Differentials and duality

Let S be a base noetherian scheme. Unless otherwise specified, all schemes and varieties are assumed to be defined and of finite type over S.

1 The sheaves of differentials

Definition 1. Let X be an S-scheme. The *sheaf of differentials* of X over S, denoted by $\Omega_{X/S}$, is the $\mathcal{O}_X$-module representing the functor from the category of $\mathcal{O}_X$-modules to the category of sets that sends an $\mathcal{O}_X$-module $\mathcal{F}$ to the set of S-derivations from $\mathcal{O}_X$ to $\mathcal{F}$, i.e.,

$$\operatorname{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \cong \operatorname{Der}_S(\mathcal{O}_X, \mathcal{F}).$$

Equivalently, $\Omega_{X/S}$ is the sheaf associated to the presheaf that sends an open subset $U \subseteq X$ to the module of Kähler differentials $\Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(f(U))}$, where $f : X \to S$ is the structure morphism.
Yang: To be revised.

Theorem 2 (Euler sequence for projective bundle)**.**

Definition 3. Let X be a normal variety over $\Bbbk$ of dimension n. If X is smooth, then the *canonical divisor* K_X is defined to be $c_1(\omega_X)$. In general, let $U \subseteq X$ be the smooth locus of X and $i : U \hookrightarrow X$ be the inclusion map. Then the *canonical divisor* K_X is defined to be any Weil divisor on X such that $\mathcal{O}_X(K_X) \cong i_*\omega_U$. Note that U is big in X since X is normal, so such a Weil divisor always exists and is unique up to linear equivalence.

2 Fundamental sequence

Theorem 4 (The first fundamental sequence of differentials)**.** Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Suppose further that f is smooth. Then the above sequence is also exact on the left. Yang: ... it will be exact.

Theorem 5 (The second fundamental sequence of differentials)**.** Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{J}/\mathcal{J}^2 \to \Omega_{X/S}|_Z \to \Omega_{Z/S} \to 0.$$

Suppose further that Z is locally a complete intersection in X. Then the above sequence is also exact on the left. Yang: ... it will be exact.

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3 Application of fundamental sequences

Theorem 6 (Ramification formula)**.** Let $f : X \to Y$ be a finite morphism of normal varieties. Then

$$K_X = f^* K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} \left(\mathrm{Mult}_D\, f^*\big(f(D)\big) - 1\right) D$$

is the ramification divisor of f.

Theorem 7 (Adjunction formula)**.** Let X be a smooth variety and $Z \subseteq X$ be a smooth subvariety of codimension 1. Then

$$K_Z = (K_X + Z)|_Z.$$

4 Serre duality

Theorem 8 (Serre duality)**.** Let X be a proper variety over $\Bbbk$ and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a natural isomorphism

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee,$$

where ω_X is the canonical sheaf on X and $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual sheaf. Yang: there are some errors. Need to be revised

5 Logarithm version