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# *Surfaces*



“仿造的又如何，当不成真正的勇者也无妨，即便如此，我也是勇者！”

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## 1 The first properties of surfaces

Let  $\mathbb{k}$  be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over  $\mathbb{k}$ .

### 1.1 Basic concepts

### 1.2 Riemann-Roch Theorem for surfaces

### 1.3 Hodge Index Theorem

## 2 Birational geometry on surfaces

Let  $\mathbb{k}$  be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over  $\mathbb{k}$ .

## 2.1 Castelnuovo's Theorem and Run the MMP

**Theorem 2.1** (Castelnuovo's contractibility criterion). Let  $X$  be a smooth projective surface over an algebraically closed field  $\mathbb{k}$ . Let  $C \subseteq X$  be an irreducible curve. Then there exists a birational morphism  $f : X \rightarrow Y$  contracting  $C$  to a smooth point if and only if  $C \cong \mathbb{P}^1$  and  $C^2 = -1$ .

## 3 Coarse classification of surfaces

Let  $\mathbb{k}$  be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over  $\mathbb{k}$ .

Let  $X$  be a smooth projective surface over an algebraically closed field  $\mathbb{k}$ . We want to classify  $X$  up to birational equivalence. Let  $K_X$  be the canonical divisor of  $X$ .

**Theorem 3.1.** Let  $X$  be a smooth projective surface over an algebraically closed field  $\mathbb{k}$ . Suppose that the Kodaira dimension  $\kappa(X) \geq 0$ . Then the linear system  $|12K_X|$  is base point free. Yang: To be checked.

### 3.1 Classification

**Theorem 3.2** (Enriques-Kodaira classification). Let  $X$  be a smooth projective surface over  $\mathbb{k}$ . Then  $X$  is birational to a unique minimal model  $X'$ , unless  $X$  is birational to a ruled surface. Moreover, the minimal model  $X'$  falls into one of the following classes:

- (a)  $\kappa(X') = -\infty$ :  $X' \cong \mathbb{P}^2$  or  $X'$  is a ruled surface;
- (b)  $\kappa(X') = 0$ :  $X'$  is a K3 surface, an abelian surface or their quotients;
- (c)  $\kappa(X') = 1$ :  $X'$  is an elliptic surface;
- (d)  $\kappa(X') = 2$ :  $X'$  is a surface of general type.

Yang: To be checked.

## 4 Ruled Surface

In this section, fix an algebraically closed field  $\mathbb{k}$ . This section is mainly based on [Har77, Chapter V.2].

### 4.1 Minimal Section and Classification

**Definition 4.1** (Ruled surface). A *ruled surface* is a smooth projective surface  $X$  together with a surjective morphism  $\pi : X \rightarrow C$  to a smooth curve  $C$  such that all geometric fibers of  $\pi$  are isomorphic to  $\mathbb{P}^1$ .

Let  $\pi : X \rightarrow C$  be a ruled surface over a smooth curve  $C$  of genus  $g$ .

**Lemma 4.2.** There exists a section of  $\pi$ .

*Proof.* Yang: To be continued... □

**Proposition 4.3.** Then there exists a vector bundle  $\mathcal{E}$  of rank 2 on  $\mathcal{C}$  such that  $X \cong \mathbb{P}_{\mathcal{C}}(\mathcal{E})$  over  $\mathcal{C}$ .

*Proof.* Let  $\sigma : \mathcal{C} \rightarrow X$  be a section of  $\pi$  and  $D$  be its image. Let  $\mathcal{L} = \mathcal{O}_X(D)$  and  $\mathcal{E} = \pi_*\mathcal{L}$ . Since  $D$  is a section of  $\pi$ ,  $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$  for any  $t \in \mathcal{C}$ , whence  $h^0(X_t, \mathcal{L}|_{X_t}) = 2$  for any  $t \in \mathcal{C}$ . By Miracle Flatness (??),  $f$  is flat. By Grauert's Theorem (??),  $\mathcal{E}$  is a vector bundle of rank 2 on  $\mathcal{C}$  and we have a natural isomorphism  $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$  for any  $t \in \mathcal{C}$ .

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{C}}} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every  $x \in X$ , we have

$$\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{C}}} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

Yang: The left side coincides with  $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$  naturally. Hence by Nakayama's Lemma, the natural homomorphism  $\pi^*\mathcal{E} \rightarrow \mathcal{L}$  is surjective.

