

# Introduction to Moduli Problems

## 1 Setup

Let  $X$  be a smooth projective curve of genus  $g$  over an algebraically closed field  $\mathbb{k}$  of characteristic 0.

We are interested in the moduli space of vector bundles on  $X$ .

## 2 Moduli functors

Let  $S$  be a noetherian scheme and  $T$  is a scheme of finite type over  $S$ .

**Theorem 1** (Yoneda lemma). There is a full and faithful functor

$$h : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \text{Fun}((\mathbf{Sch}_S)^{\text{op}}, \mathbf{Set}), \quad T \mapsto h_T(S) := \text{Hom}_{\mathbf{Sch}_S}(T, S).$$

**Definition 2.** A functor  $F : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \mathbf{Set}$  is representable if there exists a scheme  $M$  over  $S$  such that  $F \cong h_M$ .

We say that  $M$  is *the fine moduli space* of  $F$ .

**Remark 3.** If  $F$  is representable by  $M$ , then there is a universal object  $\mathcal{U} \in F(M)$  given by  $\text{id}_M \in h_M(M)$  satisfying the following universal property: for any  $T \in \mathbf{Sch}_S$  and any  $\xi \in F(T)$ , there exists a unique morphism  $f : T \rightarrow M$  such that  $F(f)(\mathcal{U}) = \xi$ .

**Example 4.** Let  $S$  be a noetherian scheme,  $\pi : X \rightarrow S$  a projective morphism,  $\mathcal{L}$  a relatively ample line bundle on  $X$ ,  $\mathcal{F}$  a coherent sheaf on  $X$ , and  $P \in \mathbb{Q}[t]$  a polynomial. We define a functor

$$\text{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}} : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \mathbf{Set}$$

$$T \mapsto \{\pi_T^* \mathcal{F} \twoheadrightarrow \mathcal{Q} \mid \mathcal{Q} \text{ is flat over } T, \forall t \in T, \mathcal{Q}_t \text{ has Hilbert polynomial } P\} / \sim,$$

where **Yang**:

By Grothendieck's Quot scheme theorem,  $\text{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}}$  is representable by a projective  $S$ -scheme  $\text{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}}$ .

## 3 Moduli functor of vector bundles

Consider the functor

$$\tilde{\mathcal{M}}_{r,d} : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \mathbf{Set}$$

$$T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a vector bundle on } X \times T \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\} / \sim,$$

**Yang:** To be continued...

Unfortunately,  $\tilde{\mathcal{M}}_{r,d}$  is not representable. There are two main reasons:

- unboundedness and
- jumping phenomenon.

**Definition 5.** A family of vector bundles on  $X$  is *bounded* if there exists a scheme  $S$  of finite type over  $\mathbb{k}$  and a vector bundle  $\mathcal{E}$  on  $X \times S$  such that every vector bundle in the family is isomorphic to  $\mathcal{E}_s$  for some  $s \in S$ .

If  $\tilde{\mathcal{M}}_{r,d}$  is representable by a scheme  $M$  of finite type over  $\mathbb{k}$ , then the family of vector bundles parametrized by  $M$  is bounded. This is impossible since if so,  $\{h^0(X, \mathcal{E}) \mid \mathcal{E} \in \tilde{\mathcal{M}}_{r,d}(\mathbb{k})\}$  is bounded by semicontinuity theorem, which is not true.

**Example 6.** Let  $\mathcal{E}$  be a vector bundle on  $X$  of rank  $r$  and degree  $d$  with a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}.$$

Let  $\mathcal{F} := \bigoplus_{i=1}^r \mathcal{E}_i / \mathcal{E}_{i-1}$  be a coherent sheaf defined over  $X \times \mathbb{A}^1$ . We have

$$\mathcal{F}_t \cong \begin{cases} \mathcal{E}, & t \neq 0, \\ \bigoplus_{i=1}^r \mathcal{E}_i / \mathcal{E}_{i-1}, & t = 0. \end{cases}$$

If  $\tilde{\mathcal{M}}_{r,d}$  is representable by a scheme  $M$ , then the family of vector bundles parametrized by  $M$  does not have jumping phenomenon. **Yang: To be continued...**

To fix the above problems, we need to

- restrict to a smaller family of vector bundles,
- kill jumping phenomenon, and
- weaken the notion of representability.

