Schemes and Varieties

“宇宙战警行为规范第四十条，遇见违反社会公德的行为，发现违法犯罪的行为，要见义勇为，勇于斗争，善于斗争。第十四条，不说谎，不骗人，不弄虚做假，知错就改，诚实守信，言行一致，答应别人的事要做到。……榴莲，不用跟任何人道歉，你跟这儿的人不一样，你是来让世界变好的。”

Contents

Date: October 7, 2025, **Author**: Tianle Yang, My Homepage

0 Locally Ringed Space

0.1 Locally Ringed Space

Definition 0.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Example 0.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 0.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0, 1) \cup (2, 3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0, 1), (2, 3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0, 1))$ and $s_2 = 2 \in \mathcal{C}((2, 3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 0.4. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 0.5. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Example 0.6 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Example 0.7 (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i \in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i \in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^\sharp(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ_i' satisfying the same properties, then by gluing $\psi_i' \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

1 The First Properties of Schemes

If you learn the following content for the first time, it is recommended to skip all the proofs in this section and focus on the examples, remarks and the statements of propositions and theorems.

1.1 Schemes

Let R be a ring. Recall that the *spectrum* of R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set*.

Proposition 1.1. Let R be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

Proof. Yang: To be continued ☐

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 1.2. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i\in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.
These data form a category, denoted by $\mathbf{Sch}$. If we fix a base scheme S, then an S*-scheme* is a scheme X together with a morphism $X \to S$. The category of S-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

Theorem 1.3. The functor $\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. Yang: To be continued

Definition 1.4. A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if f induces an isomorphism of X onto an open (resp. closed) subscheme of Y. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. Yang: To be continued

Example 1.5. Let R be a graded ring. The *projective scheme* $\operatorname{Proj} R$ is defined as the scheme associated to the sheaf of rings

$$\mathcal{O}_{\operatorname{Proj} R} = \bigoplus_{d\geq 0} R_d.$$

It can be covered by open affine subschemes of the form $\operatorname{Spec} R_f$ for homogeneous elements $f \in R$. Yang: To be checked.

Example 1.6 (Glue open subschemes)**.** The construction in Example 0.7 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Example 0.7. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

Definition 1.7. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme Z of Y such that f factors through Z. More precisely, if $Y = \operatorname{Spec} A$ is affine, then the scheme theoretic image of f is $\operatorname{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover Y by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of f. Yang: To be checked.

1.2 Fiber product

Definition 1.8. Let $\mathcal{C}$ be a category and $X, Y, S \in \mathsf{Obj}(\mathcal{C})$ with morphisms $f : X \to S$ and $g : Y \to S$. A *fiber product* of X and Y over S is an object $Z \in \mathsf{Obj}(\mathcal{C})$ together with morphisms $p : Z \to X$ and $q : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

and satisfies the universal property that for any object $W \in \mathsf{Obj}(\mathcal{C})$ with morphisms $u : W \to X$ and $v : W \to Y$ such that $f \circ u = g \circ v$, there exists a unique morphism $h : W \to Z$ such that $p \circ h = u$ and $q \circ h = v$.
If a fiber product exists, it is unique up to a unique isomorphism. We denote the fiber product by $X \times_S Y$. Yang: To be checked.

Example 1.9. In the category of sets, the fiber product $X \times_S Y$ is given by

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\},$$

with the projections $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ being the restrictions of the natural projections. Yang: To be checked.

Remark 1.10. If one reverses the arrows in Definition 1.8, one gets the notion of *fiber coproduct*. It is also called the *pushout* or *amalgamated sum* in some literature. We denote the fiber coproduct of X and Y over S by $X \amalg_S Y$. Note that in the category of rings, the fiber coproduct $A \amalg_R B$ of R-algebras A and B over R is given by the tensor product $A \otimes_R B$. Dually, one can expect that fiber products of affine schemes correspond to tensor products of rings.

Theorem 1.11. The category of schemes admits fiber products. More precisely, given morphisms of schemes $f : X \to S$ and $g : Y \to S$, there exists a scheme Z together with morphisms $p : Z \to X$ and $q : Z \to Y$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

commutes and satisfies the universal property of the fiber product. We denote this scheme by $X \times_S Y$.
Yang: To be continued

Definition 1.12. Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a point. The *scheme theoretic fiber* of f over y is the fiber product $X_y = X \times_Y \mathsf{Spec}\, \kappa(y)$, where $\kappa(y)$ is the residue field of the local ring $\mathcal{O}_{Y,y}$. Yang: To be checked.

