
Notes in Algebraic Geometry



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Chapter 1

The First Properties

1.1 Setup and the first examples

1.1.1 Notations

All schemes are assumed to be separated. For a “scheme” which is not separated, we will use the term “prescheme”.

Let A be a ring. We denote by $\operatorname{Spec} A$ the spectrum of A . For an ideal $I \subset A$, we use $V(I)$ to denote the closed subscheme of $\operatorname{Spec} A$ defined by I .

Let S be $\operatorname{Spec} \mathbf{k}$, $\operatorname{Spec} \mathcal{O}_K$ or an algebraic variety. An S -variety is an integral scheme X which is of finite type and flat over S . For an algebraic variety, we mean a \mathbf{k} -variety.

We will use \mathbf{k}, \mathbf{K} to denote fields, and \mathbb{k}, \mathbb{K} to denote their algebraic closure relatively.

Let X be an integral scheme. We denote by $\mathcal{K}(X)$ the function field of X . For a closed point $x \in X$, we denote by $\kappa(x)$ the residue field of x .

We denote the category of S -varieties by \mathbf{Var}_S . We denote by $X(T)$ the set of T -points of X , that is, the set of morphisms $T \rightarrow X$.

Let X be an algebraic variety over \mathbf{k} . A geometrical point is referred a morphism $\operatorname{Spec} \mathbb{k} \rightarrow X$.

When refer a point (may not be closed) in a scheme, we will use the notation $\xi \in X$. We use Z_ξ to denote the Zariski closure of $\{\xi\}$ in X . When we talk about a closed point on an algebraic variety, we will use the notation $x \in X(\mathbb{k})$.

Separated and proper morphisms

1.1.2 Examples

Chapter 2

Birational Geometry

2.1 Bend and Break

2.1.1 Preliminary

Definition 2.1.1 (Frobenius morphism). Let X be a variety over a field \mathbb{k} of characteristic $p > 0$. Denote the structure morphism by $\pi : X \rightarrow \operatorname{Spec} \mathbb{k}$. The *absolute Frobenius morphism* is the morphism given by $\mathcal{O}_X \rightarrow \mathcal{O}_X, f \mapsto f^p$, denoted by $\operatorname{Frob}_{X/\mathbb{F}_p}$. The *relative Frobenius morphism* is the morphism $\operatorname{Frob}_{X/\mathbb{k}}$ given by the following commutative diagram:

$$\begin{array}{ccccc}
 X & & & & \\
 \searrow^{\operatorname{Frob}_{X/\mathbb{k}}} & & \operatorname{Frob}_{X/\mathbb{F}_p} & \searrow & \\
 X \times_{\mathbb{k}} \operatorname{Spec} \mathbb{k} & \xrightarrow{\quad} & X & & \\
 \downarrow \pi & & \downarrow \pi & & \\
 \operatorname{Spec} \mathbb{k} & \xrightarrow{\operatorname{Frob}_{\mathbb{k}/\mathbb{F}_p}} & \operatorname{Spec} \mathbb{k} & &
 \end{array}$$

We usually denote $X \times_{\mathbb{k}} \operatorname{Spec} \mathbb{k}$ appearing above by $X^{(p)}$.

Proposition 2.1.2. Let X be a variety of dimension d over a field \mathbb{k} of characteristic $p > 0$. Then the relative Frobenius morphism $\operatorname{Frob}_{X/\mathbb{k}} : X \rightarrow X^{(p)}$ is a finite morphism of degree p^d over \mathbb{k} .

2.1.2 Deformation of curves

Theorem 2.1.3 (ref. [Kol96, Chapter II, Theorem 1.2]). Let C be a smooth projective curve of genus g and X a smooth projective variety of dimension n . Let $f : C \rightarrow X$ be a non-constant morphism. Then every irreducible component of $\operatorname{Mor}(C, X)$ containing f has dimension at least

$$-K_Y \cdot f(C) + (1 - g)n.$$

Proposition 2.1.4. Let X be a projective variety and $f : C \rightarrow X$ a non-constant morphism from a pointed smooth projective curve $p_0 \in C$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t : C \rightarrow X$ for $t \in T$ such that $f_0 = f$ and $f_t(p_0) = x_0$ for some point $x_0 \in X$ and all t . Then there exist some rational curves $\Gamma_1, \dots, \Gamma_m \subset X$ such that

$$(a) \quad x_0 \in \bigcup_{i=1}^m \Gamma_i;$$

$$(b) \quad \text{there is a morphism } g : C \rightarrow X \text{ such that } f(C) \equiv_{alg} g(C) + \sum_{i=1}^m a_i \Gamma_i \text{ with } a_i > 0 \text{ for all } i.$$

Proposition 2.1.5. Let X be a projective variety and $f : \mathbb{P}^1 \rightarrow X$ a non-constant morphism with $f(0) = x_0, f(\infty) = x_\infty$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t : \mathbb{P}^1 \rightarrow X$ for $t \in T$ such that $f_0 = f$ and $f_t(0) = x_0, f_t(\infty) = x_\infty$ for all t . Then there exists a curve $C \subset X$ such that $f(\mathbb{P}^1) \equiv_{alg} aC$ with $a > 1$.

2.1.3 Find rational curves

Theorem 2.1.6. Let X be a smooth Fano variety. Then for any $x \in X(\mathbb{k})$, there is a rational curve C passing through x with

$$0 < -C \cdot K_X \leq \dim X + 1.$$

Proof. To be completed. □

Theorem 2.1.7. Let X be a smooth projective variety such that $K_X \cdot C < 0$ for some irreducible curve $C \subset X$. Let H be an ample divisor on X . Then there exists a rational curve Γ such that

$$-(K_X \cdot C) \cdot \frac{H \cdot \Gamma}{H \cdot C} \leq -K_X \cdot \Gamma \leq \dim X + 1.$$

Proof. To be completed. □

Theorem 2.1.8. Let (X, B) be a projective klt pair and $f : X \rightarrow Y$ a birational projective morphism. Suppose that $K_{(X,B)}$ is f -ample. Then the exceptional locus of f is covered by rational curves Γ with

$$0 < -K_{(X,B)} \cdot \Gamma \leq 2 \dim X.$$

Theorem 2.1.9. Let X be a smooth projective variety of dimension n and H, H_1, \dots, H_{n-1} ample divisors on X . Suppose that $K_X \cdot H_1 \cdots H_{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H \cdot H_1 \cdots H_{n-1}}{K_X \cdot H_1 \cdots H_{n-1}}.$$

Proposition 2.1.10. Let X be a normal projective variety of dimension n and H an ample divisor on X . Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

2.2 Kodaira Vanishing Theorem

2.2.1 Preliminary

Theorem 2.2.1 (Serre Duality). Let X be a Cohen-Macaulay projective variety of dimension n over \mathbf{k} and D a divisor on X . Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \dots, n.$$

Theorem 2.2.2 (Log Resolution of Singularities). Let X be an irreducible reduced algebraic variety over \mathbb{C} (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z . Then there is a smooth variety (or analytic space) Y and a projective morphism $f : Y \rightarrow X$ such that

- (a) f is an isomorphism over $X - (\text{Sing}(X) \cup \text{Supp } Z)$,
- (b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and
- (c) $\text{Exc}(f) \cup D$ is an snc divisor.

Theorem 2.2.3 (Lefschetz Hyperplane Theorem). Let X be a smooth projective variety of dimension n over \mathbb{C} and Y a hyperplane section of X . Then the restriction map

$$H^k(X, \mathbb{C}) \rightarrow H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 2.2.4 (Hodge Decomposition). Let X be a smooth projective variety of dimension n over \mathbb{C} . Then for any k , there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 2.2.3 and Theorem 2.2.4, we have the following lemma.

Lemma 2.2.5. Let X be a smooth projective variety of dimension n over \mathbb{C} and Y a hyperplane

section of X . Then the restriction map $r_k : H^k(X, \mathbb{C}) \rightarrow H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \rightarrow H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p+q < n-1$ and an injection for $p+q = n-1$. In particular,

$$H^p(X, \mathcal{O}_X) \rightarrow H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n-1$ and an injection for $p = n-1$.

Theorem 2.2.6 (Leray spectral sequence). Let $f : Y \rightarrow X$ be a morphism of varieties and \mathcal{F} a coherent sheaf on Y . Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(Y, \mathcal{F}).$$

2.2.2 Kodaira Vanishing Theorem

Lemma 2.2.7. Let X be a smooth projective variety over \mathbf{k} and \mathcal{L} a line bundle on X . Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \rightarrow X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D . It induces a homomorphism $\mathcal{L}^{-m} \rightarrow \mathcal{O}_X$. Consider the \mathcal{O}_X -algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} \right) / (\mathcal{L}^{-m} \rightarrow \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then \mathcal{A} is a finite \mathcal{O}_X -algebra. Let $Y := \text{Spec}_X \mathcal{A}$. Then Y is a finite \mathcal{O}_X -scheme and the natural morphism $f : Y \rightarrow X$ is finite and surjective.

For every $x \in X$, let \mathcal{L} locally generated by t near x . Then \mathcal{O}_Y locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y . Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. \square

Remark 2.2.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D'_i = f^*(D_i)$. Then $D' + \sum_{i=1}^k D'_i$ is snc on Y by considering the local regular system of parameters.

Lemma 2.2.9. Let $f : Y \rightarrow X$ be a finite surjective morphism of projective varieties and \mathcal{L} a line bundle on X . Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^* \mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^* \mathcal{L}) \cong H^i(X, f_* \mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$ splits by the trace map $(1/n) \text{Tr}_{Y/X}$. Thus we have $f_* \mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_* \mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. \square

Theorem 2.2.10 (Kodaira Vanishing Theorem). Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0 and A an ample divisor on X . Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 2.2.7 and 2.2.9, after taking a multiple of A , we can assume that A is effective. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(-A) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_A \rightarrow 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \rightarrow H^{i-1}(X, \mathcal{O}_A) \rightarrow H^i(X, \mathcal{O}_X(-A)) \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 2.2.5 and Serre duality (Theorem 2.2.1). \square

2.2.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

Theorem 2.2.11 (Kawamata-Viehweg Vanishing Theorem I). Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0 and D a nef and big \mathbf{r} -divisor on X . Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Theorem 2.2.12 (Kawamata-Viehweg Vanishing Theorem II). Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0 and D a nef and big \mathbb{Q} -divisor on X . Suppose that $[D] - D$ has snc support. Then

$$H^i(X, K_X + [D]) = 0, \quad \forall i > 0.$$

Theorem 2.2.13 (Kawamata-Viehweg Vanishing Theorem III). Let (X, B) be a klt pair over \mathbf{k} of characteristic 0. Let D be a nef \mathbb{Q} -divisor on X such that $D + K_{(X, B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X, B)} + D) = 0, \quad \forall i > 0.$$

If we replace the assumption "nef and big" of D by "ample" in II and III, we denote them as "II(ample)" and "III(ample)" respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \Rightarrow \text{II(ample)} \Rightarrow \text{III(ample)} \Rightarrow \text{I} \Rightarrow \text{II} \Rightarrow \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

Proof of II (Theorem 2.2.12). Set $M := [D]$. Let

$$B := \sum_{i=1}^k b_i B_i := [D] - D = M - A, \quad b_i \in (0, 1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k . When $k = 0$, the conclusion follows from Theorem 2.2.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 2.2.10.)) Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 2.2.15 and 2.2.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 2.2.7 on B_k , we can find a finite surjective morphism $f : X' \rightarrow X$ such that $f^*B_k = cB'_k$, $B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^k B'_i$ is an snc divisor on X' . Let $B' = \sum_{i=1}^{k-1} B'_i$, $A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$ vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^* \mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 2.2.9. \square

Proof of III (Theorem 2.2.13). Let $f : \tilde{X} \rightarrow X$ be a resolution such that $\text{Supp } f^*B \cup \text{Exc } f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X}, \tilde{B})} + f^*D,$$

where $\tilde{B} \in (0, 1)$ has snc support and E is an effective exceptional divisor.

By Lemma 2.2.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X}, \tilde{B})} + f^*D) = H^i(X, f_* \mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 2.2.12 in either case relative to the assumption of D . \square

Proof of I (Theorem 2.2.11). By Lemma 2.2.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor A . Then the theorem comes down to Theorem 2.2.13. \square

Lemma 2.2.14. Let $f : Y \rightarrow X$ be a birational morphism of projective varieties with Y smooth and X has only rational singularities. Let E be an effective exceptional divisor on Y and D a divisor on X . Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

Proof. I am unable to proof this lemma. \square

Lemma 2.2.15. Let X be a projective variety, \mathcal{L} a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \rightarrow X$ and a line bundle \mathcal{L}' on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X , then we can take f such that f^*D is an snc divisor on Y .

Proof. We can assume that \mathcal{L} is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & \mathbb{P}^N \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{\varphi_{\mathcal{L}}} & \mathbb{P}^N \end{array},$$

where $g : [x_0 : \dots : x_N] \mapsto [x_0^m : \dots : x_N^m]$. The morphism f is finite and surjective since so is g . Let $\mathcal{L}' := \psi^* \mathcal{L}$.

For smoothness, we can compose g with a general automorphism of \mathbb{P}^N . Then the conclusion follows from [Har77, Chapter III, Theorem 10.8]. \square

Lemma 2.2.16 (ref. [KM98, Theorem 5.10, 5.22]). Let (X, B) be a klt pair over \mathbf{k} of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Lemma 2.2.17. Let X be a projective variety of dimension n and D a nef and big divisor on X . Then there exists an effective divisor B such that for every k , there is an ample divisor A_k such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k} B.$$

Proof. By **definition** of big divisor, there exists an ample divisor A_1 and effective divisor B such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k} B.$$

Since A is ample and D is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. \square

2.3 Cone Theorem

2.3.1 Preliminary

Theorem 2.3.1 (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27]). Let X be a projective variety and ℓ an semiample line bundle on X . Then there exists a fibration $\varphi : X \rightarrow Y$ of projective varieties such that for any $m \gg 0$ with ℓ^m base point free, we have that the morphism φ_{ℓ^m} induced by ℓ^m is isomorphic to φ . Such a fibration is called the *Iitaka fibration* associated to ℓ .

Theorem 2.3.2 (Rigidity Lemma, ref. [Deb01, Lemma 1.15]). Let $\pi_i : X \rightarrow Y_i$ be proper morphisms of varieties over a field \mathbf{k} for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Theorem 2.3.3. Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

Proposition 2.3.4. Let X be a normal projective variety of dimension n and H an ample divisor on X . Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

Sketch of proof. Take a resolution $f : Y \rightarrow X$, then f^*H is nef on Y and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor H_Y on Y closed enough to f^*H such that $K_Y \cdot H_Y^{n-1} < 0$. By [MM86, Theorem 5] and take limit for H_Y . \square

Lemma 2.3.5 (ref. [Kaw91, Lemma]). Let (X, B) be a projective klt pair and $f : X \rightarrow Y$ a birational projective morphism. Let E be an irreducible component of dimension d of the exceptional locus of f and $\nu : E^\nu \rightarrow X$ the normalization of E . Suppose that $f(E)$ is a point. Then for any ample divisor H on X , we have

$$K_{E^\nu} \cdot \nu^*H^{d-1} \leq K_{(X,B)}|_{E^\nu} \cdot \nu^*H^{d-1}.$$

2.3.2 Non-vanishing Theorem

Theorem 2.3.6 (Non-vanishing Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X . Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

2.3.3 Base Point Free Theorem

Theorem 2.3.7 (Base Point Free Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X . Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, mD is base point free.

Remark 2.3.8. In general, we say that a Cartier divisor D is *semiample* if there exists a positive integer m such that mD is base point free. The statement in Base Point Free Theorem (Theorem 2.3.7) is strictly stronger than the semiample condition. For example, let ℓ be a torsion line bundle, then ℓ is semiample but there exists no positive integer M such that $m\ell$ is base point free for all $m > M$.

2.3.4 Rationality Theorem

Lemma 2.3.9 (ref. [KM98, Theorem 1.36]). Let X be a proper variety of dimension n and D_1, \dots, D_m Cartier divisors on X . Then the Euler characteristic $\chi(n_1 D_1, \dots, n_m D_m)$ is a polynomial in (n_1, \dots, n_m) of degree at most n .

Theorem 2.3.10 (Rationality Theorem). Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X . Let

$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H . Then $t = v/u \in \mathbb{Q}$ and

$$0 \leq v \leq a(X) \cdot (\dim X + 1).$$

Proof. For every $r \in \mathbb{R}_{>0}$, let

$$v(r) := \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We need to show that $v(t) \leq a(\dim X + 1)$. For every $(p, q) \in \mathbb{Z}_{>0}^2$, set $D(p, q) := paK_{(X,B)} + qH$. If $(p, q) \in \mathbb{Z}_{>0}^2$ with $0 < atp - q < t$, then we have $D(p, q)$ is not nef and $D(p, q) - K_{(X,B)}$ is ample.

Step 1. We show that a polynomial $P(x, y) \neq 0 \in \mathbb{Q}[x, y]$ of degree at most n is not identically zero on the set

$$\{(p, q) \in \mathbb{Z}^2 : p, q > M, 0 < atp - q < t\varepsilon\}, \quad \forall M > 0,$$

if $v(t)\varepsilon > a(n + 1)$.

If $v(t) = \infty$, for any n , we show that we can find infinitely many lines L such that $\#L \cap \Lambda \geq n + 1$. If so, Λ is Zariski dense in \mathbb{Q}^2 . Since $1/at \in \mathbb{R} \setminus \mathbb{Q}$, there exist $p_0, q_0 > M$ such that

$$0 < \frac{p_0}{q_0} - \frac{1}{at} < \frac{\varepsilon}{(n+1)a} \cdot \frac{1}{q_0}, \text{ i.e. } 0 < atp_0 - q_0 < \frac{\varepsilon t}{n+1}.$$

Then $(ip_0, iq_0) \in \Lambda \cap \{p_0 y = q_0 x\}$ for $i = 1, \dots, n+1$. Since M is arbitrary, there are infinitely many such lines L .

Suppose $v(t) = v < \infty$ and $t = v/u$. Then the inequality is equivalent to $0 < aup - vq < \varepsilon v$. Note that $\gcd(au, v) | a$, then $aup - vq = ai$ has integer solutions for $i = 1, \dots, n+1$. Since $v(t)\varepsilon > a(n+1)$, there are at least $n+1$ lines which intersect Λ in infinitely many points. This enforces any polynomial which vanishes on Λ has degree at least $n+1$.

Step 2. There exists an index set $\Lambda \subset \mathbb{Z}^2$ such that Λ contains all sufficiently large (p, q) with $0 \leq atp - q \leq t$ and

$$Z := \text{Bs } |D(p, q)| = \text{Bs } |D(p', q')| \neq \emptyset, \quad \forall (p, q), (p', q') \in \Lambda.$$

For every $(p, q) \in \mathbb{Z}_{>0}^2$ with $0 < atp - q < t$, choose $k \in \mathbb{Z}_{>0}$ such that $k(atp - q) > t$. Then

for all $p', q' > kp$ with $0 < atp' - q' < t$, we have

$$p' - kp \geq 0, \quad q' - kp > t(p' - kp).$$

It follows that

To be completed.

Step 3. Suppose the contradiction that $v(t) > a(\dim X + 1)$. Then we show that $H^0(X, D(p, q)) \neq 0$ for all $(p, q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem (Theorem 2.3.7).

Let $P(x, y) := \chi(D(x, y))$ be the Hilbert polynomial of $D(x, y)$. Note that $P(0, n) = \chi(nH) \neq 0$ since H is ample. Then $P(x, y) \neq 0$ and $\deg P \leq \dim X$. By Step 1, P is not identically zero on Λ . Note that $D(p, q) - K_{(X, B)}$ is ample for all $(p, q) \in \Lambda$, then $h^i(X, D(p, q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem (Theorem 2.2.13). Then

$$P(p, q) = \chi(D(p, q)) = h^0(X, D(p, q)) \neq 0$$

for some $(p, q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p, q)) \neq 0$ for all $(p, q) \in \Lambda$.

Step 4. We follow the same line of the proof of Base Point Free Theorem (Theorem 2.3.7) to show that there is a section which does not vanish on Z .

Fix $(p, q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n + 1)/u$. Let $f : Y \rightarrow X$ be a resolution such that

- (a) $K_{Y, B_Y} = f^*K_{(X, B)} + E_Y$ for some effective exceptional divisor E_Y , and Y, B_Y is a klt pair;
- (b) $f^*[D(p, q)] = [L] + F$ for some effective divisor F and a base point free divisor L , and $f(\text{Supp } F) = Z$;
- (c) $f^*D(p, q) - f^*K_{(X, B)} - E_0$ is ample for some effective \mathbb{Q} -divisor $E_0 \in (0, 1)$, and coefficients of E_0 are sufficiently small;
- (d) $B_Y + E_Y + F + E_0$ has snc support.

Such resolution exists by [KM98].

