

Associated prime ideals

1 Associated prime ideals

Definition 1 (Associated prime ideals). Let A be a noetherian ring and M an A -module. The *associated prime ideals* of M are the prime ideals \mathfrak{p} of form $\text{Ann}(x)$ for some $x \in M$. The set of associated prime ideals of M is denoted by $\text{Ass}(M)$.

Example 2. Let $A = \mathbb{k}[x, y]/(xy)$ and $M = A$. First we see that $(x) = \text{Ann } y, (y) = \text{Ann } x \in \text{Ass } M$. Then we check other prime ideals. For (x, y) , if $xf = yf = 0$, then $f \in (x) \cap (y) = (0)$. If $(x - a) = \text{Ann } f$ for some f , note that $y \in (x - a)$ for $a \in \mathbb{k}^*$, then $f \in (x)$. Hence $f = 0$. Therefore $\text{Ass } M = \{(x), (y)\}$.

Example 3. Let $A = \mathbb{k}[x, y]/(x^2, xy)$ and $M = A$. The underlying space of $\text{Spec } A$ is the y -axis since $\sqrt{(x^2, xy)} = (x)$. First note that $(x) = \text{Ann } y, (x, y) = \text{Ann } x \in \text{Ass } M$. For $(x, y - a)$ with $a \in \mathbb{k}^*$, easily see that $xf = (y - a)f = 0$ implies $f = 0$ since $A = \mathbb{k} \cdot x \oplus \mathbb{k}[y]$ as \mathbb{k} -vector space. Hence $\text{Ass } M = \{(x), (x, y)\}$.

Lemma 4. Let A be a noetherian ring and M an A -module. Then the maximal element of the set

$$\{\text{Ann } x : x \in M, x \neq 0\}$$

belongs to $\text{Ass } M$.

Proof. We just need to show that such $\text{Ann } x$ is prime. Otherwise, there exist $a, b \in A$ such that $ab \in \text{Ann } x$ but $a, b \notin \text{Ann } x$. It follows that $\text{Ann } x \subsetneq \text{Ann } ax$ since $b \in \text{Ann } ax \setminus \text{Ann } x$. This contradicts the maximality of $\text{Ann } x$. \square

An element $a \in A$ is called a zero divisor for M if $M \rightarrow aM, m \mapsto am$ is not injective.

Corollary 5. Let A be a noetherian ring and M an A -module. Then

$$\{\text{zero divisors for } M\} = \bigcup_{\mathfrak{p} \in \text{Ass } M} \mathfrak{p}.$$

Lemma 6. Let A be a noetherian ring and M an A -module. Then $\mathfrak{p} \in \text{Ass}_A M$ iff $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Proof. Suppose $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $\mathfrak{p}A_{\mathfrak{p}} = \text{Ann } y_0/c$ with $y_0 \in M$ and $c \in A \setminus \mathfrak{p}$. For $a \in \text{Ann } y_0$, $ay_0 = 0$. Then $a/1 \in \mathfrak{p}A_{\mathfrak{p}}$. It follows that $a \in \mathfrak{p}$. Hence $\text{Ann } y_0 \subset \mathfrak{p}$.

Inductively, if $\text{Ann } y_n \subsetneq \mathfrak{p}$, then there exists $b_n \in A \setminus \mathfrak{p}$ such that $y_{n+1} := b_n y_n$, $\text{Ann } y_{n+1} \subset \mathfrak{p}$ and $\text{Ann } y_n \subsetneq \text{Ann } y_{n+1}$. To see this, choose $a_n \in \mathfrak{p} \setminus \text{Ann } y_n$. Then $(a_n/1)y_n = 0$ since $a_n/1 \in \mathfrak{p}A_{\mathfrak{p}}$. By definition, there exist $b_n \in A \setminus \mathfrak{p}$ such that $a_n b_n y_n = 0$. This process must terminate since A is noetherian. Thus $\text{Ann } y_n = \mathfrak{p}$ for some n . Hence $\mathfrak{p} \in \text{Ass}_A M$.