Definition 1.13. Let X be a scheme and $Z_1, Z_2 \subset X$ be closed subschemes defined by quasi-coherent sheaves of ideals $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{O}_X$, respectively. The *scheme theoretic intersection* of Z_1 and Z_2 is the closed subscheme $Z_1 \cap Z_2$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}_1 + \mathcal{I}_2$. Yang: To be checked.

1.3 Noetherian and finite type

Definition 1.14. A scheme X is *Noetherian* if it admits a finite open cover $\{U_i\}_{i=1}^n$ such that each U_i is an affine scheme $\operatorname{Spec} A_i$ with A_i a Noetherian ring. Yang: To be checked.

Proposition 1.15. A Noetherian scheme is quasi-compact. Yang: To be checked.

Definition 1.16. Let S be a scheme. A scheme X is of *finite type* over S if there exists a finite open cover $\{U_i\}_{i=1}^n$ of S such that for each i, $f^{-1}(U_i)$ can be covered by finitely many affine open subsets $\{V_{ij}\}_{j=1}^{m_i}$ with $f(V_{ij}) \subseteq U_i$ and the induced morphism $f|_{V_{ij}} : V_{ij} \to U_i$ corresponds to a finitely generated algebra over the ring of global sections of U_i.
A scheme is called *Noetherian* if it is of finite type over $\operatorname{Spec} \mathbb{Z}$. Yang: To be checked.

1.4 Integral, reduced and irreducible

Definition 1.17. A topological space X is *irreducible* if it is non-empty and cannot be expressed as the union of two proper closed subsets. Equivalently, every non-empty open subset of X is dense in X. Yang: To be checked.

Proposition 1.18. Let X be a topological space satisfying the descending chain condition on closed subsets. Then X can be written as a finite union of irreducible closed subsets, called the *irreducible components* of X. Moreover, this decomposition is unique up to permutation of the components. Yang: To be checked.

Definition 1.19. A scheme X is *reduced* if its structure sheaf $\mathcal{O}_X$ has no nilpotent elements. Yang: To be checked.

Proposition 1.20. A scheme X is reduced if and only if for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a reduced ring. Yang: To be checked.

Proposition 1.21. Let X be a scheme. There exists a unique closed subscheme X of X such that X is reduced and has the same underlying topological space as X. Moreover, for any morphism of schemes $f : Y \to X$ with Y reduced, f factors uniquely through the inclusion $X \to X$. Yang: To be checked.

Definition 1.22. A scheme X is *integral* if it is both reduced and irreducible. Yang: To be checked.

Proposition 1.23. A scheme X is integral if and only if for every open affine subset $U = \operatorname{Spec} A \subset X$, the ring A is an integral domain. Yang: To be checked.

1.5 Dimension

Definition 1.24. The *Krull dimension* of a topological space X, denoted by $\dim X$, is the supremum of the lengths n of chains of distinct irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X. If no such finite supremum exists, we say that X has infinite dimension. Yang: To be checked.

1.6 Separated and proper

Definition 1.25. A morphism of schemes $f : X \to Y$ is *separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. A scheme X is *separated* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. Yang: To be checked.

Proposition 1.26. Any affine scheme is separated. More generally, any morphism between affine schemes is separated. Yang: To be checked.

Proposition 1.27. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any scheme T and any pair of morphisms $g_1, g_2 : T \to X$ such that $f \circ g_1 = f \circ g_2$, the equalizer of g_1 and g_2 is a closed subscheme of T. Yang: To be checked.

Proposition 1.28. A scheme X is separated if and only if for any pair of affine open subschemes $U, V \subset X$, the intersection $U \cap V$ is also an affine open subscheme. Yang: To be checked.

Proposition 1.29. The composition of separated morphisms is separated. Moreover, separatedness is stable under base change, i.e., if $f : X \to Y$ is a separated morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also separated. Yang: To be checked.

Proposition 1.30. A morphism of schemes $f : X \to Y$ is separated if and only if for every commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists at most one morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. Yang: To be checked.

Definition 1.31. A morphism of schemes $f : X \to Y$ is *universally closed* if for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map. Yang: To be checked.

