Some Singular Surfaces

In this section, fix an algebraically closed field $\Bbbk$. Everything is over $\Bbbk$ unless otherwise specified.

1 Projective cone over smooth projective curve

Let $C \subset \mathbb{P}^n$ be a smooth projective curve. The *projective cone* over C is the projective variety $X \subset \mathbb{P}^{n+1}$ defined by the same homogeneous equations as C. The variety X is singular at the vertex of the cone, which corresponds to the point $[0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}$.

2 Du Val singularites

Du Val singularities (also known as rational double points, or ADE singularities) are a class of surface singularities that arise in algebraic geometry and complex surface theory. They are characterized by their resolution properties and can be classified according to the ADE classification of simply laced Dynkin diagrams.

A Du Val singularity can be locally described by one of the following equations in $\mathbb{C}^3$:

- A_n singularity: $x^2 + y^2 + z^{n+1} = 0$ for $n \geq 1$
- D_n singularity: $x^2 + y^{n-1} + yz^2 = 0$ for $n \geq 4$
- E_6 singularity: $x^2 + y^3 + z^4 = 0$
- E_7 singularity: $x^2 + y^3 + yz^3 = 0$
- E_8 singularity: $x^2 + y^3 + z^5 = 0$

These singularities are important in the study of algebraic surfaces, particularly in the context of minimal models and the classification of surfaces. They also appear in various areas of mathematics and theoretical physics, including string theory and mirror symmetry.

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3 Quotient singularities

Consider μ_n, the group of n-th roots of unity, acting on $\mathbb{A}^2$ by

$$\zeta \cdot (x, y) = (\zeta x, \zeta^m y)$$

for a fixed integer m with $\gcd(m, n) = 1$.

$(a_1, \ldots, a_n)/r$-singularity is the singularity obtained by taking the quotient of $\mathbb{A}^n$ by the action of μ_r defined by

$$\zeta \cdot (x_1, \ldots, x_n) = (\zeta^{a_1} x_1, \ldots, \zeta^{a_n} x_n)$$

where ζ is a primitive r-th root of unity.

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A_n singularity is the quotient singularity of type $(1,-1)/(n+1)$.

Its minimal resolution has exceptional locus consisting of a chain of n smooth rational curves, each with self-intersection -2. Looks like:

$$\bullet \;\text{———}\; \bullet \;\text{———}\; \bullet \;\text{———}\; \cdots \;\text{———}\; \bullet$$

Du Val singularities can be got by deforming A_n singularities.(general fiber A_n, special fiber Du Val).