First properties of algebraic groups

Let $\mathbf{k}$ be a field and $\bar{\mathbb{k}}$ its algebraic closure. All varieties are defined over $\mathbf{k}$ unless otherwise specified.

1 Basic concepts

Definition 1. A *group scheme* over S is an S-scheme G together with morphisms *multiplication* $\mu : G \times G \to G$, *identity* $\varepsilon : S \to G$ and *inversion* $\iota : G \to G$ over S such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd} & G \times G \times G \arrow[ld, "\mathrm{id}_G\times\mu"'] \arrow[rd, "\mu\times\mathrm{id}_G"] & \\ G \times G \arrow[rd, "\mu"] & & G \times G \arrow[ld, "\mu"'] \\ & G & \end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd} G \times S \arrow[r, "\mathrm{id}_G\times\varepsilon"] \arrow[rd, "\cong"'] & G \times G \arrow[d, "\mu"] & S \times G \arrow[l, "\varepsilon\times\mathrm{id}_G"'] \arrow[ld, "\cong"] \\ & G & \end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd} & G \arrow[ld, "\mathrm{id}_G\times i"'] \arrow[d] \arrow[rd, "i\times\mathrm{id}_G"] & \\ G \times G \arrow[rd, "\mu"'] & S \arrow[d, "\varepsilon"] & G \times G \arrow[ld, "\mu"] \\ & G & \end{tikzcd}.$$

In other words, a group scheme is a group object in the category of schemes.

Definition 2. An *algebraic group* is a $\mathbf{k}$-group scheme G which is reduced, separated and of finite type over a field $\mathbf{k}$.

Definition 3. Let G be an algebraic group and $x \in G(\mathbf{k})$ a $\mathbf{k}$-point. The *left translation* by x is the morphism

$$l_x : G \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times G \xrightarrow{x\times\mathrm{id}_G} G \times G \xrightarrow{\mu} G.$$

Similar definition applies to the right translation r_x.

Remark 4. In the context of algebraic groups, we often use multiplicative notation for the group law. That is, for $g, h \in G(\mathbf{k})$, we write gh instead of $\mu(g, h)$ and g^{-1} instead of $\iota(g)$. The identity element ε is often denoted by e.

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Sometimes we also abuse the notation by $\mu : G \times \cdots \times G \to G$ to denote the multiplication of multiple elements, i.e. $\mu(g_1, \dots, g_n) = g_1 \cdots g_n$ for $g_1, \dots, g_n \in G(\mathbf{k})$.

Remark 5. Since algebraic groups are almost varieties over an arbitrary field $\mathbf{k}$, we often identify an algebraic group G with its set of closed points $G(\Bbbk)$ when there is no confusion.

Proposition 6. Let G be an algebraic group. Then G is smooth over $\mathbf{k}$.

Proof. Since G is reduced and of finite type over a field, it is generically regular. Let $g \in G(\Bbbk)$ be a regular point. Then the left translation $l_{gh^{-1}} : G \to G$ is an isomorphism, hence G is regular at $h \in G(\Bbbk)$. It follows that G is regular at every $\Bbbk$-point, hence G is smooth over $\mathbf{k}$. □

Remark 7. Let G be an algebraic group. Then the irreducible components of G coincide with the connected components of G. We will use the term "connected" to refer to both concepts since "irreducible" has other meanings in the theory of representations.

Example 8. The *additive group* $\mathbb{G}_a$ is defined to be the affine line $\mathbb{A}^1$ with the group law given by addition. Concretely, we can write $\mathbb{G}_a = \operatorname{Spec} \mathbf{k}[T]$ with the group law given by the morphism

$$\begin{aligned} \mu &: \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a, \quad (x, y) \mapsto x + y, \\ \iota &: \mathbb{G}_a \to \mathbb{G}_a, \quad x \mapsto -x, \\ \varepsilon &: \operatorname{Spec} \mathbf{k} \to \mathbb{G}_a, \quad * \mapsto 0. \end{aligned}$$

Example 9. The *multiplicative group* $\mathbb{G}_m$ is defined to be the affine variety $\mathbb{A}^1 \setminus \{0\}$ with the group law given by multiplication. Concretely, we can write $\mathbb{G}_m = \operatorname{Spec} \mathbf{k}[T, T^{-1}]$ with the group law given by the morphism

$$\begin{aligned} \mu &: \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m, \quad (x, y) \mapsto xy, \\ \iota &: \mathbb{G}_m \to \mathbb{G}_m, \quad x \mapsto x^{-1}, \\ \varepsilon &: \operatorname{Spec} \mathbf{k} \to \mathbb{G}_m, \quad * \mapsto 1. \end{aligned}$$

Example 10. The *general linear group* GL_n is defined to be the open subvariety of $\mathbb{A}^{n^2}$ consisting of invertible matrices, with the group law given by matrix multiplication. Concretely, we can write $\mathrm{GL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}, \det(T_{ij})^{-1}]$ where $1 \le i, j \le n$ and the group law is given by the morphism

$$\begin{aligned} \mu &: \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, \quad (A, B) \mapsto AB, \\ \iota &: \mathrm{GL}_n \to \mathrm{GL}_n, \quad A \mapsto A^{-1}, \\ \varepsilon &: \operatorname{Spec} \mathbf{k} \to \mathrm{GL}_n, \quad * \mapsto I_n. \end{aligned}$$

Example 11. An abelian variety is an algebraic group that is also a proper variety.

