
Commutative Algebra



“南淮者，人间之胜境。无饥馑灾荒之属，里巷中常闻笑声，灯火彻夜夏不闭户，唯少年顽皮，是为一害……每春来之际，辄有窃花者、弹雀者、钓鱼者……”

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1 Dimension and Depth

There are three numbers measuring the “size” of a local ring (A, \mathfrak{m}) :

- $\dim A$: the Krull dimension of A .
- $\text{depth } A$: the depth of A .
- $\dim_{\kappa(\mathfrak{m})} T_{A, \mathfrak{m}}$: the dimension of Zariski tangent space $T_{A, \mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$ -vector space.

Somehow the Krull dimension is “homological” and the depth is “cohomological”.

Definition 1.1. Let A be a noetherian ring. The *height of a prime ideal* \mathfrak{p} in A is defined as the maximum length of chains of prime ideals contained in \mathfrak{p} , that is,

$$\text{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as

$$\dim A := \max_{\mathfrak{p} \in \text{Spec } A} \text{ht}(\mathfrak{p}).$$

Example 1.2. Let A be a PID. For every two non-zero prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 , if $\mathfrak{p}_1 = t_1 A \subset \mathfrak{p}_2 = t_2 A$, then $t_2 \mid t_1$ and hence $\mathfrak{p}_1 = \mathfrak{p}_2$. It follows that $\dim A = 1$. Consequently, the ring of integers \mathbb{Z} and the polynomial ring $\mathbf{k}[T]$ in one variable over a field have Krull dimension 1.

Definition 1.3. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A -module. A sequence $t_1, \dots, t_n \in I$ is called an *M -regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \dots, t_{i-1})M$ for all i .

Example 1.4. Let $A = \mathbf{k}[x, y]/(x^2, xy)$ and $I = (x, y)$. Then $\text{depth}_I A = 0$.

Definition 1.5. Let A be a noetherian ring. For every $\mathfrak{p} \in \text{Spec } A$, $\mathfrak{p}/\mathfrak{p}^2$ is a vector space over $\kappa(\mathfrak{p})$. The *Zariski's tangent space* $T_{A, \mathfrak{p}}$ of A at \mathfrak{p} is defined as $(\mathfrak{p}/\mathfrak{p}^2)^\vee$, the dual $\kappa(\mathfrak{p})$ -vector space of $\mathfrak{p}/\mathfrak{p}^2$.

1.1 Artinian Rings and Length of Modules

Definition 1.6. Let A be a ring and M an A module. A *simple module filtration* of M is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i-1} is a simple module, i.e. it has no submodule except 0 and itself. If M has a simple module filtration as above, we define the *length* of M as n and say that M has *finite length*.

The following proposition guarantees the length is well-defined.

Proposition 1.7. Suppose M has a simple module filtration $M = M_{0,0} \supsetneq M_{1,0} \supsetneq \cdots \supsetneq M_{n,0} = 0$. Then for any other filtration $M = M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,m} = 0$ with $m > n$, there exist $k < m$ such that $M_{0,k} = M_{0,k+1}$.

Proof. We claim that there are at least $0 \leq k_1 < \cdots < k_{m-n} < m$ satisfies that $M_{0,k_i} = M_{0,k_i+1}$. Let $M_{i,j} := M_{i,0} \cap M_{0,j}$. Inductively on n , we can assume that there exist k_1, \dots, k_{n-m+1} such that $M_{1,k} = M_{1,k+1}$. Consider the sequence

$$M_{0,0}/M_{1,0} \supset (M_{0,1} + M_{1,0})/M_{1,0} \supset \cdots \supset (M_{0,m} + M_{1,0})/M_{1,0} = 0$$

in $M_{0,0}/M_{1,0}$. Since $M_{0,0}/M_{1,0}$ is simple, there is at most one k_i with $M_{0,k_i} + M_{1,0} \neq M_{0,k_i+1} + M_{1,0}$. And note that if $M_{0,k_i} + M_{1,0} = M_{0,k_i+1} + M_{1,0}$ and $M_{0,k_i} \cap M_{1,0} = M_{0,k_i+1} \cap M_{1,0}$, then $M_{0,k_i} = M_{0,k_i+1}$ by the Five Lemma. \square

Example 1.8. Let A be a ring and $\mathfrak{m} \in \text{mSpec } A$. Then A/\mathfrak{m} is a simple module. **To be completed.**

Proposition 1.9. Let A be a ring and M an A -module. Then M is of finite length iff it satisfies both a.c.c and d.c.c.