Let $c := \inf\{[B_Y + E_0 + tF] \neq 0\}$. Adjust the coefficients of E_0 slightly such that $[B_Y + E_0 + cF] = F_0$ for unique prime divisor F_0 with $F_0 \subset \text{Supp } F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then (Y, Δ_Y) is a klt pair.

Let

$$\begin{aligned} N(p', q') &:= f^*D(p', q') + E_Y - F_0 - K_{(Y, \Delta_Y)} \\ &= (f^*D(p', q') - (1 + c)f^*D(p, q)) + (f^*D(p, q) - f^*K_{(X, B)} - E_0) + c(f^*D(p, q) - F). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p', q') \in \Lambda : 0 < atp' - q' < atp - q, \ p', q' > (1 + c) \max\{p, q\}\},$$

the divisor $f^*D(p', q') - (1 + c)f^*D(p, q) = f^*D(p' - (1 + c)p, q' - (1 + c)q)$ is ample, and hence $N(p', q')$ is ample.

By the exact sequence

$$0 \rightarrow \sigma_Y(f^*D(p', q') + E_Y - F_0) \rightarrow \sigma_Y(f^*D(p', q') + E_Y) \rightarrow \sigma_{F_0}((f^*D(p', q') + E_Y)|_{F_0}) \rightarrow 0$$

and Kawamata-Viehweg Vanishing Theorem ([Theorem 2.2.13](#)), we get a surjective map

$$H^0(Y, f^*D(p', q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}).$$

On F_0 , consider the polynomial $\chi((f^*D(p', q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n - 1$ and by the construction of $(p, q), \Lambda_0$, similar to [Step 3](#), we can show that $\chi((f^*D(p', q') + E_Y)|_{F_0})$ is not identically zero on Λ_0 . By adjunction, we have $(f^*D(p', q') + E_Y)|_{F_0} = N(p', q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p', q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem ([Theorem 2.2.13](#)) to get

$$h^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}) = \chi(F_0, (f^*D(p', q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \text{Bs } |D(p', q')|$. \square

2.3.5 Cone Theorem and Contraction Theorem

Theorem 2.3.11 (Cone Theorem). Let (X, B) be a projective klt pair. Then there exist countably many rational curves $C_i \subset X$ with

$$0 < -K_{(X, B)} \cdot C_i \leq 2 \dim X$$

such that

(a) we have a decomposition of cones

$$\text{Psef}_1(X) = \text{Psef}_1(X)_{K_{(X, B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X , we have

$$\text{Psef}_1(X) = \text{Psef}_1(X)_{K_{(X, B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. Let $F_D := \text{Psef}_1(X) \cap D^\perp$ for a nef divisor D on X . If $\dim F_D = 1$, we also write $R_D := F_D$. Let $H_1, \dots, H_{\rho-1}$ be ample divisors on X such that they together with $K_{(X, B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} := S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

Step 5. There exists an integer N such that for every $K_{(X, B)}$ -negative extremal face F_D and for every ample divisor H , there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \neq F_{nD+rK_{(X, B)}+nH} \subset F_D$.

Let $N := (a(X)(\dim X + 1))!$, where $a(X)$ is the number in [Theorem 2.3.10](#). For every n , $nD + H$

is an ample divisor and by [Theorem 2.3.10](#), the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \geq 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since $K_{(X,B)} + (N/r_n)((n+1)D + H)$ is nef, we have $r_n \leq r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$ implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \rightarrow r \in \mathbb{Z}_{\geq 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors N_1, N_2 , we have $F_{N_1+N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists m large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

Step 6. Let $\Phi : N_1(X)_{K_{(X,B)} < 0} \rightarrow \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left(\frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha} \right).$$

We show that the image of R_D under Φ lies in a \mathbb{Z} -lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class ξ . By [Step 5](#), we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers n, r . Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of R_D under Φ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

Step 7. We show that every $K_{(X,B)}$ -negative extremal ray of $\text{Psef}_1(X)$ is of the form R_D for some nef divisor D on X .

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$ -negative extremal ray. Then R is of form $D^\perp \cap \text{Psef}_1(X)$ for some nef \mathbb{R} -divisor D on X by [Theorem 2.3.3](#). We need to show that D can be choose as a nef \mathbb{Q} -divisor. There is a sequence of nef but not ample \mathbb{Q} -divisors D_m such that $D_m \rightarrow D$ as $m \rightarrow \infty$. We adjust D_m such that $\dim F_{D_m} = 1$ for all n .

By re-choosing H_i , we can assume that $D = a_1H_1 + \dots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is D_m . Then F_{D_m} is $K_{(X,B)}$ -negative. Note that $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by [Step 5](#). If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{F_{D_m}}$. We can assume that $r_1K_{(X,B)} + NH_1$ is not identically zero on F_{D_m} . Then we can choose n large enough such that $\|r_1K_{(X,B)} + NH_1\|/n < 1/m$. Replace D_m by $D_m + (r_1K_{(X,B)} + NH_1)/n$. Inductively we construct D_m nef \mathbb{Q} -divisor with $D_m \rightarrow D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that ξ_m converges. Then $\xi_m \rightarrow \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, Φ is well-defined at ξ and the image of ξ_m under Φ is discrete. Hence $\xi = \xi_m$ for all m large enough. It follows that $R = R_{D_m}$ for a nef \mathbb{Q} -divisor D_m .

Step 8. We show that any $K_{(X,B)}$ -negative extremal ray R_D contains the class of a rational curve C with $0 < -K_{(X,B)} \cdot C \leq 2 \dim X$.

By Theorem 2.3.13, let $\varphi_D : X \rightarrow Y$ be the contraction associated to R_D (note that we do not need the step to proof Theorem 2.3.13). If $\dim Y < \dim X$, let F be a general fiber of φ_D . By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that H is ample on F . By Proposition 2.3.4. In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By Lemma 2.3.5, we can use Proposition 2.3.4 to get the result.

To be completed.

Step 9. Proof of the theorem.

Given an ample divisor H on X , note that εH has positive minimum δ on $\text{Psef}_1(X) \cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \text{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and Φ is well-defined on it. By Steps 6 and 7, there are only finitely many extremal rays on $\text{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \leq 0}$. By Step 8, we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$\mathcal{C} := \text{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum r_{\geq 0} [C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\} \subset \mathcal{C}$ such that $\alpha_n \rightarrow \alpha \in N_1(X)_{\mathbb{R}}$. Note that $\text{Psef}_1(X)$ is closed, hence $\alpha \in \text{Psef}_1(X)$. We only need to consider the case $\alpha \cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$. Then $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$ for all n large enough. Note that $\mathcal{C} \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$ is a polyhedral cone by Step 6 and hence is closed. Then $\alpha \in \mathcal{C}$ and the conclusion follows. \square

Remark 2.3.12. Thanks for my friend Qin for pointing out that the extremal ray in Theorem 2.3.11 may not be exposed.

Theorem 2.3.13 (Contraction Theorem). Let (X, B) be a projective klt pair and $F \subset \text{Psef}_1(X)$ a $K_{(X,B)}$ -negative extremal face of $\text{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \rightarrow Y$ of projective varieties such that

- (a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;
- (b) up to linearly equivalence, any Cartier divisor G with $F \subset G^\perp = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$ comes from a Cartier divisor on Y , i.e., there exists a Cartier divisor G_Y on Y such that $G \sim \varphi_F^* G_Y$.

Proof. We follow the following steps to prove the theorem.

Step 10. We show that there exists a nef divisor D on X such that $F = D^\perp \cap \text{Psef}_1(X)$. In other words, F is defined on $N_1(X)_{\mathbb{Q}}$.

We can choose an ample divisor H and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on F since $F \cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in

$N_1(X)_{\mathbb{R}}$. Then by Cone Theorem (Theorem 2.3.11), F is an extremal face of a rational polyhedral cone, namely $\text{Psef}_1(X)_{K_{(X,B)} + (1/n)H \leq 0}$. It follows that $F^\perp \subset N^1(X)_{\mathbb{R}}$ is defined on \mathbb{Q} . Since F is extremal and $K_{(X,B)} + (1/n)H$ -negative, the set $\{L \in F^\perp : L|_{\text{Psef}_1(X) \setminus F} > 0\}$ has non-empty interior in F^\perp by Theorems 2.3.3 and 2.3.11. Then there exists a Cartier divisor D such that $D \in F^\perp$ and $D|_{\text{Psef}_1(X) \setminus F} > 0$. It follows that D is nef and $F = D^\perp \cap \text{Psef}_1(X)$.

Step 11. Let $\varphi : X \rightarrow Y$ be the Iitaka fibration associated to D by Theorem 2.3.1. We show that φ is the desired fibration.

Note that $\text{Psef}_1(X)_{K_{(X,B)} \geq 0} \cap S^{\rho-1}$ is compact and D is strictly positive on it. Then there exist $a \geq 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\text{Psef}_1(X)_{K_{(X,B)} \geq 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since F is $K_{(X,B)}$ -negative. Then by Base Point Free Theorem (Theorem 2.3.7), we know that mD is base point free for all $m \gg 0$. Hence we can apply Theorem 2.3.1 to get a fibration $\varphi_D : X \rightarrow Y$.

First we show that D comes from Y . Note that mD and $(m+1)D$ induces the same fibration φ_D for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m)D_{Y,m}$ and $D_{Y,m}$ is ample. Hence D_Y is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi_D^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that C is contracted by φ_D if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let G be arbitrary Cartier divisor on X such that $F \subset G^\perp$. Since D is strictly positive on $\text{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' := mD + G$, we have $D'^\perp \cap \text{Psef}_1(X) = F$. Then by the same argument as above, we get another fibration $\varphi_{D'} : X \rightarrow Y'$ such that a curve C is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 2.3.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on Y . In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor D'_Y on Y . Then $G = D' - mD$ also comes from Y . \square

Remark 2.3.14. The Step 10 is amazing. If F is not $K_{(X,B)}$ -negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E . By [Laz04, Lemma 1.5.4], we know that $\text{Psef}_1(X)$ is a circular cone. Then we see there indeed exist some irrational extremal faces of $\text{Psef}_1(X)$.

Definition 2.3.15. Let (X, B) be a projective klt pair and R a $K_{(X,B)}$ -negative extremal ray of $\text{Psef}_1(X)$ with contraction $\varphi_R : X \rightarrow Y$. There are three types of contractions:

- (a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension one;
- (b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension at least two;
- (c) *Mori fiber space*: if $\dim X > \dim Y$.

Proposition 2.3.16. Let (X, B) be a \mathbb{Q} -factorial projective klt pair and R a $K_{(X,B)}$ -negative extremal ray of $\text{Psef}_1(X)$. Suppose that the contraction $\varphi : X \rightarrow Y$ associated to R is either divisorial or a Mori fiber space. Then Y is \mathbb{Q} -factorial.

Proof. Let D be a prime Weil divisor on Y and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\geq 0}[C]$ for an irreducible curve C contracted by φ . Set $D_X := \varphi|_{\varphi^{-1}(U)}^{-1} D$. Then D_X is a prime Weil divisor on X and hence is \mathbb{Q} -Cartier.

If φ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber F of φ . Then by Contraction Theorem (Theorem 2.3.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor D' on Y . We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence D is \mathbb{Q} -Cartier.

If φ is a divisorial contraction, let E be the exceptional divisor of φ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier \mathbb{Q} -divisor E_Y on Y). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 2.3.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor D' on Y . Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence D is \mathbb{Q} -Cartier. \square

Remark 2.3.17. If φ is a small contraction, then Y is never \mathbb{Q} -factorial. Otherwise, let B_Y be the strict transform of B on Y . Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset U . Suppose $K_{(Y,B_Y)}$ is \mathbb{Q} -Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

2.4 F-singularities

Let \mathbb{k} be an algebraically closed field of characteristic $p > 0$. Let X be a projective variety over \mathbb{k} . Let F denote the relative Frobenius morphism on X .

Definition 2.4.1. We say that X is *F-finite* if $F : X \rightarrow X^{(p)}$ is finite.

Definition 2.4.2. We say that X is *globally F-split* if $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ splits as \mathcal{O}_X -modules for some $e \geq 0$. This is equivalent to for every $e \in \mathbb{Z}_{>0}$, $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ splits as \mathcal{O}_X -modules.

Definition 2.4.3. Fix $\phi : F_*^e \mathcal{O}_X \rightarrow \mathcal{O}_X$ a splitting of $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$. Define $\phi^n : F_*^{ne} \mathcal{O}_X \rightarrow \mathcal{O}_X$ by induction:

$$\phi^n := \phi \circ F_*^e(\phi^{n-1}).$$

Theorem 2.4.4. Above ϕ^n will be stable. That is, $\mathfrak{I}\phi^n = \mathfrak{I}\phi^{n+1}$ for all $n \gg 0$.

Definition 2.4.5. Let $\sigma(X, \phi) := \mathfrak{I}\phi^n$. We say that (X, ϕ) is *F-pure* if $\sigma(X, \phi) = \mathcal{O}_X$.

Proposition 2.4.6. There is a bijection between

$$\{\text{effective } \mathbb{Q}\text{-divisor } \Delta \text{ such that } (p^e - 1)(K_X + \Delta) \text{ is Cartier}\} / \sim$$

and

$$\{\text{line bundles } \ell \text{ and } \phi : F_*^e \ell \rightarrow \mathcal{O}_X\}.$$

Proof. We have

$$F_X^e \mathcal{O}_X((1 - p^e)K_X) \rightarrow \mathcal{O}_X$$

given by $F^e \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_X(K_X)$ and reflexivity of $\mathcal{O}_X(K_X)$. Since Δ is effective, we have

$$F^e(\mathcal{O}_X((1 - p^e)(K_X + \Delta))) \rightarrow F^e \mathcal{O}_X((1 - p^e)(K_X)) \rightarrow \mathcal{O}_X.$$

The another direction is by Grothendieck's duality

$$\mathcal{H}om_{\mathcal{O}_X}(F^e \ell, \mathcal{O}_X) \cong F_*^e(\ell^{-1} \otimes \mathcal{O}_X((1 - p^e)K_X)).$$

□

Definition 2.4.7. Let $\phi_{e,\Delta} : F_*^e(\mathcal{O}_X((1 - p^e)(K_X + \Delta))) \rightarrow \mathcal{O}_X$ be the morphism corresponding to the effective \mathbb{Q} -divisor Δ .

We say that (X, Δ) is *F-pure* if $(X, \phi_{e,\Delta})$ is *F-pure*.

We say that (X, Δ) is *globally F-split* if for every Weil divisor $D \geq 0$, $\mathcal{O}_X \rightarrow F_*^e(\mathcal{O}_X([(p^e - 1)\Delta] + D))$ admits a splitting for some $e \geq 0$.

We say that (X, Δ) is *strongly F-split* if for every Weil divisor $D \geq 0$, $\mathcal{O}_X \rightarrow F_*^e(\mathcal{O}_X([(p^e - 1)\Delta] + D))$ admits a local splitting for some $e \geq 0$.

Definition 2.4.8.

Definition 2.4.9. $S^0(X, \sigma(X, \Delta) \otimes m)$

Proposition 2.4.10. Let X be a globally *F-split* projective variety. Then we have

- (a) suppose that $H^i(X, \ell^n) = 0$ for all $i > 0$ and all $n \gg 0$, then $H^i(X, \ell) = 0$ for all $i > 0$;
- (b) for every ample divisor A on X , we have $H^i(X, \mathcal{O}_X(A)) = 0$ for all $i > 0$;
- (c) suppose that X is Cohen-Macaulay and A -ample, then $H^i(X, \mathcal{O}_X(-A)) = 0$ for all $i < \dim X$;
- (d) suppose that X is normal and A -ample, then $H^i(X, \omega_X(A)) = 0$ for all $i > 0$.

Appendix A

Commutative Algebra

A.1 Elementary Results

To be completed

A.1.1 Rings and modules

In the appendix and all the note, the “ring” is always commutative and with identity. We denote by $\text{Spec } A$ the set of prime ideals of a ring A . We denote by $\text{mSpec } A$ the set of maximal ideals of A . Let $I \subset A$ be an ideal of A . We define

$$V(I) := \{\mathfrak{p} \in \text{Spec } A : I \subset \mathfrak{p}\}.$$

Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A . We define

$$(\mathfrak{a} : \mathfrak{b}) := \{a \in A : a\mathfrak{b} \subset \mathfrak{a}\}.$$

This is an ideal of A .

Let $\text{rad}(A)$ be the Jacobian radical of A , i.e., the intersection of all maximal ideals of A . Let $\text{nil}(A)$ be the nilradical of A , i.e., the ideal of A consisting of all nilpotent elements.

Proposition A.1.1. Let A be a ring. Then we have

$$\text{nil}(A) = \bigcap_{\mathfrak{p} \in \text{Spec } A} \mathfrak{p}.$$

Proof. To be completed. □

Proposition A.1.2. Let A be a ring, $\mathfrak{p}, \mathfrak{p}_i$ prime ideals of A and $\mathfrak{a}, \mathfrak{a}_i$ ideals of A .

- (a) Suppose $\mathfrak{a} \subset \bigcup_{i=1}^n \mathfrak{p}_i$. Then there exists i such that $\mathfrak{a} \subset \mathfrak{p}_i$.
- (b) Suppose $\bigcap_{i=1}^n \mathfrak{a}_i \subset \mathfrak{p}$. Then there exists i such that $\mathfrak{a}_i \subset \mathfrak{p}$.

Proof. To be completed. □

Let M be an A -module. We say that M is *finite* if there exists an exact sequence

$$A^n \rightarrow M \rightarrow 0.$$

We say that M is *finite presented* if there exists an exact sequence

$$A^m \rightarrow A^n \rightarrow M \rightarrow 0.$$

If A is a noetherian ring, then every finite A -module is finite presented.

Definition A.1.3. Let A be a ring and M an A -module. The *support* of M is defined as

$$\text{Supp } M := \{\mathfrak{p} \in \text{Spec } A : M_{\mathfrak{p}} \neq 0\}.$$

The *annihilator* of M is defined as

$$\text{Ann } M := \{a \in A : aM = 0\}.$$

This is an ideal of A .

Proposition A.1.4. Let A be a ring and M a finite A -module. Then $\text{Supp } M = V(\text{Ann } M)$. In particular, $\text{Supp } M$ is a closed subset of $\text{Spec } A$.

Proof. To be completed. □

A.1.2 Localization

Definition A.1.5. Let A be a ring and $S \subset A$ a multiplicative subset, i.e., $1 \in S$ and $s_1, s_2 \in S$ implies $s_1 s_2 \in S$. The *localization* of A at S is defined as

$$S^{-1}A := A \times S / \sim,$$

where $(a, s) \sim (b, t)$ if there exists $u \in S$ such that $u(at - bs) = 0$. To be completed.

Proposition A.1.6.

A.1.3 Chain conditions

A.1.4 Nakayama's Lemma

Theorem A.1.7 (Nakayama's Lemma). Let A be a ring and \mathfrak{M} be its Jacobi radical. Suppose M is a finitely generated A -module. If $\mathfrak{a}M = M$ for $\mathfrak{a} \subset \mathfrak{M}$, then $M = 0$.

Proof. Suppose M is generated by x_1, \dots, x_n . Since $M = \mathfrak{a}M$, formally we have $(x_1, \dots, x_n)^T = \Phi(x_1, \dots, x_n)^T$ for $\Phi \in M_n(\mathfrak{a})$. Then $(\Phi - \text{id})(x_1, \dots, x_n)^T = 0$. Note that $\det(\Phi - \text{id}) = 1 + a$ for $a \in \mathfrak{a} \subset \mathfrak{M}$. Then $\Phi - \text{id}$ is invertible and then $M = 0$. □

Remark A.1.8. The finiteness of M is crucial in Nakayama's Lemma. For example, let $\bar{\mathbb{Z}}$ be the ring of algebraic integers in $\bar{\mathbb{Q}}$. Choose a non-zero prime ideal \mathfrak{p} of $\bar{\mathbb{Z}}$. Then we have that $\mathfrak{p}\bar{\mathbb{Z}}_{\mathfrak{p}} = \mathfrak{p}^2\bar{\mathbb{Z}}_{\mathfrak{p}}$. Indeed, if $a \in \mathfrak{p}\bar{\mathbb{Z}}_{\mathfrak{p}}$, let $b = \sqrt{a} \in \bar{\mathbb{Z}}_{\mathfrak{p}}$. Then $b^2 = a \in \mathfrak{p}\bar{\mathbb{Z}}_{\mathfrak{p}}$ and whence $b \in \mathfrak{p}\bar{\mathbb{Z}}_{\mathfrak{p}}$ since \mathfrak{p} is prime. It follows that $a = b^2 \in \mathfrak{p}^2\bar{\mathbb{Z}}_{\mathfrak{p}}$.

Proposition A.1.9 (Geometric form of Nakayama's Lemma). Let $X = \operatorname{Spec} A$ be an affine scheme, $x \in X$ a closed point and \mathcal{f} a coherent sheaf on X . If $a_1, \dots, a_k \in \mathcal{f}(X)$ generate $\mathcal{f}|_x = \mathcal{f} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{f}(U)$.

