# Algebraic spaces and stacks



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# 1 Preliminaries in Category Theory

## 1.1 Sites

**Definition 1.1.** Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of sets of arrows $\{U_i \to U\}_{i\in I}$, called *covering*, for each object $U$ in $\mathbf{C}$ such that:

(a) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering;

(b) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a arrow, then the fiber product $U_i \times_U V \to V$ exists and $\{U_i \times_U V \to V\}$ is a covering of $V$;

(c) if $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ are coverings, then the collection of composition $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{j})$ where $\mathbf{C}$ is a category and $\mathcal{j}$ is a Grothendieck topology on $\mathbf{C}$.

Note that sheaf is indeed defined on a site.

**Definition 1.2.** Let $(\mathbf{C}, \mathcal{j})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{j})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ satisfying the following condition: for every object $U$ in $\mathbf{C}$ and every covering $\{U_i \to U\}_{i\in I}$ of $U$, if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every $i, j$, the pullback $s_i|_{U_i\times_U U_j}$ and $s_j|_{U_i\times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every $i$, the pullback $s|_{U_i} = s_i$.

**Definition 1.3.** Let $X$ be a scheme. The *big étale site* of $X$, denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over $X$ with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if and only if each $U_i$ is étale over $U$ and the union of their images is the whole $U$.

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Let $X$ be a scheme over $S$. By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \to \mathbf{Set}$ such that for any $S$-scheme $T$, $h_X(T) = \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$. Yang: Easy to check that $h_X$ is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

## 1.2 Fibered categories and descent conditions

**Definition 1.4.** Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. A morphism $f : b \to a$ in $\mathbf{X}$ is called *strongly Cartesian* if for every object $c \in \mathrm{Obj}(\mathbf{X})$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{X}}(c,b) \arrow[r, "f\circ-"] \arrow[d, "\mathbf{p}"'] & \mathrm{Hom}_{\mathbf{X}}(c,a) \arrow[d, "\mathbf{p}"] \\
\mathrm{Hom}_{\mathbf{S}}(w,v) \arrow[r, "\mathbf{p}(f)\circ-"] & \mathrm{Hom}_{\mathbf{S}}(w,u)
\end{tikzcd}$$

is a pullback of sets, where $u = \mathbf{p}(a), v = \mathbf{p}(b), w = \mathbf{p}(c)$.

The condition in Definition 1.4 can be interpreted as follows: for any diagram as below black part with $\mathbf{p}(g) = \mathbf{p}(f) \circ \alpha$,

$$\begin{tikzcd}
c \arrow[dd, mapsto] \arrow[dr, gray, "h"] \arrow[drr, bend left, "g"] & & \\
& b \arrow[r, "f"] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
w \arrow[r, "\alpha"] & v \arrow[r, "\mathbf{p}(f)"] & u
\end{tikzcd}$$

there exists a unique gray morphism $h : c \to a$ such that $\mathbf{p}(h) = \alpha$ and $f \circ h = g$.

**Notation 1.5.** Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. For $a, b \in \mathrm{Obj}(\mathbf{X})$ and $f \in \mathrm{Hom}_{\mathbf{X}}(a, b)$, we say that $a$ is *over* $\mathbf{p}(a)$ and $f$ is *over* $\mathbf{p}(f)$. In a diagram, we have

$$\begin{tikzcd}
\mathbf{X} \arrow[d, "\mathbf{p}"'] & a \arrow[r, "f"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\
\mathbf{S} & \mathbf{p}(a) \arrow[r, "\mathbf{p}(f)"] & \mathbf{p}(b)
\end{tikzcd}$$

**Definition 1.6.** Let $\mathbf{S}$ be a category. A category $\mathbf{X}$ over $\mathbf{S}$ via $\mathbf{p}$ is called a *category fibred* over the site $\mathbf{S}$ if for every morphism $\iota : v \to u$ in $\mathbf{S}$ and every object $a \in \mathrm{Obj}(\mathbf{X})$ over $u$, there exists an object $b \in \mathrm{Obj}(\mathbf{X})$ over $v$ and a strongly Cartesian morphism $f : b \to a$ over $\iota$. Such an object $b$ is called a *pullback* of $a$ along $\iota$, and is often denoted by $\iota^* a$.

