# Birational Geometry

“要知道你为什么出枪，你的心里有闷烧的火，那是大地上燃烧的煤矿，它的火焰终有一天烧破地面去点燃天空。你会吼叫，因为你若是不吐出那火焰，它会烧穿你的胸膛，它像是愤怒，又像是高亢的歌，龙虎的吼声让时间停止。”

# Contents



Yang: This note is full of errors. Do not believe anything it says.

# 1 Kodaira Vanishing Theorem

## 1.1 Preliminary

**Theorem 1.1** (Serre Duality)**.** Let $X$ be a Cohen-Macaulay projective variety of dimension $n$ over $\mathbf{k}$ and $D$ a divisor on $X$. Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \dots, n.$$

**Theorem 1.2** (Log Resolution of Singularities)**.** Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) $Z$. Then there is a

**Date**: August 1, 2025, **Author**: Tianle Yang, My Homepage smooth variety (or analytic space) $Y$ and a projective morphism $f : Y \to X$ such that

(a) $f$ is an isomorphism over $X - (\mathrm{Sing}(X) \cup \mathrm{Supp}\, Z)$,

(b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(c) $\mathrm{Exc}(f) \cup D$ is an snc divisor.

**Theorem 1.3** (Lefschetz Hyperplane Theorem)**.** Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$ and $Y$ a hyperplane section of $X$. Then the restriction map

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

**Theorem 1.4** (Hodge Decomposition)**.** Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$. Then for any $k$, there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 1.3 and Theorem 1.4, we have the following lemma.

**Lemma 1.5.** Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$ and $Y$ a hyperplane section of $X$. Then the restriction map $r_k : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \to H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \to H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n - 1$ and an injection for $p = n - 1$.

**Theorem 1.6** (Leray spectral sequence)**.** Let $f : Y \to X$ be a morphism of varieties and $\mathcal{F}$ a coherent sheaf on $Y$. Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{F}) \implies H^{p+q}(Y, \mathcal{F}).$$

## 1.2 Kodaira Vanishing Theorem

**Lemma 1.7.** Let $X$ be a smooth projective variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on $X$. Suppose there is an integer $m$ and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \to X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

*Proof.* Let $s \in \mathcal{L}^m$ be the section defining $D$. It induces a homomorphism $\mathcal{L}^{-m} \to \mathcal{O}_X$. Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}\right) / (\mathcal{L}^{-m} \to \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then $\mathcal{A}$ is a finite $\mathcal{O}_X$-algebra. Let $Y \coloneqq \operatorname{Spec}_X \mathcal{A}$. Then $Y$ is a finite $\mathcal{O}_X$-scheme and the natural morphism $f : Y \to X$ is finite and surjective.

For every $x \in X$, let $\mathcal{L}$ locally generated by $t$ near $x$. Then $\mathcal{O}_Y$ locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let $D'$ be the divisor locally given by $t = 0$ on $Y$. Since $X$ and $D$ are smooth, then $Y$ is a smooth variety and $D'$ is smooth. Since $f$ is finite, it is proper. Then $Y$ is proper and hence $Y$ is projective. □

**Remark 1.8.** Let $D_i$ be reduced effective divisors on $X$ such that $D + \sum_{i=1}^k D_i$ is snc. Set $D'_i = f^*(D_i)$. Then $D' + \sum_{i=1}^k D'_i$ is snc on $Y$ by considering the local regular system of parameters.

**Lemma 1.9.** Let $f : Y \to X$ be a finite surjective morphism of projective varieties and $\mathcal{L}$ a line bundle on $X$. Suppose that $X$ is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

*Proof.* Since $f$ is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since $X$ are normal, the inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits by the trace map $(1/n)\operatorname{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. □

**Theorem 1.10** (Kodaira Vanishing Theorem)**.** Let $X$ be a smooth projective variety of dimension $n$ over $\mathbf{k}$ of characteristic $0$ and $A$ an ample divisor on $X$. Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

*Proof.* By Lemma 1.7 and 1.9, after taking a multiple of $A$, we can assume that $A$ is effective. Then we have an exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X,\mathcal{O}_X)\to H^{i-1}(X,\mathcal{O}_A)\to H^i(X,\mathcal{O}_X(-A))\to H^i(X,\mathcal{O}_X)\to H^i(X,\mathcal{O}_A).$$

Then the conclusion follows from Lemma 1.5 and Serre duality (Theorem 1.1). □

## 1.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

**Theorem 1.11** (Kawamata-Viehweg Vanishing Theorem I)**.** Let $X$ be a smooth projective variety of dimension $n$ over $\mathbf{k}$ of characteristic $0$ and $D$ a nef and big $\mathbb{R}$-divisor on $X$. Then

$$H^i(X,K_X+D)=0,\quad \forall i>0.$$

**Theorem 1.12** (Kawamata-Viehweg Vanishing Theorem II)**.** Let $X$ be a smooth projective variety of dimension $n$ over $\mathbf{k}$ of characteristic $0$ and $D$ a nef and big $\mathbb{Q}$-divisor on $X$. Suppose that $\lceil D\rceil-D$ has snc support. Then

$$H^i(X,K_X+\lceil D\rceil)=0,\quad \forall i>0.$$

**Theorem 1.13** (Kawamata-Viehweg Vanishing Theorem III)**.** Let $(X,B)$ be a klt pair over $\mathbf{k}$ of characteristic $0$. Let $D$ be a nef $\mathbb{Q}$-divisor on $X$ such that $D+K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X,K_{(X,B)}+D)=0,\quad \forall i>0.$$

If we replace the assumption "nef and big" of $D$ by "ample" in II and III, we denote them as "II(ample)" and "III(ample)" respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \implies \text{II(ample)} \implies \text{III(ample)} \implies \text{I} \implies \text{II} \implies \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

*Proof of II (Theorem 1.12).* Set $M:=\lceil D\rceil$. Let

$$B:=\sum_{i=1}^k b_iB_i:=\lceil D\rceil-D=M-A,\quad b_i\in(0,1)\cap\mathbb{Q}.$$

We do not require that $B_i$ are irreducible but we require that $B_i$ are smooth.

We induct on $k$. When $k=0$, the conclusion follows from Theorem 1.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 1.10.)) Let $b_k=a/c$ with lowest terms. Then $a<c$. By Lemma 1.15 and 1.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 1.7 on $B_k$, we can find a finite surjective morphism $f:X'\to X$ such that $f^*B_k=cB'_k, B'_i=f^*B_i$ for $i<k$ and $\sum_{i=1}^k B'_i$ is an snc divisor on $X'$. Let $B'=\sum_{i=1}^{k-1}B'_i, A'=f^*A$ and $M'=f^*M$. Then $A'+B'=M'-aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X',-A'-B')$

vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^*\mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 1.9. □

*Proof of III (Theorem 1.13).* Let $f : \tilde{X} \to X$ be a resolution such that $\operatorname{Supp} f^*B \cup \operatorname{Exc} f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X},\tilde{B})} + f^*D,$$

where $\tilde{B} \in (0,1)$ has snc support and $E$ is an effective exceptional divisor.

By Lemma 1.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X},\tilde{B})} + f^*D) = H^i(X, f_*\mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 1.12 in either case relative to the assumption of $D$. □

*Proof of I (Theorem 1.11).* By Lemma 1.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor $A$. Then the theorem comes down to Theorem 1.13. □

**Lemma 1.14.** Let $f : Y \to X$ be a birational morphism of projective varieties with $Y$ smooth and $X$ has only rational singularities. Let $E$ be an effective exceptional divisor on $Y$ and $D$ a divisor on $X$. Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

*Proof.* Yang: I am unable to proof this lemma. □

**Lemma 1.15.** Let $X$ be a projective variety, $\mathcal{L}$ a line bundle on $X$ and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \to X$ and a line bundle $\mathcal{L}'$ on $Y$ such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If $X$ is smooth, then we can take $Y$ to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on $X$, then we can take $f$ such that $f^*D$ is an snc divisor on $Y$.

*Proof.* We can assume that $\mathcal{L}$ is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "g"] \\ X \arrow[r, "\varphi_{\mathcal{L}}"] & \mathbb{P}^N \end{tikzcd},$$

where $g : [x_0 : \ldots : x_N] \mapsto [x_0^m : \ldots : x_N^m]$. The morphism $f$ is finite and surjective since so is $g$. Let $\mathcal{L}' := \psi^*\mathcal{L}$.

For smoothness, we can compose $g$ with a general automorphism of $\mathbb{P}^N$. Then the conclusion follows from [Har77, Chapter III, Theorem 10.8]. □

**Lemma 1.16** (ref. [KM98, Theorem 5.10, 5.22])**.** Let $(X, B)$ be a klt pair over $\mathbf{k}$ of characteristic $0$. Then $X$ has rational singularities and is Cohen-Macaulay.

**Lemma 1.17.** Let $X$ be a projective variety of dimension $n$ and $D$ a nef and big divisor on $X$. Then there exists an effective divisor $B$ such that for every $k$, there is an ample divisor $A_k$ such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k} B.$$

*Proof.* By Yang: definition of big divisor, there exists an ample divisor $A_1$ and effective divisor $B$ such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k} B.$$

Since $A$ is ample and $D$ is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. □

# 2 Cone Theorem

## 2.1 Preliminary

**Theorem 2.1** (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27])**.** Let $X$ be a projective variety and $\ell$ an semiample line bundle on $X$. Then there exists a fibration $\varphi : X \to Y$ of projective varieties such that for any $m \gg 0$ with $\ell^m$ base point free, we have that the morphism $\varphi_{\ell^m}$ induced by $\ell^m$ is isomorphic to $\varphi$. Such a fibration is called the *Iitaka fibration* associated to $\ell$.