By ??, we have a morphism  $\varphi : X \rightarrow \mathbb{P}_{\mathcal{C}}(\mathcal{E})$  over  $\mathcal{C}$  such that  $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_{\mathcal{C}}(\mathcal{E})}(1)$ . Since  $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$  for any  $t \in \mathcal{C}$ ,  $\varphi|_{X_t} : X_t \rightarrow \mathbb{P}_{\mathcal{C}}(\mathcal{E})_t$  is an isomorphism for any  $t \in \mathcal{C}$ . Hence  $\varphi$  is bijection on the underlying sets. Yang: Here is a serious gap. Why fiberwise isomorphism implies isomorphism? □

**Lemma 4.4.** It is possible to write  $X \cong \mathbb{P}_{\mathcal{C}}(\mathcal{E})$  such that  $H^0(\mathcal{C}, \mathcal{E}) \neq 0$  but  $H^0(\mathcal{C}, \mathcal{E} \otimes \mathcal{L}) = 0$  for any line bundle  $\mathcal{L}$  on  $\mathcal{C}$  with  $\deg \mathcal{L} < 0$ . Such a vector bundle  $\mathcal{E}$  is called a *normalized vector bundle*. In particular, if  $\mathcal{E}$  is normalized, then  $e = -\deg c_1(\mathcal{E})$  is an invariant of the ruled surface  $X$ .

*Proof.* We can suppose that  $\mathcal{E}$  is globally generated since we can always twist  $\mathcal{E}$  by a sufficiently ample line bundle on  $\mathcal{C}$ . Then for all line bundle  $\mathcal{L}$  of degree sufficiently large,  $\mathcal{L}$  is very ample and hence  $H^0(\mathcal{C}, \mathcal{E} \otimes \mathcal{L}) \neq 0$ . By Lemma 4.2 and ??,  $\mathcal{E}$  is an extension of line bundles. Then for all line bundle  $\mathcal{L}$  of degree sufficiently negative,  $H^0(\mathcal{C}, \mathcal{E} \otimes \mathcal{L}) = 0$  since line bundles of negative degree have no global sections. Hence we can find a line bundle  $\mathcal{M}$  on  $\mathcal{C}$  of lowest degree such that  $H^0(\mathcal{C}, \mathcal{E} \otimes \mathcal{M}) \neq 0$ . Replacing  $\mathcal{E}$  by  $\mathcal{E} \otimes \mathcal{M}$ , we are done. □

**Remark 4.5.** The invariant  $e$  is unique but the normalization of  $\mathcal{E}$  is not unique. For example, if  $\mathcal{E}$  is normalized, then so is  $\mathcal{E} \otimes \mathcal{L}$  for any line bundle  $\mathcal{L}$  on  $\mathcal{C}$  of degree 0. Yang: To be continued...

Suppose that  $X \cong \mathbb{P}_{\mathcal{C}}(\mathcal{E})$  where  $\mathcal{E}$  is a normalized vector bundle of rank 2 on  $\mathcal{C}$ . Since  $H^0(\mathcal{C}, \mathcal{E}) \neq 0$ , choosing a non-zero section  $s$ , we get an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{C}} \xrightarrow{s} \mathcal{E} \rightarrow \mathcal{E}/\mathcal{O}_{\mathcal{C}} \rightarrow 0.$$

We claim that  $\mathcal{E}/\mathcal{O}_{\mathcal{C}}$  is a line bundle on  $\mathcal{C}$ . Since  $\mathcal{C}$  is a curve, we only need to check that  $\mathcal{E}/\mathcal{O}_{\mathcal{C}}$  is torsion-free.

Yang: To be continued...

**Definition 4.6.** A section  $\mathcal{C}_0$  of  $\pi$  is called a *minimal section* if Yang: to be continued...

**Lemma 4.7.** Let  $X = \mathbb{P}_C(\mathcal{E}) \rightarrow C$  be a ruled surface over a smooth curve  $C$  of genus  $g$  with invariant  $e$  and normalized  $\mathcal{E}$ .

- (a) If  $\mathcal{E}$  is decomposable, then  $e \geq 0$  and  $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$  where  $\mathcal{L}$  is a line bundle on  $C$  with  $\deg \mathcal{L} = -e$ .
- (b) If  $\mathcal{E}$  is indecomposable, then  $-2g \leq e \leq 2g - 2$ .

*Proof.* If  $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$  is decomposable, we can assume that  $H^0(C, \mathcal{L}_1) \neq 0$ . If  $\deg \mathcal{L}_1 > 0$ , then  $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$ , contradicting the normalization of  $\mathcal{E}$ . Similarly  $\deg \mathcal{L}_2 \leq 0$ . Then  $\mathcal{L}_1 \cong \mathcal{O}_C$ . And hence  $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$ .