Conversely, suppose $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$. If $(a/s)(x/1) = 0 \in M_{\mathfrak{p}}$, there exist $t \in A \setminus \mathfrak{p}$ such that $tax = 0$. It follows that $ta \in \mathfrak{p}$ and then $(a/s) \in \mathfrak{p}A_{\mathfrak{p}}$. Hence $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. \square

Proposition 7. We have $\text{Ass } M \subset \text{Supp } M$. Moreover, if $\mathfrak{p} \in \text{Supp } M$ satisfies $V(\mathfrak{p})$ is an irreducible component of $\text{Supp } M$, then $\mathfrak{p} \in \text{Ass } M$.

Proof. For any $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$, we have $A/\mathfrak{p} \cong A \cdot x \subset M$. Tensoring with $A_{\mathfrak{p}}$ gives $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$ since $A_{\mathfrak{p}}$ is flat. Hence $M_{\mathfrak{p}} \neq 0$ and $\mathfrak{p} \in \text{Supp } M$.

Now suppose $\mathfrak{p} \in \text{Supp } M$ and $V(\mathfrak{p})$ is an irreducible component of $\text{Supp } M$. First we show that $\mathfrak{p} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $x \in M_{\mathfrak{p}}$ such that $\text{Ann } x$ is maximal in the set

$$\{\text{Ann } x : x \in M_{\mathfrak{p}}, x \neq 0\}.$$

Then we claim that $\text{Ann } x = \mathfrak{p}A_{\mathfrak{p}}$. First, $\text{Ann } x$ is prime by Lemma 4. If $\text{Ann } x \neq \mathfrak{p}$, then $V(\text{Ann } x) \supset V(\mathfrak{p})$. This implies that $\text{Ann } x \notin \text{Supp } M_{\mathfrak{p}}$ since $\text{Supp } M_{\mathfrak{p}} = \text{Supp } M \cap \text{Spec } A_{\mathfrak{p}}$. This is a contradiction. Thus $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. By Lemma 6, we have $\mathfrak{p} \in \text{Ass } M$. \square

Remark 8. The existence of irreducible component is guaranteed by Zorn's Lemma.

Definition 9. A prime ideal $\mathfrak{p} \in \text{Ass } M$ is called *embedded* if $V(\mathfrak{p})$ is not an irreducible component of $\text{Supp } M$.

Example 10. For $M = A = \mathbb{k}[x, y]/(x^2, xy)$, the origin (x, y) is an embedded point.

Proposition 11. If we have exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$, then $\text{Ass } M_2 \subset \text{Ass } M_1 \cup \text{Ass } M_3$.

Proof. Let $\mathfrak{p} = \text{Ann } x \in \text{Ass } M_2 \setminus \text{Ass } M_1$. Then the image $[x]$ of x in M_3 is not equal to 0. We have that $\text{Ann } x \subset \text{Ann}[x]$. If $a \in \text{Ann}[x] \setminus \text{Ann } x$, then $ax \in M_1$. Since $\text{Ann } x \subsetneq \text{Ann } ax$, there is $b \in \text{Ann } ax \setminus \text{Ann } x$. However, it implies $ba \in \text{Ann } x$, and then $a \in \text{Ann } x$ since $\text{Ann } x$ is prime, which is a contradiction. \square

Corollary 12. If M is finitely generated, then the set $\text{Ass } M$ is finite.

