Algebraic Groups

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Contents

1 First properties of algebraic groups

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. Everything are defined over $\mathbf{k}$ unless otherwise specified.

1.1 Basic concepts

Definition 1.1. A *group scheme* over S is an S-scheme G together with morphisms *multiplication* $\mu : G \times G \to G$, *identity* $\varepsilon : S \to G$ and *inversion* $\iota : G \to G$ over S such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd} & G \times G \times G \arrow[dl, "\mathrm{id}_G \times \mu"'] \arrow[dr, "\mu \times \mathrm{id}_G"] & \\ G \times G \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd};$$

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(b) (Identity)

$$\begin{tikzcd} G \times S \arrow[r, "\mathrm{id}_G \times \varepsilon"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] & S \times G \arrow[l, "\varepsilon \times \mathrm{id}_G"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd} & G \arrow[dl, "\mathrm{id}_G \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_G"] & \\ G \times G \arrow[dr, "\mu"'] & S \arrow[d, "\varepsilon"] & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd}.$$

In other words, an S-group scheme is a group object in the category $\mathbf{Sch}_S$.

Definition 1.2. An *algebraic group* is a $\mathbf{k}$-group scheme G which is reduced, separated and of finite type over a field $\mathbf{k}$.

Remark 1.3. Even if we work over $\Bbbk$ and just consider the closed points $G(\Bbbk)$ of an algebraic group G, $G(\Bbbk)$ is not a topological group with respect to the Zariski topology in general. The reason is that the topology on $G(\Bbbk) \times G(\Bbbk)$ is not the product topology of the topologies on $G(\Bbbk)$.

Definition 1.4. Let G be an algebraic group and $x \in G(\mathbf{k})$ a $\mathbf{k}$-point. The *left translation* by x is the morphism

$$l_x : G \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{\mu} G.$$

Similar definition applies to the right translation r_x.

Remark 1.5. In the context of algebraic groups, we often use multiplicative notation for the group law. That is, for $g, h \in G(\mathbf{k})$, we write gh instead of $\mu(g, h)$ and g^{-1} instead of $\iota(g)$.

Sometimes we also abuse the notation by $\mu : G \times \cdots \times G \to G$ to denote the multiplication of multiple elements, i.e. $\mu(g_1, \dots, g_n) = g_1 \cdots g_n$ for $g_1, \dots, g_n \in G(\mathbf{k})$.

Proposition 1.6. Let G be an algebraic group. Then G is smooth over $\mathbf{k}$.

Proof. Since G is reduced and of finite type over a field, it is generically regular. Let $g \in G(\Bbbk)$ be a regular point. Then the left translation $l_{gh^{-1}} : G \to G$ is an isomorphism, hence G is regular at $h \in G(\Bbbk)$. It follows that G is regular at every $\Bbbk$-point, hence G is smooth over $\mathbf{k}$. □

Remark 1.7. Let G be an algebraic group. Then the irreducible components of G coincide with the connected components of G. We will use the term "connected" to refer to both concepts since "irreducible" has other meanings in the theory of representations.

Example 1.8. The *additive group* $\mathbb{G}_a$ is defined to be the affine line $\mathbb{A}^1$ with the group law given by addition. Concretely, we can write $\mathbb{G}_a = \operatorname{Spec} \mathbf{k}[T]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a, &\quad (x,y) \mapsto x+y, \\ \iota : \mathbb{G}_a \to \mathbb{G}_a, &\quad x \mapsto -x, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_a, &\quad * \mapsto 0. \end{aligned}$$

Example 1.9. The *multiplicative group* $\mathbb{G}_m$ is defined to be the affine variety $\mathbb{A}^1 \setminus \{0\}$ with the group law given by multiplication. Concretely, we can write $\mathbb{G}_m = \operatorname{Spec} \mathbf{k}[T, T^{-1}]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m, &\quad (x,y) \mapsto xy, \\ \iota : \mathbb{G}_m \to \mathbb{G}_m, &\quad x \mapsto x^{-1}, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_m, &\quad * \mapsto 1. \end{aligned}$$

