Formal Completion

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Contents

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1 Formal completion of rings and modules

Definition 1. Let A be a ring and $\mathcal{T}$ a topology on A. We say that $(A, \mathcal{T})$ is a *topological ring* if the operations of addition and multiplication are continuous with respect to the topology $\mathcal{T}$.
Given a topological ring A. A *topological A-module* is a pair $(M, \mathcal{T}_M)$ where M is an A-module and $\mathcal{T}_M$ is a topology on M such that the addition and scalar multiplication is continuous. The morphisms of topological A-modules are the continuous A-linear maps. They form a category denoted by $\mathbf{TopMod}_A$.

Definition 2. Let A be a ring, I an ideal of A and M an A-module. The *I-adic topology* on M is the topology defined by the basis of open sets $x + I^k M$ for all $x \in M, k \geq 0$.

Example 3. Let $A = \mathbb{Z}$ be the ring of integers and p a prime number. The p-adic topology on $\mathbb{Z}$ is defined by the metric

$$d(x, y) \coloneqq \|x - y\|_p \coloneqq p^{-v(x-y)},$$

where v is the valuation defined by the ideal $p\mathbb{Z}$.

Let M be an A-module equipped with the I-adic topology. Note that M is Hausdorff as a topological space if and only if $\bigcap_{n \geq 0} I^n M = \{0\}$. In this case, we say that M is *I-adically separated*.

When M is I-adically separated, we can see that M is indeed a metric space. Fix $r \in (0, 1)$. For every $x \neq y \in M$, there is a unique $k \geq 0$ such that $x - y \in I^k M$ but $x - y \notin I^{k+1} M$. We can define a metric on M by

$$d(x, y) \coloneqq r^k.$$

This metric induces the I-adic topology on M.

To analyze the I-adic separation property of M, the following Artin-Rees Lemma is particularly useful.

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Theorem 4 (Artin-Rees Lemma)**.** Let A be a noetherian ring, I an ideal of A, M a finite A-module and N a submodule of M. Then there exists an integer N such that for all $n \geq 0$, we have

$$(I^{N+n}M) \cap N = I^n(I^N M \cap N).$$

Proof. Clearly $I^n(I^N M \cap N) \subset (I^{N+n}M) \cap N$. Yang: To be completed. □

Corollary 5. Let A be a noetherian ring, I an ideal of A, M a finite A-module and N a submodule of M. Then the subspace topology on N induced by $N \subset M$ coincides with the I-adic topology on N.

Proof. This is a direct consequence of the Artin-Rees Lemma. □

Corollary 6. Let A be a noetherian ring, I an ideal of A, and M a finite A-module. Let $N = \bigcap_{n \geq 0} I^n M$. Then $IN = N$. In particular, if $I \subset \mathrm{rad}(A)$, then M is I-adically separated.

Proof. Yang: To be completed. □

Lemma 7. Let A be a ring, I an ideal of A and M an A-module. Then the inverse limit

$$\widehat{M} := \varprojlim(\cdots \to M/I^n M \to M/I^{n-1}M \to \cdots \to M/IM)$$

exists in the category of A-modules. Moreover, $\widehat{A}$ is an A-algebra and $\widehat{M}$ is an $\widehat{A}$-module.

Proof. Yang: To be completed. □

Definition 8 (Formal Completion)**.** Let A be a ring, I an ideal of A and M an A-module. The *formal completion* of M with respect to I, denoted by $\widehat{M}$, is defined as

$$\widehat{M} := \varprojlim(\cdots \to M/I^n M \to M/I^{n-1}M \to \cdots \to M/IM),$$

where the maps are the natural projections $M/I^n M \to M/I^{n-1}M$.

Example 9. Let $A = \mathbb{Z}$ be the ring of integers and $I = p\mathbb{Z}$. The formal completion of $\mathbb{Z}$ with respect to $p\mathbb{Z}$ is the ring of p-adic integers, denoted by $\mathbb{Z}_p$. The elements of $\mathbb{Z}_p$ can be represented as infinite series of the form

$$a_0 + a_1 p + a_2 p^2 + \cdots,$$

where $a_i \in \{0, 1, \ldots, p-1\}$.

Example 10. Let R be a ring, $A = R[X_1, \ldots, X_n]$ and $I = (X_1, \ldots, X_n)$. The formal completion of A with respect to I is the ring of formal power series $R[[X_1, \ldots, X_n]]$. The elements of $R[[X_1, \ldots, X_n]]$ can be represented as infinite series of the form

$$\sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n},$$

where $a_{i_1, \ldots, i_n} \in R$ and the multi-index $(i_1, \ldots, i_n)$ runs over all non-negative integers.