Proof. To be completed. □

Corollary A.1.10. Let X be a scheme and \mathcal{f} a coherent sheaf on X . Then the function $x \mapsto \dim_{\kappa(x)} \mathcal{f}|_x$ is upper semicontinuous.

Proof. To be completed. □

A.1.5 Nullstellensatz

Theorem A.1.11 (Noether's Normalization Lemma). Let A be a \mathbf{k} -algebra of finite type. Then there is an injection $\mathbf{k}[T_1, \dots, T_d] \hookrightarrow A$ such that A is finite over $\mathbf{k}[T_1, \dots, T_d]$.

Remark A.1.12. Here A does not need to be integral. For example,

Theorem A.1.13 (Hilbert's Nullstellensatz). Let A be a

A.2 Associated prime ideals

A.2.1 Associated prime ideals

Definition A.2.1 (Associated prime ideals). Let A be a noetherian ring and M an A -module. The *associated prime ideals* of M are the prime ideals \mathfrak{p} of form $\operatorname{Ann}(x)$ for some $x \in M$. The set of associated prime ideals of M is denoted by $\operatorname{Ass}(M)$.

Example A.2.2. Let $A = \mathbb{k}[x, y]/(xy)$ and $M = A$. First we see that $(x) = \operatorname{Ann} y$, $(y) = \operatorname{Ann} x \in \operatorname{Ass} M$. Then we check other prime ideals. For (x, y) , if $xf = yf = 0$, then $f \in (x) \cap (y) = (0)$. If $(x - a) = \operatorname{Ann} f$ for some f , note that $y \in (x - a)$ for $a \in \mathbb{k}^*$, then $f \in (x)$. Hence $f = 0$. Therefore $\operatorname{Ass} M = \{(x), (y)\}$.

Example A.2.3. Let $A = \mathbb{k}[x, y]/(x^2, xy)$ and $M = A$. The underlying space of $\operatorname{Spec} A$ is the y -axis since $\sqrt{(x^2, xy)} = (x)$. First note that $(x) = \operatorname{Ann} y$, $(x, y) = \operatorname{Ann} x \in \operatorname{Ass} M$. For $(x, y - a)$ with $a \in \mathbb{k}^*$, easily see that $xf = (y - a)f = 0$ implies $f = 0$ since $A = \mathbb{k} \cdot x \oplus \mathbb{k}[y]$ as \mathbb{k} -vector space. Hence $\operatorname{Ass} M = \{(x), (x, y)\}$.

Lemma A.2.4. Let A be a noetherian ring and M an A -module. Then the maximal element of the set

$$\{\text{Ann } x : x \in M_{\mathfrak{p}}, x \neq 0\}$$

belongs to $\text{Ass } M$.

Proof. We just need to show that such $\text{Ann } x$ is prime. Otherwise, there exist $a, b \in A$ such that $ab \in \text{Ann } x$ but $a, b \notin \text{Ann } x$. It follows that $\text{Ann } x \subsetneq \text{Ann } ax$ since $b \in \text{Ann } ax \setminus \text{Ann } x$. This contradicts the maximality of $\text{Ann } x$. \square

An element $a \in A$ is called a zero divisor for M if $M \rightarrow aM, m \mapsto am$ is not injective.

Corollary A.2.5. Let A be a noetherian ring and M an A -module. Then

$$\{\text{zero divisors for } M\} = \bigcup_{\mathfrak{p} \in \text{Ass } M} \mathfrak{p}.$$

Lemma A.2.6. Let A be a noetherian ring and M an A -module. Then $\mathfrak{p} \in \text{Ass}_A M$ iff $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Proof. Suppose $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $\mathfrak{p}A_{\mathfrak{p}} = \text{Ann } y_0/c$ with $y_0 \in M$ and $c \in A \setminus \mathfrak{p}$. For $a \in \text{Ann } y_0$, $ay_0 = 0$. Then $a/1 \in \mathfrak{p}A_{\mathfrak{p}}$. It follows that $a \in \mathfrak{p}$. Hence $\text{Ann } y_0 \subset \mathfrak{p}$.

Inductively, if $\text{Ann } y_n \subsetneq \mathfrak{p}$, then there exists $b_n \in A \setminus \mathfrak{p}$ such that $y_{n+1} := b_n y_n$, $\text{Ann } y_{n+1} \subset \mathfrak{p}$ and $\text{Ann } y_n \subsetneq \text{Ann } y_{n+1}$. To see this, choose $a_n \in \mathfrak{p} \setminus \text{Ann } y_n$. Then $(a_n/1)y_n = 0$ since $a_n/1 \in \mathfrak{p}A_{\mathfrak{p}}$. By definition, there exist $b_n \in A \setminus \mathfrak{p}$ such that $a_n b_n y_n = 0$. This process must terminate since A is noetherian. Thus $\text{Ann } y_n = \mathfrak{p}$ for some n . Hence $\mathfrak{p} \in \text{Ass}_A M$.

Conversely, suppose $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$. If $(a/s)(x/1) = 0 \in M_{\mathfrak{p}}$, there exist $t \in A \setminus \mathfrak{p}$ such that $tax = 0$. It follows that $ta \in \mathfrak{p}$ and then $(a/s) \in \mathfrak{p}A_{\mathfrak{p}}$. Hence $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. \square

Proposition A.2.7. We have $\text{Ass } M \subset \text{Supp } M$. Moreover, if $\mathfrak{p} \in \text{Supp } M$ satisfies $V(\mathfrak{p})$ is an irreducible component of $\text{Supp } M$, then $\mathfrak{p} \in \text{Ass } M$.

Proof. For any $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$, we have $A/\mathfrak{p} \cong A \cdot x \subset M$. Tensoring with $A_{\mathfrak{p}}$ gives $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$ since $A_{\mathfrak{p}}$ is flat. Hence $M_{\mathfrak{p}} \neq 0$ and $\mathfrak{p} \in \text{Supp } M$.

Now suppose $\mathfrak{p} \in \text{Supp } M$ and $V(\mathfrak{p})$ is an irreducible component of $\text{Supp } M$. First we show that $\mathfrak{p} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $x \in M_{\mathfrak{p}}$ such that $\text{Ann } x$ is maximal in the set

$$\{\text{Ann } x : x \in M_{\mathfrak{p}}, x \neq 0\}.$$

Then we claim that $\text{Ann } x = \mathfrak{p}A_{\mathfrak{p}}$. First, $\text{Ann } x$ is prime by Lemma A.2.4. If $\text{Ann } x \neq \mathfrak{p}$, then $V(\text{Ann } x) \supset V(\mathfrak{p})$. This implies that $\text{Ann } x \notin \text{Supp } M_{\mathfrak{p}}$ since $\text{Supp } M_{\mathfrak{p}} = \text{Supp } M \cap \text{Spec } A_{\mathfrak{p}}$. This is a contradiction. Thus $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. By Lemma A.2.6, we have $\mathfrak{p} \in \text{Ass } M$. \square

Remark A.2.8. The existence of irreducible component is guaranteed by Zorn's Lemma.

Definition A.2.9. A prime ideal $\mathfrak{p} \in \text{Ass } M$ is called *embedded* if $V(\mathfrak{p})$ is not an irreducible component of $\text{Supp } M$.

Example A.2.10. For $M = A = \mathbb{k}[x, y]/(x^2, xy)$, the origin (x, y) is an embedded point.

Proposition A.2.11. If we have exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$, then $\text{Ass } M_2 \subset \text{Ass } M_1 \cup \text{Ass } M_3$.

Proof. Let $\mathfrak{p} = \text{Ann } x \in \text{Ass } M_2 \setminus \text{Ass } M_1$. Then the image $[x]$ of x in M_3 is not equal to 0 . We have that $\text{Ann } x \subset \text{Ann}[x]$. If $a \in \text{Ann}[x] \setminus \text{Ann } x$, then $ax \in M_1$. Since $\text{Ann } x \subsetneq \text{Ann } ax$, there is $b \in \text{Ann } ax \setminus \text{Ann } x$. However, it implies $ba \in \text{Ann } x$, and then $a \in \text{Ann } x$ since $\text{Ann } x$ is prime, which is a contradiction. \square

Corollary A.2.12. If M is finitely generated, then the set $\text{Ass } M$ is finite.

Proof. For $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$, we know that the submodule M_1 generated by x is isomorphic to A/\mathfrak{p} . Inductively, we can choose M_n be the preimage of a submodule of M/M_{n-1} which is isomorphic to A/\mathfrak{q} for some $\mathfrak{q} \in \text{Ass } M/M_{n-1}$. We can take an ascending sequence $0 = M_0 \subset M_1 \subset \dots \subset M_n \subset \dots$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime \mathfrak{p}_i . Since M is finitely generated, this is a finite sequence. Then the conclusion follows by Proposition A.2.11. \square

A.2.2 Primary decomposition

Definition A.2.13. An A -module is called *co-primary* if $\text{Ass } M$ has a single element. Let M be an A -module and $N \subset M$ a submodule. Then N is called *primary* if M/N is co-primary. If $\text{Ass } M/N = \{\mathfrak{p}\}$, then N is called \mathfrak{p} -primary.

Remark A.2.14. This definition coincide with primary ideals in the case $M = A$. Recall an ideal $\mathfrak{q} \subset A$ is called *primary* if $\forall ab \in \mathfrak{p}, a \notin \mathfrak{q}$ implies $b^n \in \mathfrak{q}$ for some n .

Let \mathfrak{q} be a \mathfrak{q} -primary ideal. Since $\text{Supp } A/\mathfrak{q} = \{\mathfrak{p}\}$, $\mathfrak{p} \in \text{Ass } A/\mathfrak{q}$. Suppose $\text{Ann}[a] \in \text{Ass } A/\mathfrak{q}$. Then $\mathfrak{p} \subset \text{Ann}[a]$ since $V(\mathfrak{p}) = \text{Supp } A/\mathfrak{q}$. If $b \in \text{Ann}[a]$, then $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Hence $b^n \in \mathfrak{q}$, and then $b \in \mathfrak{p}$. This shows that $\text{Ass } A/\mathfrak{q} = \{\mathfrak{p}\}$ and \mathfrak{q} is \mathfrak{p} -primary as an A -submodule.

Let $\mathfrak{q} \subset A$ be a \mathfrak{p} -primary A -submodule. First we have $\mathfrak{p} = \sqrt{\mathfrak{q}}$ since $V(\mathfrak{p})$ is the unique irreducible component of $\text{Supp } A/\mathfrak{q}$. Suppose $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Then $b \in \text{Ann}[a] \subset \mathfrak{p}$ since \mathfrak{p} is the unique maximal element in $\{\text{Ann}[c] : c \in A \setminus \mathfrak{q}\}$. This implies that $b^n \in \mathfrak{q}$.

Definition A.2.15. Let A be a noetherian ring, M an A -module and $N \subset M$ a submodule. A *minimal primary decomposition* of N in M is a finite set of primary submodules $\{Q_i\}_{i=1}^n$ such that

$$N = \bigcap_{i=1}^n Q_i,$$

no Q_i can be omitted and $\text{Ass } M/Q_i$ are pairwise distinct. For $\text{Ass } M/Q_i = \{\mathfrak{p}\}$, Q_i is called belonging to \mathfrak{p} .

Indeed, if $N \subset M$ admits a minimal primary decomposition $N = \bigcap Q_i$ with Q_i belonging to \mathfrak{p} , then $\text{Ass}(M/N) = \{\mathfrak{p}_i\}$. For given i , consider $N_i := \bigcap_{j \neq i} Q_j$, then $N_i/N \cong (N_i + Q_i)/Q_i$. Since $N_i \neq N$, $\text{Ass } N_i/N \neq \emptyset$. On the other hand, $\text{Ass } N_i/N \subset \text{Ass } M/Q_i = \{\mathfrak{p}\}$. It follows that $\text{Ass } N_i/N = \{\mathfrak{p}_i\}$,

whence $\mathfrak{p}_i \in \text{Ass } M/N$. Conversely, we have an injection $M/N \hookrightarrow \bigoplus M/Q_i$, so $\text{Ass } M/N \subset \bigcup \text{Ass } M/Q_i$. Due to this, if Q_i belongs to \mathfrak{p} , we also say that Q_i is the \mathfrak{p} -component of N .

Proposition A.2.16. Suppose $N \subset M$ has a minimal primary decomposition. If $\mathfrak{p} \in \text{Ass } M/N$ is not embedded, then the \mathfrak{p} component of N is unique. Explicitly, we have $Q = \nu^{-1}(N_{\mathfrak{p}})$, where $\nu : M \rightarrow M_{\mathfrak{p}}$.

Proof. First we show that $Q = \nu^{-1}(Q_{\mathfrak{p}})$. Clearly $Q \subset \nu^{-1}(Q_{\mathfrak{p}})$. Suppose $x \in \nu^{-1}(Q_{\mathfrak{p}})$. Then there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in Q$. That is, $[sx] = 0 \in M/Q$. If $[x] \neq 0$, we have $s \in \text{Ann}[x] \subset \mathfrak{p}$. This contradiction enforces $Q = \nu^{-1}(Q_{\mathfrak{p}})$.

Then we show that $N_{\mathfrak{p}} = Q_{\mathfrak{p}}$. Just need to show that for $\mathfrak{p}' \neq \mathfrak{p}$ and the \mathfrak{p}' component Q' of N , $Q'_{\mathfrak{p}} = M_{\mathfrak{p}}$. Since \mathfrak{p} is not embedded, $\mathfrak{p}' \not\subset \mathfrak{p}$. Then $\mathfrak{p} \notin V(\mathfrak{p}) = \text{Supp } M/Q'$. So $M_{\mathfrak{p}}/Q'_{\mathfrak{p}} = 0$. \square

Example A.2.17. If \mathfrak{p} is embedded, then its components may not be unique. For example, let $M = A = \mathbb{k}[x, y]/(x^2, xy)$. Then for every $n \in \mathbb{Z}_{\geq 1}$, $(x) \cap (x^2, xy, y^n)$ is a minimal primary decomposition of $(0) \subset M$.

Let A be a noetherian ring and $\mathfrak{p} \subset A$ a prime ideal. We consider the \mathfrak{p} component of \mathfrak{p}^n , which is called n -th symbolic power of \mathfrak{p} , denoted by $\mathfrak{p}^{(n)}$. We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A$. In general, $\mathfrak{p}^{(n)}$ is not equal to \mathfrak{p}^n ; see below example.

Example A.2.18. Let $A = \mathbf{k}[x, y, z, w]/(y^2 - zx^2, yz - xw)$ and $\mathfrak{p} = (y, z, w)$. We have $z = y^2/x^2, w = yz/x \in \mathfrak{p}^2 A_{\mathfrak{p}}$, whence $\mathfrak{p}^2 A_{\mathfrak{p}} = (z, w) \neq \mathfrak{p}^2$.

Theorem A.2.19. Let A be a noetherian ring and M an A -module. Then for every $\mathfrak{p} \in \text{Ass } M$, there is a \mathfrak{p} -primary submodule $Q(\mathfrak{p})$ such that

$$(0) = \bigcap_{\mathfrak{p} \in \text{Ass } M} Q(\mathfrak{p}).$$

Proof. Consider the set

$$\mathbf{n} := \{N \subset M : \mathfrak{p} \notin \text{Ass } N\}.$$

Note that $\text{Ass } \bigcup N_i = \bigcup \text{Ass } N_i$ by definition of associated prime ideals. Then it is easy to check that \mathbf{n} satisfies the conditions of Zorn's Lemma. Hence \mathbf{n} has a maximal element $Q(\mathfrak{p})$. We claim that $Q(\mathfrak{p})$ is \mathfrak{p} -primary. If there is $\mathfrak{p}' \neq \mathfrak{p} \in \text{Ass } M/Q(\mathfrak{p})$, then there is a submodule $N' \cong A/\mathfrak{p}'$. Let N'' be the preimage of N' in M . We have $Q(\mathfrak{p}) \subsetneq N''$ and $N'' \in \mathbf{n}$. This is a contradiction. By the fact $\text{Ass } \bigcap N_i = \bigcap \text{Ass } N_i$, we get the conclusion. \square

Corollary A.2.20. Let A be a noetherian ring and M a finite A -module. Then every submodule of M has a minimal primary decomposition.

A.3 Dimension and Depth

There are three numbers measuring the “size” of a local ring (A, \mathfrak{m}) :

- $\dim A$: the Krull dimension of A .
- $\text{depth } A$: the depth of A .
- $\dim_{\kappa(\mathfrak{m})} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$ -vector space.

Somehow the Krull dimension is “homological” and the depth is “cohomological”.

Definition A.3.1. Let A be a noetherian ring. The *height of a prime ideal* \mathfrak{p} in A is defined as the maximum length of chains of prime ideals contained in \mathfrak{p} , that is,

$$\text{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as

$$\dim A := \max_{\mathfrak{p} \in \text{Spec } A} \text{ht}(\mathfrak{p}).$$

Example A.3.2. Let A be a PID. For every two non-zero prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 , if $\mathfrak{p}_1 = t_1 A \subset \mathfrak{p}_2 = t_2 A$, then $t_2 \mid t_1$ and hence $\mathfrak{p}_1 = \mathfrak{p}_2$. It follows that $\dim A = 1$. Consequently, the ring of integers \mathbb{Z} and the polynomial ring $\mathbf{k}[T]$ in one variable over a field have Krull dimension 1.

Definition A.3.3. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A -module. A sequence $t_1, \dots, t_n \in I$ is called an *M -regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \dots, t_{i-1})M$ for all i .

Example A.3.4. Let $A = \mathbf{k}[x, y]/(x^2, xy)$ and $I = (x, y)$. Then $\text{depth}_I A = 0$.

Definition A.3.5. Let A be a noetherian ring. For every $\mathfrak{p} \in \text{Spec } A$, $\mathfrak{p}/\mathfrak{p}^2$ is a vector space over $\kappa(\mathfrak{p})$. The *Zariski’s tangent space* $T_{A,\mathfrak{p}}$ of A at \mathfrak{p} is defined as $(\mathfrak{p}/\mathfrak{p}^2)^\vee$, the dual $\kappa(\mathfrak{p})$ -vector space of $\mathfrak{p}/\mathfrak{p}^2$.

A.3.1 Artinian Rings and Length of Modules

Definition A.3.6. Let A be a ring and M an A module. A *simple module filtration* of M is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i-1} is a simple module, i.e. it has no submodule except 0 and itself. If M has a simple module filtration as above, we define the *length of M* as n and say that M has *finite length*.

The following proposition guarantees the length is well-defined.

Proposition A.3.7. Suppose M has a simple module filtration $M = M_{0,0} \supsetneq M_{1,0} \supsetneq \cdots \supsetneq M_{n,0} = 0$. Then for any other filtration $M = M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,m} = 0$ with $m > n$, there exist $k < m$ such that $M_{0,k} = M_{0,k+1}$.

Proof. We claim that there are at least $0 \leq k_1 < \cdots < k_{m-n} < m$ satisfies that $M_{0,k_i} = M_{0,k_i+1}$. Let $M_{i,j} := M_{i,0} \cap M_{0,j}$. Inductively on n , we can assume that there exist k_1, \dots, k_{n-m+1} such that $M_{1,k} = M_{1,k+1}$. Consider the sequence

$$M_{0,0}/M_{1,0} \supset (M_{0,1} + M_{1,0})/M_{1,0} \supset \cdots \supset (M_{0,m} + M_{1,0})/M_{1,0} = 0$$

in $M_{0,0}/M_{1,0}$. Since $M_{0,0}/M_{1,0}$ is simple, there is at most one k_i with $M_{0,k_i} + M_{1,0} \neq M_{0,k_i+1} + M_{1,0}$. And note that if $M_{0,k_i} + M_{1,0} = M_{0,k_i+1} + M_{1,0}$ and $M_{0,k_i} \cap M_{1,0} = M_{0,k_i+1} \cap M_{1,0}$, then $M_{0,k_i} = M_{0,k_i+1}$ by the Five Lemma. \square

Example A.3.8. Let A be a ring and $\mathfrak{m} \in \text{mSpec } A$. Then A/\mathfrak{m} is a simple module. **To be completed.**

Proposition A.3.9. Let A be a ring and M an A -module. Then M is of finite length iff it satisfies both a.c.c and d.c.c.

Proof. Note that if M has either a strictly ascending chain or a strictly descending chain, M is of infinite length. Conversely, d.c.c guarantee M has a simple submodule and a.c.c guarantee the sequence terminates. \square

Proposition A.3.10. The length $l(-)$ is an additive function for modules of finite length. That is, if we have an exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ with M_i of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

Proof. The simple module filtrations of M_1 and M_3 will give a simple module filtration of M_2 . \square

Proposition A.3.11. Let (A, \mathfrak{m}) be a local ring. Then A is artinian iff $\mathfrak{m}^n = 0$ for some $n \geq 0$.

Proof. Suppose A is artinian. Then the sequence $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ is stable. It follows that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n . By the Nakayama's Lemma A.1.7, $\mathfrak{m}^n = 0$.

