

Template for the class "Note for
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Finite Algebra and Normality **Yang: To be completed**

Definition 1. An integral domain A is called *normal* if it is integrally closed in its field of fractions $\text{Frac}(A)$.

Lemma 2. Let $A \subset C$ be rings and B the integral closure of A in C , S a multiplicatively closed subset of A . Then the integral closure of $S^{-1}A$ in $S^{-1}C$ is $S^{-1}B$.

Proof. For every $b \in B$ and $\forall s \in S$, there exists $a_i \in A$ s.t.

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then

$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s^1} \left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0.$$

Hence b/s is integral over $S^{-1}A$, $S^{-1}B$ is integral over $S^{-1}A$.

If $c/s \in S^{-1}C$ is integral over $S^{-1}A$, then $\exists a_i \in S^{-1}A$ s.t.

$$\left(\frac{c}{s}\right)^n + a_1 \left(\frac{c}{s}\right)^{n-1} + \cdots + a_n = 0.$$

Then

$$c^n + a_1 s c^{n-1} + \cdots + a_n s^n = 0 \in S^{-1}C$$

Then $\exists t \in S$ s.t.

$$t(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0 \in C.$$

Then

$$(ct)^n + a_1 st(ct)^{n-1} + \cdots + a_n s^n t^n = t^n(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0.$$

Hence ct is integral over A , then $ct \in B$. Then $c/s = (ct)/(st) \in S^{-1}B$. This completes the proof. \square

Proposition 3. Normality is a local property. That is, for an integral domain A , TFAE:

- (i) A is normal.
- (ii) For any prime ideal $\mathfrak{p} \in \text{Spec } A$, the localization $A_{\mathfrak{p}}$ is normal.
- (iii) For any maximal ideal $\mathfrak{m} \in \text{mSpec } A$, the localization $A_{\mathfrak{m}}$ is normal.

Proof. When A is normal, $A_{\mathfrak{p}}$ is normal by Lemma 2.

Assume that $A_{\mathfrak{m}}$ is normal for every $\mathfrak{m} \in \text{mSpec } A$. If A is not normal, let \tilde{A} be the integral closure of A in $\text{Frac } A$, \tilde{A}/A is a nonzero A -module. Suppose $\mathfrak{p} \in \text{Supp } \tilde{A}/A$ and $\mathfrak{p} \subset \mathfrak{m}$. We have $\tilde{A}_{\mathfrak{m}}/A_{\mathfrak{m}} = 0$ and $\tilde{A}_{\mathfrak{p}}/A_{\mathfrak{p}} = (\tilde{A}_{\mathfrak{m}}/A_{\mathfrak{m}})_{\mathfrak{p}} \neq 0$. This is a contradiction. \square

Proposition 4. Let A be a normal ring. Then $A[X]$ is also normal.

Definition 5. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,\xi}$ is normal for any point $\xi \in X$. A ring A is called *normal* if $\text{Spec } A$ is normal.

Remark 6. For a general ring A , let $S := A \setminus (\bigcup_{\mathfrak{p} \in \text{Ass } A} \mathfrak{p}) = \bigcap_{\mathfrak{p} \in \text{Ass } A} A \setminus \mathfrak{p}$. Then S is a multiplicative set. The localization $S^{-1}A$ is called *the total ring of fractions* of A .

Suppose A is reduced and $\text{Ass } A = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Denote its total ring of fractions by Q . Note that elements in Q are either unit or zero divisor. Hence any maximal ideal \mathfrak{m} is contained in $\bigcup \mathfrak{p}_i Q$, whence contained in some $\mathfrak{p}_i Q$. Thus $\mathfrak{p}_i Q$ are maximal ideals. And we have $\bigcap \mathfrak{p}_i Q = 0$. By the Chinese Remainder Theorem, we have $Q = \prod Q/\mathfrak{p}_i Q = \prod A_{\mathfrak{p}_i}$. Let A be a reduced ring with total ring of fractions Q . Then A is normal iff A is integral closed in Q . If A is normal, then for every $\mathfrak{p} \in \text{Spec } A$, $A_{\mathfrak{p}}$ is integral. Then there is unique minimal prime ideal $\mathfrak{p}_i \subset \mathfrak{p}$. In particular, any two minimal prime ideal are relatively prime. By the Chinese Remainder Theorem, $A = \prod A/\mathfrak{p}_i$. Just need to check A/\mathfrak{p}_i is integral closed in $A_{\mathfrak{p}_i}$. This is clear by check pointwise.