**Definition 1.7.** Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a category fibred over $\mathbf{S}$. For every object $u \in \mathrm{Obj}(\mathbf{S})$, the *fiber* of $\mathbf{X}$ over $u$ is the category $\mathbf{X}_u$ given by

$$\mathrm{Obj}(\mathbf{X}_u) = \{a \in \mathrm{Obj}(\mathbf{X}) \mid \mathbf{p}(a) = u\}, \quad \mathrm{Hom}_{\mathbf{X}_u}(a,b) = \{f \in \mathrm{Hom}_{\mathbf{X}}(a,b) \mid \mathbf{p}(f) = \mathrm{id}_u\}.$$

**Remark 1.8.** Note that in Definition 1.6, the pullback $r^* b$ of an object $b$ along a morphism $r$ is not necessarily unique. Yang: To be continued.

**Example 1.9.** Let $\mathbf{S}$ be a category and $\mathcal{F} : \mathbf{S}^{op} \to \mathbf{Set}$ be a presheaf on $\mathbf{S}$ taking values in $\mathbf{Set}$. We can construct a category $\mathbf{F}$ fibred over $\mathbf{S}$ as follows:

- The objects of $\mathbf{F}$ are pairs $(U, x)$ where $U \in \mathrm{Obj}(\mathbf{S})$ and $x \in \mathcal{F}(U)$;
- morphisms from $(V, y)$ to $(U, x)$ in $\mathbf{F}$ are morphisms $\iota : V \to U$ in $\mathbf{S}$ such that $\mathcal{F}(\iota)(x) = y$, denoted by $\mathrm{res}_\iota$.

The functor $\mathbf{p} : \mathbf{F} \to \mathbf{S}$ is defined by $\mathbf{p}(U, x) = U$ on objects and $\mathbf{p}(\iota) = \iota$ on morphisms. If one has the diagram

$$\begin{tikzcd}
(W,z) \arrow[d, mapsto] \arrow[rrrr, bend left, "\mathrm{res}_\tau"] & & (V,y) \arrow[rr, "\mathrm{res}_\iota"] \arrow[d, mapsto] & & (U,x) \arrow[d, mapsto] \\
W \arrow[rr, "\sigma"] & & V \arrow[rr, "\iota"] & & U
\end{tikzcd}$$

with $\mathbf{p}(\mathrm{res}_\tau) = \iota \circ \sigma$. By definition, we have $\tau = \iota \circ \sigma$ and $\mathcal{F}(\tau)(x) = z, \mathcal{F}(\iota)(x) = y$. Thus, we have $\mathcal{F}(\sigma)(y) = z$. This verifies that $\mathrm{res}_\sigma$ is a strongly Cartesian morphism. Note that the fiber of $\mathbf{F}$ over an $U \in \mathrm{Obj}(\mathbf{S})$ is the discrete category associated to the set $\mathcal{F}(U)$. Therefore, presheaves of sets can be viewed as categories fibred in sets.

Conversely, given a category $\mathbf{F}$ fibred in sets over $\mathbf{S}$ via $\mathbf{p} : \mathbf{F} \to \mathbf{S}$, one can construct a presheaf of sets $\mathcal{F} : \mathbf{S}^{op} \to \mathbf{Set}$ by defining $\mathcal{F}(U) = \mathrm{Obj}(\mathbf{F}_U)$ for each $U \in \mathrm{Obj}(\mathbf{S})$, and for each morphism $\iota : V \to U$ in $\mathbf{S}$, defining $\mathcal{F}(\iota) : \mathcal{F}(U) \to \mathcal{F}(V)$ by sending an object $x \in \mathcal{F}(U)$ to its pullback $\iota^* x \in \mathcal{F}(V)$ along $\iota$. This establishes an equivalence between presheaves of sets on $\mathbf{S}$ and categories fibred in sets over $\mathbf{S}$.

**Example 1.10.** Yang: case $\mathbf{S} = set, group$. To be added.

**Slogan** *Presheaves of sets are categories fibered in sets.*

In following, we describe categories fibered in groupoids.

