Regularity and Smoothness

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1 Modules of differentials and derivations

In this subsection, let R be a ring and A an R-algebra.

Definition 1 (Derivation)**.** A *derivation* of A over R is an R-linear map $\partial : A \to M$ with an A-module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module M, the set of all derivations of A over R into M forms an A-module, denoted by $\mathrm{Der}_R(A, M)$.

Given a module homomorphism $f : M \to N$ of A-modules and a derivation $\partial \in \mathrm{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of A over R into N.

Proposition 2. The functor $\mathrm{Der}_R(A, -)$ is representable. The representing object is denoted by $\Omega_{A/R}$, which is called the *module of differentials* of A over R.

Proof. First suppose A is a free R-algebra with a set of generators $a_\lambda, \lambda \in \Lambda$. Then an R-derivation $\partial \in \mathrm{Der}_R(A, M)$ is uniquely determined by its values on the generators a_λ. Let

$$\Omega_{A/R} \coloneqq \bigoplus_{\lambda \in \Lambda} A \cdot \mathrm{d}a_\lambda$$

and $\mathrm{d} : A \to \Omega_{A/R}$ be the R-derivation defined by $a_\lambda \mapsto \mathrm{d}a_\lambda$. For any R-derivation $\partial \in \mathrm{Der}_R(A, M)$, we can define a unique A-module homomorphism $\Phi_\partial : \Omega_{A/R} \to M$ by sending $\mathrm{d}a_\lambda$ to $\partial(a_\lambda)$ such that $\partial = \Phi_\partial \circ \mathrm{d}$. This gives a bijection

$$\mathrm{Der}_R(A, M) \cong \mathrm{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_\partial.$$

Now suppose $A = F/I$ is an arbitrary R-algebra, where F is a free R-algebra and I is an ideal of F. Then we can define the module of differentials

$$\Omega_{A/R} \coloneqq \big(\Omega_{F/R} \otimes_F A\big) \Big/ \sum_{f \in I} A \cdot \mathrm{d}f.$$

The R-linear map $\mathrm{d}_A : F \otimes_F A \xrightarrow{\mathrm{d}_F} \Omega_{F/R} \otimes_F A \to \Omega_{A/R}$ is a derivation of A over R.

For any R-derivation $\partial \in \mathrm{Der}_R(A, M)$, note that $F \to A \xrightarrow{\partial} M$ is an R-derivation of F over R into M. Then we get an F-module homomorphism $\Omega_F \to M$. It gives an A-module homomorphism $\Omega_F \otimes_F A \to M, \mathrm{d}f \otimes 1 \mapsto \partial f$. This map factors into $\Omega_F \otimes_F A \to \Omega_{A/R}$ and $\Phi_\partial : \Omega_{A/R} \to M$. Since Φ_∂ is A-linear and $\Omega_{A/R}$ is generated by $\mathrm{d}a_\lambda$ as A-module, such Φ_∂ is unique. □

Corollary 3. Suppose A is of finite type over R. Then the module of differentials $\Omega_{A/R}$ is a finitely generated A-module.

Remark 4. Let B be an A-algebra, M an A-module and N a B-module. If there is a homomorphism of A-modules $M \to N$, then we can extend it to a homomorphism of B-modules $M \otimes_A B \to N$ by sending $m \otimes b$ to $m \cdot b$. And such extension is unique in the sense of following commutative diagram:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \, . \\ M \otimes_A B \arrow[ur, "\exists!"'] & \end{tikzcd}$$

Hence we get a natural bijection

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(M \otimes_A B, N).$$

Proposition 5. Let A, R' be R-algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

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Proof. We check the universal property of $\Omega_{A/R} \otimes_A A'$. First, the map

$$\mathrm{d}_{A'} : A \otimes_R R' \to \Omega_{A/R} \otimes_R R' \cong \Omega_{A/R} \otimes_A A', \quad a \otimes r \mapsto \mathrm{d}a \otimes r$$

is an R'-derivation of A' into $\Omega_{A/R} \otimes_A A'$. For any R'-derivation $\partial' : A' \to M$ into an A'-module M, we can compose it with the homomorphism $A' \to A$ and get an R-derivation $\partial : A \to M$. By the universal property of $\Omega_{A/R}$, there is a unique A-module homomorphism $\Phi : \Omega_{A/R} \to M$ such that $\partial = \Phi \circ \mathrm{d}_A$. Then we can extend it to an A'-module homomorphism $\Phi' : \Omega_{A/R} \otimes_A A' \to M$ by Remark 4. By the construction, we have $\Phi' \circ \mathrm{d}_{A'} = \partial'$. □

