Notes in Algebraic Geometry

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Chapter 1

Schemes and Varieties

1.0 Locally Ringed Space

1.0.1 Sheaves

Definition 1.0.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

For two open sets $V \subset U \subset X$, the morphism $\mathcal{F}(U) \to \mathcal{F}(V)$, often denoted by res^U_V, is called the *restriction map*.

Example 1.0.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 1.0.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U \to \mathbb{R}\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0, 1) \cup (2, 3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0, 1), (2, 3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0, 1))$ and $s_2 = 2 \in \mathcal{C}((2, 3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 1.0.4. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be presheaves on X with values in the same category (e.g., $\mathbf{Set}$, $\mathbf{Ab}$, $\mathbf{Ring}$, etc.). A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$. In other words, for every open set $U \subset X$, there is a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every inclusion of open sets $V \subset U$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}^U_V"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V). \end{tikzcd}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then φ is called a *morphism of sheaves.*

Fix a topological space X and a category $\mathbf{C}$. The sheaves (resp. presheaves) on X with values in $\mathbf{C}$ form a category, denoted by $\mathbf{Sh}(X, \mathbf{C})$ (resp. $\mathbf{PSh}(X, \mathbf{C})$), where the objects are sheaves (resp. presheaves) on X with values in $\mathbf{C}$ and the morphisms are morphisms of sheaves (resp. presheaves).

Definition 1.0.5. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in a category $\mathbf{C}$. For a point $x \in X$, the *stalk* of $\mathcal{F}$ at x, denoted by $\mathcal{F}_x$, is defined as the colimit

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the colimit is taken over all open neighborhoods U of x. An element of $\mathcal{F}_x$ is called a *germ* of sections of $\mathcal{F}$ at x.

More concretely, we have

$$\mathcal{F}_x = \{(U, s) : U \in \mathbf{Open}(X), U \ni x, s \in \mathcal{F}(U)\}/\sim,$$

where $(U, s) \sim (V, t)$ if there exists an open neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$.

Definition 1.0.6. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in $\mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.). A *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}^\dagger$ on X together with a morphism of presheaves $\eta : \mathcal{F} \to \mathcal{F}^+$ such that for every sheaf $\mathcal{G}$ on X and every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \eta$.
In other words, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & \mathcal{F}^\dagger \arrow[d, "\varphi^\dagger"] \\ & \mathcal{G}. \end{tikzcd}$$

To be checked.

The concrete describe of sheafification.

Definition 1.0.7. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The morphism φ is called *injective* (resp. *surjective*) if for every $x \in X$, the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).

Proposition 1.0.8. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is injective if and only if for every open set $U \subset X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. To be checked.

Remark 1.0.9. The surjectivity on stalks cannot imply the surjectivity on sections. A counterexample is given by the exponential map $\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^*$ defined by $\exp(f) = e^f$, where $\mathcal{O}_{\mathbb{C}}$ is the sheaf of holomorphic functions on $\mathbb{C}$ and $\mathcal{O}_{\mathbb{C}}^*$ is the sheaf of non-vanishing holomorphic functions on $\mathbb{C}$. The induced map on stalks $\exp_z : \mathcal{O}_{\mathbb{C},z} \to \mathcal{O}_{\mathbb{C},z}^*$ is surjective for every $z \in \mathbb{C}$ by the existence of logarithm locally. However, the map on global sections $\exp(\mathbb{C}) : \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}^*(\mathbb{C})$ is not surjective since there is no entire function f such that $e^{f(z)} = z$ for all $z \in \mathbb{C}^*$. To be continued. This is wrong, need to be revised.

Proposition 1.0.10. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is an isomorphism if and only if it is injective and surjective.

Now we consider sheaves with values in an abelian category.

Definition 1.0.11. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The *kernel* of φ, denoted by $\ker\varphi$, is the sheaf defined by

$$(\ker\varphi)(U) := \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$.
The *cokernel* of φ, denoted by $\mathsf{coker}\,\varphi$, is the sheafification of the presheaf defined by

$$(\mathsf{coker}\,\varphi)_{\mathrm{pre}}(U) := \mathsf{coker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$. To be continued.

Theorem 1.0.12. Let X be a topological space and $\mathbf{C}$ be an abelian category (e.g., $\mathbf{Ab}$). Then the category of sheaves on X with values in $\mathbf{C}$ is an abelian category.

Proof. To be continued. □

To be checked and continuous.

1.0.2 Locally ringed space

Definition 1.0.13. Let $f : X \to Y$ be a continuous map between topological spaces. The *push-forward* functor $f_* : \mathbf{Sh}(X, \mathbf{C}) \to \mathbf{Sh}(Y, \mathbf{C})$ is defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

for every open set $V \subset Y$ and sheaf $\mathcal{F} \in \mathbf{Sh}(X, \mathbf{C})$.

Definition 1.0.14. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 1.0.15. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Construction 1.0.16 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Construction 1.0.17 (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i\in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i\in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^{\sharp}(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ_i' satisfying the same properties, then by gluing $\psi_i' \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

Definition 1.0.18. A morphism of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a *closed immersion* (resp. *open immersion*) if f induces a homeomorphism from X to a closed (resp. open) subset of Y and the map $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective (resp. an isomorphism). To be checked.

1.0.3 Manifolds as locally ringed spaces

1.0.4 Vector bundles and $\mathcal{O}_X$-modules

Let $(X, \mathcal{O}_X)$ be a manifold (real or complex) and $(\mathcal{E}, \pi, X)$ a vector bundle over X.

It can regard as a sheaf on X.

Definition 1.0.19. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F}$ of abelian groups on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\mathsf{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear, where the $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(V)$ is induced by the restriction map $\mathsf{res}_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.
A *morphism of $\mathcal{O}_X$-modules* is a morphism of sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ such that for every open set $U \subseteq X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. To be checked...

Definition 1.0.20. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is said to be *locally free of rank r* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_U^r$, where $\mathcal{O}_U^r$ is the direct sum of r copies of $\mathcal{O}_U$. To be continued.

1.1 The First Properties of Schemes

If you learn the following content for the first time, it is recommended to skip all the proofs in this section and focus on the examples, remarks and the statements of propositions and theorems.

1.1.1 Schemes

Let R be a ring. Recall that the *spectrum* of R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set*.

Proposition 1.1.1. Let R be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

Proof. To be continued □

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 1.1.2. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i\in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.
These data form a category, denoted by **Sch**. If we fix a base scheme S, then an S*-scheme* is a scheme X together with a morphism $X \to S$. The category of S-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

Theorem 1.1.3. The functor $\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. To be continued

Definition 1.1.4. A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if f induces an isomorphism of X onto an open (resp. closed) subscheme of Y. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. To be continued

Construction 1.1.5. Let R be a graded ring. The *projective scheme* $\operatorname{Proj} R$ is defined as the scheme associated to the sheaf of rings

$$\mathcal{O}_{\operatorname{Proj} R} = \bigoplus_{d\geq 0} R_d.$$

It can be covered by open affine subschemes of the form $\operatorname{Spec} R_f$ for homogeneous elements $f \in R$. To be checked.

Construction 1.1.6 (Glue open subschemes)**.** The construction in Construction 1.0.17 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Construction 1.0.17. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

Definition 1.1.7. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme Z of Y such that f factors through Z. More precisely, if $Y = \operatorname{Spec} A$ is affine, then the scheme theoretic image of f is $\operatorname{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover Y by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of f. To be checked.

1.1.2 Fiber product and base change

Definition 1.1.8. Let $\mathbf{C}$ be a category and $X, Y, S \in \operatorname{Obj}(\mathbf{C})$ with morphisms $f : X \to S$ and $g : Y \to S$. A *fiber product* of X and Y over S is an object $X \times_S Y \in \operatorname{Obj}(\mathbf{C})$ together with morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ such that $f \circ p = g \circ q$ and satisfies the universal property that for any object $W \in \operatorname{Obj}(\mathbf{C})$ with morphisms $u : W \to X$ and $v : W \to Y$ such that $f \circ u = g \circ v$, there exists a unique morphism $h = (u, v) : W \to X \times_S Y$ such that $p \circ h = u$ and $q \circ h = v$.

$$\begin{tikzcd} W \arrow[dr, "h"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & X \times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & S. \end{tikzcd}$$

To be checked.

Example 1.1.9. In the category of sets, the fiber product $X \times_S Y$ is given by

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\},$$

with the projections $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ being the restrictions of the natural projections. To be checked.

Remark 1.1.10. If one reverses the arrows in Definition 1.1.8, one gets the notion of *fiber coproduct*. It is also called the *pushout* or *amalgamated sum* in some literature. We denote the fiber coproduct of X and Y over S by $X \amalg_S Y$. Note that in the category of rings, the fiber coproduct $A \amalg_R B$ of R-algebras A and B over R is given by the tensor product $A \otimes_R B$. Dually, one can expect that fiber products of affine schemes correspond to tensor products of rings.

Theorem 1.1.11. The category of schemes admits fiber products. To be continued

Definition 1.1.12. Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a point. The *scheme theoretic fiber* of f over y is the fiber product $X_y = X \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y)$ is the residue field of the local ring $\mathcal{O}_{Y,y}$. To be checked.

Definition 1.1.13. Let X be a scheme and $Z_1, Z_2 \subset X$ be closed subschemes of X with inclusion morphisms $i_1 : Z_1 \to X$ and $i_2 : Z_2 \to X$. The *scheme theoretic intersection* of Z_1 and Z_2 is the fiber product $Z_1 \times_X Z_2$. To be checked.

1.1.3 Noetherian schemes and morphisms of finite type

Definition 1.1.14. A scheme X is *noetherian* if it admits a finite open cover $\{U_i\}_{i=1}^n$ such that each U_i is an affine scheme $\operatorname{Spec} A_i$ with A_i a noetherian ring. To be checked.

Proposition 1.1.15. A noetherian scheme is quasi-compact. To be checked.