**Definition 1.11.** Let $\mathbf{X}$ be a category fibred over a category $\mathbf{S}$ via $\mathbf{p} : \mathbf{X} \to \mathbf{S}$. For every $u \in \mathrm{Obj}(\mathbf{S})$ and every pair of objects $a, b$ over $u$, we define the *presheaf of morphisms* $\mathrm{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ by

$$\mathrm{Hom}_{\mathbf{X}}(a, b)(\iota : v \to u) = \mathrm{Hom}_{\mathbf{X}_v}(\iota^* a, \iota^* b)$$

for every morphism $\iota : v \to u$ in $\mathbf{S}/u$. For a morphism $\alpha : w \to v$ in $\mathbf{S}/u$, the restriction map

$$\mathrm{Hom}_{\mathbf{X}}(a, b)(\iota) \to \mathrm{Hom}_{\mathbf{X}}(a, b)(\iota \circ \alpha)$$

is given by sending a morphism $f : \iota^* a \to \iota^* b$ in $\mathbf{X}_v$ to the pullback morphism Yang: $\alpha^* f : (\iota \circ \alpha)^* a \to (\iota \circ \alpha)^* b$ need to conjugate with a natural transformation. in $\mathbf{X}_w$. Yang: To be checked.

In a diagram, the presheaf of morphisms can be visualized as follows:

$$
\begin{tikzcd}
& & & \iota^*a \arrow[rr] \arrow[dr, bend right] \arrow[dr, "f"] \arrow[ddd, mapsto] & & a \arrow[ddd, mapsto] & \\
& \tau^*a \arrow[dr, bend right] \arrow[dr, "f|_w"] \arrow[ddd, mapsto] & & & \iota^*b \arrow[rr] \arrow[ddl, mapsto] & & b \arrow[ddl, mapsto] \\
& & \tau^*b \arrow[ddl, mapsto] & & & & \\
& & & v \arrow[rr, "\iota"] & & u & \\
& w \arrow[urr, "\alpha"] \arrow[urrrr, "\tau"'] & & & & &
\end{tikzcd}
$$

**Proposition 1.12.** Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a category fibred over $\mathbf{S}$. Then $\mathbf{X}$ is a category fibred in groupoids if and only if for every object $u \in \mathsf{Obj}(\mathbf{S})$ and every pair of objects $a, b$ over $u$, the presheaf of morphisms $\mathrm{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf. Yang: To be checked.

**Definition 1.13.** Let $\mathbf{S}$ be a category. A category $\mathbf{X}$ fibred over $\mathbf{S}$ via $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *category fibred in groupoids* over $\mathbf{S}$ if for every object $u \in \mathsf{Obj}(\mathbf{S})$ and every pair of objects $a, b$ over $u$, the presheaf of morphisms $\mathrm{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf. Yang: To be checked.

Now let us discuss how sheaves fit into the framework of fibered categories. Of course, we need assume the base category $\mathbf{S}$ is a site. The glued condition for sheaves can be interpreted in terms of descent data in fibered categories.

**Definition 1.14.** Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. Let $U \in \mathsf{Obj}(\mathbf{S})$ and $\{U_i \to U\}$ be a covering in $\mathbf{S}$. A *descent datum* for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ consists of

- a collection of objects $a_i \in \mathsf{Obj}(\mathbf{X}_{U_i})$ for each $i$,
- a collection of isomorphisms $\varphi_{ij} : a_j|_{U_{ij}} \to a_i|_{U_{ij}}$ in $\mathbf{X}_{U_{ij}}$ for each pair $(i, j)$, where $U_{ij} = U_i \times_U U_j$,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}}$$

holds for all triples $(i, j, k)$, where $U_{ijk} = U_i \times_U U_j \times_U U_k$. Yang: To be checked.

**Example 1.15.** Yang: To be added.

**Definition 1.16.** Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. A descent datum $(\{a_i\}, \{\varphi_{ij}\})$ for objects of $\mathbf{X}$ relative to a covering $\{U_i \to U\}$ in $\mathbf{S}$ is called *effective* if there exists an object $a \in \mathsf{Obj}(\mathbf{X}_U)$ and isomorphisms $\psi_i : a|_{U_i} \to a_i$ in $\mathbf{X}_{U_i}$ such that for all pairs $(i, j)$, the diagram

$$
\begin{tikzcd}
a|_{U_{ij}} \arrow[r, "\psi_j|_{U_{ij}}"] \arrow[d, "\psi_i|_{U_{ij}}"'] & a_j|_{U_{ij}} \arrow[d, "\varphi_{ij}"] \\
a_i|_{U_{ij}} \arrow[r, "\varphi_{ij}"'] & a_j|_{U_{ij}}
\end{tikzcd}
$$

commutes. Yang: To be checked.