Definition 1.32. A morphism of schemes $f : X \to Y$ is *proper* if it is of finite type, separated, and universally closed (i.e., for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map). A scheme X is *proper* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is proper. Yang: To be checked.

Theorem 1.33. Any projective morphism is proper. In particular, any projective scheme is proper. Yang: To be checked.

Proposition 1.34. The composition of proper morphisms is proper. Moreover, properness is stable under base change, i.e., if $f : X \to Y$ is a proper morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also proper. Yang: To be checked.

Proposition 1.35. A morphism of schemes $f : X \to Y$ is proper if and only if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists a unique morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. Yang: To be checked.

2 Category of sheaves of modules

2.1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 2.1. Let X be a ringed space with structure sheaf $\mathcal{O}_X$. A **sheaf of (left) $\mathcal{O}_X$-modules** is a sheaf $\mathcal{F}$ on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ in the sense that for every $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$, we have

$$\rho_{UV}(s \cdot m) = \rho_{UV}(s) \cdot \rho_{UV}(m).$$

Yang: To be continued...

Example 2.2. Let X be a scheme. The structure sheaf $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-modules. More generally, any quasi-coherent sheaf (to be defined later) is a sheaf of $\mathcal{O}_X$-modules. In particular, if $X = \operatorname{Spec} A$ is an affine scheme, then for any A-module M, the associated sheaf $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules. Yang: To be continued...

Definition 2.3. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **quasi-coherent** if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets. Yang: To be continued...

Definition 2.4. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **coherent** if it is quasi-coherent and for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$

is isomorphic to the cokernel of a morphism of finite free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0,$$

where m, n are finite integers. Yang: To be continued...

2.2 As abelian categories

Theorem 2.5. Let X be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules is an abelian category. Yang: To be continued...

Theorem 2.6. Let X be a scheme. The category of quasi-coherent sheaves on X is an abelian category. Yang: To be continued...

Theorem 2.7. Let X be a noetherian scheme. The category of coherent sheaves on X is an abelian category. Yang: To be continued...

2.3 Relevant functors

Theorem 2.8. Let X be a ringed space. The global sections functor

$$\Gamma(X, -) : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\mathcal{O}_X(X)\text{-modules})$$

is left exact. Yang: To be continued...

Theorem 2.9. Let $f : X \to Y$ be a morphism of ringed spaces. The direct image functor

$$f_* : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_Y\text{-modules})$$

is left exact. Yang: To be continued...

Theorem 2.10. Let $f : X \to Y$ be a morphism of ringed spaces. The inverse image functor

$$f^* : (\text{Sheaves of } \mathcal{O}_Y\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_X\text{-modules})$$

is right exact. Yang: To be continued...

2.4 Locally free sheaves and vector bundles

Definition 2.11. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **locally free** if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to a finite free $\mathcal{O}_U$-module, i.e., there exists an isomorphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{F}|_U \cong \mathcal{O}_U^n,$$

where n is a finite integer called the **rank** of $\mathcal{F}$ at x. Yang: To be continued...

Example 2.12. A **line bundle** on a scheme X is a locally free sheaf of rank 1. The sheaf of differentials $\Omega_{X/k}$ on a smooth variety X over a field k is a locally free sheaf of rank equal to the dimension of X. Yang: To be continued...

Theorem 2.13. Let X be a scheme. There is an equivalence of categories between the category of locally free sheaves of finite rank on X and the category of vector bundles on X. Yang: To be continued...

2.5 Cohomological theory

Theorem 2.14. Let X be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

Theorem 2.15. Let X be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

Theorem 2.16. Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

3 Normal, Cohen-Macaulay, and regular schemes

3.1 Tangent spaces

There are many description of the tangent space of a scheme at a point. Here we give one of them.

Let X be a scheme over a field $\mathbf{k}$, and let $x \in X(\mathbf{k})$.

Proposition 3.1. Let $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$ be the spectrum of the ring of dual numbers over $\mathbf{k}$ with point $* : \operatorname{Spec} \mathbf{k} \to \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$. The tangent space $T_x X$ is naturally isomorphic to the set of morphisms $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X$ that send $*$ to x, i.e.

$$T_x X \cong \{f : \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X \mid f(*) = x\}.$$

Proof. Yang: To be filled. □

4 Line Bundles and Divisors

4.1 Cartier Divisors

4.2 Line Bundles and Picard Group

Definition 4.1. Let X be a scheme. The *Picard group* of X is defined to be $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. The group operation is given by the tensor product of line bundles.