Definition 1.1.16. Let $f : X \to S$ be a morphism of schemes. We say that f is *of finite type*, or X is *of finite type* over S, if there exists a finite affine cover $\{U_i\}_{i=1}^n$ of S such that for each i, $f^{-1}(U_i)$ can be covered by finitely many affine open subsets $\{V_{ij}\}_{j=1}^{m_i}$ with $f(V_{ij}) \subseteq U_i$ and the induced morphism $f|_{V_{ij}} : V_{ij} \to U_i$ corresponds to a finitely generated algebra over the ring of global sections of U_i. Given S, the category consisted of S-scheme of finite type over S, together with morphisms of S-schemes, is denoted by $\mathbf{sch}_S$. To be checked.

1.1.4 Integral, reduced and irreducible schemes

Definition 1.1.17. A topological space X is *irreducible* if it is non-empty and cannot be expressed as the union of two proper closed subsets. Equivalently, every non-empty open subset of X is dense in X. To be checked.

Proposition 1.1.18. Let X be a topological space satisfying the descending chain condition on closed subsets. Then X can be written as a finite union of irreducible closed subsets, called the *irreducible components* of X. Moreover, this decomposition is unique up to permutation of the components. To be checked.

Definition 1.1.19. A scheme X is *reduced* if its structure sheaf $\mathcal{O}_X$ has no nilpotent elements. To be checked.

Proposition 1.1.20. A scheme X is reduced if and only if for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a reduced ring. To be checked.

Proposition 1.1.21. Let X be a scheme. There exists a unique closed subscheme X of X such that X is reduced and has the same underlying topological space as X. Moreover, for any morphism of schemes $f : Y \to X$ with Y reduced, f factors uniquely through the inclusion $X \to X$. To be checked.

Definition 1.1.22. A scheme X is *integral* if it is both reduced and irreducible. To be checked.

Proposition 1.1.23. A scheme X is integral if and only if for every open affine subset $U = \operatorname{Spec} A \subset X$, the ring A is an integral domain. To be checked.

Corollary 1.1.24. Let $\Bbbk$ be an algebraically closed field and $n \geq 1$ be an integer. Then the polynomial $\det(x_{ij}) \in k[x_{ij} : 1 \leq i, j \leq n]$ is irreducible. To be checked.

1.1.5 Dimension

Definition 1.1.25. The *Krull dimension* of a topological space X, denoted by $\dim X$, is the supremum of the lengths n of chains of distinct irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X. If no such finite supremum exists, we say that X has infinite dimension. To be checked.

Definition 1.1.26. Let $\xi \in X$ be a point in a scheme X. The *local dimension* of X at ξ, denoted by $\dim_\xi X$, is defined as the infimum of the dimensions of all open neighborhoods U of ξ:

$$\dim_\xi X = \inf\{\dim U : U \text{ is an open neighborhood of } \xi\}.$$

To be checked.

1.1.6 Separated, proper and projective morphisms

Definition 1.1.27. A morphism of schemes $f : X \to Y$ is *separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. A scheme X is *separated* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. To be checked.

Proposition 1.1.28. Any affine scheme is separated. More generally, any morphism between affine schemes is separated. To be checked.

Proposition 1.1.29. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any scheme T and any pair of morphisms $g_1, g_2 : T \to X$ such that $f \circ g_1 = f \circ g_2$, the equalizer of g_1 and g_2 is a closed subscheme of T. To be checked.

Proposition 1.1.30. A scheme X is separated if and only if for any pair of affine open subschemes $U, V \subset X$, the intersection $U \cap V$ is also an affine open subscheme. To be checked.

Proposition 1.1.31. The composition of separated morphisms is separated. Moreover, separatedness is stable under base change, i.e., if $f : X \to Y$ is a separated morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also separated. To be checked.

Proposition 1.1.32. A morphism of schemes $f : X \to Y$ is separated if and only if for every commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[d] \arrow[dr] & \\
\operatorname{Spec} R \arrow[r] \arrow[dr] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists at most one morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. To be checked.

Definition 1.1.33. A morphism of schemes $f : X \to Y$ is *universally closed* if for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map. To be checked.

Definition 1.1.34. A morphism of schemes $f : X \to Y$ is *proper* if it is of finite type, separated, and universally closed. A scheme X is *proper* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is proper. To be checked.

Theorem 1.1.35. Any projective morphism is proper. In particular, any projective scheme is proper. To be checked.

Proposition 1.1.36. The composition of proper morphisms is proper. Moreover, properness is stable under base change, i.e., if $f : X \to Y$ is a proper morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also proper. To be checked.

Proposition 1.1.37. A morphism of schemes $f : X \to Y$ is proper if and only if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists a unique morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. To be checked.

1.1.7 Varieties

Definition 1.1.38. Let $\Bbbk$ be a algebraically closed field. A *variety over* $\Bbbk$ is an integral scheme of finite type over $\operatorname{Spec} \Bbbk$. The category of varieties over $\Bbbk$ is denoted by $\mathbf{Var}_{\Bbbk}$. To be checked.

Let X be a variety over $\Bbbk$. The closed points $X(\Bbbk)$ is a locally ringed subspace of X with the induced topology and structure sheaf. We denote the category of such locally ringed spaces by $\mathbf{ClaVar}_{\Bbbk}$, meaning the category of *classical varieties* over $\Bbbk$.

Theorem 1.1.39. Let X be a variety over $\Bbbk$. Then there is an equivalence of categories between $\mathbf{Var}_{\Bbbk}$ and $\mathbf{ClaVar}_{\Bbbk}$.

Slogan *Closed points determine varieties.*

Proof. To be continued. □

1.2 Category of sheaves of modules

Mostly results in this section fits into the context of ringed spaces.

1.2.1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 1.2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets.

Definition 1.2.2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that there exists a surjective morphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

Remark 1.2.3. There are many versions of "local" properties for sheaves of $\mathcal{O}_X$-modules. To be continued.

Definition 1.2.4. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it is finitely generated, and for every open set $U \subseteq X$ and every morphism of sheaves of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of φ is finitely generated.

Slogan

$$\mathbf{Sh}_X(\mathbf{Ab}) \supseteq \mathbf{Mod}_{\mathcal{O}_X} \supseteq \mathbf{QCoh}_X \supseteq \mathbf{Coh}_X.$$

Definition 1.2.5. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a coherent sheaf on X. The *support* of $\mathcal{F}$ is defined to be the set

$$\mathrm{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\},$$

where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x. To be checked.

1.2.2 As abelian categories

Theorem 1.2.6. Let $(X, \mathcal{O}_X)$ be a ringed space. All of $\mathbf{Sh}_X(\mathbf{Ab}), \mathbf{Mod}(\mathcal{O}_X), \mathbf{QCoh}_X, \mathbf{Coh}_X$ are abelian categories.

Theorem 1.2.7. Let $(X, \mathcal{O}_X)$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules has enough injectives. To be checked.

Remark 1.2.8. The category of sheaves of $\mathcal{O}_X$-modules generally does not have enough projectives. To be checked.

Theorem 1.2.9. Let X be a noetherian, integral, separated, regular scheme. Then every coherent sheaf on X admits a finite resolution by locally free sheaves.

Proof. To be continued. □

1.2.3 Relevant functors

Definition 1.2.10. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *sheaf Hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf of abelian groups defined as follows: for an open set $U \subseteq X$, we define

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of morphisms of sheaves of $\mathcal{O}_U$-modules from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. For an inclusion of open sets $V \subseteq U$, the restriction map

$$\mathrm{res}_{UV} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(V)$$

is defined by sending a morphism $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ to its restriction $\varphi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. To be continued.

Definition 1.2.11. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *dual sheaf* $\mathcal{F}^\vee$ is defined to be

$$\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$$

To be continued.

Definition 1.2.12. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules defined as follows: for an open set $U \subseteq X$, we define

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

where $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is the tensor product of $\mathcal{O}_X(U)$-modules. For an inclusion of open sets $V \subseteq U$, the restriction map
To be continued.

Definition 1.2.13. Let $f : X \to Y$ be a morphism of ringed spaces. The *pull-back functor* $f^* : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$ is defined as follows: for an $\mathcal{O}_Y$-module $\mathcal{F}$, we define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

where $f^{-1}\mathcal{F}$ is the inverse image sheaf of $\mathcal{F}$. For a morphism of $\mathcal{O}_Y$-modules $\varphi : \mathcal{F} \to \mathcal{G}$, we define

$$f^*\varphi : f^*\mathcal{F} \to f^*\mathcal{G}$$

to be the morphism induced by the morphism of sheaves of abelian groups $f^{-1}\varphi : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$.
To be continued.

Definition 1.2.14. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $Z \subseteq X$ be a closed subset. The *functor of sections with support in* Z is defined as follows: for an $\mathcal{O}_X$-module $\mathcal{F}$, we define

$$\Gamma_Z(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) \mid \mathrm{Supp}(s) \subseteq Z\},$$

where $\mathrm{Supp}(s)$ is the support of the section s. To be checked.

1.2.4 Cohomological theory

Definition 1.2.15. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *sheaf cohomology* $H^i(X, \mathcal{F})$ is defined as the i-th right derived functor of the global section functor $\Gamma(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ applied to $\mathcal{F}$, i.e.,

$$H^i(X, \mathcal{F}) := \mathrm{R}^i\Gamma(X, \mathcal{F}).$$

To be checked.

Definition 1.2.16. Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *i-th higher direct image* $\mathrm{R}^i f_*\mathcal{F}$ is defined as the i-th right derived functor of the direct image functor $f_* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ applied to $\mathcal{F}$, i.e.,

$$\mathsf{R}^i f_* \mathcal{F} := \mathsf{R}^i (f_* \mathcal{F}).$$

To be checked.

Definition 1.2.17. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The i*-th sheaf Ext functor* $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is defined as the i-th right derived functor of the sheaf Hom functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$ applied to $\mathcal{G}$, i.e.,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \mathsf{R}^i \, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

To be checked.

Proposition 1.2.18. Let $f : X \to Y$ be a morphism of ringed spaces, and let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. Then there are long exact sequences of $\mathcal{O}_Y$-modules

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to \mathsf{R}^1 f_*\mathcal{F} \to \mathsf{R}^1 f_*\mathcal{G} \to \mathsf{R}^1 f_*\mathcal{H} \to \mathsf{R}^2 f_*\mathcal{F} \to \cdots$$

To be checked.

Theorem 1.2.19 (Affine criterion by Serre)**.** Let X be a scheme. Then X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i > 0$. To be checked.