**Slogan** *Descent data are like gluing data for objects, and effectiveness means that the glued object exists.*

## 1.3 Prestacks and stacks

**Definition 1.17.** A *prestack* over the site $\mathbf{S}$ is a category $\mathbf{X}$ fibered in groupoids over $\mathbf{S}$.

**Slogan** *Prestacks are "presheaf remembering automorphisms".*

**Example 1.18.** presheaf is a prestack. Yang: To be added.

**Example 1.19.** The moduli problem of classifying algebraic curves of a fixed genus $g$ can be formulated as a prestack over the site of schemes. Consider the category $\mathbf{M}_g$ whose objects are families of smooth projective curves of genus $g$ over schemes, and whose morphisms are isomorphisms of such families. The functor $\mathbf{p} : \mathbf{M}_g \to \mathbf{Sch}$ sending a family of curves to its base scheme makes $\mathbf{M}_g$ a category fibred in groupoids over $\mathbf{Sch}$. For each scheme $S$, the fiber category $\mathbf{M}_{g,S}$ consists of families of smooth projective curves of genus $g$ over $S$ and their isomorphisms. The descent data for objects in $\mathbf{M}_g$ relative to a covering of schemes correspond to gluing families of curves along isomorphisms on overlaps, which is effective due to the nature of algebraic curves. Thus, $\mathbf{M}_g$ is a prestack over the site of schemes. Yang: To be revised.

**Proposition 1.20.** Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$, $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$, and $\mathbf{r} : \mathbf{Z} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Let $\Phi : \mathbf{X} \to \mathbf{Z}$ and $\Psi : \mathbf{Y} \to \mathbf{Z}$ be morphisms of prestacks over $\mathbf{S}$. Then the fiber product $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ exists in the category of prestacks over $\mathbf{S}$. Yang: To be checked.

**Definition 1.21.** Let $\mathbf{S}$ be a site. A prestack $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *stack* over the site $\mathbf{S}$ if for every object $U \in \mathrm{Obj}(\mathbf{S})$ and every covering $\{U_i \to U\}$ in $\mathbf{S}$, the descent data for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ are effective. Yang: To be revised.

**Definition 1.22.** Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ and $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$ be stacks over $\mathbf{S}$. A *morphism of stacks* $F : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$ is a functor $F : \mathbf{X} \to \mathbf{Y}$ such that $\mathbf{q} \circ F = \mathbf{p}$. Yang: To be checked.

**Slogan** *Stacks are to prestacks as sheaves are to presheaves.*

**Example 1.23.** Let $X$ be a scheme over a base noetherian scheme $S$. The functor of points $h_X : (\mathbf{Sch}/S)^{\mathrm{op}}_{\text{ét}} \to \mathbf{Set}$ is a sheaf, and thus a stack.

**Construction 1.24.** Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ be a prestack over $\mathbf{S}$. There exists a stack $\mathbf{p}^+ : \mathbf{X}^+ \to \mathbf{S}$ over $\mathbf{S}$ together with a morphism of prestacks $F : \mathbf{X} \to \mathbf{X}^+$ over $\mathbf{S}$ satisfying the following universal property: for every stack $\mathbf{p}' : \mathbf{Y} \to \mathbf{S}$ over $\mathbf{S}$ and every morphism of prestacks $G : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$, there exists a unique morphism of stacks $G^+ : \mathbf{X}^+ \to \mathbf{Y}$ over $\mathbf{S}$ such that $G = G^+ \circ F$. The stack $\mathbf{X}^+$ is called the *stackification* of the prestack $\mathbf{X}$. Yang: To be checked.

**Notation 1.25.** As Example 1.9, we can associate a prestack $\mathbf{X}$ over a $\mathbf{S}$ to a functor $\mathcal{X} : \mathbf{S}^{\mathrm{op}} \to \mathbf{Grpd}$ by setting $\mathbf{X}_u = \mathcal{X}(u)$ for each $u \in \mathrm{Obj}(\mathbf{S})$ and defining the pullback functors accordingly. In particular, we can talk about representability of such prestacks. Yang: To be revised. Yang: Why do not we just talk about sheaves of groupoid?

