Notes in Algebraic Geometry

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Contents

Chapter 1

Schemes and Varieties

1.1 Definition and First Properties of Schemes

1.1.1 Locally Ringed Space

Definition 1.1.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Example 1.1.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 1.1.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0, 1) \cup (2, 3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0, 1), (2, 3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0, 1))$ and $s_2 = 2 \in \mathcal{C}((2, 3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 1.1.4. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 1.1.5. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Example 1.1.6 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Example 1.1.7 (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i\in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i\in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^{\sharp}(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ_i' satisfying the same properties, then by gluing $\psi_i' \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

1.1.2 Schemes

Example 1.1.8 (Glue open subschemes)**.** The construction in Example 1.1.7 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Example 1.1.7. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

1.1.3 Integral, reduced and irreducible

1.1.4 Fiber product

1.1.5 Dimension

1.1.6 Noetherian and finite type

1.1.7 Separated and proper

1.2 Schemes as functors

1.2.1 The functor of points

Let X be a scheme over a base scheme S. The *functor of points* of X is the functor $h_X(-) : (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ defined by $T \mapsto h_X(T) = \mathrm{Hom}_S(T, X)$.

1.2.2 What is a scheme?

For a scheme X over S, we will often identify X with its functor of points h_X. In this way, we can think of a scheme as a functor from $(\mathbf{Sch}/S)^{\mathrm{op}}$ to $\mathbf{Set}$.

The underlying topological space of X can be recovered from the functor of points h_X as follows: The points of X correspond to the morphisms from the spectrum of a field to X.

The structure sheaf of X can also be recovered from the functor of points h_X.

1.3 Line Bundles and Divisors

1.3.1 Cartier Divisors

1.3.2 Line Bundles and Picard Group

Definition 1.3.1. Let X be a scheme. The *Picard group* of X is defined to be $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. The group operation is given by the tensor product of line bundles.

Definition 1.3.2. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X\times\{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X\times\{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\mathrm{alg}} \mathcal{L}'$. To be checked.

1.3.3 Weil Divisors and Reflexive Sheaves

1.4 Line bundles induce morphisms

1.4.1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 1.4.1. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. $\square$

Proposition 1.4.2. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. To be continued. $\square$

Example 1.4.3. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \dots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \dots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the d*-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 1.4.4. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_iS_j = \pi_1^*T_i \otimes \pi_2^*S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}_A^{(m+1)(n+1)-1}$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_iy_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 1.4.5. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. To be continued.

Example 1.4.6. Let

Example 1.4.7.

Definition 1.4.8. Let $\mathcal{L}$ be a line bundle on a scheme X. To be continued.

Definition 1.4.9. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. To be continued.

Definition 1.4.10. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. To be continued.

Theorem 1.4.11. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

To be continued.

Proposition 1.4.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

To be continued.

Proof. To be continued. □

Proposition 1.4.13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then $\mathcal{L}$ is very ample if and only if the following two conditions hold:

(a) (separate points) for any two distinct points $x, y \in X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $s(y) \neq 0$;

(b) (separate tangent vectors) for any point $x \in X$ and non-zero tangent vector $v \in T_x X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $v(s) \neq 0$.

To be continued.

1.4.2 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 1.4.14. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

Definition 1.4.15. A *linear system* on a scheme X is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on X and $V \subseteq \Gamma(X, \mathcal{L})$ is a subspace. The dimension of the linear system is $\dim V - 1$. A linear system is *base-point free* if V is base-point free. A linear system is *complete* if $V = \Gamma(X, \mathcal{L})$. To be continued.

Definition 1.4.16. Let $\mathcal{L}$ be a line bundle on a scheme X and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace. The *base locus* of V is the closed subset

$$\mathrm{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If $\mathrm{Bs}(V) = \varnothing$, we say that V is *base-point free*. To be continued.

1.4.3 Asymptotic behavior

Definition 1.4.17. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. To be continued.

Definition 1.4.18. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. To be continued.

Theorem 1.4.19. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \operatorname{Proj} R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. To be continued.

Definition 1.4.20. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large n. To be continued.

Example 1.4.21. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. To be continued.

Chapter 2

Surfaces

2.1 Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$. This section is mainly based on [**Har77**].

2.1.1 Preliminaries

Let S be a variety over $\Bbbk$ and $\mathcal{E}$ a vector bundle of rank $r+1$ on S.

Proposition 2.1.1. The S-varieties $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on S.

Theorem 2.1.2. Let $\pi : X = \mathbb{P}_S(\mathcal{E}) \to S$ be the projective bundle associated to a vector bundle $\mathcal{E}$ of rank $r+1$ on S. Then there is an exact sequence of vector bundles on $\mathbb{P}_S(\mathcal{E})$

$$0 \to \Omega_{\mathbb{P}_S(\mathcal{E})/S} \to \pi^*(\mathcal{E})(-1) \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \to 0.$$

In particular, $K_X \sim \pi^*(K_S + \det \mathcal{E}) - (r+1)\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. To be continued...

Theorem 2.1.3 (Tsen's Theorem, [**Stacks**])**.** Let C be a smooth curve over an algebraically closed field $\Bbbk$. Then $\mathbf{K} = \Bbbk(C)$ is a C_1 field, i.e., every degree d hypersurface in $\mathbb{P}^n_{\mathbf{K}}$ has a $\mathbf{K}$-rational point provided $d \leq n$.

Theorem 2.1.4 (Grauert's Theorem, [**Har77**])**.** Let $f : X \to S$ be a projective morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf on X which is flat over S. Suppose that S is integral and the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$ is constant on S for some $i \geq 0$. Then $\mathrm{R}^i f_* \mathcal{F}$ is locally free and the base change homomorphism

$$\varphi^i_s : \mathrm{R}^i f_* \mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s) \to H^i(X_s, \mathcal{F}_s)$$

is an isomorphism for all $s \in S$.

Theorem 2.1.5 (Miracle Flatness, [**Mat89**]). Let $f : X \to Y$ be a morphism of noetherian schemes. Assume that Y is regular and X is Cohen-Macaulay. If all fibers of f have the same dimension $d = \dim X - \dim Y$, then f is flat.

Proposition 2.1.6 (Geometric form of Nakayama's Lemma). Let X be a variety, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proposition 2.1.7. Let S be a noetherian scheme and $\mathcal{E}$ a vector bundle of rank $r+1$ on S. Denote by $\pi : \mathbb{P}_S(\mathcal{E}) \to S$ the projection. Let X be an S-scheme via a morphism $g : X \to S$. Then there is a bijection

$$\left\{ \begin{array}{l} S\text{-morphisms} \\ X \to \mathbb{P}_S(\mathcal{E}) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \mathcal{L} \in \operatorname{Pic}(X) \text{ and surjective} \\ \text{homomorphisms } g^*\mathcal{E} \to \mathcal{L} \end{array} \right\}.$$

Proof. We have a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ by the definition of $\mathbb{P}_S(\mathcal{E})$. If we have a morphism $f : X \to \mathbb{P}_S(\mathcal{E})$ over S, then we have a surjective homomorphism $f^*\pi^*\mathcal{E} \to f^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$.