Theorem 1.2.20 (Leray spectral sequence)**.** Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there exists a spectral sequence

$$E_2^{p,q} = H^p(Y, \mathsf{R}^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

To be checked.

1.3 Line bundles and divisors

1.3.1 Cartier divisors

Definition 1.3.1. Let X be a scheme. A *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of X. Equivalently, a Cartier divisor D can be represented by an open covering $\{U_i\}$ of X and a collection of rational functions $f_i \in \mathcal{K}_X^*(U_i)$ such that for any i, j, the function $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. We denote a Cartier divisor by $D = \{(U_i, f_i)\}$.

1.3.2 Line bundles and Picard group

Definition 1.3.2. Let X be a scheme. A *line bundle* on X is a locally free sheaf of $\mathcal{O}_X$-modules of rank 1.

Example 1.3.3. Let $X = \mathbb{P}^n_A = \operatorname{Proj} A[T_0, T_1, \dots, T_n] = \operatorname{Proj} B$ be the projective n-space over a ring A. For each integer $d \in \mathbb{Z}$, the sheaf $\mathcal{O}_X(d)$, defined by

$$\{f \neq 0\} \mapsto B(d)_{(f)},$$

is a line bundle on X, called the *twisted line bundle* of degree d. Recall that here $B(d)_{(f)}$ is the degree-zero part of the localization of the shifted graded ring $B(d)$ at the multiplicative set generated by f, and $B(d)$ is defined by $B(d)_m = B_{m+d}$ for all $m \in \mathbb{Z}$.

Let us verify this by direct computation. On the standard open subset $U_i = D_+(T_i) = \operatorname{Spec} B_i$, where $B_i = A\,[T_0/T_i, \dots, T_n/T_i]$, write $t_{j,i} = T_j/T_i$. We have

$$\mathcal{O}_X(d)(U_i) = B(d)^0_{(T_i)} = \left\{ \frac{f}{T_i^k} \middle| f \in B, \deg f = k + d \right\} = B_i \cdot T_i^d =: B_i \cdot e_i,$$

where we denote $e_i = T_i^d$. Hence $\mathcal{O}_X(d)(U_i)$ is a free B_i-module of rank 1 and thus $\mathcal{O}_X(d)$ is locally free of rank 1.

In the language of bundles, on $U_{ij} = U_i \cap U_j$, we have

$$e_i = t_{i,j}^d \cdot e_j.$$

Thus the transition functions of $\mathcal{O}_X(d)$ are given by $\{(U_{ij},\ t_{i,j}^d : U_{ij} \to \mathbb{G}_m)\}$.

Proposition 1.3.4. Let X be a scheme and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. Then

(a) the tensor product $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is also a line bundle on X;

(b) the dual $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also a line bundle on X;

(c) there is a natural isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$.

Proof. □

Definition 1.3.5. Let X be a scheme. The *Picard group* of X is defined to be the group of isomorphism classes of line bundles on X with the group operation given by the tensor product. It is denoted by $\operatorname{Pic}(X)$.

Definition 1.3.6. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\mathbb{k})$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X \times \{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X \times \{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\text{alg}} \mathcal{L}'$. To be checked.

1.3.3 Weil divisors and reflexive sheaves

To talk about Weil divisors, we need to work with normal schemes.

Definition 1.3.7. Let X be a normal integral scheme. A *Weil divisor* on X is a formal sum

$$D = \sum_{Z} n_Z Z,$$

where the sum runs over all prime divisors Z of X (i.e., integral closed subschemes of codimension 1) and $n_Z \in \mathbb{Z}$, such that for any affine open subset $U = \operatorname{Spec} A \subseteq X$, only finitely many Z intersecting U have nonzero coefficients n_Z. The group of Weil divisors on X is denoted by $\mathrm{WDiv}(X)$.

Definition 1.3.8. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. The sheaf $\mathcal{F}$ is called *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

Proposition 1.3.9. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. If $\mathcal{F}$ is reflexive, then it is determined by its restriction to any open subset $U \subseteq X$ whose complement has codimension at least 2, i.e., $\mathcal{F} \cong i_*(\mathcal{F}|_U)$, where $i : U \hookrightarrow X$ is the inclusion map. To be checked.

Proof. To be continued. □

Theorem 1.3.10. Let X be a normal integral scheme. There is a one-to-one correspondence between the set of isomorphism classes of reflexive sheaves of rank 1 on X and the Weil divisor class group $\mathrm{WDiv}(X)$ of X. Under this correspondence, a Weil divisor D corresponds to the reflexive sheaf $\mathcal{O}_X(D)$. To be checked.

Proof. To be continued. □

1.3.4 The first Chern class

Definition 1.3.11. Let X be a normal scheme and $\mathcal{L}$ a vector bundle on X. The *first Chern class* of $\mathcal{L}$, denoted by $c_1(\mathcal{L})$, is a Weil divisor class defined as follows:
To be completed.

Definition 1.3.12. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. On X_{reg}, the regular locus of X, $\mathcal{F}|_{X_{\mathrm{reg}}}$ admits a finite resolution by vector bundles

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F}|_{X_{\mathrm{reg}}} \to 0.$$

The *first Chern class* of $\mathcal{F}$, denoted by $c_1(\mathcal{F})$, is defined to be

$$c_1(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i c_1(\mathcal{E}_i).$$

To be revised.

Proposition 1.3.13. Let X be a normal scheme and $\mathcal{F}$ a torsion sheaf on X. Then

$$c_1(\mathcal{F}) = \sum_Z \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{F}_Z) \cdot Z,$$

where the sum runs over all prime divisors Z of X and $\mathcal{F}_Z$ is the stalk of $\mathcal{F}$ at the generic point of Z. To be checked.

1.4 Projective morphisms and "positive" line bundles

1.4.1 Ample line bundles

Definition 1.4.1. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. To be continued.

Theorem 1.4.2 (Serre Vanishing)**.** Let X be a projective scheme over a field k and $\mathcal{L}$ an ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that for all $n \geq N$, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

Definition 1.4.3. Let $(X, \mathcal{O}_X(1))$ be a projective scheme over a field k and $\mathcal{F}$ a coherent sheaf on X. The *Hilbert polynomial* of $\mathcal{F}$ with respect to $\mathcal{O}_X(1)$ is the polynomial

$$P_{\mathcal{F}}(n) = \chi(X, \mathcal{F} \otimes \mathcal{O}_X(n)) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F} \otimes \mathcal{O}_X(n)).$$

To be continued.
Let $Z \subseteq X$ be a closed subscheme with ideal sheaf $\mathcal{I}_Z$. The *Hilbert polynomial* of Z with respect to $\mathcal{O}_X(1)$ is defined as $P_Z(n) = P_{\mathcal{O}_X/\mathcal{I}_Z}(n)$. To be revised.

1.4.2 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 1.4.4. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$. We need a more "functorial" expression.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^* x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Proposition 1.4.5. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. To be continued. □

Example 1.4.6. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0, \dots, i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 1.4.7. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}^{(m+1)(n+1)-1}_A$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 1.4.8. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. To be continued.

Example 1.4.9. Let

Example 1.4.10.

Definition 1.4.11. Let $\mathcal{L}$ be a line bundle on a scheme X. To be continued.

Definition 1.4.12. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. To be continued.

Theorem 1.4.13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

To be continued.

Proposition 1.4.14. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

To be continued.

Proof. To be continued. □

1.4.3 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 1.4.15. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

1.4.4 Asymptotic behavior

Definition 1.4.16. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. To be continued.

Definition 1.4.17. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. To be continued.

Theorem 1.4.18. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \operatorname{Proj} R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. To be continued.

Definition 1.4.19. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq Cn^{\dim X}$$

for all sufficiently large n. To be continued.

Example 1.4.20. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. To be continued.

1.5 Finite morphisms and fibrations

Theorem 1.5.1 (Zariski's Main Theorem)**.** Let $f : Y \to X$ be a quasi-finite and separated morphism of schemes. Then there exists a factorization

Theorem 1.5.2 (Stein factorization)**.** Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then there exists a factorization

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where g is a proper morphism with connected fibers and h is a finite morphism. Moreover, this factorization is unique up to isomorphism. To be checked.

1.5.1 Finite morphisms

Theorem 1.5.3. Let $f : Y \to X$ be a finite morphism of schemes. If $\mathcal{L}$ is an ample line bundle on X, then $f^*\mathcal{L}$ is an ample line bundle on Y. If and only if.

1.5.2 Fibrations

Definition 1.5.4. Let $f : Y \to X$ be a proper morphism of noetherian schemes. We say that f is a fibration if for every point $x \in X$, the fiber $f^{-1}(x)$ is a geometrically connected scheme.

Proposition 1.5.5. Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then f is a fibration if and only if the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is an isomorphism. In particular, if X is an algebraically closed field and f is a fibration, then the fibers $f^{-1}(x)$ are also algebraically closed in the function field $K(X)$. To be revised

Definition 1.5.6. Let $f : Y \dashrightarrow X$ be a rational map of noetherian schemes. We say that f is a fibration if there exists an open subset $U \subseteq Y$ such that the restriction $f|_U : U \to X$ is a fibration.

1.6 Differentials and duality

Let S be a base noetherian scheme, $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over S and all varieties are assumed to be defined over $\Bbbk$.

1.6.1 The sheaves of differentials

Definition 1.6.1. Let $f : X \to S$ be an S-scheme. The *sheaf of differentials* of X over S, denoted by $\Omega_{X/S}$, is the $\mathcal{O}_X$-module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$.

Proposition 1.6.2. Let X and T be S-schemes and $X_T := X \times_S T$ be the base change of X along $T \to S$. Let $p : X_T \to X$ be the projection morphism. Then there is a natural isomorphism of $\mathcal{O}_{X_T}$-modules

$$\Omega_{X_T/T} \cong p^*\Omega_{X/S}.$$

Proof. Given by algebras, see ref. To be continued. □

Proposition 1.6.3. Let X be an S-scheme and $U \subseteq X$ be an open subscheme. Then there is a natural isomorphism of $\mathcal{O}_U$-modules

$$\Omega_{U/S} \cong \Omega_{X/S}|_U.$$

Furthermore, let $\xi \in X$, then there is a natural isomorphism of $\mathcal{O}_{X,\xi}$-modules

$$\Omega_{X/S,\xi} \cong \Omega_{\mathcal{O}_{X,\xi}/\mathcal{O}_{S,f(\xi)}}.$$

To be checked.