Example 1.10. The *general linear group* GL_n is defined to be the open subvariety of $\mathbb{A}^{n^2}$ consisting of invertible matrices, with the group law given by matrix multiplication. Concretely, we can write $\mathrm{GL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}, \det(T_{ij})^{-1}]$ where $1 \le i,j \le n$ and the group law is given by the morphism

$$\begin{aligned} \mu : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, &\quad (A,B) \mapsto AB, \\ \iota : \mathrm{GL}_n \to \mathrm{GL}_n, &\quad A \mapsto A^{-1}, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathrm{GL}_n, &\quad * \mapsto I_n. \end{aligned}$$

Example 1.11. An abelian variety is an algebraic group that is also a proper variety.

Example 1.12. Let G and H be algebraic groups. The *product* $G \times H$ is an algebraic group with the group law defined by

$$\begin{aligned} \mu_{G\times H} = \mu_G \times \mu_H &: (G \times H) \times (G \times H) \cong (G \times G) \times (H \times H) \to G \times H, \\ \varepsilon_{G\times H} = \varepsilon_G \times \varepsilon_H &: \operatorname{Spec} \mathbf{k} \cong \operatorname{Spec} \mathbf{k} \times \operatorname{Spec} \mathbf{k} \to G \times H, \\ \iota_{G\times H} = \iota_G \times \iota_H &: G \times H \to G \times H. \end{aligned}$$

Example 1.13. Let G be an algebraic group over $\mathbf{k}$ and $\mathbf{K}/\mathbf{k}$ a field extension. The base change $G_{\mathbf{K}} = G \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} \mathbf{K}$ is an algebraic group over $\mathbf{K}$ with the group law defined by the base change of the original group law of G to $\mathbf{K}$.

Definition 1.14. A *homomorphism* of algebraic groups is a morphism of schemes that is also a group homomorphism. Explicitly, a morphism $f : G \to H$ between algebraic groups G and H is a homomorphism if the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

where μ_G and μ_H are the group laws of G and H, respectively.

Definition 1.15. An *algebraic subgroup* of an algebraic group G is a closed subscheme $H \subseteq G$ that is also a subgroup of G. More precisely, H is an algebraic subgroup and the inclusion morphism $H \hookrightarrow G$ is compatible with the group laws.
An algebraic subgroup H of G is called *normal* if for any $\mathbf{k}$-scheme S, the subgroup $H(S)$ is a normal subgroup of the abstract group $G(S)$.

Remark 1.16. To check $H < G$ whether H is a normal subgroup of G, it suffices to check that $H(\Bbbk)$ is normal in $G(\Bbbk)$. Yang: To be continued.

Example 1.17. The *special linear group* SL_n is defined to be the closed subvariety of GL_n defined by the equation $\det = 1$. It is an algebraic subgroup of GL_n.

Proposition 1.18. Let G be an algebraic group and S is a closed subgroup of $G(\Bbbk)$. Then there exists a unique algebraic subgroup H of G such that $H(\Bbbk) = S$.

Proof. Yang: To be continued... □

Remark 1.19. By Proposition 1.18, we often identify an algebraic group G with its set of closed points $G(\Bbbk)$ when there is no confusion.

Remark 1.20. If one replaces $\Bbbk$ by $\mathbf{k}$ in Proposition 1.18, the statement may not hold. For example, let $\mathbf{k} = \mathbb{Q}$ and G be the elliptic curve defined by $X^3+Y^3 = Z^3$ in $\mathbb{P}^2$. It is well-known that $\#G(\mathbb{Q}) = 3$. Let S be the disjoint union of the three $\mathbb{Q}$-points of G endowed with the reduced subscheme structure and the group structure induced from G. Then S is a proper closed subgroup of G and we have $S(\mathbb{Q}) = G(\mathbb{Q})$. This contradicts the uniqueness in Proposition 1.18.

