Schemes and Varieties

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1 Definition and First Properties of Schemes

1.1 Locally Ringed Space

Definition 1.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Example 1.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 1.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0, 1) \cup (2, 3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0, 1), (2, 3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0, 1))$ and $s_2 = 2 \in \mathcal{C}((2, 3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 1.4. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 1.5. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Example 1.6 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Example 1.7 (Glue locally ringed space). We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i \in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i \in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^{\sharp}(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ_i' satisfying the same properties, then by gluing $\psi_i' \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

1.2 Schemes

Example 1.8 (Glue open subschemes). The construction in Example 1.7 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Example 1.7. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

1.3 Integral, reduced and irreducible

1.4 Fiber product

1.5 Dimension

1.6 Noetherian and finite type

1.7 Separated and proper

2 Category of sheaves of modules

2.1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 2.1. Let X be a ringed space with structure sheaf $\mathcal{O}_X$. A **sheaf of (left) $\mathcal{O}_X$-modules** is a sheaf $\mathcal{F}$ on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ in the sense that for every $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$, we have

$$\rho_{UV}(s \cdot m) = \rho_{UV}(s) \cdot \rho_{UV}(m).$$

Yang: To be continued...

Example 2.2. Let X be a scheme. The structure sheaf $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-modules. More generally, any quasi-coherent sheaf (to be defined later) is a sheaf of $\mathcal{O}_X$-modules. In particular, if $X = \operatorname{Spec} A$ is an affine scheme, then for any A-module M, the associated sheaf $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules. Yang: To be continued...

Definition 2.3. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **quasi-coherent** if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets. Yang: To be continued...

Definition 2.4. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **coherent** if it is quasi-coherent and for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of finite free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0,$$

where m, n are finite integers. Yang: To be continued...

2.2 As abelian categories

Theorem 2.5. Let X be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules is an abelian category. Yang: To be continued...

Theorem 2.6. Let X be a scheme. The category of quasi-coherent sheaves on X is an abelian category. Yang: To be continued...

Theorem 2.7. Let X be a noetherian scheme. The category of coherent sheaves on X is an abelian category. Yang: To be continued...

2.3 Relevant functors

Theorem 2.8. Let X be a ringed space. The global sections functor

$$\Gamma(X, -) : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\mathcal{O}_X(X)\text{-modules})$$

is left exact. Yang: To be continued...

Theorem 2.9. Let $f : X \to Y$ be a morphism of ringed spaces. The direct image functor

$$f_* : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_Y\text{-modules})$$

is left exact. Yang: To be continued...

Theorem 2.10. Let $f : X \to Y$ be a morphism of ringed spaces. The inverse image functor

$$f^* : (\text{Sheaves of } \mathcal{O}_Y\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_X\text{-modules})$$

is right exact. Yang: To be continued...

2.4 Locally free sheaves and vector bundles

Definition 2.11. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **locally free** if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to a finite free $\mathcal{O}_U$-module, i.e., there exists an isomorphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{F}|_U \cong \mathcal{O}_U^n,$$

where n is a finite integer called the **rank** of $\mathcal{F}$ at x. Yang: To be continued...

Example 2.12. A **line bundle** on a scheme X is a locally free sheaf of rank 1. The sheaf of differentials $\Omega_{X/k}$ on a smooth variety X over a field k is a locally free sheaf of rank equal to the dimension of X. Yang: To be continued...

Theorem 2.13. Let X be a scheme. There is an equivalence of categories between the category of locally free sheaves of finite rank on X and the category of vector bundles on X. Yang: To be continued...

2.5 Cohomological theory

Theorem 2.14. Let X be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

Theorem 2.15. Let X be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

Theorem 2.16. Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

3 Normal, Cohen-Macaulay, and regular schemes

4 Line Bundles and Divisors

4.1 Cartier Divisors

4.2 Line Bundles and Picard Group

4.3 Weil Divisors and Reflexive Sheaves

5 Line bundles induce morphisms

5.1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 5.1. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Proposition 5.2. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subsetneq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. Yang: To be continued. □

Example 5.3. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 5.4. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}_A^{(m+1)(n+1)-1}$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 5.5. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. Yang: To be continued.

Example 5.6. Let

Example 5.7.

Definition 5.8. Let $\mathcal{L}$ be a line bundle on a scheme X. Yang: To be continued.

Definition 5.9. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

Definition 5.10. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. Yang: To be continued.

Theorem 5.11. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

Proposition 5.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

Yang: To be continued.

Proof. Yang: To be continued. □

Proposition 5.13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then $\mathcal{L}$ is very ample if and only if the following two conditions hold:

(a) (separate points) for any two distinct points $x, y \in X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $s(y) \neq 0$;

(b) (separate tangent vectors) for any point $x \in X$ and non-zero tangent vector $v \in T_xX$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $v(s) \neq 0$.

