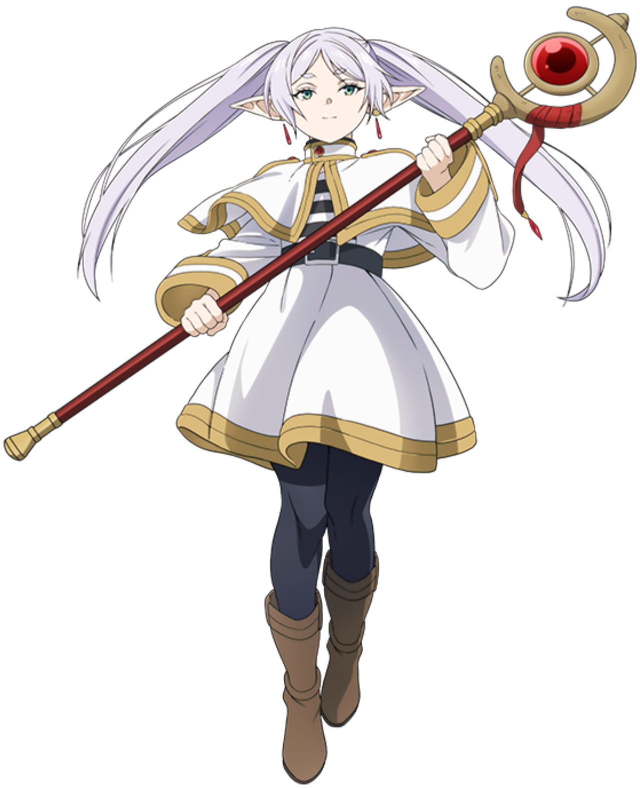
Abelian Varieties



“如果是勇者辛美尔，他一定会这么做的！”

Contents

1 The First Properties of Abelian Varieties

1.1 Definition and examples of Abelian Varieties

Theorem 1.1 (Rigidity Lemma)**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field k for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Definition 1.2. Let S be a scheme. An *abelian scheme over* S is a group object in the category $\mathbf{Sch}_S$ such that the structure morphism is proper, smooth and a fibration. If $S = \operatorname{Spec} \mathsf{k}$ for some field k, then it is called an *abelian variety over* k.

Example 1.3.

Example 1.4.

Example 1.5.

In the following, we will always assume that A is an abelian variety over a field k of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A.

Proposition 1.6. Let A be an abelian variety. Then A is smooth.

Proof. Note that there is an open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a. □

Proposition 1.7. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Date: July 21, 2025, **Author**: Tianle Yang, My Website

Proof. Yang: To be completed. □

Lemma 1.8. Let $p : X \times Y \to Z$ be a proper morphism of varieties over k such that p contracts $\{x_0\} \times Y$ for some point $x_0 \in X$. Then there exists a unique morphism $f : Y \to Z$ such that $p = f \circ p_Y$.

Proof. Yang: To be completed. □

Theorem 1.9. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism.

Proof. Yang: To be completed. □

Proposition 1.10. Let A be an abelian variety. Then A is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 1.9. □

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{N}^*$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 1.11. Let A be an abelian variety over $\mathbf{k}$ and n a positive integer. Then the multiplication by n morphism $[n]_A : A \to A$ is finite surjective and étale.

Proof. Yang: To be completed. □

1.2 Complex abelian varieties

Theorem 1.12. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if Λ Yang: To be completed.

2 Picard Groups of Abelian Varieties

2.1 Pullback along group operations

Theorem 2.1 (Seesaw Theorem)**.** Let A be an abelian variety over $\mathbf{k}$.

Theorem 2.2 (Theorem of the cube)**.** Let X, Y, Z be completed varieties over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\}\times Y\times Z}$, $\mathcal{L}|_{X\times\{y\}\times Z}$ and $\mathcal{L}|_{X\times Y\times\{z\}}$ are trivial. Then $\mathcal{L}$ is trivial.

Proof. Yang: To be completed. □

Remark 2.3. If we assume the existence of the Picard scheme, then the theorem of the cube can be deduced from the Rigidity Lemma. Yang: To be completed.

Proposition 2.4. Let A be an abelian variety over $\mathbf{k}$, $f, g, h : X \to A$ morphisms from a variety X to A and $\mathcal{L}$ a line bundle on A. Then

$$(f+g+h)^*\mathcal{L} \cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof. Yang: To be completed. □

Proposition 2.5. Let A be an abelian variety over $\mathbf{k}$, $n \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle on A. Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes \frac{1}{2}(n^2-n)}.$$

Proof. Yang: To be completed. □

Theorem 2.6 (Theorem of the square)**.** Let A be an abelian variety over $\mathbf{k}$, $x, y \in A(\mathbf{k})$ two points and $\mathcal{L}$ a line bundle on A. Then

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Remark 2.7. We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \to \mathrm{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary $\mathcal{L}$, the map

$$\Phi_{\square} : \mathrm{Pic}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \mathrm{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is a group homomorphism. For any $x \in A(\mathbf{k})$, we have

$$\Phi_{t_x^*\mathcal{L}} = \Phi_{\mathcal{L}}.$$

In the other words,

$$\Phi_{\mathcal{L}}(x) \in \mathrm{Ker}\,\Phi_{\square}, \quad \forall \mathcal{L} \in \mathrm{Pic}(A), x \in A(\mathbf{k}).$$

Yang: To be completed.

If we assume the scheme structure on $\mathrm{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\mathrm{Pic}^0(A)$. Let $K(\mathcal{L}) \coloneqq \mathrm{Ker}\,\Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A. We give another description of $K(\mathcal{L})$. From this point, we can recover the dual abelian variety $A^\vee = \mathrm{Pic}^0(A)$ as the quotient $A/K(\mathcal{L})$. Yang: To be completed.

2.2 Positivity

Theorem 2.8. Let A be an abelian variety over k. Then A is projective.

Proof. Yang: To be completed. □

2.3 Isogenies and finite subgroups

Theorem 2.9. Let A be an abelian variety of dimension d over $\mathbf{k}$. Then the subgroup $A[n]$ of n torsion points is finite and we have

(a) if n is coprime to $\mathrm{char}(\mathsf{k})$, then $A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2d}$;

(b) if $n = p^k$ for $p = \mathrm{char}(\mathsf{k}) > 0$

Proof. Yang: To be completed. □

2.4 Dual abelian varieties

Theorem 2.10. Let A be an abelian variety over k. Then $\mathrm{Pic}^0(A)$ has a natural structure of an abelian variety, called the *dual abelian variety* of A, denoted by $A^\vee$.

Proposition 2.11.