Suppose we have a surjective homomorphism $g^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ where $\mathcal{L}$ is a line bundle on X. Take an affine cover $\{U_i\}$ of S such that $\mathcal{E}|_{U_i}$ is trivial. On U_i, choose a basis $e_0^{(i)}, \dots, e_r^{(i)}$ of $\mathcal{E}|_{U_i}$. Suppose $\mathbb{P}_S(\mathcal{E})$ is given by gluing $\mathbb{P}^r_{U_i}$ via φ_{ij} induced by the transition functions of $\mathcal{E}$.

The surjection $g^*\mathcal{E}|_{U_i} \twoheadrightarrow \mathcal{L}|_{X_{U_i}}$ gives a unique morphism $f_i : X_{U_i} \to \mathbb{P}^r_{U_i}$ by Theorem 1.4.1. On $X_{U_i \cap U_j}$, f_i and f_j agree since we have

$$\begin{tikzcd} X_{U_i\cap U_j} \arrow[r, "="] \arrow[d, "f_i"'] & X_{U_i\cap U_j} \arrow[d, "f_j"] \\ \mathbb{P}_{U_i\cap U_j}(\bigoplus \mathcal{O}_{U_i\cap U_j} e_k^{(i)}) \arrow[r, "\varphi_{ij}"] & \mathbb{P}_{U_i\cap U_j}(\bigoplus \mathcal{O}_{U_i\cap U_j} e_k^{(j)}) \end{tikzcd}$$

and the bottom arrow is identical to the identity map on $\mathbb{P}_S(\mathcal{E})_{U_i\cap U_j}$. Gluing f_i gives a morphism $f : X \to \mathbb{P}_S(\mathcal{E})$ over S. In particular, we have $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. □

Definition 2.1.8. An *extension* of a coherent sheaf $\mathcal{F}$ by a coherent sheaf $\mathcal{G}$ on a scheme X is an exact sequence of coherent sheaves

$$S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0).$$

Two extensions S and S' are *equivalent* if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{G}}"] & \mathcal{E} \arrow[r] \arrow[d, "\cong"] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{F}}"] & 0 \\ 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0. \end{tikzcd}$$

Proposition 2.1.9. Let X be a scheme and $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. Then there is a one-to-one correspondence between equivalence classes of extensions

$$S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0)$$

and elements of $\mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})$ given by

$$S \mapsto \delta(\mathrm{id}_{\mathcal{F}})$$

where $\delta : \mathrm{Hom}_X(\mathcal{F},\mathcal{F}) \to \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})$ is the connecting homomorphism.

Proof. Take an exact sequence

$$0 \to \mathcal{G} \to \mathcal{I} \xrightarrow{\varphi} \mathcal{C} \to 0$$

with $\mathcal{I}$ injective. Applying $\mathrm{Hom}_X(\mathcal{F}, -)$ gives a long exact sequence

$$0 \to \mathrm{Hom}_X(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_X(\mathcal{F},\mathcal{I}) \to \mathrm{Hom}_X(\mathcal{F},\mathcal{C}) \xrightarrow{\delta} \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}) \to 0.$$

For $a \in \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})$, choose a lifting $\alpha \in \mathrm{Hom}_X(\mathcal{F},\mathcal{C})$ of a. Let $\mathcal{E} := \mathrm{Ker}(\mathcal{I} \oplus \mathcal{F} \to \mathcal{C}, (i,f) \mapsto \varphi(i) - \alpha(f))$.

Let $\mathcal{E} \to \mathcal{F}$ be the projection to the second factor. It is surjective since φ is surjective. Consider the inclusion $\mathcal{G} \to \mathcal{I} \to \mathcal{I} \oplus \mathcal{F}$, which factors through $\mathcal{E}$. On the other hand, if $e \in \mathcal{E}$ maps to 0 in $\mathcal{F}$, then $e \in \mathcal{I}$ and $\varphi(e) = 0$, whence $e \in \mathcal{G}$. Hence we have an extension $S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0)$.

To be continued... □

2.1.2 Minimal Section and Classification

Definition 2.1.10 (Ruled surface)**.** A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all geometric fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 2.1.11. There exists a section of π.

Proof. To be continued... □

Proposition 2.1.12. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Proof. Let $\sigma : C \to X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π, $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in C$. By Miracle Flatness (Theorem 2.1.5), f is flat. By Grauert's Theorem (Theorem 2.1.4), $\mathcal{E}$ is a vector bundle of rank 2 on C and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in C$.

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every $x \in X$, we have

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \to \mathcal{L}$ is surjective.

By Proposition 2.1.7, we have a morphism $\varphi : X \to \mathbb{P}_C(\mathcal{E})$ over C such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Since $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, $\varphi|_{X_t} : X_t \to \mathbb{P}_C(\mathcal{E})_t$ is an isomorphism for any $t \in C$. Hence

φ is bijection on the underlying sets. Here is a serious gap. Why fiberwise isomorphism implies isomorphism? $\square$

Lemma 2.1.13. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*. In particular, if $\mathcal{E}$ is normalized, then $e = -\deg c_1(\mathcal{E})$ is an invariant of the ruled surface X.

Proof. We can suppose that $\mathcal{E}$ is globally generated since we can always twist $\mathcal{E}$ by a sufficiently ample line bundle on C. Then for all line bundle $\mathcal{L}$ of degree sufficiently large, $\mathcal{L}$ is very ample and hence $H^0(C, \mathcal{E} \otimes \mathcal{L}) \neq 0$. By Lemma 2.1.11 and Proposition 2.1.7, $\mathcal{E}$ is an extension of line bundles. Then for all line bundle $\mathcal{L}$ of degree sufficiently negative, $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ since line bundles of negative degree have no global sections. Hence we can find a line bundle $\mathcal{M}$ on C of lowest degree such that $H^0(C, \mathcal{E} \otimes \mathcal{M}) \neq 0$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{M}$, we are done. $\square$

Remark 2.1.14. The invariant e is unique but the normalization of $\mathcal{E}$ is not unique. For example, if $\mathcal{E}$ is normalized, then so is $\mathcal{E} \otimes \mathcal{L}$ for any line bundle $\mathcal{L}$ on C of degree 0. To be continued...

Suppose that $X \cong \mathbb{P}_C(\mathcal{E})$ where $\mathcal{E}$ is a normalized vector bundle of rank 2 on C. Since $H^0(C, \mathcal{E}) \neq 0$, choosing a non-zero section s, we get an exact sequence

$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{E} \to \mathcal{E}/\mathcal{O}_C \to 0.$$

We claim that $\mathcal{E}/\mathcal{O}_C$ is a line bundle on C. Since C is a curve, we only need to check that $\mathcal{E}/\mathcal{O}_C$ is torsion-free.

To be continued...

Definition 2.1.15. A section C_0 of π is called a *minimal section* if to be continued...

Lemma 2.1.16. Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface over a smooth curve C of genus g with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on C with $\deg \mathcal{L} = -e$.

(b) If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.

Proof. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable, we can assume that $H^0(C, \mathcal{L}_1) \neq 0$. If $\deg \mathcal{L}_1 > 0$, then $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$, contradicting the normalization of $\mathcal{E}$. Similarly $\deg \mathcal{L}_2 \leq 0$. Then $\mathcal{L}_1 \cong \mathcal{O}_C$. And hence $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$.