**Theorem 2.2** (Rigidity Lemma, ref. [Deb01, Lemma 1.15])**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that $\pi_1$ is a fibration and $\pi_2$ contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and $\varphi$ is well-defined near $Y_1 \setminus \{y_0\}$.

**Theorem 2.3.** Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

**Proposition 2.4.** Let $X$ be a normal projective variety of dimension $n$ and $H$ an ample divisor on $X$. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve $\Gamma$ passing through $x$ such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

*Schetch of proof.* Take a resolution $f : Y \to X$, then $f^*H$ is nef on $Y$ and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor $H_Y$ on $Y$ closed enough to $f^*H$ such that $K_Y \cdot H_Y^{n-1} < 0$. By [MM86, Theorem 5] and take limit for $H_Y$. □

**Lemma 2.5** (ref. [Kaw91, Lemma]). Let $(X, B)$ be a projective klt pair and $f : X \to Y$ a birational projective morphism. Let $E$ be an irreducible component of dimension $d$ of the exceptional locus of $f$ and $\nu : E^\nu \to X$ the normalization of $E$. Suppose that $f(E)$ is a point. Then for any ample divisor $H$ on $X$, we have

$$K_{E^\nu} \cdot \nu^* H^{d-1} \leq K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

## 2.2 Non-vanishing Theorem

**Theorem 2.6** (Non-vanishing Theorem). Let $(X, B)$ be a projective klt pair and $D$ a Cartier divisor on $X$. Suppose that $D$ is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

## 2.3 Base Point Free Theorem

**Theorem 2.7** (Base Point Free Theorem). Let $(X, B)$ be a projective klt pair and $D$ a Cartier divisor on $X$. Suppose that $D$ is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, $mD$ is base point free.

**Remark 2.8.** In general, we say that a Cartier divisor $D$ is *semiample* if there exists a positive integer $m$ such that $mD$ is base point free. The statement in Base Point Free Theorem (Theorem 2.7) is strictly stronger than the semiample condition. For example, let $\ell$ be a torsion line bundle, then $\ell$ is semiample but there exists no positive integer $M$ such that $m\ell$ is base point free for all $m > M$.

## 2.4 Rationality Theorem

**Lemma 2.9** (ref. [KM98, Theorem 1.36]). Let $X$ be a proper variety of dimension $n$ and $D_1, \dots, D_m$ Cartier divisors on $X$. Then the Euler characteristic $\chi(n_1 D_1, \dots, n_m D_m)$ is a polynomial in $(n_1, \cdots, n_m)$ of degree at most $n$.

**Theorem 2.10** (Rationality Theorem). Let $(X, B)$ be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and $H$ an ample divisor on $X$. Let

$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of $(X, B)$ with respect to $H$. Then $t = v/u \in \mathbb{Q}$ and

$$0 \leq v \leq a(X) \cdot (\dim X + 1).$$

*Proof.* For every $r \in \mathbb{R}_{>0}$, let

$$v(r) := \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We need to show that $v(t) \le a(\dim X + 1)$. For every $(p,q) \in \mathbb{Z}^2_{>0}$, set $D(p,q) := paK_{(X,B)} + qH$. If $(p,q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, then we have $D(p,q)$ is not nef and $D(p,q) - K_{(X,B)}$ is ample.

**Step 1.** We show that a polynomial $P(x,y) \ne 0 \in \mathbb{Q}[x,y]$ of degree at most $n$ is not identically zero on the set

$$\{(p,q) \in \mathbb{Z}^2 : p,q > M, 0 < atp - q < t\varepsilon\}, \quad \forall M > 0,$$

if $v(t)\varepsilon > a(n+1)$.

If $v(t) = \infty$, for any $n$, we show that we can find infinitely many lines $L$ such that $\#L \cap \Lambda \ge n+1$. If so, $\Lambda$ is Zariski dense in $\mathbb{Q}^2$. Since $1/at \in \mathbb{R} \setminus \mathbb{Q}$, there exist $p_0, q_0 > M$ such that

$$0 < \frac{p_0}{q_0} - \frac{1}{at} < \frac{\varepsilon}{(n+1)a} \cdot \frac{1}{q_0}, \text{ i.e. } 0 < atp_0 - q_0 < \frac{\varepsilon t}{n+1}.$$

Then $(ip_0, iq_0) \in \Lambda \cap \{p_0 y = q_0 x\}$ for $i = 1, \cdots, n+1$. Since $M$ is arbitrary, there are infinitely many such lines $L$.

Suppose $v(t) = v < \infty$ and $t = v/u$. Then the inequality is equivalent to $0 < aup - vq < \varepsilon v$. Note that $\gcd(au, v) | a$, then $aup - vq = ai$ has integer solutions for $i = 1, \cdots, n+1$. Since $v(t)\varepsilon > a(n+1)$, there are at least $n+1$ lines which intersect $\Lambda$ in infinitely many points. This enforce any polynomial which vanishes on $\Lambda$ has degree at least $n+1$.

**Step 2.** There exists an index set $\Lambda \subset \mathbb{Z}^2$ such that $\Lambda$ contains all sufficiently large $(p,q)$ with $0 \le atp - q \le t$ and

$$Z := \operatorname{Bs}|D(p,q)| = \operatorname{Bs}|D(p',q')| \ne \emptyset, \quad \forall (p,q), (p',q') \in \Lambda.$$

For every $(p,q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, choose $k \in \mathbb{Z}_{>0}$ such that $k(atp - q) > t$. Then for all $p', q' > kp$ with $0 < atp' - q' < t$, we have

$$p' - kp \ge 0, \quad q' - kp > t(p' - kp).$$

It follows that

Yang: To be completed.

**Step 3.** Suppose the contradiction that $v(t) > a(\dim X + 1)$. Then we show that $H^0(X, D(p,q)) \ne 0$ for all $(p,q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem (Theorem 2.7).

Let $P(x,y) := \chi(D(x,y))$ be the Hilbert polynomial of $D(x,y)$. Note that $P(0,n) = \chi(nH) \ne 0$ since $H$ is ample. Then $P(x,y) \ne 0$ and $\deg P \le \dim X$. By Step 1, $P$ is not identically zero on $\Lambda$. Note that $D(p,q) - K_{(X,B)}$ is ample for all $(p,q) \in \Lambda$, then $h^i(X, D(p,q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem (Theorem 1.13). Then

$$P(p,q) = \chi(D(p,q)) = h^0(X, D(p,q)) \ne 0$$

for some $(p,q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$.

**Step 4.** We follow the same line of the proof of Base Point Free Theorem (Theorem 2.7) to show that there is a section which does not vanish on $Z$.

Fix $(p,q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n+1)/u$. Let $f : Y \to X$ be a resolution such that

(a) $K_{Y,B_Y} = f^*K_{(X,B)} + E_Y$ for some effective exceptional divisor $E_Y$, and $Y, B_Y$ is a klt pair;

(b) $f^*|D(p,q)| = |L| + F$ for some effective divisor $F$ and a base point free divisor $L$, and $f(\operatorname{Supp} F) = Z$;

(c) $f^*D(p,q) - f^*K_{(X,B)} - E_0$ is ample for some effective $\mathbb{Q}$-divisor $E_0 \in (0,1)$, and coefficients of $E_0$ are sufficiently small;

(d) $B_Y + E_Y + F + E_0$ has snc support.

Yang: Such resolution exists by [KM98].

Let $c := \inf\{\lfloor B_Y + E_0 + tF \rfloor \neq 0\}$. Adjust the coefficients of $E_0$ slightly such that $\lfloor B_Y + E_0 + cF \rfloor = F_0$ for unique prime divisor $F_0$ with $F_0 \subset \operatorname{Supp} F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then $(Y, \Delta_Y)$ is a klt pair.

Let

$$\begin{aligned} N(p',q') &:= f^*D(p',q') + E_Y - F_0 - K_{(Y,\Delta_Y)} \\ &= \Big(f^*D(p',q') - (1+c)f^*D(p,q)\Big) + \Big(f^*D(p,q) - f^*K_{(X,B)} - E_0\Big) + c\Big(f^*D(p,q) - F\Big). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p',q') \in \Lambda : 0 < atp' - q' < atp - q,\ p', q' > (1+c)\max\{p,q\}\},$$

the divisor $f^*D(p',q') - (1+c)f^*D(p,q) = f^*D(p' - (1+c)p, q' - (1+c)q)$ is ample, and hence $N(p',q')$ is ample.