Indeed, in this chapter, despite working over an arbitrary field $\mathbf{k}$, we mostly consider the closed points of algebraic groups over $\Bbbk$.

Definition 1.21. Let G be an algebraic group. The *neutral component* G^0 is the connected component of G containing the identity element ε.

Proposition 1.22. The neutral component G^0 is a closed, normal algebraic subgroup of G.

Proof. Yang: To be continued... □

Proposition 1.23. Let G be an algebraic group and $H \subseteq G(\Bbbk)$ a subgroup (not necessarily closed). Then the Zariski closure $\overline{H}$ of H in G is an algebraic subgroup of G. If $H \subset G(\Bbbk)$ is constructible, then $H = \overline{H}(\Bbbk)$.

Proof. Yang: To be continued... □

Remark 1.24. In general, we can only expect the image of a morphism of varieties to be a constructible subset. This is not sufficient to guarantee that the image is closed, even if the original variety is closed. However, the group structure provides additional constraints that ensure the constructible subgroup is indeed closed. Example 1.25 provides an example where the product of two closed algebraic subgroups is not closed, illustrating that the importance of the subgroup condition.

Yang: To be continued...

Example 1.25. Let $G = \mathrm{SL}_2$ over $\Bbbk$, $T = \{\mathrm{diag}(t, t^{-1}) | t \in \Bbbk^\times\}$ and $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Set $S = gTg^{-1}$. Then both T and S are closed algebraic subgroups of $G(\Bbbk)$, but the product TS is not closed in $G(\Bbbk)$. By direct computation, we have

$$S = \left\{ \begin{pmatrix} s & s^{-1} - s \\ 0 & s^{-1} \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

Then

$$TS = \left\{ \begin{pmatrix} ts & t(s^{-1} - s) \\ 0 & (ts)^{-1} \end{pmatrix} \middle| t, s \in \Bbbk^\times \right\}.$$

We have

$$TS \cap \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \Bbbk \right\} = \left\{ \begin{pmatrix} 1 & s^{-2} - 1 \\ 0 & 1 \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

The right hand side is not closed in $\mathrm{SL}_2(\Bbbk)$ since it does not contain the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Hence TS is not closed in $G(\Bbbk)$.

Proposition 1.26. Let G be an algebraic group, X_i varieties over $\mathbf{k}$ and $f_i : X_i \to G$ morphisms for $i = 1, \dots, n$ with images $Y_i = f_i(X_i)$. Suppose that Y_i pass through the identity element of G. Let H be the closed subgroup of G generated by $Y_1, \dots, Y_n$, i.e. the smallest closed subgroup of G containing $Y_1, \dots, Y_n$. Then H is connected and $H = Y_{a_1}^{e_1} \cdots Y_{a_m}^{e_m}$ for some $a_1, \dots, a_m \in \{1, \dots, n\}$ and $e_1, \dots, e_m \in \{\pm 1\}$.

Proof. Yang: To be continued... □

Remark 1.27. We can take $m \leq 2 \dim G$ in Proposition 1.26.

1.2 Action and representations

Definition 1.28. An *action* of an algebraic group G on a variety X is a morphism

$$\sigma : G \times X \to X$$

such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \sigma"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}\,\mathbf{k} \times X \arrow[r, "\varepsilon \times \mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

where μ is the group law of G and ε is the identity element of G. In other words, for any $\mathbf{k}$-scheme S, the induced map $G(S) \times X(S) \to X(S)$ defines a group action of the abstract group $G(S)$ on the set $X(S)$.
For simplicity, we often write $g.x$ instead of $\sigma(g, x)$ for $g \in G(\mathbf{k})$ and $x \in X(\mathbf{k})$.

