Quot Functor

Definition 1. Let X be a projective variety over $\mathbf{k}$ with a very ample line bundle $\mathcal{H}$. A coherent sheaf $\mathcal{F}$ on X is said to be m-*regular* (with respect to $\mathcal{H}$) if

$$H^i(X, \mathcal{F} \otimes \mathcal{H}^{\otimes(m-i)}) = 0 \quad \text{for all } i > 0.$$

Yang: To be continued.

1 Boundness of Quotients

Lemma 2. Let $\mathcal{F}$ be a coherent sheaf of dimension d. Let $s_1, \cdots, s_d$ be an $\mathcal{F}$-regular sequence of global sections of $\mathcal{O}_X(1)$. Let $H_i = V(s_i)$ be the corresponding hyperplane sections. Let $\mathcal{F}_i = \mathcal{F}|_{H_1 \cap \cdots \cap H_i}$ be the restriction of $\mathcal{F}$ to the intersection. Assume that $h^0(\mathcal{F}_i) \le b_i$. Then

$$h^0(\mathcal{F} \otimes \mathcal{H}^{\otimes m}) \le \sum_{i=0}^{d} \binom{m+i-1}{i} b_i.$$

Proposition 3. There are universal polynomials $P_i \in \mathbb{Q}[T_0, \cdots, T_i]$ such that the following holds. Let $\mathcal{F}$ be a coherent sheaf of dimension d. Let $s_1, \cdots, s_d$ be an $\mathcal{F}$-regular sequence of global sections of $\mathcal{O}_X(1)$ and $H_i = V(s_i)$ the corresponding hyperplane sections. Set $\mathcal{F}_i = \mathcal{F}|_{H_1 \cap \cdots \cap H_i}$ and $h^0(\mathcal{F}_i) \le b_i$. Suppose that $\chi(\mathcal{F}_i) = a_i$. Then $\mathcal{F}$ is $P_d(a_0 - b_0, \cdots, a_d - b_d)$-regular.

Theorem 4. Let S be a noetherian scheme, X a projective scheme over S with a relatively very ample line bundle $\mathcal{O}_X(1)$. Let $P \in \mathbb{Q}[T]$ be a polynomial of degree d. Let Σ be a set-theoretic family of coherent sheaves on the fibers of X/S with Hilbert polynomial P. TFAE:

(a) Σ is bounded;

(b) there are constant C_i such that for any $\mathcal{F} \in \Sigma$, there is an $\mathcal{F}$-regular sequence of global sections $s_1, \cdots, s_d$ of $\mathcal{O}_X(1)$ such that for each i,

$$h^0(\mathcal{F}|_{H_1 \cap \cdots \cap H_i}) \le C_i;$$

(c) there is m such that any $\mathcal{F} \in \Sigma$ is m-regular;

(d) there is a coherent sheaf $\mathcal{E}$ on X such that any $\mathcal{F} \in \Sigma$ is a quotient of $\mathcal{E}_t$ for some $t \in S$.

(e) there is a scheme T of finite type over S and a coherent sheaf $\mathcal{G}$ on X_T such that any $\mathcal{F} \in \Sigma$ is isomorphic to $\mathcal{G}_t$ for some $t \in T$.

Yang: To be checked.

Date: October 22, 2025, **Author**: Tianle Yang, My Homepage