Notes in Algebraic Geometry

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Notes in Algebraic Geometry

Author: Tianle Yang

Email: loveandjustice@88.com

Homepage: tianleyang.com

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Contents

Chapter 1

The First Properties

1.1 Setup and the first examples

1.1.1 Notations

All schemes are assumed to be separated. For a "scheme" which is not separated, we will use the term "prescheme".

Let A be a ring. We denote by $\operatorname{Spec} A$ the spectrum of A. For an ideal $I \subset A$, we use $V(I)$ to denote the closed subscheme of $\operatorname{Spec} A$ defined by I.

Let S be $\operatorname{Spec} \mathsf{k}$, $\operatorname{Spec} \mathcal{O}_K$ or an algebraic variety. An S-variety is an integral scheme X which is of finite type and flat over S. For an algebraic variety, we mean a k-variety.

We will use k, K to denote fields, and $\mathbf{k}, \mathbf{K}$ to denote their algebraically closure relatively.

Let X be an integral scheme. We denote by $\mathscr{K}(X)$ the function field of X. For a closed point $x \in X$, we denote by $\kappa(x)$ the residue field of x.

We denote the category of S-varieties by $\mathbf{Var}_S$. We denote by $X(T)$ the set of T-points of X, that is, the set of morphisms $T \to X$.

Let X be an algebraic variety over k. A geometrical point is referred a morphism $\operatorname{Spec} \mathbf{k} \to X$.

When refer a point (may not be closed) in a scheme, we will use the notation $\xi \in X$. We use Z_ξ to denote the Zariski closure of $\{\xi\}$ in X. When we talk about a closed point on an algebraic variety, we will use the notation $x \in X(\mathbf{k})$.

Separated and proper morphisms

1.1.2 Examples

Appendix A

Commutative Algebra

A.1 Elementary ResultsYang: To be completed

A.1.1 Notations

Proposition A.1.1. Let A be a ring, $\mathfrak{p}, \mathfrak{p}_i$ prime ideals of A and $\mathfrak{a}, \mathfrak{a}_i$ ideals of A.

(a) Suppose $\mathfrak{a} \subset \bigcup_{i=1}^n \mathfrak{p}_i$. Then there exists i such that $\mathfrak{a} \subset \mathfrak{p}_i$.

(b) Suppose $\bigcap_{i=1}^n \mathfrak{a}_i \subset \mathfrak{p}$. Then there exists i such that $\mathfrak{a}_i \subset \mathfrak{p}$.

Definition A.1.2. Let A be a ring and M an A-module. The *support* of M is defined as

$$\operatorname{Supp} M := \{\mathfrak{p} \in \operatorname{Spec} A : M_{\mathfrak{p}} \neq 0\}.$$

Proposition A.1.3. Let A be a ring and M a finite A-module. Then $\operatorname{Supp} M = V(\operatorname{Ann} M)$. In particular, $\operatorname{Supp} M$ is a closed subset of $\operatorname{Spec} A$.

Proof. Yang: To be completed. □

A.1.2 Nakayama's Lemma

Theorem A.1.4 (Nakayama's Lemma)**.** Let A be a ring and $\mathfrak{M}$ be its Jacobi radical. Suppose M is a finitely generated A-module. If $\mathfrak{a}M = M$ for $\mathfrak{a} \subset \mathfrak{M}$, then $M = 0$.

Proof. Suppose M is generated by $x_1, \cdots, x_n$. Since $M = \mathfrak{a}M$, formally we have $(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(\mathfrak{a})$. Then $(\Phi - \mathrm{id})(x_1, \cdots, x_n)^T = 0$. Note that $\det(\Phi - \mathrm{id}) = 1 + a$ for $a \in \mathfrak{a} \subset \mathfrak{M}$. Then $\Phi - \mathrm{id}$ is invertible and then $M = 0$. □

Remark A.1.5. The finiteness of M is crucial in Nakayama's Lemma. For example, let $\overline{\mathbb{Z}}$ be the ring of algebraic integers in $\overline{\mathbb{Q}}$. Choose a non-zero prime ideal $\mathfrak{p}$ of $\overline{\mathbb{Z}}$. Then we have that $\mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}} = \mathfrak{p}^2\overline{\mathbb{Z}}_{\mathfrak{p}}$. Indeed, if $a \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$, let $b = \sqrt{a} \in \overline{\mathbb{Z}}_{\mathfrak{p}}$. Then $b^2 = a \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$ and whence $b \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$ since $\mathfrak{p}$ is prime. It follows that $a = b^2 \in \mathfrak{p}^2\overline{\mathbb{Z}}_{\mathfrak{p}}$.

Proposition A.1.6 (Geometric form of Nakayama's Lemma)**.** Let $X = \operatorname{Spec} A$ be an affine scheme, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proof. Yang: To be completed. □

Corollary A.1.7.

Proof. Yang: To be completed. □

A.1.3 Nullstellensatz

Theorem A.1.8 (Noether's Normalization Lemma)**.** Let A be a k-algebra of finite type. Then there is an injection $\mathsf{k}[T_1, \cdots, T_d] \hookrightarrow A$ such that A is finite over $\mathsf{k}[T_1, \cdots, T_d]$.

Remark A.1.9. Here A does not need to be integral. For example,

Theorem A.1.10 (Hilbert's Nullstellensatz)**.** Let A be a

A.2 Associated prime ideals

A.2.1 Associated prime ideals

Definition A.2.1 (Associated prime ideals)**.** Let A be a noetherian ring and M an A-module. The *associated prime ideals* of M are the prime ideals $\mathfrak{p}$ of form $\operatorname{Ann}(x)$ for some $x \in M$. The set of associated prime ideals of M is denoted by $\operatorname{Ass}(M)$.

Example A.2.2. Let $A = \mathbf{k}[x, y]/(xy)$ and $M = A$. First we see that $(x) = \operatorname{Ann} y, (y) = \operatorname{Ann} x \in \operatorname{Ass} M$. Then we check other prime ideals. For (x, y), if $xf = yf = 0$, then $f \in (x) \cap (y) = (0)$. If $(x - a) = \operatorname{Ann} f$ for some f, note that $y \in (x - a)$ for $a \in \mathbf{k}^*$, then $f \in (x)$. Hence $f = 0$. Therefore $\operatorname{Ass} M = \{(x), (y)\}$.

Example A.2.3. Let $A = \mathbf{k}[x, y]/(x^2, xy)$ and $M = A$. The underlying space of $\operatorname{Spec} A$ is the y-axis since $\sqrt{(x^2, xy)} = (x)$. First note that $(x) = \operatorname{Ann} y, (x, y) = \operatorname{Ann} x \in \operatorname{Ass} M$. For $(x, y - a)$ with $a \in \mathbf{k}^*$, easily see that $xf = (y - a)f = 0$ implies $f = 0$ since $A = \mathbf{k} \cdot x \oplus \mathbf{k}[y]$ as $\mathbf{k}$-vector space. Hence $\operatorname{Ass} M = \{(x), (x, y)\}$.

Lemma A.2.4. Let A be a noetherian ring and M an A-module. Then the maximal element of the set

$$\{\operatorname{Ann} x \colon x \in M_{\mathfrak{p}}, x \neq 0\}$$

belongs to $\operatorname{Ass} M$.

Proof. We just need to show that such $\operatorname{Ann} x$ is prime. Otherwise, there exist $a, b \in A$ such that $ab \in \operatorname{Ann} x$ but $a, b \notin \operatorname{Ann} x$. It follows that $\operatorname{Ann} x \subsetneq \operatorname{Ann} ax$ since $b \in \operatorname{Ann} ax \setminus \operatorname{Ann} x$. This contradicts the maximality of $\operatorname{Ann} x$. □

An element $a \in A$ is called a zero divisor for M if $M \to aM, m \mapsto am$ is not injective.

Corollary A.2.5. Let A be a noetherian ring and M an A-module. Then

$$\{\text{zero divisors for } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}.$$

Lemma A.2.6. Let A be a noetherian ring and M an A-module. Then $\mathfrak{p} \in \operatorname{Ass}_A M$ iff $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Proof. Suppose $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Ann} y_0/c$ with $y_0 \in M$ and $c \in A \setminus \mathfrak{p}$. For $a \in \operatorname{Ann} y_0$, $ay_0 = 0$. Then $a/1 \in \mathfrak{p}A_{\mathfrak{p}}$. It follows that $a \in \mathfrak{p}$. Hence $\operatorname{Ann} y_0 \subset \mathfrak{p}$.
Inductively, if $\operatorname{Ann} y_n \subsetneq \mathfrak{p}$, then there exists $b_n \in A \setminus \mathfrak{p}$ such that $y_{n+1} := b_n y_n$, $\operatorname{Ann} y_{n+1} \subset \mathfrak{p}$ and $\operatorname{Ann} y_n \subsetneq \operatorname{Ann} y_{n+1}$. To see this, choose $a_n \in \mathfrak{p} \setminus \operatorname{Ann} y_n$. Then $(a_n/1)y_n = 0$ since $a_n/1 \in \mathfrak{p}A_{\mathfrak{p}}$. By definition, there exist $b_n \in A \setminus \mathfrak{p}$ such that $a_n b_n y_n = 0$. This process must terminate since A is noetherian. Thus $\operatorname{Ann} y_n = \mathfrak{p}$ for some n. Hence $\mathfrak{p} \in \operatorname{Ass}_A M$.
Conversely, suppose $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$. If $(a/s)(x/1) = 0 \in M_{\mathfrak{p}}$, there exist $t \in A \setminus \mathfrak{p}$ such that $tax = 0$. It follows that $ta \in \mathfrak{p}$ and then $(a/s) \in \mathfrak{p}A_{\mathfrak{p}}$. Hence $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. □

Proposition A.2.7. We have $\operatorname{Ass} M \subset \operatorname{Supp} M$. Moreover, if $\mathfrak{p} \in \operatorname{Supp} M$ satisfies $V(\mathfrak{p})$ is an irreducible component of $\operatorname{Supp} M$, then $\mathfrak{p} \in \operatorname{Ass} M$.

Proof. For any $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$, we have $A/\mathfrak{p} \cong A \cdot x \subset M$. Tensoring with $A_\mathfrak{p}$ gives $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \hookrightarrow M_\mathfrak{p}$ since $A_\mathfrak{p}$ is flat. Hence $M_\mathfrak{p} \neq 0$ and $\mathfrak{p} \in \operatorname{Supp} M$.
Now suppose $\mathfrak{p} \in \operatorname{Supp} M$ and $V(\mathfrak{p})$ is an irreducible component of $\operatorname{Supp} M$. First we show that $\mathfrak{p} \in \operatorname{Ass}_{A_\mathfrak{p}} M_\mathfrak{p}$. Let $x \in M_\mathfrak{p}$ such that $\operatorname{Ann} x$ is maximal in the set

$$\{\operatorname{Ann} x \colon x \in M_\mathfrak{p}, x \neq 0\}.$$

Then we claim that $\operatorname{Ann} x = \mathfrak{p}A_\mathfrak{p}$. First, $\operatorname{Ann} x$ is prime by Lemma A.2.4. If $\operatorname{Ann} x \neq \mathfrak{p}$, then $V(\operatorname{Ann} x) \supset V(\mathfrak{p})$. This implies that $\operatorname{Ann} x \notin \operatorname{Supp} M_\mathfrak{p}$ since $\operatorname{Supp} M_\mathfrak{p} = \operatorname{Supp} M \cap \operatorname{Spec} A_\mathfrak{p}$. This is a contradiction. Thus $\mathfrak{p}A_\mathfrak{p} \in \operatorname{Ass}_{A_\mathfrak{p}} M_\mathfrak{p}$. By Lemma A.2.6, we have $\mathfrak{p} \in \operatorname{Ass} M$. □

Remark A.2.8. The existence of irreducible component is guaranteed by Zorn's Lemma.

Definition A.2.9. A prime ideal $\mathfrak{p} \in \operatorname{Ass} M$ is called *embedded* if $V(\mathfrak{p})$ is not an irreducible component of $\operatorname{Supp} M$.

Example A.2.10. For $M = A = \mathbf{k}[x, y]/(x^2, xy)$, the origin (x, y) is an embedded point.