Yang: To be continued.

5.2 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 5.14. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X,\mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X,\mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X,\mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X,\mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X,\mathcal{L}))$.

Definition 5.15. A *linear system* on a scheme X is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on X and $V \subseteq \Gamma(X,\mathcal{L})$ is a subspace. The dimension of the linear system is $\dim V - 1$. A linear system is *base-point free* if V is base-point free. A linear system is *complete* if $V = \Gamma(X,\mathcal{L})$. Yang: To be continued.

Definition 5.16. Let $\mathcal{L}$ be a line bundle on a scheme X and $V \subseteq \Gamma(X,\mathcal{L})$ a subspace. The *base locus* of V is the closed subset

$$\mathrm{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If $\mathrm{Bs}(V) = \varnothing$, we say that V is *base-point free*. Yang: To be continued.

5.3 Asymptotic behavior

Definition 5.17. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X,\mathcal{L}) = \bigoplus_{n\geq 0} \Gamma(X,\mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

Definition 5.18. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. Yang: To be continued.

Theorem 5.19. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \mathrm{Proj}\, R(X,\mathcal{L})$ and f is induced by the natural map $R(X,\mathcal{L}) \to \Gamma(X,\mathcal{L}^{\otimes n})$. Yang: To be continued.

Definition 5.20. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large n. Yang: To be continued.

Example 5.21. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. Yang: To be continued.

6 Differentials and duality

7 Flat, smooth and étale morphisms

8 Relative objects

8.1 Relative schemes

Definition 8.1. Let X be a scheme. An $\mathcal{O}_X$-algebra is a sheaf . Yang: To be continued...

Definition 8.2. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. The relative Spec of $\mathcal{A}$, denoted by $\operatorname{Spec}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Spec} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. Yang: To be continued...

Proposition 8.3. Let X be a scheme and $\mathcal{E}$ be a locally free sheaf of finite rank on X. Then the relative Spec of the symmetric algebra of $\mathcal{E}$, denoted by $\mathbb{V}(\mathcal{E}) = \operatorname{Spec}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}$, is called the geometric vector bundle associated to $\mathcal{E}$. The projection morphism $\pi : \mathbb{V}(\mathcal{E}) \to X$ is affine and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Spec} \operatorname{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$. Yang: To be continued...

Definition 8.4. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_0 = \mathcal{O}_X$ and $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{O}_X$-algebra. The relative Proj of $\mathcal{A}$, denoted by $\operatorname{Proj}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Proj} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. The projection morphism $\pi : \operatorname{Proj}_X \mathcal{A} \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{A}(U)$. Yang: To be continued...

8.2 Blowing up

Definition 8.5. Let X be a scheme and $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blowing up of X along $\mathcal{J}$, denoted by $\mathrm{Bl}_{\mathcal{J}} X$, is defined to be the relative Proj of the Rees algebra of $\mathcal{J}$:

$$\mathrm{Bl}_{\mathcal{J}} X = \mathrm{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{J}^n.$$

The projection morphism $\pi : \mathrm{Bl}_{\mathcal{J}} X \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \mathrm{Bl}_{\mathcal{J}(U)} U$. The exceptional divisor of the blowing up is defined to be the closed subscheme $E = \pi^{-1}(V(\mathcal{J}))$ of $\mathrm{Bl}_{\mathcal{J}} X$. Yang: To be continued...

8.3 Relative ampleness and relative morphisms

9 Finite morphisms and fibrations

10 Varieties in more general settings

10.1 Varieties

Definition 10.1. A *variety* over an algebraically closed field $\Bbbk$ is an integral separated scheme of finite type over $\operatorname{Spec} \mathbf{k}$.

Yang: Suppose that $\mathbf{k}$ is not algebraically closed, let $\mathbf{k}'$ be an algebraic extension of $\mathbf{k}$. What is the relation between X, $X_{\mathbf{k}'}$, $X(\mathbf{k}')$ and $X_{\mathbf{k}'}(\mathbf{k}')$?

10.2 Geometric properties

10.3 Points in varieties

Proposition 10.2. Let $\mathscr{k}$ be a field and ℓ an extension of $\mathscr{k}$. Let X be a variety over $\mathscr{k}$. Then we have the following:

(a) there is a natural bijection between $X(\ell)$ and $X_\ell(\ell)$;

(b) let m/ℓ be an extension, then there is a natural inclusion $X(\ell) \subseteq X(m)$;

(c) suppose that $X = \operatorname{Spec} \mathscr{k}[T_1, \dots, T_n]/I$ is an affine variety, then there is a natural bijection between $X(\ell)$ and the set $\{(x_1, \dots, x_n) \in \ell^n | f(x_1, \dots, x_n) = 0, \forall f \in I\}$.