Example 1.29. There are three natural actions of an algebraic group G on itself:

(a) Left translation: $g.h = l_g(h) = gh$;

(b) Right translation: $g.h = r_g(h) = hg^{-1}$;

(c) Conjugation: $g.h = \mathrm{Ad}_g(h) = ghg^{-1}$.

All of them are morphisms of varieties since they are defined by the group law and inversion of G.

Example 1.30. The general linear group GL_n acts on the affine space $\mathbb{A}^n$ by matrix multiplication. It is given by polynomials, hence is a morphism of varieties.

Example 1.31. The general linear group GL_{n+1} acts on the projective space $\mathbb{P}^n$ by

$$A \cdot [x_0 : \cdots : x_n] = [y_0 : \cdots : y_n], \quad \text{where } (y_0, \ldots, y_n)^T = A(x_0, \ldots, x_n)^T.$$

Let U_i be the standard affine open subset of $\mathbb{P}^n$ defined by $x_i \neq 0$. The map is given by polynomials on the principal open subset of $\mathrm{GL}_{n+1} \times U_i$ defined by $y_j \neq 0$ for any j. Hence it is a morphism of varieties.

Definition 1.32. A *linear representation* of an algebraic group G on a finite-dimensional vector space V over $\mathbb{k}$ is an abstract group representation $\rho : G(\mathbb{k}) \to \mathrm{GL}(V)$ such that if we identify V with $\mathbb{A}^n$ for some n, then the map $G(\mathbb{k}) \times \mathbb{A}^n(\mathbb{k}) \to \mathbb{A}^n(\mathbb{k})$ is is a morphism of varieties.

Definition 1.33. Let G be an algebraic group acting on a variety X. For any $x \in X(\mathbf{k})$, the *orbit* of x is the locally closed subvariety $G \cdot x = \sigma(G \times \{x\})$ of X.

Proposition 1.34. Let G be an algebraic group acting on a variety X. Then for any $x \in X(\mathbf{k})$, the orbit $G \cdot x$ is a locally closed subvariety of X, and $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of strictly smaller dimension.

Proof. Yang: To be continued... □

Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For $x \in G(\mathbf{k})$, we have the left translation of functions $\tau_x : A \to A$ defined by $\tau_x(f)(y) = f(x^{-1}y)$ for $y \in X(\mathbf{k})$.

Lemma 1.35. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For any finite-dimensional subspace $V \subseteq A$, there exists a finite-dimensional G-invariant subspace $W \subseteq A$ containing V.

Proof. Yang: To be continued... □

Theorem 1.36. Any affine algebraic group is linear, i.e. is isomorphic to a closed algebraic subgroup of some GL_n.

Proof. Yang: To be continued... □

1.3 Lie algebra of an algebraic group

Let G be an algebraic group. The *Lie algebra* of G is defined to be the tangent space of G at the identity element ε:

$$\mathrm{Lie}(G) = T_\varepsilon G.$$

It is a finite-dimensional vector space over $\mathbf{k}$.

Proposition 1.37. The group law $\mu : G \times G \to G$ induces the plus map on $\mathrm{Lie}(G)$:

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)} : T_{(\varepsilon,\varepsilon)}(G \times G) \cong T_\varepsilon G \oplus T_\varepsilon G \to T_\varepsilon G, \quad (v, w) \mapsto v + w.$$

Proof. We have

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v, w) = \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v, 0) + \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(0, w) = (\mathrm{d}\mu \circ (\mathrm{id}_G \times \varepsilon))_\varepsilon(v) + (\mathrm{d}\mu \circ (\varepsilon \times \mathrm{id}_G))_\varepsilon(w) = v + w.$$

□

Proposition 1.38. Let G be an algebraic group and n a positive integer which is not divisible by $\mathrm{char}\,\mathbf{k}$. Then the power map $p_n : G \to G$ is generically finite.

