First properties of algebraic groups

Let $\mathbf{k}$ be a field and $\bar{\mathbb{k}}$ its algebraic closure. All varieties are defined over $\mathbf{k}$ unless otherwise specified.

1 Basic concepts

Definition 1. An *algebraic group* is a group object in the category of algebraic varieties, i.e. an algebraic variety G together with morphisms Yang: To be continued...

Proposition 2. Let G be an algebraic group. Then G is a smooth variety over $\mathbf{k}$.

Example 3. The *additive group* $\mathbb{G}_a$ is defined to be the affine line $\mathbb{A}^1$ with the group law given by addition. Concretely, we can write $\mathbb{G}_a = \operatorname{Spec} \mathbf{k}[T]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a &\quad \mathbf{k}[T] \to \mathbf{k}[T] \otimes_{\mathbf{k}} \mathbf{k}[T], \quad T \mapsto T \otimes 1 + 1 \otimes T. \\ \iota : \mathbb{G}_a \to \mathbb{G}_a &\quad \mathbf{k}[T] \to \mathbf{k}[T], \quad T \mapsto -T. \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_a &\quad \mathbf{k}[T] \to \mathbf{k}, \quad T \mapsto 0. \end{aligned}$$

Yang: To be continued...

Example 4. The *multiplicative group* $\mathbb{G}_m$ is defined to be the affine variety $\mathbb{A}^1 \setminus \{0\}$ with the group law given by multiplication. Concretely, we can write $\mathbb{G}_m = \operatorname{Spec} \mathbf{k}[T, T^{-1}]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m &\rightsquigarrow \mathbf{k}[T, T^{-1}] \to \mathbf{k}[T, T^{-1}] \otimes_{\mathbf{k}} \mathbf{k}[T, T^{-1}], \quad T \mapsto T \otimes T. \\ \iota : \mathbb{G}_m \to \mathbb{G}_m &\rightsquigarrow \mathbf{k}[T, T^{-1}] \to \mathbf{k}[T, T^{-1}], \quad T \mapsto T^{-1}. \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_m &\rightsquigarrow \mathbf{k}[T, T^{-1}] \to \mathbf{k}, \quad T \mapsto 1. \end{aligned}$$

Yang: To be continued...

Example 5. The *general linear group* GL_n is defined to be the open subvariety of $\mathbb{A}^{n^2}$ consisting of invertible matrices, with the group law given by matrix multiplication. Concretely, we can write $\mathrm{GL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}, \det(T_{ij})^{-1}]$ where $1 \le i, j \le n$ and the group law is given by the morphism Yang: To be continued...

Example 6. An *elliptic curve* is a smooth projective curve of genus 1 with a specified point O. Given an elliptic curve E, we can define a group law on E using the chord-tangent process. Concretely, for any two points $P, Q \in E$, we can define their sum $P + Q$ as follows:

- If $P \neq Q$, then let L be the line passing through P and Q. Since E is a cubic curve, L intersects E at a third point R. Then we define $P + Q$ to be the point obtained by reflecting R across the x-axis (i.e., if $R = (x, y)$, then $P + Q = (x, -y)$).
- If $P = Q$, then let L be the tangent line to E at P. Again, since E is a cubic curve, L intersects E at a second point R. Then we define $2P = P + P$ to be the point obtained by reflecting R across the x-axis.

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- The identity element is the specified point O.
- The inverse of a point $P = (x, y)$ is given by $-P = (x, -y)$.

This group law makes E into an algebraic group. We can identify the $\Bbbk$-points of E with the set of points on the elliptic curve defined over $\Bbbk$. Yang: To be continued...

Definition 7. Let G be an algebraic group and $x \in G(\mathbf{k})$ a $\mathbf{k}$-point. The *left translation* by x is the morphism

$$L_x : G \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{\mu} G,$$

Definition 8. An *algebraic subgroup* of an algebraic group G is a closed subvariety $H \subseteq G$ that is also a subgroup of G. In other words, the inclusion morphism $H \hookrightarrow G$ is a morphism of algebraic groups.

Example 9. The *special linear group* SL_n is defined to be the closed subvariety of GL_n consisting of matrices with determinant equal to 1, with the group law given by matrix multiplication. Concretely, we can write $\mathrm{SL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}]/(\det(T_{ij}) - 1)$ where $1 \leq i, j \leq n$ and the group law is given by the morphism Yang: To be continued...

Definition 10. Let G and H be algebraic groups. The *product* $G \times H$ is an algebraic group with the group law defined by

$$\mu_{G \times H} = (\mu_G, \mu_H) : (G \times H) \times (G \times H) \cong (G \times G) \times (H \times H) \to G \times H,$$

where μ_G and μ_H are the group laws of G and H, respectively. Yang: To be continued...

Definition 11. Let G be an algebraic group. The *neutral component* G^0 is the connected component of G containing the identity element ε. Yang: To be continued...

Proposition 12. The neutral component G^0 is a closed, normal algebraic subgroup of G of finite index. Moreover, each closed subgroup H of finite index contains G^0.

Proof. Yang: To be continued... □

Definition 13. A *homomorphism* of algebraic groups is a morphism of varieties that is also a group homomorphism. In other words, a morphism $f : G \to H$ between algebraic groups G and H is a homomorphism if the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f \times f"] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

where μ_G and μ_H are the group laws of G and H, respectively. Yang: To be continued...

Proposition 14. Let G be an algebraic group and $H \subseteq G$ a subgroup (not necessarily closed). Then the Zariski closure $\overline{H}$ of H in G is an algebraic subgroup of G. Yang: To be continued...

Proof. Yang: To be continued... □

Proposition 15. Let G be an algebraic group, Y_i irreducible constructible subsets of G containing the identity element for $i = 1, \dots, n$. Then the closed subgroup Yang: To be continued...

Proof. Yang: To be continued... □

Remark 16. We can take $n \leq 2 \dim G$. Yang: To be continued...

2 Action and representations

Definition 17. An *action* of an algebraic group G on a variety X is a morphism

$$\sigma : G \times X \to X$$

such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu\times\mathrm{id}_X"] \arrow[d, "\mathrm{id}_G\times\sigma"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}\mathbf{k} \times X \arrow[r, "\varepsilon\times\mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

where μ is the group law of G and ε is the identity element of G. In other words, for any field extension $K/\mathbf{k}$, the induced map $G(K) \times X(K) \to X(K)$ defines a group action of the abstract group $G(K)$ on the set $X(K)$. We say that X is a G-variety. Yang: To be continued...

Example 18. A *linear representation* of an algebraic group G on a finite-dimensional vector space V over $\mathbf{k}$ is an action of G on the affine space associated to V, i.e. a morphism

$$\rho : G \times V \to V$$

such that for any field extension $K/\mathbf{k}$, the induced map $G(K) \times V(K) \to V(K)$ defines a group homomorphism from the abstract group $G(K)$ to the general linear group of the vector space $V(K)$. In other words, for any $g \in G(K)$, the map $\rho_g : V(K) \to V(K)$ defined by $\rho_g(v) = \rho(g, v)$ is a linear automorphism of $V(K)$. We say that V is a G-module. Yang: To be continued...