If $\mathcal{E}$ is indecomposable, we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

which is a non-trivial extension, with $\mathcal{L}$ a line bundle on C of degree $-e$. Hence by Proposition 2.1.9, we have $0 \neq \mathrm{Ext}^1_C(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$. By Serre duality, we have $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$. Hence $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$.

On the other hand, let $\mathcal{M}$ be a line bundle on C of degree -1. Twist the above exact sequence by $\mathcal{M}$ and take global sections, we have an equation

$$h^0(\mathcal{M}) - h^0(\mathcal{E} \otimes \mathcal{M}) + h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{M}) + h^1(\mathcal{E} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = 0.$$

Since $\deg \mathcal{M} < 0$ and $\mathcal{E}$ is normalized, we have $h^0(\mathcal{M}) = h^0(\mathcal{E} \otimes \mathcal{M}) = 0$. By Riemann-Roch, we have $h^1(\mathcal{M}) = g$ and $h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = -e - 1 + 1 - g$. Hence

$$h^1(\mathcal{E} \otimes \mathcal{M}) = e + 2g \geq 0.$$

This gives $e \geq -2g$. □

Theorem 2.1.17. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Proof. This is a direct consequence of Lemma 2.1.16. □

Example 2.1.18. Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$.

Let C be covered by two standard affine charts U_0, U_1 with coordinate u on U_0 and v on U_1 such that $u = 1/v$ on $U_0 \cap U_1$. On U_i, let $\mathcal{O}(-e)|_{U_i}$ be generated by s_i for $i = 0, 1$. We have $s_0 = u^e s_1$ on $U_0 \cap U_1$.

On $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$, let $[x_0 : x_1]$ and $[y_0 : y_1]$ be the homogeneous coordinates of $\mathbb{P}^1$ on X_0 and X_1 respectively. Then the transition function on $X_0 \cap X_1$ is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

Remark 2.1.19. The surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ is also called the *Hirzebruch surface*.

Theorem 2.1.20. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and for each e there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on E with $\deg \mathcal{L} = -e$.

Proof. Only the indecomposable case needs a proof. By Lemma 2.1.16, we have $-2 \leq e \leq 0$ and a non-trivial extension

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle on E of degree $-e$.

Case 1. $e = 0$.

In this case, $\mathcal{L}$ is of degree 0 and $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \neq 0$. Hence $\mathcal{L} \cong \mathcal{O}_E$. To be continued...

Case 2. $e = -1$.

In this case, $\mathcal{L}$ is of degree 1 and $H^1(E, \mathcal{L}) \cong H^0(E, \mathcal{L}^{-1}) = 0$. By Riemann-Roch, we have $h^0(E, \mathcal{L}) = 1$.

Case 3. $e = -2$.

To be continued...

□

Example 2.1.21. To be continued...

2.1.3 The Néron-Severi Group of Ruled Surfaces

Proposition 2.1.22. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and F a fiber of π. Then $\mathrm{Pic}(X) \cong \mathbb{Z}[C_0] \oplus \pi^* \mathrm{Pic}(C)$.

Proof. Let D be any divisor on X with $D.F = a \in \mathbb{Z}$. Then $D - aC_0$ is numerically trivial on the fibers of π. Let $\mathcal{L} = \mathcal{O}_X(D - aC_0)$. Then $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$ for any $t \in C$. By Grauert's Theorem (Theorem 2.1.4), $\pi_* \mathcal{L}$ is a line bundle on C and the natural map $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ is an isomorphism. □

Proposition 2.1.23. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let F be a fiber of π. Then $K_X \sim -2C_0 + \pi^*(K_C - c_1(\mathcal{E}))$. Numerically, we have $K_X \equiv -2C_0 + (2g - 2 - e)F$ where e is the invariant of X. Check this carefully.

Proof. To be continued. □

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

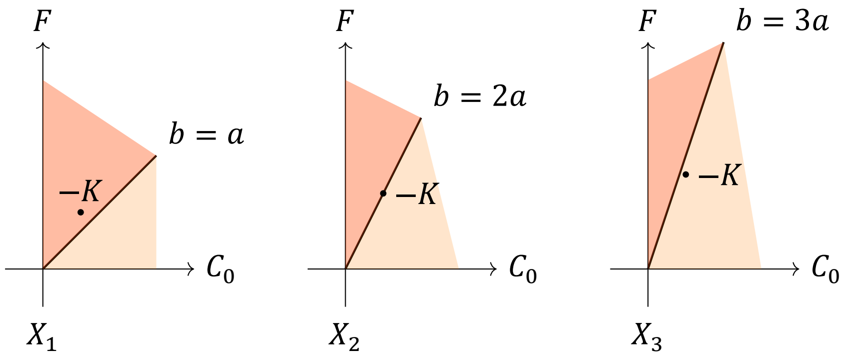
Theorem 2.1.24. Let $\pi : X \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with invariant e. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \sim aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a, b \geq 0$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$.

Proof. To be continued... □

Example 2.1.25. Here we draw the Néron-Severi group of the rational ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 1, 2, 3$.



We have $-K_{X_e} \equiv 2C_0 + (2 + e)F$. For $e = 1$, $-K$ is ample and hence X_1 is a del Pezzo surface. For $e = 2$, $-K$ is nef and big but not ample. For $e \geq 3$, $-K$ is big but not nef.

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

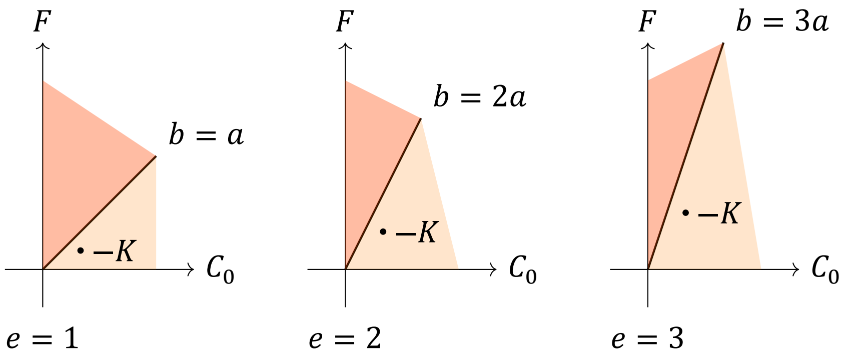
Theorem 2.1.26. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is decomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$.

Proof. To be continued... □

Example 2.1.27. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with decomposable normalized $\mathcal{E}$ for $e = 1, 2, 3$.



In this case, $-K \equiv 2C_0 + eF$ is always big but not nef.

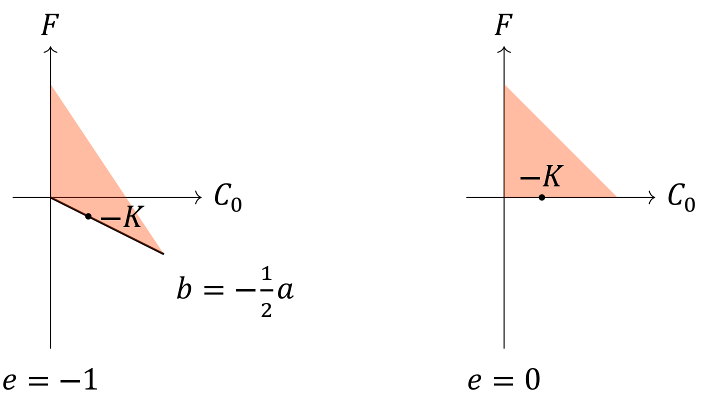
Theorem 2.1.28. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is indecomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq \frac{1}{2}ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > \frac{1}{2}ae$.