Proposition A.2.11. If we have exact sequence $0 \to M_1 \to M_2 \to M_3$, then $\operatorname{Ass} M_2 \subset \operatorname{Ass} M_1 \cup \operatorname{Ass} M_3$.

Proof. Let $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M_2 \setminus \operatorname{Ass} M_1$. Then the image $[x]$ of x in M_3 is not equal to 0. We have that $\operatorname{Ann} x \subset \operatorname{Ann}[x]$. If $a \in \operatorname{Ann}[x] \setminus \operatorname{Ann} x$, then $ax \in M_1$. Since $\operatorname{Ann} x \subsetneq \operatorname{Ann} ax$, there is $b \in \operatorname{Ann} ax \setminus \operatorname{Ann} x$. However, it implies $ba \in \operatorname{Ann} x$, and then $a \in \operatorname{Ann} x$ since $\operatorname{Ann} x$ is prime, which is a contradiction. □

Corollary A.2.12. If M is finitely generated, then the set $\operatorname{Ass} M$ is finite.

Proof. For $\mathfrak{p} = \operatorname{Ann} x \in \operatorname{Ass} M$, we know that the submodule M_1 generated by x is isomorphic to $A/\mathfrak{p}$. Inductively, we can choose M_n be the preimage of a submodule of M/M_{n-1} which is isomorphic to $A/\mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass} M/M_{n-1}$. We can take an ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$. Since M is finitely generated, this is a finite sequence. Then the conclusion follows by Proposition A.2.11. □

A.2.2 Primary decomposition

Definition A.2.13. An A-module is called *co-primary* if $\operatorname{Ass} M$ has a single element. Let M be an A-module and $N \subset M$ a submodule. Then N is called *primary* if M/N is co-primary. If $\operatorname{Ass} M/N = \{\mathfrak{p}\}$, then N is called $\mathfrak{p}$-primary.

Remark A.2.14. This definition coincide with primary ideals in the case $M = A$. Recall an ideal $\mathfrak{q} \subset A$ is called *primary* if $\forall ab \in \mathfrak{p}$, $a \notin \mathfrak{q}$ implies $b^n \in \mathfrak{q}$ for some n.
Let $\mathfrak{q}$ be a $\mathfrak{q}$-primary ideal. Since $\operatorname{Supp} A/\mathfrak{q} = \{\mathfrak{p}\}$, $\mathfrak{p} \in \operatorname{Ass} A/\mathfrak{q}$. Suppose $\operatorname{Ann}[a] \in \operatorname{Ass} A/\mathfrak{q}$. Then $\mathfrak{p} \subset \operatorname{Ann}[a]$ since $V(\mathfrak{p}) = \operatorname{Supp} A/\mathfrak{q}$. If $b \in \operatorname{Ann}[a]$, then $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Hence $b^n \in \mathfrak{q}$, and then $b \in \mathfrak{p}$. This shows that $\operatorname{Ass} A/\mathfrak{q} = \{\mathfrak{p}\}$ and $\mathfrak{q}$ is $\mathfrak{p}$-primary as an A-submodule.
Let $\mathfrak{q} \subset A$ be a $\mathfrak{p}$-primary A-submodule. First we have $\mathfrak{p} = \sqrt{\mathfrak{q}}$ since $V(\mathfrak{p})$ is the unique irreducible component of $\operatorname{Supp} A/\mathfrak{q}$. Suppose $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Then $b \in \operatorname{Ann}[a] \subset \mathfrak{p}$ since $\mathfrak{p}$ is the unique maximal element in $\{\operatorname{Ann}[c] \colon c \in A \setminus \mathfrak{q}\}$. This implies that $b^n \in \mathfrak{q}$.

Definition A.2.15. Let A be a noetherian ring, M an A-module and $N \subset M$ a submodule. A *minimal primary decomposition* of N in M is a finite set of primary submodules $\{Q_i\}_{i=1}^n$ such that

$$N = \bigcap_{i=1}^{n} Q_i,$$

no Q_i can be omitted and $\operatorname{Ass} M/Q_i$ are pairwise distinct. For $\operatorname{Ass} M/Q_i = \{\mathfrak{p}\}$, Q_i is called belonging to $\mathfrak{p}$.

Indeed, if $N \subset M$ admits a minimal primary decomposition $N = \bigcap Q_i$ with Q_i belonging to $\mathfrak{p}$, then $\operatorname{Ass}(M/N) = \{\mathfrak{p}_i\}$. For given i, consider $N_i := \bigcap_{j \neq i} Q_j$, then $N_i/N \cong (N_i + Q_i)/Q_i$. Since $N_i \neq N$, $\operatorname{Ass} N_i/N \neq \emptyset$. On the other hand, $\operatorname{Ass} N_i/N \subset \operatorname{Ass} M/Q_i = \{\mathfrak{p}\}$. It follows that $\operatorname{Ass} N_i/N = \{\mathfrak{p}_i\}$, whence $\mathfrak{p}_i \in \operatorname{Ass} M/N$. Conversely, we have an injection $M/N \hookrightarrow \bigoplus M/Q_i$, so $\operatorname{Ass} M/N \subset \bigcup \operatorname{Ass} M/Q_i$. Due to this, if Q_i belongs to $\mathfrak{p}$, we also say that Q_i is the $\mathfrak{p}$-component of N.

Proposition A.2.16. Suppose $N \subset M$ has a minimal primary decomposition. If $\mathfrak{p} \in \operatorname{Ass} M/N$ is not embedded, then the $\mathfrak{p}$ component of N is unique. Explicitly, we have $Q = \nu^{-1}(N_\mathfrak{p})$, where $\nu : M \to M_\mathfrak{p}$.

Proof. First we show that $Q = \nu^{-1}(Q_\mathfrak{p})$. Clearly $Q \subset \nu^{-1}(Q_\mathfrak{p})$. Suppose $x \in \nu^{-1}(Q_\mathfrak{p})$. Then there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in Q$. That is, $[sx] = 0 \in M/Q$. If $[x] \neq 0$, we have $s \in \operatorname{Ann}[x] \subset \mathfrak{p}$. This contradiction enforces $Q = \nu^{-1}(Q_\mathfrak{p})$.
Then we show that $N_\mathfrak{p} = Q_\mathfrak{p}$. Just need to show that for $\mathfrak{p}' \neq \mathfrak{p}$ and the $\mathfrak{p}'$ component Q' of N, $Q'_\mathfrak{p} = M_\mathfrak{p}$. Since $\mathfrak{p}$ is not embedded, $\mathfrak{p}' \not\subset \mathfrak{p}$. Then $\mathfrak{p} \notin V(\mathfrak{p}) = \operatorname{Supp} M/Q'$. So $M_\mathfrak{p}/Q'_\mathfrak{p} = 0$. ☐

Example A.2.17. If $\mathfrak{p}$ is embedded, then its components may not be unique. For example, let $M = A = \mathsf{k}[x,y]/(x^2, xy)$. Then for every $n \in \mathbb{Z}_{\geq 1}$, $(x) \cap (x^2, xy, y^n)$ is a minimal primary decomposition of $(0) \subset M$.

Let A be a noetherian ring and $\mathfrak{p} \subset A$ a prime ideal. We consider the $\mathfrak{p}$ component of $\mathfrak{p}^n$, which is called n-th symbolic power of $\mathfrak{p}$, denoted by $\mathfrak{p}^{(n)}$. We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_\mathfrak{p} \cap A$. In general, $\mathfrak{p}^{(n)}$ is not equal to $\mathfrak{p}^n$; see below example.

Example A.2.18. Let $A = \mathsf{k}[x,y,z,w]/(y^2 - zx^2, yz - xw)$ and $\mathfrak{p} = (y,z,w)$. We have $z = y^2/x^2, w = yz/x \in \mathfrak{p}^2 A_\mathfrak{p}$, whence $\mathfrak{p}^2 A_\mathfrak{p} = (z,w) \neq \mathfrak{p}^2$.

Theorem A.2.19. Let A be a noetherian ring and M an A-module. Then for every $\mathfrak{p} \in \operatorname{Ass} M$, there is a $\mathfrak{p}$-primary submodule $Q(\mathfrak{p})$ such that

$$(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M} Q(\mathfrak{p}).$$

Proof. Consider the set

$$\mathcal{N} := \{N \subset M : \mathfrak{p} \notin \operatorname{Ass} N\}.$$

Note that $\operatorname{Ass} \bigcup N_i = \bigcup \operatorname{Ass} N_i$ by definition of associated prime ideals. Then it is easy to check that $\mathcal{N}$ satisfies the conditions of Zorn's Lemma. Hence $\mathcal{N}$ has a maximal element $Q(\mathfrak{p})$. We claim that $Q(\mathfrak{p})$ is $\mathfrak{p}$-primary. If there is $\mathfrak{p}' \neq \mathfrak{p} \in \operatorname{Ass} M/Q(\mathfrak{p})$, then there is a submodule $N' \cong A/\mathfrak{p}$. Let N'' be the preimage of N' in M. We have $Q(\mathfrak{p}) \subsetneq N''$ and $N'' \in \mathcal{N}$. This is a contradiction. By the fact $\operatorname{Ass} \bigcap N_i = \bigcap \operatorname{Ass} N_i$, we get the conclusion. ☐

Corollary A.2.20. Let A be a noetherian ring and M a finitely generated A-module. Then every submodule of M has a minimal primary decomposition.

A.3 Dimension and Depth

There are three numbers measuring the "size" of a local ring $(A, \mathfrak{m})$:

- $\dim A$: the Krull dimension of A.
- $\operatorname{depth} A$: the depth of A.
- $\dim_{\kappa(\mathfrak{m})} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$-vector space.

Somehow the Krull dimension is "homological" and the depth is "cohomological".

Definition A.3.1. Let A be a noetherian ring. The *height of a prime ideal* $\mathfrak{p}$ in A is defined as the maximum length of chains of prime ideals contained in $\mathfrak{p}$, that is,

$$\operatorname{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as

$$\dim A := \max_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht}(\mathfrak{p}).$$

Example A.3.2. Let A be a PID. For every two non-zero prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$, if $\mathfrak{p}_1 = t_1 A \subset \mathfrak{p}_2 = t_2 A$, then $t_2 \mid t_1$ and hence $\mathfrak{p}_1 = \mathfrak{p}_2$. It follows that $\dim A = 1$. Consequently, the ring of integers $\mathbb{Z}$ and the polynomial ring $\mathsf{k}[T]$ in one variable over a field have Krull dimension 1.

Definition A.3.3. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A-module. A sequence $t_1, \cdots, t_n \in I$ is called an *M-regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \cdots, t_{i-1})M$ for all i.

Example A.3.4. Let $A = \mathsf{k}[x,y]/(x^2, xy)$ and $I = (x,y)$. Then $\operatorname{depth}_I A = 0$.

Definition A.3.5. Let A be a noetherian ring. For every $\mathfrak{p} \in \operatorname{Spec} A$, $\mathfrak{p}/\mathfrak{p}^2$ is a vector space over $\kappa(\mathfrak{p})$. The *Zariski's tangent space* $T_{A,\mathfrak{p}}$ of A at $\mathfrak{p}$ is defined as $(\mathfrak{p}/\mathfrak{p}^2)^\vee$, the dual $\kappa(\mathfrak{p})$-vector space of $\mathfrak{p}/\mathfrak{p}^2$.

A.3.1 Artinian Rings and Length of Modules

Definition A.3.6. Let A be a ring and M an A module. *A simple module filtration of M* is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i-1} is a simple module, i.e. it has no submodule except 0 and itself. If M has a simple module filtration as above, we define the *length of M* as n and say that M has *finite length*.

The following proposition guarantees the length is well-defined.

Proposition A.3.7. Suppose M has a simple module filtration $M = M_{0,0} \supsetneq M_{1,0} \supsetneq \cdots \supsetneq M_{n,0} = 0$. Then for any other filtration $M = M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,m} = 0$ with $m > n$, there exist $k < m$ such that $M_{0,k} = M_{0,k+1}$.