Proof. To be continued. □

Proposition 1.6.4. Let X be a regular variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a locally free sheaf of rank n.

Proof. To be continued. □

Proposition 1.6.5. Let X be a normal variety over $\Bbbk$ of dimension n. Then $\Omega_{X/\Bbbk}$ is a reflexive sheaf of rank n.

Proof. To be continued. □

Definition 1.6.6. Let X be a normal variety over $\Bbbk$. The *canonical divisor* K_X of X is defined to be the Weil divisor class $c_1(\Omega_{X/\Bbbk})$.

Theorem 1.6.7 (Euler sequence for projective bundle)**.** Let X be a normal variety over $\Bbbk$ and $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on X. Let $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projective bundle associated to $\mathcal{E}$. Then there is an exact sequence of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$-modules

$$0 \to \Omega_{\mathbb{P}_X(\mathcal{E})/X} \xrightarrow{\phi} \pi^*\mathcal{E}(-1) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \to 0.$$

Here $\pi^*\mathcal{E}(-1)$ is twisted by the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-1)$.

Proof.

Step 1. First assume that $X = \operatorname{Spec} A$ is affine and $\mathcal{E}$ is free. Under this assumption, find expressions for ϕ and ψ.

Fix a basis $T_0, \dots, T_r$ of the free A-module $\mathcal{E}(X)$. On the standard open subset $U_i = \{T_i \neq 0\} = \operatorname{Spec} B_i \subseteq \mathbb{P}_X(\mathcal{E})$, we have coordinates $t_{j,i} := T_j/T_i$ for $j \neq i$. The exact sequence becomes

$$0 \to \bigoplus_{k\neq i} B_i \mathrm{d}t_{k,i} \xrightarrow{\phi} \bigoplus_{k=0}^{r} B_i e_i \cdot T_k \xrightarrow{\psi} B_i \to 0.$$

Here e_i is the local generator of $\mathcal{O}_{\mathbb{P}_A(\mathcal{E})}(-1)$ on U_i, symbolically satisfying $e_i T_i = 1$.

Recall that on the overlap $U_{ij} = U_i \cap U_j$, the coordinates are related by

$$t_{i,j} e_i = e_j, \quad \mathrm{d}t_{k,i} = t_{j,i}\mathrm{d}t_{k,j} - t_{k,i}t_{j,i}\mathrm{d}t_{i,j}.$$

Here we set $t_{l,l} := 1$ for convenience. Symbolically, we have

$$\text{“ } \mathrm{d}t_{k,i} = \frac{T_i \mathrm{d}T_k - T_k \mathrm{d}T_i}{T_i^2} = e_i \mathrm{d}T_k - t_{k,i} e_i \mathrm{d}T_i \text{ ”.}$$

On the overlap U_{ij}, it transitions as

$$\begin{aligned} \text{“}\mathrm{d}t_{k,i} &= t_{j,i}\mathrm{d}t_{k,j} - t_{k,i}t_{j,i}\mathrm{d}t_{i,j} \\ &= t_{j,i}e_j\mathrm{d}T_k - t_{j,i}t_{k,j}e_j\mathrm{d}T_j - t_{k,i}t_{j,i}\left(e_j\mathrm{d}T_i - t_{i,j}e_j\mathrm{d}T_j\right) \\ &= e_i\mathrm{d}T_k - t_{k,i}e_i\mathrm{d}T_i\text{”.} \end{aligned}$$

To make sense of the above symbolic expressions, we define ϕ and ψ locally on each U_i by

$$\phi(\mathrm{d}t_{k,i}) = e_i T_k - t_{k,i} e_i T_i, \quad \psi(e_i T_k) = t_{k,i}.$$

Step 2. Verify that ϕ and ψ are well-defined and the sequence is exact.

By computations in Step 1, ϕ is well-defined on the overlaps U_{ij}. For ψ, on the overlap U_{ij}, we have

$$\psi(e_j T_k) = \psi(t_{i,j} e_i T_k) = t_{i,j} t_{k,i} = t_{k,j}.$$

Thus ψ is also well-defined. It is clear that $\psi \circ \phi = 0$. Consider the matrix representation of ϕ with respect to the bases $\{\mathrm{d}t_{k,i}\}_{k\neq i}$ and $\{e_i T_k\}_{k=0}^r$:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ -t_{0,i} & -t_{1,i} & \cdots & -t_{i-1,i} & -t_{i+1,i} & \cdots & -t_{r,i} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

It has rank r, ϕ is injective and $\ker\psi = \sqrt{-1}\phi$. Thus the sequence is exact.

Step 3. General case: glue the local exact sequences on affine open subsets of X.

In the local case, choose a different basis $S_0, \dots, S_r$ of $\mathcal{E}(X)$ given by the transition matrix $g \in \mathrm{GL}_{r+1}(A)$. For simplicity, we just look at on the open subset $U = \{T_0 \neq 0, S_0 \neq 0\}$. Set B_U be the localization of $B = A[T_0, \dots, T_r]$ at the multiplicative set generated by T_0 and S_0. It is still a graded algebra.

Note that ϕ is formally given by differentials in $A[T_0, \dots, T_r]$ and then sending the symbol $\mathrm{d}T_i$ to T_i and $1/T_0$ to e_0. The differentials are intrinsic and linear over A, and the assignment of $1/T_0$ to e_0 is just a change of notation. Thus ϕ is independent of the choice of basis. For ψ, it is indeed given by multiplying $B_U(-1)$ by the linear part of B and then taking the degree 0 part. It is also independent of the choice of basis.

Therefore, after changing basis, ϕ and ψ remain the same. This allows us to glue the local exact sequences on each affine open subset of X to obtain a global exact sequence. □

Corollary 1.6.8. Let $\mathbf{k}$ be a field. We have

$$\omega_{\mathbb{P}^n_{\mathbf{k}}/\mathbf{k}} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbf{k}}}(-(n+1)) \quad \text{and} \quad K_{\mathbb{P}^n_{\mathbf{k}}} \sim -(n+1)H,$$

where H is a hyperplane in $\mathbb{P}^n_{\mathbf{k}}$.

1.6.2 Fundamental sequences

Theorem 1.6.9 (The first fundamental sequence of differentials)**.** Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

Proof. To be completed. □

Proposition 1.6.10. Let $f : X \to Y$ be a surjective and generically finite morphism of normal varieties over $\Bbbk$. Then the first fundamental sequence of differentials is exact on the left.

Proof. To be completed. □

Corollary 1.6.11 (Ramification formula)**.** Let $f : X \to Y$ be a finite morphism of normal varieties. Then

$$K_X = f^* K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} \big(\mathrm{Mult}_D\, f^*\big(f(D)\big) - 1\big) D$$

is the ramification divisor of f. To be checked. definition of ramification divisor needs to be checked.

Proof. To be completed. □

Theorem 1.6.12 (The second fundamental sequence of differentials)**.** Let $Z \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}|_Z \to \Omega_{Z/S} \to 0.$$

Suppose further that $Z \to X$ is a regular immersion. Then the above sequence is also exact on the left.

Proof. To be completed. □

Corollary 1.6.13 (Adjunction formula)**.** Let X be a normal variety and $Z \subseteq X$ be a prime Cartier divisor which is normal as variety. Then

$$K_Z = (K_X + Z)|_Z.$$

Proof. Since both X and Z are normal, they are smooth in codimension 1. Removing the singular locus of X and Z, we may assume that both X and Z are smooth varieties. This is valid since the canonical divisor is determined by the smooth locus.

Since Z is Cartier, it is a local complete intersection in X. By Theorem 1.6.12, we have the exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/\Bbbk}|_Z \to \Omega_{Z/\Bbbk} \to 0.$$

Note that Z is of codimension 1 in X, so $\mathcal{I}_Z \cong \mathcal{O}_X(-Z)$ and thus $\mathcal{I}_Z/\mathcal{I}_Z^2 \cong \mathcal{O}_X(-Z)|_Z$. Taking c_1, we obtain

$$c_1(\Omega_X)|_Z = c_1(\Omega_Z) + c_1(\mathcal{O}_X(-Z))|_Z.$$

That is,

$$K_X|_Z = K_Z - Z|_Z.$$

Rearranging gives the desired result. To be revised. restriction of Weil divisors needs to be clarified. □

1.6.3 Serre duality

Definition 1.6.14 (Dualizing sheaf)**.** Let X be a proper scheme of dimension n over $\Bbbk$. A *dualizing sheaf* on X is a coherent sheaf ω_X° together with a trace map $\mathrm{tr}_X : H^n(X, \omega_X^\circ) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\operatorname{tr}_X} \Bbbk$$

induces an isomorphism

$$\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

Theorem 1.6.15. Let X be a projective scheme of dimension n over $\Bbbk$. Then there exists a dualizing sheaf ω_X° on X up to isomorphism. Moreover, if X is smooth, $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\Bbbk}$.

Proof. To be completed. □

Theorem 1.6.16 (Serre duality)**.** Let X be a projective, Cohen-Macaulay variety of dimension n over $\Bbbk$ with dualizing sheaf ω_X°. Then for every coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphism

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. To be completed. □

When $\mathcal{F}$ is locally free, we have $\operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^i(X, \omega_X^\circ \otimes \mathcal{F}^\vee)$.

Corollary 1.6.17. Let X be a projective, normal variety of dimension n over $\Bbbk$. Then for every integer m and $0 \le i \le n$, there is a natural isomorphism To be completed.

1.6.4 Logarithm version

1.7 Flat, smooth and étale morphisms

1.7.1 Flat families

Definition 1.7.1. Let $f : X \to Y$ be a morphism of schemes. For a point $\xi \in X$, we say that f is *flat at* ξ if the local ring $\mathcal{O}_{X,\xi}$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that f is *flat* if it is flat at every point $\xi \in X$.