Conversely, we have

$$\mathfrak{m} \subset \mathfrak{N} \subset \bigcap_{\text{minimal prime ideal}} \mathfrak{p},$$

whence \mathfrak{m} is minimal. \square

Proposition A.3.12. Let A be a ring. Then A is artinian iff A is of finite length.

Proof. First we show that A has only finite maximal ideal. Otherwise, consider the set $\{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k\}$. It has a minimal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ and for any maximal ideal \mathfrak{m} , $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}$. It follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i . Let $\mathfrak{M} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ be the Jacobi radical of A . Consider the sequence $\mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots$ and by Nakayama's Lemma, we have $\mathfrak{M}^k = 0$ for some k . Consider the filtration

$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1^k \supset \mathfrak{m}_1^k \mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0).$$

We have $\mathfrak{m}_1^k \cdots \mathfrak{m}_i^j / \mathfrak{m}_1^k \cdots \mathfrak{m}_i^{j+1}$ is an A/\mathfrak{m}_i -vector space. It is artinian and then of finite length. Hence A is of finite length. \square

Theorem A.3.13. Let A be a ring. Then A is artinian iff A is noetherian and of dimension 0.

Proof. Suppose A is artinian. Then A is noetherian by Proposition A.3.12. Let $\mathfrak{p} \in \text{Spec } A$. Then A/\mathfrak{p} is an artinian integral domain. If there is $a \in A/\mathfrak{p}$ is not invertible, consider $(a) \supset (a^2) \supset \dots$, we see $a = 0$. Hence \mathfrak{p} is maximal and $\dim A = 0$.

Suppose that A is noetherian and of dimension 0. Then every maximal ideal is minimal. In particular, A has only finite maximal ideal $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Let \mathfrak{q}_i be the \mathfrak{p}_i -component of (0) . Then we have $A \hookrightarrow \bigoplus_i A/\mathfrak{q}_i$. We just need to show that A/\mathfrak{q}_i is of finite length as A -module. If $\mathfrak{q}_i \subset \mathfrak{p}_j$, take radical we get $\mathfrak{p}_i \subset \mathfrak{p}_j$ and hence $i = j$. So A/\mathfrak{q}_i is a local ring with maximal ideal $\mathfrak{p}_i A/\mathfrak{q}_i$. Then every element in $\mathfrak{p}_i A/\mathfrak{q}_i$ is nilpotent. Since \mathfrak{p}_i is finitely generated, $(\mathfrak{p}_i A/\mathfrak{q}_i)^k = 0$ for some k . Then A/\mathfrak{q}_i is artinian and then of finite length as A/\mathfrak{q}_i -module. Then the conclusion follows. \square

A.3.2 Dedekind Domains

To be completed

A.3.3 Krull's Principal Ideal Theorem

Theorem A.3.14 (Krull's Principal Ideal Theorem). Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let \mathfrak{p} be a minimal prime ideal among those containing f . Then $\text{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_{\mathfrak{p}}$, we may assume A is local with maximal ideal \mathfrak{p} . Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.

Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \dots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is \mathfrak{q} -primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1} A_{\mathfrak{q}}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_{\mathfrak{q}} = 0$. It follows that $\mathfrak{q} A_{\mathfrak{q}}$ is minimal, whence $A_{\mathfrak{q}}$ is artinian. Therefore, \mathfrak{q} is minimal in A . \square

Corollary A.3.15. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n . Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.

Choose \mathfrak{q} be a minimal prime ideal among those containing (\mathfrak{p}_{k-1}, f) and contained in \mathfrak{p}_{k+1} . Then by Krull's Principal Ideal Theorem A.3.14, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace \mathfrak{p}_k by \mathfrak{q}_k , we have $f \in \mathfrak{q}_k \setminus \mathfrak{p}_{k-1}$.

Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \dots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \dots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.

Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$ has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A . It follows that $\dim A/(f) + 1 \leq \dim A$. \square

Proposition A.3.16. Let (A, \mathfrak{m}) be a local noetherian ring with residue field \mathbf{k} . Then the following inequalities hold:

$$\text{depth } A \leq \dim A \leq \dim_{\mathbf{k}} T_{A, \mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary A.3.15.

Let t_1, \dots, t_n be a $\kappa(\mathfrak{m})$ -basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \dots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \dots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \dots, t_n))$. It follows that $\mathfrak{m} = (t_1, \dots, t_n)$ by Nakayama's Lemma. By Corollary A.3.15,

$$n + \dim A/(t_1, \dots, t_n) \geq n - 1 + \dim A/(t_1, \dots, t_{n-1}) \geq \dots \geq 1 + \dim A/(t_1) \geq \dim A.$$

We conclude the result. \square

Definition A.3.17. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property (R_k)* or *is regular in codimension k* if $\forall \xi \in X$ with $\text{codim } Z_\xi \leq k$,

$$\dim_{\kappa(\xi)} T_{X, \xi} = \dim \mathcal{O}_{X, \xi}.$$

We say that X *verifies property (S_k)* if $\forall \xi \in X$ with $\text{depth } \mathcal{O}_{X, \xi} < k$,

$$\text{depth } \mathcal{O}_{X, \xi} = \dim \mathcal{O}_{X, \xi}.$$

Example A.3.18. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.

Suppose A verifies (S_1) . If $\mathfrak{p} \in \text{Ass } A$, every element in \mathfrak{p} is a zero divisor. Then $\text{depth } A_{\mathfrak{p}} = 0$. It follows that $\dim A_{\mathfrak{p}} = 0$ and then \mathfrak{p} is minimal.

Suppose A has no embedded point. Let $\mathfrak{p} \in \text{Spec } A$ with $\text{depth } A_{\mathfrak{p}} = 0$. This means every element in $\mathfrak{p}A_{\mathfrak{p}}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Proposition A.1.2, $\mathfrak{p} = \mathfrak{q}$ for some minimal \mathfrak{q} , whence $\dim A_{\mathfrak{p}} = 0$.

Example A.3.19. Let A be a noetherian ring. Then A is reduced iff it verifies (R_0) and (S_1) .

Suppose A is reduced. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all minimal prime ideals of A . We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where \mathfrak{N} is the nilradical of A . Hence A has no embedded point. Since $A_{\mathfrak{p}}$ is artinian, local and reduced, $A_{\mathfrak{p}}$ is a field and hence regular.

Conversely, let $\text{Ass } A$ be equal to $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Then every \mathfrak{p}_i is minimal by (S_1) . Let f be in \mathfrak{N} . Then the image of f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0) , $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where \mathfrak{q}_i is the \mathfrak{p}_i component of (0) in A . Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

A.3.4 Cohen-Macaulay rings

Definition A.3.20 (Cohen-Macaulay). A noetherian local ring (A, \mathfrak{m}) is called *Cohen-Macaulay* if $\dim A = \text{depth } A$. A noetherian ring A is called *Cohen-Macaulay* if for every prime ideal $\mathfrak{p} \in \text{Spec } A$, the localization $A_{\mathfrak{p}}$ is Cohen-Macaulay. This is equivalent to that A verifies (S_k) for all $k \geq 0$.

Definition A.3.21. Let (A, \mathfrak{m}) be a noetherian local ring of dimension d . A sequence $t_1, \dots, t_d \in \mathfrak{m}$ is called a *system of parameters* if **To be completed.**

Example A.3.22 (Non Cohen-Macaulay rings). **To be completed.**

Corollary A.3.23. Let A be a noetherian ring, M a finite A -module and $a \in A$ an M -regular element. Then $\text{depth } M = \text{depth } M/aM + 1$.

Corollary A.3.24. Let A be a noetherian ring $a \in A$ a nonzero divisor. Then A verifies (S_d) iff A/aA verifies (S_{d-1}) .

Definition A.3.25. An ideal I of a noetherian ring A is called *unmixed* if

$$\text{ht}(I) = \text{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \text{Ass}(A/I).$$

Here $\text{ht}(I)$ is defined as

$$\text{ht}(I) := \inf\{\text{ht}(\mathfrak{p}) : I \subset \mathfrak{p}\}.$$

We say that *the unmixedness theorem holds for a noetherian ring A* if any ideal $I \subset A$ generated by $\text{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme X* if $\mathcal{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Theorem A.3.26. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Proof. We can assume that $X = \text{Spec } A$ is affine.

Suppose X is Cohen-Macaulay. Let $I \subset A$ be an ideal generated by a_1, \dots, a_r with $r = \text{ht}(I)$. We claim that a_1, \dots, a_r is an A -regular sequence. If so, we get that the unmixedness theorem holds for A by applying Example A.3.18 on A/I . Since $\text{ht}(a_1, \dots, a_{r-1}) \leq r - 1$ by Krull's Principal Ideal Theorem A.3.14 and $\text{ht}(a_1, \dots, a_r) = r \leq \text{ht}(a_1, \dots, a_{r-1}) + 1$, we have $\text{ht}(a_1, \dots, a_{r-1}) = r - 1$. By induction on r , we can assume that a_1, \dots, a_{r-1} is an A -regular sequence. Hence any prime ideal $\mathfrak{p} \in \text{Ass } A/(a_1, \dots, a_{r-1})$ has height $r - 1$. Now suppose a_r is a zero divisor in $A/(a_1, \dots, a_{r-1})$. Then there exists a prime ideal $\mathfrak{p} \in \text{Ass } A/(a_1, \dots, a_{r-1})$ such that $a_r \in \mathfrak{p}$. Then $I \subset \mathfrak{p}$ and $\text{ht}(I) \leq r - 1$. This contradicts that $\text{ht}(I) = r$.

Suppose the unmixedness theorem holds for A . Let $\mathfrak{p} \in \text{Spec } A$ be a prime ideal with $\text{ht}(\mathfrak{p}) = r$. Then $\mathfrak{p} \in \text{Ass } A$ if and only if $\text{ht}(\mathfrak{p}) = 0$. If $r > 0$, there is a nonzero divisor $a \in \mathfrak{p}$. By Krull's Principal Ideal Theorem A.3.14, $\text{ht}(\mathfrak{p}A/aA) = r - 1$. Inductively, we can find a regular sequence a_1, \dots, a_r in \mathfrak{p} . Then $\text{depth } A_{\mathfrak{p}} = r$. \square

Theorem A.3.27. Let X be a locally noetherian scheme. Suppose that X is Cohen-Macaulay. Let $F \subset X$ be a closed subset of codimension $\geq k$. Then the restriction $H^i(X, \mathcal{O}_X) \rightarrow H^i(X \setminus F, \mathcal{O}_X)$ is an isomorphism.

Proof. To be completed. □

A.3.5 Regular rings

Definition A.3.28. A noetherian ring A is said to be *regular at* $\mathfrak{p} \in \operatorname{Spec} A$ if we have

$$\dim_{\kappa(\mathfrak{p})} T_{A,\mathfrak{p}} = \dim A_{\mathfrak{p}},$$

where $\dim A_{\mathfrak{p}}$ is the Krull dimension of the local ring $A_{\mathfrak{p}}$.

A noetherian ring A is said to be *regular* if it is regular at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. This is equivalent to the condition that A verifies (R_k) for all $k \geq 0$.

Remark A.3.29. A noetherian ring A is regular if and only if it is regular at every maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$. The proof uses homological tools; see Theorem B.3.17 and Corollary B.3.18.

Definition A.3.30. Let A be a noetherian ring that is regular at $\mathfrak{p} \in \operatorname{Spec} A$. A sequence $t_1, \dots, t_n \in \mathfrak{p}$ is called a *regular system of parameters* at \mathfrak{p} if their images form a basis of the $\kappa(\mathfrak{p})$ -vector space $\mathfrak{p}/\mathfrak{p}^2$.

Proposition A.3.31. Let (A, \mathfrak{m}) be a noetherian local ring that is regular at \mathfrak{m} . Let t_1, \dots, t_n be a regular system of parameters at \mathfrak{m} , $\mathfrak{p}_i = (t_1, \dots, t_i)$ and $\mathfrak{p}_0 = (0)$. Then \mathfrak{p}_i is a prime ideal of height i , and A/\mathfrak{p}_i is a regular local ring for all i . In particular, regular local ring is integral, and the regular system of parameters t_1, \dots, t_n is a regular sequence in A .

Proof. By the Krull's Principal Ideal Theorem A.3.14, we have

$$n - 1 = \dim A - 1 \leq \dim A/(t_1) \leq \dim_{\kappa(\mathfrak{m}/(t_1))} T_{A/(t_1), \mathfrak{m}/(t_1)} \leq n - 1.$$

Hence $\dim A/(t_1) = n - 1$ and $\operatorname{ht}(t_1) = 1$. Since t_2, \dots, t_n generate $\mathfrak{m}/(t_1)$, we have that $A/(t_1)$ is regular at $\mathfrak{m}/(t_1)$ and the images of t_2, \dots, t_n form a regular system of parameters.

For integrality, we induct on the dimension of A . If $\dim A = 0$, then A is a field and hence integral. Suppose $\dim A > 0$, let \mathfrak{q} be a minimal prime ideal of A . Then $t_1 \notin \mathfrak{q}$. We have

$$n - 1 = \dim A - 1 \leq \dim A/(\mathfrak{q} + t_1 A) \leq \dim_{\kappa(\mathfrak{q}/(t_1))} T_{A/(\mathfrak{q} + t_1 A), \mathfrak{q}/(t_1)} \leq n - 1.$$

By similar arguments, we have $A/(\mathfrak{q} + t_1 A)$ is regular at $\mathfrak{m}/(\mathfrak{q} + t_1 A)$. By induction hypothesis, both of $A/t_1 A$ and $A/(\mathfrak{q} + t_1 A)$ are integral and of dimension $n - 1$. Hence $t_1 A = t_1 A + \mathfrak{q}$, i.e. $\mathfrak{q} \subset t_1 A$. For every $a = bt_1 \in \mathfrak{q}$, we have $b \in \mathfrak{q}$ since $t_1 \notin \mathfrak{q}$. Then $\mathfrak{q} \subset t_1 \mathfrak{q} \subset \mathfrak{m} \mathfrak{q}$. By Nakayama's Lemma, $\mathfrak{q} = 0$, whence A is integral. □

Corollary A.3.32. A regular noetherian ring is Cohen-Macaulay.

Corollary A.3.33. A regular noetherian ring is normal.

Remark A.3.34. Indeed we can show a stronger result: a noetherian regular local ring is a UFD; see [ref.](#)

A.4 Finite Algebra and Normality

Let R be a ring and A be an R -algebra. We say that A is of *finite type* over R if there exists a surjective R -algebra homomorphism $R[T_1, \dots, T_n] \rightarrow A$ for some $n \geq 0$. We say that A is *finite* over R if it is finite as an R -module.

A.4.1 Finite algebra

Let A be a ring and B a finite A -algebra.

Example A.4.1. Let K be a number field. Then \mathcal{O}_K is a finite \mathbb{Z} -algebra. [To be completed.](#)

Lemma A.4.2. Let $A \subset B$ be noetherian rings such that B is finite over A . Then the induced morphism $\text{Spec } B \rightarrow \text{Spec } A$ is surjective.

Proof. For $\mathfrak{p} \in \text{Spec } A$, let $S := A - \mathfrak{p}$ and denote $S^{-1}B$ by $B_{\mathfrak{p}}$. Then we have $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$ is finite over $A_{\mathfrak{p}}$. Let $\mathfrak{P}B_{\mathfrak{p}}$ be a maximal ideal of $B_{\mathfrak{p}}$. We claim that $\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$ is maximal. Indeed, consider $A_{\mathfrak{p}}/(\mathfrak{P} \cap A_{\mathfrak{p}}) \hookrightarrow B_{\mathfrak{p}}/\mathfrak{P}B_{\mathfrak{p}}$, the latter is finite over the former. This enforces $A_{\mathfrak{p}}/(\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}})$ to be a field. Hence $\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, and then $\mathfrak{P} \cap A = \mathfrak{p}$. \square

Proposition A.4.3. Let $A \subset B$ be noetherian rings such that B is finite over A . Then $\dim A = \dim B$.

Proof. If we have a sequence $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ of prime ideals in B , then there exists $f \in \mathfrak{P}_2 \setminus \mathfrak{P}_1$. Since B is finite over A , there exist $a_1, \dots, a_n \in A$ such that

$$f^n + a_1 f^{n-1} + \dots + a_n = 0.$$

Then $a_n \in \mathfrak{P}_2 \cap A$. If $a_n \in \mathfrak{P}_1$, $f^{n-1} + \dots + a_{n-1}f \in \mathfrak{P}_1$ since $f \notin \mathfrak{P}_1$. Then $a_{n-1} \in \mathfrak{P}_2$. Repeat the process, it will terminate, whence $\mathfrak{P}_1 \cap A \subsetneq \mathfrak{P}_2 \cap A$. Otherwise, we have $f^n \in a_1 B + \dots + a_n B \subset \mathfrak{P}_1$.

Conversely, suppose we have $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Spec } A$ with $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Choose $\mathfrak{P}_1 \in \text{Spec } B$ such that $\mathfrak{P}_1 \cap A = \mathfrak{p}_1$, then we have $A/\mathfrak{p}_1 \subset B/\mathfrak{P}_1$. Let \mathfrak{P}_2 be the preimage of the prime ideal in B/\mathfrak{P}_1 which is over image of \mathfrak{p}_2 in A/\mathfrak{p}_1 . Proposition [A.4.2](#) guarantees that such \mathfrak{P}_2 exists. Then we get $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$. Repeat this progress, we get $\dim B \geq \dim A$. \square

[To be completed](#)

Definition A.4.4. An integral domain A is called *normal* if it is integrally closed in its field of fractions $\text{Frac}(A)$.

Lemma A.4.5. Let $A \subset C$ be rings and B the integral closure of A in C , S a multiplicatively closed subset of A . Then the integral closure of $S^{-1}A$ in $S^{-1}C$ is $S^{-1}B$.

Proof. For every $b \in B$ and $\forall s \in S$, there exists $a_i \in A$ s.t.

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then

$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s^1} \left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0.$$

Hence b/s is integral over $S^{-1}A$, $S^{-1}B$ is integral over $S^{-1}A$.

If $c/s \in S^{-1}C$ is integral over $S^{-1}A$, then $\exists a_i \in S^{-1}A$ s.t.

$$\left(\frac{c}{s}\right)^n + a_1 \left(\frac{c}{s}\right)^{n-1} + \cdots + a_n = 0.$$

Then

$$c^n + a_1 s c^{n-1} + \cdots + a_n s^n = 0 \in S^{-1}C$$

Then $\exists t \in S$ s.t.

$$t(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0 \in C.$$

Then

$$(ct)^n + a_1 st(ct)^{n-1} + \cdots + a_n s^n t^n = t^n(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0.$$

Hence ct is integral over A , then $ct \in B$. Then $c/s = (ct)/(st) \in S^{-1}B$. This completes the proof. \square

Proposition A.4.6. Normality is a local property. That is, for an integral domain A , TFAE:

- (i) A is normal.
- (ii) For any prime ideal $\mathfrak{p} \in \text{Spec } A$, the localization $A_{\mathfrak{p}}$ is normal.
- (iii) For any maximal ideal $\mathfrak{m} \in \text{mSpec } A$, the localization $A_{\mathfrak{m}}$ is normal.

Proof. When A is normal, $A_{\mathfrak{p}}$ is normal by Lemma A.4.5.

Assume that $A_{\mathfrak{m}}$ is normal for every $\mathfrak{m} \in \text{mSpec } A$. If A is not normal, let \tilde{A} be the integral closure of A in $\text{Frac } A$, \tilde{A}/A is a nonzero A -module. Suppose $\mathfrak{p} \in \text{Supp } \tilde{A}/A$ and $\mathfrak{p} \subset \mathfrak{m}$. We have $\tilde{A}_{\mathfrak{m}}/A_{\mathfrak{m}} = 0$ and $\tilde{A}_{\mathfrak{p}}/A_{\mathfrak{p}} = (\tilde{A}_{\mathfrak{m}}/A_{\mathfrak{m}})_{\mathfrak{p}} \neq 0$. This is a contradiction. \square

Proposition A.4.7. Let A be a normal ring. Then $A[X]$ is also normal.

Definition A.4.8. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,\xi}$ is normal for any point $\xi \in X$. A ring A is called *normal* if $\text{Spec } A$ is normal.

Remark A.4.9. For a general ring A , let $S := A \setminus (\bigcup_{\mathfrak{p} \in \text{Ass } A} \mathfrak{p}) = \bigcap_{\mathfrak{p} \in \text{Ass } A} A \setminus \mathfrak{p}$. Then S is a multiplicative set. The localization $S^{-1}A$ is called *the total ring of fractions* of A .

Suppose A is reduced and $\text{Ass } A = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Denote its total ring of fractions by Q . Note that

elements in Q are either unit or zero divisor. Hence any maximal ideal \mathfrak{m} is contained in $\bigcup \mathfrak{p}_i Q$, whence contained in some $\mathfrak{p}_i Q$. Thus $\mathfrak{p}_i Q$ are maximal ideals. And we have $\bigcap \mathfrak{p}_i Q = 0$. By the Chinese Remainder Theorem, we have $Q = \prod Q/\mathfrak{p}_i Q = \prod A_{\mathfrak{p}_i}$.