Proof. We claim that there are at least $0 \leq k_1 < \cdots < k_{m-n} < m$ satisfies that $M_{0,k_i} = M_{0,k_i+1}$. Let $M_{i,j} := M_{i,0} \cap M_{0,j}$. Inductively on n, we can assume that there exist $k_1, \cdots, k_{n-m+1}$ such that $M_{1,k} = M_{1,k+1}$. Consider the sequence

$$M_{0,0}/M_{1,0} \supset (M_{0,1} + M_{1,0})/M_{1,0} \supset \cdots \supset (M_{0,m} + M_{1,0})/M_{1,0} = 0$$

in $M_{0,0}/M_{1,0}$. Since $M_{0,0}/M_{1,0}$ is simple, there is at most one k_i with $M_{0,k_i} + M_{1,0} \neq M_{0,k_i+1} + M_{1,0}$. And note that if $M_{0,k_i} + M_{1,0} = M_{0,k_i+1} + M_{1,0}$ and $M_{0,k_i} \cap M_{1,0} = M_{0,k_i} \cap M_{1,0}$, then $M_{0,k_i} = M_{0,k_i+1}$ by the Five Lemma. □

Example A.3.8. Let A be a ring and $\mathfrak{m} \in \operatorname{mSpec} A$. Then $A/\mathfrak{m}$ is a simple module. Yang: To be completed.

Proposition A.3.9. Let A be a ring and M an A-module. Then M is of finite length iff it satisfies both a.c.c and d.c.c.

Proof. Note that if M has either a strictly ascending chain or a strictly descending chain, M is of infinite length. Conversely, d.c.c guarantee M has a simple submodule and a.c.c guarantee the sequence terminates. □

Proposition A.3.10. The length $l(-)$ is an additive function for modules of finite length. That is, if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ with M_i of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

Proof. The simple module filtrations of M_1 and M_3 will give a simple module filtration of M_2. □

Proposition A.3.11. Let $(A, \mathfrak{m})$ be a local ring. Then A is artinian iff $\mathfrak{m}^n = 0$ for some $n \geq 0$.

Proof. Suppose A is artinian. Then the sequence $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ is stable. It follows that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n. By the Nakayama's Lemma A.1.4, $\mathfrak{m}^n = 0$.
Conversely, we have

$$\mathfrak{m} \subset \mathfrak{N} \subset \bigcap_{\text{minimal prime ideal}} \mathfrak{p},$$

whence $\mathfrak{m}$ is minimal. □

Proposition A.3.12. Let A be a ring. Then A is artinian iff A is of finite length.

Proof. First we show that A has only finite maximal ideal. Otherwise, consider the set $\{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k\}$. It has a minimal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ and for any maximal ideal $\mathfrak{m}$, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}$. It follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i. Let $\mathfrak{M} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ be the Jacobi radical of A. Consider the sequence $\mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots$ and by Nakayama's Lemma, we have $\mathfrak{M}^k = 0$ for some k. Consider the filtration

$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1^k \supset \mathfrak{m}_1^k\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0).$$

We have $\mathfrak{m}_1^k \cdots \mathfrak{m}_i^j/\mathfrak{m}_1^k \cdots \mathfrak{m}_i^{j+1}$ is an $A/\mathfrak{m}_i$-vector space. It is artinian and then of finite length. Hence A is of finite length. □

Theorem A.3.13. Let A be a ring. Then A is artinian iff A is noetherian and of dimension 0.

Proof. Suppose A is artinian. Then A is noetherian by Proposition A.3.12. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A/\mathfrak{p}$ is an artinian integral domain. If there is $a \in A/\mathfrak{p}$ is not invertible, consider $(a) \supset (a^2) \supset \cdots$, we see $a = 0$. Hence $\mathfrak{p}$ is maximal and $\dim A = 0$.
Suppose that A is noetherian and of dimension 0. Then every maximal ideal is minimal. In particular, A has only finite maximal ideal $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$. Let $\mathfrak{q}_i$ be the $\mathfrak{p}_i$-component of (0). Then we have $A \hookrightarrow \bigoplus_i A/\mathfrak{q}_i$. We just need to show that $A/\mathfrak{q}_i$ is of finite length as A-module. If $\mathfrak{q}_i \subset \mathfrak{p}_j$, take radical we get $\mathfrak{p}_i \subset \mathfrak{q}_j$ and hence $i = j$. So $A/\mathfrak{q}_i$ is a local ring with maximal ideal $\mathfrak{p}_i A/\mathfrak{q}_i$. Then every element in $\mathfrak{p}_i A/\mathfrak{q}_i$ is nilpotent. Since $\mathfrak{p}_i$ is finitely generated, $(\mathfrak{p}_i A/\mathfrak{q}_i)^k = 0$ for some k. Then $A/\mathfrak{q}_i$ is artinian and then of finite length as $A/\mathfrak{q}_i$-module. Then the conclusion follows. □

A.3.2 Dedekind DomainsYang: To be completed

A.3.3 Krull's Principal Ideal Theorem

Theorem A.3.14 (Krull's Principal Ideal Theorem)**.** Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let $\mathfrak{p}$ be a minimal prime ideal among those containing f. Then $\operatorname{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_{\mathfrak{p}}$, we may assume A is local with maximal ideal $\mathfrak{p}$. Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.
Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is $\mathfrak{q}$-primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1} A_{\mathfrak{q}}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_{\mathfrak{q}} = 0$. It follows that $\mathfrak{q} A_{\mathfrak{q}}$ is minimal, whence $A_{\mathfrak{q}}$ is artinian. Therefore, $\mathfrak{q}$ is minimal in A. □

Corollary A.3.15. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n. Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.
Choose $\mathfrak{q}$ be a minimal prime ideal among those containing $(\mathfrak{p}_{k-1}, f)$ and contained in $\mathfrak{p}_{k+1}$. Then by Krull's Principal Ideal Theorem A.3.14, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace $\mathfrak{p}_k$ by $\mathfrak{q}_k$, we have $f \in \mathfrak{q}_k \setminus \mathfrak{p}_{k-1}$
Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.
Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$ has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A. It follows that $\dim A/(f) + 1 \leq \dim A$. □

Proposition A.3.16. Let $(A, \mathfrak{m})$ be a local noetherian ring with residue field k. Then the following inequalities hold:

$$\operatorname{depth} A \leq \dim A \leq \dim_{\mathsf{k}} T_{A,\mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary A.3.15.
Let $t_1, \cdots, t_n$ be a $\kappa(\mathfrak{m})$-basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \cdots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \cdots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \cdots, t_n))$. It follows that $\mathfrak{m} = (t_1, \cdots, t_n)$ by Nakayama's Lemma. By Corollary A.3.15,

$$n + \dim A/(t_1, \cdots, t_n) \geq n - 1 + \dim A/(t_1, \cdots, t_{n-1}) \geq \cdots \geq 1 + \dim A/(t_1) \geq \dim A.$$

We conclude the result. □

Definition A.3.17. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property* (R_k) or *is regular in codimension* k if $\forall \xi \in X$ with $\operatorname{codim} Z_\xi \leq k$,

$$\dim_{\kappa(\xi)} T_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

We say that X *verifies property* (S_k) if $\forall \xi \in X$ with $\operatorname{depth} \mathcal{O}_{X,\xi} < k$,

$$\operatorname{depth} \mathcal{O}_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

Example A.3.18. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.
Suppose A verifies (S_1). If $\mathfrak{p} \in \operatorname{Ass} A$, every element in $\mathfrak{p}$ is a zero divisor. Then $\operatorname{depth} A_{\mathfrak{p}} = 0$. It follows that $\dim A_{\mathfrak{p}} = 0$ and then $\mathfrak{p}$ is minimal.

Suppose A has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_{\mathfrak{p}} = 0$. This means every element in $\mathfrak{p}A_{\mathfrak{p}}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Proposition A.1.1, $\mathfrak{p} = \mathfrak{q}$ for some minimal $\mathfrak{q}$, whence $\dim A_{\mathfrak{p}} = 0$.

Example A.3.19. Let A be a noetherian ring. Then A is reduced iff it verifies (R_0) and (S_1).
Suppose A is reduced. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be all minimal prime ideals of A. We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where $\mathfrak{N}$ is the nilradical of A. Hence A has no embedded point. Since $A_{\mathfrak{p}}$ is artinian, local and reduced, $A_{\mathfrak{p}}$ is a field and hence regular.
Conversely, let $\operatorname{Ass} A$ be equal to $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Then every $\mathfrak{p}_i$ is minimal by (S_1). Let f be in $\mathfrak{N}$. Then the image of f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0), $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where $\mathfrak{q}_i$ is the $\mathfrak{p}_i$ component of (0) in A. Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

A.3.4 Cohen-Macaulay rings

Definition A.3.20 (Cohen-Macaulay)**.** A noetherian local ring $(A, \mathfrak{m})$ is called *Cohen-Macaulay* if $\dim A = \operatorname{depth} A$. A noetherian ring A is called *Cohen-Macaulay* if for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is Cohen-Macaulay. This is equivalent to that A verifies (S_k) for all $k \geq 0$.

Example A.3.21 (Non Cohen-Macaulay rings)**.** Yang: To be completed.

Corollary A.3.22. Let A be a noetherian ring, M a finite A-module and $a \in A$ an M-regular element. Then $\operatorname{depth} M = \operatorname{depth} M/aM + 1$.

Corollary A.3.23. Let A be a noetherian ring $a \in A$ a nonzero divisor. Then A verifies (S_d) iff A/aA verifies (S_{d-1}).

Definition A.3.24. An ideal I of a noetherian ring A is called *unmixed* if

$$\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \operatorname{Ass}(A/I).$$

Here $\operatorname{ht}(I)$ is defined as

$$\operatorname{ht}(I) := \inf\{\operatorname{ht}(\mathfrak{p}) : I \subset \mathfrak{p}\}.$$

We say that *the unmixedness theorem holds for a noetherian ring* A if any ideal $I \subset A$ generated by $\operatorname{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme* X if $\mathcal{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Theorem A.3.25. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Proof. We can assume that $X = \operatorname{Spec} A$ is affine.
Suppose X is Cohen-Macaulay. Let $I \subset A$ be an ideal generated by $a_1, \cdots, a_r$ with $r = \operatorname{ht}(I)$. We claim that $a_1, \cdots, a_r$ is an A-regular sequence. If so, we get that the unmixedness theorem holds for A by applying Example A.3.18 on A/I. Since $\operatorname{ht}(a_1, \cdots, a_{r-1}) \leq r-1$ by Krull's Principal Ideal Theorem A.3.14 and $\operatorname{ht}(a_1, \cdots, a_r) = r \leq \operatorname{ht}(a_1, \cdots, a_{r-1})+1$, we have $\operatorname{ht}(a_1, \cdots, a_{r-1}) = r-1$. By induction on r, we can assume that $a_1, \cdots, a_{r-1}$ is an A-regular sequence. Hence any prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1, \cdots, a_{r-1})$ has height $r-1$. Now suppose a_r is a zero divisor in $A/(a_1, \cdots, a_{r-1})$. Then there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1, \cdots, a_{r-1})$ such that $a_r \in \mathfrak{p}$. Then $I \subset \mathfrak{p}$ and $\operatorname{ht}(I) \leq r-1$. This contradicts that $\operatorname{ht}(I) = r$.
Suppose the unmixedness theorem holds for A. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal with $\operatorname{ht}(\mathfrak{p}) = r$. Then $\mathfrak{p} \in \operatorname{Ass} A$ if and only if $\operatorname{ht}(\mathfrak{p}) = 0$. If $r > 0$, there is a nonzero divisor $a \in \mathfrak{p}$. By Krull's Principal Ideal Theorem A.3.14, $\operatorname{ht}(\mathfrak{p}A/aA) = r-1$. Inductively, we can find a regular sequence $a_1, \cdots, a_r$ in $\mathfrak{p}$. Then $\operatorname{depth} A_{\mathfrak{p}} = r$. □

Theorem A.3.26. Let X be a locally noetherian scheme. Suppose that X is Cohen-Macaulay. Let $F \subset X$ be a closed subset of codimension $\geq k$. Then the restriction $H^i(X, \mathcal{O}_X) \to H^i(X \setminus F, \mathcal{O}_X)$ is an isomorphism.