By the exact sequence

$$0 \to \mathcal{O}_Y(f^*D(p',q') + E_Y - F_0) \to \mathcal{O}_Y(f^*D(p',q') + E_Y) \to \mathcal{O}_{F_0}((f^*D(p',q') + E_Y)|_{F_0}) \to 0$$

and Kawamata-Viehweg Vanishing Theorem (Theorem 1.13), we get a surjective map

$$H^0(Y, f^*D(p',q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p',q') + E_Y)|_{F_0}).$$

On $F_0$, consider the polynomial $\chi((f^*D(p',q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n - 1$ and by the construction of $(p,q), \Lambda_0$, similar to Step 3, we can show that $\chi((f^*D(p',q') + E_Y)|_{F_0})$ is not identically zero on $\Lambda_0$. By adjunction, we have $(f^*D(p',q') + E_Y)|_{F_0} = N(p',q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p',q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (Theorem 1.13) to get

$$h^0(F_0, (f^*D(p',q') + E_Y)|_{F_0}) = \chi(F_0, (D(p',q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \operatorname{Bs}|D(p',q')|$. $\square$

## 2.5 Cone Theorem and Contraction Theorem

**Theorem 2.11** (Cone Theorem)**.** Let $(X,B)$ be a projective klt pair. Then there exist countably many rational curves $C_i \subset X$ with

$$0 < -K_{(X,B)} \cdot C_i \le 2\dim X$$

such that

(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)}\ge 0} + \sum \mathbb{R}_{\ge 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor $H$ on $X$, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H\ge 0} + \sum_{\text{finite}} \mathbb{R}_{\ge 0}[C_i].$$

*Proof.* Let $F_D \coloneqq \mathrm{Psef}_1(X) \cap D^{\perp}$ for a nef divisor $D$ on $X$. If $\dim F_D = 1$, we also write $R_D \coloneqq F_D$. Let $H_1, \cdots, H_{\rho-1}$ be ample divisors on $X$ such that they together with $K_{(X,B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} \coloneqq S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

**Step 1.** There exists an integer $N$ such that for every $K_{(X,B)}$-negative extremal face $F_D$ and for every ample divisor $H$, there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \ne F_{nD+rK_{(X,B)}+NH} \subset F_D$.

Let $N \coloneqq (a(X)(\dim X + 1))!$, where $a(X)$ is the number in Theorem 2.10. For every $n$, $nD + H$ is an ample divisor and by Theorem 2.10, the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \ge 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\ge 0}.$$

Since $K_{(X,B)} + (N/r_n)((n+1)D + H)$ is nef, we have $r_n \le r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \ge 0$ implies that

$$r_n \le -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \to r \in \mathbb{Z}_{\ge 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors $N_1, N_2$, we have $F_{N_1+N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists $m$ large enough such that

$$\{0\} \ne F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

**Step 2.** Let $\Phi : N_1(X)_{K_{(X,B)}<0} \to \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left( \frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha} \right).$$

We show that the image of $R_D$ under $\Phi$ lies in a $\mathbb{Z}$-lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class $\xi$. By Step 1, we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers $n, r$. Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of $R_D$ under $\Phi$ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

**Step 3.** We show that every $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ is of the form $R_D$ for some nef divisor $D$ on $X$.

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$-negative extremal ray. Yang: Then $R$ is of form $D^{\perp} \cap \mathrm{Psef}_1(X)$ for some nef $\mathbb{R}$-divisor $D$ on $X$ by Theorem 2.3. We need to show that $D$ can be choose as a nef $\mathbb{Q}$-divisor. There is a sequence of nef but not ample $\mathbb{Q}$-divisors $D_m$ such that $D_m \to D$ as $m \to \infty$. We adjust $D_m$ such that $\dim F_{D_m} = 1$ for all $n$.

By re-choosing $H_i$, we can assume that $D = a_1H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is $D_m$. Then $F_{D_m}$ is $K_{(X,B)}$-negative. Note that $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by Step 1. If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{D_m}$. We can assume that$r_1K_{(X,B)}+NH_1$ is not identically zero on $F_{D_m}$. Then we can choose $n$ large enough such that $\|r_1K_{(X,B)} + NH_1\|/n < 1/m$. Replace $D_m$ by $D_m + (r_1K_{(X,B)} + NH_1)/n$. Inductively we construct $D_m$ nef $\mathbb{Q}$-divisor with $D_m \to D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that $\xi_m$ converges. Then $\xi_m \to \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, $\Phi$ is well-defined at $\xi$ and the image of $\xi_m$ under $\Phi$ is discrete. Hence $\xi = \xi_m$ for all $m$ large enough. It follows that $R = R_{D_m}$ for a nef $\mathbb{Q}$-divisor $D_m$.

**Step 4.** We show that any $K_{(X,B)}$-negative extremal ray $R_D$ contains the class of a rational curve $C$ with $0 < -K_{(X,B)} \cdot C \leq 2\dim X$.

By Theorem 2.13, let $\varphi_D : X \to Y$ be the contraction associated to $R_D$ (note that we do not need the step to proof Theorem 2.13). If $\dim Y < \dim X$, let $F$ be a general fiber of $\varphi_D$. Yang: By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that $H$ is ample on $F$. By Proposition 2.4. Yang: In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By Lemma 2.5, we can use Proposition 2.4 to get the result.

Yang: To be completed.

**Step 5.** Proof of the theorem.

Given an ample divisor $H$ on $X$, note that $\varepsilon H$ has positive minimum $\delta$ on $\mathrm{Psef}_1(X) \cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \mathrm{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and $\Phi$ is well-defined on it. By Steps 2 and 3, there are only finitely many extremal rays on $\mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H \leq 0}$. By Step 4, we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$\mathcal{C} := \mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\} \subset \mathcal{C}$ such that $\alpha_n \to \alpha \in N_1(X)_{\mathbb{R}}$. Note that $\mathrm{Psef}_1(X)$ is closed, hence $\alpha \in \mathrm{Psef}_1(X)$. We only need to consider the case $\alpha \cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$. Then $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$ for all $n$ large enough. Note that $\mathcal{C} \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$ is a polyhedral cone by Step 2 and hence is closed. Then $\alpha \in \mathcal{C}$ and the conclusion follows. □

**Remark 2.12.** Yang: Thanks for my friend Qin for pointing out that the extremal ray in Theorem 2.11 may not be exposed.

**Theorem 2.13** (Contraction Theorem)**.** Let $(X, B)$ be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by $\varphi_F$ if and only if $[C] \in F$;

(b) up to linearly equivalence, any Cartier divisor $G$ with $F \subset G^{\perp} = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$ comes from a Cartier divisor on $Y$, i.e., there exists a Cartier divisor $G_Y$ on $Y$ such that $G \sim \varphi_F^* G_Y$.

*Proof.* We follow the following steps to prove the theorem.

**Step 1.** We show that there exists a nef divisor $D$ on $X$ such that $F = D^{\perp} \cap \mathrm{Psef}_1(X)$. In other words, $F$ is defined on $N_1(X)_{\mathbb{Q}}$.

We can choose an ample divisor $H$ and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on $F$ since $F \cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in $N_1(X)_{\mathbb{R}}$. Then by Cone Theorem (Theorem 2.11), $F$ is an extremal face of a rational polyhedral cone, namely $\mathrm{Psef}_1(X)_{K_{(X,B)}+(1/n)H\leq 0}$. It follows that $F^{\perp} \subset N^1(X)_{\mathbb{R}}$ is defined on $\mathbb{Q}$. Since $F$ is extremal and $K_{(X,B)} + (1/n)H$-negative, the set $\{L \in F^{\perp} : L|_{\mathrm{Psef}_1(X)\setminus F} > 0\}$ has non-empty interior in $F^{\perp}$ by Theorems 2.3 and 2.11. Then there exists a Cartier divisor $D$ such that $D \in F^{\perp}$ and $D|_{\mathrm{Psef}_1(X)\setminus F} > 0$. It follows that $D$ is nef and $F = D^{\perp} \cap \mathrm{Psef}_1(X)$.

**Step 2.** Let $\varphi : X \to Y$ be the Iitaka fibration associated to $D$ by Theorem 2.1. We show that $\varphi$ is the desired fibration.

Note that $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$ is compact and $D$ is strictly positive on it. Then there exist $a \geq 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since $F$ is $K_{(X,B)}$-negative. Then by Base Point Free Theorem (Theorem 2.7), we know that $mD$ is base point free for all $m \gg 0$. Hence we can apply Theorem 2.1 to get a fibration $\varphi_D : X \to Y$.

First we show that $D$ comes from $Y$. Note that $mD$ and $(m+1)D$ induces the same fibration $\varphi_D$ for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m)D_{Y,m}$ and $D_{Y,m}$ is ample. Hence $D_Y$ is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that $C$ is contracted by $\varphi_D$ if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let $G$ be arbitrary Cartier divisor on $X$ such that $F \subset G^\perp$. Since $D$ is strictly positive on $\mathrm{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' := mD + G$, we have $D'^\perp \cap \mathrm{Psef}_1(X) = F$. Then by the same argument as above, we get an other fibration $\varphi_{D'} : X \to Y'$ such that a curve $C$ is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 2.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on $Y$. In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor $D'_Y$ on $Y$. Then $G = D' - mD$ also comes from $Y$. □

**Remark 2.14.** The Step 1 is amazing. If $F$ is not $K_{(X,B)}$-negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve $E$. By [Laz04, Lemma 1.5.4], we know that $\mathrm{Psef}_1(X)$ is a circular cone. The we see there indeed exist some irrational extremal faces of $\mathrm{Psef}_1(X)$.

**Definition 2.15.** Let $(X, B)$ be a projective klt pair and $R$ a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ with contraction $\varphi_R : X \to Y$. There are three types of contractions:

(a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of $\varphi_R$ is of codimension one;

(b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of $\varphi_R$ is of codimension at least two;

(c) *Mori fiber space*: if $\dim X > \dim Y$.

**Proposition 2.16.** Let $(X, B)$ be a $\mathbb{Q}$-factorial projective klt pair and $R$ a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Suppose that the contraction $\varphi : X \to Y$ associated to $R$ is either divisorial or a Mori fiber space. Then $Y$ is $\mathbb{Q}$-factorial.