Proof. Yang: To be added. □

Corollary 1.39. Let G be a connected algebraic group and H a closed subgroup of $G(\Bbbk)$ with finite index. Then $H = G(\Bbbk)$.

Proof. Yang: To be added. □

Corollary 1.40. Let G be an algebraic group and H a closed subgroup of $G(\Bbbk)$. Suppose that there exists a positive integer n which is not divisible by $\mathrm{char}\,\mathbf{k}$ such that $h^n = e$ for all $h \in H$. Then H is finite. Yang: To be completed.

Remark 1.41. Thanks for my mathematical brother Zelong Chen for telling me this. Yang: To be revised

Remark 1.42. The classical Burnside theorem states that a finite exponent subgroup of $\mathrm{GL}_n(\mathbb{C})$ is finite. Corollary 1.40 can be viewed as a generalization of the classical Burnside theorem to arbitrary algebraic groups over arbitrary fields.

2 Quotient by algebraic group

Everything in this section is over an arbitrary field $\mathbf{k}$ unless otherwise specified.

2.1 Quotient

Definition 2.1. Let G be an algebraic group acting on a variety X. A *quotient* of X by G is a variety Y together with a morphism $\pi : X \to Y$ such that

(a) π is G-invariant, i.e., $\pi(g \cdot x) = \pi(x)$ for all $g \in G$ and $x \in X$.

(b) For any variety Z and any G-invariant morphism $f : X \to Z$, there exists a unique morphism $\overline{f} : Y \to Z$ such that $f = \overline{f} \circ \pi$.

In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\overline{f}"] \\ & Z \end{tikzcd}$$

If a quotient exists, it is unique up to a unique isomorphism. Yang: To be continued...

Such a quotient does not always exist.

Theorem 2.2. Let G be an affine algebraic group acting on a variety X. Then there exists a variety Y and a rational morphism $\pi : X \dashrightarrow Y$ with commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "\overline{f}"] \\ & Z \end{tikzcd}$$

satisfying the following universal property: If a quotient exists, it is unique up to a unique isomorphism.
Furthermore, if all orbits of G in X are closed, then π is a morphism (i.e., defined everywhere). Yang: To be continued... Yang: Ref?

2.2 Quotient of affine algebraic group by closed subgroup

Lemma 2.3. Let V be a finite-dimensional vector space over $\mathbf{k}$ and G an abstract group acting linearly on V. Let $W \subseteq V$ be a subspace of dimension m. Then $G.W = W$ if and only if $G. \wedge^m W = \wedge^m W$.

Proof. Yang: To be filled. □

Lemma 2.4. Let G be an affine algebraic group and H a closed subgroup. Then there exists a finite-dimensional linear representation V of G and a one-dimensional subspace $L \subseteq V$ such that H is the stabilizer of L.

Proof. Yang: To be filled. □

Theorem 2.5. Let G be an affine algebraic group and H a closed subgroup. Then the quotient G/H exists as a quasi-projective variety.

Proof. Yang: To be filled. □

3 Decomposition of algebraic groups

Theorem 3.1 (Chavellaye Decomposition)**.** Let G be an algebraic group. Then there exists a unique maximal connected affine normal algebraic subgroup G_{aff} of G such that the quotient G/G_{aff} is an abelian variety. This subgroup is called the *affine part* of G. Yang: To be continued...

Theorem 3.2 (Rosenlicht Decomposition)**.** Let G be an algebraic group. Then there exists a smallest normal connected algebraic subgroup G_{ant} of G such that the quotient G/G_{ant} is affine. This subgroup is called the *anti-affine part* of G. Moreover, G_{ant} is contained in the center of G^0. Yang: To be continued...

3.1

4 Structure of linear algebraic groups

4.1 Jordan-Chevalley Decomposition of elements

Recall that for a linear operator $T : V \to V$ of finite-dimensional $\Bbbk$-vector space V is called *semisimple* if it is diagonalizable, and *unipotent* if $T - \mathrm{id}_V$ is nilpotent.