Let A be a reduced ring with total ring of fractions Q . Then A is normal iff A is integral closed in Q . If A is normal, then for every $\mathfrak{p} \in \text{Spec } A$, $A_{\mathfrak{p}}$ is integral. Then there is unique minimal prime ideal $\mathfrak{p}_i \subset \mathfrak{p}$. In particular, any two minimal prime ideal are relatively prime. By the Chinese Remainder Theorem, $A = \prod A/\mathfrak{p}_i$. Just need to check A/\mathfrak{p}_i is integral closed in $A_{\mathfrak{p}_i}$. This is clear by check pointwise.

Conversely, suppose A is integral closed in Q . Let e_i be the unit element of $A_{\mathfrak{p}_i}$. It belongs to A since $e_i^2 - e_i = 0$. Since $1 = e_1 + \cdots + e_n$ and $e_i e_j = \delta_{ij}$, we have $A = \prod A e_i$. Since $A e_i$ is integral closed in $A_{\mathfrak{p}_i}$, it is normal. Hence A is normal.

Lemma A.4.10. Let A be a normal ring. Then A verifies (R_1) and (S_2) .

Proof. Since all properties are local, we can assume A is integral and local.

For (S_2) , by Example ??, we only need to show that $\text{Ass}_A A/f$ has no embedded point. Let $\mathfrak{p} = (f : g) \in \text{Ass}_A A/fA$ and $t := f/g \in \text{Frac } A$. After Replacing A by $A_{\mathfrak{p}}$, we can assume that \mathfrak{p} is maximal. By definition, $t^{-1}\mathfrak{p} \subset A$. If $t^{-1}\mathfrak{p} \subset \mathfrak{p}$, suppose \mathfrak{p} is generated by (x_1, \dots, x_n) and $t^{-1}(x_1, \dots, x_n)^T = \Phi(x_1, \dots, x_n)^T$ for $\Phi \in M_n(A)$. There is a monic polynomial $\chi(T) \in A[T]$ vanishing Φ . Then $\chi(t^{-1}) = 0$ and $t^{-1} \in A$. This is impossible by definition of t . Then $t^{-1}\mathfrak{p} = A$, and $\mathfrak{p} = (t)$ is principal. By Krull's Principal Ideal Theorem A.3.14, $\text{ht}(\mathfrak{p}) = 1$.

Now we show that A verifies (R_1) . Suppose (A, \mathfrak{m}) is local of dimension 1. Choosing $a \in \mathfrak{m}$, A/a is of dimension 0. Then by A.3.11, $\mathfrak{m}^n \subset aA$ for some $n \geq 1$. Suppose $\mathfrak{m}^{n-1} \not\subset aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$ and let $t = a/b$. By construction, $t^{-1} \notin A$ and $t^{-1}\mathfrak{m} \subset A$. After similar argument, we see that $\mathfrak{m} = tA$, whence A is regular. \square

Lemma A.4.11. Let (A, \mathfrak{m}) be a noetherian local ring of dimension 1. Then A is normal iff A is regular.

Proof. By lemma A.4.10, we just need to show that regularity implies normality.

Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since A is regular, $\mathfrak{m} = (t)$. Let $I \subset \mathfrak{m}$ be an ideal. If $I \subset \bigcap_n \mathfrak{m}^n$, then for every $a \in I$, there exists a_n such that $a = a_n t^n$. Then we get an ascending chain of ideals $(a_1) \subset (a_2) \subset \cdots$. Hence $a = 0$ by Nakayama's Lemma. Suppose I is not zero. Then there is some n such that $I \subset \mathfrak{m}^n$ and $I \not\subset \mathfrak{m}^{n+1}$. For every $at^n \in I \setminus \mathfrak{m}^{n+1}$, $a \notin \mathfrak{m}$, whence a is a unit in A . Then $I = (t^n)$. Hence A is PID and hence normal. \square

Proposition A.4.12. Let A be a noetherian integral domain of dimension ≥ 1 verifying (S_2) . Then

$$A = \bigcap_{\mathfrak{p} \in \text{Spec } A, \text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

Proof. Clearly $A \subset \bigcap A_{\mathfrak{p}}$. Let $t = f/g \in \bigcap A_{\mathfrak{p}}$. Since $f \in gA_{\mathfrak{p}}$ and we have $gA = \bigcap (gA_{\mathfrak{p}} \cap A)$, $f \in gA$. It follows that $t \in A$. \square

Theorem A.4.13 (Serre's criterion for normality). Let X be a locally noetherian scheme. Then X is normal if and only if it verifies (R_1) and (S_2) .

Proof. One direction has been proved in Lemma A.4.10. Suppose X verifies (R_1) and (S_2) . Again we can assume $X = \operatorname{Spec} A$ is affine and A is local. By Remark A.4.9, we just need to show that A is integral closed in its total ring of fractions Q . Suppose we have

$$\left(\frac{a}{b}\right)^n + c_1 \left(\frac{a}{b}\right)^{n-1} + \cdots + c_n = 0 \in Q.$$

Since A verifies (S_2) , $bA = \bigcap \nu_{\mathfrak{p}}^{-1}(b_{\mathfrak{p}}A_{\mathfrak{p}})$. So it is sufficient to show that $a_{\mathfrak{p}} \in b_{\mathfrak{p}}A_{\mathfrak{p}}$ with $\operatorname{ht}(\mathfrak{p}) = 1$. Note that $A_{\mathfrak{p}}$ is regular and hence normal by Lemma A.4.11. Then above equation gives us desired result. \square

A.5 Smoothness

A.5.1 Modules of differentials and derivations

In this subsection, let R be a ring and A an R -algebra.

Definition A.5.1 (Derivation). A *derivation* of A over R is an R -linear map $\partial : A \rightarrow M$ with an A -module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module M , the set of all derivations of A over R into M forms an A -module, denoted by $\operatorname{Der}_R(A, M)$.

Given a module homomorphism $f : M \rightarrow N$ of A -modules and a derivation $\partial \in \operatorname{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of A over R into N .

Proposition A.5.2. The functor $\operatorname{Der}_R(A, -)$ is representable. The representing object is denoted by $\Omega_{A/R}$, which is called the *module of differentials* of A over R .

Proof. First suppose A is a free R -algebra with a set of generators $a_{\lambda}, \lambda \in \Lambda$. Then an R -derivation $\partial \in \operatorname{Der}_R(A, M)$ is uniquely determined by its values on the generators a_{λ} . Let

$$\Omega_{A/R} := \bigoplus_{\lambda \in \Lambda} A \cdot da_{\lambda}$$

and $d : A \rightarrow \Omega_{A/R}$ be the R -derivation defined by $a_{\lambda} \mapsto da_{\lambda}$. For any R -derivation $\partial \in \operatorname{Der}_R(A, M)$, we can define a unique A -module homomorphism $\Phi_{\partial} : \Omega_{A/R} \rightarrow M$ by sending da_{λ} to $\partial(a_{\lambda})$ such that $\partial = \Phi_{\partial} \circ d$. This gives a bijection

$$\operatorname{Der}_R(A, M) \cong \operatorname{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_{\partial}.$$

Now suppose $A = F/I$ is an arbitrary R -algebra, where F is a free R -algebra and I is an ideal

of F . Then we can define the module of differentials

$$\Omega_{A/R} := (\Omega_{F/R} \otimes_F A) / \sum_{f \in I} A \cdot df.$$

The R -linear map $d_A : F \otimes_F A \xrightarrow{d_F} \Omega_{F/R} \otimes_F A \rightarrow \Omega_{A/R}$ is a derivation of A over R .

For any R -derivation $\partial \in \text{Der}_R(A, M)$, note that $F \rightarrow A \xrightarrow{\partial} M$ is an R -derivation of F over R into M . Then we get an F -module homomorphism $\Omega_F \rightarrow M$. It gives an A -module homomorphism $\Omega_F \otimes_F A \rightarrow M, df \otimes 1 \mapsto \partial f$. This map factors into $\Omega_F \otimes_F A \rightarrow \Omega_{A/R}$ and $\Phi_\partial : \Omega_{A/R} \rightarrow M$. Since Φ_∂ is A -linear and $\Omega_{A/R}$ is generated by da_λ as A -module, such Φ_∂ is unique. \square

Corollary A.5.3. Suppose A is of finite type over R . Then the module of differentials $\Omega_{A/R}$ is a finitely generated A -module.

Remark A.5.4. Let B be an A -algebra, M an A -module and N a B -module. If there is a homomorphism of A -modules $M \rightarrow N$, then we can extend it to a homomorphism of B -modules $M \otimes_A B \rightarrow N$ by sending $m \otimes b$ to $m \cdot b$. And such extension is unique in the sense of following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\quad} & N \\ \downarrow & \nearrow \exists! & \\ M \otimes_A B & & \end{array}$$

Hence we get a natural bijection

$$\text{Hom}_A(M, N) \cong \text{Hom}_B(M \otimes_A B, N).$$

Proposition A.5.5. Let A, R' be R -algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

Proof. We check the universal property of $\Omega_{A/R} \otimes_A A'$. First, the map

$$d_{A'} : A \otimes_R R' \rightarrow \Omega_{A/R} \otimes_R R' \cong \Omega_{A/R} \otimes_A A', \quad a \otimes r \mapsto da \otimes r$$

is an R' -derivation of A' into $\Omega_{A/R} \otimes_A A'$. For any R' -derivation $\partial' : A' \rightarrow M$ into an A' -module M , we can compose it with the homomorphism $A' \rightarrow A$ and get an R -derivation $\partial : A \rightarrow M$. By the universal property of $\Omega_{A/R}$, there is a unique A -module homomorphism $\Phi : \Omega_{A/R} \rightarrow M$ such that $\partial = \Phi \circ d_A$. Then we can extend it to an A' -module homomorphism $\Phi' : \Omega_{A/R} \otimes_A A' \rightarrow M$ by Remark A.5.4. By the construction, we have $\Phi' \circ d_{A'} = \partial'$. \square

Proposition A.5.6. Let A be an R -algebra and S a multiplicative set of A . Then we have an isomorphism

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

Proof. Let

$$d_{S^{-1}A} : S^{-1}A \rightarrow S^{-1}\Omega_{A/R}, \quad \frac{a}{s} \mapsto \frac{sda - ads}{s^2}.$$

By direct computation, $d_{S^{-1}A}$ is an R -derivation of $S^{-1}A$ over R into $S^{-1}\Omega_{A/R}$. For any R -derivation

$\partial : S^{-1}A \rightarrow M$ into an $S^{-1}A$ -module M , we can get an $S^{-1}A$ -module homomorphism $\Phi' : S^{-1}\Omega_{A/R} \rightarrow M$ as proof of Proposition A.5.5. We have

$$\partial(s \cdot \frac{a}{s}) = s\partial(\frac{a}{s}) + \frac{a}{s}\partial s.$$

It follows that

$$\partial(\frac{a}{s}) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(da) - a\Phi'(ds)}{s^2} = \Phi'(\frac{sda - ads}{s^2}).$$

Thus, $\Phi' \circ d_{S^{-1}A} = \partial$. □

Theorem A.5.7. Let A be an R -algebra and B an A -algebra. Then there is a natural short exact sequence

$$\Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow \Omega_{B/A} \rightarrow 0$$

of B -modules.

Proof. Let $d_{A/R} : A \rightarrow \Omega_{A/R}$ be the R -derivation of A over R . The map $A \rightarrow B \xrightarrow{d_{B/R}} \Omega_{B/R}$ induces a B -linear map

$$u : \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R}, \quad d_{A/R}(a) \otimes b \mapsto bd_{B/R}(a).$$

The map $d_{B/A}$ is an A -derivation and hence R -derivation. Then it induces a B -linear map

$$v : \Omega_{B/R} \rightarrow \Omega_{B/A}, \quad d_{B/R}(b) \mapsto d_{B/A}(b).$$

Since $\Omega_{B/A}$ is generated by elements of the form $d_{B/A}(b)$ for $b \in B$, the map v is surjective. And clearly $d_{B/A}(a) = ad_{B/A}(1) = 0$ for $a \in A$.

Consider the composition $B \xrightarrow{d_{B/R}} \Omega_{B/R} \rightarrow \Omega_{B/R}/\mathfrak{I}u$. For every $a \in A, b \in B$, we have

$$[d_{B/R}(ab)] = [bd_{B/R}(a) + ad_{B/R}(b)] = [bd_{B/R}(a)] + [ad_{B/A}(b)] = [ad_{B/A}(b)].$$

Hence it is indeed an A -derivation of B . Then it induces a B -linear map

$$\varphi : \Omega_{B/A} \rightarrow \Omega_{B/R}/\mathfrak{I}u, \quad d_{B/A}(b) \mapsto [d_{B/R}(b)].$$

The map φ is surjective since $\Omega_{B/R}$ is generated by elements of the form $d_{B/R}(b)$ for $b \in B$. Note that the composition

$$\Omega_{B/A} \xrightarrow{\varphi} \Omega_{B/R}/\mathfrak{I}u \rightarrow \Omega_{B/A}/\text{Ker } v$$

is the identity map. Thus, φ is injective and hence an isomorphism. In particular, we have $\text{Ker } v = \mathfrak{I}u$. □

Remark A.5.8. The exact sequence in Theorem A.5.7 is left exact if and only if every R -derivation of A into B -module extends to an R -derivation of B into B -module.

To be completed.

Theorem A.5.9. Let A be an R -algebra and I an ideal of A . Set $B := A/I$. Then there is a natural

short exact sequence

$$I/I^2 \rightarrow \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow 0$$

of B -modules.

Proof. Suppose $A = F/\mathfrak{b}$ for some free R -algebra F and an ideal \mathfrak{b} of F . Let \mathfrak{a} be the preimage of I in F . Let $d\mathfrak{b}$ (resp. $d\mathfrak{a}$) denote the image of \mathfrak{b} (resp. \mathfrak{a}) in $\Omega_{F/R}$. Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B / (d\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B / (d\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \rightarrow (d\mathfrak{a} \otimes_F B) / (d\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows. \square

Definition A.5.10. Let \mathbf{k} be a field and A an integral \mathbf{k} -algebra of finite type of dimension n . We say A is *smooth at* $\mathfrak{p} \in \text{Spec } A$ if the module of differentials $\Omega_{A,\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module of rank n .

Example A.5.11. Let \mathbf{K}/\mathbf{k} be a finite generated field extension and \mathbf{k}' be the algebraic closure of \mathbf{k} in \mathbf{K} . Then

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \text{trdeg}(\mathbf{K}/\mathbf{k}) + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}},$$

and $\dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}} = 0$ if and only if \mathbf{k}' is separable over \mathbf{k} .

First suppose $\mathbf{K} = \mathbf{k}'$ is algebraic over \mathbf{k} . Suppose \mathbf{k}'/\mathbf{k} is separable. For every $\alpha \in \mathbf{k}'$, suppose $f(\alpha) = 0$ for $f \in \mathbf{k}[T]$. Then $df(\alpha) = f'(\alpha)d\alpha = 0$. By the separability of \mathbf{k}'/\mathbf{k} , we have $f'(\alpha) \neq 0$. It follows that $d\alpha = 0$. Conversely, let $\alpha \in \mathbf{k}'$ be an inseparable element over \mathbf{k} . Since $\mathbf{k}[\alpha] \rightarrow \mathbf{k}[\alpha], \alpha^n \mapsto n\alpha^{n-1}$ is a non-zero R -derivation, we have $\Omega_{\mathbf{k}[\alpha]/\mathbf{k}} \neq 0$. By induction on number of generated elements, choosing a middle field $\mathbf{k} \subset \mathbf{k}'' \subset \mathbf{k}'$, at least one of $\Omega_{\mathbf{k}''/\mathbf{k}}$ and $\Omega_{\mathbf{k}'/\mathbf{k}''}$ is non-zero. Then $\Omega_{\mathbf{K}/\mathbf{k}} \neq 0$ by Theorem A.5.7.

Then suppose $\mathbf{k}' = \mathbf{k}$. By the Noether's Normalization Lemma, we can find a finite set of elements $T_1, \dots, T_n \in \mathbf{K}$ such that \mathbf{K} is algebraic over $\mathbf{k}'(T_1, \dots, T_n)$. Note that we can choose T_i such that $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$ is separable. To see this, if $\alpha \in \mathbf{K}$ is an inseparable element over $\mathbf{k}'(T_1, \dots, T_n)$, then by replacing a suitable T_i with α , we reduce the inseparable degree of $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$.

Since $\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)$ is finite, every \mathbf{k} -derivation of $\mathbf{k}'(T_1, \dots, T_n)$ into \mathbf{K} -module extends to a \mathbf{k} -derivation of \mathbf{K} into \mathbf{K} -module. Then by Remark A.5.8, we have

$$0 \rightarrow \Omega_{\mathbf{k}'(T_1, \dots, T_n)/\mathbf{k}} \otimes_{\mathbf{k}'(T_1, \dots, T_n)} \mathbf{K} \rightarrow \Omega_{\mathbf{K}/\mathbf{k}} \rightarrow \Omega_{\mathbf{K}/\mathbf{k}'(T_1, \dots, T_n)} \rightarrow 0.$$

Finally, note that every \mathbf{k} -derivation ∂ of \mathbf{k}' into \mathbf{K} -module can be extended to $\mathbf{k}'[T_1, \dots, T_n]$ by setting $\partial T_i = 0$. Thus, we have

$$0 \rightarrow \Omega_{\mathbf{k}'/\mathbf{k}} \otimes_{\mathbf{k}'} \mathbf{k}'[T_1, \dots, T_n] \rightarrow \Omega_{\mathbf{k}'[T_1, \dots, T_n]/\mathbf{k}} \rightarrow \Omega_{\mathbf{k}'[T_1, \dots, T_n]/\mathbf{k}'} \rightarrow 0.$$

This follows that

$$\dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}} = \dim_{\mathbf{K}} \Omega_{\mathbf{K}/\mathbf{k}'} + \dim_{\mathbf{k}'} \Omega_{\mathbf{k}'/\mathbf{k}}.$$

A.5.2 Applications to affine varieties

Let \mathbf{k} be arbitrary field, $A = \mathbf{k}[T_1, \dots, T_n]$ and \mathfrak{m} a maximal ideal of A such that $\kappa(\mathfrak{m})$ is separable over \mathbf{k} . We try to give an explanation of Zariski's tangent space at \mathfrak{m} using the language of derivation. We know that $\Omega_{A/\mathbf{k}} = \bigoplus_{i=1}^n A dT_i$, thus $\Omega_{A_{\mathfrak{m}}/\mathbf{k}} \cong \bigoplus_{i=1}^n A_{\mathfrak{m}} dT_i$. Then

$$\mathrm{Der}_{\mathbf{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}}) \cong \mathrm{Hom}_{\mathbf{k}}(\Omega_{A_{\mathfrak{m}}/\mathbf{k}}, A_{\mathfrak{m}}) \cong \bigoplus_{i=1}^n A_{\mathfrak{m}} \partial_i,$$

where $\partial_i \in \mathrm{Der}_{\mathbf{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ is the derivation defined by $dT_i \mapsto 1$ and $dT_j \mapsto 0$ for $j \neq i$. It coincides with the usual derivation $f \mapsto \partial f / \partial T_i$. Consider the restriction of ∂_i to \mathfrak{m} and take values in the residue field $\kappa(\mathfrak{m})$, we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1, \dots, \partial_n)^T} A_{\mathfrak{m}}^n \rightarrow \kappa(\mathfrak{m})^n.$$

Since $\kappa(\mathfrak{m})$ is separable over \mathbf{k} , we claim that $\mathrm{Ker} \Phi = \mathfrak{m}^2$. Indeed, by Remark A.5.12, we can write every $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ as $\sum_i a_i g_i$. Then

$$\frac{\partial f}{\partial T_i} = a_i \frac{\partial g_i}{\partial T_i} + g_i \frac{\partial a_i}{\partial T_i}.$$

Since g_i is separable, the image of $\partial g_i / \partial T_i$ in $\kappa(\mathfrak{m})$ is not zero. Hence $\Phi(f) \neq 0$. By the claim, Φ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$ of $\kappa(\mathfrak{m})$ -vector spaces. Then we get

$$T_{A, \mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where $x \in \mathbb{A}_{\mathbf{k}}^n$ is the point corresponding to \mathfrak{m} . This coincides with the usual tangent space at x in language of differential geometry.

Remark A.5.12. Let \mathbf{k} be arbitrary field, $A = \mathbf{k}[T_1, \dots, T_n]$ and g_i irreducible polynomials in one variable T_i over \mathbf{k} . Then for every $f \in A$, we can write

$$f = \sum_{I=(i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \leq \deg g_i.$$

This is called the *Taylor expansion of f with respect to g_1, \dots, g_n* .

