The First Properties of Abelian Varieties

1 Definition and examples of Abelian Varieties

Theorem 1 (Rigidity Lemma)**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Definition 2. Let S be a scheme. An *abelian scheme over* S is a group object in the category $\mathbf{Sch}_S$ such that the structure morphism is proper, smooth and a fibration. If $S = \operatorname{Spec} \mathbf{k}$ for some field $\mathbf{k}$, then it is called an *abelian variety over* $\mathbf{k}$.

Example 3.

Example 4.

Example 5.

In the following, we will always assume that A is an abelian variety over a field $\mathbf{k}$ of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A.

Proposition 6. Let A be an abelian variety. Then A is smooth.

Proof. Note that there is an open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a. □

Proposition 7. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Yang: To be completed. □

Lemma 8. Let $p : X \times Y \to Z$ be a proper morphism of varieties over $\mathbf{k}$ such that p contracts $\{x_0\} \times Y$ for some point $x_0 \in X$. Then there exists a unique morphism $f : Y \to Z$ such that $p = f \circ p_Y$.

Proof. Yang: To be completed. □

Theorem 9. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism.

Proof. Yang: To be completed. □

Proposition 10. Let A be an abelian variety. Then A is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 9. □

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition mor-

Date: August 28, 2025, **Author**: Tianle Yang, My Website phism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 11. Let A be an abelian variety over $\Bbbk$ and n a positive integer. Then the multiplication by n morphism $[n]_A : A \to A$ is finite surjective and étale.

Proof. Yang: To be completed. □

2 Complex abelian varieties

Theorem 12. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if Λ Yang: To be completed.