Definitions and examples

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Let G be a reductive group over $\Bbbk$ acting on a variety X over $\Bbbk$.

Definition 1. A *categorical quotient* of X by G is a variety Y together with a G-invariant morphism $\pi : X \to Y$ such that for any G-invariant morphism $\varphi : X \to Z$ to a variety Z, there exists a unique morphism $\psi : Y \to Z$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & Y \arrow[d, "\psi"] \\ & Z \end{tikzcd}$$

Definition 2. A *geometric quotient* of X by G is a variety Y together with a G-invariant morphism $\pi : X \to Y$ satisfying the following conditions:

(a) The morphism π is surjective, and the fibers of π are precisely the G-orbits in X.

(b) The topology on Y is the quotient topology induced by π, i.e., a subset $U \subseteq Y$ is open if and only if $\pi^{-1}(U)$ is open in X.

(c) The structure sheaf $\mathcal{O}_Y$ is given by the sheaf of G-invariant regular functions on X, i.e., for any open subset $U \subseteq Y$,

$$\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}(U))^G.$$

Appendix

Date: January 25, 2026, **Author**: Tianle Yang, My Homepage