Proof. To be continued... □

Example 2.1.29. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with indecomposable normalized $\mathcal{E}$ for $e = -1, 0$.



In this case, $-K \equiv 2C_0 + eF$ is always nef but not big.

Proposition 2.1.30. Let $\pi : X \to C$ be a ruled surface over a smooth curve C. Then every nef divisor on X is semi-ample. Check this carefully.

2.2 Some Singular Surfaces

In this section, fix an algebraically closed field $\Bbbk$. Everything is over $\Bbbk$ unless otherwise specified.

2.2.1 Projective cone over smooth projective curve

Let $C \subset \mathbb{P}^n$ be a smooth projective curve. The *projective cone* over C is the projective variety $X \subset \mathbb{P}^{n+1}$ defined by the same homogeneous equations as C. The variety X is singular at the vertex of the cone, which corresponds to the point $[0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}$.

Chapter 3

Birational Geometry

3.1 Bend and Break

3.1.1 Preliminary

Definition 3.1.1 (Frobinius morphism)**.** Let X be a variety over a field $\mathbb{k}$ of characteristic $p > 0$. Denote the structure morphism by $\pi : X \to \operatorname{Spec}\mathbb{k}$. The *absolute Frobenius morphism* is the morphism given by $o_X \to o_X, f \mapsto f^p$, denoted by $\operatorname{Frob}_{X/\mathbb{f}_p}$. The *relative Frobenius morphism* is the morphism $\operatorname{Frob}_{X/\mathbb{k}}$ given by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, "\operatorname{Frob}_{X/\mathbb{k}}"] \arrow[drr, bend left, "\operatorname{Frob}_{X/\mathbb{f}_p}"] \arrow[ddr, bend right, "\pi"'] & & \\
& X \times_{\mathbb{k}} \operatorname{Spec}\mathbb{k} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& \operatorname{Spec}\mathbb{k} \arrow[r, "\operatorname{Frob}_{\mathbb{k}/\mathbb{f}_p}"] & \operatorname{Spec}\mathbb{k}
\end{tikzcd}.$$

We usually denote $X \times_{\mathbb{k}} \operatorname{Spec}\mathbb{k}$ appearing above by $X^{(p)}$.

Proposition 3.1.2. Let X be a variety of dimension d over a field $\mathbb{k}$ of characteristic $p > 0$. Then the relative Frobenius morphism $\operatorname{Frob}_{X/\mathbb{k}} : X \to X^{(p)}$ is a finite morphism of degree p^d over $\mathbb{k}$.

3.1.2 Deformation of curves

Theorem 3.1.3 (ref. [**Kol96**])**.** Let C be a smooth projective curve of genus g and X a smooth projective variety of dimension n. Let $f : C \to X$ be a non-constant morphism. Then every irreducible component of $\operatorname{Mor}(C, X)$ containing f has dimension at least

$$-K_Y \cdot f(C) + (1 - g)n.$$

Proposition 3.1.4. Let X be a projective variety and $f : C \to X$ a non-constant morphism from a pointed smooth projective curve $p_0 \in C$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t : C \to X$ for $t \in T$ such that $f_0 = f$ and $f_t(p_0) = x_0$ for some point $x_0 \in X$ and all t. Then there exist some rational curves $\Gamma_1, \dots, \Gamma_m \subset X$ such that

(a) $x_0 \in \bigcup_{i=1}^m \Gamma_i$;

(b) there is a morphism $g : C \to X$ such that $f(C) \equiv_{alg} g(C) + \sum_{i=1}^m a_i \Gamma_i$ with $a_i > 0$ for all i.

Proposition 3.1.5. Let X be a projective variety and $f : \mathbb{P}^1 \to X$ a non-constant morphism with $f(0) = x_0, f(\infty) = x_\infty$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t : \mathbb{P}^1 \to X$ for $t \in T$ such that $f_0 = f$ and $f_t(0) = x_0, f_t(\infty) = x_\infty$ for all t. Then there exists a curve $C \subset X$ such that $f(\mathbb{P}^1) \equiv_{alg} aC$ with $a > 1$.

3.1.3 Find rational curves

Theorem 3.1.6. Let X be a smooth Fano variety. Then for any $x \in X(\Bbbk)$, there is a rational curve C passing through x with

$$0 < -C \cdot K_X \leq \dim X + 1.$$

Proof. To be completed. □

Theorem 3.1.7. Let X be a smooth projective variety such that $K_X \cdot C < 0$ for some irreducible curve $C \subset X$. Let H be an ample divisor on X. Then there exists a rational curve Γ such that

$$-(K_X \cdot C) \cdot \frac{H \cdot \Gamma}{H \cdot C} \leq -K_X \cdot \Gamma \leq \dim X + 1.$$

Proof. To be completed. □

Theorem 3.1.8. Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Suppose that $K_{(X,B)}$ is f-ample. Then the exceptional locus of f is covered by rational curves Γ with

$$0 < -K_{(X,B)} \cdot \Gamma \leq 2 \dim X.$$

Theorem 3.1.9. Let X be a smooth projective variety of dimension n and $H, H_1, \cdots, H_{n-1}$ ample divisors on X. Suppose that $K_X \cdot H_1 \cdots H_{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H \cdot H_1 \cdots H_{n-1}}{K_X \cdot H_1 \cdots H_{n-1}}.$$

3.2 Kodaira Vanishing Theorem

3.2.1 Preliminary

Theorem 3.2.1 (Serre Duality)**.** Let X be a Cohen-Macaulay projective variety of dimension n over $\mathbf{k}$ and D a divisor on X. Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \dots, n.$$

Theorem 3.2.2 (Log Resolution of Singularities)**.** Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there is a smooth variety (or analytic space) Y and a projective morphism $f : Y \to X$ such that

(a) f is an isomorphism over $X - (\mathrm{Sing}(X) \cup \mathrm{Supp}\, Z)$,

(b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(c) $\mathrm{Exc}(f) \cup D$ is an snc divisor.

Theorem 3.2.3 (Lefschetz Hyperplane Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 3.2.4 (Hodge Decomposition)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Then for any k, there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 3.2.3 and Theorem 3.2.4, we have the following lemma.

Lemma 3.2.5. Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map $r_k : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \to H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \to H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n - 1$ and an injection for $p = n - 1$.

Theorem 3.2.6 (Leray spectral sequence)**.** Let $f : Y \to X$ be a morphism of varieties and $\mathcal{F}$ a coherent sheaf on Y. Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{F}) \implies H^{p+q}(Y, \mathcal{F}).$$

3.2.2 Kodaira Vanishing Theorem

Lemma 3.2.7. Let X be a smooth projective variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on X. Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \to X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D. It induces a homomorphism $\mathcal{L}^{-m} \to \mathcal{O}_X$. Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} \right) \bigg/ (\mathcal{L}^{-m} \to \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then $\mathcal{A}$ is a finite $\mathcal{O}_X$-algebra. Let $Y \coloneqq \operatorname{Spec}_X \mathcal{A}$. Then Y is a finite $\mathcal{O}_X$-scheme and the natural morphism $f : Y \to X$ is finite and surjective.

