Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$.

1 Preliminaries on Projective Bundles

Let S be a variety over $\Bbbk$ and $\mathcal{E}$ a vector bundle of rank $r+1$ on S.

Proposition 1. The S-varieties $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on S.

Theorem 2. Let $\pi : X = \mathbb{P}_S(\mathcal{E}) \to S$ be the projective bundle associated to a vector bundle $\mathcal{E}$ of rank $r+1$ on S. Then there is an exact sequence of vector bundles on $\mathbb{P}_S(\mathcal{E})$

$$0 \to \Omega_{\mathbb{P}_S(\mathcal{E})/S} \to \pi^*(\mathcal{E})(-1) \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \to 0.$$

In particular, $K_X \sim \pi^*(K_S + \det \mathcal{E}) - (r+1)\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. Yang: To be continued...

Theorem 3 (Tsen's Theorem, [Stacks, Tag 03RD])**.** Let C be a smooth curve over an algebraically closed field $\Bbbk$. Then $\mathbf{K} = \Bbbk(C)$ is a C_1 field, i.e., every degree d hypersurface in $\mathbb{P}^n_{\mathbf{K}}$ has a $\mathbf{K}$-rational point provided $d \le n$. Yang: Need a reference.

Theorem 4 (Cohomology and Base Change, [Har77, Theorem 12.11])**.** Let $f : X \to S$ be a projective morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf on X which is flat over S. Then for each $i \ge 0$ and each point $s \in S$ there is a natural base change homomorphism

$$\varphi^i_s : \mathrm{R}^i f_*\mathcal{F} \otimes \kappa(s) \to H^i(X_s, \mathcal{F}_s).$$

Suppose that φ^i_s is surjective. Then

(a) there exists an open neighborhood U of s such that $\varphi^i_{s'}$ is an isomorphism for all $s' \in U$;

(b) TFAE:

  (i) φ^{i-1}_s is surjective;

  (ii) $\mathrm{R}^i f_*\mathcal{F}$ is locally free on an open neighborhood of s.

Theorem 5 (Grauert's Theorem, [Har77, Corollary 12.9])**.** Let $f : X \to S$ be a projective morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf on X which is flat over S. Suppose that S is integral and the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$ is constant on S for some $i \ge 0$. Then $\mathrm{R}^i f_*\mathcal{F}$ is locally free and the base change homomorphism

$$\varphi^i_s : \mathrm{R}^i f_*\mathcal{F} \otimes \kappa(s) \to H^i(X_s, \mathcal{F}_s)$$

Date: August 26, 2025, **Author**: Tianle Yang, My Homepage

is an isomorphism for all $s \in S$.

Theorem 6 (Miracle Flatness, [Mat89, Theorem 23.1])**.** Let $f : X \to Y$ be a morphism of noetherian schemes. Assume that Y is regular and X is Cohen-Macaulay. If all fibers of f have the same dimension $d = \dim X - \dim Y$, then f is flat.

Proposition 7. Let X be a noetherian scheme and $\mathcal{E}$ a vector bundle of rank $r+1$ on X. Let Y be a X-scheme via a morphism $g : Y \to X$. Then there is a bijection

$$\{X\text{-morphisms } Y \to \mathbb{P}_X(\mathcal{E})\} \leftrightarrow \left\{ \begin{array}{l} \text{surjective morphisms } g^*\mathcal{E} \to \mathcal{L} \\ \text{where } \mathcal{L} \text{ is a line bundle on } Y \end{array} \right\}.$$

Yang: Need to check.

2 Minimal Section and Classification

Definition 8 (Ruled surface)**.** A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 9. There exists a section of π.

Proof. □

Proposition 10. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Proof. Let $\sigma : C \to X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π, $\mathcal{L}|_{X_c} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $c \in C$, whence $h^0(X_c, \mathcal{L}|_{X_c}) = 2$ for any $c \in C$. By Miracle Flatness (Theorem 6), f is flat. By Grauert's Theorem (Theorem 5), $\mathcal{E}$ is a vector bundle of rank 2 on C. □

Lemma 11. There is a one-to-one correspondence between sections of π and quotient line bundles of $\mathcal{E}$.

Proof. □

Lemma 12. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*.

Proof. □

Definition 13. A section C_0 of π is called a *minimal section* if Yang: to be continued...

Theorem 14. Let

Theorem 15. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Theorem 16. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and for each e there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on E with $\deg \mathcal{L} = -e$.

3 The Néron-Severi Group of Ruled Surfaces

Proposition 17. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let f be a fiber of π. Then $K_X \sim -2C_0 + (K_C -)f$ where $e = -C_0^2$. Yang: Check this carefully.

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9 (cit. on p. 1).

[Mat89] Hideyuki Matsumura. *Commutative ring theory*. 8. Cambridge university press, 1989 (cit. on p. 2).

[Stacks] The Stacks Project Authors. *Stacks Project*. URL: https://stacks.math.columbia.edu/ (cit. on p. 1).