Proof. For $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$, we know that the submodule M_1 generated by x is isomorphic to A/\mathfrak{p} . Inductively, we can choose M_n be the preimage of a submodule of M/M_{n-1} which is isomorphic to A/\mathfrak{q} for some $\mathfrak{q} \in \text{Ass } M/M_{n-1}$. We can take an ascending sequence $0 = M_0 \subset M_1 \subset \dots \subset M_n \subset \dots$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime \mathfrak{p}_i . Since M is finitely generated, this is a finite sequence. Then the conclusion follows by Proposition 11. \square

2 Primary decomposition

Definition 13. An A -module is called *co-primary* if $\text{Ass } M$ has a single element. Let M be an A -module and $N \subset M$ a submodule. Then N is called *primary* if M/N is co-primary. If $\text{Ass } M/N = \{\mathfrak{p}\}$, then N is called \mathfrak{p} -primary.

Remark 14. This definition coincide with primary ideals in the case $M = A$. Recall an ideal $\mathfrak{q} \subset A$ is called *primary* if $\forall ab \in \mathfrak{q}, a \notin \mathfrak{q}$ implies $b^n \in \mathfrak{q}$ for some n .

Let \mathfrak{q} be a \mathfrak{q} -primary ideal. Since $\text{Supp } A/\mathfrak{q} = \{\mathfrak{p}\}$, $\mathfrak{p} \in \text{Ass } A/\mathfrak{q}$. Suppose $\text{Ann}[a] \in \text{Ass } A/\mathfrak{q}$. Then $\mathfrak{p} \subset \text{Ann}[a]$ since $V(\mathfrak{p}) = \text{Supp } A/\mathfrak{q}$. If $b \in \text{Ann}[a]$, then $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Hence $b^n \in \mathfrak{q}$, and then $b \in \mathfrak{p}$. This shows that $\text{Ass } A/\mathfrak{q} = \{\mathfrak{p}\}$ and \mathfrak{q} is \mathfrak{p} -primary as an A -submodule.

Let $\mathfrak{q} \subset A$ be a \mathfrak{p} -primary A -submodule. First we have $\mathfrak{p} = \sqrt{\mathfrak{q}}$ since $V(\mathfrak{p})$ is the unique irreducible component of $\text{Supp } A/\mathfrak{q}$. Suppose $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Then $b \in \text{Ann}[a] \subset \mathfrak{p}$ since \mathfrak{p} is the unique maximal element in $\{\text{Ann}[c] : c \in A \setminus \mathfrak{q}\}$. This implies that $b^n \in \mathfrak{q}$.

Definition 15. Let A be a noetherian ring, M an A -module and $N \subset M$ a submodule. A *minimal primary decomposition* of N in M is a finite set of primary submodules $\{Q_i\}_{i=1}^n$ such that

$$N = \bigcap_{i=1}^n Q_i,$$

no Q_i can be omitted and $\text{Ass } M/Q_i$ are pairwise distinct. For $\text{Ass } M/Q_i = \{\mathfrak{p}\}$, Q_i is called belonging to \mathfrak{p} .

Indeed, if $N \subset M$ admits a minimal primary decomposition $N = \bigcap Q_i$ with Q_i belonging to \mathfrak{p} , then $\text{Ass}(M/N) = \{\mathfrak{p}_i\}$. For given i , consider $N_i := \bigcap_{j \neq i} Q_j$, then $N_i/N \cong (N_i + Q_i)/Q_i$. Since $N_i \neq N$, $\text{Ass } N_i/N \neq \emptyset$. On the other hand, $\text{Ass } N_i/N \subset \text{Ass } M/Q_i = \{\mathfrak{p}\}$. It follows that $\text{Ass } N_i/N = \{\mathfrak{p}_i\}$, whence $\mathfrak{p}_i \in \text{Ass } M/N$. Conversely, we have an injection $M/N \hookrightarrow \bigoplus M/Q_i$, so $\text{Ass } M/N \subset \bigcup \text{Ass } M/Q_i$. Due to this, if Q_i belongs to \mathfrak{p} , we also say that Q_i is the \mathfrak{p} -component of N .

Proposition 16. Suppose $N \subset M$ has a minimal primary decomposition. If $\mathfrak{p} \in \text{Ass } M/N$ is not embedded, then the \mathfrak{p} component of N is unique. Explicitly, we have $Q = \nu^{-1}(N_{\mathfrak{p}})$, where $\nu : M \rightarrow M_{\mathfrak{p}}$.