For every $x \in X$, let $\mathcal{L}$ locally generated by t near x. Then $\mathcal{O}_Y$ locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y. Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. □

Remark 3.2.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D_i' = f^*(D_i)$. Then $D' + \sum_{i=1}^k D_i'$ is snc on Y by considering the local regular system of parameters.

Lemma 3.2.9. Let $f : Y \to X$ be a finite surjective morphism of projective varieties and $\mathcal{L}$ a line bundle on X. Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits by the trace map $(1/n)\operatorname{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. □

Theorem 3.2.10 (Kodaira Vanishing Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and A an ample divisor on X. Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 3.2.7 and 3.2.9, after taking a multiple of A, we can assume that A is effective.

Then we have an exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \to H^{i-1}(X, \mathcal{O}_A) \to H^i(X, \mathcal{O}_X(-A)) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 3.2.5 and Serre duality (Theorem 3.2.1). □

3.2.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

Theorem 3.2.11 (Kawamata-Viehweg Vanishing Theorem I)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{R}$-divisor on X. Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Theorem 3.2.12 (Kawamata-Viehweg Vanishing Theorem II)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{Q}$-divisor on X. Suppose that $\lceil D \rceil - D$ has snc support. Then

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall i > 0.$$

Theorem 3.2.13 (Kawamata-Viehweg Vanishing Theorem III)**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Let D be a nef $\mathbb{Q}$-divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

If we replace the assumption "nef and big" of D by "ample" in II and III, we denote them as "II(ample)" and "III(ample)" respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \implies \text{II(ample)} \implies \text{III(ample)} \implies \text{I} \implies \text{II} \implies \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

Proof of II (Theorem 3.2.12). Set $M := \lceil D \rceil$. Let

$$B := \sum_{i=1}^{k} b_i B_i := \lceil D \rceil - D = M - A, \quad b_i \in (0,1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k. When $k = 0$, the conclusion follows from Theorem 3.2.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 3.2.10.)) Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 3.2.15 and 3.2.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 3.2.7 on B_k, we can find a finite surjective morphism $f : X' \to X$ such that $f^*B_k = cB'_k, B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^k B'_i$ is an snc divisor on X'. Let $B' = \sum_{i=1}^{k-1} B'_i, A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$ vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^*\mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 3.2.9. □

Proof of III (Theorem 3.2.13). Let $f : \tilde{X} \to X$ be a resolution such that $\operatorname{Supp} f^*B \cup \operatorname{Exc} f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X},\tilde{B})} + f^*D,$$

where $\tilde{B} \in (0,1)$ has snc support and E is an effective exceptional divisor.

By Lemma 3.2.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X},\tilde{B})} + f^*D) = H^i(X, f_*\mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 3.2.12 in either case relative to the assumption of D. □

Proof of I (Theorem 3.2.11). By Lemma 3.2.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor A. Then the theorem comes down to Theorem 3.2.13. □

Lemma 3.2.14. Let $f : Y \to X$ be a birational morphism of projective varieties with Y smooth and X has only rational singularities. Let E be an effective exceptional divisor on Y and D a divisor on X. Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

Proof. I am unable to proof this lemma. □

Lemma 3.2.15. Let X be a projective variety, $\mathcal{L}$ a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \to X$ and a line bundle $\mathcal{L}'$ on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X, then we can take f such that f^*D is an snc divisor on Y.

Proof. We can assume that $\mathcal{L}$ is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "g"] \\ X \arrow[r, "\varphi_{\mathcal{L}}"] & \mathbb{P}^N \end{tikzcd},$$

where $g : [x_0 : \ldots : x_N] \mapsto [x_0^m : \ldots : x_N^m]$. The morphism f is finite and surjective since so is g. Let $\mathcal{L}' := \psi^*\mathcal{L}$.

For smoothness, we can compose g with a general automorphism of $\mathbb{P}^N$. Then the conclusion follows from [**Har77**]. □

Lemma 3.2.16 (ref. [**KM98**]). Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Lemma 3.2.17. Let X be a projective variety of dimension n and D a nef and big divisor on X. Then there exists an effective divisor B such that for every k, there is an ample divisor A_k such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k}B.$$

Proof. By definition of big divisor, there exists an ample divisor A_1 and effective divisor B such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k}B.$$

Since A is ample and D is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. □

3.3 Cone Theorem

3.3.1 Preliminary

Theorem 3.3.1 (Iitaka fibration, semiample case, ref. [**Laz04a**])**.** Let X be a projective variety and $\mathcal{L}$ an semiample line bundle on X. Then there exists a fibration $\varphi : X \to Y$ of projective varieties such that for any $m \gg 0$ with $\mathcal{L}^m$ base point free, we have that the morphism $\varphi_{\mathcal{L}^m}$ induced by $\mathcal{L}^m$ is isomorphic to φ. Such a fibration is called the *Iitaka fibration* associated to $\mathcal{L}$.

Theorem 3.3.2 (Rigidity Lemma, ref. [**Deb01**])**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Theorem 3.3.3. Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

Proposition 3.3.4. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

Schetch of proof. Take a resolution $f : Y \to X$, then f^*H is nef on Y and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor H_Y on Y closed enough to f^*H such that $K_Y \cdot H_Y^{n-1} < 0$. By [**MM86**] and take limit for H_Y. □

Lemma 3.3.5 (ref. [**Kaw91**])**.** Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Let E be an irreducible component of dimension d of the exceptional locus of f and $\nu : E^\nu \to X$ the normalization of E. Suppose that $f(E)$ is a point. Then for any ample divisor

H on X, we have

$$K_{E^\nu} \cdot \nu^* H^{d-1} \leq K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

3.3.2 Non-vanishing Theorem

Theorem 3.3.6 (Non-vanishing Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

Proof. To be completed. □

3.3.3 Base Point Free Theorem

Theorem 3.3.7 (Base Point Free Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, mD is base point free.

Proof. To be completed. □

Remark 3.3.8. In general, we say that a Cartier divisor D is *semiample* if there exists a positive integer m such that mD is base point free. The statement in Base Point Free Theorem (Theorem 3.3.7) is strictly stronger than the semiample condition. For example, let $\mathcal{L}$ be a torsion line bundle, then $\mathcal{L}$ is semiample, but there exists no positive integer M such that $m\mathcal{L}$ is base point free for all $m > M$.

3.3.4 Rationality Theorem

Lemma 3.3.9 (ref. [**KM98**])**.** Let X be a proper variety of dimension n and $D_1, \ldots, D_m$ Cartier divisors on X. Then the Euler characteristic $\chi(n_1 D_1, \ldots, n_m D_m)$ is a polynomial in $(n_1, \cdots, n_m)$ of degree at most n.