Proof. Yang: To be completed. □

A.3.5 Regular rings

Definition A.3.27. A noetherian ring A is said to be *regular at* $\mathfrak{p} \in \operatorname{Spec} A$ if we have

$$\dim_{\kappa(\mathfrak{p})} T_{A,\mathfrak{p}} = \dim A_{\mathfrak{p}},$$

where $\dim A_{\mathfrak{p}}$ is the Krull dimension of the local ring $A_{\mathfrak{p}}$.
A noetherian ring A is said to be *regular* if it is regular at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. This is equivalent to the condition that A verifies (R_k) for all $k \geq 0$.

Definition A.3.28. Let A be a noetherian ring that is regular at $\mathfrak{p} \in \operatorname{Spec} A$. A sequence $t_1, \cdots, t_n \in \mathfrak{p}$ is called a *regular system of parameters* at $\mathfrak{p}$ if their images form a basis of the $\kappa(\mathfrak{p})$-vector space $\mathfrak{p}/\mathfrak{p}^2$.

Proposition A.3.29. Let $(A, \mathfrak{m})$ be a noetherian local ring that is regular at $\mathfrak{m}$. Let $t_1, \cdots, t_n$ be a regular system of parameters at $\mathfrak{m}$, $\mathfrak{p}_i = (t_1, \cdots, t_i)$ and $\mathfrak{p}_0 = (0)$. Then $\mathfrak{p}_i$ is a prime ideal of height i, and $A/\mathfrak{p}_i$ is a regular local ring for all i. In particular, regular local ring is integral, and the regular system of parameters $t_1, \cdots, t_n$ is a regular sequence in A.

Proof. By the Krull's Principal Ideal Theorem A.3.14, we have

$$n - 1 = \dim A - 1 \leq \dim A/(t_1) \leq \dim_{\kappa(\mathfrak{m}/(t_1))} T_{A/(t_1),\mathfrak{m}/(t_1)} \leq n - 1.$$

Hence $\dim A/(t_1) = n - 1$ and $\operatorname{ht}(t_1) = 1$. Since $t_2, \cdots, t_n$ generate $\mathfrak{m}/(t_1)$, we have that $A/(t_1)$ is regular at $\mathfrak{m}/(t_1)$ and the images of $t_2, \cdots, t_n$ form a regular system of parameters.
For integrality, we induct on the dimension of A. If $\dim A = 0$, then A is a field and hence integral. Suppose $\dim A > 0$, let $\mathfrak{q}$ be a minimal prime ideal of A. Then $t_1 \notin \mathfrak{q}$. We have

$$n - 1 = \dim A - 1 \leq \dim A/(\mathfrak{q} + t_1 A) \leq \dim_{\kappa(\mathfrak{q}/(t_1))} T_{A/(\mathfrak{q}+t_1 A),\mathfrak{q}/(t_1)} \leq n - 1.$$

By similar arguments, we have $A/(\mathfrak{q} + t_1 A)$ is regular at $\mathfrak{m}/(\mathfrak{q} + t_1 A)$. By induction hypothesis, both of $A/t_1 A$ and $A/(\mathfrak{q} + t_1 A)$ are integral and of dimension $n - 1$. Hence $t_1 A = t_1 A + \mathfrak{q}$, i.e. $\mathfrak{q} \subset t_1 A$. For every $a = bt_1 \in \mathfrak{q}$, we have $b \in \mathfrak{q}$ since $t_1 \notin \mathfrak{q}$. Then $\mathfrak{q} \subset t_1 \mathfrak{q} \subset \mathfrak{m}\mathfrak{q}$. By Nakayama's Lemma, $\mathfrak{q} = 0$, whence A is integral. □

Corollary A.3.30. A regular ring is Cohen-Macaulay.

Corollary A.3.31. A regular ring is normal.

Remark A.3.32. A noetherian ring A is regular if and only if it is regular at every maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$. The proof uses homomorphism tools; see Theorem B.3.17 and Corollary B.3.18.

A.4 Finite Algebra and Normality

Yang: To be completed

Definition A.4.1. An integral domain A is called *normal* if it is integrally closed in its field of fractions $\operatorname{Frac}(A)$.

Lemma A.4.2. Let $A \subset C$ be rings and B the integral closure of A in C, S a multiplicatively closed subset of A. Then the integral closure of $S^{-1}A$ in $S^{-1}C$ is $S^{-1}B$.

Proof. For every $b \in B$ and $\forall s \in S$, there exists $a_i \in A$ s.t.

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then

$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s^1}\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0.$$

Hence b/s is integral over $S^{-1}A$, $S^{-1}B$ is integral over $S^{-1}A$.

If $c/s \in S^{-1}C$ is integral over $S^{-1}A$, then $\exists a_i \in S^{-1}A$ s.t.

$$\left(\frac{c}{s}\right)^n + a_1\left(\frac{c}{s}\right)^{n-1} + \cdots + a_n = 0.$$

Then

$$c^n + a_1 s c^{n-1} + \cdots + a_n s^n = 0 \in S^{-1}C$$

Then $\exists t \in S$ s.t.

$$t(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0 \in C.$$

Then

$$(ct)^n + a_1 st(ct)^{n-1} + \cdots + a_n s^n t^n = t^n(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0.$$

Hence ct is integral over A, then $ct \in B$. Then $c/s = (ct)/(st) \in S^{-1}B$. This completes the proof. □

Proposition A.4.3. Normality is a local property. That is, for an integral domain A, TFAE:

(i) A is normal.

(ii) For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_\mathfrak{p}$ is normal.

(iii) For any maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$, the localization $A_\mathfrak{m}$ is normal.

Proof. When A is normal, $A_\mathfrak{p}$ is normal by Lemma A.4.2.
Assume that $A_\mathfrak{m}$ is normal for every $\mathfrak{m} \in \operatorname{mSpec} A$. If A is not normal, let $\tilde{A}$ be the integral closure of A in $\operatorname{Frac} A$, $\tilde{A}/A$ is a nonzero A-module. Suppose $\mathfrak{p} \in \operatorname{Supp} \tilde{A}/A$ and $\mathfrak{p} \subset \mathfrak{m}$. We have $\tilde{A}_\mathfrak{m}/A_\mathfrak{m} = 0$ and $\tilde{A}_\mathfrak{p}/A_\mathfrak{p} = (\tilde{A}_\mathfrak{m}/A_\mathfrak{m})_\mathfrak{p} \neq 0$. This is a contradiction. □

Proposition A.4.4. Let A be a normal ring. Then $A[X]$ is also normal.

Definition A.4.5. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,\xi}$ is normal for any point $\xi \in X$. A ring A is called *normal* if $\operatorname{Spec} A$ is normal.

Remark A.4.6. For a general ring A, let $S := A \setminus (\bigcup_{\mathfrak{p}\in \operatorname{Ass} A} \mathfrak{p}) = \bigcap_{\mathfrak{p}\in\operatorname{Ass} A} A \setminus \mathfrak{p}$. Then S is a multiplicative set. The localization $S^{-1}A$ is called *the total ring of fractions* of A.
Suppose A is reduced and $\operatorname{Ass} A = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Denote its total ring of fractions by Q. Note that elements in Q are either unit or zero divisor. Hence any maximal ideal $\mathfrak{m}$ is contained in $\bigcup \mathfrak{p}_i Q$, whence contained in some $\mathfrak{p}_i Q$. Thus $\mathfrak{p}_i Q$ are maximal ideals. And we have $\bigcap \mathfrak{p}_i Q = 0$. By the Chinese Remainder Theorem, we have $Q = \prod Q/\mathfrak{p}_i Q = \prod A_{\mathfrak{p}_i}$.
Let A be a reduced ring with total ring of fractions Q. Then A is normal iff A is integral closed in Q. If A is normal, then for every $\mathfrak{p} \in \operatorname{Spec} A$, $A_\mathfrak{p}$ is integral. Then there is unique minimal prime ideal $\mathfrak{p}_i \subset \mathfrak{p}$. In particular, any two minimal prime ideal are relatively prime. By the Chinese Remainder Theorem, $A = \prod A/\mathfrak{p}_i$. Just need to check $A/\mathfrak{p}_i$ is integral closed in $A_{\mathfrak{p}_i}$. This is clear by check pointwise.
Conversely, suppose A is integral closed in Q. Let e_i be the unit element of $A_{\mathfrak{p}_i}$. It belongs to A since $e_i^2 - e_i = 0$. Since $1 = e_1 + \cdots + e_n$ and $e_i e_j = \delta_{ij}$, we have $A = \prod Ae_i$. Since Ae_i is integral closed in $A_{\mathfrak{p}_i}$, it is normal. Hence A is normal.

Lemma A.4.7. Let A be a normal ring. Then A verifies (R_1) and (S_2).

Proof. Since all properties are local, we can assume A is integral and local.
For (S_2), by Example **??**, we only need to show that $\operatorname{Ass}_A A/f$ has no embedded point. Let $\mathfrak{p} = (f : g) =\in \operatorname{Ass}_A A/fA$ and $t := f/g \in \operatorname{Frac} A$. After Replacing A by $A_\mathfrak{p}$, we can assume that $\mathfrak{p}$ is maximal. By definition, $t^{-1}\mathfrak{p} \subset A$. If $t^{-1}\mathfrak{p} \subset \mathfrak{p}$, suppose $\mathfrak{p}$ is generated by $(x_1, \cdots, x_n)$ and $t^{-1}(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(A)$. There is a monic polynomial $\chi(T) \in A[T]$ vanishing Φ. Then $\chi(t^{-1}) = 0$ and $t^{-1} \in A$. This is impossible by definition of t. Then $t^{-1}\mathfrak{p} = A$, and $\mathfrak{p} = (t)$ is principal. By Krull's Principal Ideal Theorem A.3.14, $\operatorname{ht}(\mathfrak{p}) = 1$.
Now we show that A verifies (R_1). Suppose $(A, \mathfrak{m})$ is local of dimension 1. Choosing $a \in \mathfrak{m}$, A/a is of dimension 0. Then by A.3.11, $\mathfrak{m}^n \subset aA$ for some $n \geq 1$. Suppose $\mathfrak{m}^{n-1} \not\subset aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$ and let $t = a/b$. By construction, $t^{-1} \notin A$ and $t^{-1}\mathfrak{m} \subset A$. After similar argument, we see that $\mathfrak{m} = tA$, whence A is regular. □

Lemma A.4.8. Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension 1. Then A is normal iff A is regular.

Proof. By lemma A.4.7, we just need to show that regularity implies normality.
Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since A is regular, $\mathfrak{m} = (t)$. Let $I \subset \mathfrak{m}$ be an ideal. If $I \subset \bigcap_n \mathfrak{m}^n$, then for every $a \in I$, there exists a_n such that $a = a_n t^n$. Then we get an ascending chain of ideals $(a_1) \subset (a_2) \subset \cdots$. Hence $a = 0$ by Nakayama's

Lemma. Suppose I is not zero. Then there is some n such that $I \subset \mathfrak{m}^n$ and $I \not\subset \mathfrak{m}^{n+1}$. For every $at^n \in I \setminus \mathfrak{m}^{n+1}$, $a \notin \mathfrak{m}$, whence a is a unit in A. Then $I = (t^n)$. Hence A is PID and hence normal. □

Proposition A.4.9. Let A be a noetherian integral domain of dimension ≥ 1 verifying (S_2). Then

$$A = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A, \operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

Proof. Clearly $A \subset \bigcap A_{\mathfrak{p}}$. Let $t = f/g \in \bigcap A_{\mathfrak{p}}$. Since $f \in gA_{\mathfrak{p}}$ and we have $gA = \bigcap (gA_{\mathfrak{p}} \cap A)$, $f \in gA$. It follows that $t \in A$. □

Theorem A.4.10 (Serre's criterion for normality)**.** Let X be a locally noetherian scheme. Then X is normal if and only if it verifies (R_1) and (S_2).