Proposition 11. The formal completion $\widehat{A}$ of a ring A with respect to an ideal I is a complete topological ring with respect to the I-adic topology. That is, every Cauchy sequence in $\widehat{A}$ converges to an element in $\widehat{A}$.

Yang: To be completed.

Yang: When is the homomorphism $M \to N$ continuous?

By the universal property of the inverse limit, we get a covariant functor from the category of A-modules to the category of topological $\widehat{A}$-modules, which sends an A-module M to $\widehat{M}$ and a morphism $f : M \to N$ to the induced morphism $\widehat{f} : \widehat{M} \to \widehat{N}$.

Lemma 12. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of finite A-modules. Then the sequence of $\widehat{A}$-modules

$$0 \to \widehat{M_1} \to \widehat{M_2} \to \widehat{M_3} \to 0$$

is still exact.

Proof. Yang: To be completed. □

Lemma 13. Let $\widehat{A}$ be the formal completion of a noetherian ring A with respect to an ideal I. Suppose that I is generated by $a_1, ..., a_n$. Then we have an isomorphism of topological rings

$$\widehat{A} \cong A[[X_1, \ldots, X_n]]/(X_1 - a_1, \cdots, X_n - a_n).$$

Proof. Yang: To be completed. □

Proposition 14. Let A be a noetherian ring and I an ideal of A. Then the formal completion $\widehat{A}$ of A with respect to I is a noetherian ring.

Proof. Yang: To be completed. □

Proposition 15. Let A be a noetherian ring and I an ideal of A. Then the formal completion $\widehat{A}$ of A with respect to I is a flat A-module.

Proof. Yang: To be completed. □

Proposition 16. Let $\widehat{A}$ be completion of a noetherian ring A with respect to an ideal I and M a finite A-module. Then the natural map $M \otimes_A \widehat{A} \to \widehat{M}$ is an isomorphism.

Proof. Yang: To be completed. □

Proposition 17. Let A be a noetherian ring and $\mathfrak{m}$ a maximal ideal of A. Then the formal completion $\widehat{A}$ of A with respect to $\mathfrak{m}$ is a local ring with maximal ideal $\mathfrak{m}\widehat{A}$.

Proof. Yang: To be completed. □

2 Complete local rings

Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian complete local ring. We say that A is *of equal characteristic* if $\operatorname{char} A = \operatorname{char} \mathsf{k}$, and *of mixed characteristic* if $\operatorname{char} A \neq \operatorname{char} \mathsf{k}$. In latter case, $\operatorname{char} \mathsf{k} = p$ and $\operatorname{char} A = 0$ or $\operatorname{char} A = p^k$.

The goal of this subsection is the following structure theorem for noetherian complete local rings due to Cohen.

Theorem 18 (Cohen Structure Theorem)**.** Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian complete local ring of dimension d. Then

(a) A is a quotient of a noetherian regular complete local ring;

(b) if A is regular and of equal characteristic, then $A \cong \mathsf{k}[[X_1, \ldots, X_d]]$;

(c) if A is regular, of mixed characteristic $(0, p)$ and $p \notin \mathfrak{m}^2$, then $A \cong D[[X_1, \ldots, X_{d-1}]]$, where (D, p, k) is a complete DVR;

(d) if A is regular, of mixed characteristic $(0, p)$ and $p \in \mathfrak{m}^2$, then $A \cong D[[X_1, \ldots, X_d]]/(f)$, where (D, p, k) is a complete DVR and f a regular parameter.

2.1 Some facts about complete local rings

To prove the Cohen Structure Theorem, we first list some preliminary results on complete local rings. Most of them are independently important and can be used in other contexts.

Theorem 19 (Hensel's Lemma)**.** Let $(A, \mathfrak{m}, \mathsf{k})$ be a complete local ring, $f \in A[X]$ a monic polynomial and $\overline{f} \in \mathsf{k}[X]$ its reduction modulo $\mathfrak{m}$. Suppose that $\overline{f} = \overline{g} \cdot \overline{h}$ for some polynomials $\overline{g}, \overline{h} \in \mathsf{k}[X]$ such that $\gcd(\overline{g}, \overline{h}) = 1$. Then the factorization lifts to a unique factorization $f = g \cdot h$ in $A[X]$ such that g and h are monic polynomials.

Proof. Yang: To be completed. ☐

Remark 20. Note that the Hensel's Lemma does not require A to be noetherian.

Lemma 21. Let k be a field of characteristic p. Then there exists a complete DVR (D, p, k) of mixed characteristic $(0, p)$.