Theorem 3.3.10 (Rationality Theorem)**.** Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let

$$t \coloneqq \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H. Then $t = v/u \in \mathbb{Q}$ and

$$0 \leq v \leq a(X) \cdot (\dim X + 1).$$

Proof. For every $r \in \mathbb{R}_{>0}$, let

$$v(r) \coloneqq \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We need to show that $v(t) \leq a(\dim X + 1)$. For every $(p, q) \in \mathbb{Z}^2_{>0}$, set $D(p, q) \coloneqq paK_{(X,B)} + qH$. If $(p, q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, then we have $D(p, q)$ is not nef and $D(p, q) - K_{(X,B)}$ is ample.

Step 1. We show that a polynomial $P(x,y) \neq 0 \in \mathbb{Q}[x,y]$ of degree at most n is not identically zero on the set

$$\{(p,q) \in \mathbb{Z}^2 : p,q > M, 0 < atp - q < t\varepsilon\}, \quad \forall M > 0,$$

if $v(t)\varepsilon > a(n+1)$.

If $v(t) = \infty$, for any n, we show that we can find infinitely many lines L such that $\#L \cap \Lambda \geq n+1$. If so, Λ is Zariski dense in $\mathbb{Q}^2$. Since $1/at \in \mathbb{R} \setminus \mathbb{Q}$, there exist $p_0, q_0 > M$ such that

$$0 < \frac{p_0}{q_0} - \frac{1}{at} < \frac{\varepsilon}{(n+1)a} \cdot \frac{1}{q_0}, \text{ i.e. } 0 < atp_0 - q_0 < \frac{\varepsilon t}{n+1}.$$

Then $(ip_0, iq_0) \in \Lambda \cap \{p_0 y = q_0 x\}$ for $i = 1, \cdots, n+1$. Since M is arbitrary, there are infinitely many such lines L.

Suppose $v(t) = v < \infty$ and $t = v/u$. Then the inequality is equivalent to $0 < aup - vq < \varepsilon v$. Note that $\gcd(au, v)|a$, then $aup - vq = ai$ has integer solutions for $i = 1, \cdots, n+1$. Since $v(t)\varepsilon > a(n+1)$, there are at least $n+1$ lines which intersect Λ in infinitely many points. This enforce any polynomial which vanishes on Λ has degree at least $n+1$.

Step 2. There exists an index set $\Lambda \subset \mathbb{Z}^2$ such that Λ contains all sufficiently large (p,q) with $0 \leq atp - q \leq t$ and

$$Z := \mathrm{Bs}\,|D(p,q)| = \mathrm{Bs}\,|D(p',q')| \neq \varnothing, \quad \forall (p,q),(p',q') \in \Lambda.$$

For every $(p,q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, there exists $M > 0$ such that

$$D(\alpha,\beta) = \alpha a K_{(X,B)} + \beta H$$

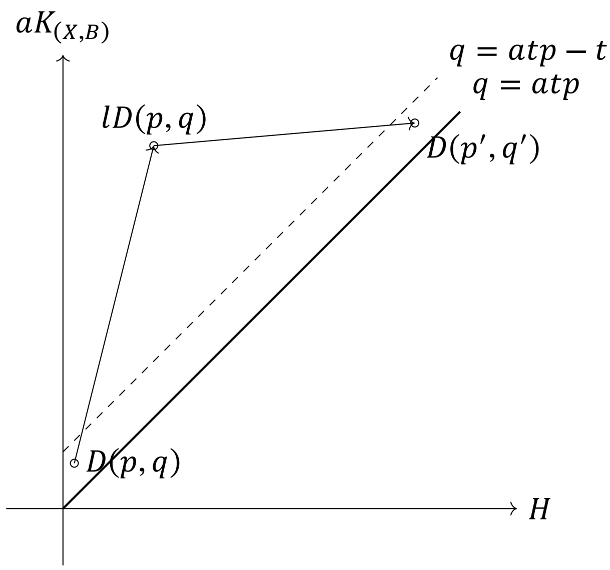
is base point free for all $\alpha = 0, \cdots, p$ and $\beta > M$. Choose M' large enough such that for all $(p',q') \in \mathbb{Z}^2_{>0}$ with $p', q' > M'$ and $0 < atp' - q' < t$, write

$$p' = lp + p_0, \quad q' = lq + q_0$$

for some $l \in \mathbb{Z}_{\geq 0}$ and $0 \leq p_0 < p$, we have $q_0 > M$. The existence of such M' follows from the estimate

$$q_0 = q' - lq = q' - \frac{p' - p_0}{p} q > q' - (p' - p_0)(at - \delta) > p'\delta,$$

where $\delta > 0$ is a small enough number such that $at - \delta > q/p$.



Then $D(p',q') - lD(p,q) = D(p_0,q_0)$ is base point free. It follows that $\operatorname{Bs}|D(p',q')| \subseteq \operatorname{Bs}|D(p,q)|$. By noetherian induction, there exists an index set Λ such that $\operatorname{Bs}|D(p,q)| = Z$ for all $(p,q) \in \Lambda$.

Step 3. Suppose the contradiction that $v(t) > a(\dim X+1)$. Then we show that $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem (Theorem 3.3.7).

Let $P(x,y) := \chi(D(x,y))$ be the Hilbert polynomial of $D(x,y)$. Note that $P(0,n) = \chi(nH) \neq 0$ since H is ample. Then $P(x,y) \neq 0$ and $\deg P \leq \dim X$. By Step 1, P is not identically zero on Λ. Note that $D(p,q) - K_{(X,B)}$ is ample for all $(p,q) \in \Lambda$, then $h^i(X, D(p,q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem (Theorem 3.2.13). Then

$$P(p,q) = \chi(D(p,q)) = h^0(X, D(p,q)) \neq 0$$

for some $(p,q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$.

Step 4. We follow the same line of the proof of Base Point Free Theorem (Theorem 3.3.7) to show that there is a section which does not vanish on Z.

Fix $(p,q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n+1)/u$. Let $f : Y \to X$ be a resolution such that

(a) $K_{Y,B_Y} = f^*K_{(X,B)} + E_Y$ for some effective exceptional divisor E_Y, and Y, B_Y is a klt pair;

(b) $f^*|D(p,q)| = |L| + F$ for some effective divisor F and a base point free divisor L, and $f(\operatorname{Supp} F) = Z$;

(c) $f^*D(p,q) - f^*K_{(X,B)} - E_0$ is ample for some effective $\mathbb{Q}$-divisor $E_0 \in (0,1)$, and coefficients of E_0 are sufficiently small;

(d) $B_Y + E_Y + F + E_0$ has snc support.

Such resolution exists by [**KM98**].

Let $c := \inf\{\lfloor B_Y + E_0 + tF \rfloor \neq 0\}$. Adjust the coefficients of E_0 slightly such that $\lfloor B_Y + E_0 + cF \rfloor = F_0$ for unique prime divisor F_0 with $F_0 \subset \operatorname{Supp} F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then (Y, Δ_Y) is a klt pair.

Let

$$\begin{aligned} N(p',q') &:= f^*D(p',q') + E_Y - F_0 - K_{(Y,\Delta_Y)} \\ &= \Big(f^*D(p',q') - (1+c)f^*D(p,q)\Big) + \Big(f^*D(p,q) - f^*K_{(X,B)} - E_0\Big) + c\Big(f^*D(p,q) - F\Big). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p',q') \in \Lambda : 0 < atp' - q' < atp - q,\ p', q' > (1+c)\max\{p,q\}\},$$

the divisor $f^*D(p',q') - (1+c)f^*D(p,q) = f^*D(p' - (1+c)p, q' - (1+c)q)$ is ample, and hence $N(p',q')$ is ample.