*Proof.* Let $D$ be a prime Weil divisor on $Y$ and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\geq 0}[C]$ for an irreducible curve $C$ contracted by $\varphi$. Set $D_X := \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$. Then $D_X$ is a prime Weil divisor on $X$ and hence is $\mathbb{Q}$-Cartier.

If $\varphi$ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber $F$ of $\varphi$. Then by Contraction Theorem (Theorem 2.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor $D'$ on $Y$. We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence $D$ is $\mathbb{Q}$-Cartier.

If $\varphi$ is a divisorial contraction, let $E$ be the exceptional divisor of $\varphi$ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier $\mathbb{Q}$-divisor $E_Y$ on $Y$). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 2.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor $D'$ on $Y$. Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence $D$ is $\mathbb{Q}$-Cartier. □

**Remark 2.17.** If $\varphi$ is a small contraction, then $Y$ is never $\mathbb{Q}$-factorial. Otherwise, let $B_Y$ be the strict transform of $B$ on $Y$. Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset $U$. Suppose $K_{(Y,B_Y)}$ is $\mathbb{Q}$-Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

# 3 Basepoint Free Theorem on Positive Characteristic

This section refers to [Kee99]. For site and algebraic space, we refer to [Knu71], [Art70], [Stacks] and [FGA05]. Throughout this section, all schemes (or algebraic space) are of finite type over a base scheme $S$ with $S$ noetherian.

## 3.1 Preliminaries

**Theorem 3.1** (Serre vanishing in relative setting, ref. [Laz04, Theorem 1.7.6])**.** Let $f : X \to S$ be a proper morphism of schemes, $\mathscr{L}$ a line bundle and $\mathscr{F}$ a coherent sheaf on $X$. Suppose that $\mathscr{L}$ is relatively ample. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the higher direct image sheaves $R^i f_* \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ are zero for all $i > 0$.

**Theorem 3.2** (ref. [Laz04, Proposition 1.4.37])**.** Let $X$ be a projective scheme over a field $\mathbb{k}$. Then there exists a scheme $T$ of finite type over $\mathbb{k}$ and a line bundle $\mathscr{L}$ on $X \times T$ such that every numerically trivial line bundle on $X$ arises as the restriction $\mathscr{L}|_{X \times \{t\}}$ for some $t \in T$.

**Theorem 3.3** (Theorem on Formal Functions, ref. [Har77, Chapter III, Theorem 11.1])**.** Let $f : X \to Y$ be a projective morphism of noetherian schemes, let $\mathscr{F}$ be a coherent sheaf on $X$, and let $y \in Y$. Then the natural map

$$(R^i f_* \mathscr{F})^\wedge_y \to \varprojlim H^i(X_n, \mathscr{F}_n)$$

is an isomorphism for all $i \geq 0$, where $X_n = X \times_Y \operatorname{Spec} \mathscr{O}_{Y,y}/\mathfrak{m}_y^n$ and $\mathscr{F}_n = \mathscr{F}|_{X_n}$.

**Definition 3.4.** Let $X$ be a proper variety and $\mathscr{L}$ a nef line bundle on $X$. A closed subvariety $Z \subseteq X$ is called the *exceptional* for $\mathscr{L}$ if $\mathscr{L}^{\dim Z} \cdot Z = 0$. The *exceptional locus* of $\mathscr{L}$, denoted by $\operatorname{Exc} \mathscr{L}$, is defined as the closure of the union of all exceptional subvarieties of $\mathscr{L}$.

If $\mathscr{L}$ is semiample, then $\operatorname{Exc} \mathscr{L} = \operatorname{Exc} \varphi$ for the fibration $\varphi : X \to Y$ induced by $\mathscr{L}$.

**Definition 3.5.** Let $X$ be a proper scheme and $\mathscr{L}$ a nef line bundle on $X$. We say that $\mathscr{L}$ is *endowed with a map (EWM)* if there is a proper morphism $\varphi : X \to Y$ to a proper algebraic space such that

$\dim Z > \dim f(Z)$ if and only if $Z$ is an exceptional subvariety of $\ell$. If such a morphism is a fibration, then it is unique, called the *fibration associated to* $\ell$.

**Proposition 3.6.** Let $X$ be a proper variety and $\ell$ a nef line bundle on $X$ endowed with a map. Let $\varphi : X \to Y$ be the associated fibration. Then TFAE:

(a) $\ell$ is semiample;

(b) $\ell^{\otimes m}$ is pulled back from an ample line bundle on $Y$ for some $m \in \mathbb{Z}_{>0}$;

(c) $\ell^{\otimes m}$ is pulled back from a line bundle on $Y$ for some $m \in \mathbb{Z}_{>0}$;

*Proof.* (a) $\Leftrightarrow$ (b) $\implies$ (c) is clear. Replacing $\ell$ by $\ell^{\otimes m}$ for some $m \in \mathbb{Z}_{>0}$, suppose that $\ell = \varphi^*\ell_Y$ for some line bundle $\ell_Y$ on $Y$. We show that $\ell_Y$ is ample. Indeed, for all closed subvarieties $Z \subset Y$, we can find $Z' \subset X$ such that $Z' \twoheadrightarrow Z$ and $\dim Z' = \dim Z$. Then

$$\ell_Y^{\dim Z} \cdot Z = d\ell^{\dim Z'} \cdot Z' > 0$$

where $d = \deg(Z' \to Z)$. Hence $\ell_Y$ is ample. □

**Definition 3.7.** A morphism $f : X \to Y$ of schemes is called a *universal homeomorphism* if for every $Y$-scheme $Y'$, the base change $X \times_Y Y' \to Y'$ is a homeomorphism between the underlying topological spaces.

**Example 3.8.** Let $X$ be a scheme of finite type over $\mathbb{k}$. Then the natural morphism $X_{\text{red}} \to X$ is a universal homeomorphism.

Let $X$ be a scheme over $S$ of characteristic $p$. Then the absolute and relative Frobenius morphisms are universal homeomorphisms. Yang: To be completed.

The morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ is not a universal homeomorphism.

**Lemma 3.9.** Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes with $g$ finite. Let $\mathcal{F}$ be a coherent sheaf on $X$. Then the we have

$$R^i(g \circ f)_*\mathcal{F} = g_*(R^i f_*\mathcal{F}).$$

*Proof.* Yang: This is a simple application of the Grothendieck spectral sequence. However, I do not know anything about it. □

## 3.2 Algebraic space

**Definition 3.10.** Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of sets of arrows $\{U_i \to U\}_{i\in I}$, called *covering*, for each object $U$ in $\mathbf{C}$ such that:

(a) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering;

(b) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a arrow, then the fiber product $U_i \times_U V \to V$ exists and $\{U_i \times_U V \to V\}$ is a covering of $V$;

(c) if $\{U_i \to U\}_{i \in I}$ and $\{U_{ij} \to U_i\}_{j \in J_i}$ are coverings, then the collection of composition $\{U_{ij} \to U_i \to U\}_{i \in I, j \in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{J})$ where $\mathbf{C}$ is a category and $\mathcal{J}$ is a Grothendieck topology on $\mathbf{C}$.

Note that sheaf is indeed defined on a site.

**Definition 3.11.** Let $(\mathbf{C}, \mathcal{J})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{J})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ satisfying the following condition: for every object $U$ in $\mathbf{C}$ and every covering $\{U_i \to U\}_{i \in I}$ of $U$, if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every $i, j$, the pullback $s_i|_{U_i \times_U U_j}$ and $s_j|_{U_i \times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every $i$, the pullback $s|_{U_i} = s_i$.

**Definition 3.12.** Let $X$ be a scheme. The *big étale site* of $X$, denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over $X$ with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \to U\}_{i \in I}$ is a covering if and only if each $U_i$ is étale over $U$ and the union of their images is the whole $U$.

Let $X$ be a scheme over $S$. By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \to \mathbf{Set}$ such that for any $S$-scheme $T$, $h_X(T) = \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$. Yang: Easy to check that $h_X$ is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

**Definition 3.13.** Let $U$ be a scheme over a base scheme $S$. An *étale equivalence relation* on $U$ is a morphism $R \to U \times_S U$ between schemes over $S$ such that:

(a) the projections in two factors $R \to U$ are étale and surjective;

(b) for every $S$-scheme $T$, $h_R(T) \to h_U(T) \times h_U(T)$ gives an equivalence relation on $h_U(T)$ set-theoretically.

**Definition 3.14.** An *algebraic space* $X$ over a base scheme $S$ is an $S$-scheme $U$ together with an étale equivalence relation $R \to U \times_S U$.

Let $X = (U, R)$ be an algebraic space over $S$. We explain $X$ as a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. For any scheme $T$ over $S$, $h_R(T)$ is an equivalence relation on $h_U(T)$. The rule sending $T$ to the set of equivalence classes of $h_R(T)$ gives a presheaf on the site $(\mathbf{Sch}/S)_{\text{ét}}$. The sheafification of this presheaf is the sheaf associated to the algebraic space $X$. Explicitly, we have

$$X(T) := \left\{ f = (f_i) \,\middle|\, \begin{array}{c} \{T_i \to T\} \text{ a covering, } f_i \in h_U(T_i) \text{ such} \\ \text{that } (f_i|_{T_i \times_T T_j}, f_j|_{T_i \times_T T_j}) \in h_R(T_i \times_T T_j) \end{array} \right\} / \sim,$$

where

$$\alpha \sim \beta \quad \text{if } \exists \{S_i \to T\} \text{ such that } (\alpha|_{S_i}, \beta|_{S_i}) \in h_R(S_i).$$

**Definition 3.15.** An *algebraic space* over a base scheme $S$ is a sheaf $F$ on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$ such that

(a) the diagonal morphism $F \to F \times_S F$ is representable;

(b) there exists a scheme $U$ over $S$ and a map $h_U \to F$ which is surjective and étale.