If  $\mathcal{E}$  is indecomposable, we have an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$$

which is a non-trivial extension, with  $\mathcal{L}$  a line bundle on  $C$  of degree  $-e$ . Hence by ??, we have  $0 \neq \text{Ext}_C^1(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$ . By Serre duality, we have  $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$ . Hence  $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$ .

On the other hand, let  $\mathcal{M}$  be a line bundle on  $C$  of degree  $-1$ . Twist the above exact sequence by  $\mathcal{M}$  and take global sections, we have an equation

$$h^0(\mathcal{M}) - h^0(\mathcal{E} \otimes \mathcal{M}) + h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{M}) + h^1(\mathcal{E} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = 0.$$

Since  $\deg \mathcal{M} < 0$  and  $\mathcal{E}$  is normalized, we have  $h^0(\mathcal{M}) = h^0(\mathcal{E} \otimes \mathcal{M}) = 0$ . By Riemann-Roch, we have  $h^1(\mathcal{M}) = g$  and  $h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = -e - 1 + 1 - g$ . Hence

$$h^1(\mathcal{E} \otimes \mathcal{M}) = e + 2g \geq 0.$$

This gives  $e \geq -2g$ . □

**Theorem 4.8.** Let  $\pi : X \rightarrow C$  be a ruled surface over  $C = \mathbb{P}^1$  with invariant  $e$ . Then  $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$ .

*Proof.* This is a direct consequence of Lemma 4.7. □

**Example 4.9.** Here we give an explicit description of the ruled surface  $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$  for  $e \geq 0$ .

Let  $C$  be covered by two standard affine charts  $U_0, U_1$  with coordinate  $u$  on  $U_0$  and  $v$  on  $U_1$  such that  $u = 1/v$  on  $U_0 \cap U_1$ . On  $U_i$ , let  $\mathcal{O}(-e)|_{U_i}$  be generated by  $s_i$  for  $i = 0, 1$ . We have  $s_0 = u^e s_1$  on  $U_0 \cap U_1$ .

On  $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$ , let  $[x_0 : x_1]$  and  $[y_0 : y_1]$  be the homogeneous coordinates of  $\mathbb{P}^1$  on  $X_0$  and  $X_1$  respectively. Then the transition function on  $X_0 \cap X_1$  is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

**Remark 4.10.** The surface  $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$  is also called the *Hirzebruch surface*.

**Theorem 4.11.** Let  $\pi : X = \mathbb{P}_E(\mathcal{E}) \rightarrow E$  be a ruled surface over an elliptic curve  $E$  with invariant  $e$  and normalized  $\mathcal{E}$ .

- (a) If  $\mathcal{E}$  is indecomposable, then  $e = 0$  or  $-1$ , and for each  $e$  there exists a unique such ruled surface up to isomorphism.
- (b) If  $\mathcal{E}$  is decomposable, then  $e \geq 0$  and  $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$  where  $\mathcal{L}$  is a line bundle on  $E$  with  $\deg \mathcal{L} = -e$ .

*Proof.* Only the indecomposable case needs a proof. By Lemma 4.7, we have  $-2 \leq e \leq 0$  and a non-trivial extension

$$0 \rightarrow \mathcal{O}_E \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$$

where  $\mathcal{L}$  is a line bundle on  $E$  of degree  $-e$ .

**Case 1.**  $e = 0$ .

In this case,  $\mathcal{L}$  is of degree 0 and  $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \neq 0$ . Hence  $\mathcal{L} \cong \mathcal{O}_E$ .

Yang: To be continued...

**Case 2.**  $e = -1$ .

In this case,  $\mathcal{L}$  is of degree 1 and  $H^1(E, \mathcal{L}) \cong H^0(E, \mathcal{L}^{-1}) = 0$ . By Riemann-Roch, we have  $h^0(E, \mathcal{L}) = 1$ .

**Case 3.**  $e = -2$ .

Yang: To be continued...

□

**Example 4.12.** Yang: To be continued...

## 4.2 The Néron-Severi Group of Ruled Surfaces

**Proposition 4.13.** Let  $\pi : X \rightarrow \mathcal{C}$  be a ruled surface over a smooth curve  $\mathcal{C}$  of genus  $g$ . Let  $C_0$  be a minimal section of  $\pi$  and  $F$  a fiber of  $\pi$ . Then  $\text{Pic}(X) \cong \mathbb{Z}[C_0] \oplus \pi^* \text{Pic}(\mathcal{C})$ .