**Definition 1.26.** Let $\mathbf{S}$ be a site, and let $\mathbf{X}, \mathbf{Y}$ be prestacks over $\mathbf{S}$. A morphism of prestacks $F : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$ is called *representable* if for every $\mathbf{Z} \to \mathbf{Y}$ over $\mathbf{S}$ with $\mathbf{Z}$ representable in $\mathbf{S}$, the fiber product $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}$ is representable in $\mathbf{S}$.

# 2 Algebraic spaces

**Definition 2.1.** Let $U$ be a scheme over a base scheme $S$. An *étale equivalence relation* on $U$ is a morphism $R \to U \times_S U$ between schemes over $S$ such that:

(a) the projections in two factors $R \to U$ are étale and surjective;

(b) for every $S$-scheme $T$, $h_R(T) \to h_U(T) \times h_U(T)$ gives an equivalence relation on $h_U(T)$ set-theoretically.

**Definition 2.2.** An *algebraic space* $X$ over a base scheme $S$ is an $S$-scheme $U$ together with an étale equivalence relation $R \to U \times_S U$.

Let $X = (U, R)$ be an algebraic space over $S$. We explain $X$ as a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. For any scheme $T$ over $S$, $h_R(T)$ is an equivalence relation on $h_U(T)$. The rule sending $T$ to the set of equivalence classes of $h_R(T)$ gives a presheaf on the site $(\mathbf{Sch}/S)_{\text{ét}}$. The sheafification of this presheaf is the sheaf associated to the algebraic space $X$. Explicitly, we have

$$X(T) := \left\{ f = (f_i) \,\middle|\, \begin{array}{c} \{T_i \to T\} \text{ a covering, } f_i \in h_U(T_i) \text{ such} \\ \text{that } (f_i|_{T_i \times_T T_j}, f_j|_{T_i \times_T T_j}) \in h_R(T_i \times_T T_j) \end{array} \right\} \Big/ \sim,$$

where

$$\alpha \sim \beta \quad \text{if } \exists \{S_i \to T\} \text{ such that } (\alpha|_{S_i}, \beta|_{S_i}) \in h_R(S_i).$$

**Definition 2.3.** An *algebraic space* over a base scheme $S$ is a sheaf $F$ on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$ such that

(a) the diagonal morphism $F \to F \times_S F$ is representable;

(b) there exists a scheme $U$ over $S$ and a map $h_U \to F$ which is surjective and étale.

The *morphism between algebraic spaces* $F_1, F_2$ is defined as a natural transformation of functors $F_1, F_2$.

**Remark 2.4.** By Yoneda's Lemma, given a morphism $h_U \to F$ between sheaves is the same as giving an element of $F(U)$. We may abuse the notation.

**Definition 2.5.** Let $\mathscr{p}$ be a property of morphisms of schemes satisfying the following conditions:

(a) is preserved under any base change;

(b) is étale local on the base. Yang: In [Stacks], this requires that "fppf local".

Let $\alpha : F \to G$ be a representable morphism of sheaves on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. We say that $\alpha$ has property $\mathscr{p}$ if for every $h_T \to G$, the base change $h_T \times_G F \to F$ has property $\mathscr{p}$.

**Remark 2.6.** The fiber product $F_1 \times_F F_2$ is just defined as $F_1 \times_F F_2(T) := F_1(T) \times_{F(T)} F_2(T)$ for any object $T \in \mathrm{Obj}(\mathbf{Sch}_S)$. We say that a morphism $f : F_1 \to F_2$ of sheaves is *representable* if for every $T \in \mathrm{Obj}(\mathbf{Sch}/S)$ and every $\xi \in F_2(T)$, the sheaf $F_1 \times_{F_2} h_T$ is representable as a functor. Here $h_T \to F_2$ is given by

$$h_T(U) \to F_2(U), \quad f \in \mathrm{Hom}(U,T) \mapsto F_2(f)(\xi) \in F_2(U).$$

In our case, given an arbitrary $h_U \to F \times F$ is equivalent to giving morphisms $h_{U_i} \to F$ for $i = 1, 2$. And the fiber product $F \times_{F \times F} (h_{U_1} \times h_{U_2})$ is just the fiber product $h_{U_1} \times_F h_{U_2}$. Hence the first condition in Definition 2.3 is equivalent to that $h_{U_1} \times_F h_{U_2}$ is representable for any $U_1, U_2$ over $F$. This implies that $h_U \to F$ is representable, whence the second condition in Definition 2.3 makes sense.