Definition 4.1. Let G be a linear algebraic group and $g \in G(\Bbbk)$. We say that g is *semisimple* (resp. *unipotent*) if its image under some (equivalently, any) faithful linear representation of G is a semisimple (resp. unipotent) linear operator.

Lemma 4.2. The notion of semisimple and unipotent elements in Definition 4.1 does not depend on the choice of faithful linear representation.

Proof. Yang: To be added. □

Theorem 4.3 (Jordan-Chevalley Decomposition)**.** Let G be a linear algebraic group and $g \in G(\Bbbk)$. Then there exist unique commuting elements $g_s, g_u \in G(\Bbbk)$ such that $g = g_s g_u$, where g_s is semisimple and g_u is unipotent.
Moreover, this decomposition is functorial in the sense that for any homomorphism of linear algebraic groups $\varphi : G \to H$, we have $\varphi(g)_s = \varphi(g_s)$ and $\varphi(g)_u = \varphi(g_u)$. Yang: To be checked

Proof. Yang: To be continued. □

4.2 Solvable groups and Borel subgroups

Definition 4.4. A group G is said to be *solvable* if there exists a finite sequence of algebraic subgroups

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = \{e\}$$

such that each G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is commutative for all $0 \leq i < n$. Yang: to be checked.

Theorem 4.5. Let G be a solvable linear algebraic group acting on a proper variety X. Then there exists a fixed point $x \in X(\mathbb{k})$ such that $g \cdot x = x$ for all $g \in G(\mathbb{k})$.

Corollary 4.6 (Lie-Kolchin Theorem)**.** Let $G < \mathrm{GL}_n(\mathbb{k})$ be a solvable linear algebraic group over an algebraically closed field $\mathbb{k}$. Then there exists a basis of $\mathbb{k}^n$ such that G is contained in the group of upper triangular matrices with respect to this basis.

Theorem 4.7. Let G be a linear algebraic group of dimension 1 over an algebraically closed field $\mathbb{k}$. Then G is isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$.

4.3 Decomposition of linear algebraic groups

Definition 4.8. Let G be a linear algebraic group over a field $\mathbb{k}$. The *radical* of G, denoted by $\mathrm{rad}(G)$, is defined to be the unique maximal connected normal solvable subgroup of G.

Yang: Well-defined?

Definition 4.9. Let G be a linear algebraic group. The *unipotent radical* of G, denoted by $\mathrm{rad}_u(G)$, is defined to be the subgroup of $\mathrm{rad}(G)$ consisting of all unipotent elements.

Yang: Why a group?

Definition 4.10. Let G be a linear algebraic group over a field $\mathbb{k}$. We say that G is *semisimple* if $\mathrm{rad}(G)$ is trivial.

Definition 4.11. Let G be a linear algebraic group over a field $\mathbb{k}$. We say that G is *reductive* if the unipotent radical of G is trivial.

Slogan

$$\begin{array}{ccc} \text{“\textit{unipotent radical}”} & \rightarrow\leftarrow & \text{“\textit{reductive}”} \\ \Downarrow & & \Uparrow \\ \text{“\textit{solvable radical}”} & \rightarrow\leftarrow & \text{“\textit{semisimple}”} \end{array}$$

Theorem 4.12 (Levi Decomposition)**.** Let G be a linear algebraic group over an algebraically closed field $\mathbb{k}$. Then there exists a reductive subgroup H of G such that the multiplication map $\mathrm{rad}_u(G) \rtimes H \to G$ is an isomorphism of algebraic groups. Such a subgroup H is called a *Levi subgroup* of G. Yang: To be checked.

Proof. Yang: To be continued. □

4.4 Semisimple and reductive algebraic groups

5 Weil regularization theorem

6 Application: birational group of varieties of general type

In this section, we apply the results from the previous sections to study the birational automorphism groups of varieties of general type.