Example 12. Let G and H be algebraic groups. The *product* $G \times H$ is an algebraic group with the group law defined by

$$\begin{aligned} \mu_{G\times H} = \mu_G \times \mu_H &: (G \times H) \times (G \times H) \cong (G \times G) \times (H \times H) \to G \times H, \\ \varepsilon_{G\times H} = \varepsilon_G \times \varepsilon_H &: \operatorname{Spec} \mathbf{k} \cong \operatorname{Spec} \mathbf{k} \times \operatorname{Spec} \mathbf{k} \to G \times H, \\ \iota_{G\times H} = \iota_G \times \iota_H &: G \times H \to G \times H. \end{aligned}$$

Example 13. Let G be an algebraic group over $\mathbf{k}$ and $\mathbf{K}/\mathbf{k}$ a field extension. The base change $G_{\mathbf{K}} = G \times_{\operatorname{Spec}\mathbf{k}} \operatorname{Spec}\mathbf{K}$ is an algebraic group over $\mathbf{K}$ with the group law defined by the base change of the original group law of G to $\mathbf{K}$.

Definition 14. A *homomorphism* of algebraic groups is a morphism of schemes that is also a group homomorphism. Explicitly, a morphism $f : G \to H$ between algebraic groups G and H is a homomorphism if the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

where μ_G and μ_H are the group laws of G and H, respectively.

Definition 15. An *algebraic subgroup* of an algebraic group G is a closed subscheme $H \subseteq G$ that is also a subgroup of G. More precisely, H is an algebraic subgroup and the inclusion morphism $H \hookrightarrow G$ is compatible with the group laws.

Yang: I need the definition of normal subgroup here.

Example 16. The *special linear group* SL_n is defined to be the closed subvariety of GL_n defined by the equation $\det = 1$. It is an algebraic subgroup of GL_n.

Definition 17. Let G be an algebraic group. The *neutral component* G^0 is the connected component of G containing the identity element ε.

Proposition 18. The neutral component $G^0(\Bbbk)$ is a closed, normal algebraic subgroup of $G(\Bbbk)$ of finite index. Moreover, each closed subgroup H of finite index contains $G^0(\Bbbk)$.

Proof. Yang: To be continued... □

Proposition 19. Let G be an algebraic group and $H \subseteq G(\Bbbk)$ a subgroup (not necessarily closed). Then the Zariski closure $\overline{H}$ of H in G is an algebraic subgroup of G. If $H \subset G(\Bbbk)$ is constructible, then $H = \overline{H}(\Bbbk)$.

Proof. Yang: To be continued... □

Example 20. Let $G = \mathrm{SL}_2$ over $\Bbbk$, $T = \{\operatorname{diag}(t, t^{-1}) | t \in \Bbbk^\times\}$ and $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Set $S = gTg^{-1}$. Then both T and S are closed algebraic subgroups of $G(\Bbbk)$, but the product TS is not closed in $G(\Bbbk)$. By direct computation, we have

$$S = \left\{ \begin{pmatrix} s & s^{-1} - s \\ 0 & s^{-1} \end{pmatrix} \middle| \, s \in \Bbbk^\times \right\}.$$

Then

$$TS = \left\{ \begin{pmatrix} ts & t(s^{-1} - s) \\ 0 & (ts)^{-1} \end{pmatrix} \middle| \, t, s \in \Bbbk^\times \right\}.$$

We have
$$TS\cap\left\{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\,\middle|\, a\in \Bbbk\right\}=\left\{\begin{pmatrix}1 & s^{-2}-1\\ 0 & 1\end{pmatrix}\,\middle|\, s\in \Bbbk^{\times}\right\}.$$
The right hand side is not closed in $\mathrm{SL}_2(\Bbbk)$ since it does not contain the matrix $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$. Hence TS is not closed in $G(\Bbbk)$.

Proposition 21. Let G be an algebraic group, X_i varieties over $\mathbf{k}$ and $f_i : X_i \to G$ morphisms for $i = 1, \dots, n$ with images $Y_i = f_i(X_i)$. Suppose that Y_i pass through the identity element of G. Let H be the closed subgroup of G generated by $Y_1, \dots, Y_n$, i.e. the smallest closed subgroup of G containing $Y_1, \dots, Y_n$. Then H is connected and $H = Y_{a_1}^{e_1} \cdots Y_{a_m}^{e_m}$ for some $a_1, \dots, a_m \in \{1, \dots, n\}$ and $e_1, \dots, e_m \in \{\pm 1\}$.