Definition 1.7.2. Let X be Y-scheme via a morphism $f : X \to Y$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *flat over* Y *at* $\xi \in X$ if the stalk $\mathcal{F}_\xi$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that $\mathcal{F}$ is *flat over* Y if it is flat over Y at every point $\xi \in X$.

Proposition 1.7.3. We have the following fundamental properties of flat morphisms:

(a) open immersions are flat;

(b) the composition of flat morphisms is flat;

(c) flatness is preserved under base change;

(d) a coherent sheaf $\mathcal{F}$ on a noetherian scheme X is flat over X iff it is locally free.

To be checked.

Proposition 1.7.4. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. Then the set of points $\xi \in X$ at which f is flat is open in X. To be checked.

Proposition 1.7.5. Let X be a regular integral scheme of dimension 1 and $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is flat over X iff it is torsion-free, i.e., for every non-zero-divisor $s \in \mathcal{O}_{X,x}$, the multiplication map

$$s : \mathcal{F} \to \mathcal{F}$$

is injective. To be checked.

Proposition 1.7.6. Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field $\mathbf{k}$. Then for every point $\xi \in X$, we have

$$\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}.$$

To be checked.

Theorem 1.7.7 (Miracle flatness)**.** Let $f : X \to Y$ be a morphism between noetherian schemes. Suppose that X is Cohen–Macaulay and that Y is regular. Then f is flat at $\xi \in X$ iff $\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}$. To be checked.

Theorem 1.7.8. Let T be a integral noetherian scheme and $f : X \to T$ be a projective morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix a relatively ample line bundle H on X over T. Then $\mathcal{F}$ is flat over T iff the Hilbert polynomials

$$P_{(X_t, H_t; \mathcal{F}_t)}(n) = \chi(X_t, \mathcal{F}_t \otimes H_t^{\otimes n})$$

are independent of $t \in T$. To be checked.

To be added: deformation, algebraic families...

1.7.2 Base change and semicontinuity

Theorem 1.7.9 (Grauert's theorem)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the sheaf $R^i f_* \mathcal{F}$ is coherent on Y, and there exists an open subset $U \subseteq Y$ such that for every point $y \in U$, the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism. To be checked.

Theorem 1.7.10 (Cohomology and base change)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. For each integer $i \geq 0$, the following are equivalent:

(a) the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism for all points $y \in Y$;

(b) the sheaf $R^i f_* \mathcal{F}$ is locally free on Y.

To be checked.

Theorem 1.7.11 (Semicontinuity of cohomology)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the function

$$h^i : Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is upper semicontinuous on Y.

To be checked.

1.7.3 Smooth morphisms

Definition 1.7.12. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. For $\xi \in X$ with image $\zeta = f(\xi) \in Y$, set $\overline{\zeta} : \operatorname{Spec} \overline{\kappa(\zeta)} \to Y$ to be the geometric point over ζ and $X_{\overline{\zeta}}$ be the geometric fiber over ζ. We say that f is *smooth at* ξ if f is flat at ξ and the geometric fiber $X_{\overline{\zeta}}$ is regular over $\overline{\kappa(\zeta)}$ at every point lying over ξ. We say that f is *smooth* if it is smooth at every point $\xi \in X$.

To be checked.

1.7.4 Étale morphisms

Definition 1.7.13. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. We say that f is *étale at* ξ if f is smooth and finite at ξ. We say that f is *étale* if it is étale at every point $\xi \in X$.

To be checked.

Chapter 2

Surfaces

2.1 The first properties of surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

2.1.1 Basic concepts

Definition 2.1.1. A *surface* is a two-dimensional integral scheme of finite type over an algebraically closed field $\Bbbk$. A *projective surface* is a surface that is projective over $\Bbbk$. A *smooth surface* is a surface that is smooth over $\Bbbk$. To be checked.

2.1.2 Riemann-Roch Theorem for surfaces

2.1.3 Hodge Index Theorem

2.2 Birational geometry on surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

2.2.1 Birational morphisms on surfaces

Let X be a smooth projective surface, $0 \in X(\Bbbk)$ and $\pi : \tilde{X} = \mathrm{Bl}_0 X \to X$ the blow-up of X at 0. Denote by E the exceptional divisor of π.

Proposition 2.2.1. We have $E^2 = -1$.

Proof. To be continued $\square$

Proposition 2.2.2. We have $K_{\tilde{X}} = \pi^* K_X + E$.

Proof. We have the exact sequence

$$\Omega_{\tilde{X}} \to \pi^*\Omega_X \to \Omega_{\tilde{X}/X} \to 0.$$

Since both $\tilde{X}$ and X are smooth, $\Omega_{\tilde{X}}$ and Ω_X are locally free sheaves of rank 2. The kernel of the first map is of rank 0 and torsion, thus it is zero. Therefore, we have the short exact sequence

$$0 \to \Omega_{\tilde{X}} \to \pi^*\Omega_X \to \Omega_{\tilde{X}/X} \to 0.$$

By taking c_1, we only need to show that $c_1(\Omega_{\tilde{X}/X}) = E$.

For $\eta \in \tilde{X}$ of codimension 1, if $\eta \notin E$, then $(\Omega_{\tilde{X}/X})_\eta = \Omega_{\mathcal{O}_{\tilde{X},\eta}/\mathcal{O}_{X,\pi(\eta)}} = 0$. Hence we only need to consider the case $\overline{\{\eta\}} = E$. To be continued ☐

Corollary 2.2.3. We have $K_{\tilde{X}}^2 = K_X^2 - 1$.

Proof. By Proposition 2.2.2, we have

$$K_{\tilde{X}}^2 = (\pi^*K_X + E)^2 = (\pi^*K_X)^2 + 2\pi^*K_X \cdot E + E^2 = K_X^2 + 0 - 1 = K_X^2 - 1.$$

☐

Theorem 2.2.4. Let $f : X \to Y$ be a birational morphism between two smooth projective surfaces. Then f can be decomposed as a finite sequence of blow-ups at points.

Proof. To be continued ☐

2.2.2 Castelnuovo's Theorem

Definition 2.2.5. A (-1)-*curve* on a smooth projective surface X is an irreducible curve $C \subseteq X$ such that $C \cong \mathbb{P}^1$ and $C^2 = -1$.

Remark 2.2.6. Let C be a (-1)-curve on a smooth projective surface X. Then its numerical class $[C] \in \mathrm{N}_1(X)$ spans an extremal ray of $\mathrm{Psef}_1(X)$ such that $K_X \cdot C < 0$. To be revised.

Theorem 2.2.7 (Castelnuovo's contractibility criterion)**.** Let X be a smooth projective surface and $C \subseteq X$ an irreducible curve. Then there exists a birational morphism $f : X \to Y$ contracting C to a smooth point if and only if C is a (-1)-curve.

Proof. To be continued ☐

Definition 2.2.8. A *minimal surface* is a smooth projective surface that does not contain any (-1)-curves. To be checked.

2.2.3 Resolution of singularities on surface

Definition 2.2.9. A *resolution of singularities* of a projective surface X is a smooth projective surface $\tilde{X}$ together with a birational and proper morphism $\pi : \tilde{X} \to X$ such that π is an isomorphism over the smooth locus of X. To be checked.

Theorem 2.2.10 (Resolution of singularities on surfaces)**.** Let X be a projective surface over an algebraically closed field $\Bbbk$. Then X admits a resolution of singularities. To be checked.

Definition 2.2.11. Let X be a projective surface. A *minimal resolution* of X is a resolution of singularities $\pi : \widetilde{X} \to X$ such that for any other resolution of singularities $\pi' : \widetilde{X}' \to X$, there exists a morphism $f : \widetilde{X}' \to \widetilde{X}$ such that π' factors as $\pi' = \pi \circ f$.

Proposition 2.2.12. Let X be a projective surface. Then X admits a unique minimal resolution of singularities.

Proof. To be continued □

2.3 Coarse classification of surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. We want to classify X up to birational equivalence. Let K_X be the canonical divisor of X.

Theorem 2.3.1. Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. Suppose that the Kodaira dimension $\kappa(X) \geq 0$. Then the linear system $|12K_X|$ is base point free. To be checked.

2.3.1 Classification

Theorem 2.3.2 (Enriques-Kodaira classification)**.** Let X be a smooth projective surface over $\Bbbk$. Then X is birational to a unique minimal model X', unless X is birational to a ruled surface. Moreover, the minimal model X' falls into one of the following classes:

(a) $\kappa(X') = -\infty$: $X' \cong \mathbb{P}^2$ or X' is a ruled surface;

(b) $\kappa(X') = 0$: X' is a K3 surface, an abelian surface or their quotients;

(c) $\kappa(X') = 1$: X' is an elliptic surface;

(d) $\kappa(X') = 2$: X' is a surface of general type.

To be checked.

2.4 Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$. This section is mainly based on [Har77, Chapter V.2].

2.4.1 Minimal Section and Classification

Definition 2.4.1 (Ruled surface)**.** A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all geometric fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 2.4.2. There exists a section of π.

Proof. To be continued... □

Proposition 2.4.3. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Proof. Let $\sigma : C \to X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π, $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in C$. By Miracle Flatness (??), f is flat. By Grauert's Theorem (Theorem 1.7.9), $\mathcal{E}$ is a vector bundle of rank 2 on C and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in C$.

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every $x \in X$, we have

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \to \mathcal{L}$ is surjective.

By ??, we have a morphism $\varphi : X \to \mathbb{P}_C(\mathcal{E})$ over C such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Since $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, $\varphi|_{X_t} : X_t \to \mathbb{P}_C(\mathcal{E})_t$ is an isomorphism for any $t \in C$. Hence φ is bijection on the underlying sets. Here is a serious gap. Why fiberwise isomorphism implies isomorphism? □

Lemma 2.4.4. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*. In particular, if $\mathcal{E}$ is normalized, then $e = -\deg c_1(\mathcal{E})$ is an invariant of the ruled surface X.