Proposition 6. Let A be an R-algebra and S a multiplicative set of A. Then we have an isomorphism

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

Proof. Let

$$\mathrm{d}_{S^{-1}A} : S^{-1}A \to S^{-1}\Omega_{A/R}, \quad \frac{a}{s} \mapsto \frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}.$$

By direct computation, $\mathrm{d}_{S^{-1}A}$ is an R-derivation of $S^{-1}A$ over R into $S^{-1}\Omega_{A/R}$. For any R-derivation $\partial : S^{-1}A \to M$ into an $S^{-1}A$-module M, we can get an $S^{-1}A$-module homomorphism $\Phi' : S^{-1}\Omega_{A/R} \to M$ as proof of Proposition 5. We have

$$\partial(s \cdot \frac{a}{s}) = s\partial(\frac{a}{s}) + \frac{a}{s}\partial s.$$

It follows that

$$\partial(\frac{a}{s}) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(\mathrm{d}a) - a\Phi'(\mathrm{d}s)}{s^2} = \Phi'(\frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}).$$

Thus, $\Phi' \circ \mathrm{d}_{S^{-1}A} = \partial$. □

Theorem 7. Let A be an R-algebra and B an A-algebra. Then there is a natural short exact sequence

$$\Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to \Omega_{B/A} \to 0$$

of B-modules.

Proof. Let $\mathrm{d}_{A/R} : A \to \Omega_{A/R}$ be the R-derivation of A over R. The map $A \to B \xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R}$ induces a B-linear map

$$u : \Omega_{A/R} \otimes_A B \to \Omega_{B/R}, \quad \mathrm{d}_{A/R}(a) \otimes b \mapsto b\mathrm{d}_{B/R}(a).$$

The map $\mathrm{d}_{B/A}$ is an A-derivation and hence R-derivation. Then it induces a B-linear map

$$v : \Omega_{B/R} \to \Omega_{B/A}, \quad \mathrm{d}_{B/R}(b) \mapsto \mathrm{d}_{B/A}(b).$$

Since $\Omega_{B/A}$ is generated by elements of the form $\mathrm{d}_{B/A}(b)$ for $b \in B$, the map v is surjective. And clearly $\mathrm{d}_{B/A}(a) = a\mathrm{d}_{B/A}(1) = 0$ for $a \in A$.

Consider the composition $B \xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R} \to \Omega_{B/R}/\operatorname{Im} u$. For every $a \in A, b \in B$, we have

$$[\mathrm{d}_{B/R}(ab)] = [b\mathrm{d}_{B/R}(a) + a\mathrm{d}_{B/R}(b)] = [b\mathrm{d}_{B/R}(a)] + [a\mathrm{d}_{B/A}(b)] = [a\mathrm{d}_{B/A}(b)].$$

Hence it is indeed an A-derivation of B. Then it induces a B-linear map

$$\varphi : \Omega_{B/A} \to \Omega_{B/R}/\operatorname{Im} u, \quad \mathrm{d}_{B/A}(b) \mapsto [\mathrm{d}_{B/R}(b)].$$

The map φ is surjective since $\Omega_{B/R}$ is generated by elements of the form $\mathrm{d}_{B/R}(b)$ for $b \in B$. Note that the composition

$$\Omega_{B/A} \xrightarrow{\varphi} \Omega_{B/R}/\operatorname{Im} u \to \Omega_{B/A}/\operatorname{Ker} v$$

is the identity map. Thus, φ is injective and hence an isomorphism. In particular, we have $\operatorname{Ker} v = \operatorname{Im} u$. □

Theorem 8. Let A be an R-algebra and I an ideal of A. Set $B \coloneqq A/I$. Then there is a natural short exact sequence

$$I/I^2 \to \Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to 0$$

of B-modules.

Proof. Suppose $A = F/\mathfrak{b}$ for some free R-algebra F and an ideal $\mathfrak{b}$ of F. Let $\mathfrak{a}$ be the preimage of I in F. Let $\mathrm{d}\mathfrak{b}$ (resp. $\mathrm{d}\mathfrak{a}$) denote the image of $\mathfrak{b}$ (resp. $\mathfrak{a}$) in $\Omega_{F/R}$. Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \to (\mathrm{d}\mathfrak{a} \otimes_F B)/(\mathrm{d}\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows. □

Definition 9. Let k be a field and A an integral k-algebra of finite type of dimension n. We say A is *smooth at* $\mathfrak{p} \in \operatorname{Spec} A$ if the module of differentials $\Omega_{A,\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank n.