Proof. One direction has been proved in Lemma A.4.7. Suppose X verifies (R_1) and (S_2). Again we can assume $X = \operatorname{Spec} A$ is affine and A is local. By Remark A.4.6, we just need to show that A is integral closed in its total ring of fractions Q. Suppose we have

$$\left(\frac{a}{b}\right)^n + c_1 \left(\frac{a}{b}\right)^{n-1} + \cdots + c_n = 0 \in Q.$$

Since A verifies (S_2), $bA = \bigcap \nu_{\mathfrak{p}}^{-1}(b_{\mathfrak{p}} A_{\mathfrak{p}})$. So it is sufficient to show that $a_{\mathfrak{p}} \in b_{\mathfrak{p}} A_{\mathfrak{p}}$ with $\operatorname{ht}(\mathfrak{p}) = 1$. Note that $A_{\mathfrak{p}}$ is regular and hence normal by Lemma A.4.8. Then above equation gives us desired result. □

A.5 Smoothness

A.5.1 Modules of differentials and derivations

In this subsection, let R be a ring and A an R-algebra.

Definition A.5.1 (Derivation)**.** A *derivation* of A over R is an R-linear map $\partial : A \to M$ with an A-module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module M, the set of all derivations of A over R into M forms an A-module, denoted by $\operatorname{Der}_R(A, M)$.

Given a module homomorphism $f : M \to N$ of A-modules and a derivation $\partial \in \operatorname{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of A over R into N.

Proposition A.5.2. The functor $\operatorname{Der}_R(A, -)$ is representable. The representing object is denoted by $\Omega_{A/R}$, which is called the *module of differentials* of A over R.

Proof. First suppose A is a free R-algebra with a set of generators $a_\lambda, \lambda \in \Lambda$. Then an R-derivation $\partial \in \operatorname{Der}_R(A, M)$ is uniquely determined by its values on the generators a_λ. Let

$$\Omega_{A/R} \coloneqq \bigoplus_{\lambda \in \Lambda} A \cdot \mathrm{d}a_\lambda$$

and $\mathrm{d} : A \to \Omega_{A/R}$ be the R-derivation defined by $a_\lambda \mapsto \mathrm{d}a_\lambda$. For any R-derivation $\partial \in \operatorname{Der}_R(A, M)$, we can define a unique A-module homomorphism $\Phi_\partial : \Omega_{A/R} \to M$ by sending $\mathrm{d}a_\lambda$ to $\partial(a_\lambda)$ such that $\partial = \Phi_\partial \circ \mathrm{d}$. This gives a bijection

$$\operatorname{Der}_R(A, M) \cong \operatorname{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_\partial.$$

Now suppose $A = F/I$ is an arbitrary R-algebra, where F is a free R-algebra and I is an ideal of F. Then we can define the module of differentials

$$\Omega_{A/R} \coloneqq \left(\Omega_{F/R} \otimes_F A\right) \Big/ \sum_{f \in I} A \cdot \mathrm{d}f.$$

The R-linear map $\mathrm{d}_A : F \otimes_F A \xrightarrow{\mathrm{d}_F} \Omega_{F/R} \otimes_F A \to \Omega_{A/R}$ is a derivation of A over R.

For any R-derivation $\partial \in \operatorname{Der}_R(A, M)$, note that $F \to A \xrightarrow{\partial} M$ is an R-derivation of F over R into M. Then we get an F-module homomorphism $\Omega_F \to M$. It gives an A-module homomorphism $\Omega_F \otimes_F A \to M, \mathrm{d}f \otimes 1 \mapsto \partial f$. This map factors into $\Omega_F \otimes_F A \to \Omega_{A/R}$ and $\Phi_\partial : \Omega_{A/R} \to M$. Since Φ_∂ is A-linear and $\Omega_{A/R}$ is generated by $\mathrm{d}a_\lambda$ as A-module, such Φ_∂ is unique. □

Corollary A.5.3. Suppose A is of finite type over R. Then the module of differentials $\Omega_{A/R}$ is a finitely generated A-module.

Remark A.5.4. Let B be an A-algebra, M an A-module and N a B-module. If there is a homomorphism of A-modules $M \to N$, then we can extend it to a homomorphism of B-modules $M \otimes_A B \to N$ by sending $m \otimes b$ to $m \cdot b$. And such extension is unique in the sense of following commutative diagram:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N . \\ M \otimes_A B \arrow[ur, "\exists!"'] & \end{tikzcd}$$

Hence we get a natural bijection

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(M \otimes_A B, N).$$

Proposition A.5.5. Let A, R' be R-algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

Proof. We check the universal property of $\Omega_{A/R} \otimes_A A'$. First, the map

$$\mathrm{d}_{A'} : A \otimes_R R' \to \Omega_{A/R} \otimes_R R' \cong \Omega_{A/R} \otimes_A A', \quad a \otimes r \mapsto \mathrm{d}a \otimes r$$

is an R'-derivation of A' into $\Omega_{A/R} \otimes_A A'$. For any R'-derivation $\partial' : A' \to M$ into an A'-module M, we can compose it with the homomorphism $A' \to A$ and get an R-derivation $\partial : A \to M$. By the universal property of $\Omega_{A/R}$, there is a unique A-module homomorphism $\Phi : \Omega_{A/R} \to M$ such that $\partial = \Phi \circ \mathrm{d}_A$. Then we can extend it to an A'-module homomorphism $\Phi' : \Omega_{A/R} \otimes_A A' \to M$ by Remark A.5.4. By the construction, we have $\Phi' \circ \mathrm{d}_{A'} = \partial'$. □

Proposition A.5.6. Let A be an R-algebra and S a multiplicative set of A. Then we have an isomorphism

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

Proof. Let

$$\mathrm{d}_{S^{-1}A} : S^{-1}A \to S^{-1}\Omega_{A/R}, \quad \frac{a}{s} \mapsto \frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}.$$

By direct computation, $\mathrm{d}_{S^{-1}A}$ is an R-derivation of $S^{-1}A$ over R into $S^{-1}\Omega_{A/R}$. For any R-derivation $\partial : S^{-1}A \to M$ into an $S^{-1}A$-module M, we can get an $S^{-1}A$-module homomorphism $\Phi' : S^{-1}\Omega_{A/R} \to M$ as proof of Proposition A.5.5. We have

$$\partial(s \cdot \frac{a}{s}) = s\partial(\frac{a}{s}) + \frac{a}{s}\partial s.$$

It follows that

$$\partial(\frac{a}{s}) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(\mathrm{d}a) - a\Phi'(\mathrm{d}s)}{s^2} = \Phi'(\frac{s\mathrm{d}a - a\mathrm{d}s}{s^2}).$$

Thus, $\Phi' \circ \mathrm{d}_{S^{-1}A} = \partial$. □

Theorem A.5.7. Let A be an R-algebra and B an A-algebra. Then there is a natural short exact sequence

$$\Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to \Omega_{B/A} \to 0$$

of B-modules.

Proof. Let $\mathrm{d}_{A/R} : A \to \Omega_{A/R}$ be the R-derivation of A over R. The map $A \to B \xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R}$ induces a B-linear map

$$u : \Omega_{A/R} \otimes_A B \to \Omega_{B/R}, \quad \mathrm{d}_{A/R}(a) \otimes b \mapsto b\mathrm{d}_{B/R}(a).$$

The map $\mathrm{d}_{B/A}$ is an A-derivation and hence R-derivation. Then it induces a B-linear map

$$v : \Omega_{B/R} \to \Omega_{B/A}, \quad \mathrm{d}_{B/R}(b) \mapsto \mathrm{d}_{B/A}(b).$$

Since $\Omega_{B/A}$ is generated by elements of the form $\mathrm{d}_{B/A}(b)$ for $b \in B$, the map v is surjective. And clearly $\mathrm{d}_{B/A}(a) = a\mathrm{d}_{B/A}(1) = 0$ for $a \in A$.

Consider the composition $B \xrightarrow{\mathrm{d}_{B/R}} \Omega_{B/R} \to \Omega_{B/R}/\operatorname{Im} u$. For every $a \in A, b \in B$, we have

$$[\mathrm{d}_{B/R}(ab)] = [b\mathrm{d}_{B/R}(a) + a\mathrm{d}_{B/R}(b)] = [b\mathrm{d}_{B/R}(a)] + [a\mathrm{d}_{B/A}(b)] = [a\mathrm{d}_{B/A}(b)].$$

Hence it is indeed an A-derivation of B. Then it induces a B-linear map

$$\varphi : \Omega_{B/A} \to \Omega_{B/R}/\operatorname{Im} u, \quad \mathrm{d}_{B/A}(b) \mapsto [\mathrm{d}_{B/R}(b)].$$

The map φ is surjective since $\Omega_{B/R}$ is generated by elements of the form $\mathrm{d}_{B/R}(b)$ for $b \in B$. Note that the composition

$$\Omega_{B/A} \xrightarrow{\varphi} \Omega_{B/R}/\operatorname{Im} u \to \Omega_{B/A}/\operatorname{Ker} v$$

is the identity map. Thus, φ is injective and hence an isomorphism. In particular, we have $\operatorname{Ker} v = \operatorname{Im} u$. □

Remark A.5.8. The exact sequence in Theorem A.5.7 is left exact if and only if every R-derivation of A into B-module extends to an R-derivation of B into B-module.
Yang: To be completed.

Theorem A.5.9. Let A be an R-algebra and I an ideal of A. Set $B := A/I$. Then there is a natural short exact sequence

$$I/I^2 \to \Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to 0$$

of B-modules.

Proof. Suppose $A = F/\mathfrak{b}$ for some free R-algebra F and an ideal $\mathfrak{b}$ of F. Let $\mathfrak{a}$ be the preimage of I in F. Let $\mathrm{d}\mathfrak{b}$ (resp. $\mathrm{d}\mathfrak{a}$) denote the image of $\mathfrak{b}$ (resp. $\mathfrak{a}$) in $\Omega_{F/R}$. Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B/(\mathrm{d}\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \to (\mathrm{d}\mathfrak{a} \otimes_F B)/(\mathrm{d}\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows. □

Definition A.5.10. Let k be a field and A an integral k-algebra of finite type of dimension n. We say A is *smooth at* $\mathfrak{p} \in \operatorname{Spec} A$ if the module of differentials $\Omega_{A,\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank n.

Example A.5.11. Let K/k be a finite generated field extension and k' be the algebraic closure of k in K. Then

$$\dim_{\mathsf{K}} \Omega_{\mathsf{K}/\mathsf{k}} = \operatorname{trdeg}(\mathsf{K}/\mathsf{k}) + \dim_{\mathsf{k}'} \Omega_{\mathsf{k}'/\mathsf{k}},$$

and $\dim_{\mathsf{k}'} \Omega_{\mathsf{k}'/\mathsf{k}} = 0$ if and only if k' is separable over k.
First suppose $\mathsf{K} = \mathsf{k}'$ is algebraic over k. Suppose k'/k is separable. For every $\alpha \in \mathsf{k}'$, suppose $f(\alpha) = 0$ for $f \in \mathsf{k}[T]$. Then $\mathrm{d}f(\alpha) = f'(\alpha)\mathrm{d}\alpha = 0$. By the separability of k'/k, we have $f'(\alpha) \neq 0$. It follows that $\mathrm{d}\alpha = 0$. Conversely, let $\alpha \in \mathsf{k}'$ be a inseparable element over k. Since $\mathsf{k}[\alpha] \to \mathsf{k}[\alpha], \alpha^n \mapsto n\alpha^{n-1}$ is a non-zero R-derivation, we have $\Omega_{\mathsf{k}[\alpha]/\mathsf{k}} \neq 0$. By induction on number of generated elements, choosing a middle field $\mathsf{k} \subset \mathsf{k}'' \subset \mathsf{k}'$, at least one of $\Omega_{\mathsf{k}''/\mathsf{k}}$ and $\Omega_{\mathsf{k}'/\mathsf{k}''}$ is non-zero. Then $\Omega_{\mathsf{K}/\mathsf{k}} \neq 0$ by Theorem A.5.7.
Then suppose $\mathsf{k}' = \mathsf{k}$. By the Noether's Normalization Lemma, we can find a finite set of elements $T_1, \cdots, T_n \in \mathsf{K}$ such that K is algebraic over $\mathsf{k}'(T_1, \ldots, T_n)$. Note that we can choose T_i such that $\mathsf{K}/\mathsf{k}'(T_1, \cdots, T_n)$ is separable. To see this, if $\alpha \in \mathsf{K}$ is an inseparable element over $\mathsf{k}'(T_1, \cdots, T_n)$, then by replacing a suitable T_i with α, we reduce the inseparable degree of $\mathsf{K}/\mathsf{k}'(T_1, \cdots, T_n)$.
Since $\mathsf{K}/\mathsf{k}'(T_1, \cdots, T_n)$ is finite, every k-derivation of $\mathsf{k}'(T_1, \cdots, T_n)$ into K-module extends to a k-derivation of K into K-module. Then by Remark A.5.8, we have

$$0 \to \Omega_{\mathsf{k}'(T_1, \cdots, T_n)/\mathsf{k}} \otimes_{\mathsf{k}'(T_1, \cdots, T_n)} \mathsf{K} \to \Omega_{\mathsf{K}/\mathsf{k}} \to \Omega_{\mathsf{K}/\mathsf{k}'(T_1, \cdots, T_n)} \to 0.$$