The *morphism between algebraic spaces* $F_1, F_2$ is defined as a natural transformation of functors $F_1, F_2$.

**Remark 3.16.** By Yoneda's Lemma, given a morphism $h_U \to F$ between sheaves is the same as giving an element of $F(U)$. We may abuse the notation.

**Definition 3.17.** Let $\mathcal{p}$ be a property of morphisms of schemes satisfying the following conditions:

(a) is preserved under any base change;

(b) is étale local on the base.Yang: In [Stacks], this requires that "fppf local".

Let $\alpha : F \to G$ be a representable morphism of sheaves on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. We say that $\alpha$ has property $\mathcal{p}$ if for every $h_T \to G$, the base change $h_T \times_G F \to F$ has property $\mathcal{p}$.

**Remark 3.18.** The fiber product $F_1 \times_F F_2$ is just defined as $F_1 \times_F F_2(T) := F_1(T) \times_{F(T)} F_2(T)$ for any object $T \in \mathrm{Obj}(\mathbf{Sch}_S)$. We say that a morphism $f : F_1 \to F_2$ of sheaves is *representable* if for every $T \in \mathrm{Obj}(\mathbf{Sch}/S)$ and every $\xi \in F_2(T)$, the sheaf $F_1 \times_{F_2} h_T$ is representable as a functor. Here $h_T \to F_2$ is given by

$$h_T(U) \to F_2(U), \quad f \in \mathrm{Hom}(U, T) \mapsto F_2(f)(\xi) \in F_2(U).$$

In our case, given an arbitrary $h_U \to F \times F$ is equivalent to giving morphisms $h_{U_i} \to F$ for $i = 1, 2$. And the fiber product $F \times_{F \times F} (h_{U_1} \times h_{U_2})$ is just the fiber product $h_{U_1} \times_F h_{U_2}$. Hence the first condition in Definition 3.15 is equivalent to that $h_{U_1} \times_F h_{U_2}$ is representable for any $U_1, U_2$ over $F$. This implies that $h_U \to F$ is representable, whence the second condition in Definition 3.15 makes sense.

**Definition 3.19.** Let $X$ be an algebraic space over a base scheme $S$. Two two morphisms form field $\operatorname{Spec} k_i \to X$ is called equivalent if there is a common extension $K \supset k_1, k_2$ such that we have $\operatorname{Spec} K \to \operatorname{Spec} k_i \to X$ are the same for $i = 1, 2$. The *underlying point set* of $X$, denote by $|X|$, is defined as the set of equivalence classes of morphisms $\operatorname{Spec} k \to X$ for all field $k$ over the base field $\Bbbk$.

This definition coincides with the underlying set of a scheme. Let $\alpha : X \to Y$ be a morphism of algebraic spaces. It induces a map $|\alpha| : |X| \to |Y|$ by $x \mapsto \alpha \circ x$ (vertical composition).

**Proposition 3.20** (ref. [Stacks, Lemma 66.4.6])**.** There is a unique topology on $|X|$ such that

(a) if $X$ is a scheme, then the topology coincides with the usual topology.

(b) every morphism of algebraic spaces $f : X \to Y$ induces a continuous map $|f| : |X| \to |Y|$.

(c) if $U$ is a scheme and $U \to X$ is étale, then the induced map $|U| \to |X|$ is open.

This topology is called the *Zariski topology* on $|X|$.

**Definition 3.21.** Let $X$ be an algebraic space over a base scheme $S$. All étale morphisms $U \to X$ with $U$ scheme form a small site $X_{\text{ét}}$. All étale morphisms $U \to X$ with $U$ algebraic space form a small site $X_{\text{sp, ét}}$. The *structure sheaf* $\mathscr{O}_X$ of $X$ is given by $U \mapsto \Gamma(U, \mathscr{O}_U)$ for every étale morphism $U \to X$ from a scheme. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

**Example 3.22.** Let $U = \mathbb{A}^1_{\mathbb{C}}$ and $R \subset U \times U$ given by $y = x + n, n \in \mathbb{Z}$. Then $R$ is a disjoint union of lines in $U \times U$. Write $R = \coprod_{n \in \mathbb{Z}} R_n$ with $R_n = \{(x, x+n) : x \in \mathbb{C}\}$. Then the projection is given by

$$\begin{aligned} \pi_1|_{R_n} &: R_n \to U, \quad (x, x+n) \mapsto x, \\ \pi_2|_{R_n} &: R_n \to U, \quad (x, x+n) \mapsto x+n. \end{aligned}$$

Easily see that the projection $\pi_i : R \to U$ is étale and surjective for $i = 1, 2$. Let $r_{ij} : R \times U \to U \times U \times U$ be the morphism which maps $((x, y), u)$ to $(a_1, a_2, a_3)$ where $a_i = x$, $a_j = y$ and $a_k = u$ for $k \neq i, j$. Since $\Delta_U \to U \times U$ factors through $R$, $(\pi_1, \pi_2) = (\pi_2, \pi_1)$ and $r_{12} \times_{(U \times U \times U)} r_{23}$ factors through $r_{13}$, we have that $h_R(T)$ is an equivalence relation on $h_U(T)$ for all $T$ over $S$. Then $X := (U, R)$ is an algebraic space.

We do not check the representability here but give an example. Let $U \to X$ be the natural morphism given by $\mathrm{id}_U \in X(U)$. For any scheme $T$ over $\mathbb{C}$, we have

$$(U \times_X U)(T) = \{(f, g) \in h_{U \times U}(T) : \exists \{T_i \to T\} \text{ s.t. } (f_i, g_i) \in h_R(T_i)\} = h_R(T).$$

Hence the fiber product $h_U \times_X h_U$ is represented by $R$.

We show that $X \ncong \mathbb{C}^\times$ by computing the the global sections. Consider the covering $U \to X$, a section $s \in \mathscr{O}_X(X)$ is given by a section $s \in \Gamma(U, \mathscr{O}_U) = \mathbb{C}[t]$ such that $\pi_1^* s = \pi_2^* s$ in $\Gamma(R, \mathscr{O}_R)$. This means that $s(x+n) = s(x)$ for all $n \in \mathbb{Z}$. Hence $s$ is a constant function. In particular, $\mathscr{O}_X(X) = \mathbb{C} \neq \mathbb{C}[t, t^{-1}]$.

The underlying set $|X|$ is union of the quotient set $\mathbb{C}/\mathbb{Z}$ and a generic point. The Zariski topology on $|X|$ is the trivial topology.

In following, we will use the technique of *local construction* to construct many scheme-like objects on algebraic spaces. For local construction, see [Knu71]. Roughly speaking, for every étale morphism $U \to X$ with $U$ a scheme, we construct a scheme-theoretic object on $U$ which is compatible under base change. Then we glue these objects together to get a global object on $X$.

**Definition 3.23.** Let $X$ be an algebraic space over a base scheme $S$. A *coherent sheaf* on $X$ is a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ such that for every covering $\{U_i \to X\}$ with $U_i$ schemes, the sheaf $\mathcal{F}|_{U_i}$ is coherent for every $i$. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.
An *ideal sheaf* on $X$ is a coherent sheaf $\mathcal{I} \subset \mathcal{O}_X$. It defines a closed subspace $V(\mathcal{I}) \subset X$ by Yang: to be completed. And every closed subspace $Y \subset X$ is defined by an ideal sheaf $\mathcal{I}_Y$ such that $V(\mathcal{I}_Y) = Y$.

**Definition 3.24.** Let $X$ be an algebraic space over a base scheme $S$. A *line bundle* on $X$ is a coherent sheaf $\mathcal{L}$ on $X$ such that for every covering $\{U_i \to X\}$ with $U_i$ schemes, the sheaf $\mathcal{L}|_{U_i}$ is a line bundle on $U_i$. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

**Theorem 3.25** (ref. [Stacks, Theorem 76.36.4]). Let $f : X \to Y$ be a proper morphism of algebraic spaces over a base scheme $S$. Then there exists a factorization

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y,$$

where $f_1$ has geometrically connected fibers and $(f_1)_*\mathcal{O}_X = \mathcal{O}_Z$ and $f_2$ is finite.

**Definition 3.26.** Let $X$ be an algebraic space over a base scheme $S$ and $Y$ a closed subset of $|X|$. The *formal completion* of $X$ along $Y$, denoted by $\mathfrak{X}$, is
Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined as $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$ where $\mathcal{I}$ is the ideal sheaf of $Y$ in $\mathcal{O}_X$. Yang: to be completed.

**Definition 3.27.** Let $X$ be an algebraic space and $Y$ a closed subset of $X$. A *modification* of $X$ along $Y$ is a proper morphism $f : X' \to X$ and a closed subset $Y' \subset X'$ such that $X' \setminus Y' \to X \setminus Y$ is an isomorphism and $f^{-1}(Y) = Y'$.

**Theorem 3.28** (ref. [Art70, Theorem 3.1]). Let $Y'$ be a closed subset of an algebraic space $X'$ of finite type over $\Bbbk$. Let $\mathfrak{X}'$ be the formal completion of $X'$ along $Y'$. Suppose that there is a formal modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$. Then there is a unique modification

$$f : X' \to X, \quad Y \subset X$$

such that the formal completion of $X$ along $Y$ is isomorphic to $\mathfrak{X}$ and the induced morphism $\mathfrak{X}' \to \mathfrak{X}$ is isomorphic to $\mathfrak{f}$.

