. Then E is an abelian variety of dimension 1.

Example 4.

Example 5.

In the following, we will always assume that A is an abelian variety over a field $\mathbf{k}$ of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The left translation by $a \in A(\mathbf{k})$ is defined

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as

$$l_a : A \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times A \xrightarrow{a \times \mathrm{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a.

Proposition 6. Let A be an abelian variety. Then A is smooth.

Proof. Note that there is an open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a. □

Proposition 7. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. Yang: But how to show it is a morphism of varieties? Yang: To be completed. □

Lemma 8. Let $p : X \times Y \to Z$ be a proper morphism of varieties over $\mathbf{k}$ such that p contracts $\{x_0\} \times Y$ for some point $x_0 \in X$. Then there exists a unique morphism $f : Y \to Z$ such that $p = f \circ p_Y$.

Proof. Yang: To be completed. □

Theorem 9. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism.

Proof. Yang: To be completed. □

Proposition 10. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 9. □

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 11. Let A be an abelian variety over $\mathbb{k}$ and n a positive integer. Then the multiplication by n morphism $[n]_A : A \to A$ is finite surjective and étale.

Proof. Yang: To be completed. □

2 Complex abelian varieties

Theorem 12. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if Λ Yang: To be completed.