Definitions and examples

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Let G be an algebraic group over $\Bbbk$ acting on a variety X over $\Bbbk$.

Definition 1. A *categorical quotient* of X by G is a variety Y with trivial G-action together with a G-invariant morphism $\pi : X \to Y$ such that for any G-invariant morphism $\varphi : X \to Z$ to a variety Z with trivial G-action, there exists a unique morphism $\psi : Y \to Z$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & Y \arrow[d, "\psi"] \\ & Z \end{tikzcd}$$

Definition 2. A *geometric quotient* of X by G is a variety Y together with a G-invariant morphism $\pi : X \to Y$ satisfying the following conditions:

(a) The morphism π is surjective, and $(\sigma, p_2) : G \times X \to X \times X$ factors through the fiber product $X \times_Y X \to X \times X$ and $G \times X \to X \times_Y X$ is surjective;

(b) The topology on Y is the quotient topology induced by π, i.e., a subset $U \subseteq Y$ is open if and only if $\pi^{-1}(U)$ is open in X;

(c) The structure sheaf $\mathcal{O}_Y$ is given by the sheaf of G-invariant regular functions on X, i.e., for any open subset $U \subseteq Y$,

$$\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}(U))^G.$$

Yang: geometric quotient must be categorical quotient

Proposition 3. A geometric quotient is a categorical quotient.

Example 4. Let $G = \mathbb{G}_m$ and $X = \mathbb{A}^1$ with the action of G on X given by multiplication. Yang: To be continous

Yang: Scheme and universal and uniform quotient.

Proposition 5. Let $\pi : X \to Y$ be a categorical quotient of X by G.

(a) If X is separated, then Y is separated.

(b) If X is normal, then Y is normal.

(c) If X is smooth, then Y is smooth.

Yang: To be revised

Proposition 6. Let $\pi : X \to Y$ be a categorical quotient of X by G. If for every $y \in Y$, the fiber $\pi^{-1}(y)$ is a single G-orbit, then $\pi : X \to Y$ is a geometric quotient. Yang: To be checked.

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Appendix