Normal, Cohen-Macaulay, and regular schemes

1 Normal scheme

Definition 1. A scheme X is called *normal* if for every open affine subset $U = \operatorname{Spec} A$ of X, the ring A is an integrally closed domain. Yang: To be checked.

Definition 2. The *normalization* of a scheme X is a normal scheme $\widetilde{X}$ together with a finite birational morphism $\pi : \widetilde{X} \to X$ such that for every normal scheme Y and every birational morphism $f : Y \to X$, there exists a unique morphism $g : Y \to \widetilde{X}$ such that $f = \pi \circ g$. Yang: To be checked.

Theorem 3. Let X be a scheme. Then there exists a normalization $\widetilde{X}$ of X.

Theorem 4 (Hartog's phenomenon)**.** Let X be a normal integral scheme, and let $U \subseteq X$ be an open subset whose complement has codimension at least 2. Then every regular function on U extends uniquely to a regular function on X, i.e., the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is an isomorphism. Yang: To be checked.

Proposition 5. Let X be a normal scheme, and let $x \in X$ be a point with codimension 1. Then X is regular at x.

2 Cohen-Macaulay scheme

Definition 6. Let X be a scheme, and let $Z \subseteq X$ be a closed subset. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the *local cohomology* $H^i_Z(X, \mathcal{F})$ is defined as the i-th right derived functor of the functor $\Gamma_Z(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ that sends a sheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ to the abelian group of sections of $\mathcal{G}$ with support in Z, i.e.,

$$H^i_Z(X, \mathcal{F}) := \mathrm{R}^i\Gamma_Z(X, \mathcal{F}).$$

Yang: To be checked.

Definition 7. A scheme X is called *Cohen-Macaulay* if for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a Cohen-Macaulay ring. Yang: To be checked.

Theorem 8. Let X be a Cohen-Macaulay scheme, and let $Z \subseteq X$ be a closed subset of codimension at least 2. Then for every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the local cohomology $H^i_Z(X, \mathcal{F}) = 0$ for every $i < 2$. Yang: To be checked.

3 Regular scheme

We first define the tangent space of a scheme at a point.

There are many descriptions of the tangent space of a scheme at a point. Here we give one of them.

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Let X be a scheme over a field $\mathbf{k}$, and let $x \in X(\mathbf{k})$.

Proposition 9. Let $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$ be the spectrum of the ring of dual numbers over $\mathbf{k}$ with point $* : \operatorname{Spec} \mathbf{k} \to \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$. The tangent space $T_x X$ is naturally isomorphic to the set of morphisms $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X$ that send $*$ to x, i.e.

$$T_x X \cong \{f : \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X \mid f(*) = x\}.$$

Proof. Yang: To be filled. □