Theorem 6.1. Let X be a projective variety of general type over an algebraically closed field $\Bbbk$ of characteristic zero. Then the group of birational automorphisms $\mathrm{Bir}(X)$ is finite.

Proof. We will prove this theorem in several steps. By replacing X with its resolution of singularities, we may assume that X is smooth.

Step 1. For every $m \geq 1$, $\mathrm{Bir}(X)$ linearly acts on $H^0(X, mK_X)$ via pull-back of functions (as abstract group).

Let $\mathscr{K}(X)$ be the function field of X. Then for every $g \in \mathrm{Bir}(X)$, g induces an automorphism of $\mathscr{K}(X)$ over $\Bbbk$, which we denote by g^*. In particular we know that g^* is injective and $\Bbbk$-linear. By definition, $H^0(X, mK_X) = \{s \in \mathscr{K}(X) \mid \mathrm{div}(s) + mK_X \geq 0\}$. We only need to show that for every $s \in H^0(X, mK_X)$, $g^*(s) \in H^0(X, mK_X)$ since $\dim_{\Bbbk} H^0(X, mK_X) < \infty$. Consider the commutative diagram

$$\begin{tikzcd} \Gamma \arrow[d, "p"'] \arrow[dr, "q"] & \\ X \arrow[r, dashed, "g"] & X \end{tikzcd}$$

with Γ smooth and p, q birational morphisms. Then we have

$$K_\Gamma = p^*K_X + E_p = q^*K_X + E_q,$$

where E_p and E_q are p- and q-exceptional divisors respectively. Moreover, E_p and E_q are effective since X is smooth. For every $s \in H^0(X, mK_X)$, we have

$$\mathrm{div}(q^*s) + mK_\Gamma = q^*(\mathrm{div}(s) + mK_X) + mE_q \geq 0.$$

Then

$$\begin{aligned} \mathrm{div}(g^*s) + mK_X &= p_*p^*(\mathrm{div}(g^*s) + mK_X) \\ &= p_*\left(\mathrm{div}(q^*s) + mK_\Gamma - mE_p\right) \\ &= p_*\left(\mathrm{div}(q^*s) + mK_\Gamma\right) \geq 0. \end{aligned}$$

It follows that $g^*(s) \in H^0(X, mK_X)$.

Note this action $g \mapsto g^*$ is contravariant, i.e., for every $g_1, g_2 \in \mathrm{Bir}(X)$, we have $(g_1 \circ g_2)^* = g_2^* \circ g_1^*$.

Step 2. The group $\mathrm{Bir}(X)$ is a linear algebraic group by identifying it with a closed subgroup of $\mathrm{Aut}(\mathbb{P}(V))$ for some finite-dimensional $\Bbbk$-vector space V (subspace of $H^0(X, mK_X)$ for some $m > 0$). Moreover, its rational action on X is algebraic.

By ??, there exists an integer $m > 0$ such that the map $\psi : X \dashrightarrow \mathbb{P}(H^0(X, mK_X))$ is birational onto its image Y. Let V be the subspace of $H^0(X, mK_X)$ spanned by the affine cone over Y. Since $\mathrm{Bir}(X)$ linearly acts on $H^0(X, mK_X)$ by Step 1, it also linearly acts on V. we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "g"] \arrow[d, dashed, "\psi"] & X \arrow[d, dashed, "\psi"] \\
Y \arrow[r, "\varphi_g|_Y"] \arrow[d, hook] & Y \arrow[d, hook] \\
\mathbb{P}(V) \arrow[r, "\varphi_g"] & \mathbb{P}(V)
\end{tikzcd}$$

for every $g \in \mathrm{Bir}(X)$, where φ_g is the induced automorphism of $\mathbb{P}(V)$.