**Definition 2.7.** Let $X$ be an algebraic space over a base scheme $S$. Two two morphisms form field $\mathrm{Spec}\, k_i \to X$ is called equivalent if there is a common extension $K \supset k_1, k_2$ such that we have $\mathrm{Spec}\, K \to \mathrm{Spec}\, k_i \to X$ are the same for $i = 1, 2$. The *underlying point set* of $X$, denote by $|X|$, is defined as the set of equivalence classes of morphisms $\mathrm{Spec}\, k \to X$ for all field $k$ over the base field $\Bbbk$.

This definition coincides with the underlying set of a scheme. Let $\alpha : X \to Y$ be a morphism of algebraic spaces. It induces a map $|\alpha| : |X| \to |Y|$ by $x \mapsto \alpha \circ x$ (vertical composition).

**Proposition 2.8** (ref. [Stacks, Lemma 66.4.6])**.** There is a unique topology on $|X|$ such that

(a) if $X$ is a scheme, then the topology coincides with the usual topology.

(b) every morphism of algebraic spaces $f : X \to Y$ induces a continuous map $|f| : |X| \to |Y|$.

(c) if $U$ is a scheme and $U \to X$ is étale, then the induced map $|U| \to |X|$ is open.

This topology is called the *Zariski topology* on $|X|$.

**Definition 2.9.** Let $X$ be an algebraic space over a base scheme $S$. All étale morphisms $U \to X$ with $U$ scheme form a small site $X_{\text{ét}}$. All étale morphisms $U \to X$ with $U$ algebraic space form a small site $X_{\text{sp, ét}}$. The *structure sheaf* $\mathcal{O}_X$ of $X$ is given by $U \mapsto \Gamma(U, \mathcal{O}_U)$ for every étale morphism $U \to X$ from a scheme. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

**Example 2.10.** Let $U = \mathbb{A}^1_{\mathbb{C}}$ and $R \subset U \times U$ given by $y = x + n, n \in \mathbb{Z}$. Then $R$ is a disjoint union of lines in $U \times U$. Write $R = \coprod_{n \in \mathbb{Z}} R_n$ with $R_n = \{(x, x + n) : x \in \mathbb{C}\}$. Then the projection is given by

$$\pi_1|_{R_n} : R_n \to U, \quad (x, x+n) \mapsto x,$$
$$\pi_2|_{R_n} : R_n \to U, \quad (x, x+n) \mapsto x+n.$$

Easily see that the projection $\pi_i : R \to U$ is étale and surjective for $i = 1, 2$. Let $r_{ij} : R \times U \to U \times U \times U$ be the morphism which maps $((x, y), u)$ to $(a_1, a_2, a_3)$ where $a_i = x$, $a_j = y$ and $a_k = u$ for $k \neq i, j$. Since $\Delta_U \to U \times U$ factors through $R$, $(\pi_1, \pi_2) = (\pi_2, \pi_1)$ and $r_{12} \times_{(U\times U\times U)} r_{23}$ factors through $r_{13}$, we have that $h_R(T)$ is an equivalence relation on $h_U(T)$ for all $T$ over $S$. Then $X := (U, R)$ is an algebraic space.

We do not check the representability here but give an example. Let $U \to X$ be the natural morphism given by $\mathrm{id}_U \in X(U)$. For any scheme $T$ over $\mathbb{C}$, we have

$$(U \times_X U)(T) = \{(f, g) \in h_{U\times U}(T) : \exists \{T_i \to T\} \text{ s.t. } (f_i, g_i) \in h_R(T_i)\} = h_R(T).$$

Hence the fiber product $h_U \times_X h_U$ is represented by $R$.

We show that $X \ncong \mathbb{C}^\times$ by computing the the global sections. Consider the covering $U \to X$, a section $s \in \mathcal{O}_X(X)$ is given by a section $s \in \Gamma(U, \mathcal{O}_U) = \mathbb{C}[t]$ such that $\pi_1^* s = \pi_2^* s$ in $\Gamma(R, \mathcal{O}_R)$. This means that $s(x+n) = s(x)$ for all $n \in \mathbb{Z}$. Hence $s$ is a constant function. In particular, $\mathcal{O}_X(X) = \mathbb{C} \neq \mathbb{C}[t, t^{-1}]$.