Proof. First we show that $Q = \nu^{-1}(Q_{\mathfrak{p}})$. Clearly $Q \subset \nu^{-1}(Q_{\mathfrak{p}})$. Suppose $x \in \nu^{-1}(Q_{\mathfrak{p}})$. Then there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in Q$. That is, $[sx] = 0 \in M/Q$. If $[x] \neq 0$, we have $s \in \text{Ann}[x] \subset \mathfrak{p}$. This contradiction enforces $Q = \nu^{-1}(Q_{\mathfrak{p}})$.

Then we show that $N_{\mathfrak{p}} = Q_{\mathfrak{p}}$. Just need to show that for $\mathfrak{p}' \neq \mathfrak{p}$ and the \mathfrak{p}' component Q' of N , $Q'_{\mathfrak{p}} = M_{\mathfrak{p}}$. Since \mathfrak{p} is not embedded, $\mathfrak{p}' \not\subset \mathfrak{p}$. Then $\mathfrak{p} \notin V(\mathfrak{p}) = \text{Supp } M/Q'$. So $M_{\mathfrak{p}}/Q'_{\mathfrak{p}} = 0$. \square

Example 17. If \mathfrak{p} is embedded, then its components may not be unique. For example, let $M = A = \mathbb{k}[x, y]/(x^2, xy)$. Then for every $n \in \mathbb{Z}_{\geq 1}$, $(x) \cap (x^2, xy, y^n)$ is a minimal primary decomposition of $(0) \subset M$.

Let A be a noetherian ring and $\mathfrak{p} \subset A$ a prime ideal. We consider the \mathfrak{p} component of \mathfrak{p}^n , which is called n -th symbolic power of \mathfrak{p} , denoted by $\mathfrak{p}^{(n)}$. We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A$. In general, $\mathfrak{p}^{(n)}$ is not equal to \mathfrak{p}^n ; see below example.

Example 18. Let $A = \mathbb{k}[x, y, z, w]/(y^2 - zx^2, yz - xw)$ and $\mathfrak{p} = (y, z, w)$. We have $z = y^2/x^2, w = yz/x \in \mathfrak{p}^2 A_{\mathfrak{p}}$, whence $\mathfrak{p}^2 A_{\mathfrak{p}} = (z, w) \neq \mathfrak{p}^2$.

Theorem 19. Let A be a noetherian ring and M an A -module. Then for every $\mathfrak{p} \in \text{Ass } M$, there is

a \mathfrak{p} -primary submodule $Q(\mathfrak{p})$ such that

$$(0) = \bigcap_{\mathfrak{p} \in \text{Ass } M} Q(\mathfrak{p}).$$

Proof. Consider the set

$$\mathfrak{n} := \{N \subset M : \mathfrak{p} \notin \text{Ass } N\}.$$

Note that $\text{Ass } \bigcup N_i = \bigcup \text{Ass } N_i$ by definition of associated prime ideals. Then it is easy to check that \mathfrak{n} satisfies the conditions of Zorn's Lemma. Hence \mathfrak{n} has a maximal element $Q(\mathfrak{p})$. We claim that $Q(\mathfrak{p})$ is \mathfrak{p} -primary. If there is $\mathfrak{p}' \neq \mathfrak{p} \in \text{Ass } M/Q(\mathfrak{p})$, then there is a submodule $N' \cong A/\mathfrak{p}'$. Let N'' be the preimage of N' in M . We have $Q(\mathfrak{p}) \subsetneq N''$ and $N'' \in \mathfrak{n}$. This is a contradiction. By the fact $\text{Ass } \bigcap N_i = \bigcap \text{Ass } N_i$, we get the conclusion. \square

Corollary 20. Let A be a noetherian ring and M a finite A -module. Then every submodule of M has a minimal primary decomposition.