Birational geometry on surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

1 Birational morphisms on surfaces

Let X be a smooth projective surface, $0 \in X(\Bbbk)$ and $\pi : \tilde{X} = \mathrm{Bl}_0 X \to X$ the blow-up of X at 0. Denote by E the exceptional divisor of π.

Proposition 1. We have $E^2 = -1$.

Proof. Yang: To be continued □

Proposition 2. We have $K_{\tilde{X}} = \pi^* K_X + E$.

Proof. We have the exact sequence

$$\Omega_{\tilde{X}} \to \pi^* \Omega_X \to \Omega_{\tilde{X}/X} \to 0.$$

Since both $\tilde{X}$ and X are smooth, $\Omega_{\tilde{X}}$ and Ω_X are locally free sheaves of rank 2. The kernel of the first map is of rank 0 and torsion, thus it is zero. Therefore, we have the short exact sequence

$$0 \to \Omega_{\tilde{X}} \to \pi^* \Omega_X \to \Omega_{\tilde{X}/X} \to 0.$$

Yang: To be continued □

Corollary 3. We have $K_{\tilde{X}}^2 = K_X^2 - 1$.

Proof. By Proposition 2, we have

$$K_{\tilde{X}}^2 = (\pi^* K_X + E)^2 = (\pi^* K_X)^2 + 2\pi^* K_X \cdot E + E^2 = K_X^2 + 0 - 1 = K_X^2 - 1.$$

□

Theorem 4. Let $f : X \to Y$ be a birational morphism between two smooth projective surfaces. Then f can be decomposed as a finite sequence of blow-ups at points.

Proof. Yang: To be continued □

2 Castelnuovo's Theorem

Definition 5. A (-1)-*curve* on a smooth projective surface X is an irreducible curve $C \subseteq X$ such that $C \cong \mathbb{P}^1$ and $C^2 = -1$.

Remark 6. Let C be a (-1)-curve on a smooth projective surface X. Then its numerical class $[C] \in \mathrm{N}_1(X)$ spans an extremal ray of $\mathrm{Psef}_1(X)$ such that $K_X \cdot C < 0$. Yang: To be revised.

Date: December 31, 2025, **Author**: Tianle Yang, My Homepage

Theorem 7 (Castelnuovo's contractibility criterion)**.** Let X be a smooth projective surface and $C \subseteq X$ an irreducible curve. Then there exists a birational morphism $f : X \to Y$ contracting C to a smooth point if and only if C is a (-1)-curve.

Proof. Yang: To be continued □

Definition 8. A *minimal surface* is a smooth projective surface that does not contain any (-1)-curves. Yang: To be checked.

3 Resolution of singularities on surface

Definition 9. A *resolution of singularities* of a projective surface X is a smooth projective surface $\widetilde{X}$ together with a birational and proper morphism $\pi : \widetilde{X} \to X$ such that π is an isomorphism over the smooth locus of X. Yang: To be checked.

Theorem 10 (Resolution of singularities on surfaces)**.** Let X be a projective surface over an algebraically closed field $\Bbbk$. Then X admits a resolution of singularities. Yang: To be checked.

Definition 11. Let X be a projective surface. A *minimal resolution* of X is a resolution of singularities $\pi : \widetilde{X} \to X$ such that for any other resolution of singularities $\pi' : \widetilde{X}' \to X$, there exists a morphism $f : \widetilde{X}' \to \widetilde{X}$ such that π' factors as $\pi' = \pi \circ f$.

Proposition 12. Let X be a projective surface. Then X admits a unique minimal resolution of singularities.

Proof. Yang: To be continued □