Finally, note that every k-derivation ∂ of k' into K-module can be extended to $\mathsf{k}'[T_1, \cdots, T_n]$ by setting $\partial T_i = 0$. Thus, we have

$$0 \to \Omega_{\mathsf{k}'/\mathsf{k}} \otimes_{\mathsf{k}'} \mathsf{k}'[T_1, \cdots, T_n] \to \Omega_{\mathsf{k}'[T_1, \cdots, T_n]/\mathsf{k}} \to \Omega_{\mathsf{k}'[T_1, \cdots, T_n]/\mathsf{k}'} \to 0.$$

This follows that

$$\dim_{\mathsf{K}} \Omega_{\mathsf{K}/\mathsf{k}} = \dim_{\mathsf{K}} \Omega_{\mathsf{K}/\mathsf{k}'} + \dim_{\mathsf{k}'} \Omega_{\mathsf{k}'/\mathsf{k}}.$$

A.5.2 Applications to affine varieties

Let k be arbitrary field, $A = \mathsf{k}[T_1, \ldots, T_n]$ and $\mathfrak{m}$ a maximal ideal of A such that $\kappa(\mathfrak{m})$ is separable over k. We try to give an explanation of Zariski's tangent space at $\mathfrak{m}$ using the language of derivation. We know that $\Omega_{A/\mathsf{k}} = \bigoplus_{i=1}^n A\mathrm{d}T_i$, thus

$\Omega_{A_\mathfrak{m}/\mathsf{k}} \cong \bigoplus_{i=1}^n A_\mathfrak{m} \mathrm{d}T_i$. Then

$$\mathrm{Der}_\mathsf{k}(A_\mathfrak{m}, A_\mathfrak{m}) \cong \mathrm{Hom}_\mathsf{k}(\Omega_{A_\mathfrak{m}/\mathsf{k}}, A_\mathfrak{m}) \cong \bigoplus_{i=1}^n A_\mathfrak{m}\partial_i,$$

where $\partial_i \in \mathrm{Der}_\mathsf{k}(A_\mathfrak{m}, A_\mathfrak{m})$ is the derivation defined by $\mathrm{d}T_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. It coincides with the usual derivation $f \mapsto \partial f/\partial T_i$. Consider the restriction of ∂_i to $\mathfrak{m}$ and take values in the residue field $\kappa(\mathfrak{m})$, we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1,\cdots,\partial_n)^T} A_\mathfrak{m}^n \to \kappa(\mathfrak{m})^n.$$

Since $\kappa(\mathfrak{m})$ is separable over k, we claim that $\mathrm{Ker}\,\Phi = \mathfrak{m}^2$. Indeed, by Remark A.5.12, we can write every $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ as $\sum_i a_i g_i$. Then

$$\frac{\partial f}{\partial T_i} = a_i \frac{\partial g_i}{\partial T_i} + g_i \frac{\partial a_i}{\partial T_i}.$$

Since g_i is separable, the image of $\partial g_i/\partial T_i$ in $\kappa(\mathfrak{m})$ is not zero. Hence $\Phi(f) \neq 0$. By the claim, Φ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$ of $\kappa(\mathfrak{m})$-vector spaces. Then we get

$$T_{A,\mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where $x \in \mathbb{A}_\mathsf{k}^n$ is the point corresponding to $\mathfrak{m}$. This coincides with the usual tangent space at x in language of differential geometry.

Remark A.5.12. Let k be arbitrary field, $A = \mathsf{k}[T_1, \cdots, T_n]$ and g_i irreducible polynomials in one variable T_i over k. Then for every $f \in A$, we can write

$$f = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{Z}_{\geq 0}^n} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \leq \deg g_i.$$

This is called the *Taylor expansion of f with respect to $g_1, \cdots, g_n$*.
When $n = 1$, it follows from division algorithm. For $n > 1$, we can use induction on n. Let $\mathsf{K} = \mathsf{k}(T_1, \cdots, T_{n-1})$. Then we can write f as

$$f = \sum_{i=0}^r a_i g_n^i, \quad a_i \in \mathsf{K}[T_n], \quad \deg a_i < \deg g_n.$$

. Comparing the coefficients of two sides from the highest degree of T_n to the lowest degree, we see that

$$a_i \in \mathsf{k}[T_1, \cdots, T_{n-1}].$$

By induction hypothesis, the conclusion follows.

Let $B = A/I$ be a k of finite type, $I = (F_1, \ldots, F_m) \subset \mathfrak{m}$ and $\mathfrak{n}$ the image of $\mathfrak{m}$ in B. We have an exact sequence of $\kappa(\mathfrak{m})$-vector spaces

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} : \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of $F_1, \ldots, F_m$ is the $m \times n$ matrix

$$J(F_1, \ldots, F_m) := \left(\frac{\partial F_i}{\partial T_j}\right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

with entries in B.

Theorem A.5.13. Setting as above. Then B is regular at $\mathfrak{n}$ if and only if the Jacobian matrix J has maximal rank $n - \dim B_\mathfrak{n}$ after taking values in the residue field $\kappa(\mathfrak{m})$.

Proof. We have an exact sequence

$$0 \to T_{B,\mathfrak{n}} \to T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa^m \to 0,$$

where Ψ sends $\partial \in T_{A,\mathfrak{m}}$ to $(\partial(F_1), \ldots, \partial(F_m))^T$. Note that the matrix of Ψ is just J^T, the transpose of the Jacobian matrix. Hence

$$\mathrm{rank}\, J = n - \dim_\kappa T_{B,\mathfrak{n}} \leq n - \dim B_\mathfrak{n}$$

and the equality holds if and only if B is regular at $\mathfrak{n}$. ∎

Remark A.5.14. If $\kappa(\mathfrak{m})$ is not separable over k, then we still have the inequality

$$\operatorname{rank} J \leq n - \dim B_{\mathfrak{n}}.$$

Indeed, in any case, we have an exact sequence

$$0 \to I/(I \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Hence $\dim_\kappa I/(I \cap \mathfrak{m}^2) = n - \dim B_{\mathfrak{n}}$. There is a $\kappa(\mathfrak{m})$-linear map

$$I/(I \cap \mathfrak{m}^2) \to \kappa(\mathfrak{m})^n, \quad [f] \mapsto (\partial_1(f), \dots, \partial_n(f))^T,$$

and every row of the Jacobian matrix J is in the image of this map. Thus, the rank of J is at most $n - \dim B_{\mathfrak{n}}$. Hence if $\operatorname{rank} J = n - \dim B_{\mathfrak{n}}$, we can still see that B is regular at $\mathfrak{n}$. However, the converse does not hold in general.

Proposition A.5.15. Let k be a field, $\mathbf{k}$ the algebraic closure of k, A a k-algebra of finite type and $A_{\mathbf{k}} := A \otimes_{\mathsf{k}} \mathsf{k}$. Yang: Suppose $A_{\mathbf{k}}$ is integral. Let $\mathfrak{m} \in \operatorname{mSpec} A$ and $\mathfrak{m}'$ be a maximal ideal of $A_{\mathbf{k}}$ lying over $\mathfrak{m}$. Then

(a) If $A_{\mathbf{k}}$ is regular at $\mathfrak{m}'$, then A is regular at $\mathfrak{m}$;

(b) suppose $\kappa(\mathfrak{m})$ is separable over k, the converse holds.

Proof. Regarding $J_{\mathfrak{m}}$ and $J_{\mathfrak{m}'}$ as matrices with entries in $\mathbf{k}$, they are the same and hence have the same rank. If $A_{\mathbf{k}}$ is regular at $\mathfrak{m}'$, since $\kappa(\mathfrak{m}) = \mathbf{k}$, then $\operatorname{rank} J_{\mathfrak{m}'} = n - \dim A_{\mathbf{k},\mathfrak{m}'}$. Note that $\dim A_{\mathbf{k},\mathfrak{m}'} = \operatorname{trdeg}(\mathscr{K}(A_{\mathbf{k}})/\mathbf{k}) = \operatorname{trdeg}(\mathscr{K}(A)/\mathsf{k}) = \dim A_{\mathfrak{m}}$, we have $\operatorname{rank} J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence A is regular at $\mathfrak{m}$.
Conversely, suppose A is regular at $\mathfrak{m}$ and $\kappa(\mathfrak{m})$ is separable over k. Then $\operatorname{rank} J_{\mathfrak{m}} = n - \dim A_{\mathfrak{m}}$. Hence $A_{\mathbf{k}}$ is regular at $\mathfrak{m}'$. □

Proposition A.5.16. Let k be a field and A an integral k-algebra of finite type and of dimension n. Let $\mathbf{k}$ be the algebraic closure of k and $A_{\mathbf{k}} := A \otimes_{\mathsf{k}} \mathbf{k}$. Then A is smooth at $\mathfrak{p} \in \operatorname{Spec} A$ if and only if $A_{\mathbf{k}}$ is regular at every $\mathfrak{m}'$ over $\mathfrak{m}$.

Proof. Since $\Omega_{A_{\mathbf{k}}/\mathbf{k}} \cong \Omega_{A/\mathsf{k}} \otimes_A A_{\mathbf{k}}$ is free of rank n if and only if $\Omega_{A/\mathsf{k}}$ is free of rank n, we can assume that $\mathsf{k} = \mathbf{k}$. If A is smooth at $\mathfrak{p}$, then $\Omega_{A_{\mathfrak{p}}/\mathsf{k}} \cong \bigoplus A_{\mathfrak{p}} \mathrm{d}f_i$ is free of rank n. Let $\mathfrak{P}_i \in \operatorname{Der}_{\mathbf{k}}(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ be the derivation defined by $\mathrm{d}f_i \mapsto 1$ and $\mathrm{d}T_j \mapsto 0$ for $j \neq i$. Then we have $\partial_i f_j = \delta_{ij}$ for $1 \leq i, j \leq n$. Then similar to above argument, we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{(\partial_1, \dots, \partial_n)^T} \mathbf{k}^n.$$

This shows that $A_{\mathbf{k}}$ is regular at $\mathfrak{m}$.
Conversely, suppose $A_{\mathbf{k}}$ is regular at $\mathfrak{m}$. Note that $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A,\mathbf{k}} \otimes_A \mathbf{k}$ is surjective since $\Omega_{A_{\mathbf{k}}/\mathbf{k}} = 0$. Then by Nakayama's lemma, $\Omega_{A_{\mathfrak{m}}/\mathbf{k}}$ is generated by n elements as an $A_{\mathfrak{m}}$-module.
Note that $\dim_{\mathscr{K}(A)} \Omega_{\mathscr{K}(A)/\mathsf{k}} = \operatorname{trdeg}(\mathscr{K}(A)/\mathsf{k}) = \dim A_{\mathfrak{m}} = n$. Yang: By induction on transcendental degree.
Yang: By Nakayama's Lemma, $\Omega_{A_{\mathfrak{m}}/\mathsf{k}}$ is free of rank n as an $A_{\mathfrak{m}}$-module.
Yang: To be completed. □

Example A.5.17. Let k be an imperfect field of characteristic $p > 2$. Suppose $\alpha = \beta^p \in \mathsf{k}$ and β is not in k. Let $A = \mathsf{k}[x, y]/(x^2 - y^p - \alpha)$ and $\mathfrak{m} = (x, y^p - \alpha) = (x)$. Note that $\mathfrak{m}$ is principal, so A is regular at $\mathfrak{m}$. However,

$$J_{\mathfrak{m}} = \left(\frac{\partial}{\partial x}(x^2 - y^p - \alpha), \frac{\partial}{\partial y}(y^p - \alpha) \right) = (2x, 0) = (0, 0) \in M_{1\times 2}(\kappa(\mathfrak{m})).$$

Thus, A is not smooth at $\mathfrak{m}$. From the view of differentials, we have

$$\Omega_{A_{\mathfrak{m}}/\mathsf{k}} = A_{\mathfrak{m}} \mathrm{d}x \oplus A_{\mathfrak{m}} \mathrm{d}y / A_{\mathfrak{m}} \cdot x \mathrm{d}x = \kappa(\mathfrak{m}) \mathrm{d}x \oplus A_{\mathfrak{m}} \mathrm{d}y,$$

which is not free as an $A_{\mathfrak{m}}$-module.