Proof. We can suppose that $\mathcal{E}$ is globally generated since we can always twist $\mathcal{E}$ by a sufficiently ample line bundle on C. Then for all line bundle $\mathcal{L}$ of degree sufficiently large, $\mathcal{L}$ is very ample and hence $H^0(C, \mathcal{E} \otimes \mathcal{L}) \neq 0$. By Lemma 2.4.2 and ??, $\mathcal{E}$ is an extension of line bundles. Then for all line bundle $\mathcal{L}$ of degree sufficiently negative, $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ since line bundles of negative degree have no global sections. Hence we can find a line bundle $\mathcal{M}$ on C of lowest degree such that $H^0(C, \mathcal{E} \otimes \mathcal{M}) \neq 0$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{M}$, we are done. □

Remark 2.4.5. The invariant e is unique but the normalization of $\mathcal{E}$ is not unique. For example, if $\mathcal{E}$ is normalized, then so is $\mathcal{E} \otimes \mathcal{L}$ for any line bundle $\mathcal{L}$ on C of degree 0. To be continued...

Suppose that $X \cong \mathbb{P}_C(\mathcal{E})$ where $\mathcal{E}$ is a normalized vector bundle of rank 2 on C. Since $H^0(C, \mathcal{E}) \neq 0$, choosing a non-zero section s, we get an exact sequence

$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{E} \to \mathcal{E}/\mathcal{O}_C \to 0.$$

We claim that $\mathcal{E}/\mathcal{O}_C$ is a line bundle on C. Since C is a curve, we only need to check that $\mathcal{E}/\mathcal{O}_C$ is torsion-free.

To be continued...

Definition 2.4.6. A section C_0 of π is called a *minimal section* if to be continued...

Lemma 2.4.7. Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface over a smooth curve C of genus g with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on C with $\deg \mathcal{L} = -e$.

(b) If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.

Proof. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable, we can assume that $H^0(C, \mathcal{L}_1) \neq 0$. If $\deg \mathcal{L}_1 > 0$, then $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$, contradicting the normalization of $\mathcal{E}$. Similarly $\deg \mathcal{L}_2 \leq 0$. Then $\mathcal{L}_1 \cong \mathcal{O}_C$. And hence $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$.

If $\mathcal{E}$ is indecomposable, we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

which is a non-trivial extension, with $\mathcal{L}$ a line bundle on C of degree $-e$. Hence by ??, we have $0 \neq \operatorname{Ext}^1_C(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$. By Serre duality, we have $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$. Hence $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$.

On the other hand, let $\mathcal{M}$ be a line bundle on C of degree -1. Twist the above exact sequence by $\mathcal{M}$ and take global sections, we have an equation

$$h^0(\mathcal{M}) - h^0(\mathcal{E} \otimes \mathcal{M}) + h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{M}) + h^1(\mathcal{E} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = 0.$$

Since $\deg \mathcal{M} < 0$ and $\mathcal{E}$ is normalized, we have $h^0(\mathcal{M}) = h^0(\mathcal{E} \otimes \mathcal{M}) = 0$. By Riemann-Roch, we have $h^1(\mathcal{M}) = g$ and $h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = -e - 1 + 1 - g$. Hence

$$h^1(\mathcal{E} \otimes \mathcal{M}) = e + 2g \geq 0.$$

This gives $e \geq -2g$. □

Theorem 2.4.8. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Proof. This is a direct consequence of Lemma 2.4.7. □

Example 2.4.9. Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$.

Let C be covered by two standard affine charts U_0, U_1 with coordinate u on U_0 and v on U_1 such that $u = 1/v$ on $U_0 \cap U_1$. On U_i, let $\mathcal{O}(-e)|_{U_i}$ be generated by s_i for $i = 0, 1$. We have $s_0 = u^e s_1$ on $U_0 \cap U_1$.

On $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$, let $[x_0 : x_1]$ and $[y_0 : y_1]$ be the homogeneous coordinates of $\mathbb{P}^1$ on X_0 and X_1 respectively. Then the transition function on $X_0 \cap X_1$ is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

Remark 2.4.10. The surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ is also called the *Hirzebruch surface*.

Theorem 2.4.11. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and for each e there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on E with $\deg \mathcal{L} = -e$.

Proof. Only the indecomposable case needs a proof. By Lemma 2.4.7, we have $-2 \leq e \leq 0$ and a non-trivial extension

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle on E of degree $-e$.

Case 1. $e = 0$.

In this case, $\mathcal{L}$ is of degree 0 and $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \neq 0$. Hence $\mathcal{L} \cong \mathcal{O}_E$. To be continued...

Case 2. $e = -1$.

In this case, $\mathcal{L}$ is of degree 1 and $H^1(E, \mathcal{L}) \cong H^0(E, \mathcal{L}^{-1}) = 0$. By Riemann-Roch, we have $h^0(E, \mathcal{L}) = 1$.

Case 3. $e = -2$.

To be continued...

□

Example 2.4.12. To be continued...

2.4.2 The Néron-Severi Group of Ruled Surfaces

Proposition 2.4.13. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and F a fiber of π. Then $\mathrm{Pic}(X) \cong \mathbb{Z}[C_0] \oplus \pi^* \mathrm{Pic}(C)$.

Proof. Let D be any divisor on X with $D.F = a \in \mathbb{Z}$. Then $D - aC_0$ is numerically trivial on the fibers of π. Let $\mathcal{L} = \mathcal{O}_X(D - aC_0)$. Then $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$ for any $t \in C$. By Grauert's Theorem (Theorem 1.7.9), $\pi_* \mathcal{L}$ is a line bundle on C and the natural map $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ is an isomorphism. □

Proposition 2.4.14. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let F be a fiber of π. Then $K_X \sim -2C_0 + \pi^*(K_C - c_1(\mathcal{E}))$. Numerically, we have $K_X \equiv -2C_0 + (2g - 2 - e)F$ where e is the invariant of X. Check this carefully.

Proof. To be continued. □

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

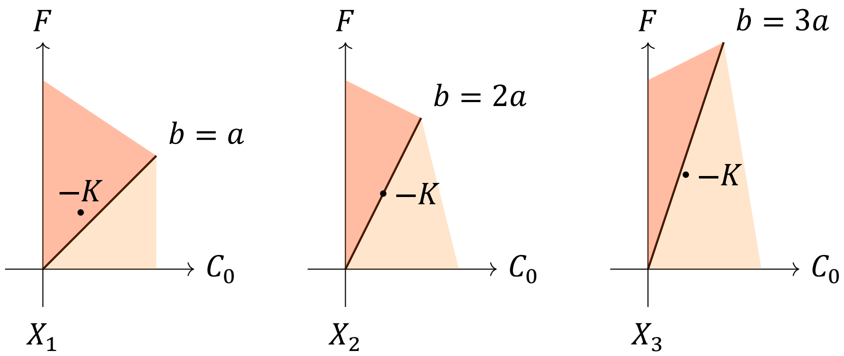
Theorem 2.4.15. Let $\pi : X \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with invariant e. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \sim aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a, b \geq 0$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$.

Proof. To be continued... □

Example 2.4.16. Here we draw the Néron-Severi group of the rational ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 1, 2, 3$.



We have $-K_{X_e} \equiv 2C_0 + (2+e)F$. For $e = 1$, $-K$ is ample and hence X_1 is a del Pezzo surface. For $e = 2$, $-K$ is nef and big but not ample. For $e \geq 3$, $-K$ is big but not nef.

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

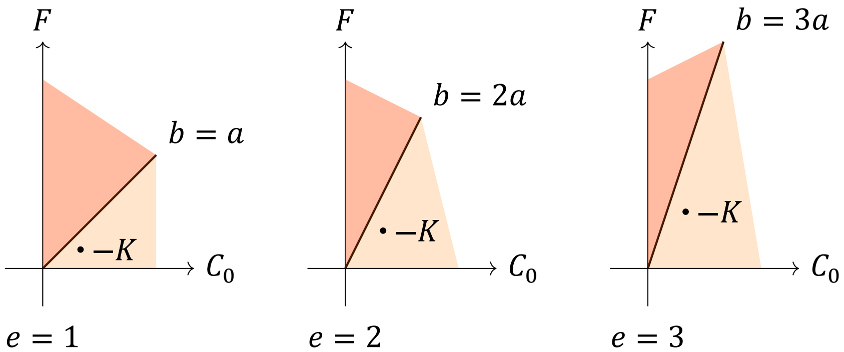
Theorem 2.4.17. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is decomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$.

Proof. To be continued... □

Example 2.4.18. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with decomposable normalized $\mathcal{E}$ for $e = 1, 2, 3$.



In this case, $-K \equiv 2C_0 + eF$ is always big but not nef.

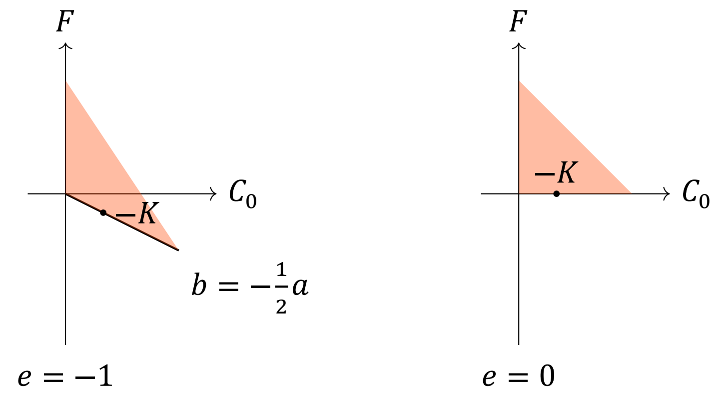
Theorem 2.4.19. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is indecomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq \frac{1}{2}ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > \frac{1}{2}ae$.

Proof. To be continued... □

Example 2.4.20. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with indecomposable normalized $\mathcal{E}$ for $e = -1, 0$.



In this case, $-K \equiv 2C_0 + eF$ is always nef but not big.

Proposition 2.4.21. Let $\pi : X \to C$ be a ruled surface over a smooth curve C. Then every nef divisor on X is semi-ample. Check this carefully.

2.5 K3 surface

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

2.5.1 The first properties

Definition 2.5.1. A *K3 surface* is a smooth, projective surface X with trivial canonical bundle $K_X \cong \mathcal{O}_X$ and irregularity $q(X) = h^1(X, \mathcal{O}_X) = 0$.

