Projective morphisms and "positive" line bundles

1 Ample line bundles

Definition 1. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. Yang: To be continued.

Theorem 2 (Serre Vanishing)**.** Let X be a projective scheme over a field k and $\mathcal{L}$ an ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that for all $n \geq N$, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

Definition 3. Let $(X, \mathcal{O}_X(1))$ be a projective scheme over a field k and $\mathcal{F}$ a coherent sheaf on X. The *Hilbert polynomial* of $\mathcal{F}$ with respect to $\mathcal{O}_X(1)$ is the polynomial

$$P_{\mathcal{F}}(n) = \chi(X, \mathcal{F} \otimes \mathcal{O}_X(n)) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F} \otimes \mathcal{O}_X(n)).$$

Yang: To be continued.

Let $Z \subseteq X$ be a closed subscheme with ideal sheaf $\mathcal{I}_Z$. The *Hilbert polynomial* of Z with respect to $\mathcal{O}_X(1)$ is defined as $P_Z(n) = P_{\mathcal{O}_X/\mathcal{I}_Z}(n)$. Yang: To be revised.

2 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 4. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Proposition 5. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. Yang: To be continued. □

Example 6. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 7. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}_A^{(m+1)(n+1)-1}$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 8. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. Yang: To be continued.

Example 9. Let

Example 10.

Definition 11. Let $\mathcal{L}$ be a line bundle on a scheme X. Yang: To be continued.

Definition 12. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

Theorem 13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

Proposition 14. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

Yang: To be continued.

Proof. Yang: To be continued. □

3 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 15. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

4 Asymptotic behavior

Definition 16. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

Definition 17. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. Yang: To be continued.

Theorem 18. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \mathrm{Proj}\, R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. Yang: To be continued.

Definition 19. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq Cn^{\dim X}$$

for all sufficiently large n. Yang: To be continued.

Example 20. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. Yang: To be continued.