When $n = 1$, it follows from division algorithm. For $n > 1$, we can use induction on n . Let $\mathbf{K} = \mathbf{k}(T_1, \dots, T_{n-1})$. Then we can write f as

$$f = \sum_{i=0}^r a_i g_n^i, \quad a_i \in \mathbf{K}[T_n], \quad \deg a_i < \deg g_n.$$

Comparing the coefficients of two sides from the highest degree of T_n to the lowest degree, we see that

$$a_i \in \mathbf{k}[T_1, \dots, T_{n-1}].$$

By induction hypothesis, the conclusion follows.

Let $B = A/I$ be a \mathbf{k} of finite type, $I = (F_1, \dots, F_m) \subset \mathfrak{m}$ and \mathfrak{n} the image of \mathfrak{m} in B . We have an exact sequence of $\kappa(\mathfrak{m})$ -vector spaces

$$0 \rightarrow I/(I \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} : \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of F_1, \dots, F_m is the $m \times n$ matrix

$$J(F_1, \dots, F_m) := \left(\frac{\partial F_i}{\partial T_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

with entries in B .

Theorem A.5.13. Setting as above. Then B is regular at \mathfrak{n} if and only if the Jacobian matrix J has maximal rank $n - \dim B_{\mathfrak{n}}$ after taking values in the residue field $\kappa(\mathfrak{m})$.

Proof. We have an exact sequence

$$0 \rightarrow T_{B,\mathfrak{n}} \rightarrow T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa^m \rightarrow 0,$$

where Ψ sends $\partial \in T_{A,\mathfrak{m}}$ to $(\partial(F_1), \dots, \partial(F_m))^T$. Note that the matrix of Ψ is just J^T , the transpose of the Jacobian matrix. Hence

$$\text{rank } J = n - \dim_{\kappa} T_{B,\mathfrak{n}} \leq n - \dim B_{\mathfrak{n}}$$

and the equality holds if and only if B is regular at \mathfrak{n} . □

Remark A.5.14. If $\kappa(\mathfrak{m})$ is not separable over \mathbf{k} , then we still have the inequality

$$\text{rank } J \leq n - \dim B_{\mathfrak{n}}.$$

Indeed, in any case, we have an exact sequence

$$0 \rightarrow I/(I \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0.$$

Hence $\dim_{\kappa} I/(I \cap \mathfrak{m}^2) = n - \dim B_{\mathfrak{n}}$. There is a $\kappa(\mathfrak{m})$ -linear map

$$I/(I \cap \mathfrak{m}^2) \rightarrow \kappa(\mathfrak{m})^n, \quad [f] \mapsto (\partial_1(f), \dots, \partial_n(f))^T,$$

and every row of the Jacobian matrix J is in the image of this map. Thus, the rank of J is at most $n - \dim B_{\mathfrak{n}}$.

Hence if $\text{rank } J = n - \dim B_{\mathfrak{n}}$, we can still see that B is regular at \mathfrak{n} . However, the converse does not hold in general.

Proposition A.5.15. Let \mathbf{k} be a field, \mathbb{k} the algebraic closure of \mathbf{k} , A a \mathbf{k} -algebra of finite type and $A_{\mathbb{k}} := A \otimes_{\mathbf{k}} \mathbb{k}$. Suppose $A_{\mathbb{k}}$ is integral. Let $\mathfrak{m} \in \text{mSpec } A$ and \mathfrak{m}' be a maximal ideal of $A_{\mathbb{k}}$ lying over \mathfrak{m} . Then

- (a) If $A_{\mathbb{k}}$ is regular at \mathfrak{m}' , then A is regular at \mathfrak{m} ;
- (b) suppose $\kappa(\mathfrak{m})$ is separable over \mathbf{k} , the converse holds.

Proof. Regarding $J_{\mathfrak{m}}$ and $J_{\mathfrak{m}'}$ as matrices with entries in \mathbb{k} , they are the same and hence have the same rank. If $A_{\mathbb{k}}$ is regular at \mathfrak{m}' , since $\kappa(\mathfrak{m}) = \mathbb{k}$, then $\text{rank } J_{\mathfrak{m}'} = n - \dim A_{\mathbb{k}, \mathfrak{m}'}$. Note that $\dim A_{\mathbb{k}, \mathfrak{m}'} = \text{trdeg}(\mathcal{K}(A_{\mathbb{k}})/\mathbb{k}) = \text{trdeg}(\mathcal{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}}$, we have $\text{rank } J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence A is regular at \mathfrak{m} .

Conversely, suppose A is regular at \mathfrak{m} and $\kappa(\mathfrak{m})$ is separable over \mathbf{k} . Then $\text{rank } J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence $A_{\mathbb{k}}$ is regular at \mathfrak{m}' . **To be modified.** □

Proposition A.5.16. Let \mathbf{k} be a field and A an integral \mathbf{k} -algebra of finite type and of dimension n . Let \mathbb{k} be the algebraic closure of \mathbf{k} and $A_{\mathbb{k}} := A \otimes_{\mathbf{k}} \mathbb{k}$. Then A is smooth at $\mathfrak{p} \in \text{Spec } A$ if and only if $A_{\mathbb{k}}$ is regular at every \mathfrak{m}' over \mathfrak{m} .

Proof. Since $\Omega_{A_{\mathbb{k}}/\mathbb{k}} \cong \Omega_{A/\mathbf{k}} \otimes_A A_{\mathbb{k}}$ is free of rank n if and only if $\Omega_{A/\mathbf{k}}$ is free of rank n , we can assume that $\mathbf{k} = \mathbb{k}$. If A is smooth at \mathfrak{p} , then $\Omega_{A_{\mathfrak{p}}/\mathbf{k}} \cong \bigoplus A_{\mathfrak{p}} df_i$ is free of rank n . Let $\mathfrak{P}_i \in \text{Der}_{\mathbb{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ be the derivation defined by $df_i \mapsto 1$ and $dT_j \mapsto 0$ for $j \neq i$. Then we have $\partial_i f_j = \delta_{ij}$ for $1 \leq i, j \leq n$. Then similar to above argument, we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{(\partial_1, \dots, \partial_n)^T} \mathbb{k}^n.$$

This shows that $A_{\mathbb{k}}$ is regular at \mathfrak{m} .

Conversely, suppose $A_{\mathbb{k}}$ is regular at \mathfrak{m} . Note that $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \Omega_{A_{\mathbb{k}}/\mathbb{k}} \otimes_A \mathbb{k}$ is surjective since $\Omega_{A_{\mathbb{k}}/\mathbb{k}} = 0$. Then by Nakayama's lemma, $\Omega_{A_{\mathfrak{m}}/\mathbb{k}}$ is generated by n elements as an $A_{\mathfrak{m}}$ -module.

Note that $\dim_{\mathcal{K}(A)} \Omega_{\mathcal{K}(A)/\mathbf{k}} = \text{trdeg}(\mathcal{K}(A)/\mathbf{k}) = \dim A_{\mathfrak{m}} = n$. **By induction on transcendental degree.**

By Nakayama's Lemma, $\Omega_{A_{\mathfrak{m}}/\mathbf{k}}$ is free of rank n as an $A_{\mathfrak{m}}$ -module.

To be completed. □

Example A.5.17. Let \mathbf{k} be an imperfect field of characteristic $p > 2$. Suppose $\alpha = \beta^p \in \mathbf{k}$ and β is not in \mathbf{k} . Let $A = \mathbf{k}[x, y]/(x^2 - y^p - \alpha)$ and $\mathfrak{m} = (x, y^p - \alpha) = (x)$. Note that \mathfrak{m} is principal, so A is regular at \mathfrak{m} . However,

$$J_{\mathfrak{m}} = \left(\frac{\partial}{\partial x}(x^2 - y^p - \alpha), \frac{\partial}{\partial y}(y^p - \alpha) \right) = (2x, 0) = (0, 0) \in M_{1 \times 2}(\kappa(\mathfrak{m})).$$

Thus, A is not smooth at \mathfrak{m} . From the view of differentials, we have

$$\Omega_{A_{\mathfrak{m}}/\mathbf{k}} = A_{\mathfrak{m}} dx \oplus A_{\mathfrak{m}} dy / A_{\mathfrak{m}} \cdot x dx = \kappa(\mathfrak{m}) dx \oplus A_{\mathfrak{m}} dy,$$

which is not free as an $A_{\mathfrak{m}}$ -module.

A.6 Formal Completion

A.6.1 Formal completion of rings and modules

Definition A.6.1. Let A be a ring and \mathcal{T} a topology on A . We say that (A, \mathcal{T}) is a *topological ring* if the operations of addition and multiplication are continuous with respect to the topology \mathcal{T} .

Given a topological ring A . A *topological A -module* is a pair (M, \mathcal{T}_M) where M is an A -module and \mathcal{T}_M is a topology on M such that the addition and scalar multiplication is continuous. The morphisms of topological A -modules are the continuous A -linear maps. They form a category denoted by **TopMod $_A$** .

Definition A.6.2. Let A be a ring, I an ideal of A and M an A -module. The *I -adic topology* on M is the topology defined by the basis of open sets $x + I^k M$ for all $x \in M, k \geq 0$.

Example A.6.3. Let $A = \mathbb{Z}$ be the ring of integers and p a prime number. The p -adic topology on \mathbb{Z} is defined by the metric

$$d(x, y) := \|x - y\|_p := p^{-v(x-y)},$$

where v is the valuation defined by the ideal $p\mathbb{Z}$.

Note that for I -adic topology, any homomorphism $f : M \rightarrow N$ of A -modules is continuous since $f(x + I^k N) \subset f(x) + I^k M$ for all $x \in M$ and $k \geq 0$. Hence the forgotten functor **TopMod $_A$** \rightarrow **Mod $_A$** gives an equivalence of categories.

Let M be an A -module equipped with the I -adic topology. Note that M is Hausdorff as a topological space if and only if $\bigcap_{n \geq 0} I^n M = \{0\}$. In this case, we say that M is *I -adically separated*.

When M is I -adically separated, we can see that M is indeed a metric space. Fix $r \in (0, 1)$. For every $x \neq y \in M$, there is a unique $k \geq 0$ such that $x - y \in I^k M$ but $x - y \notin I^{k+1} M$. We can define a metric on M by

$$d(x, y) := r^k.$$

This metric induces the I -adic topology on M .

To analyze the I -adic separation property of M , the following Artin-Rees Lemma is particularly useful.

Theorem A.6.4 (Artin-Rees Lemma). Let A be a noetherian ring, I an ideal of A , M a finite A -module and N a submodule of M . Then there exists an integer r such that for all $n \geq 0$, we have

$$(I^{r+n} M) \cap N = I^n (I^r M \cap N).$$

Proof. Let

$$A' := A \oplus IX \oplus I^2 X^2 \oplus \cdots \subset A[X]$$

be a graded A -algebra. Note that if $I = (a_1, \dots, a_k)$, then $A' = A[a_1 X, \dots, a_k X]$. Hence A' is a noetherian ring. Let

$$M' := M \oplus IMX \oplus I^2 MX^2 \oplus \cdots$$

be a graded A' -module. Then M' is a finite A' -module since it is generated by M and M is finite over A . Let

$$N' := N \oplus (IM \cap N)X \oplus (I^2M \cap N)X^2 \oplus \dots$$

be a graded submodule of M' . Then N' is finite over A' . Suppose $N' = \sum A'x_i$ with $x_i \in I^{d_i}M \cap N$. Choose $r \geq d_i$ for all i . Then the degree $n+r$ part of N' is equal to degree n part of A' timing the degree r part of N' . That is, for all $n \geq 0$, $I^{n+r}M \cap N = I^n(I^rM \cap N)$. \square

Corollary A.6.5. Let A be a noetherian ring, I an ideal of A , M a finite A -module and N a submodule of M . Then the subspace topology on N induced by $N \subset M$ coincides with the I -adic topology on N .

Proof. This is a direct consequence of the Artin-Rees Lemma. \square

Corollary A.6.6. Let A be a noetherian ring, I an ideal of A , and M a finite A -module. Let $N = \bigcap_{n \geq 0} I^n M$. Then $IN = N$. In particular, if $I \subset \text{rad}(A)$, then M is I -adically separated.

Proof. We have that

$$N = I^{n+r}M \cap N = I^n(I^rM \cap N) = I^nN \subset IN \subset N.$$

The latter conclusion follows from the Nakayama's Lemma. \square

Definition A.6.7. Let A be a ring, I an ideal of A and M an A -module. We say that M is *complete* (with respect to I -adic topology) if M is I -adically separated and complete as a metric space with respect to the metric induced by the I -adic topology.

Lemma A.6.8. Let A be a ring, I an ideal of A and M an A -module. Then the inverse limit

$$\hat{M} := \varprojlim (\dots \rightarrow M/I^n M \rightarrow M/I^{n-1} M \rightarrow \dots \rightarrow M/IM)$$

exists in the category of A -modules. Moreover, \hat{A} is an A -algebra and \hat{M} is an \hat{A} -module.

Proof. Let

$$\hat{M} := \left\{ (x_n) \in \prod_{n \geq 0} M/I^n M \mid x_{n+1} \mapsto x_n \right\}.$$

We claim that \hat{M} is that we desired. **To be completed.** \square

Definition A.6.9 (Formal Completion). Let A be a ring, I an ideal of A and M an A -module. The *formal completion* of M with respect to I , denoted by \hat{M} , is defined as

$$\hat{M} := \varprojlim (\dots \rightarrow M/I^n M \rightarrow M/I^{n-1} M \rightarrow \dots \rightarrow M/IM),$$

where the maps are the natural projections $M/I^n M \rightarrow M/I^{n-1} M$.

Example A.6.10. Let $A = \mathbb{Z}$ be the ring of integers and $I = p\mathbb{Z}$. The formal completion of \mathbb{Z} with respect to $p\mathbb{Z}$ is the ring of p -adic integers, denoted by \mathbb{Z}_p . The elements of \mathbb{Z}_p can be represented

as infinite series of the form

$$a_0 + a_1p + a_2p^2 + \cdots,$$

where $a_i \in \{0, 1, \dots, p-1\}$.

Example A.6.11. Let R be a ring, $A = R[X_1, \dots, X_n]$ and $I = (X_1, \dots, X_n)$. The formal completion of A with respect to I is the ring of formal power series $R[[X_1, \dots, X_n]]$. The elements of $R[[X_1, \dots, X_n]]$ can be represented as infinite series of the form

$$\sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} X_1^{i_1} \cdots X_n^{i_n},$$

where $a_{i_1, \dots, i_n} \in R$ and the multi-index (i_1, \dots, i_n) runs over all non-negative integers.

Proposition A.6.12. The formal completion \widehat{M} of a A -module M is complete, and image of M is dense in \widehat{M} . Moreover, \widehat{M} is uniquely characterized by above properties.

Proof. To be completed. □

By the universal property of the inverse limit, we get a covariant functor from the category of A -modules to the category of topological \widehat{A} -modules, which sends an A -module M to \widehat{M} and a morphism $f : M \rightarrow N$ to the induced morphism $\widehat{f} : \widehat{M} \rightarrow \widehat{N}$.

Lemma A.6.13. Let

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

be an exact sequence of finite A -modules. Then the sequence of \widehat{A} -modules

$$0 \rightarrow \widehat{M}_1 \rightarrow \widehat{M}_2 \rightarrow \widehat{M}_3 \rightarrow 0$$

is still exact.

Proof. To be completed. □

Proposition A.6.14. Let \widehat{A} be completion of a noetherian ring A with respect to an ideal I and M a finite A -module. Then the natural map $M \otimes_A \widehat{A} \rightarrow \widehat{M}$ is an isomorphism.

Proof. Since A is noetherian and M is finite, we have an exact sequence

$$A^m \rightarrow A^n \rightarrow M \rightarrow 0.$$

By Lemma A.6.13, we have an exact sequence

$$\widehat{A}^m \rightarrow \widehat{A}^n \rightarrow \widehat{M} \rightarrow 0.$$

On the other hand, we have

$$A^m \otimes_A \widehat{A} \rightarrow A^n \otimes_A \widehat{A} \rightarrow M \otimes_A \widehat{A} \rightarrow 0$$

by right exactness of the tensor product. Since the inverse limit commutes with finite direct sums, we complete the proof by the Five Lemma. □

Proposition A.6.15. Let A be a noetherian ring and I an ideal of A . Then the formal completion \hat{A} of A with respect to I is a flat A -module.

Proof. This is a direct consequence of Lemma A.6.13 and Proposition A.6.14. \square

Lemma A.6.16. Let \hat{A} be the formal completion of a noetherian ring A with respect to an ideal I . Suppose that I is generated by a_1, \dots, a_n . Then we have an isomorphism of topological rings

$$\hat{A} \cong A[[X_1, \dots, X_n]]/(X_1 - a_1, \dots, X_n - a_n).$$

Proof. To be completed. \square

Proposition A.6.17. Let A be a noetherian ring and I an ideal of A . Then the formal completion \hat{A} of A with respect to I is a noetherian ring.

Proof. Note that $A[[X_1, \dots, X_n]]$ is noetherian by Hilbert's Basis Theorem. Then the conclusion follows from Lemma A.6.16. \square

Proposition A.6.18. Let A be a noetherian ring and \mathfrak{m} a maximal ideal of A . Then the formal completion \hat{A} of A with respect to \mathfrak{m} is a local ring with maximal ideal $\mathfrak{m}\hat{A}$.

Proof. To be completed. \square

A.6.2 Complete local rings

Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring with respect to the \mathfrak{m} -adic topology. We say that A is *of equal characteristic* if $\text{char } A = \text{char } \mathbf{k}$, and *of mixed characteristic* if $\text{char } A \neq \text{char } \mathbf{k}$. In latter case, $\text{char } \mathbf{k} = p$ and $\text{char } A = 0$ or $\text{char } A = p^k$.

The goal of this subsection is the following structure theorem for noetherian complete local rings due to Cohen.

Theorem A.6.19 (Cohen Structure Theorem). Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring of dimension d . Then

- (a) A is a quotient of a noetherian regular complete local ring;
- (b) if A is regular and of equal characteristic, then $A \cong \mathbf{k}[[X_1, \dots, X_d]]$;
- (c) if A is regular, of mixed characteristic $(0, p)$ and $p \notin \mathfrak{m}^2$, then $A \cong D[[X_1, \dots, X_{d-1}]]$, where (D, p, \mathbf{k}) is a complete DVR;
- (d) if A is regular, of mixed characteristic $(0, p)$ and $p \in \mathfrak{m}^2$, then $A \cong D[[X_1, \dots, X_d]]/(f)$, where (D, p, \mathbf{k}) is a complete DVR and f a regular parameter.

To prove the Cohen Structure Theorem, we first list some preliminary results on complete local rings. They are independently important and can be used in other contexts.

Theorem A.6.20 (Hensel's Lemma). Let $(A, \mathfrak{m}, \mathbf{k})$ be a complete local ring, $f \in A[X]$ a monic polynomial and $\bar{f} \in \mathbf{k}[X]$ its reduction modulo \mathfrak{m} . Suppose that $\bar{f} = \bar{g} \cdot \bar{h}$ for some monic polynomials $\bar{g}, \bar{h} \in \mathbf{k}[X]$ such that $\gcd(\bar{g}, \bar{h}) = 1$. Then the factorization lifts to a unique factorization $f = g \cdot h$ in $A[X]$ such that g and h are monic polynomials.

Proof. Lift \bar{g} and \bar{h} to monic polynomials $g_1, h_1 \in A[X]$. We inductively construct a sequence of monic polynomials $g_n, h_n \in A[X]$ such that $\Delta_n = f - g_n h_n \in \mathfrak{m}^n[X]$ and $g_n - g_{n+1}, h_n - h_{n+1} \in \mathfrak{m}^n[X]$ for all $n \geq 1$. Suppose that g_n and h_n are constructed. Let $g_{n+1} = g_n + \varepsilon_n$ and $h_{n+1} = h_n + \eta_n$ for $\varepsilon_n, \eta_n \in \mathfrak{m}^n[X]$. Then we have

$$f - g_{n+1}h_{n+1} = \Delta_n - (\varepsilon_n h_n + \eta_n g_n) + \varepsilon_n \eta_n.$$

Hence we just need to choose ε_n and η_n such that

$$\varepsilon_n h_n + \eta_n g_n \equiv \Delta_n \pmod{\mathfrak{m}^{n+1}}, \quad \deg \varepsilon_n < \deg g_n, \quad \deg \eta_n < \deg h_n.$$

Since $\gcd(\bar{g}, \bar{h}) = 1$, there exist $\bar{u}, \bar{v} \in \mathbf{k}[X]$ such that $\bar{u}\bar{g} + \bar{v}\bar{h} = 1$ and $\deg \bar{u} < \deg \bar{g}$, $\deg \bar{v} < \deg \bar{h}$. Lift \bar{u} and \bar{v} to $u, v \in A[X]$ preserving the degrees. Then we have $u g_n + v h_n \equiv 1 \pmod{\mathfrak{m}}$. Let $\varepsilon_n = u \Delta_n$ and $\eta_n = v \Delta_n$. Then we get the desired equation. \square

Proposition A.6.21. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring and M an A -module that is \mathfrak{m} -adically separated. Suppose $\dim_{\mathbf{k}} M/\mathfrak{m}M < \infty$. Then the basis of $M \otimes_A \mathbf{k}$ as \mathbf{k} -vector space can be lifted to a generating set of M as an A -module.