Proof. Note that if M has either a strictly ascending chain or a strictly descending chain, M is of infinite length. Conversely, d.c.c guarantee M has a simple submodule and a.c.c guarantee the sequence terminates. \square

Proposition 1.10. The length $l(-)$ is an additive function for modules of finite length. That is, if we have an exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ with M_i of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

Proof. The simple module filtrations of M_1 and M_3 will give a simple module filtration of M_2 . \square

Proposition 1.11. Let (A, \mathfrak{m}) be a local ring. Then A is artinian iff $\mathfrak{m}^n = 0$ for some $n \geq 0$.

Proof. Suppose A is artinian. Then the sequence $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ is stable. It follows that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n . By the Nakayama's Lemma ??, $\mathfrak{m}^n = 0$.

Conversely, we have

$$\mathfrak{m} \subset \mathfrak{N} \subset \bigcap_{\text{minimal prime ideal}} \mathfrak{p},$$

whence \mathfrak{m} is minimal. \square

Proposition 1.12. Let A be a ring. Then A is artinian iff A is of finite length.

Proof. First we show that A has only finite maximal ideal. Otherwise, consider the set $\{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k\}$. It has a minimal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ and for any maximal ideal \mathfrak{m} , $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}$. It follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i . Let $\mathfrak{M} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ be the Jacobi radical of A . Consider the sequence $\mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots$ and by Nakayama's Lemma, we have $\mathfrak{M}^k = 0$ for some k . Consider the filtration

$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1^k \supset \mathfrak{m}_1^k \mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0).$$

We have $\mathfrak{m}_1^k \cdots \mathfrak{m}_i^j / \mathfrak{m}_1^k \cdots \mathfrak{m}_i^{j+1}$ is an A/\mathfrak{m}_i -vector space. It is artinian and then of finite length. Hence A is of finite length. \square

Theorem 1.13. Let A be a ring. Then A is artinian iff A is noetherian and of dimension 0.

Proof. Suppose A is artinian. Then A is noetherian by Proposition 1.12. Let $\mathfrak{p} \in \text{Spec } A$. Then A/\mathfrak{p} is an artinian integral domain. If there is $a \in A/\mathfrak{p}$ is not invertible, consider $(a) \supset (a^2) \supset \cdots$, we see $a = 0$. Hence \mathfrak{p} is maximal and $\dim A = 0$.

Suppose that A is noetherian and of dimension 0. Then every maximal ideal is minimal. In particular, A has only finite maximal ideal $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Let \mathfrak{q}_i be the \mathfrak{p}_i -component of (0) . Then we have $A \hookrightarrow \bigoplus_i A/\mathfrak{q}_i$. We just need to show that A/\mathfrak{q}_i is of finite length as A -module. If $\mathfrak{q}_i \subset \mathfrak{p}_j$, take radical we get $\mathfrak{p}_i \subset \mathfrak{p}_j$ and hence $i = j$. So A/\mathfrak{q}_i is a local ring with maximal ideal $\mathfrak{p}_i A/\mathfrak{q}_i$. Then every element in $\mathfrak{p}_i A/\mathfrak{q}_i$ is nilpotent. Since \mathfrak{p}_i is finitely generated, $(\mathfrak{p}_i A/\mathfrak{q}_i)^k = 0$ for some k . Then A/\mathfrak{q}_i is artinian and then of finite length as A/\mathfrak{q}_i -module. Then the conclusion follows. \square

1.2 Dedekind Domains

To be completed

1.3 Krull's Principal Ideal Theorem

Theorem 1.14 (Krull's Principal Ideal Theorem). Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let \mathfrak{p} be a minimal prime ideal among those containing f . Then $\text{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_{\mathfrak{p}}$, we may assume A is local with maximal ideal \mathfrak{p} . Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.

Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is \mathfrak{q} -primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1} A_{\mathfrak{q}}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_{\mathfrak{q}} = 0$. It follows that $\mathfrak{q} A_{\mathfrak{q}}$ is minimal, whence $A_{\mathfrak{q}}$ is artinian. Therefore, \mathfrak{q} is minimal in A . \square

Corollary 1.15. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n . Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.

Choose \mathfrak{q} be a minimal prime ideal among those containing (\mathfrak{p}_{k-1}, f) and contained in \mathfrak{p}_{k+1} . Then by Krull's Principal Ideal Theorem 1.14, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace \mathfrak{p}_k by \mathfrak{q}_k , we have $f \in \mathfrak{q}_k \setminus \mathfrak{p}_{k-1}$.

Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.

Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$ has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A . It follows that $\dim A/(f) + 1 \leq \dim A$. \square

Proposition 1.16. Let (A, \mathfrak{m}) be a local noetherian ring with residue field \mathbf{k} . Then the following inequalities hold:

$$\text{depth } A \leq \dim A \leq \dim_{\mathbf{k}} T_{A, \mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary 1.15.

Let t_1, \dots, t_n be a $\kappa(\mathfrak{m})$ -basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \dots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \dots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \dots, t_n))$. It follows that $\mathfrak{m} = (t_1, \dots, t_n)$ by Nakayama's Lemma. By Corollary 1.15,

$$n + \dim A/(t_1, \dots, t_n) \geq n - 1 + \dim A/(t_1, \dots, t_{n-1}) \geq \cdots \geq 1 + \dim A/(t_1) \geq \dim A.$$

We conclude the result. \square

Definition 1.17. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property* (R_k) or *is regular in codimension* k if $\forall \xi \in X$ with $\text{codim } Z_\xi \leq k$,

$$\dim_{\kappa(\xi)} T_{X, \xi} = \dim \mathcal{O}_{X, \xi}.$$

We say that X *verifies property* (S_k) if $\forall \xi \in X$ with $\text{depth } \mathcal{O}_{X, \xi} < k$,

$$\text{depth } \mathcal{O}_{X, \xi} = \dim \mathcal{O}_{X, \xi}.$$

Example 1.18. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.

Suppose A verifies (S_1) . If $\mathfrak{p} \in \text{Ass } A$, every element in \mathfrak{p} is a zero divisor. Then $\text{depth } A_{\mathfrak{p}} = 0$. It follows that $\dim A_{\mathfrak{p}} = 0$ and then \mathfrak{p} is minimal.

Suppose A has no embedded point. Let $\mathfrak{p} \in \text{Spec } A$ with $\text{depth } A_{\mathfrak{p}} = 0$. This means every element in $\mathfrak{p}A_{\mathfrak{p}}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Proposition ??, $\mathfrak{p} = \mathfrak{q}$ for some minimal \mathfrak{q} , whence $\dim A_{\mathfrak{p}} = 0$.

Example 1.19. Let A be a noetherian ring. Then A is reduced iff it verifies (R_0) and (S_1) .

Suppose A is reduced. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all minimal prime ideals of A . We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where \mathfrak{N} is the nilradical of A . Hence A has no embedded point. Since $A_{\mathfrak{p}}$ is artinian, local and reduced, $A_{\mathfrak{p}}$ is a field and hence regular.

Conversely, let $\text{Ass } A$ be equal to $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Then every \mathfrak{p}_i is minimal by (S_1) . Let f be in \mathfrak{N} . Then the image of f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0) , $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where \mathfrak{q}_i is the \mathfrak{p}_i component of (0) in A . Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

1.4 Cohen-Macaulay rings

Definition 1.20 (Cohen-Macaulay). A noetherian local ring (A, \mathfrak{m}) is called *Cohen-Macaulay* if $\dim A = \text{depth } A$. A noetherian ring A is called *Cohen-Macaulay* if for every prime ideal $\mathfrak{p} \in \text{Spec } A$, the localization $A_{\mathfrak{p}}$ is Cohen-Macaulay. This is equivalent to that A verifies (S_k) for all $k \geq 0$.