**Theorem 3.29** (ref. [Art70, Theorem 6.2]). Let $\mathfrak{X}'$ be a formal algebraic space and $Y' = V(\mathcal{I}')$ with $\mathcal{I}'$ the defining ideal sheaf of $\mathfrak{X}'$. Let $f : Y' \to Y$ be a proper morphism. Suppose that

(a) for every coherent sheaf $\mathcal{F}$ on $\mathfrak{X}'$, we have

$$R^1 f_* \mathcal{I}'^n \mathcal{F}/\mathcal{I}'^{n+1}\mathcal{F} = 0, \quad \forall n \gg 0;$$

(b) for every $n$, the homomorphism

$$f_*(\mathscr{O}_{\mathfrak{X}'}/i'^n) \otimes_{f_*\mathscr{O}_{Y'}} \mathscr{O}_Y \to \mathscr{O}_Y$$

is surjective.

Then there exists a modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ and a defining ideal sheaf $i$ of $\mathfrak{X}$ such that $V(i) = Y$ and $\mathfrak{f}$ induces $f$ on $Y$.

**Theorem 3.30** (ref. [Art70, Theorem 6.1])**.** Let $Y'$ be a closed algebraic subspace of an algebraic space $X'$ and $f_0 : Y' \to Y$ a finite morphism. Then there exists a modification $f : X' \to X$ whose restriction to $Y'$ is $f_0$. It is the amalgamated sum $X = X' \amalg_{Y'} Y$ in the category of algebraic spaces **AlgSp**.

**Example 3.31.** Let $X = \mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y]$ and $Y = V(y)$ be the $x$-axis. Let $f_0 : Y' = \mathbb{A}^1 \to Y, x \mapsto x^2$. Then there exists a modification $f : X' \to X$ such that the restriction $f|_{Y'} : Y' \to Y$ is $f_0$. Yang: To be completed.

## 3.3 A sufficient and necessary condition for EWM

In this and next subsection, we assume that all schemes (algebraic spaces) are of finite type over a field $\Bbbk$ with characteristic $p > 0$.

**Lemma 3.32.** Let $f : X \to Y$ be a finite morphism of algebraic space which is of finite type over $\Bbbk$. Suppose that $f$ is a universal homeomorphism. Then there exists $q = p^n$ such that the relative Frobinius morphism $\mathrm{Frob}^n_{X/\Bbbk}$ factors as

$$\mathrm{Frob}^n_{X/\Bbbk} : X \xrightarrow{f} Y \to X^{(q)}.$$

*Proof.* Yang: I can only prove this for schemes. Suppose that $X, Y$ are affine. Factor it as $A \twoheadrightarrow B \hookrightarrow C$ with $A, B, C$ $\Bbbk$-algebras.

For $A \twoheadrightarrow B$, let $I$ be the kernel of the surjection. Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is finite universal homeomorphism, we have that $I$ is a nilpotent ideal. Hence there exists $q$ such that $I^q = 0$. Let $a, a' \in A$ with the same image $b$ in $B$. Then we have $a^q - a'^q \in I^q = 0$. Hence $a^q = a'^q$ in $A$. This gives a map $B^q \to A, b^q \mapsto a^q$.

For $B \hookrightarrow C$, we induct on the dimension. If $C$ is artinian, then $0 = C^q \subset B \subset C$. In general case, this shows that $B \cdot C^{q_1} \subset C$ is an isomorphism at generic points. Let $I \coloneqq \operatorname{Ann}(B \cdot C^q/B) \subset B$. This is the conductor of extension $B \cdot C^{q_1} \subset C$, whence also an ideal of $B \cdot C^{q_1}$. To see this, for every $x \in B \cdot C^{q_1}$, $b \in I$, we have $xbB \cdot C^{q_1} = bB \cdot C^{q_1} \subset B$. By induction hypothesis, we have $(BC^{q_1}/I)^{q_2} \subset B/I$. For $x \in BC^{q_1}$, there exists $b \in B$ and $\delta \in I \subset B$ such that $x^{q_2} = b + \delta \in B$. Hence we have $(BC^{q_1})^{q_2} \subset B$. In particular, we have $C^{q_1 q_2} \subset (B \cdot C^{q_1})^{q_2} \subset B$.

In general case, we have

$$\begin{tikzcd} C^{q_1q_2} \arrow[r] & A' \arrow[r, two heads] \arrow[d] & C^{q_1} \arrow[d] & \\ & A \arrow[r, two heads] & B \arrow[r, hook] & C \end{tikzcd},$$

where $A'$ is the preimage of $C^{q_1}$ in $A$. One we have $C^q \to A \to C$, note that $A \to C$ is over $\Bbbk$, then it gives

$$C^q \to C^{(q)} \to A \to C.$$

□

**Corollary 3.33.** Let $Z \to X$ be a finite universal homeomorphism of algebraic spaces and $Z \to Y$ any finite morphism of algebraic spaces. Suppose that $X, Y, Z$ are all of finite type over $\Bbbk$. Then the amalgamated sum $X \amalg_Z Y$ exists in the category of algebraic spaces. Moreover, $Y \to X \amalg_Z Y$ is a finite universal homeomorphism.

*Proof.* By Lemma 3.32, we have a diagram

$$\begin{tikzcd} Y^{(q)} & Y \arrow[l] \\ Z^{(q)} \arrow[u] & \\ X \arrow[u] & Z \arrow[l, "f"'] \arrow[uu, "g"] \end{tikzcd}.$$

Denote $X \to Y^{(q)}$ by $f$. Let

$$\mathscr{A} := \mathrm{Ker}(\mathscr{O}_X \times \mathscr{O}_Y \to \mathscr{O}_Z, \quad (s,t) \mapsto f^*s - g^*t).$$

Then $\mathscr{A}$ is an $\mathscr{O}_{Y^{(q)}}$-algebra. Set $W := \mathrm{Spec}_{Y^{(q)}} \mathscr{A}$. Then $W = X \amalg_Z Y$ is the amalgamated sum in the category of algebraic spaces. Yang: The most important point is that $Z \to W$ is finite. Yang: At least in the cat of schemes. □

**Proposition 3.34.** Let $g : X' \to X$ be a proper, finite universal homeomorphism between algebraic spaces. Then a line bundle $\ell$ on $X$ is endowed with a map if and only if $g^*\ell$ is endowed with a map.

*Proof.* Let $f : X' \to Z$ be the map endowed on $g^*\ell$. By Lemma 3.32, we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[dd, "f"] & X \arrow[d] \\ & X'^{(q)} \arrow[d] \\ Z \arrow[r] & Z^{(q)} \end{tikzcd}.$$

Easy to check that $X \to Z^{(q)}$ is a map associated to $\ell$. □

**Proposition 3.35.** Let $X$ be a projective scheme and $\ell$ a nef line bundle on $X$. Assume that $X = X_1 \cup X_2$ for closed subsets $X_1$ and $X_2$. Suppose that $\ell|_{X_i}$ is endowed with a fibration $g_i : X_i \to Z_i$ for $i = 1, 2$. Then $\ell$ is endowed with a map $g : X \to Z$.

*Proof.* Let $X_{12} := X_1 \cap X_2$. Let $X_{12} \to Z_{12}$ be the Stein factorization of the map $g_1|_{X_{12}}$. Then by Yang: Rigidity Lemma, it is also the Stein factorization of the map $g_2|_{X_{12}}$. Denote $Y_i$ be the image of $Z_{12}$ in $Z_i$ for $i = 1, 2$. Then we have a commutative diagram

$$\begin{tikzcd}
& & Z_1 & \\
& X & X_1 \arrow[l] \arrow[u] & Y_1 \arrow[ul, "h'"] \\
Z_2 & X_2 \arrow[l] \arrow[u] & X_{12} \arrow[l] \arrow[u] \arrow[dr] & \\
& Y_2 \arrow[ul] & & Z_{12} \arrow[l] \arrow[uu, "h"'] \arrow[ulll, "f"]
\end{tikzcd}.$$

Consider the sub-diagram

$$\begin{tikzcd}
& Z_1 \\
& Y_1 \arrow[u, "h'"] \\
Z_2 & Z_{12} \arrow[l, "f"] \arrow[u, "h"]
\end{tikzcd}.$$

Here $f$ is finite, $h$ is finite universal homeomorphism and $h'$ is a closed immersion. By Corollary 3.33, we have the amalgamated sum $Z' := Y_1 \sqcup_{Z_{12}} Z_2$ exists in the category of algebraic spaces. Since $f$ is finite, so is the induced morphism $Y_1 \to Z'$. Then by Theorem 3.30, the amalgamated sum $Z := Z' \sqcup_{Y_1} Z_1$ exists in the category of algebraic spaces.

Then we have a commutative diagram

$$\begin{tikzcd}
Z & & Z_1 \arrow[ll] \\
& X \arrow[ul, "g"] & X_1 \arrow[l] \arrow[u] \\
Z_2 \arrow[uu] & X_2 \arrow[l] \arrow[u] & X_{12} \arrow[l] \arrow[u]
\end{tikzcd}.$$

Directly check shows that $g$ is a map associated to $\ell$. $\square$

**Proposition 3.36.** Let $X$ be a projective scheme and $D$ a nef and big divisor on $X$. Then we can write $D = A + E$ where $A$ is an ample divisor and $E$ is an effective divisor. Then $D$ is endowed with a map iff $D|_{E_{red}}$ is endowed with a map.