*Proof.* Let  $D$  be any divisor on  $X$  with  $D.F = a \in \mathbb{Z}$ . Then  $D - aC_0$  is numerically trivial on the fibers of  $\pi$ . Let  $\mathcal{L} = \mathcal{O}_X(D - aC_0)$ . Then  $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$  for any  $t \in \mathcal{C}$ . By Grauert's Theorem (??),  $\pi_* \mathcal{L}$  is a line bundle on  $\mathcal{C}$ . Yang: and the natural map  $\pi^* \pi_* \mathcal{L} \rightarrow \mathcal{L}$  is an isomorphism. □

**Proposition 4.14.** Let  $\pi : X \rightarrow \mathcal{C}$  be a ruled surface over a smooth curve  $\mathcal{C}$  of genus  $g$ . Let  $C_0$  be a minimal section of  $\pi$  and let  $F$  be a fiber of  $\pi$ . Then  $K_X \sim -2C_0 + \pi^*(K_{\mathcal{C}} - c_1(\mathcal{E}))$ . Numerically, we have  $K_X \equiv -2C_0 + (2g - 2 - e)F$  where  $e$  is the invariant of  $X$ . Yang: Check this carefully.

*Proof.* Yang: To be continued. □

**Rational case.** Let  $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \rightarrow \mathbb{P}^1$  be a ruled surface over  $\mathbb{P}^1$  with  $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$  for some  $e \geq 0$ .

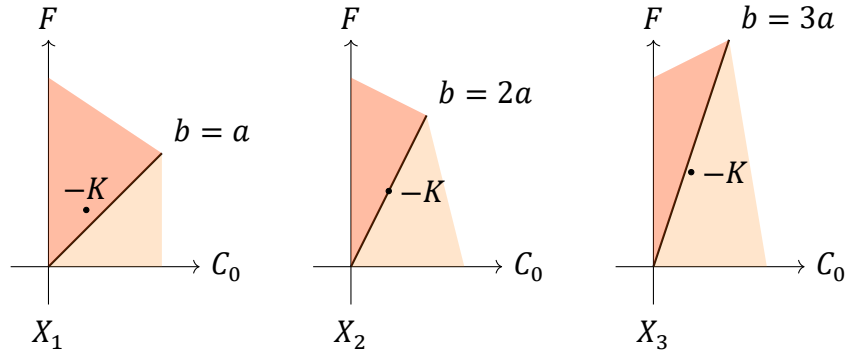
**Theorem 4.15.** Let  $\pi : X \rightarrow \mathbb{P}^1$  be a ruled surface over  $\mathbb{P}^1$  with invariant  $e$ . Let  $C_0$  be a minimal section of  $\pi$  and let  $F$  be a fiber of  $\pi$ . Let  $D \sim aC_0 + bF$  be a divisor on  $X$  with  $a, b \in \mathbb{Z}$ .

- (a)  $D$  is effective  $\iff a, b \geq 0$ ;

(b)  $D$  is ample  $\iff D$  is very ample  $\iff a > 0$  and  $b > ae$ .

*Proof.* Yang: To be continued... □

**Example 4.16.** Here we draw the Néron-Severi group of the rational ruled surface  $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$  for  $e = 1, 2, 3$ .



We have  $-K_{X_e} \equiv 2C_0 + (2+e)F$ . For  $e = 1$ ,  $-K$  is ample and hence  $X_1$  is a del Pezzo surface. For  $e = 2$ ,  $-K$  is nef and big but not ample. For  $e \geq 3$ ,  $-K$  is big but not nef.

**Elliptic case.** Let  $\pi : X = \mathbb{P}_E(\mathcal{E}) \rightarrow E$  be a ruled surface over an elliptic curve  $E$  with  $\mathcal{E}$  a normalized vector bundle of rank 2 and degree  $-e$ .