## 4 Coarse moduli space

Let  $F : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \mathbf{Set}$  be a functor,  $M$  a scheme over  $S$ , and  $\eta : F \rightarrow h_M$  a natural transformation. We say that  $(M, \eta)$  *corepresents*  $F$  if it satisfies the following universal property: for any scheme  $N$  over  $S$  and any natural transformation  $\eta' : F \rightarrow h_N$ , there exists a unique morphism  $f : M \rightarrow N$  such that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\eta} & h_M \\ \eta' \downarrow & & \downarrow h_f \\ h_N & \xrightarrow{\sim} & N. \end{array}$$

**Definition 7.** A scheme  $M$  over  $S$  is called the *coarse moduli space* of  $F$  if

- there exists a natural transformation  $\eta : F \rightarrow h_M$  such that  $(M, \eta)$  corepresents  $F$ ;
- $\eta_{\mathbb{k}} : F(\mathbb{k}) \rightarrow M(\mathbb{k})$  is a bijection.

For a vector bundle  $\mathcal{E}$  of rank  $r$  and degree  $d$  on  $X$ , we define its slope to be  $\mu(\mathcal{E}) := d/r$ . We say that  $\mathcal{E}$  is *stable* (resp. *semistable*) if for any proper sub-bundle  $\mathcal{F} \subset \mathcal{E}$ , we have  $\mu(\mathcal{F}) < \mu(\mathcal{E})$  (resp.  $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ ).

**Proposition 8.** If  $\mathcal{E}$  is a stable vector bundle on  $X$  and  $\mathcal{F}$  is a semistable vector bundle on  $X$  with  $\mu(\mathcal{E}) \geq \mu(\mathcal{F})$ , then any non-zero homomorphism  $\varphi : \mathcal{E} \rightarrow \mathcal{F}$  is an isomorphism.

If  $\mathcal{E}$  and  $\mathcal{F}$  are stable vector bundles on  $X$  with  $\mu(\mathcal{E}) \geq \mu(\mathcal{F})$ , then any non-zero homomorphism  $\varphi : \mathcal{E} \rightarrow \mathcal{F}$  is an isomorphism. In particular, if  $\mathcal{E}$  is a stable vector bundle, then  $\text{End}(\mathcal{E}) \cong \mathbb{k}$ . **Yang:** To be continued...

**Corollary 9.** A stable vector bundle is simple.

**Lemma 10.** Let  $\mathcal{E}$  be a semistable vector bundle on  $X$ .

- (a) if  $\mu(\mathcal{E}) > 2g - 2$ , then  $H^1(X, \mathcal{E}) = 0$ ;
- (b) if  $\mu(\mathcal{E}) > 2g - 1$ , then  $\mathcal{E}$  is globally generated.

Let  $\mathcal{S}_{r,d}$  be set of isomorphism classes of semistable vector bundles on  $X$  of rank  $r$  and degree  $d$ .

**Proposition 11.** The family  $\mathcal{S}_{r,d}$  is bounded.

**Definition 12** (Jordan-Hölder filtration). Let  $\mathcal{E}$  be a semistable vector bundle on  $X$ . A *Jordan-Hölder filtration* of  $\mathcal{E}$  is a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

such that each successive quotient  $\mathcal{E}_i/\mathcal{E}_{i-1}$  is a stable vector bundle with slope  $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E})$ . The associated graded object of  $F$  is defined to be

$$\text{gr}(\mathcal{E}) := \bigoplus_{i=1}^n \mathcal{E}_i/\mathcal{E}_{i-1}.$$

Any semistable vector bundle on  $X$  admits a Jordan-Hölder filtration, and the associated graded object is unique up to isomorphism. **Yang:** To be continued...

**Definition 13** (S-equivalence). Two semistable vector bundles  $\mathcal{E}$  and  $\mathcal{F}$  on  $X$  are S-equivalent if their associated graded objects  $\text{gr}(\mathcal{E})$  and  $\text{gr}(\mathcal{F})$  (from their Jordan-Hölder filtrations) are isomorphic.

**Definition 14.** We define a functor

$$\mathcal{M}_{r,d} : (\mathbf{Sch}_{\mathbb{k}})^{\text{op}} \rightarrow \mathbf{Set}$$

$$T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a family of semistable vector bundles on } X \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\} / \sim,$$

where  $\mathcal{E} \sim \mathcal{E}'$  if for any  $t \in T$ , the vector bundles  $\mathcal{E}_t$  and  $\mathcal{E}'_t$  are S-equivalent. **Yang:** To be continued...