Proof. Yang: To be continued... □

Remark 22. We can take $m \leq 2\dim G$ in Proposition 21.

2 Action and representations

Definition 23. An *action* of an algebraic group G on a variety X is a morphism
$$\sigma : G\times X \to X$$
such that the following diagrams commute:
$$\begin{tikzcd} G\times G\times X \arrow[r, "\mu\times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G\times\sigma"] & G\times X \arrow[d, "\sigma"] \\ G\times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}\mathbf{k}\times X \arrow[r, "\varepsilon\times \mathrm{id}_X"] \arrow[dr, "\cong"'] & G\times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$
where μ is the group law of G and ε is the identity element of G. In other words, for any $\mathbf{k}$-scheme S, the induced map $G(S)\times X(S) \to X(S)$ defines a group action of the abstract group $G(S)$ on the set $X(S)$.

Definition 24. An *rational action* of an algebraic group G on a variety X is a rational map
$$\sigma : G\times X \dashrightarrow X$$
such that the following diagrams commute wherever the maps are defined:
$$\begin{tikzcd} G\times G\times X \arrow[r, "\mu\times \mathrm{id}_X"] \arrow[d, dashed, "\mathrm{id}_G\times\sigma"] & G\times X \arrow[d, dashed, "\sigma"] \\ G\times X \arrow[r, dashed, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}\mathbf{k}\times X \arrow[r, "\varepsilon\times \mathrm{id}_X"] \arrow[dr, "\cong"'] & G\times X \arrow[d, dashed, "\sigma"] \\ & X \end{tikzcd}$$
where μ is the group law of G and ε is the identity element of G. Yang: To be checked.

Definition 25. Let G be an algebraic group acting on a variety X. For any $x \in X(\mathbf{k})$, the *orbit* of x is the locally closed subvariety $G \cdot x = \sigma(G \times \{x\})$ of X. Yang: To be checked.

Proposition 26. Let G be an algebraic group acting on a variety X. Then for any $x \in X(\mathbf{k})$, the orbit $G \cdot x$ is a locally closed subvariety of X, and $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of strictly smaller dimension.

Proof. Yang: To be continued... □

Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For $x \in G(\mathbf{k})$, we have the left translation of functions $\tau_x : A \to A$ defined by $\tau_x(f)(y) = f(x^{-1}y)$ for $y \in X(\mathbf{k})$.

Lemma 27. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For any finite-dimensional subspace $V \subseteq A$, there exists a finite-dimensional G-invariant subspace $W \subseteq A$ containing V.

Proof. Yang: To be continued... □

Theorem 28. Any affine algebraic group is isomorphic to a closed algebraic subgroup of some GL_n.

Proof. Yang: To be continued... □

3 Lie algebra of an algebraic group

Let G be an algebraic group. The *Lie algebra* of G is defined to be the tangent space of G at the identity element ε:

$$\mathrm{Lie}(G) = T_\varepsilon G.$$

It is a finite-dimensional vector space over $\mathbf{k}$.

Proposition 29. The group law $\mu : G \times G \to G$ induces the plus map on $\mathrm{Lie}(G)$:

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)} : T_{(\varepsilon,\varepsilon)}(G \times G) \cong T_\varepsilon G \oplus T_\varepsilon G \to T_\varepsilon G, \quad (v, w) \mapsto v + w.$$

Proof. We have

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v,w) = \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v,0) + \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(0,w) = (\mathrm{d}\mu \circ (\mathrm{id}_G \times \varepsilon))_\varepsilon(v) + (\mathrm{d}\mu \circ (\varepsilon \times \mathrm{id}_G))_\varepsilon(w) = v + w.$$

□

Preliminaries

Definition 30. Let X be a scheme with underlying topological space $|X|$. The family $\mathfrak{C}$ of constructible sets in $|X|$ is the smallest family of subsets of $|X|$ that contains all open subsets and is closed under finite intersections, finite unions, and complements. A subset $E \subseteq |X|$ is called a *constructible set* if $E \in \mathfrak{C}$.

Theorem 31. Let $f : X \to Y$ be a morphism of varieties. Then the image of f is a constructible set in Y.

Lemma 32. Let X and Y be varieties over a field $\mathbf{k}$. For any point $x \in X(\mathbf{k})$ and $y \in Y(\mathbf{k})$, there is a natural isomorphism of $\mathbf{k}$-vector spaces

$$T_{(x,y)}(X \times Y) \cong T_x X \oplus T_y Y$$

given by $v \mapsto (d\pi_1(v), d\pi_2(v))$, where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projection morphisms.

Proof. The inverse map is given by $(u, w) \mapsto d(\iota_1)(u) + d(\iota_2)(w)$, where $\iota_1 : X \cong X \times \{y\} \to X \times Y$ and $\iota_2 : Y \cong \{x\} \times Y \to X \times Y$ are the natural inclusions. $\square$