Proof. Yang: To be completed. ☐

Lemma 22. Given k a field of characteristic p, there exists a unique complete local ring (R, pR, k) of mixed characteristic (p^k, p).

Proof. Yang: To be completed. ☐

Proposition 23. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian complete local ring and M an A-module that is $\mathfrak{m}$-adically separated. Suppose $\dim_{\mathsf{k}} M/\mathfrak{m}M < \infty$. Then the basis of $M \otimes_A \mathsf{k}$ as k-vector space can be lifted to a generating set of M as an A-module.

Proof. Yang: To be completed. □

2.2 Existence of coefficient rings

Definition 24 (Coefficient rings)**.** Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian complete local ring.
When A is equal-characteristic, the coefficient ring (or coefficient field) is a homomorphism of rings $\mathsf{k} \to A$ such that $\mathsf{k} \to A \to A/\mathfrak{m}$ is an isomorphism.
When A is mixed-characteristic, the coefficient ring is a complete DVR (D, t) with a local homomorphism of rings $D \hookrightarrow A$ such that the induced homomorphism $D/(t) \to A/\mathfrak{m}$ is an isomorphism.

Remark 25. Recall that a homomorphism of local rings $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is said to be local if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

This subsubsection we show the following existence result of coefficient rings due to Cohen.

Theorem 26. Every noetherian complete local ring $(A, \mathfrak{m}, \mathsf{k})$ has a coefficient ring.

Proof of Theorem 26 in characteristic 0. Note that for any $n \in \mathbb{Z}$, $n \notin \mathfrak{m}$. Hence $\mathbb{Q} \subset A$. Let $\Sigma := \{$ subfield in $A\}$ and K a maximal element in Σ with respect to the inclusion. The set Σ is nonempty since $\mathbb{Q} \in \Sigma$. By Zorn's Lemma, K exists. Then K is a subfield of k by $K \hookrightarrow A \twoheadrightarrow A/\mathfrak{m} \cong \mathsf{k}$. We claim that K is a coefficient field of A.

Suppose there is $\bar{t} \in \mathsf{k} \setminus K$. If $\bar{t}$ is transcendent over K, lift $\bar{t}$ to an element $t \in A$. Then for any polynomial $f \neq 0 \in K[T]$, we have $f(\bar{t}) \neq 0 \in \mathsf{k}$. Hence $f(t) \notin \mathfrak{m}$. This implies that $1/f(t) \in A$, whence $K(t) \subset A$. This contradicts the maximality of K. If $\bar{t}$ is algebraic over K, let $f \in K[T]$ be the minimal polynomial of $\bar{t}$. Then f is irreducible in $K[T]$ and $f(\bar{t}) = 0$. Regard f as a polynomial in $A[T]$ by $K \hookrightarrow A$. Note that $\operatorname{char} A = 0$ implies that f is separable. By Hensel's Lemma (Theorem 19), we can lift the root $\bar{t}$ to an element $t \in A$ such that $f(t) = 0$. Then $K(t)$ is a field extension of K and $K(t) \subset A$. This contradicts the maximality of K again. □

The same strategy does not work when $\operatorname{char} \mathsf{k} = p > 0$ since there might be inseparable extensions. To fix this, we need to introduce the notion of p-basis.

Definition 27. Let k be a field of characteristic p. A finite set $\{t_1, \ldots, t_n\} \subset \mathsf{k} \setminus \mathsf{k}^p$ is called *p-independent* if $[\mathsf{k}(t_1, \ldots, t_n) : \mathsf{k}] = p^n$. A set $\Theta \subset \mathsf{k} \setminus \mathsf{k}^p$ is called a *p-independent* if its any finite subset is p-independent. A *p-basis* for k is a maximal p-independent set $\Theta \subset \mathsf{k} \setminus \mathsf{k}^p$.

Proof of Theorem 26 in characteristic p. Choose $\Theta \subset A$ such that its image in $A/\mathfrak{m}$ is a p-basis for k.

Yang: To be completed. □

Proof of Theorem 26 in mixed characteristic. Yang: To be completed. □

2.3 Proof of Cohen Structure Theorem

3 Unique factorization of regular local rings

Theorem 28 (Weierstrass Preparation Theorem)**.** Let $(A, \mathfrak{m})$ be a noetherian complete local ring, $f = \sum_{n=0}^{\infty} a_n X^n \in A[[X]]$ a power series with $f \not\equiv 0 \mod \mathfrak{m}$. Then there exists a unique factorization of the form $f = ug$, where u is a unit in $A[[X]]$ and g is a polynomial of the form

$$g = X^d + b_{d-1}X^{d-1} + \cdots + b_0,$$

where $b_i \in \mathfrak{m}$ for all i.