Conversely, suppose A is integral closed in Q . Let e_i be the unit element of $A_{\mathfrak{p}_i}$. It belongs to A since $e_i^2 - e_i = 0$. Since $1 = e_1 + \cdots + e_n$ and $e_i e_j = \delta_{ij}$, we have $A = \prod A e_i$. Since $A e_i$ is integral closed in $A_{\mathfrak{p}_i}$, it is normal. Hence A

A is normal.

Lemma 7. Let A be a normal ring. Then A verifies (R_1) and (S_2) .

Proof. Since all properties are local, we can assume A is integral and local.

For (S_2) , by Example ??, we only need to show that $\text{Ass}_A A/fA$ has no embedded point. Let $\mathfrak{p} = (f : g) \in \text{Ass}_A A/fA$ and $t := f/g \in \text{Frac } A$. After Replacing A by $A_{\mathfrak{p}}$, we can assume that \mathfrak{p} is maximal. By definition, $t^{-1}\mathfrak{p} \subset A$. If $t^{-1}\mathfrak{p} \subset \mathfrak{p}$, suppose \mathfrak{p} is generated by (x_1, \dots, x_n) and $t^{-1}(x_1, \dots, x_n)^T = \Phi(x_1, \dots, x_n)^T$ for $\Phi \in M_n(A)$. There is a monic polynomial $\chi(T) \in A[T]$ vanishing Φ . Then $\chi(t^{-1}) = 0$ and $t^{-1} \in A$. This is impossible by definition of t . Then $t^{-1}\mathfrak{p} = A$, and $\mathfrak{p} = (t)$ is principal. By Krull's Principal Ideal Theorem ??, $\text{ht}(\mathfrak{p}) = 1$.

Now we show that A verifies (R_1) . Suppose (A, \mathfrak{m}) is local of dimension 1. Choosing $a \in \mathfrak{m}$, A/a is of dimension 0. Then by ??, $\mathfrak{m}^n \subset aA$ for some $n \geq 1$. Suppose $\mathfrak{m}^{n-1} \not\subset aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$ and let $t = a/b$. By construction, $t^{-1} \notin A$ and $t^{-1}\mathfrak{m} \subset A$. After similar argument, we see that $\mathfrak{m} = tA$, whence A is regular. \square

Lemma 8. Let (A, \mathfrak{m}) be a noetherian local ring of dimension 1. Then A is normal iff A is regular.

Proof. By lemma 7, we just need to show that regularity implies normality.

Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since A is regular, $\mathfrak{m} = (t)$. Let $I \subset \mathfrak{m}$ be an ideal. If $I \subset \bigcap_n \mathfrak{m}^n$, then for every $a \in I$, there exists a_n such that $a = a_n t^n$. Then we get an ascending chain of ideals $(a_1) \subset (a_2) \subset \dots$. Hence $a = 0$ by Nakayama's Lemma. Suppose I is not zero. Then there is some n such that $I \subset \mathfrak{m}^n$ and $I \not\subset \mathfrak{m}^{n+1}$. For every $at^n \in I \setminus \mathfrak{m}^{n+1}$, $a \notin \mathfrak{m}$, whence a is a unit in A . Then $I = (t^n)$. Hence A is PID and hence normal. \square

Proposition 9. Let A be a noetherian integral domain of dimension ≥ 1 verifying (S_2) . Then

$$A = \bigcap_{\mathfrak{p} \in \text{Spec } A, \text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

Proof. Clearly $A \subset \bigcap A_{\mathfrak{p}}$. Let $t = f/g \in \bigcap A_{\mathfrak{p}}$. Since $f \in gA_{\mathfrak{p}}$ and we have $gA = \bigcap (gA_{\mathfrak{p}} \cap A)$, $f \in gA$. It follows that $t \in A$. \square

Theorem 10 (Serre's criterion for normality). Let X be a locally noetherian scheme. Then X is normal if and only if it verifies (R_1) and (S_2) .

Proof. One direction has been proved in Lemma 7. Suppose X verifies (R_1) and (S_2) . Again we can assume $X = \text{Spec } A$ is affine and A is local. By Remark 6, we just need to show that A is integral closed in its total ring of fractions Q . Suppose we have

$$\left(\frac{a}{b}\right)^n + c_1 \left(\frac{a}{b}\right)^{n-1} + \dots + c_n = 0 \in Q.$$

Since A verifies (S_2) , $bA = \bigcap \nu_{\mathfrak{p}}^{-1}(b_{\mathfrak{p}}A_{\mathfrak{p}})$. So it is sufficient to show that $a_{\mathfrak{p}} \in b_{\mathfrak{p}}A_{\mathfrak{p}}$ with $\text{ht}(\mathfrak{p}) = 1$. Note that $A_{\mathfrak{p}}$ is regular and hence normal by Lemma 8. Then above equation gives us desired result. \square