Proof. Let $t_1, \dots, t_n \in M$ such that their images in $M/\mathfrak{m}M$ form a basis of $M/\mathfrak{m}M$ as a \mathbf{k} -vector space. Then $M = t_1 A + \dots + t_n A + \mathfrak{m}M$. For every $x \in M$, we can write

$$x = a_{0,1}t_1 + \dots + a_{0,n}t_n + m_1$$

for some $a_{0,i} \in A$ and $m_1 \in \mathfrak{m}M$. Inductively, we have $\mathfrak{m}^k M = t_1 \mathfrak{m}^k + \dots + t_n \mathfrak{m}^k + \mathfrak{m}^{k+1}M$. Suppose that we have constructed $m_k \in \mathfrak{m}^k M$. Then we can write

$$m_k = a_{k,1}t_1 + \dots + a_{k,n}t_n + m_{k+1}.$$

Note that $\sum_{k \geq 0} a_{k,i}$ converges in A , denote its limit by a_i . Then we have

$$x - a_1 t_1 - \dots - a_n t_n = \sum_{i=1}^n \sum_{r \geq k} a_{r,i} t_i + m_k \in \mathfrak{m}^k M$$

for all k . Since M is \mathfrak{m} -adically separated, $x = a_1 t_1 + \dots + a_n t_n$. It follows that $M = \sum A t_i$. \square

The key to prove the Cohen Structure Theorem is the existence of coefficient rings.

Definition A.6.22 (Coefficient rings). Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring.

When A is equal-characteristic, the coefficient ring (or coefficient field) is a homomorphism of rings $\mathbf{k} \rightarrow A$ such that $\mathbf{k} \rightarrow A \rightarrow A/\mathfrak{m}$ is an isomorphism.

When A is mixed-characteristic, the coefficient ring is a complete local ring (R, pR, \mathbf{k}) with a local homomorphism of rings $R \hookrightarrow A$ such that the induced homomorphism $R/pR \rightarrow A/\mathfrak{m}$ is an isomorphism.

Remark A.6.23. Recall that a homomorphism of local rings $f : (A, \mathfrak{m}_A) \rightarrow (B, \mathfrak{m}_B)$ is said to be local if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Theorem A.6.24. Every noetherian complete local ring $(A, \mathfrak{m}, \mathbf{k})$ has a coefficient ring.

Assume the existence of coefficient rings, we can prove the Cohen Structure Theorem.

Proof of Cohen Structure Theorem. Let R be a coefficient ring of A and $\mathfrak{m} = (f_1, \dots, f_d)$ a minimal generating set of \mathfrak{m} . Then we have a homomorphism of complete local rings

$$\Phi : R[[X_1, \dots, X_d]] \rightarrow A, \quad X_i \mapsto f_i.$$

Let \mathfrak{n} be the maximal ideal of $R[[X_1, \dots, X_d]]$. Then $\mathfrak{n}A = \mathfrak{m}$. By Proposition A.6.21, A is generated by 1 as an $R[[X_1, \dots, X_d]]$ -module. This implies that Φ is surjective and (a) follows.

If A is regular of equal characteristic, then \mathfrak{m} is generated by a regular sequence. By consider the dimension of $R[[X_1, \dots, X_d]]$ and A , we have that Φ is an isomorphism. This proves (b).

Note that if A is regular of mixed characteristic $(0, p)$ and $p \notin \mathfrak{m}^2$, then \mathfrak{m} is generated by p, f_1, \dots, f_{d-1} . Then consider the homomorphism of complete local rings

$$R[[X_1, \dots, X_{d-1}]] \rightarrow A, \quad X_i \mapsto f_i.$$

By the same argument as above, we have that it is an isomorphism. This proves (c).

For (d), we have that $\ker \Phi$ is of height 1 by the dimension argument. Since regular local rings are UFDs, we can write $\ker \Phi = (f)$ for some $f \in R[[X_1, \dots, X_d]]$. Then we finish. \square

Existence of coefficient rings

Proof of Theorem A.6.24 in characteristic 0. Note that for any $n \in \mathbb{Z}$, $n \notin \mathfrak{m}$. Hence $\mathbb{Q} \subset A$. Let $\Sigma := \{\text{subfield in } A\}$ and K a maximal element in Σ with respect to the inclusion. The set Σ is non-empty since $\mathbb{Q} \in \Sigma$. By Zorn's Lemma, K exists. Then K is a subfield of \mathbf{k} by $K \hookrightarrow A \twoheadrightarrow A/\mathfrak{m} \cong \mathbf{k}$. We claim that K is a coefficient field of A .

Suppose there is $\bar{t} \in \mathbf{k} \setminus K$. If \bar{t} is transcendental over K , lift \bar{t} to an element $t \in A$. Then for any polynomial $f \neq 0 \in K[T]$, we have $f(\bar{t}) \neq 0 \in \mathbf{k}$. Hence $f(t) \notin \mathfrak{m}$. This implies that $1/f(t) \in A$, whence $K(t) \subset A$. This contradicts the maximality of K . If \bar{t} is algebraic over K , let $f \in K[T]$ be the minimal polynomial of \bar{t} . Then f is irreducible in $K[T]$ and $f(\bar{t}) = 0$. Regard f as a polynomial in $A[T]$ by $K \hookrightarrow A$. Note that $\text{char } A = 0$ implies that f is separable. By Hensel's Lemma (Theorem A.6.20), we can lift the root \bar{t} to an element $t \in A$ such that $f(t) = 0$. Then $K(t)$ is a field extension of K and $K(t) \subset A$. This contradicts the maximality of K again. \square

The same strategy does not work when $\text{char } \mathbf{k} = p > 0$ since there might be inseparable extensions. To fix this, we need to introduce the notion of p -basis.

Definition A.6.25. Let \mathbf{k} be a field of characteristic p . A finite set $\{t_1, \dots, t_n\} \subset \mathbf{k} \setminus \mathbf{k}^p$ is called p -independent if $[\mathbf{k}(t_1, \dots, t_n) : \mathbf{k}] = p^n$. A set $\Theta \subset \mathbf{k} \setminus \mathbf{k}^p$ is called a p -independent if its any finite subset is p -independent. A p -basis for \mathbf{k} is a maximal p -independent set $\Theta \subset \mathbf{k} \setminus \mathbf{k}^p$.

By definition, we have that $\mathbf{k} = \mathbf{k}^p[\Theta]$ for any p -basis Θ of \mathbf{k} . For any $a \in \mathbf{k}$ and $\theta \in \Theta$, we can write a as a polynomial in Θ with coefficients in \mathbf{k}^p . The degree of θ in such polynomial representation is at most $p - 1$. Such polynomial representation is unique by definition of p -independence.

Applying the Frobenius map n times, we have that $\mathbf{k}^{p^n} = \mathbf{k}^{p^{n+1}}[\Theta^{p^n}]$. This follows that $\mathbf{k} = \mathbf{k}^{p^n}[\Theta]$ for all n . Moreover, for any $a \in \mathbf{k}$ and $\theta \in \Theta$, we can write a as a polynomial in Θ with coefficients in \mathbf{k}^{p^n} and the degree of θ is at most $p^n - 1$. Such polynomial representation is unique.

Let \mathbf{k} be a perfect field of characteristic p . If there is $a \in \mathbf{k} \setminus \mathbf{k}^p$, then $\mathbf{k}(a^{1/p})/\mathbf{k}$ is an inseparable extension. This contradicts the perfectness of \mathbf{k} . Hence $\mathbf{k} = \mathbf{k}^p$ and \mathbf{k} has no nonempty p -basis.

Example A.6.26. Let $\mathbf{k} = \mathbb{F}_p(t_1, \dots, t_n)$. Then $\mathbf{k}^p = \mathbb{F}_p(t_1^p, \dots, t_n^p)$. The set $\{t_1, \dots, t_n\}$ is a p -basis for \mathbf{k} .

Proof of Theorem A.6.24 in characteristic p . Choose $\Theta \subset A$ such that its image in A/\mathfrak{m} is a p -basis for \mathbf{k} . Let $A_n := A^{p^n} = \{a^{p^n} : a \in A\}$ and $K := \bigcap_{n \geq 0} (A_n[\Theta])$. Then we claim that K is a coefficient field of A .

First we show that $A_n[\Theta] \cap \mathfrak{m} \subset \mathfrak{m}^{p^n}$. For every $a \in A_n[\Theta]$, if the degree of θ in the polynomial representation of a is more than $p^n - 1$, we can write $\theta^k = \theta^{ap^n} \cdot \theta^b$ for some $b < p^n$. Regard $\theta^{ap^n} \in A^{p^n}$ as coefficients. Now assume that $a \in A_n[\Theta] \cap \mathfrak{m}$. Then consider the image of a in A/\mathfrak{m} . The image of a equals 0 implies every coefficient of a is in \mathfrak{m} . Such coefficients are of form b^{p^n} for some $b \in A$, whence $b \in \mathfrak{m}$. Hence $a \in \mathfrak{m}^{p^n}$. This implies that $K \cap \mathfrak{m} = \bigcap_{n \geq 0} (A_n[\Theta] \cap \mathfrak{m}) \subset \bigcap_{n \geq 0} \mathfrak{m}^{p^n} = \{0\}$. Then K is a field and hence a subfield of \mathbf{k} .

For any $\bar{a} \in \mathbf{k}$, note that $\mathbf{k} = \mathbf{k}^p[\bar{\Theta}] = \mathbf{k}^{p^2}[\bar{\Theta}] = \dots = \mathbf{k}^{p^n}[\bar{\Theta}] = \dots$. For every n , write

$$\bar{a} = \sum_{\mu_n} \bar{c}_{\mu_n}^{p^n} \mu_n =: P_{\bar{a},n}(\bar{c}_{\mu_n}),$$

where μ_n runs over all monomials in $\bar{\Theta}$ with degree at most $p^n - 1$ and $\bar{c}_{\mu_n} \in \mathbf{k}$. We call this representation the p^n -development of \bar{a} with respect to $\bar{\Theta}$. Plug the p^m -development of c_{μ_n} into $P_{\bar{a},n}$, we get the p^{n+m} -development of \bar{a} . In formula, that is,

$$P_{\bar{a},n}(P_{\bar{a},m}(\bar{c}_{\mu_{n+m}})) = P_{\bar{a},n+m}(\bar{c}_{\mu_{n+m}}).$$

Lift \bar{c}_{μ_n} to $c_{\mu_n} \in A$ for all μ_n . Let $a_n := P_{\bar{a},n}(c_{\mu_n}) = \sum_{\mu_n} c_{\mu_n}^{p^n} \mu_n \in A_n[\Theta]$. For $m \geq n$, we have $a_n - a_m \in A_n[\Theta] \cap \mathfrak{m} \subset \mathfrak{m}^{p^n}$. Hence a_n converges to an element $a \in A$. Now we show that $a \in K$. For every μ_k , let $b_{\mu_k,n} \in A$ be the element getting by plugging $c_{\mu_{n+k}}$ into the $P_{\bar{a},n}$. Then $b_{\mu_k,n}$ converges to an element $b_{\mu_k} \in A$. By construction, we have

$$a = \lim_{n \rightarrow \infty} P_{\bar{a},n+k}(c_{\mu_{n+k}}) = \lim_{n \rightarrow \infty} P_{\bar{a},k}(b_{\mu_k,n}) = P_{\bar{a}}(b_{\mu_k}) = \sum_{\mu_k} b_{\mu_k}^{p^k} \mu_k \in A_k[\Theta], \quad \forall k.$$

It follows that $a \in K$. □

Lemma A.6.27. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian complete local ring of mixed characteristic. Suppose that $\mathfrak{m}^n = 0$ for some $n \geq 1$. Then there exists a complete local ring (R, pR, \mathbf{k}) with $R \subset A$.

Proof. Fix a p -basis of \mathbf{k} and lift it to $\Theta \subset R$. Let $q = p^{n-1}$ and

$$m := \left\{ \theta_1^{k_1} \cdots \theta_d^{k_d} \mid \theta_i \in \Theta, k_i \leq q-1 \right\}, \quad S := \left\{ \sum_{\mu \in m, \text{ finite}} a_\mu \mu \mid a_\mu \in R^q \right\}.$$

For any $a, b \in A$, we claim that $a \equiv b \pmod{\mathfrak{m}}$ if and only if $a^q \equiv b^q \pmod{\mathfrak{m}^n}$. If $a \equiv b \pmod{\mathfrak{m}}$, write $a = b + m$ for some $m \in \mathfrak{m}$. Then $a^p = b^p + pb^{q-1}m + \cdots + m^q$. Hence $a^p \equiv b^p \pmod{\mathfrak{m}^2}$. Inductively, we have $a^q \equiv b^q \pmod{\mathfrak{m}^n}$. Conversely, if $a^q \equiv b^q \pmod{\mathfrak{m}^n}$, then $a^q - b^q \in \mathfrak{m}^n \subset \mathfrak{m}$. Note that the Frobenius map $x \mapsto x^q$ is injective on A/\mathfrak{m} . It follows that $a \equiv b \pmod{\mathfrak{m}}$. By the claim, S maps to $\mathbf{k}^q[\Theta] = \mathbf{k}$ bijectively.

Let

$$R := S + pS + p^2S + \cdots + p^{n-1}S.$$

We claim that R is a subring of A . If so, $R/pR \cong \mathbf{k}$ and we get a complete local ring (R, pR, \mathbf{k}) .

Take $a, b \in A$. We have

$$a^q + b^q = (a + b)^q + pc \in A^q + pA.$$

Inductively, we have

$$a^q + b^q \in A^q + pA^q + \cdots + p^{n-1}A^q.$$

This implies that R is closed under addition. Note that $\theta^a = \theta^{aq} \cdot \theta^b$ with $b < q$. Then for any $\mu, \nu \in m$, we have $\mu\nu \in S$. Hence R is closed under multiplication. \square

Lemma A.6.28. Let \mathbf{k} be a field of characteristic p . Then there exists a DVR (D, pD, \mathbf{k}) of mixed characteristic $(0, p)$.

Proof. Fix a well order \leq on \mathbf{k} and for any $a \in \mathbf{k}$, set \mathbf{k}_a be the subfield of \mathbf{k} generated by all elements $b \in \mathbf{k}$ such that $b \leq a$. Then $\mathbf{k} = \bigcup_{a \in \mathbf{k}} \mathbf{k}_a$. We construct DVRs D_a with residue field \mathbf{k}_a such that $D_a \subset D_b$ for $a \leq b$. Begin from $\mathbf{k}_0 = \mathbb{F}_p$ and let $D_0 = \mathbb{Z}_{(p)}$. Suppose that D_a is constructed for all $a < b$. If $\mathbf{k}_b/\mathbf{k}_a$ is transcendental, then let D_b be the localization of $D_a[b]$ at the prime ideal generated by p .

If $\mathbf{k}_b/\mathbf{k}_a$ is algebraic, then let $\bar{f} \in \mathbf{k}_a[T]$ be the monic minimal polynomial of b . Let $\mathbf{K}_a = \text{Frac}(D_a)$ and $K_b = \mathbf{K}_a[T]/(f)$, where f is a monic lift of \bar{f} to $D_a[T]$. Note that f is irreducible since \bar{f} is irreducible. Let D_b be the integral closure of D_a in K_b . In general, D_b is a Dedekind domain. Consider the prime factorization $pD_b = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_k^{e_k}$ in D_b . For every i , D_b/\mathfrak{p}_i is a field extension of \mathbf{k}_a and \bar{f} has a root in D_b/\mathfrak{p}_i . Suppose $\deg \bar{f} = \deg f = d$. It follows that $[(D_b/\mathfrak{p}_i) : \mathbf{k}_a] = d$. Note that we have $\sum_{i=1}^k e_i f_i = [K_b : \mathbf{K}_a] = d$. Hence $k = 1$ and $e_1 = 1$. It follows that pD_b is prime and D_b is a DVR with residue field \mathbf{k}_b .

Let $D = \bigcup_{a \in \mathbf{k}} D_a$. Then (D, pD, \mathbf{k}) is the desired DVR. \square

Example A.6.29. Let $\mathbf{k} = \mathbb{F}_p(t)$. Then $D = \mathbb{Z}[t]_{(p)}$ is a DVR satisfying the condition in Lemma A.6.28.

Let $\mathbf{k} = \overline{\mathbb{F}_p}$. For any $n \geq 1$, let $K_n = K_{n-1}(\zeta_{p^{n-1}})$ and $K_0 = \mathbb{Q}$. Let $D_n := \mathcal{O}_{K_n, \mathfrak{p}_n}$ be the localization of the ring of integers of K_n at the prime \mathfrak{p}_n lying above \mathfrak{p}_{n-1} . Then $D := \bigcup_n D_n$ is a DVR with residue field \mathbf{k} .

Lemma A.6.30. Given \mathbf{k} a field of characteristic p , there exists a unique complete local ring (R, pR, \mathbf{k}) of mixed characteristic (p^n, p) .

Proof. The existence follows from Lemma A.6.28. To show the uniqueness, suppose that (R', pR', \mathbf{k}) is another complete local ring of mixed characteristic (p^n, p) . Fix a p -basis of \mathbf{k} and lift it to $\Theta \subset R$ and $\Theta' \subset R'$ relatively. Let $q = p^{n-1}$ and

$$m := \{\theta_1^{k_1} \cdots \theta_d^{k_d} \mid \theta_i \in \Theta, k_i \leq q-1\}, \quad S := \left\{ \sum_{\mu \in m, \text{ finite}} a_\mu \mu \mid a_\mu \in R^q \right\}.$$

Define m', S' similarly with Θ' and R' . Since $S \rightarrow R \rightarrow \mathbf{k}$ and $S' \rightarrow R' \rightarrow \mathbf{k}$ are bijections, we can define a bijective map $\Phi : S \rightarrow S'$.

Note that any element in S can be written as $s + pr$ with $s \in S$ and $r \in R$ uniquely since $S \rightarrow \mathbf{k}$ is bijective. Inductively, we can write any element in R as

$$r = s + ps_1 + p^2s_2 + \cdots + p^{n-1}s_{n-1},$$

where $s_i \in S$. The similarly for R' . Extend Φ to R and we get a bijection between R and R' . Note that by construction, Φ preserves addition and multiplication. Hence we get a ring isomorphism $\Phi : R \rightarrow R'$. \square

Proof of Theorem A.6.24 in mixed characteristic. Since A is complete, we have $A = \varprojlim_n A/\mathfrak{m}^n$. By Lemma A.6.27, there is a complete local ring (R_n, pR_n, \mathbf{k}) with $R_n \subset A/\mathfrak{m}^n$. By Lemma A.6.30, such R_n is unique up to isomorphism. It follows that $R_n \cong R_m/p^{k_n}$ for $m \geq n$. We get an inverse system

$$\cdots \rightarrow R_n \rightarrow R_{n-1} \rightarrow \cdots \rightarrow R_1 \cong \mathbf{k}.$$

Let $R := \varprojlim_n R_n$. Then (R, pR, \mathbf{k}) is a complete local ring. The homomorphisms $R_n \hookrightarrow A/\mathfrak{m}^n$ induce a homomorphism of complete local rings $R \hookrightarrow A$. This concludes the proof. \square

Appendix B

Homological Algebra

B.1 Complexes and Homology

Definition B.1.1. Let A_\bullet and B_\bullet be two complexes in \mathcal{A} and $\varphi_\bullet, \psi_\bullet : A_\bullet \rightarrow B_\bullet$ be two morphisms of complexes. A *homotopy* between φ_\bullet and ψ_\bullet is a collection of morphisms $h_n : A_n \rightarrow B_{n-1}$ such that

$$\varphi_n - \psi_n = d_{B_{n+1}} \circ h_n + h_{n-1} \circ d_{A_n}.$$

In diagram, we have

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \xrightarrow{d_{A_n}} & A_{n-1} \longrightarrow \cdots \\ & & \searrow h_n & & \downarrow \psi_n & & \swarrow \varphi_n \\ \cdots & \longrightarrow & B_{n+1} & \xrightarrow{d_{B_{n+1}}} & B_n & \longrightarrow & B_{n-1} \longrightarrow \cdots \\ & & \swarrow \psi_{n+1} & & \downarrow \varphi_n & & \swarrow h_{n-1} \end{array}$$

B.2 Derived Functors

In this section, fix an abelian category \mathcal{A} .

B.2.1 Resolution

Definition B.2.1 (Resolution). Let $A \in \mathcal{A}$. A *projective resolution* (resp. *flat resolution*, *free resolution*) of A is an exact sequence

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0,$$

where each P_i is a projective (resp. flat, free) object in \mathcal{A} .