Appendix B
Homological Algebra

B.1 Complexes and Homology

Definition B.1.1. Let $A_\bullet$ and $B_\bullet$ be two complexes in $\mathcal{A}$ and $\varphi_\bullet, \psi_\bullet : A_\bullet \to B_\bullet$ be two morphisms of complexes. A *homotopy* between $\varphi_\bullet$ and $\psi_\bullet$ is a collection of morphisms $h_n : A_n \to B_{n-1}$ such that

$$\varphi_n - \psi_n = \mathrm{d}_{B_{n+1}} \circ h_n + h_{n-1} \circ \mathrm{d}_{A_n}.$$

In diagram, we have

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r, "\mathrm{d}_{A_n}"] \arrow[dl, "h_n"'] \arrow[d, shift right, "\psi_n"'] \arrow[d, shift left, "\varphi_n"] & A_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] & \cdots . \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\mathrm{d}_{B_{n+1}}"] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

B.2 Derived Functors

In this section, fix an abelian category $\mathcal{A}$.

B.2.1 Resolution

Definition B.2.1 (Resolution)**.** Let $A \in \mathcal{A}$. A *projective resolution* (resp. *flat resolution, free resolution*) of A is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0,$$

where each P_i is a projective (resp. flat, free) object in $\mathcal{A}$.
An *injective resolution* of A is an exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots \to I^n \to \cdots ,$$

where each I^i is an injective object in $\mathcal{A}$.

Proposition B.2.2. Let $P_\bullet : \cdots \to P_1 \to P_0 \to A \to 0$ and $Q_\bullet : \cdots \to Q_1 \to Q_0 \to B \to 0$ be complexes in $\mathcal{A}$ such that P_i is projective and $Q_\bullet$ is exact. Given a morphism $f : A \to B$, there exists a morphism of complexes $f_\bullet : P_\bullet \to Q_\bullet$ such that $f_0 = f$. In particular, any two such morphism of complexes are homotopic.
Dually, let $I^\bullet : 0 \to A \to I^0 \to I^1 \to \cdots$ and $J^\bullet : 0 \to B \to J^0 \to J^1 \to \cdots$ be complexes in $\mathcal{A}$ such that J^i is injective and $I^\bullet$ is exact. Given a morphism $f : A \to B$, there exists a morphism of complexes $f^\bullet : I^\bullet \to J^\bullet$ such that $f^0 = f$. In particular, any two such morphism of complexes are homotopic.

Proof. Yang: To be completed. □

Definition B.2.3. For an object $A \in \mathcal{A}$, the *projective dimension* of A, denoted $\operatorname{proj.dim} A$, is the smallest integer n such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

of A of length n. If no such n exists, we set $\operatorname{proj.dim} A = \infty$.
Dually, the *injective dimension* of A, denoted $\operatorname{inj.dim} A$, is the smallest integer n such that there exists an injective resolution

$$0 \to A \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to 0$$

of A of length n. If no such n exists, we set $\operatorname{inj.dim} A = \infty$.

B.3 Applications to Commutative Algebra

B.3.1 Homological dimension

Lemma B.3.1. Let A be a ring and M an A-module. Then

$$\sup_M \operatorname{proj.dim} M = \sup_N \operatorname{inj.dim} N.$$

Proof. Note that

$$\operatorname{proj.dim} M \leq n$$

if and only if

$$\operatorname{Ext}^A_{n+1}(M, N) = 0, \quad \forall N.$$

And this is equivalent to

$$\operatorname{inj.dim} N \leq n.$$

□

Remark B.3.2. In fact, for fix N, we have

$$\operatorname{inj.dim} N \leq n$$

if and only if

$$\operatorname{Ext}^A_{n+1}(A/I, N) = 0, \quad \forall I$$

By Lemma Yang: ?. Hence we have

$$\sup_{M \text{ finite}} \operatorname{proj.dim} M = \sup_M \operatorname{proj.dim} M = \sup_N \operatorname{inj.dim} N.$$

Definition B.3.3. Let A be a ring. The *homological dimension* of A, denoted $\operatorname{hl.dim} A$, is defined as

$$\operatorname{hl.dim} A := \sup_M \operatorname{proj.dim} M = \sup_M \operatorname{inj.dim} M.$$

Lemma B.3.4. Let A be a noetherian ring, B a flat A-algebra and M a finite A-module. Then we have

$$\operatorname{Ext}^i_A(M, N) \otimes B \cong \operatorname{Ext}^i_B(M \otimes B, N \otimes M), \quad \forall N.$$

Proof. Yang: To be completed. □

Proposition B.3.5. Let A be a noetherian ring. Then

$$\operatorname{hl.dim} A = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{hl.dim} A_{\mathfrak{p}}.$$

Proof. Compute homological dimension of A using $\operatorname{Ext}^i_A(M, N)$ for finite M. The conclusion follows from Propostion B.3.5. □

Definition B.3.6. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring. We say that a homomorphism of A-modules $f : M \to N$ is *minimal* if the induced map $M \otimes \mathsf{k} \to N \otimes \mathsf{k}$ is an isomorphism. Equivalently, f is minimal if and only if f is surjective and $\operatorname{Ker} f \subset \mathfrak{m}M$.

Definition B.3.7. Let A be a noetherian local ring and M a finite A-module. A *minimal projective resolution* of M is a projective resolution

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

such that each homomorphism $P_i \to \operatorname{Ker} d_{i-1}$ is minimal.

Proposition B.3.8. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring and M a finite A-module. Then M has a minimal projective resolution. Moreover, any two minimal projective resolutions of M are isomorphic.

Proof. Suppose $M \otimes_A \mathsf{k} = \bigoplus \mathsf{k} \cdot \overline{x_i}$. Lift x_i to elements of M. Then we have a minimal homomorphism $\mathrm{d}_0 : \bigoplus A \cdot x_i \to M$. Similarly choose minimal homomorphisms $\mathrm{d}_k : A^{n_i} \to \operatorname{Ker} \mathrm{d}_{i-1}$ for $i = 1, 2, \cdots$. This gives a minimal projective resolution.
Suppose we have two minimal homomorphism $f, g : A^n \to M$. After tensoring with k, we have isomorphisms between $f \otimes \mathsf{k}$ and $g \otimes \mathsf{k}$. Lifting to A, we get an homomorphism $\varphi : f \to g$. Here homomorphism between f, g means a homomorphism $A^n \to A^n$ such that $f = g \circ \varphi$. The homomorphism φ is represented by a matrix T. We have $\det T \notin \mathfrak{m}$, whence φ is an isomorphism. $\square$

Proposition B.3.9. Let $L_\bullet \to M$ be a minimal projective resolution and $P_\bullet$ be an arbitrary projective resolution of M. Then we have $P_\bullet \cong L_\bullet \oplus P'_\bullet$ for some exact complexes $P'_\bullet$.

Proof. By Propostion B.2.2, we have homomorphism

$$L_\bullet \xrightarrow{\varphi_\bullet} P_\bullet \xrightarrow{\psi_\bullet} L_\bullet.$$

between complexes. By Propostion B.2.2 again, $T_\bullet := \psi_\bullet \circ \varphi_\bullet$ is homotopic to the identity by $h_\bullet$. Suppose $T_\bullet$ is represented by a matrix. Since $L_\bullet$ is minimal, we have

$$(T - \mathrm{id})(L_n) = (\mathrm{d}_{n+1} \circ h_n + h_{n-1} \circ \mathrm{d}_n)(L_n) \subset \mathfrak{m}L_n.$$

Then $\det T \notin \mathfrak{m}$ and hence $T_\bullet$ is an isomorphism. It follows that $\psi_\bullet$ is surjective, whence it splits $P_\bullet$ into a direct sum $L \oplus P'_\bullet$ since $L_\bullet$ is projective. By the Five Lemma, we see that $P'_\bullet$ is exact. $\square$

Lemma B.3.10. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring and M a finite A-module. Then $\operatorname{proj.dim} M \leq n$ if and only if $\operatorname{Tor}^A_{n+1}(M, \mathsf{k}) = 0$.

Proof. The necessity is clear. For the sufficiency, we have a minimal projective resolution

$$\cdots \to P_{n+1} \xrightarrow{\mathrm{d}_{n+1}} P_n \xrightarrow{\mathrm{d}_n} P_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \cdots \to P_1 \xrightarrow{\mathrm{d}_1} P_0 \xrightarrow{\mathrm{d}_0} M \to 0.$$

Let $C := \operatorname{Im} \mathrm{d}_n$. Then we have

$$0 \to P_{n+1} \xrightarrow{\mathrm{d}_{n+1}} P_n \xrightarrow{\mathrm{d}_n} C \to 0.$$

Hence $\operatorname{Tor}^A_1(C, \mathsf{k}) \cong \operatorname{Tor}^A_{n+1}(M, \mathsf{k}) = 0$. Let $K = \operatorname{Ker} \mathrm{d}_n$. Then we have the short exact sequence

$$0 \to K \to P_n \to C \to 0.$$

Since $\operatorname{Tor}^A_1(C, \mathsf{k}) = 0$, there is an exact sequence

$$0 \to K \otimes_A \mathsf{k} \to P_n \otimes_A \mathsf{k} \to C \otimes_A \mathsf{k} \to 0.$$

Since $P_n \to C$ is minimal, we have $K \otimes_A \mathsf{k} = 0$. By the Nakayama's lemma, $K = 0$. This implies that $\operatorname{proj.dim} C \leq 0$ and hence $\operatorname{proj.dim} M \leq n$. $\square$

Proposition B.3.11. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring. Then $\operatorname{hl.dim} A = \operatorname{proj.dim} \mathsf{k}$ (finite or infinite).

Proof. The inequality $\operatorname{hl.dim} A \geq \operatorname{proj.dim} \mathsf{k}$ is by definition. Conversely, we can compute $\operatorname{Tor}^A_{n+1}(M, \mathsf{k})$ using minimal projective resolution of k for any finite A-module M. By Lemma B.3.10, we have $\operatorname{proj.dim} M \leq n$ if and only if $\operatorname{Tor}^A_{n+1}(M, \mathsf{k}) = 0$. This implies that $\operatorname{proj.dim} M \leq n$ for all finite A-modules M if $\operatorname{proj.dim} \mathsf{k} = n$. By Remark B.3.2, we have $\operatorname{hl.dim} A \leq n$. $\square$

Proposition B.3.12. Let $(A,\mathfrak{m})$ be a noetherian local ring and M a finite A-module. Let $a\in\mathfrak{m}$ be an M-regular element. Then $\operatorname{proj.dim} M/aM = \operatorname{proj.dim} M + 1$. Here we set $\infty+1=\infty$.

