

Differentials and duality

Let S be a base noetherian scheme, \mathbb{k} be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over S and all varieties are assumed to be defined over \mathbb{k} .

1 The sheaves of differentials

Definition 1. Let $f : X \rightarrow S$ be an S -scheme. The *sheaf of differentials* of X over S , denoted by $\Omega_{X/S}$, is the \mathcal{O}_X -module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$.

Proposition 2. Let X and T be S -schemes and $X_T := X \times_S T$ be the base change of X along $T \rightarrow S$. Let $p : X_T \rightarrow X$ be the projection morphism. Then there is a natural isomorphism of \mathcal{O}_{X_T} -modules

$$\Omega_{X_T/T} \cong p^* \Omega_{X/S}.$$

| *Proof.* Yang: To be continued. □

Proposition 3. Let X be an S -scheme and $U \subseteq X$ be an open subscheme. Then there is a natural isomorphism of \mathcal{O}_U -modules

$$\Omega_{U/S} \cong \Omega_{X/S}|_U.$$

Furthermore, let $\xi \in X$, then there is a natural isomorphism of $\mathcal{O}_{X,\xi}$ -modules

$$\Omega_{X/S,\xi} \cong \Omega_{\mathcal{O}_{X,\xi}/\mathcal{O}_{S,f(\xi)}}.$$

| Yang: To be checked.

| *Proof.* Yang: To be continued. □

Proposition 4. Let X be a regular variety over \mathbb{k} of dimension n . Then $\Omega_{X/\mathbb{k}}$ is a locally free sheaf of rank n .

| *Proof.* Yang: To be continued. □

Proposition 5. Let X be a normal variety over \mathbb{k} of dimension n . Then $\Omega_{X/\mathbb{k}}$ is a reflexive sheaf of rank n .

| *Proof.* Yang: To be continued. □

Theorem 6 (Euler sequence for projective bundle). Let X be a normal variety over \mathbb{k} and \mathcal{E} be a locally free sheaf of rank $r + 1$ on X . Let $\pi : \mathbb{P}_X(\mathcal{E}) \rightarrow X$ be the projective bundle associated to \mathcal{E} .

Then there is an exact sequence of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$ -modules

$$0 \rightarrow \Omega_{\mathbb{P}_X(\mathcal{E})/X} \rightarrow \pi^*\mathcal{E}(-1) \rightarrow \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \rightarrow 0.$$

Here $\pi^*\mathcal{E}(-1)$ is twisted by the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-1)$.

| *Proof.* Yang: To be continued. □

Definition 7. Let X be a normal variety over \mathbb{k} . The *canonical divisor* K_X of X is defined to be the Weil divisor class $c_1(\Omega_{X/\mathbb{k}})$.

Example 8. Let $\mathbb{P}_{\mathbb{k}}^n$ be the projective space of dimension n over \mathbb{k} . Then the canonical divisor $K_{\mathbb{P}_{\mathbb{k}}^n} \sim -(n+1)H$, where H is a hyperplane in $\mathbb{P}_{\mathbb{k}}^n$. Yang: To be checked.

2 Fundamental sequences

Theorem 9 (The first fundamental sequence of differentials). Let $f : X \rightarrow Y$ be a morphism of schemes. Then there is a natural exact sequence of \mathcal{O}_X -modules

$$f^*\Omega_{Y/S} \rightarrow \Omega_{X/S} \rightarrow \Omega_{X/Y} \rightarrow 0.$$

| *Proof.* Yang: To be completed. □

Proposition 10. Let $f : X \rightarrow Y$ be a surjective and generically finite morphism of normal varieties over \mathbb{k} . Then the first fundamental sequence of differentials is exact on the left.

| *Proof.* Yang: To be completed. □

Corollary 11 (Ramification formula). Let $f : X \rightarrow Y$ be a finite morphism of normal varieties. Then

$$K_X = f^*K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} (\text{Mult}_D f^*(f(D)) - 1) D$$

is the ramification divisor of f . Yang: To be checked. definition of ramification divisor needs to be checked.

| *Proof.* Yang: To be completed. □

Theorem 12 (The second fundamental sequence of differentials). Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of \mathcal{O}_X -modules

$$\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/S}|_Z \rightarrow \Omega_{Z/S} \rightarrow 0.$$

Suppose further that $Z \rightarrow X$ is a regular immersion. Then the above sequence is also exact on the left.

| *Proof.* Yang: To be completed. □

Corollary 13 (Adjunction formula). Let X be a normal variety and $Z \subseteq X$ be a prime Cartier divisor which is normal as variety. Then

$$K_Z = (K_X + Z)|_Z.$$

Proof. Since both X and Z are normal, they are smooth in codimension 1. Removing the singular locus of X and Z , we may assume that both X and Z are smooth varieties. This is valid since the canonical divisor is determined by the smooth locus.

Since Z is Cartier, it is a local complete intersection in X . By [Theorem 12](#), we have the exact sequence

$$0 \rightarrow \mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow \Omega_{X/\mathbb{k}}|_Z \rightarrow \Omega_{Z/\mathbb{k}} \rightarrow 0.$$

Note that Z is of codimension 1 in X , so $\mathcal{I}_Z \cong \mathcal{O}_X(-Z)$ and thus $\mathcal{I}_Z/\mathcal{I}_Z^2 \cong \mathcal{O}_X(-Z)|_Z$. Taking c_1 , we obtain

$$c_1(\Omega_X)|_Z = c_1(\Omega_Z) + c_1(\mathcal{O}_X(-Z))|_Z.$$

That is,

$$K_X|_Z = K_Z - Z|_Z.$$

Rearranging gives the desired result. *Yang: To be revised. restriction of Weil divisors needs to be clarified.* \square

3 Serre duality

Definition 14 (Dualizing sheaf). Let X be a proper scheme of dimension n over \mathbb{k} . A *dualizing sheaf* on X is a coherent sheaf ω_X° together with a trace map $\text{tr}_X : H^n(X, \omega_X^\circ) \rightarrow \mathbb{k}$ such that for every coherent sheaf \mathcal{F} on X , the natural pairing

$$\text{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \rightarrow H^n(X, \omega_X^\circ) \xrightarrow{\text{tr}_X} \mathbb{k}$$

induces an isomorphism

$$\text{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

Theorem 15. Let X be a projective scheme of dimension n over \mathbb{k} . Then there exists a dualizing sheaf ω_X° on X up to isomorphism. Moreover, if X is smooth, $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\mathbb{k}}$.

Proof. *Yang: To be completed.* \square

Theorem 16 (Serre duality). Let X be a projective, Cohen-Macaulay variety of dimension n over \mathbb{k} with dualizing sheaf ω_X° . Then for every coherent sheaf \mathcal{F} on X , there is a natural isomorphism

$$\text{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. *Yang: To be completed.* \square

Yang: When \mathcal{F} is locally free, we have $\text{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^i(X, \omega_X^\circ \otimes \mathcal{F}^\vee)$.

Corollary 17. Let X be a projective, normal variety of dimension n over \mathbb{k} . Then for every integer m and $0 \leq i \leq n$, there is a natural isomorphism [Yang: To be completed.](#)

4 Logarithm version