The underlying set $|X|$ is union of the quotient set $\mathbb{C}/\mathbb{Z}$ and a generic point. The Zariski topology on $|X|$ is the trivial topology.

In following, we will use the technique of *local construction* to construct many scheme-like objects on algebraic spaces. For local construction, see [Knu71]. Roughly speaking, for every étale morphism $U \to X$ with $U$ a scheme, we construct a scheme-theoretic object on $U$ which is compatible under base change. Then we glue these objects together to get a global object on $X$.

**Definition 2.11.** Let $X$ be an algebraic space over a base scheme $S$. A *coherent sheaf* on $X$ is a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ such that for every covering $\{U_i \to X\}$ with $U_i$ schemes, the sheaf $\mathcal{F}|_{U_i}$ is coherent for every $i$. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.
An *ideal sheaf* on $X$ is a coherent sheaf $\mathcal{I} \subset \mathcal{O}_X$. It defines a closed subspace $V(\mathcal{I}) \subset X$ by Yang: to be completed. And every closed subspace $Y \subset X$ is defined by an ideal sheaf $\mathcal{I}_Y$ such that $V(\mathcal{I}_Y) = Y$.

**Definition 2.12.** Let $X$ be an algebraic space over a base scheme $S$. A *line bundle* on $X$ is a coherent sheaf $\mathcal{L}$ on $X$ such that for every covering $\{U_i \to X\}$ with $U_i$ schemes, the sheaf $\mathcal{L}|_{U_i}$ is a line bundle on $U_i$. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

**Theorem 2.13** (ref. [Stacks, Theorem 76.36.4]). Let $f : X \to Y$ be a proper morphism of algebraic spaces over a base scheme $S$. Then there exists a factorization

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y,$$

where $f_1$ has geometrically connected fibers and $(f_1)_* \mathcal{O}_X = \mathcal{O}_Z$ and $f_2$ is finite.

**Definition 2.14.** Let $X$ be an algebraic space over a base scheme $S$ and $Y$ a closed subset of $|X|$. The *formal completion* of $X$ along $Y$, denoted by $\mathfrak{X}$, is
Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined as $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$ where $\mathcal{I}$ is the ideal sheaf of $Y$ in $\mathcal{O}_X$. Yang: to be completed.

**Definition 2.15.** Let $X$ be an algebraic space and $Y$ a closed subset of $X$. A *modification* of $X$ along $Y$ is a proper morphism $f : X' \to X$ and a closed subset $Y' \subset X'$ such that $X' \setminus Y' \to X \setminus Y$ is an isomorphism and $f^{-1}(Y) = Y'$.

**Theorem 2.16** (ref. [Art70, Theorem 3.1])**.** Let $Y'$ be a closed subset of an algebraic space $X'$ of finite type over $\Bbbk$. Let $\mathfrak{X}'$ be the formal completion of $X'$ along $Y'$. Suppose that there is a formal modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$. Then there is a unique modification

$$f : X' \to X, \quad Y \subset X$$

such that the formal completion of $X$ along $Y$ is isomorphic to $\mathfrak{X}$ and the induced morphism $\mathfrak{X}' \to \mathfrak{X}$ is isomorphic to $\mathfrak{f}$.

**Theorem 2.17** (ref. [Art70, Theorem 6.2])**.** Let $\mathfrak{X}'$ be a formal algebraic space and $Y' = V(\mathcal{J}')$ with $\mathcal{J}'$ the defining ideal sheaf of $\mathfrak{X}'$. Let $f : Y' \to Y$ be a proper morphism. Suppose that

(a) for every coherent sheaf $\mathcal{F}$ on $\mathfrak{X}'$, we have

$$R^1 f_* \mathcal{J}'^n \mathcal{F}/\mathcal{J}'^{n+1}\mathcal{F} = 0, \quad \forall n \gg 0;$$

(b) for every $n$, the homomorphism

$$f_*(\mathcal{O}_{\mathfrak{X}'}/\mathcal{J}'^n) \otimes_{f_*\mathcal{O}_{Y'}} \mathcal{O}_Y \to \mathcal{O}_Y$$

is surjective.