Example 2.5.2. A smooth quartic surface $X \subseteq \mathbb{P}^3$ is a K3 surface. Indeed, by the adjunction formula, we have

$$K_X = (K_{\mathbb{P}^3} + X)|_X = (-4H + 4H)|_X = 0,$$

where H is a hyperplane in $\mathbb{P}^3$. Moreover, by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0,$$

we have long exact sequence in cohomology

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(X, \mathcal{O}_X) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \to \cdots.$$

Since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$ and $H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$, we get $H^1(X, \mathcal{O}_X) = 0$.

2.5.2 Hodge Structure and Moduli of K3 surfaces

2.5.3 Neron-Severi group of K3 surfaces

2.6 Elliptic surfaces

2.6.1 The first properties

Definition 2.6.1. An *elliptic surface* is a smooth projective surface S together with a surjective morphism $\pi : S \to C$ to a smooth projective curve C such that the generic fiber of π is a smooth curve of genus 1, and π has a section $s : C \to S$. To be continued...

2.6.2 Classification of singular fibers

2.6.3 Mordell-Weil group and Neron-Severi group

2.7 Some Singular Surfaces

In this section, fix an algebraically closed field $\Bbbk$. Everything is over $\Bbbk$ unless otherwise specified.

2.7.1 Projective cone over smooth projective curve

Let $C \subset \mathbb{P}^n$ be a smooth projective curve. The *projective cone* over C is the projective variety $X \subset \mathbb{P}^{n+1}$ defined by the same homogeneous equations as C. The variety X is singular at the vertex of the cone, which corresponds to the point $[0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}$.

Chapter 3

Moduli Spaces

3.1 Introduction to moduli problems

3.1.1 Moduli problem by representable functors

Moduli space is a geometric space whose points represent isomorphism classes of certain geometric objects. For example, fix a field $\Bbbk$, all elliptic curves over $\Bbbk$ can be classified by the j-invariant, which gives a bijection between isomorphism classes of elliptic curves and elements of $\Bbbk$. However, this classification is just a set-theoretic one. We would like to have a geometric "parameter space" such that we can "deform" the objects continuously. This is the initial motivation for moduli spaces.

In algebraic geometry, "deforming objects continuously" is usually described by flat families. Hence the most perfect object to represent this moduli problem is such a flat family $\mathcal{X} \to M$, where M is a variety parameterizing all elliptic curves, and the fiber over each point $m \in M(\Bbbk)$ is the elliptic curve corresponding to m. Furthermore, we hope that this family is universal, i.e., any other flat family $\mathcal{Y} \to B$ of elliptic curves is obtained by pulling back $\mathcal{X} \to M$ along a unique morphism $B \to M$. (Despite in this example, such a perfect family does not exist; we will discuss this later.) It can be described by the language of functors. Consider the functor

$$\mathcal{M} : \mathbf{Var}_{\Bbbk}^{\mathrm{op}} \to \mathbf{Set}, \quad X \mapsto \{\text{flat families of elliptic curves over } X\}/\sim,$$

where $\sim$ is the equivalence relation given by isomorphisms of families over X. If the family $\mathcal{X} \to M$ above exists, then it represents the functor $\mathcal{M}$. In this case, we say that the moduli problem $\mathcal{M}$ is *representable*, M is called a *fine moduli space* and $\mathcal{X} \to M$ is called a *universal object*.

Hence, to study a moduli problem, we follow the steps:

1. Define the moduli functor $\mathcal{M}$.

2. Check whether $\mathcal{M}$ is representable.

Slogan *To solve a moduli problem is to find a representable functor which describes "continuous deformations".*

Definition 3.1.1. Let S be a noetherian scheme and $\mathcal{M} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ be a functor. If $\mathcal{M}$ is representable by a scheme M of finite type over S, then we say that M is a *fine moduli space* for the moduli problem $\mathcal{M}$. The object $\mathcal{U} \in \mathcal{M}(M)$ corresponding to the identity morphism $\mathrm{id}_M \in \mathrm{Hom}_{\mathbf{Sch}_S}(M, M)$ is called a *universal object* over M.

We give a simple example of a fine moduli space.

Example 3.1.2. Consider the moduli problem of lines in the projective plane $\mathbb{P}^2_{\Bbbk}$ over a field $\Bbbk$. Define the moduli functor

$$\mathcal{G} : \mathbf{Sch}_{\Bbbk}^{\mathrm{op}} \to \mathbf{Set}, \quad X \mapsto \{L \subset \mathbb{P}^2 \times X | L \text{ is flat over } X, L_x \text{ is a line in } \mathbb{P}^2 \text{ for all } x \in X(\Bbbk)\}.$$

We claim that the dual projective plane $G = \mathbb{P}^2_{\Bbbk}$ is a fine moduli space for the moduli problem $\mathcal{G}$. The universal object is given by

$$U = \{([x : y : z], [a : b : c]) \in \mathbb{P}^2_{\Bbbk} \times G | ax + by + cz = 0\} \subset \mathbb{P}^2_{\Bbbk} \times G.$$

For every $X \in \mathbf{Sch}_{\Bbbk}$ and $f \in G(X)$, we can give a family of lines $L = U \times_G X \in \mathcal{G}(X)$ by pulling back the universal object U along $\mathrm{id}_{\mathbb{P}^2_{\Bbbk}} \times f : \mathbb{P}^2_{\Bbbk} \times X \to \mathbb{P}^2_{\Bbbk} \times G$. The difficult part is to construct the inverse map, i.e., given a family of lines $L \in \mathcal{G}(X)$, we need to construct a morphism $f : X \to G$ such that L is obtained by pulling back U along $\mathrm{id}_{\mathbb{P}^2_{\Bbbk}} \times f$.

We need a more "functorial" way to describe the dual projective plane G. Set $V = H^0(\mathcal{O}_{\mathbb{P}^2_{\Bbbk}}(1))$, let $G = \mathrm{Proj}(\mathrm{Sym}^{\bullet} V^{\vee})$ be the dual projective plane. To give a morphism $f : X \to G$, it is equivalent to giving a line bundle $\mathcal{L}$ on X and a surjective morphism $\mathcal{O}_X \otimes_{\Bbbk} V^{\vee} \twoheadrightarrow \mathcal{L}$ by To be added..

Let $\mathcal{I}_L$ be the ideal sheaf of L in $\mathbb{P}^2_X = \mathbb{P}^2 \times X$. Consider the short exact sequence on $\mathbb{P}^2_X$:

$$0 \to \mathcal{I}_L(1) \to \mathcal{O}_{\mathbb{P}^2_X}(1) \to \mathcal{O}_L(1) \to 0.$$

Since L is a family of lines, we have $\mathcal{I}_L(1)|_{\mathbb{P}^2 \times \{x\}} \cong \mathcal{O}_{\mathbb{P}^2}$ for all $x \in X(\Bbbk)$. By Theorem of Cohomology and Base Change (To be added.), we have $\mathcal{L}^{\vee} := (\mathrm{pr}_X)_*(\mathcal{I}_L(1))$ is a line bundle on X and $\mathrm{R}^1(\mathrm{pr}_X)_*\mathcal{I}_L(1) = 0$. Then, pushing forward the above short exact sequence along the projection $\mathrm{pr}_X : \mathbb{P}^2_X \to X$ gives a short exact sequence

$$0 \to \mathcal{L}^{\vee} \to \mathcal{O}_X \otimes_{\Bbbk} V \to (\mathrm{pr}_X)_*\mathcal{O}_L(1) \to 0.$$

Dualizing it gives a surjective morphism $\mathcal{O}_X \otimes_{\Bbbk} V^{\vee} \twoheadrightarrow \mathcal{L}$. In particular, if $X = G$ and $L = U$ is the universal family, we get the line bundle $\mathcal{U} = \mathcal{O}_G(1)$ on G and the surjective morphism $\mathcal{O}_G \otimes_{\Bbbk} V^{\vee} \twoheadrightarrow \mathcal{O}_G(1)$ corresponding to the identity morphism id_G.

Let $f : X \to G$ be the morphism induced by the surjective morphism $\mathcal{O}_X \otimes_{\Bbbk} V^{\vee} \twoheadrightarrow \mathcal{L}$. Then we have the following commutative diagram:(Why?)

$$\begin{tikzcd}
f^*(\mathcal{O}_G \otimes_{\Bbbk} V^{\vee}) \arrow[r] \arrow[d, "\cong"] & f^*\mathcal{U} \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathcal{O}_X \otimes_{\Bbbk} V^{\vee} \arrow[r] & \mathcal{L} \arrow[r] & 0.
\end{tikzcd}$$

Take duals and pull back along $\mathrm{pr}_X : \mathbb{P}^2_X \to X$, we get a commutative diagram: (To be added.)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_X^* f^*\mathcal{U}^{\vee} \arrow[r] \arrow[d, "\cong"] & \mathrm{pr}_X^* f^*\mathcal{O}_G \otimes_{\Bbbk} V \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{pr}_X^*\mathcal{L}^{\vee} \arrow[r] & \mathrm{pr}_X^*\mathcal{O}_X \otimes_{\Bbbk} V.
\end{tikzcd}$$

Note that $\mathcal{L}^\vee = (\mathrm{pr}_X)_*(\mathcal{I}_L(1))$ and $\mathcal{U}^\vee = (\mathrm{pr}_G)_*(\mathcal{I}_U(1))$. Hence, by cohomology commuting with flat base change To be added., we have

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{pr}_X^*(\mathrm{pr}_X)_*\Big((\mathrm{id}_{\mathbb{P}^2} \times f)^*\mathcal{I}_U(1)\Big) \arrow[r] \arrow[d, "\cong"] & \mathrm{pr}_X^*\Big(f^*\mathcal{O}_G \otimes_{\Bbbk} V\Big) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathrm{pr}_X^*(\mathrm{pr}_X)_*\Big(\mathcal{I}_L(1)\Big) \arrow[r] & \mathrm{pr}_X^*\Big(\mathcal{O}_X \otimes_{\Bbbk} V\Big). \end{tikzcd}$$

Note that $(\mathrm{pr}_X)_*\mathcal{I}_L(1)$ is a line bundle and hence we have $\mathrm{pr}_X^*(\mathrm{pr}_X)_*(\mathcal{I}_L(1)) \cong \mathcal{I}_L(1)$. Together with the natural surjective homomorphism $\mathrm{pr}_X^*(\mathcal{O}_X \otimes_{\Bbbk} V) \to \mathcal{O}_{\mathbb{P}^2_X}(1)$, we have

$$\begin{tikzcd} & & \mathrm{pr}_X^*\Big(f^*\mathcal{O}_G \otimes_{\Bbbk} V\Big) \arrow[dr, two heads] & \\ 0 \arrow[r] & (\mathrm{id}_{\mathbb{P}^2} \times f)^*\mathcal{I}_U(1) \arrow[ur] \arrow[rr] \arrow[d, "\cong"] & & \mathcal{O}_{\mathbb{P}^2_X}(1) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathcal{I}_L(1) \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^2_X}(1). \\ & & \mathrm{pr}_X^*\Big(\mathcal{O}_X \otimes_{\Bbbk} V\Big) \arrow[ur, two heads] & \end{tikzcd}$$

Why this homomorphism injective and why it is the natural inclusion? After identifying the last vertical isomorphism, we have the equality of sheaf ideals $(\mathrm{id}_{\mathbb{P}^2} \times f)^*\mathcal{I}_U = \mathcal{I}_L$. It follows that $L = U \times_G X$ and we are done.