2 Zariski's tangent space and regularity

Let k be arbitrary field, $A = \mathsf{k}[T_1, \ldots, T_n]$ and $\mathfrak{m}$ a maximal ideal of A such that $\kappa(\mathfrak{m})$ is separable over k. We try to give an explanation of Zariski's tangent space at $\mathfrak{m}$ using the language of derivation. We know that $\Omega_{A/\mathsf{k}} = \bigoplus_{i=1}^n A\mathrm{d}T_i$, thus $\Omega_{A_\mathfrak{m}/\mathsf{k}} \cong \bigoplus_{i=1}^n A_\mathfrak{m}\mathrm{d}T_i$. Then

$$\operatorname{Der}_\mathsf{k}(A_\mathfrak{m}, A_\mathfrak{m}) \cong \operatorname{Hom}_\mathsf{k}(\Omega_{A_\mathfrak{m}/\mathsf{k}}, A_\mathfrak{m}) \cong \bigoplus_{i=1}^n A_\mathfrak{m}\partial_i,$$

where $\partial_i \in \operatorname{Der}_\mathsf{k}(A_\mathfrak{m}, A_\mathfrak{m})$ is the derivation defined by $\mathrm{d}T_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. It coincides with the usual derivation $f \mapsto \partial f/\partial T_i$. Consider the restriction of ∂_i to $\mathfrak{m}$ and take values in the residue field $\kappa(\mathfrak{m})$, we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1, \cdots, \partial_n)^T} A_\mathfrak{m}^n \to \kappa(\mathfrak{m})^n.$$

Since $\kappa(\mathfrak{m})$ is separable over k, the map $\operatorname{Ker}\Phi = \mathfrak{m}^2$. Hence Φ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$ of $\kappa(\mathfrak{m})$-vector spaces. Then we get

$$T_{A,\mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where $x \in \mathbb{A}_\mathsf{k}^n$ is the point corresponding to $\mathfrak{m}$. This coincides with the usual tangent space at x in language of differential geometry.

Let $B = A/I$ be a k of finite type, $I = (F_1, \ldots, F_m) \subset \mathfrak{m}$ and $\mathfrak{n}$ the image of $\mathfrak{m}$ in B. We have an exact sequence of $\kappa(\mathfrak{m})$-vector spaces

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} \colon \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of $F_1, \ldots, F_m$ is the $m \times n$ matrix

$$J(F_1, \ldots, F_m) \coloneqq \left(\frac{\partial F_i}{\partial T_j}\right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

with entries in B.

Theorem 10. Setting as above. Then B is regular at $\mathfrak{n}$ if and only if the Jacobian matrix J has maximal rank $n - \dim B_\mathfrak{n}$ after taking values in the residue field $\kappa(\mathfrak{m})$.

Proof. We have an exact sequence

$$0 \to T_{B,\mathfrak{n}} \to T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa^m \to 0,$$

where Ψ sends $\partial \in T_{A,\mathfrak{m}}$ to $(\partial(F_1), \ldots, \partial(F_m))^T$. Note that the matrix of Ψ is just J^T, the transpose of the Jacobian matrix. Hence

$$\operatorname{rank} J = n - \dim_\kappa T_{B,\mathfrak{n}} \leq n - \dim B_\mathfrak{n}$$

and the equality holds if and only if B is regular at $\mathfrak{n}$. □

Remark 11. If $\kappa(\mathfrak{m})$ is not separable over k, then we still have the inequality

$$\operatorname{rank} J \leq n - \dim B_\mathfrak{n}.$$

Indeed, in any case, we have an exact sequence

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Hence $\dim_\kappa I/(I\cap\mathfrak{m}^2) = n - \dim B_\mathfrak{n}$. There is a $\kappa(\mathfrak{m})$-linear map

$$I/(I\cap\mathfrak{m}^2) \to \kappa(\mathfrak{m})^n, \quad [f]\mapsto (\partial_1(f),\dots,\partial_n(f))^T,$$

and every row of the Jacobian matrix J is in the image of this map. Thus, the rank of J is at most $n-\dim B_\mathfrak{n}$. Hence if $\operatorname{rank} J = n - \dim B_\mathfrak{n}$, we can still see that B is regular at $\mathfrak{n}$. However, the converse does not hold in general.