Then there exists a modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ and a defining ideal sheaf $\mathcal{J}$ of $\mathfrak{X}$ such that $V(\mathcal{J}) = Y$ and $\mathfrak{f}$ induces $f$ on $Y$.

**Theorem 2.18** (ref. [Art70, Theorem 6.1])**.** Let $Y'$ be a closed algebraic subspace of an algebraic space $X'$ and $f_0 : Y' \to Y$ a finite morphism. Then there exists a modification $f : X' \to X$ whose restriction to $Y'$ is $f_0$. It is the amalgamated sum $X = X' \amalg_{Y'} Y$ in the category of algebraic spaces **AlgSp**.

**Example 2.19.** Let $X = \mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y]$ and $Y = V(y)$ be the $x$-axis. Let $f_0 : Y' = \mathbb{A}^1 \to Y, x \mapsto x^2$. Then there exists a modification $f : X' \to X$ such that the restriction $f|_{Y'} : Y' \to Y$ is $f_0$. Yang: To be completed.

# 3 Algebraic stacks

## 3.1 Definitions

**Conventions** Throughout this section, we fix a base noetherian scheme $S$. All schemes are viewed as its associated functor of points over $S$. In other words, we work in the category $\mathrm{Fun}((\mathbf{Sch}/S)^{\mathrm{op}}, \mathbf{Grpd})$. On the base category $\mathbf{Sch}/S$, we consider the étale topology unless otherwise specified.

**Definition 3.1.** A morphism $f : X \to Y$ of stacks is said to be *representable (by schemes)* if for every morphism of schemes $U \to Y$, the fiber product $X \times_Y U$ is a scheme.

**Definition 3.2.** Let $P$ be a property of morphisms of schemes which is stable under base change, for example, being étale, smooth, flat, surjective, etc. A representable morphism of stacks $f : X \to Y$ is said to *satisfy property* $P$ if for every morphism of schemes $U \to Y$, the projection morphism $X \times_Y U \to U$ satisfies property $P$.

**Definition 3.3.** A *Deligne-Mumford stack* over $S$ is a stack $\mathcal{X}$ over $S$ such that

(a) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, and

(b) there exists a scheme $U$ over $S$ and an étale surjective morphism $U \to \mathcal{X}$.

**Definition 3.4.** An *algebraic stack* over $S$ is a stack $\mathcal{X}$ over $S$ such that

(a) the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, and

(b) there exists a scheme $U$ over $S$ and a smooth surjective morphism $U \to \mathcal{X}$.

**Construction 3.5.** Let $G$ be a group scheme over $S$ acting on a scheme $X$ over $S$ via a morphism $\sigma : G \times_S X \to X$. The *quotient stack* $[X/G]$ is defined as following:

- For each scheme $U$ over $S$, the objects of $[X/G](U)$ are pairs $(P, f)$ where $P$ is a $G$-torsor over $U$ and $f : P \to X$ is a $G$-equivariant morphism over $S$.
- Morphisms between two objects $(P, f)$ and $(P', f')$ in $[X/G](U)$ are given by $G$-equivariant morphisms $\varphi : P \to P'$ over $U$ such that $f' \circ \varphi = f$.

The assignment $U \mapsto [X/G](U)$ defines a stack over the site $(\mathbf{Sch}/S)_{\text{ét}}$. This stack captures the quotient of $X$ by the action of $G$ in a way that respects the group action and the torsor structure. Yang: To be added.

**Example 3.6.** Let $\Bbbk$ be a field. Consider the projective plane $\mathbb{P}^2_{\Bbbk}$ over $\Bbbk$ and all cubic curve $C \subseteq \mathbb{P}^2_{\Bbbk}$. Its moduli stack $\mathcal{M}$ of cubic curves is an algebraic stack. Yang: To be revised.

# References

[Art70] Michael Artin. "Algebraization of formal moduli: II. Existence of modifications". In: *Annals of Mathematics* 91.1 (1970), pp. 88–135 (cit. on p. 9).

[Knu71] Donald Knutson. *Algebraic Spaces*. Vol. 203. Lecture Notes in Mathematics. Berlin, Heidelberg, New York: Springer-Verlag, 1971. ISBN: 978-3-540-05496-2 (cit. on p. 8).

[Stacks] The Stacks Project Authors. *Stacks Project*. URL: https://stacks.math.columbia.edu/ (cit. on pp. 7, 8).