

Relative objects

1 Relative schemes

Definition 1. Let X be a scheme. An \mathcal{O}_X -algebra is a sheaf . Yang: To be continued...

Definition 2. Let X be a scheme and \mathcal{A} be a quasi-coherent \mathcal{O}_X -algebra. The relative Spec of \mathcal{A} , denoted by $\text{Spec}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\text{Spec } \mathcal{A}(U) \rightarrow U$ for all affine open subsets $U \subset X$. Yang: To be continued...

Proposition 3. Let X be a scheme and \mathcal{E} be a locally free sheaf of finite rank on X . Then the relative Spec of the symmetric algebra of \mathcal{E} , denoted by $\mathbb{V}(\mathcal{E}) = \text{Spec}_X \text{Sym}_{\mathcal{O}_X} \mathcal{E}$, is called the geometric vector bundle associated to \mathcal{E} . The projection morphism $\pi : \mathbb{V}(\mathcal{E}) \rightarrow X$ is affine and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \text{Spec } \text{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$. Yang: To be continued... Yang: To be revised, need to take dual.

Definition 4. Let X be a scheme and \mathcal{A} be a quasi-coherent graded \mathcal{O}_X -algebra such that $\mathcal{A}_0 = \mathcal{O}_X$ and \mathcal{A} is generated by \mathcal{A}_1 as an \mathcal{O}_X -algebra. The relative Proj of \mathcal{A} , denoted by $\text{Proj}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\text{Proj } \mathcal{A}(U)$ for all affine open subsets $U \subset X$. The projection morphism $\pi : \text{Proj}_X \mathcal{A} \rightarrow X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \text{Proj } \mathcal{A}(U)$. Yang: To be continued...

2 Relative ampleness and projective morphisms

Definition 5. Let X be an S -scheme via a morphism $f : X \rightarrow S$. A line bundle \mathcal{L} on X is called *relatively ample* (or f -ample) if for every affine open subset $U \subset S$, the restriction $\mathcal{L}|_{f^{-1}(U)}$ is an ample line bundle on the scheme $f^{-1}(U)$. Yang: To be revised.

Definition 6. Let $f : X \rightarrow S$ be a morphism of schemes. The morphism f is called *projective* if there exists a quasi-coherent

Remark 7. There are subtle differences between various definitions of projective morphisms in the literature. (see [FGA05, abaaba])

3 Blowing up

Definition 8. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blow up of X along \mathcal{I} , denoted by $\text{Bl}_{\mathcal{I}} X$, is defined to be the relative Proj of the Rees algebra of \mathcal{I} :

$$\text{Bl}_{\mathcal{I}} X = \text{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{I}^n.$$

Yang: To be continued...

Proposition 9. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blow up morphism $\pi : \text{Bl}_{\mathcal{I}} X \rightarrow X$ is projective. Moreover, if the support of $\mathcal{O}_X/\mathcal{I}$ is nowhere dense in X , then π is birational. Yang: To be continued...

Proposition 10. Let X be a regular scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then the blow up $\text{Bl}_{\mathcal{I}} X$ is also a regular scheme. Yang: To be continued...

Proposition 11. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The exceptional locus of the blow up morphism $\pi : \text{Bl}_{\mathcal{I}} X \rightarrow X$ is equal to the inverse image of the support of $\mathcal{O}_X/\mathcal{I}$:

$$\text{Exc}(\pi) = \pi^{-1}(\text{Supp}(\mathcal{O}_X/\mathcal{I})).$$

Yang: To be continued...

References

- [FGA05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*. Vol. 123. Mathematical Surveys and Monographs. Grothendieck's FGA explained. American Mathematical Society, Providence, RI, 2005, pp. x+339. ISBN: 0-8218-3541-6. DOI: [10.1090/surv/123](https://doi.org/10.1090/surv/123). URL: <https://doi.org/10.1090/surv/123> (cit. on p. 1).