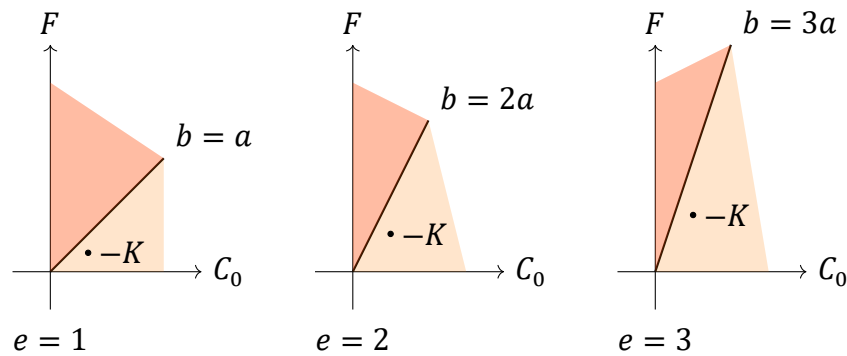
**Theorem 4.17.** Let  $\pi : X \rightarrow E$  be a ruled surface over an elliptic curve  $E$  with invariant  $e$ . Assume that  $\mathcal{E}$  is decomposable. Let  $C_0$  be a minimal section of  $\pi$  and let  $F$  be a fiber of  $\pi$ . Let  $D \equiv aC_0 + bF$  be a divisor on  $X$  with  $a, b \in \mathbb{Z}$ .

(a)  $D$  is effective  $\iff a \geq 0$  and  $b \geq ae$ ;

(b)  $D$  is ample  $\iff D$  is very ample  $\iff a > 0$  and  $b > ae$ .

*Proof.* Yang: To be continued... □

**Example 4.18.** Here we draw the Néron-Severi group of the ruled surface  $X$  over an elliptic curve  $E$  with decomposable normalized  $\mathcal{E}$  for  $e = 1, 2, 3$ .



In this case,  $-K \equiv 2C_0 + eF$  is always big but not nef.

**Theorem 4.19.** Let  $\pi : X \rightarrow E$  be a ruled surface over an elliptic curve  $E$  with invariant  $e$ . Assume that  $\mathcal{E}$  is indecomposable. Let  $C_0$  be a minimal section of  $\pi$  and let  $F$  be a fiber of  $\pi$ . Let  $D \equiv aC_0 + bF$

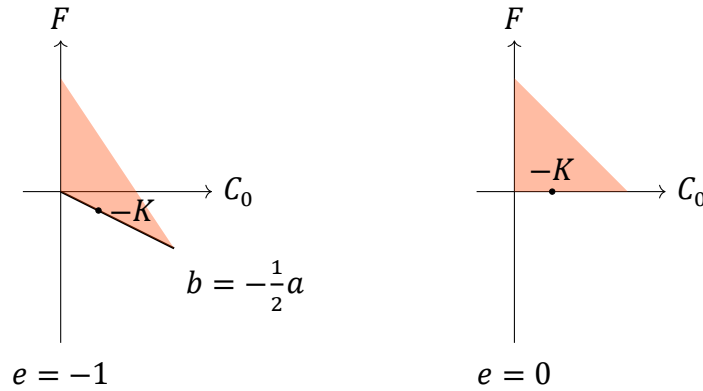
be a divisor on  $X$  with  $a, b \in \mathbb{Z}$ .

(a)  $D$  is effective  $\iff a \geq 0$  and  $b \geq \frac{1}{2}ae$ ;

(b)  $D$  is ample  $\iff D$  is very ample  $\iff a > 0$  and  $b > \frac{1}{2}ae$ .

*Proof.* Yang: To be continued... □

**Example 4.20.** Here we draw the Néron-Severi group of the ruled surface  $X$  over an elliptic curve  $E$  with indecomposable normalized  $\mathcal{E}$  for  $e = -1, 0$ .



In this case,  $-K \equiv 2C_0 + eF$  is always nef but not big.

**Proposition 4.21.** Let  $\pi : X \rightarrow \mathcal{C}$  be a ruled surface over a smooth curve  $\mathcal{C}$ . Then every nef divisor on  $X$  is semi-ample. Yang: Check this carefully.

## 5 K3 surface

Let  $\mathbb{k}$  be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over  $\mathbb{k}$ .

### 5.1

**Definition 5.1.** A *K3 surface* is a smooth, projective surface  $X$  with trivial canonical bundle  $K_X \cong \mathcal{O}_X$  and irregularity  $q(X) = h^1(X, \mathcal{O}_X) = 0$ .

**Example 5.2.** A smooth quartic surface  $X \subseteq \mathbb{P}^3$  is a K3 surface. Indeed, by the adjunction formula, we have

$$K_X = (K_{\mathbb{P}^3} + X)|_X = (-4H + 4H)|_X = 0,$$

where  $H$  is a hyperplane in  $\mathbb{P}^3$ . Moreover, by the Lefschetz hyperplane theorem, we have  $h^1(X, \mathcal{O}_X) = h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$ . Yang: To be checked.

### 5.2 Hodge Structure and Moduli of K3 surfaces