Definition 1.21. Let (A, \mathfrak{m}) be a noetherian local ring of dimension d . A sequence $t_1, \dots, t_d \in \mathfrak{m}$ is called a *system of parameters* if **To be completed.**

Example 1.22 (Non Cohen-Macaulay rings). **To be completed.**

Corollary 1.23. Let A be a noetherian ring, M a finite A -module and $a \in A$ an M -regular element. Then $\text{depth } M = \text{depth } M/aM + 1$.

Corollary 1.24. Let A be a noetherian ring $a \in A$ a nonzero divisor. Then A verifies (S_d) iff A/aA verifies (S_{d-1}) .

Definition 1.25. An ideal I of a noetherian ring A is called *unmixed* if

$$\text{ht}(I) = \text{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \text{Ass}(A/I).$$

Here $\text{ht}(I)$ is defined as

$$\text{ht}(I) := \inf\{\text{ht}(\mathfrak{p}) : I \subset \mathfrak{p}\}.$$

We say that *the unmixedness theorem holds for a noetherian ring A* if any ideal $I \subset A$ generated by $\text{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme X* if $\sigma_{X, \xi}$ is unmixed for any point $\xi \in X$.

Theorem 1.26. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Proof. We can assume that $X = \text{Spec } A$ is affine.

Suppose X is Cohen-Macaulay. Let $I \subset A$ be an ideal generated by a_1, \dots, a_r with $r = \text{ht}(I)$. We claim that a_1, \dots, a_r is an A -regular sequence. If so, we get that the unmixedness theorem holds for A by applying Example 1.18 on A/I . Since $\text{ht}(a_1, \dots, a_{r-1}) \leq r - 1$ by Krull's Principal Ideal Theorem 1.14 and $\text{ht}(a_1, \dots, a_r) = r \leq \text{ht}(a_1, \dots, a_{r-1}) + 1$, we have $\text{ht}(a_1, \dots, a_{r-1}) = r - 1$. By

induction on r , we can assume that a_1, \dots, a_{r-1} is an A -regular sequence. Hence any prime ideal $\mathfrak{p} \in \text{Ass } A/(a_1, \dots, a_{r-1})$ has height $r-1$. Now suppose a_r is a zero divisor in $A/(a_1, \dots, a_{r-1})$. Then there exists a prime ideal $\mathfrak{p} \in \text{Ass } A/(a_1, \dots, a_{r-1})$ such that $a_r \in \mathfrak{p}$. Then $I \subset \mathfrak{p}$ and $\text{ht}(I) \leq r-1$. This contradicts that $\text{ht}(I) = r$.

Suppose the unmixedness theorem holds for A . Let $\mathfrak{p} \in \text{Spec } A$ be a prime ideal with $\text{ht}(\mathfrak{p}) = r$. Then $\mathfrak{p} \in \text{Ass } A$ if and only if $\text{ht}(\mathfrak{p}) = 0$. If $r > 0$, there is a nonzero divisor $a \in \mathfrak{p}$. By Krull's Principal Ideal Theorem 1.14, $\text{ht}(\mathfrak{p}A/aA) = r-1$. Inductively, we can find a regular sequence a_1, \dots, a_r in \mathfrak{p} . Then $\text{depth } A_{\mathfrak{p}} = r$. \square

Theorem 1.27. Let X be a locally noetherian scheme. Suppose that X is Cohen-Macaulay. Let $F \subset X$ be a closed subset of codimension $\geq k$. Then the restriction $H^i(X, \mathcal{O}_X) \rightarrow H^i(X \setminus F, \mathcal{O}_X)$ is an isomorphism.

