The first order deformations

Fix a field $\mathbf{k}$ throughout this section.

1 Dual number

Definition 1. The *dual number* over $\mathbf{k}$ is the ring $\mathbf{k}[\epsilon]$ defined as

$$\mathbf{k}[\epsilon] := \mathbf{k}[x]/(x^2),$$

where ϵ is the image of x in the quotient.

Definition 2. Let X be a $\mathbf{k}$-scheme, and let $Z \subseteq X$ be a closed subscheme. A *deformation* of Z in X over the dual number $\mathbf{k}[\epsilon]$ is a closed subscheme $\mathcal{Z} \subseteq X \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} \mathbf{k}[\epsilon]$ flat over $\operatorname{Spec} \mathbf{k}[\epsilon]$ such that the special fiber $\mathcal{Z}_0 := \mathcal{Z} \times_{\operatorname{Spec} \mathbf{k}[\epsilon]} \operatorname{Spec} \mathbf{k}$ is isomorphic to Z. Yang: To be revised.

Proposition 3. Let X be a $\mathbf{k}$-scheme, and let $Z \subseteq X$ be a closed subscheme. Suppose that there exists a moduli space $\mathcal{M}$ parameterizing closed subschemes of X (e.g., the Hilbert scheme $\operatorname{Hilb}_{X/\mathbf{k}}$). Then the set of deformations of Z in X over the dual number $\mathbf{k}[\epsilon]$ is in bijection with the tangent space $T_{[Z]}\mathcal{M}$ of $\mathcal{M}$ at the point $[Z] \in \mathcal{M}$ corresponding to Z. Yang: To be checked.

Theorem 4. Let X be a $\mathbf{k}$-scheme, and let $Z \subseteq X$ be a closed subscheme. Then the deformations of Z in X over the dual number $\mathbf{k}[\epsilon]$ is in bijection with $H^0(Z, \mathcal{N}_{Z/X})$, where $\mathcal{N}_{Z/X} = (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$ is the normal sheaf of Z in X.

Theorem 5. Let X be a $\mathbf{k}$-scheme, and let $\mathcal{F}$ be a coherent sheaf on X. Then the deformations of $\mathcal{F}$ over the dual number $\mathbf{k}[\epsilon]$ is in bijection with $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$. In particular, if $\mathcal{F} \cong \mathcal{L}$ is a line bundle on X, then the deformations of $\mathcal{L}$ over the dual number $\mathbf{k}[\epsilon]$ is in bijection with $H^1(X, \mathcal{O}_X)$.

Corollary 6. Let X be a smooth projective variety over $\mathbf{k}$. Then $\dim \operatorname{Pic}^0(X) = q(X) = h^{1,0}(X)$.

Proof. Yang: To be added. □

Yang: To be checked.

Appendix

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