Definition 4.2. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a Yang: non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X\times\{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X\times\{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\mathrm{alg}} \mathcal{L}'$. Yang: To be checked.

4.3 Weil Divisors and Reflexive Sheaves

5 Line bundles induce morphisms

5.1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 5.1. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Let $U_i \coloneqq \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Proposition 5.2. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. Yang: To be continued. □

Example 5.3. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 5.4. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}^{(m+1)(n+1)-1}_A$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 5.5. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. Yang: To be continued.

Example 5.6. Let

Example 5.7.

Definition 5.8. Let $\mathcal{L}$ be a line bundle on a scheme X. Yang: To be continued.

Definition 5.9. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

Definition 5.10. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^* \mathcal{O}(1)$. Yang: To be continued.

Theorem 5.11. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

Proposition 5.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

Yang: To be continued.

Proof. Yang: To be continued. □

Proposition 5.13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then $\mathcal{L}$ is very ample if and only if the following two conditions hold:

(a) (separate points) for any two distinct points $x, y \in X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $s(y) \neq 0$;

(b) (separate tangent vectors) for any point $x \in X$ and non-zero tangent vector $v \in T_x X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $v(s) \neq 0$.

Yang: To be continued.

5.2 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 5.14. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

Definition 5.15. A *linear system* on a scheme X is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on X and $V \subseteq \Gamma(X, \mathcal{L})$ is a subspace. The dimension of the linear system is $\dim V - 1$. A linear system is *base-point free* if V is base-point free. A linear system is *complete* if $V = \Gamma(X, \mathcal{L})$. Yang: To be continued.

Definition 5.16. Let $\mathcal{L}$ be a line bundle on a scheme X and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace. The *base locus* of V is the closed subset

$$\mathrm{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If $\mathrm{Bs}(V) = \varnothing$, we say that V is *base-point free*. Yang: To be continued.

5.3 Asymptotic behavior

Definition 5.17. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

Definition 5.18. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. Yang: To be continued.

Theorem 5.19. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \mathrm{Proj}\, R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. Yang: To be continued.

Definition 5.20. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large n. Yang: To be continued.

Example 5.21. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. Yang: To be continued.

6 Differentials and duality

7 Flat, smooth and étale morphisms

8 Relative objects

8.1 Relative schemes

Definition 8.1. Let X be a scheme. An $\mathcal{O}_X$-algebra is a sheaf . Yang: To be continued...

Definition 8.2. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. The relative Spec of $\mathcal{A}$, denoted by $\operatorname{Spec}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Spec} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. Yang: To be continued...

Proposition 8.3. Let X be a scheme and $\mathcal{E}$ be a locally free sheaf of finite rank on X. Then the relative Spec of the symmetric algebra of $\mathcal{E}$, denoted by $\mathbb{V}(\mathcal{E}) = \operatorname{Spec}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}$, is called the geometric vector bundle associated to $\mathcal{E}$. The projection morphism $\pi : \mathbb{V}(\mathcal{E}) \to X$ is affine and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Spec} \operatorname{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$. Yang: To be continued...

Definition 8.4. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_0 = \mathcal{O}_X$ and $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{O}_X$-algebra. The relative Proj of $\mathcal{A}$, denoted by $\operatorname{Proj}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Proj} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. The projection morphism $\pi : \operatorname{Proj}_X \mathcal{A} \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{A}(U)$. Yang: To be continued...

8.2 Blowing up

Definition 8.5. Let X be a scheme and $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blowing up of X along $\mathcal{J}$, denoted by $\operatorname{Bl}_{\mathcal{J}} X$, is defined to be the relative Proj of the Rees algebra of $\mathcal{J}$:

$$\operatorname{Bl}_{\mathcal{J}} X = \operatorname{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{J}^n.$$

The projection morphism $\pi : \operatorname{Bl}_{\mathcal{J}} X \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Bl}_{\mathcal{J}(U)} U$. The exceptional divisor of the blowing up is defined to be the closed subscheme $E = \pi^{-1}(V(\mathcal{J}))$ of $\operatorname{Bl}_{\mathcal{J}} X$. Yang: To be continued...

8.3 Relative ampleness and relative morphisms

9 Finite morphisms and fibrations