An *injective resolution* of A is an exact sequence

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots \rightarrow I^n \rightarrow \cdots,$$

where each I^i is an injective object in \mathcal{A} .

Proposition B.2.2. Let $P_\bullet : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$ and $Q_\bullet : \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow B \rightarrow 0$ be complexes in \mathcal{A} such that P_i is projective and Q_\bullet is exact. Given a morphism $f : A \rightarrow B$, there exists a morphism of complexes $f_\bullet : P_\bullet \rightarrow Q_\bullet$ such that $f_0 = f$. In particular, any two such morphism of complexes are homotopic.

Dually, let $I^\bullet : 0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ and $J^\bullet : 0 \rightarrow B \rightarrow J^0 \rightarrow J^1 \rightarrow \cdots$ be complexes in \mathcal{A} such that J^i is injective and I^\bullet is exact. Given a morphism $f : A \rightarrow B$, there exists a morphism of complexes $f^\bullet : I^\bullet \rightarrow J^\bullet$ such that $f^0 = f$. In particular, any two such morphism of complexes are homotopic.

Proof. To be completed. □

Definition B.2.3. For an object $A \in \mathcal{A}$, the *projective dimension* of A , denoted $\text{proj. dim } A$, is the smallest integer n such that there exists a projective resolution

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

of A of length n . If no such n exists, we set $\text{proj. dim } A = \infty$.

Dually, the *injective dimension* of A , denoted $\text{inj. dim } A$, is the smallest integer n such that there exists an injective resolution

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^{n-1} \rightarrow I^n \rightarrow 0$$

of A of length n . If no such n exists, we set $\text{inj. dim } A = \infty$.

B.3 Applications to Commutative Algebra

B.3.1 Homological dimension

Lemma B.3.1. Let A be a ring and M an A -module. Then

$$\sup_M \text{proj. dim } M = \sup_N \text{inj. dim } N.$$

Proof. Note that

$$\text{proj. dim } M \leq n$$

if and only if

$$\text{Ext}_{n+1}^A(M, N) = 0, \quad \forall N.$$

And this is equivalent to

$$\text{inj. dim } N \leq n.$$

□

Remark B.3.2. In fact, for fix N , we have

$$\text{inj. dim } N \leq n$$

if and only if

$$\text{Ext}_{n+1}^A(A/I, N) = 0, \quad \forall I$$

By Lemma ?. Hence we have

$$\sup_{M \text{ finite}} \text{proj. dim } M = \sup_M \text{proj. dim } M = \sup_N \text{inj. dim } N.$$

Definition B.3.3. Let A be a ring. The *homological dimension* of A , denoted $\text{hl. dim } A$, is defined as

$$\text{hl. dim } A := \sup_M \text{proj. dim } M = \sup_M \text{inj. dim } M.$$

Lemma B.3.4. Let A be a noetherian ring, B a flat A -algebra and M a finite A -module. Then we have

$$\text{Ext}_A^i(M, N) \otimes B \cong \text{Ext}_B^i(M \otimes A, N \otimes A), \quad \forall N.$$

Proof. To be completed. □

Proposition B.3.5. Let A be a noetherian ring. Then

$$\text{hl. dim } A = \sup_{\mathfrak{p} \in \text{Spec } A} \text{hl. dim } A_{\mathfrak{p}}.$$

Proof. Compute homological dimension of A using $\text{Ext}_A^i(M, N)$ for finite M . The conclusion follows from Proposition B.3.4. □

Definition B.3.6. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring. We say that a homomorphism of A -modules $f : M \rightarrow N$ is *minimal* if the induced map $M \otimes \mathbf{k} \rightarrow N \otimes \mathbf{k}$ is an isomorphism. Equivalently, f is minimal if and only if f is surjective and $\text{Ker } f \subset \mathfrak{m}M$.

Definition B.3.7. Let A be a noetherian local ring and M a finite A -module. A *minimal projective resolution* of M is a projective resolution

$$\cdots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

such that each homomorphism $P_i \rightarrow \text{Ker } d_{i-1}$ is minimal.

Proposition B.3.8. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring and M a finite A -module. Then M has a minimal projective resolution. Moreover, any two minimal projective resolutions of M are isomorphic.

Proof. Suppose $M \otimes_A \mathbf{k} = \bigoplus \mathbf{k} \cdot \bar{x}_i$. Lift x_i to elements of M . Then we have a minimal homomorphism $d_0 : \bigoplus A \cdot x_i \rightarrow M$. Similarly choose minimal homomorphisms $d_k : A^{n_i} \rightarrow \text{Ker } d_{k-1}$ for $i = 1, 2, \dots$. This gives a minimal projective resolution.

Suppose we have two minimal homomorphism $f, g : A^n \rightarrow M$. After tensoring with \mathbf{k} , we have isomorphisms between $f \otimes \mathbf{k}$ and $g \otimes \mathbf{k}$. Lifting to A , we get an homomorphism $\varphi : f \rightarrow g$. Here homomorphism between f, g means a homomorphism $A^n \rightarrow A^n$ such that $f = g \circ \varphi$. The homomorphism φ is represented by a matrix T . We have $\det T \notin \mathfrak{m}$, whence φ is an isomorphism. \square

Proposition B.3.9. Let $L_\bullet \rightarrow M$ be a minimal projective resolution and P_\bullet be an arbitrary projective resolution of M . Then we have $P_\bullet \cong L_\bullet \oplus P'_\bullet$ for some exact complexes P'_\bullet .

Proof. By Proposition B.2.2, we have homomorphism

$$L_\bullet \xrightarrow{\varphi_\bullet} P_\bullet \xrightarrow{\psi_\bullet} L_\bullet.$$

between complexes. By Proposition B.2.2 again, $T_\bullet := \psi_\bullet \circ \varphi_\bullet$ is homotopic to the identity by h_\bullet . Suppose T_\bullet is represented by a matrix. Since L_\bullet is minimal, we have

$$(T - \text{id})(L_n) = (d_{n+1} \circ h_n + h_{n-1} \circ d_n)(L_n) \subset \mathfrak{m}L_n.$$

Then $\det T \notin \mathfrak{m}$ and hence T_\bullet is an isomorphism. It follows that ψ_\bullet is surjective, whence it splits P_\bullet into a direct sum $L \oplus P'_\bullet$ since L_\bullet is projective. By the Five Lemma, we see that P'_\bullet is exact. \square

Lemma B.3.10. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring and M a finite A -module. Then $\text{proj. dim } M \leq n$ if and only if $\text{Tor}_{n+1}^A(M, \mathbf{k}) = 0$.

Proof. The necessity is clear. For the sufficiency, we have a minimal projective resolution

$$\cdots \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0.$$

Let $C := \Im d_n$. Then we have

$$0 \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} C \rightarrow 0.$$

Hence $\text{Tor}_1^A(C, \mathbf{k}) \cong \text{Tor}_{n+1}^A(M, \mathbf{k}) = 0$. Let $K = \text{Ker } d_n$. Then we have the short exact sequence

$$0 \rightarrow K \rightarrow P_n \rightarrow C \rightarrow 0.$$

Since $\text{Tor}_1^A(C, \mathbf{k}) = 0$, there is an exact sequence

$$0 \rightarrow K \otimes_A \mathbf{k} \rightarrow P_n \otimes_A \mathbf{k} \rightarrow C \otimes_A \mathbf{k} \rightarrow 0.$$

Since $P_n \rightarrow C$ is minimal, we have $K \otimes_A \mathbf{k} = 0$. By the Nakayama's lemma, $K = 0$. This implies that $\text{proj. dim } C \leq 0$ and hence $\text{proj. dim } M \leq n$. \square

Proposition B.3.11. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring. Then $\text{hl. dim } A = \text{proj. dim } \mathbf{k}$ (finite or infinite).

Proof. The inequality $\text{hl. dim } A \geq \text{proj. dim } \mathbf{k}$ is by definition. Conversely, we can compute $\text{Tor}_{n+1}^A(M, \mathbf{k})$ using minimal projective resolution of \mathbf{k} for any finite A -module M . By Lemma B.3.10, we have $\text{proj. dim } M \leq n$ if and only if $\text{Tor}_{n+1}^A(M, \mathbf{k}) = 0$. This implies that $\text{proj. dim } M \leq n$ for all finite A -modules M if $\text{proj. dim } \mathbf{k} = n$. By Remark B.3.2, we have $\text{hl. dim } A \leq n$. \square

Proposition B.3.12. Let (A, \mathfrak{m}) be a noetherian local ring and M a finite A -module. Let $a \in \mathfrak{m}$ be an M -regular element. Then $\text{proj. dim } M/aM = \text{proj. dim } M + 1$. Here we set $\infty + 1 = \infty$.

Proof. We have an exact sequence

$$0 \rightarrow M \xrightarrow{*a} M \rightarrow M/aM \rightarrow 0.$$

Take the long exact sequence with respect to $\text{Tor}(-, \mathbf{k})$, we get

$$\cdots \rightarrow \text{Tor}_{i+1}^A(M, \mathbf{k}) \rightarrow \text{Tor}_{i+1}^A(M/aM, \mathbf{k}) \rightarrow \text{Tor}_i^A(M, \mathbf{k}) \xrightarrow{*a} \text{Tor}_i^A(M, \mathbf{k}) \rightarrow \cdots$$

Since the derived homomorphism of $*a$ is zero, we have $\text{Tor}_{i+1}^A(M/aM, \mathbf{k}) = 0$ if and only if $\text{Tor}_i^A(M, \mathbf{k}) = 0$. By Lemma B.3.10, we have $\text{proj. dim } M/aM = \text{proj. dim } M + 1$. \square

B.3.2 Depth and regularity by homological algebra

Proposition B.3.13. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring and M a finite A -module. Then

$$\text{depth } M := \inf\{i : \text{Ext}_A^i(\mathbf{k}, M) \neq 0\}.$$

Proof. Let $a \in \mathfrak{m}$ be M -regular and $N = M/aM$. Then we claim that

$$\inf\{i : \text{Ext}_A^i(\mathbf{k}, N) \neq 0\} = \inf\{i : \text{Ext}_A^i(\mathbf{k}, M) \neq 0\} - 1.$$

Indeed, we have an exact sequence

$$0 \rightarrow M \xrightarrow{a} M \rightarrow N \rightarrow 0.$$

It induces a long exact sequence

$$\cdots \rightarrow \text{Ext}_A^{i-1}(\mathbf{k}, M) \rightarrow \text{Ext}_A^{i-1}(\mathbf{k}, N) \rightarrow \text{Ext}_A^i(\mathbf{k}, M) \xrightarrow{\text{Ext}_A^i(\mathbf{k}, \text{Mult}_a)} \text{Ext}_A^i(\mathbf{k}, M) \rightarrow \cdots.$$

Note that $a \in \mathfrak{m}$, then $\text{Ext}_A^i(\mathbf{k}, \text{Mult}_a) = 0$. It follows that when $\text{Ext}_A^{i-1}(\mathbf{k}, M) = 0$, we have $\text{Ext}_A^{i-1}(\mathbf{k}, N) = 0$ iff $\text{Ext}_A^i(\mathbf{k}, M) = 0$, whence the claim.

Let $n = \inf\{i : \text{Ext}_A^i(\mathbf{k}, M) \neq 0\}$. Induct on n . Suppose first $n = 0$. Since \mathbf{k} is a simple A -module, there is an injective homomorphism $\mathbf{k} \rightarrow M$. Then $\mathfrak{m} \in \text{Ass } M$ and hence $\text{depth } M = 0$.

Suppose $n > 0$, let $a_1, \dots, a_m \in \mathfrak{m}$ be any M -regular sequence. Using the claim inductively on $M/(a_1, \dots, a_m)M$, we have $n \geq \text{depth}$. If M has no regular element, then $\mathfrak{m} \subset \bigcup_{\mathfrak{p} \in \text{Ass } M} \mathfrak{p}$. Then $\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass } M$. This show that we can find $x \neq 0 \in M$ such that $\mathfrak{p} = \text{Ann } x$. It gives a

homomorphism $\mathbf{k} = A/\mathfrak{m} \rightarrow M$. That is a contradiction and hence M has a regular element. Let a be M -regular and $N = M/aM$. Then $\text{depth } N = n - 1$ by the claim and induction hypothesis. Hence we have $\text{depth } M \geq n$. \square

Lemma B.3.14. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring. Suppose we have exact sequences

$$0 \rightarrow A^{n_r} \xrightarrow{d_r} A^{n_{r-1}} \xrightarrow{d_{r-1}} \dots \rightarrow A^{n_1} \xrightarrow{d_1} A^{n_0},$$

such that $A^{n_i} \rightarrow \text{Ker } d_{i-1}$ is minimal for all i . Then $\text{depth } A \geq r$.

Proof. Since d_r is injective and its image is contained in $\mathfrak{m}A^{n_{r-1}}$, we can choose $t \in \mathfrak{m}$ that is not a zero divisor. Denote the sequence by C_\bullet . Then we have a short exact sequence of complexes

$$0 \rightarrow C_\bullet \xrightarrow{*t} C_\bullet \rightarrow C_\bullet/tC_\bullet \rightarrow 0.$$

Consider the long exact sequence in homology

$$\dots \rightarrow H_i(C_\bullet) \xrightarrow{*t} H_i(C_\bullet) \rightarrow H_i(C_\bullet/tC_\bullet) \rightarrow H_{i-1}(C_\bullet) \xrightarrow{*t} H_{i-1}(C_\bullet) \rightarrow \dots.$$

Since C_\bullet is exact, we have $H_i(C_\bullet) = 0$ for all i . In particular, $H_i(C_\bullet/tC_\bullet) = 0$ for all $i \geq 2$. Inductively, we can choose a regular sequence of length r in \mathfrak{m} . \square

Lemma B.3.15. Let $(A, \mathfrak{m}, \mathbf{k})$ be a noetherian local ring and M a finite A -module. Suppose there is an injective homomorphism $\mathbf{k} \rightarrow M$. Then $\text{proj. dim } M \geq \dim_{\mathbf{k}} T_{A, \mathfrak{m}}$.

Proof. Let $x_1, \dots, x_n \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that their images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. Then we have a complex

$$K_\bullet := 0 \rightarrow \wedge^n A^{\oplus n} \xrightarrow{d_n} \wedge^{n-1} A^{\oplus n} \xrightarrow{d_{n-1}} \dots \rightarrow \wedge^1 A^{\oplus n} \xrightarrow{d_1} \wedge^0 A^{\oplus n} \xrightarrow{d_0} \mathbf{k} \rightarrow 0,$$

where

$$d_r : \wedge^r A^{\oplus n} \rightarrow \wedge^{r-1} A^{\oplus n}, \quad e_{i_1} \wedge \dots \wedge e_{i_r} \mapsto \sum_{k=1}^r (-1)^k x_{i_k} e_{i_1} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_r}.$$

Here $\widehat{e_{i_k}}$ means that we omit the k -th element. Let $P_\bullet \rightarrow M$ be the minimal projective resolution of M . Then we have a homomorphism of complexes

$$\varphi_\bullet : K_\bullet \rightarrow P_\bullet.$$

induced by the injective homomorphism $\mathbf{k} \rightarrow M$.

We claim that φ_i is injective and splits P_i into a direct sum $K_i \oplus F_i$ with F_i free for all $i \geq 0$. Since K_i and P_i are free, we just need to show that $\varphi_i \otimes_A \text{id}_{\mathbf{k}}$ is injective. Induct on i . For $i = 0$, note that $\mathbf{k} \rightarrow M \otimes_A \mathbf{k}$ is injective, by the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{k} \\ \varphi_0 \otimes_A \text{id}_{\mathbf{k}} \downarrow & & \downarrow \\ P_0 \otimes_A \mathbf{k} & \xrightarrow{\cong} & M \otimes_A \mathbf{k} \end{array},$$

the image of $\varphi_0 \otimes_A \text{id}_{\mathbf{k}}$ is not zero in $P_0 \otimes_A \mathbf{k}$.

For $i > 0$, since K_{i-1} and P_{i-1} are free, we have a natural isomorphism between

$$\mathfrak{m}K_{i-1} \otimes_A \mathbf{k} \rightarrow \mathfrak{m}P_{i-1} \otimes_A \mathbf{k}$$

and

$$K_{i-1} \otimes_A \mathfrak{m}/\mathfrak{m}^2 \rightarrow P_{i-1} \otimes_A \mathfrak{m}/\mathfrak{m}^2.$$

We have a commutative diagram

$$\begin{array}{ccc} K_i \otimes_A \mathbf{k} & \longrightarrow & \mathfrak{m}K_{i-1} \otimes_A \mathbf{k} \\ \downarrow & & \downarrow \\ P_i \otimes_A \mathbf{k} & \longrightarrow & \mathfrak{m}P_{i-1} \otimes_A \mathbf{k} \end{array} \quad (\text{B.1})$$

Since $P_{i-1}/K_{i-1} \cong F_{i-1}$ is free, the right vertical map in (B.1) is injective. By construction of K_\bullet , $K_i \otimes_A \mathbf{k} \rightarrow \mathfrak{m}K_{i-1} \otimes_A \mathbf{k}$ is injective. Hence the left vertical map in (B.1) is injective. This completes the proof of the claim.

By the claim, $P_i \neq 0$ for all $i \leq n$ and the conclusion follows. \square

Proposition B.3.16 (Auslander-Buchsbaum formula). Let A be a noetherian local ring and M a finite A -module. Suppose $\text{proj. dim } M < \infty$. Then $\text{proj. dim } M = \text{depth } A - \text{depth } M$.

Proof. We have a minimal projective resolution

$$0 \rightarrow A^{n_r} \rightarrow A^{n_{r-1}} \rightarrow \dots \rightarrow A^{n_1} \rightarrow A^{n_0} \rightarrow M \rightarrow 0.$$

By Lemma B.3.14, we have $\text{depth } A \geq \text{proj. dim } M$.

Induct on $\text{depth } M$. Suppose $\text{depth } M = 0$. Then by Proposition B.3.13, we have $\text{Hom}_A(\mathbf{k}, M) \neq 0$, whence there is an injective homomorphism $\mathbf{k} \rightarrow M$. By Lemma B.3.15, we have

$$\text{depth } A \geq \text{proj. dim } M \geq \dim_{\mathbf{k}} T_{A, \mathfrak{m}} \geq \text{depth } A.$$

If $\text{depth } M > 0$, choose a regular element $a \in \mathfrak{m}$ that is M -regular. Then by Proposition B.3.12, we have

$$\text{depth } M + \text{proj. dim } M = \text{depth}(M/aM) - 1 + \text{proj. dim}(M/aM) + 1 = \text{depth } A.$$

\square

Theorem B.3.17. Let (A, \mathfrak{m}) be a noetherian local ring. Then A is regular at \mathfrak{m} if and only if $\text{hl. dim } A < \infty$.

Proof. Suppose A is regular at \mathfrak{m} . Let x_1, \dots, x_n be a minimal generating set of \mathfrak{m} . Then x_1, \dots, x_n is an A -regular sequence since A is regular at \mathfrak{m} . By Proposition B.3.12, we have $\text{proj. dim } \mathbf{k} = \text{proj. dim } A/(x_1, \dots, x_n)A = n + \text{proj. dim } A = n$.

Conversely, suppose $\text{hl. dim } A < \infty$. Then by Proposition B.3.11, we have $\text{proj. dim } \mathbf{k} < \infty$. We have

$$\dim_{\mathbf{k}} T_{A, \mathfrak{m}} \leq \text{proj. dim } \mathbf{k} \leq \text{depth } A \leq \dim_{\mathbf{k}} T_{A, \mathfrak{m}}.$$

The first “ \leq ” follows from Lemma B.3.15. The second “ \leq ” follows from Proposition B.3.16. Hence we see that A is regular at \mathfrak{m} . \square

Corollary B.3.18. Let (A, \mathfrak{m}) be a noetherian local ring. Then A is regular if and only if it is regular at \mathfrak{m} .

Proof. The sufficiency is trivial. For the necessity, note that if A is regular, then $\text{hl. dim } A < \infty$ by Theorem B.3.17. For any $\mathfrak{p} \in \text{Spec } A$, we have a finite projective resolution

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A/\mathfrak{p} \rightarrow 0.$$

Tensoring with $A_{\mathfrak{p}}$, we have a finite projective resolution of $\kappa(\mathfrak{p})$. By Theorem B.3.17 again, we see that $A_{\mathfrak{p}}$ is regular at \mathfrak{p} . \square

Lemma B.3.19. Let A be a noetherian integral domain. Then A is a UFD if and only if every height 1 prime ideal of A is principal.

Proof. To be completed. \square

Lemma B.3.20. Let A be a noetherian integral domain and $(x) \subset A$ a non-zero prime ideal. Then A is a UFD if and only if $A[1/x]$ is a UFD.

Proof. To be completed. \square

Theorem B.3.21. Let A, \mathfrak{m} be a regular noetherian local ring. Then A is UFD.

Proof. To be completed. \square

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