By the exact sequence

$$0 \to o_Y(f^*D(p',q') + E_Y - F_0) \to o_Y(f^*D(p',q') + E_Y) \to o_{F_0}((f^*D(p',q') + E_Y)|_{F_0}) \to 0$$

and Kawamata-Viehweg Vanishing Theorem (Theorem 3.2.13), we get a surjective map

$$H^0(Y, f^*D(p',q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p',q') + E_Y)|_{F_0}).$$

On F_0, consider the polynomial $\chi((f^*D(p',q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n-1$ and by the construction of $(p,q), \Lambda_0$, similar to Step 3, we can show that $\chi((f^*D(p',q') + E_Y)|_{F_0})$ is not identically zero on Λ_0. By adjunction, we have $(f^*D(p',q') + E_Y)|_{F_0} = N(p',q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p',q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (Theorem 3.2.13) to get

$$h^0(F_0, (f^*D(p',q') + E_Y)|_{F_0}) = \chi(F_0, (D(p',q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \mathrm{Bs}\,|D(p',q')|$. □

3.3.5 Cone Theorem and Contraction Theorem

Theorem 3.3.11 (Cone Theorem)**.** Let (X,B) be a projective klt pair. Then there exist countably many curves $C_i \subset X$ such that

(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. Let $F_D := \mathrm{Psef}_1(X) \cap D^\perp$ for a nef divisor D on X. If $\dim F_D = 1$, we also write $R_D := F_D$. Let $H_1, \cdots, H_{\rho-1}$ be ample divisors on X such that they together with $K_{(X,B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} := S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

Step 1. There exists an integer N such that for every $K_{(X,B)}$-negative extremal face F_D and for every ample divisor H, there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \neq F_{nD + rK_{(X,B)} + NH} \subset F_D$.

Let $N := (a(X)(\dim X + 1))!$, where $a(X)$ is the number in Theorem 3.3.10. For every n, $nD + H$ is an ample divisor and by Theorem 3.3.10, the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \geq 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since $K_{(X,B)} + (N/r_n)((n+1)D + H)$ is nef, we have $r_n \leq r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$ implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \to r \in \mathbb{Z}_{\geq 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors N_1, N_2, we have $F_{N_1+N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists m large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

Step 2. Let $\Phi : N_1(X)_{K_{(X,B)}<0} \to \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left(\frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha}\right).$$

We show that the image of R_D under Φ lies in a $\mathbb{Z}$-lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class ξ. By Step 1, we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers n, r. Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of R_D under Φ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

Step 3. We show that every $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ is of the form R_D for some nef divisor D on X.

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$-negative exposed ray. Then R is of form $D^{\perp} \cap \mathrm{Psef}_1(X)$ for some nef $\mathbb{R}$-divisor D on X. We need to show that D can be choose as a nef $\mathbb{Q}$-divisor. There is a sequence of nef but not ample $\mathbb{Q}$-divisors D_m such that $D_m \to D$ as $m \to \infty$. We adjust D_m such that $\dim F_{D_m} = 1$ for all n.

By re-choosing H_i, we can assume that $D = a_1H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is D_m. Then F_{D_m} is $K_{(X,B)}$-negative. Note that $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by Step 1. If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{D_m}$. We can assume that$r_1K_{(X,B)}+NH_1$ is not identically zero on F_{D_m}. Then we can choose n large enough such that $\|r_1K_{(X,B)}+NH_1\|/n < 1/m$. Replace D_m by $D_m + (r_1K_{(X,B)} + NH_1)/n$. Inductively we construct D_m nef $\mathbb{Q}$-divisor with $D_m \to D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that ξ_m converges. Then $\xi_m \to \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, Φ is well-defined at ξ and the image of ξ_m under Φ is discrete. Hence $\xi = \xi_m$ for all m large enough. It follows that $R = R_{D_m}$ for a nef $\mathbb{Q}$-divisor D_m.

By Step 2, the $K_{(X,B)}$-negative extremal rays form a discrete set in $\{\alpha \in \mathrm{Psef}_1(X) : K_{(X,B)} \cdot \alpha < 0\}$. Hence every $K_{(X,B)}$-negative extremal ray is an exposed ray by Straszewicz's Theorem.

Step 4. Proof of the theorem.

Given an ample divisor H on X, note that εH has positive minimum δ on $\mathrm{Psef}_1(X) \cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \mathrm{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and Φ is well-defined on it. By Steps 2 and 3, there are only finitely many extremal rays on $\mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H \leq 0}$. Hence we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$c := \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\} \subset c$ such that $\alpha_n \to \alpha \in N_1(X)_{\mathbb{R}}$. Note that $\mathrm{Psef}_1(X)$ is closed, hence $\alpha \in \mathrm{Psef}_1(X)$. We only need to consider the case $\alpha \cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$. Then $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$ for all n large enough. Note that $c \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$ is a polyhedral cone by Step 2 and hence is closed. Then $\alpha \in c$ and the conclusion follows. □

Remark 3.3.12. Thanks for my friend Qin for pointing out that the extremal ray may not be exposed.

Theorem 3.3.13 (Contraction Theorem)**.** Let (X, B) be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;

(b) up to linearly equivalence, any Cartier divisor G with $F \subset G^{\perp} = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$ comes from a Cartier divisor on Y, i.e., there exists a Cartier divisor G_Y on Y such that $G \sim \varphi_F^* G_Y$.

Proof. We follow the following steps to prove the theorem.

Step 1. We show that there exists a nef divisor D on X such that $F = D^{\perp} \cap \mathrm{Psef}_1(X)$. In other words, F is defined on $N_1(X)_{\mathbb{Q}}$.

We can choose an ample divisor H and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on F since $F \cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in $N_1(X)_{\mathbb{R}}$. Then by Cone Theorem (Theorem 3.3.11), F is an extremal face of a rational polyhedral cone, namely $\mathrm{Psef}_1(X)_{K_{(X,B)}+(1/n)H\leq 0}$. It follows that $F^{\perp} \subset N^1(X)_{\mathbb{R}}$ is defined on $\mathbb{Q}$. Since F is extremal and $K_{(X,B)} + (1/n)H$-negative, the set $\{L \in F^{\perp} : L|_{\mathrm{Psef}_1(X)\setminus F} > 0\}$ has non-empty interior in $F^{\perp}$ by Theorems 3.3.3 and 3.3.11. Then there exists a Cartier divisor D such that $D \in F^{\perp}$ and $D|_{\mathrm{Psef}_1(X)\setminus F} > 0$. It follows that D is nef and $F = D^{\perp} \cap \mathrm{Psef}_1(X)$.

Step 2. Let $\varphi : X \to Y$ be the Iitaka fibration associated to D by Theorem 3.3.1. We show that φ is the desired fibration.

