Finite morphisms and fibrations

The main references for this section are [Har77], [Laz04] and [FOV99].

1 Finite morphisms

Theorem 1. Let $f : Y \to X$ be a finite morphism of schemes. If $\mathcal{L}$ is an ample line bundle on X, then $f^*\mathcal{L}$ is an ample line bundle on Y. If and only if.

2 Fibrations

Definition 2. Let S be an integral excellent scheme and X be an integral scheme over S of finite type via the structure morphism $f : X \to S$. Suppose that f is dominant. We say that f is a *fibration* if $\mathscr{K}(S)$ is algebraically closed in $\mathscr{K}(X)$.

Theorem 3. Let $f : X \to S$ be a fibration. Then for a general point $\sigma \in S$, the fiber X_σ is integral.

Proof. Yang: □

Slogan *Fibration is the relatively version of "integral".*

Proposition 4. Let S be an integral excellent scheme and X be an integral scheme over S of finite type via the structure morphism $f : X \to S$. Suppose that f is proper surjective and that S is normal. Then f is a fibration if and only if $f_*\mathcal{O}_X = \mathcal{O}_S$.

Proof. Yang: □

Proposition 5. Let $f : X \to Y$ be a fibration with Y normal. Then the fibers of f are connected.

Remark 6. If we drop the normality assumption in Proposition 4, then the condition $f_*\mathcal{O}_X = \mathcal{O}_Y$ is still sufficient to guarantee that f is a fibration. However, the converse may fail.

Yang: To be added.

Theorem 7 (Zariski's Main Theorem)**.** Let $f : Y \to X$ be a birational finite type morphism of excellent integral schemes. Suppose that X is normal. Then the fiber of f are connected.

Theorem 8 (Stein factorization)**.** Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then there exists a factorization

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where g is a proper morphism with connected fibers and h is a finite morphism. Moreover, this factorization is unique up to isomorphism. Yang: To be checked.

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Theorem 9 (Rigidity Lemma)**.** Let $f : Y \to X$ be a fibration of noetherian schemes. Let $g : Y \to Z$ be a morphism such that the restriction $g|_{f^{-1}(x)} : f^{-1}(x) \to Z$ is constant for every point $x \in X$. Then there exists a unique morphism $h : X \to Z$ such that $g = h \circ f$. Yang:

Corollary 10. Let $X \subseteq \mathbb{P}^n_k$ be an irreducible closed subvariety of dimension ≥ 2. Then for a general hyperplane $H \subseteq \mathbb{P}^n_k$, the (scheme-theoretic) intersection $X \cap H$ is integral.

Proof. Yang: □

Corollary 11. Let X be a quasi-projective variety over a field $\mathbf{k}$ of dimension ≥ 2. Given any finitely many closed points $x_1, x_2, \dots, x_r \in X(\Bbbk)$, there exists an irreducible curve $C \subseteq X$ passing through all the given points.

Proof. Yang: □

References

[FOV99] H Flenner, Liam O'Carroll, and W Vogel. *Joins and Intersections*. Springer Science & Business Media, 1999 (cit. on p. 1).

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[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: 10.1007/978-3-642-18808-4. URL: https://doi.org/10.1007/978-3-642-18808-4 (cit. on p. 1).