Proposition 12. Let k be a field, $\mathbf{k}$ the algebraic closure of k, A a k-algebra of finite type and $A_\mathbf{k} := A\otimes_\mathsf{k}\mathbf{k}$. Yang: Suppose $A_\mathbf{k}$ is integral. Let $\mathfrak{m}\in\operatorname{mSpec} A$ and $\mathfrak{m}'$ be a maximal ideal of $A_\mathbf{k}$ lying over $\mathfrak{m}$. Then

(a) If $A_\mathbf{k}$ is regular at $\mathfrak{m}'$, then A is regular at $\mathfrak{m}$;

(b) suppose $\kappa(\mathfrak{m})$ is separable over k, the converse holds.

Proof. Regarding $J_\mathfrak{m}$ and $J_{\mathfrak{m}'}$ as matrices with entries in $\mathbf{k}$, they are the same and hence have the same rank. If $A_\mathbf{k}$ is regular at $\mathfrak{m}'$, since $\kappa(\mathfrak{m}) = \mathbf{k}$, then $\operatorname{rank} J_{\mathfrak{m}'} = n - \dim A_{\mathbf{k},\mathfrak{m}'}$. Note that $\dim A_{\mathbf{k},\mathfrak{m}'} = \operatorname{trdeg}(\mathscr{K}(A_\mathbf{k})/\mathbf{k}) = \operatorname{trdeg}(\mathscr{K}(A)/\mathsf{k}) = \dim A_\mathfrak{m}$, we have $\operatorname{rank} J_\mathfrak{m} = n - \dim A_\mathfrak{m}$. Hence A is regular at $\mathfrak{m}$.
Conversely, suppose A is regular at $\mathfrak{m}$ and $\kappa(\mathfrak{m})$ is separable over k. Then $\operatorname{rank} J_\mathfrak{m} = n-\dim A_\mathfrak{m}$. Hence $A_\mathbf{k}$ is regular at $\mathfrak{m}'$. □

Proposition 13. Let k be a field and A an integral k-algebra of finite type and of dimension n. Let $\mathbf{k}$ be the algebraic closure of k and $A_\mathbf{k} := A\otimes_\mathsf{k}\mathbf{k}$. Then A is smooth at $\mathfrak{p}\in\operatorname{Spec} A$ if and only if $A_\mathbf{k}$ is regular at every $\mathfrak{P}$ over $\mathfrak{p}$.

Proof. Since $\Omega_{A_\mathbf{k}/\mathbf{k}} \cong \Omega_{A/\mathsf{k}}\otimes_A A_\mathbf{k}$ is free of rank n if and only if $\Omega_{A/\mathsf{k}}$ is free of rank n, we can assume that $\mathsf{k}=\mathbf{k}$. If A is smooth at $\mathfrak{p}$, then $\Omega_{A_\mathfrak{p}/\mathsf{k}} \cong \bigoplus A_\mathfrak{p}\mathrm{d}T_i$ is free of rank n. Let $\mathfrak{P}_i\in\operatorname{Der}_\mathsf{k}(A_\mathfrak{m}, A_\mathfrak{m})$ be the derivation defined by $\mathrm{d}T_i\mapsto 1$ and $\mathrm{d}T_j\mapsto 0$ for $j\neq i$. Then similar to above argument, we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{(\partial_1,\dots,\partial_n)^T} \mathbf{k}^n.$$

Yang: To be completed. □

Example 14. Let k be an imperfect field of characteristic $p>2$. Suppose $\alpha=\beta^p\in\mathsf{k}$ and β is not in k. Let $A=\mathsf{k}[x,y]/(x^2-y^p-\alpha)$ and $\mathfrak{m}=(x,y^p-\alpha)=(x)$. Note that $\mathfrak{m}$ is principal, so A is regular at $\mathfrak{m}$. However,

$$J_\mathfrak{m} = \left(\frac{\partial}{\partial x}(x^2-y^p-\alpha), \frac{\partial}{\partial y}(y^p-\alpha)\right) = (2x,0) = (0,0)\in M_{1\times 2}(\kappa(\mathfrak{m})).$$

Thus, A is not smooth at $\mathfrak{m}$. From the view of differentials, we have

$$\Omega_{A_\mathfrak{m}/\mathsf{k}} = A_\mathfrak{m}\mathrm{d}x\oplus A_\mathfrak{m}\mathrm{d}y/A_\mathfrak{m}\cdot x\mathrm{d}x = \kappa(\mathfrak{m})\mathrm{d}x\oplus A_\mathfrak{m}\mathrm{d}y,$$

which is not free as an $A_\mathfrak{m}$-module.