Note that $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$ is compact and D is strictly positive on it. Then there exist $a \geq 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since F is $K_{(X,B)}$-negative. Then by Base Point Free Theorem (Theorem 3.3.7), we know that mD is base point free for all $m \gg 0$. Hence we can apply Theorem 3.3.1 to get a fibration $\varphi_D : X \to Y$.

First we show that D comes from Y. Note that mD and $(m+1)D$ induces the same fibration φ_D for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m)D_{Y,m}$ and $D_{Y,m}$ is ample. Hence D_Y is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that C is contracted by φ_D if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let G be arbitrary Cartier divisor on X such that $F \subset G^{\perp}$. Since D is strictly positive on $\mathrm{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' := mD + G$, we have $D'^{\perp} \cap \mathrm{Psef}_1(X) = F$. Then by the same argument as above, we get an other fibration $\varphi_{D'} : X \to Y'$ such that a curve C is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 3.3.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on Y. In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor D'_Y on Y. Then $G = D' - mD$ also comes from Y. □

Remark 3.3.14. The Step 1 is amazing. If F is not $K_{(X,B)}$-negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E. By [**Laz04a**], we know that $\mathrm{Psef}_1(X)$ is a circular cone. The we see there indeed exist some irrational extremal faces of $\mathrm{Psef}_1(X)$.

Theorem 3.3.15 (Length of extremal rays)**.** Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Then there exists a rational curve $C \subset X$ such that $[C] \in R$ and

$$0 < -K_{(X,B)} \cdot C \le 2 \dim X.$$

Proof. By Theorem 3.3.13, let $\varphi_D : X \to Y$ be the contraction associated to R_D (note that we do not need the step to proof Theorem 3.3.13). If $\dim Y < \dim X$, let F be a general fiber of φ_D. By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that H is ample on F. By Proposition 3.3.4. In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By Lemma 3.3.5, we can use Proposition 3.3.4 to get the result. To be completed. □

Definition 3.3.16. Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ with contraction $\varphi_R : X \to Y$. There are three types of contractions:

(a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension one;

(b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension at least two;

(c) *Mori fiber space*: if $\dim X > \dim Y$.

Proposition 3.3.17. Let (X, B) be a $\mathbb{Q}$-factorial projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Suppose that the contraction $\varphi : X \to Y$ associated to R is either divisorial or a Mori fiber space. Then Y is $\mathbb{Q}$-factorial.

Proof. Let D be a prime Weil divisor on Y and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\ge 0}[C]$ for an irreducible curve C contracted by φ. Set $D_X \coloneqq \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$. Then D_X is a prime Weil divisor on X and hence is $\mathbb{Q}$-Cartier.

If φ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber F of φ. Then by Contraction Theorem (Theorem 3.3.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor D' on Y. We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence D is $\mathbb{Q}$-Cartier.

If φ is a divisorial contraction, let E be the exceptional divisor of φ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier $\mathbb{Q}$-divisor E_Y on Y). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 3.3.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor D' on Y. Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence D is $\mathbb{Q}$-Cartier. □

Remark 3.3.18. If φ is a small contraction, then Y is never $\mathbb{Q}$-factorial. Otherwise, let B_Y be the strict transform of B on Y. Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset U. Suppose $K_{(Y,B_Y)}$

is $\mathbb{Q}$-Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

Example 3.3.19. Let $X = E \times E \times \mathbb{P}^1$. To be completed.

3.4 F-singularities

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let X be a projective variety over $\Bbbk$. Let F denote the relative Frobenius morphism on X.

Definition 3.4.1. We say that X is *F-finite* if $F : X \to X^{(p)}$ is finite.

Definition 3.4.2. We say that X is *globally F-split* if $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ splits as $\mathcal{O}_X$-modules for some $e \geq 0$. This is equivalent to for every $e \in \mathbb{Z}_{>0}$, $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ splits as $\mathcal{O}_X$-modules.

Definition 3.4.3. Fix $\phi : F^e_* L \to \mathcal{O}_X$ a splitting of $\mathcal{O}_X \to F^e_* \mathcal{O}_X$. Define $\phi^n : F^{ne}_* L^{1+p^e+\cdots+p^{(n-1)e}} \to \mathcal{O}_X$ by induction:

$$\phi^n := \phi \circ F^e_*(\phi^{n-1}).$$

Theorem 3.4.4. Above ϕ^n will be stable. That is, $\Im \phi^n = \Im \phi^{n+1}$ for all $n \gg 0$.

Definition 3.4.5. Let $\sigma(X, \phi) := \Im \phi^n$. We say that (X, ϕ) is *F-pure* if $\sigma(X, \phi) = \mathcal{O}_X$.

Proposition 3.4.6. There is a bijection between

$$\{\text{effective } \mathbb{Q}\text{-divisor } \Delta \text{ such that } (p^e - 1)(K_X + \Delta) \text{ is Cartier}\}/\sim$$

and

$$\{\text{line bundles } \mathcal{L} \text{ and } \phi : F^e_* \mathcal{L} \to \mathcal{O}_X\}.$$

Proof. We have

$$F^e_X \mathcal{O}_X((1 - p^e)K_X) \to \mathcal{O}_X$$

given by $F^e \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X)$ and reflexivity of $\mathcal{O}_X(K_X)$. Since Δ is effective, we have

$$F^e(\mathcal{O}_X((1 - p^e)(K_X + \Delta))) \to F^e \mathcal{O}_X((1 - p^e)(K_X)) \to \mathcal{O}_X.$$

The another direction is by Grothendieck's duality

$$\mathcal{H}om_{\mathcal{O}_X}(F^e \mathcal{L}, \mathcal{O}_X) \cong F^e_*(\mathcal{L}^{-1} \otimes \mathcal{O}_X((1 - p^e)K_X)).$$

□

Definition 3.4.7. Let $\phi_{e,\Delta} : F^e_*(o_X((1-p^e)(K_X+\Delta))) \to o_X$ be the morphism corresponding to the effective $\mathbb{q}$-divisor Δ.
We say that (X, Δ) is *F-pure* if $(X, \phi_{e,\Delta})$ is F-pure.
We say that (X, Δ) is *globally F-split* if for every Weil divisor $D \geq 0$, $o_X \to F^e_*(o_X(\lceil (p^e-1)\Delta \rceil + D))$ admits a splitting for some $e \geq 0$.
We say that (X, Δ) is *strongly F-split* if for every Weil divisor $D \geq 0$, $o_X \to F^e_*(o_X(\lceil (p^e-1)\Delta \rceil + D))$ admits a local splitting for some $e \geq 0$.

Definition 3.4.8.

Definition 3.4.9. $S^0(X, \sigma(X, \Delta) \otimes m)$

Proposition 3.4.10. Let X be a globally F-split projective variety. Then we have

(a) suppose that $H^i(X, l^n) = 0$ for all $i > 0$ and all $n \gg 0$, then $H^i(X, l) = 0$ for all $i > 0$;

(b) for every ample divisor A on X, we have $H^i(X, o_X(A)) = 0$ for all $i > 0$;

(c) suppose that X is Cohen-Macaulay and A-ample, then $H^i(X, o_X(-A)) = 0$ for all $i < \dim X$;

(d) suppose that X is normal and A-ample, then $H^i(X, \omega_X(A)) = 0$ for all $i > 0$.