Since ψ is birational, the map $g \mapsto \varphi_g$ defines an injective group homomorphism from $\mathrm{Bir}(X)$ to $\mathrm{Aut}(\mathbb{P}(V))$. Consider the natural algebraic group structure on $\mathrm{Aut}(\mathbb{P}(V))$ and let G be the Zariski closure of the image of $\mathrm{Bir}(X)$ in $\mathrm{Aut}(\mathbb{P}(V))$. Note that $\mathrm{Bir}(X)$ fixes Y. Thus G also fixes Y. Since the affine cone over Y spans V, we conclude that any element $g \in G$ is uniquely determined by its restriction to Y. In particular, we have $G = \mathrm{Bir}(X)$. Note that $\mathrm{Aut}(\mathbb{P}(V))$ is a linear algebraic group and so is its closed subgroup $\mathrm{Bir}(X)$.

Step 3. If $\dim \mathrm{Bir}(X) > 0$, then it contains $\mathbb{G}_a$ or $\mathbb{G}_m$ as a subgroup. We show that the action of $\mathbb{G}_a$ or $\mathbb{G}_m$ on X leads to X being uniruled, which contradicts the assumption that X is of general type.

By Lemma 6.5 and Theorem 4.7, if $\dim \mathrm{Bir}(X) > 0$, then $\mathrm{Bir}(X)$ contains either $\mathbb{G}_a$ or $\mathbb{G}_m$ as a subgroup. Note that both $\mathbb{G}_a$ and $\mathbb{G}_m$ are rational varieties, without loss of generality, we may assume that $\mathrm{Bir}(X)$ contains $\mathbb{G}_m$ as a subgroup. Then we have a rational map

$$\Phi : \mathbb{G}_m \times X \dashrightarrow X.$$

Fix $x \in X$ such that $\Phi|_{\mathbb{G}_m \times \{x\}} : \mathbb{G}_m \to X$ is not constant. Choose $Z \subset X$ a closed subvariety of codimension 1 passing through x such that $\mathbb{G}_m.x \not\subseteq Z$. Then the closure of $\Phi(\mathbb{G}_m \times Z)$ in X has dimension at least $\dim Z + 1 = \dim X$. Hence we have a dominant rational map

$$\Phi : \mathbb{P}^1 \times Z \dashrightarrow X.$$

This contradicts ?? and the assumption that X is of general type. Therefore, we must have $\dim \mathrm{Bir}(X) = 0$, i.e., $\mathrm{Bir}(X)$ is finite. □

Remark 6.2. In the proof of Theorem 6.1, by $\mathbb{P}(V)$ we mean the projective space associated to the vector space V in the sense of Grothendieck, i.e., $\mathbb{P}(V) = \mathrm{Proj}(\bigoplus_{k \geq 0} \mathrm{Sym}^k V)$. Hence if one have a linear map $f : V \to W$ between two finite-dimensional $\Bbbk$-vector spaces, then it induces a morphism $\mathbb{P}(W) \to \mathbb{P}(V)$ (not $\mathbb{P}(V) \to \mathbb{P}(W)$).

Corollary 6.3. Let X be a projective variety of general type over an algebraically closed field $\Bbbk$ of characteristic zero. Then there exists a projective variety Y birational to X such that $\mathrm{Bir}(Y) = \mathrm{Aut}(Y)$.

Corollary 6.4. Let X be a smooth projective Fano variety over an algebraically closed field $\Bbbk$ of characteristic zero. Then the group of automorphisms $\mathrm{Aut}(X)$ is a linear algebraic group.

Proof. Note that for every $g \in \mathrm{Aut}(X)$, g induces an automorphism of $H^0(X, -mK_X)$ for every integer $m \geq 1$ via pull-back of functions. Then the same argument as in Step 2 shows that $\mathrm{Aut}(X)$ is a linear algebraic group. □

Lemma 6.5. Let G be a linear algebraic group over an algebraically closed field $\Bbbk$. Then G has a one-dimensional algebraic subgroup.