# Differentials and duality

Let $S$ be a base noetherian scheme, $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over $S$ and all varieties are assumed to be defined over $\Bbbk$.

## 1 The sheaves of differentials

**Definition 1.** Let $f : X \to S$ be an $S$-scheme. The *sheaf of differentials* of $X$ over $S$, denoted by $\Omega_{X/S}$, is the $\mathcal{O}_X$-module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$.

**Example 2.**

**Proposition 3.** Let $X$ be a smooth variety over $\Bbbk$ of dimension $n$. Then $\Omega_{X/\Bbbk}$ is a locally free sheaf of rank $n$.

*Proof.* Yang: To be continued. □

**Theorem 4** (Euler sequence for projective bundle)**.** Let $X$ be a normal variety over $\Bbbk$ and $\mathcal{E}$ be a locally free sheaf of rank $r + 1$ on $X$. Let $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. Then there is a natural exact sequence of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$-modules

$$0 \to \Omega_{\mathbb{P}_X(\mathcal{E})/X} \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \to 0.$$

Yang: To be checked.

*Proof.* □

**Definition 5.** Let $X$ be a normal variety over $\Bbbk$ of dimension $n$. If $X$ is smooth, then the *canonical divisor* $K_X$ is defined to be $c_1(\omega_X)$. In general, let $U \subseteq X$ be the smooth locus of $X$ and $i : U \hookrightarrow X$ be the inclusion map. Then the *canonical divisor* $K_X$ is defined to be any Weil divisor on $X$ such that $\mathcal{O}_X(K_X) \cong i_*\omega_U$. Note that $U$ is big in $X$ since $X$ is normal, so such a Weil divisor always exists and is unique up to linear equivalence.

**Example 6.** Let $\mathbb{P}^n_{\Bbbk}$ be the projective space of dimension $n$ over $\Bbbk$. Then the canonical divisor $K_{\mathbb{P}^n_{\Bbbk}} \sim -(n + 1)H$, where $H$ is a hyperplane in $\mathbb{P}^n_{\Bbbk}$. Yang: To be checked.

## 2 Fundamental sequence

**Theorem 7** (The first fundamental sequence of differentials). Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Suppose further that $f$ is smooth. Then the above sequence is also exact on the left. Yang: ... it will be exact.

*Proof.* Yang: To be completed. □

**Theorem 8** (The second fundamental sequence of differentials). Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}|_Z \to \Omega_{Z/S} \to 0.$$

Suppose further that $Z$ is a local complete intersection in $X$. Then the above sequence is also exact on the left. Yang: ... it will be exact.

*Proof.* Yang: To be completed. □

## 3 Application of fundamental sequences

**Theorem 9** (Ramification formula). Let $f : X \to Y$ be a finite morphism of normal varieties. Then

$$K_X = f^*K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} \left(\mathrm{Mult}_D f^*\big(f(D)\big) - 1\right) D$$

is the ramification divisor of $f$.

*Proof.* Yang: To be completed. □

**Theorem 10** (Adjunction formula). Let $X$ be a smooth variety and $Z \subseteq X$ be a smooth subvariety of codimension $1$. Then

$$K_Z = (K_X + Z)|_Z.$$

*Proof.* Yang: To be completed. □

## 4 Serre duality

**Definition 11** (Dualizing sheaf). Let $X$ be a proper scheme of dimension $n$ over $\Bbbk$. A *dualizing sheaf* on $X$ is a coherent sheaf $\omega_X^\circ$ together with a trace map $\mathrm{tr}_X : H^n(X, \omega_X^\circ) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$ on $X$, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}_X} \Bbbk$$

induces an isomorphism

$$\text{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

**Theorem 12.** Let $X$ be a projective scheme of dimension $n$ over $\Bbbk$. Then there exists a dualizing sheaf $\omega_X^\circ$ on $X$ up to isomorphism. Moreover, if $X$ is smooth, $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\Bbbk}$.

*Proof.* Yang: To be completed. □

**Theorem 13** (Serre duality)**.** Let $X$ be a projective, Cohen-Macaulay variety of dimension $n$ over $\Bbbk$ with dualizing sheaf $\omega_X^\circ$. Then for every coherent sheaf $\mathcal{F}$ on $X$, there is a natural isomorphism

$$\text{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

*Proof.* Yang: To be completed. □

**Corollary 14.** Let $X$ be a projective, normal variety of dimension $n$ over $\Bbbk$. Then for every integer $m$ and $0 \leq i \leq n$, there is a natural isomorphism Yang: To be completed.

## 5 Logarithm version