*Proof.* By Proposition 3.34, we may assume that $D|_E$ is endowed with a map $f : E \to Z$. Let $\mathscr{l} = \mathscr{o}_X(-E)$ be the ideal sheaf of $E$. note that $-E = A - D$ and $D$ is $f$-numerically trivial. Hence $\mathscr{l}|_E$ is $f$-ample. By Serre's vanishing, for every coherent sheaf $\mathscr{f}$ on $X$, there exists $n_0 \in \mathbb{m}$ such that for all $n \geq n_0$, we have

$$R^i f_* \mathscr{f}|_E \otimes \mathscr{l}|_E^{\otimes n} = 0$$

for all $i > 0$. In particular, let $n \in \mathbb{z}$ such that $R^i f_* \mathscr{o}_X/\mathscr{l} \otimes \mathscr{l}^{\otimes m} = 0$ for all $i > 0, m \geq n$. Set $\mathscr{i} := \mathscr{l}^{\otimes n}$. Then by the exact sequence

$$0 \to \mathscr{l}^{n-1} \otimes \mathscr{o}_X/\mathscr{l} \to \mathscr{o}_X/\mathscr{l}^n \to \mathscr{o}_X/\mathscr{l} \to 0,$$

we have that $R^i f_*(\mathscr{o}_X/\mathscr{i} \otimes \mathscr{i}^t) = 0$ for all $i > 0, t \geq 1$. This implies that $f_* \mathscr{o}_X/\mathscr{i}^t \to f_* \mathscr{o}_X/\mathscr{i}$ is surjective for all $t \geq 1$.

Let

$$\mathscr{a} := \mathscr{o}_X \oplus \mathscr{i}T \oplus \mathscr{i}^2 T^2 \oplus \cdots,$$
$$\mathscr{m} := \mathscr{f} \oplus \mathscr{i}\mathscr{f}T \oplus \mathscr{i}^2 \mathscr{f} T^2 \oplus \cdots,$$

where $T$ is a formal variable to denote the grading. Then $\mathscr{a}$ is a graded $\mathscr{o}_X$-algebra of finite type and $\mathscr{m}$ is a finite graded $\mathscr{a}$-module. We have an exact sequence of graded $\mathscr{a}$-modules

$$0 \to \mathscr{k} \to \mathscr{m} \otimes_{\mathscr{a}} \mathscr{i}T \to \mathscr{m} \to 0,$$

where $\mathscr{k} = \bigoplus \mathscr{k}_r T^r$ is a finite graded $\mathscr{a}$-module. Hence for $r \gg 1$, we have that $\mathscr{i}T \cdot \mathscr{k}_r T^r = \mathscr{k}_{r+1} T^{r+1}$. It implies that the image of $\mathscr{k}_{r+1} T^{r+1} \to \mathscr{m}_r T^r \otimes_{\mathscr{a}} \mathscr{i}T$ is contained in $\mathscr{i}\mathscr{m}_r$ for all $r \gg 1$. Tensor with $\mathscr{a} \otimes_{\mathscr{o}_X} \mathscr{o}_X/\mathscr{i}$, we have that

$$\mathscr{k}_{r+1} \otimes_{\mathscr{o}_X} \mathscr{o}_X/\mathscr{i} \to 0 \to \mathscr{m}_r \otimes_{\mathscr{o}_X} \mathscr{i} \otimes_{\mathscr{o}_X} \mathscr{o}_X/\mathscr{i} \to \mathscr{m}_{r+1} \otimes_{\mathscr{o}_X} \mathscr{o}_X/\mathscr{i} \to 0.$$

That is, $\mathscr{i}^r \mathscr{f}/\mathscr{i}^{r+1}\mathscr{f} \otimes_{\mathscr{o}_X/\mathscr{i}} \mathscr{i}/\mathscr{i}^2 \cong \mathscr{i}^{r+1}\mathscr{f}/\mathscr{i}^{r+2}\mathscr{f}$ for all $r \gg 1$. Hence we have that

$$R^i f_*(\mathscr{i}^{r-1}\mathscr{f}/\mathscr{i}^r \mathscr{f}) = 0$$

for all $i > 0, r \gg 1$.

Let $E' := V(\mathscr{i})$, we have that $D|_{E'}$ is endowed with a map $f' : E' \to Z'$ by Proposition 3.34. Moreover, we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & Z \\ \downarrow & & \downarrow g \\ E' & \xrightarrow{f'} & Z' \end{array}$$

with $g$ finite. Then by Grothendieck Spectral Sequence, we have that

$$R^i f'_*(\mathscr{i}^{r-1}\mathscr{f}/\mathscr{i}^r \mathscr{f}) = 0$$

for all $i > 0, r \gg 1$.

Then we can apply Theorems 3.28 and 3.29 to get a modification $X \to Y$. Note that $\operatorname{Exc} D \subset \operatorname{Supp} E$. It follows that $X \to Y$ is a map associated to $D$. $\square$

**Theorem 3.37.** Let $X$ be a proper variety and $\ell$ a nef line bundle on $X$. Then $\ell$ is endowed with a map if and only if $\ell|_{\mathrm{Exc}\,\ell}$ is endowed with a map.

*Proof.* By Proposition 3.35, we can assume that $\ell$ is big. Then the result follows from Proposition 3.36 and induction on dimension. ☐

## 3.4 For semiample

**Lemma 3.38.** Let $X$ be a projective scheme over $\Bbbk = \overline{\mathbb{F}_p}$. Then $\ell$ is numerically trivial if and only if $\ell$ is torsion in $\mathrm{Pic}(X)$.

*Proof.* Let $T$ be the scheme in Theorem 3.2. Then $\ell$ corresponds to a $\mathbb{F}_q$-point of $T$. Note that there are only finitely many $\mathbb{F}_q$-points in $T$. Hence $\ell$ is torsion in $\mathrm{Pic}(X)$. ☐

**Proposition 3.39.** Let $f : X \to Y$ be a finite universal homeomorphism between algebraic spaces of finite type over $\Bbbk$ and $\ell$ a line bundle on $Y$. Then there exists $q = p^n$ such that

(a) for every section $s \in H^0(X, f^*\ell)$, we have $s^q \in \Im(H^0(Y, \ell^{\otimes q}) \to H^0(X, f^*\ell^{\otimes q}))$;

(b) $\ell$ is semiample if and only if $f^*\ell$ is semiample;

(c) the map

$$f^* : \mathrm{Pic}(Y) \otimes \mathbb{Z}[1/q] \to \mathrm{Pic}(X) \otimes \mathbb{Z}[1/q]$$

is an isomorphism;

(d) if $f^*s_1 = f^*s_2$ for two sections $s_1, s_2 \in H^0(Y, \ell)$, then $s_1^q = s_2^q$ in $H^0(X, \ell^{\otimes q})$.

*Proof.* Note that $\mathrm{Frob}^*\ell \cong \ell^{\otimes p}$. Then all the properties follows from Lemma 3.32. ☐

**Proposition 3.40.** Let $X$ be a projective scheme and $\ell$ a nef line bundle on $X$. Assume that $X = X_1 \cup X_2$ for closed subsets $X_1$ and $X_2$. Suppose that $\ell|_{X_i}$ is semiample for $i = 1, 2$. Then $\ell$ is semiample.

*Proof.* Yang: To be learned. ☐

**Lemma 3.41.** Let $f : X \to Y$ be a proper map between algebraic spaces with $f_*\mathscr{O}_X = \mathscr{O}_Y$ and $\ell$ a line bundle on $X$. Let $D = V(i) \subset X$ be a closed subspace defined by an ideal sheaf $i$, $Z = f(D)$ and $D_k \coloneqq V(i^k)$. Suppose that $f$ is a modification with respect to $D, Z$ and $R^1f_*i^k/i^{k+1} = 0$ for all $k \gg 0$. Suppose for every $k$, there exists $r > 0$ such that $\ell^{\otimes r}|_{D_k}$ is pulled back from $f(D_k)$. Then $\ell^{\otimes r}$ is pulled back from $Y$ for some $r > 0$.

*Proof.* Replace $D$ by $D_k$ and $\ell$ by $\ell^{\otimes r}$ for some $k, r > 0$, we can assume that $R^1f_*i^k/i^{k+1} = 0$ for all $k$ and $\ell|_D$ is pulled back from $f(D)$. Then we show that $f_*\ell$ is a line bundle and $f^*f_*\ell \cong \ell$. Both of them are local, so we can assume that $X = \mathrm{Spec}\, B, Z = \mathrm{Spec}\, A$ are spectrum of local rings. Hence $\ell|_{D_k}$ is trivial for all $k$. By vanishing of $R^1f_*i^k/i^{k+1}$, we have a surjection $H^0(D_{k+1}, \ell|_{D_{k+1}}) \twoheadrightarrow H^0(D_k, \ell|_{D_k})$

for all $k$. This allow us to choose a section $s_k \in H^0(D_k, \ell|_{D_k})$ such that $s_k = s_{k+1}|_{D_k}$ for all $k$. Then we have a section $s \in H^0(D, \ell|_D)$ such that $s|_{D_k} = s_k$ for all $k$. By Nakayama's Lemma, we can assume that $s_k$ is nowhere vanishing. Yang: To be completed. □

**Proposition 3.42.** Let $X$ be a projective scheme and $D$ a nef and big divisor on $X$. Then we can write $D = A + E$ where $A$ is an ample divisor and $E$ is an effective divisor. Then $D$ is semiample iff $D|_{E_{red}}$ is semiample.