Proof. To be completed. \square

1.5 Regular rings

Definition 1.28. A noetherian ring A is said to be *regular at* $\mathfrak{p} \in \text{Spec } A$ if we have

$$\dim_{\kappa(\mathfrak{p})} T_{A,\mathfrak{p}} = \dim A_{\mathfrak{p}},$$

where $\dim A_{\mathfrak{p}}$ is the Krull dimension of the local ring $A_{\mathfrak{p}}$.

A noetherian ring A is said to be *regular* if it is regular at every prime ideal $\mathfrak{p} \in \text{Spec } A$. This is equivalent to the condition that A verifies (R_k) for all $k \geq 0$.

Remark 1.29. A noetherian ring A is regular if and only if it is regular at every maximal ideal $\mathfrak{m} \in \text{mSpec } A$. The proof uses homological tools; see Theorem ?? and Corollary ??.

Definition 1.30. Let A be a noetherian ring that is regular at $\mathfrak{p} \in \text{Spec } A$. A sequence $t_1, \dots, t_n \in \mathfrak{p}$ is called a *regular system of parameters* at \mathfrak{p} if their images form a basis of the $\kappa(\mathfrak{p})$ -vector space $\mathfrak{p}/\mathfrak{p}^2$.

Proposition 1.31. Let (A, \mathfrak{m}) be a noetherian local ring that is regular at \mathfrak{m} . Let t_1, \dots, t_n be a regular system of parameters at \mathfrak{m} , $\mathfrak{p}_i = (t_1, \dots, t_i)$ and $\mathfrak{p}_0 = (0)$. Then \mathfrak{p}_i is a prime ideal of height i , and A/\mathfrak{p}_i is a regular local ring for all i . In particular, regular local ring is integral, and the regular system of parameters t_1, \dots, t_n is a regular sequence in A .

Proof. By the Krull's Principal Ideal Theorem 1.14, we have

$$n-1 = \dim A - 1 \leq \dim A/(t_1) \leq \dim_{\kappa(\mathfrak{m}/(t_1))} T_{A/(t_1), \mathfrak{m}/(t_1)} \leq n-1.$$

Hence $\dim A/(t_1) = n-1$ and $\text{ht}(t_1) = 1$. Since t_2, \dots, t_n generate $\mathfrak{m}/(t_1)$, we have that $A/(t_1)$ is regular at $\mathfrak{m}/(t_1)$ and the images of t_2, \dots, t_n form a regular system of parameters.

For integrality, we induct on the dimension of A . If $\dim A = 0$, then A is a field and hence

integral. Suppose $\dim A > 0$, let \mathfrak{q} be a minimal prime ideal of A . Then $t_1 \notin \mathfrak{q}$. We have

$$n - 1 = \dim A - 1 \leq \dim A/(\mathfrak{q} + t_1 A) \leq \dim_{\kappa(\mathfrak{q}/(t_1))} T_{A/(\mathfrak{q} + t_1 A), \mathfrak{q}/(t_1)} \leq n - 1.$$

By similar arguments, we have $A/(\mathfrak{q} + t_1 A)$ is regular at $\mathfrak{m}/(\mathfrak{q} + t_1 A)$. By induction hypothesis, both of $A/t_1 A$ and $A/(\mathfrak{q} + t_1 A)$ are integral and of dimension $n - 1$. Hence $t_1 A = t_1 A + \mathfrak{q}$, i.e. $\mathfrak{q} \subset t_1 A$. For every $a = bt_1 \in \mathfrak{q}$, we have $b \in \mathfrak{q}$ since $t_1 \notin \mathfrak{q}$. Then $\mathfrak{q} \subset t_1 \mathfrak{q} \subset \mathfrak{m} \mathfrak{q}$. By Nakayama's Lemma, $\mathfrak{q} = 0$, whence A is integral. \square

Corollary 1.32. A regular noetherian ring is Cohen-Macaulay.

Corollary 1.33. A regular noetherian ring is normal.

Remark 1.34. Indeed we can show a stronger result: a noetherian regular local ring is a UFD; see [ref.](#)