Proof. We have an exact sequence
$$0\to M\xrightarrow{*a} M\to M/aM\to 0.$$
Take the long exact sequence with respect to $\operatorname{Tor}(-,\mathsf{k})$, we get
$$\cdots\to\operatorname{Tor}^A_{i+1}(M,\mathsf{k})\to\operatorname{Tor}^A_{i+1}(M/aM,\mathsf{k})\to\operatorname{Tor}^A_i(M,\mathsf{k})\xrightarrow{*a}\operatorname{Tor}^A_i(M,\mathsf{k})\to\cdots$$
Since the derived homomorphism of $*a$ is zero, we have $\operatorname{Tor}^A_{i+1}(M/aM,\mathsf{k})=0$ if and only if $\operatorname{Tor}^A_i(M,\mathsf{k})=0$. By Lemma B.3.10, we have $\operatorname{proj.dim} M/aM = \operatorname{proj.dim} M+1$. $\square$

B.3.2 Depth and regularity by homological algebra

Proposition B.3.13. Let $(A,\mathfrak{m},\mathsf{k})$ be a noetherian local ring and M a finite A-module. Then
$$\operatorname{depth} M := \inf\{i : \operatorname{Ext}^i_A(\mathsf{k},M)\neq 0\}.$$

Proof. Let $a\in\mathfrak{m}$ be M-regular and $N=M/aM$. Then we claim that
$$\inf\{i:\operatorname{Ext}^i_A(\mathsf{k},N)\neq 0\}=\inf\{i:\operatorname{Ext}^i_A(\mathsf{k},M)\neq 0\}-1.$$
Indeed, we have an exact sequence
$$0\to M\xrightarrow{a} M\to N\to 0.$$
It induces a long exact sequence
$$\cdots\to\operatorname{Ext}^{i-1}_A(\mathsf{k},M)\to\operatorname{Ext}^{i-1}_A(\mathsf{k},N)\to\operatorname{Ext}^i_A(\mathsf{k},M)\xrightarrow{\operatorname{Ext}^i_A(\mathsf{k},\operatorname{Mult}_a)}\operatorname{Ext}^i_A(\mathsf{k},M)\to\cdots.$$
Note that $a\in\mathfrak{m}$, then $\operatorname{Ext}^i_A(\mathsf{k},\operatorname{Mult}_a)=0$. It follows that when $\operatorname{Ext}^{i-1}_A(\mathsf{k},M)=0$, we have $\operatorname{Ext}^{i-1}_A(\mathsf{k},N)=0$ iff $\operatorname{Ext}^i_A(\mathsf{k},M)=0$, whence the claim.
Let $n=\inf\{i:\operatorname{Ext}^i_A(\mathsf{k},M)\neq 0\}$. Induct on n. Suppose first $n=0$. Since k is a simple A-module, there is an injective homomorphism $\mathsf{k}\to M$. Then $\mathfrak{m}\in\operatorname{Ass} M$ and hence $\operatorname{depth} M=0$.
Suppose $n>0$., let $a_1,\cdots,a_m\in\mathfrak{m}$ be any M-regular sequence. Using the claim inductively on $M/(a_1,\cdots,a_m)M$, we have $n\geq$ depth. If M has no regular element, then $\mathfrak{m}\subset\bigcup_{\mathfrak{p}\in\operatorname{Ass} M}\mathfrak{p}$. Then $\mathfrak{m}=\mathfrak{p}$ for some $\mathfrak{p}\in\operatorname{Ass} M$. This show that we can find $x\neq 0\in M$ such that $\mathfrak{p}=\operatorname{Ann} x$. It gives a homomorphism $\mathsf{k}=A/\mathfrak{m}\to M$. That is a contradiction and hence M has a regular element. Let a be M-regular and $N=M/aM$. Then $\operatorname{depth} N=n-1$ by the claim and induction hypothesis. Hence we have $\operatorname{depth} M\geq n$. $\square$

Lemma B.3.14. Let $(A,\mathfrak{m},\mathsf{k})$ be a noetherian local ring. Suppose we have exact sequences
$$0\to A^{n_r}\xrightarrow{\mathrm{d}_r} A^{n_{r-1}}\xrightarrow{\mathrm{d}_{r-1}}\cdots\to A^{n_1}\xrightarrow{\mathrm{d}_1} A^{n_0},$$
such that $A^{n_i}\to\operatorname{Ker}\mathrm{d}_{i-1}$ is minimal for all i. Then $\operatorname{depth} A\geq r$.

Proof. Since d_r is injective and its image is contained in $\mathfrak{m}A^{n_{r-1}}$, we can choose $t\in\mathfrak{m}$ that is not a zero divisor. Denote the sequence by $C_\bullet$. Then we have a short exact sequence of complexes
$$0\to C_\bullet\xrightarrow{*t} C_\bullet\to C_\bullet/tC_\bullet\to 0.$$
Consider the long exact sequence in homology
$$\cdots\to H_i(C_\bullet)\xrightarrow{*t} H_i(C_\bullet)\to H_i(C_\bullet/tC_\bullet)\to H_{i-1}(C_\bullet)\xrightarrow{*t} H_{i-1}(C_\bullet)\to\cdots.$$
Since $C_\bullet$ is exact, we have $H_i(C_\bullet)=0$ for all i. In particular, $H_i(C_\bullet/tC_\bullet)=0$ for all $i\geq 2$. Inductively, we can choose a regular sequence of length r in $\mathfrak{m}$. $\square$

Lemma B.3.15. Let $(A,\mathfrak{m},\mathsf{k})$ be a noetherian local ring and M a finite A-module. Suppose there is an injective homomorphism $\mathsf{k}\to M$. Then $\operatorname{proj.dim} M\geq\dim_{\mathsf{k}} T_{A,\mathfrak{m}}$.

Proof. Let $x_1, \cdots, x_n \subset \mathfrak{m} \setminus \mathfrak{m}^2$ such that their images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. Then we have a complex

$$K_\bullet \coloneqq 0 \to \wedge^n A^{\oplus n} \xrightarrow{\mathrm{d}_n} \wedge^{n-1} A^{\oplus n} \xrightarrow{\mathrm{d}_{n-1}} \cdots \to \wedge^1 A^{\oplus n} \xrightarrow{\mathrm{d}_1} \wedge^0 A^{\oplus n} \xrightarrow{\mathrm{d}_0} \mathsf{k} \to 0,$$

where

$$\mathrm{d}_r : \wedge^r A^{\oplus n} \to \wedge^{r-1} A^{\oplus n}, \quad e_{i_1} \wedge \cdots \wedge e_{i_r} \mapsto \sum_{k=1}^{r} (-1)^k x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_r}.$$

Here $\widehat{e_{i_k}}$ means that we omit the k-th element. Let $P_\bullet \to M$ be the minimal projective resolution of M. Then we have a homomorphism of complexes

$$\varphi_\bullet : K_\bullet \to P_\bullet$$

induced by the injective homomorphism $\mathsf{k} \to M$.
We claim that φ_i is injective and splits P_i into a direct sum $K_i \oplus F_i$ with F_i free for all $i \geq 0$. Since K_i and P_i are free, we just need to show that $\varphi_i \otimes_A \mathrm{id}_\mathsf{k}$ is injective. Induct on i. For $i = 0$, note that $\mathsf{k} \to M \otimes_A \mathsf{k}$ is injective, by the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi_0 \otimes_A \mathrm{id}_\mathsf{k}"'] & \mathsf{k} \arrow[d] \\ P_0 \otimes_A \mathsf{k} \arrow[r, "\cong"] & M \otimes_A \mathsf{k} \end{tikzcd},$$

the image of $\varphi_0 \otimes_A \mathrm{id}_\mathsf{k}$ is not zero in $P_0 \otimes_A \mathsf{k}$.
For $i > 0$, since K_{i-1} and P_{i-1} are free, we have a natural isomorphism between

$$\mathfrak{m}K_{i-1} \otimes_A \mathsf{k} \to \mathfrak{m}P_{i-1} \otimes_A \mathsf{k}$$

and

$$K_{i-1} \otimes_A \mathfrak{m}/\mathfrak{m}^2 \to P_{i-1} \otimes_A \mathfrak{m}/\mathfrak{m}^2.$$

We have a commutative diagram

$$\begin{tikzcd} K_i \otimes_A \mathsf{k} \arrow[r] \arrow[d] & \mathfrak{m}K_{i-1} \otimes_A \mathsf{k} \arrow[d] \\ P_i \otimes_A \mathsf{k} \arrow[r] & \mathfrak{m}P_{i-1} \otimes_A \mathsf{k} \end{tikzcd}. \tag{B.1}$$

Since $P_{i-1}/K_{i-1} \cong F_{i-1}$ is free, the right vertical map in (B.1) is injective. By construction of $K_\bullet$, $K_i \otimes_A \mathsf{k} \to \mathfrak{m}K_{i-1} \otimes_A \mathsf{k}$ is injective. Hence the left vertical map in (B.1) is injective. This completes the proof of the claim.
By the claim, $P_i \neq 0$ for all $i \leq n$ and the conclusion follows. □

Proposition B.3.16 (Auslander-Buchsbaum formula)**.** Let A be a noetherian local ring and M a finite A-module. Suppose $\operatorname{proj.dim} M < \infty$. Then $\operatorname{proj.dim} M = \operatorname{depth} A - \operatorname{depth} M$.

Proof. We have a minimal projective resolution

$$0 \to A^{n_r} \to A^{n_{r-1}} \to \cdots \to A^{n_1} \to A^{n_0} \to M \to 0.$$

By Lemma B.3.14, we have $\operatorname{depth} A \geq \operatorname{proj.dim} M$.
Induct on $\operatorname{depth} M$. Suppose $\operatorname{depth} M = 0$. Then by Proposition B.3.13, we have $\operatorname{Hom}_A(\mathsf{k}, M) \neq 0$, whence there is an injective homomorphism $\mathsf{k} \to M$. By Lemma B.3.15, we have

$$\operatorname{depth} A \geq \operatorname{proj.dim} M \geq \dim_\mathsf{k} T_{A,\mathfrak{m}} \geq \operatorname{depth} A.$$

If $\operatorname{depth} M > 0$, choose a regular element $a \in \mathfrak{m}$ that is M-regular. Then by Propostion B.3.12, we have

$$\operatorname{depth} M + \operatorname{proj.dim} M = \operatorname{depth}(M/aM) - 1 + \operatorname{proj.dim}(M/aM) + 1 = \operatorname{depth} A.$$

□

Theorem B.3.17. Let $(A, \mathfrak{m})$ be a noetherian local ring. Then A is regular at $\mathfrak{m}$ if and only if $\operatorname{hl.dim} A < \infty$.

Proof. Suppose A is regular at $\mathfrak{m}$. Let $x_1, \cdots, x_n$ be a minimal generating set of $\mathfrak{m}$. Then $x_1, \cdots, x_n$ is an A-regular sequence since A is regular at $\mathfrak{m}$. By Proposition B.3.12, we have $\operatorname{proj.dim} \mathsf{k} = \operatorname{proj.dim} A/(x_1, \cdots, x_n)A = n + \operatorname{proj.dim} A = n$.
Conversely, suppose $\operatorname{hl.dim} A < \infty$. Then by Proposition B.3.11, we have $\operatorname{proj.dim} \mathsf{k} < \infty$. We have

$$\dim_\mathsf{k} T_{A,\mathfrak{m}} \leq \operatorname{proj.dim} \mathsf{k} \leq \operatorname{depth} A \leq \dim_\mathsf{k} T_{A,\mathfrak{m}}.$$

The first "$\leq$" follows from Lemma B.3.15. The second "$\leq$" follows from Proposition B.3.16. Hence we see that A is regular at $\mathfrak{m}$. $\square$

Corollary B.3.18. Let $(A, \mathfrak{m})$ be a noetherian local ring. Then A is regular if and only if it is regular at $\mathfrak{m}$.

Proof. The sufficiency is trivial. For the necessity, note that if A is regular, then $\operatorname{hl.dim} A < \infty$ by Theorem B.3.17. For any $\mathfrak{p} \in \operatorname{Spec} A$, we have a finite projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A/\mathfrak{p} \to 0.$$

Tensoring with $A_{\mathfrak{p}}$, we have a finite projective resolution of $\kappa(\mathfrak{p})$. By Theorem B.3.17 again, we see that $A_{\mathfrak{p}}$ is regular at $\mathfrak{p}$. $\square$

Lemma B.3.19. Let A be a noetherian integral domain. Then A is a UFD if and only if every height 1 prime ideal of A is principal.

Proof. Yang: To be completed. $\square$

Lemma B.3.20. Let A be a noetherian integral domain and $(x) \subset A$ a non-zero prime ideal. Then A is a UFD if and only if $A[1/x]$ is a UFD.

Proof. Yang: To be completed. $\square$

Theorem B.3.21. Let $A, \mathfrak{m}$ be a regular noetherian local ring. Then A is UFD.

Proof. Yang: To be completed. $\square$

Bibliography

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