*Proof.* Yang: To be completed. □

**Theorem 3.43.** Let $X$ be a proper variety and $\ell$ a nef line bundle on $X$. Then $\ell$ is semiample if and only if $\ell|_{\mathrm{Exc}\,\ell}$ is semiample.

*Proof.* Yang: To be completed. □

## 3.5 Basepoint free theorem on positive characteristic

**Proposition 3.44** (ref. Yang: ). Let $T \subset X$ be a reduced Weil divisor on a normal variety $X$. Let $T^\nu \to T$ be the normalization, $C \subset T^\nu$ the effective Weil divisor defined by the conductor and $p : T^\nu \to T \hookrightarrow X$ the composition. Suppose that $K_X + T$ is $\mathbb{q}$-Cartier. Then there exists an effective $\mathbb{q}$-Weil divisor $D$ on $T^\nu$ such that

$$K_{T^\nu} + C + D = p^*(K_X + T).$$

**Theorem 3.45.** Let $X$ be a normal projective $\mathbb{q}$-factorial threefold and $B \in (0, 1)$ a $\mathbb{q}$-divisor. Let $\ell$ be a nef and big line bundle on $X$ such that $\ell - K_{(X,B)}$ is nef and big. Then $\ell$ is endowed with a map. Moreover, if $\mathbb{k} = \overline{\mathbb{f}}_p$, $\ell$ is semiample.

*Proof.* Let $\ell = \mathcal{O}_X(A+E)$ with $A$ an ample divisor and $E$ an effective divisor. Write $E = E_0 + E_1 + E_2$ such that the restriction of $\ell$ to every irreducible component of $E_i$ is of numerical dimension $i$. Let $S := \operatorname{Supp} E_1$ and $S = \sum S_i$ with $S_i$ irreducible components. Let $S^\nu \to S$ and $S_i^\nu \to S_i$ be the normalizations.

**Step 1.** Reduce to show that $\ell|_S$ is endowed with a map (semiample).

Yang: To be completed.

**Step 2.** Reduce to show that $\ell|_{S_i^\nu}$ is endowed with a map (semiample).

Yang: To be completed.

**Step 3.** Show that $\ell|_{S_i^\nu}$ is endowed with a map (semiample).

Yang: To be completed. □

# 4 F-singularities

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let $X$ be a projective variety over $\Bbbk$. Let $F$ denote the relative Frobenius morphism on $X$.

**Definition 4.1.** We say that $X$ is *$F$-finite* if $F : X \to X^{(p)}$ is finite.

**Definition 4.2.** We say that $X$ is *globally $F$-split* if $\mathscr{O}_X \to F^e_*\mathscr{O}_X$ splits as $\mathscr{O}_X$-modules for some $e \geq 0$. This is equivalent to for every $e \in \mathbb{Z}_{>0}$, $\mathscr{O}_X \to F^e_*\mathscr{O}_X$ splits as $\mathscr{O}_X$-modules.

**Definition 4.3.** Fix $\phi : F^e_* L \to \mathscr{O}_X$ a splitting of $\mathscr{O}_X \to F^e_*\mathscr{O}_X$. Define $\phi^n : F^{ne}_* L^{1+p^e+\cdots+p^{(n-1)e}} \to \mathscr{O}_X$ by induction:

$$\phi^n \coloneqq \phi \circ F^e_*(\phi^{n-1}).$$

**Theorem 4.4.** Above $\phi^n$ will be stable. That is, $\Im\phi^n = \Im\phi^{n+1}$ for all $n \gg 0$.

**Definition 4.5.** Let $\sigma(X, \phi) \coloneqq \Im\phi^n$. We say that $(X, \phi)$ is *$F$-pure* if $\sigma(X, \phi) = \mathscr{O}_X$.

**Proposition 4.6.** There is a bijection between

$$\{\text{effective } \mathbb{Q}\text{-divisor } \Delta \text{ such that } (p^e - 1)(K_X + \Delta) \text{ is Cartier}\}/\sim$$

and

$$\{\text{line bundles } \ell \text{ and } \phi : F^e_*\ell \to \mathscr{O}_X\}.$$

*Proof.* We have

$$F^e_X\mathscr{O}_X((1 - p^e)K_X) \to \mathscr{O}_X$$

given by $F^e\mathscr{O}_X(K_X) \to \mathscr{O}_X(K_X)$ and reflexivity of $\mathscr{O}_X(K_X)$. Since $\Delta$ is effective, we have

$$F^e(\mathscr{O}_X((1 - p^e)(K_X + \Delta))) \to F^e\mathscr{O}_X((1 - p^e)(K_X)) \to \mathscr{O}_X.$$

The another direction is by Grothendieck's duality

$$\mathscr{H}om_{\mathscr{O}_X}(F^e\ell, \mathscr{O}_X) \cong F^e_*(\ell^{-1} \otimes \mathscr{O}_X((1 - p^e)K_X)).$$

□

**Definition 4.7.** Let $\phi_{e,\Delta} : F^e_*(\mathscr{O}_X((1 - p^e)(K_X + \Delta))) \to \mathscr{O}_X$ be the morphism corresponding to the effective $\mathbb{Q}$-divisor $\Delta$.

We say that $(X, \Delta)$ is *$F$-pure* if $(X, \phi_{e,\Delta})$ is $F$-pure.

We say that $(X, \Delta)$ is *globally $F$-split* if for every Weil divisor $D \geq 0$, $\mathscr{O}_X \to F^e_*(\mathscr{O}_X(\lceil (p^e - 1)\Delta \rceil + D))$ admits a splitting for some $e \geq 0$.

We say that $(X, \Delta)$ is *strongly $F$-split* if for every Weil divisor $D \geq 0$, $\mathscr{O}_X \to F^e_*(\mathscr{O}_X(\lceil (p^e - 1)\Delta \rceil + D))$

admits a local splitting for some $e \geq 0$.

**Definition 4.8.**

**Definition 4.9.** $S^0(X, \sigma(X, \Delta) \otimes m)$

**Proposition 4.10.** Let $X$ be a globally $F$-split projective variety. Then we have

(a) suppose that $H^i(X, \ell^n) = 0$ for all $i > 0$ and all $n \gg 0$, then $H^i(X, \ell) = 0$ for all $i > 0$;

(b) for every ample divisor $A$ on $X$, we have $H^i(X, \mathcal{O}_X(A)) = 0$ for all $i > 0$;

(c) suppose that $X$ is Cohen-Macaulay and $A$-ample, then $H^i(X, \mathcal{O}_X(-A)) = 0$ for all $i < \dim X$;

(d) suppose that $X$ is normal and $A$-ample, then $H^i(X, \omega_X(A)) = 0$ for all $i > 0$.

# References

[Art70] Michael Artin. "Algebraization of formal moduli: II. Existence of modifications". In: *Annals of Mathematics* 91.1 (1970), pp. 88–135 (cit. on pp. 14, 19, 20).

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001, pp. xiv+233. ISBN: 0-387-95227-6. DOI: 10.1007/978-1-4757-5406-3. URL: https://doi.org/10.1007/978-1-4757-5406-3 (cit. on p. 6).

[FGA05] Barbara Fantechi et al. *Fundamental algebraic geometry*. Vol. 123. Mathematical Surveys and Monographs. Grothendieck's FGA explained. American Mathematical Society, Providence, RI, 2005, pp. x+339. ISBN: 0-8218-3541-6. DOI: 10.1090/surv/123. URL: https://doi.org/10.1090/surv/123 (cit. on p. 14).

[Har77] Robin Hartshorne. *Algebraic geometry*. Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9 (cit. on pp. 5, 14).

[Kaw91] Yujiro Kawamata. "On the length of an extremal rational curve". In: *Inventiones mathematicae* 105.1 (1991), pp. 609–611 (cit. on p. 7).

[Kee99] Seán Keel. "Basepoint freeness for nef and big line bundles in positive characteristic". In: *Annals of Mathematics* (1999), pp. 253–286 (cit. on p. 14).

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*. Vol. 134. Cambridge Tracts in Mathematics. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. Cambridge University Press, Cambridge, 1998, pp. viii+254. ISBN: 0-521-63277-3. DOI: 10.1017/CBO9780511662560. URL: https://doi.org/10.1017/CBO9780511662560 (cit. on pp. 6, 7, 9).

[Knu71] Donald Knutson. *Algebraic Spaces*. Vol. 203. Lecture Notes in Mathematics. Berlin, Heidelberg, New York: Springer-Verlag, 1971. ISBN: 978-3-540-05496-2 (cit. on pp. 14, 18).

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: 10.1007/978-3-642-18808-4. URL: https://doi.org/10.1007/978-3-642-18808-4 (cit. on pp. 6, 13, 14).

[MM86] Yoichi Miyaoka and Shigefumi Mori. “A numerical criterion for uniruledness”. In: *Annals of Mathematics* 124.1 (1986), pp. 65–69 (cit. on p. 6).

[Stacks] The Stacks Project Authors. *Stacks Project*. URL: https://stacks.math.columbia.edu/ (cit. on pp. 14, 17–19).

[Wik24] Wikipedia contributors. *Algebraic space — Wikipedia, The Free Encyclopedia*. [Online; accessed 28-July-2025]. 2024. URL: https://en.wikipedia.org/w/index.php?title=Algebraic_space&oldid=1248766894.