3.1.2 Coarse moduli space

Chapter 4

Sites, algebraic space and stacks

4.1 Sites

4.1.1 Grothendieck topology

Definition 4.1.1. Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of sets of arrows $\{U_i \to U\}_{i\in I}$, called *covering*, for each object U in $\mathbf{C}$ such that:

(a) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering;

(b) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a arrow, then the fiber product $U_i \times_U V \to V$ exists and $\{U_i \times_U V \to V\}$ is a covering of V;

(c) if $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ are coverings, then the collection of composition $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{J})$ where $\mathbf{C}$ is a category and $\mathcal{J}$ is a Grothendieck topology on $\mathbf{C}$.

Note that sheaf is indeed defined on a site.

Definition 4.1.2. Let $(\mathbf{C}, \mathcal{J})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{J})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ satisfying the following condition: for every object U in $\mathbf{C}$ and every covering $\{U_i \to U\}_{i\in I}$ of U, if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every i, j, the pullback $s_i|_{U_i \times_U U_j}$ and $s_j|_{U_i \times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every i, the pullback $s|_{U_i} = s_i$.

Definition 4.1.3. Let X be a scheme. The *big étale site* of X, denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over X with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if and only if each U_i is étale over U and the union of their images is the whole U.

Let X be a scheme over S. By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \to \mathbf{Set}$ such that for any S-scheme T, $h_X(T) = \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$. Easy to check that h_X is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

4.2 Stacks in category theory

4.2.1 Prestacks

Definition 4.2.1. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. A morphism $f : a \to b$ in $\mathbf{X}$ is called *strongly Cartesian* if for every object $c \in \mathsf{Obj}(\mathbf{X})$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{X}}(c,a) \arrow[r, "f\circ-"] \arrow[d, "\mathbf{p}"'] & \mathrm{Hom}_{\mathbf{X}}(c,b) \arrow[d, "\mathbf{p}"] \\ \mathrm{Hom}_{\mathbf{S}}(\mathbf{p}(c),\mathbf{p}(a)) \arrow[r, "\mathbf{p}(f)\circ-"] & \mathrm{Hom}_{\mathbf{S}}(\mathbf{p}(c),\mathbf{p}(b)) \end{tikzcd}$$

is a pullback of sets.

Notation 4.2.2. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. For $a, b \in \mathsf{Obj}(\mathbf{X})$ and $f \in \mathrm{Hom}_{\mathbf{X}}(a,b)$, we say that a is *over* $\mathbf{p}(a)$ and f is *over* $\mathbf{p}(f)$. In a diagram, we have

$$\begin{tikzcd} \mathbf{X} \arrow[d, "\mathbf{p}"'] & a \arrow[r, "f"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ \mathbf{S} & \mathbf{p}(a) \arrow[r, "\mathbf{p}(f)"] & \mathbf{p}(b) \end{tikzcd}$$

Definition 4.2.3. Let $\mathbf{S}$ be a site. A category $\mathbf{X}$ over $\mathbf{S}$ via $\mathbf{p}$ is called a *category fibred* over the site $\mathbf{S}$ if for every morphism $r : u \to v$ in $\mathbf{S}$ and every object $b \in \mathsf{Obj}(\mathbf{X})$ over v, there exists an object $a \in \mathsf{Obj}(\mathbf{X})$ over u and a strongly Cartesian morphism $f : a \to b$ over r. Such an object a is called a *pullback* of b along r, and is often denoted by r^*b.

Definition 4.2.4. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a category fibred over $\mathbf{S}$. For every object $u \in \mathsf{Obj}(\mathbf{S})$, the *fiber* of $\mathbf{X}$ over u is the category $\mathbf{X}_u$ given by

$$\mathsf{Obj}(\mathbf{X}_u) = \{a \in \mathsf{Obj}(\mathbf{X}) \mid \mathbf{p}(a) = u\}, \quad \mathrm{Hom}_{\mathbf{X}_u}(a,b) = \{f \in \mathrm{Hom}_{\mathbf{X}}(a,b) \mid \mathbf{p}(f) = \mathrm{id}_u\}.$$

Remark 4.2.5. Note that in Definition 4.2.3, the pullback r^*b of an object b along a morphism r is not necessarily unique. To be continued.

Why do we need the Cartesian morphisms exists?

Remark 4.2.6. presheaves as category fibered in set, right?

Slogan *Presheaf is a category fibered in sets.*

Definition 4.2.7. A *prestack* over the site $\mathbf{S}$ is a category $\mathbf{X}$ fibered in groupoids over $\mathbf{S}$ via $\mathbf{p} : \mathbf{X} \to \mathbf{S}$. To be revised.

Remark 4.2.8. Let $\mathbf{S}$ be a site. A presheaf of sets on $\mathbf{S}$ can be viewed as a functor $\mathbf{S}^{op} \to \mathbf{Set}$. A prestack over $\mathbf{S}$ can be viewed as a functor $\mathbf{S}^{op} \to \mathbf{Grpd}$ by associating to each object $u \in \mathsf{Obj}(\mathbf{S})$ the fiber category $\mathbf{X}_u$, which is a groupoid, and to each morphism $u \to v$ in $\mathbf{S}$ the pullback functor $\mathbf{X}_v \to \mathbf{X}_u$. Thus, prestacks can be seen as a generalization of presheaves of sets, where the values are groupoids instead of sets. To be checked.

Slogan *Prestacks are "presheaf remembering automorphisms".*

Where is the 2-category?

Theorem 4.2.9 (Yoneda 2-Lemma)**.** Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ and $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Then the functor

$$\mathrm{Fun}_{\mathbf{S}}(\mathbf{X}, \mathbf{Y}) \to (\mathbf{p}_*, \mathbf{q}_*)$$

given by $\Phi \mapsto \Phi_*$ is an equivalence of categories. To be revised.

Theorem 4.2.10. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$, $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$, and $\mathbf{r} : \mathbf{Z} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Let $\Phi : \mathbf{X} \to \mathbf{Z}$ and $\Psi : \mathbf{Y} \to \mathbf{Z}$ be morphisms of prestacks over $\mathbf{S}$. Then the fiber product $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ exists in the category of prestacks over $\mathbf{S}$. To be checked.

4.2.2 Descent conditions

Definition 4.2.11. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. Let $U \in \mathsf{Obj}(\mathbf{S})$ and $\{U_i \to U\}$ be a covering in $\mathbf{S}$. A *descent datum* for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ consists of

- a collection of objects $a_i \in \mathsf{Obj}(\mathbf{X}_{U_i})$ for each i,
- a collection of isomorphisms $\varphi_{ij} : a_j|_{U_{ij}} \to a_i|_{U_{ij}}$ in $\mathbf{X}_{U_{ij}}$ for each pair (i, j), where $U_{ij} = U_i \times_U U_j$,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}}$$

holds for all triples (i, j, k), where $U_{ijk} = U_i \times_U U_j \times_U U_k$. To be checked.

Definition 4.2.12. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. A descent datum $(\{a_i\}, \{\varphi_{ij}\})$ for objects of $\mathbf{X}$ relative to a covering $\{U_i \to U\}$ in $\mathbf{S}$ is called *effective* if there exists an object $a \in \mathsf{Obj}(\mathbf{X}_U)$ and isomorphisms $\psi_i : a|_{U_i} \to a_i$ in $\mathbf{X}_{U_i}$ such that for all pairs (i, j), the diagram

$$\begin{tikzcd}
a|_{U_{ij}} \arrow[r, "\psi_j|_{U_{ij}}"] \arrow[d, "\psi_i|_{U_{ij}}"'] & a_j|_{U_{ij}} \arrow[d, "\varphi_{ij}"] \\
a_i|_{U_{ij}} \arrow[r, "\varphi_{ij}"'] & a_j|_{U_{ij}}
\end{tikzcd}$$

commutes. To be checked.

Slogan *Descent data are like gluing data for objects, and effectiveness means that the glued object exists.*

4.2.3 Stacks

Definition 4.2.13. Let $\mathbf{S}$ be a site. A prestack $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *stack* over the site $\mathbf{S}$ if for every object $U \in \mathsf{Obj}(\mathbf{S})$ and every covering $\{U_i \to U\}$ in $\mathbf{S}$, the descent data for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ are effective. To be revised.

Slogan *Stacks to prestacks are like sheaves to presheaves.*

Definition 4.2.14. Let $\mathbf{S}$ be a site, and let G be a group object in $\mathbf{S}$ acting on an object $X \in \mathsf{Obj}(\mathbf{S})$. The *quotient stack* $[X/G]$ is the stack over $\mathbf{S}$ defined as follows:

- For each object $U \in \mathsf{Obj}(\mathbf{S})$, the groupoid $[X/G](U)$ has as objects the pairs (P, f), where P is a G-torsor over U and $f : P \to X$ is a G-equivariant morphism.
- Morphisms between two objects (P, f) and (P', f') in $[X/G](U)$ are given by G-equivariant morphisms $\varphi : P \to P'$ such that $f' \circ \varphi = f$.

The assignment $U \mapsto [X/G](U)$ defines a stack over $\mathbf{S}$. To be checked.

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