Schemes and Varieties

"宇宙战警行为规范第四十条，遇见违反社会公德的行为，发现违法犯罪的行为，要见义勇为，勇于斗争，善于斗争。第十四条，不说谎，不骗人，不弄虚做假，知错就改，诚实守信，言行一致，答应别人的事要做到。……榴莲，不用跟任何人道歉，你跟这儿的人不一样，你是来让世界变好的。"

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0.1 Sheaves

Definition 0.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

For two open sets $V \subset U \subset X$, the morphism $\mathcal{F}(U) \to \mathcal{F}(V)$, often denoted by res^U_V, is called the *restriction map*.

Example 0.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 0.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0,1) \cup (2,3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0,1), (2,3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0,1))$ and $s_2 = 2 \in \mathcal{C}((2,3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 0.4. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be presheaves on X with values in the same category (e.g., **Set**, **Ab**, **Ring**, etc.). A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$. In other words, for every open set $U \subset X$, there is a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every inclusion of open sets $V \subset U$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}^U_V"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V).
\end{tikzcd}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then φ is called a *morphism of sheaves*.

Fix a topological space X and a category $\mathbf{C}$. The sheaves (resp. presheaves) on X with values in $\mathbf{C}$ form a category, denoted by $\mathbf{Sh}(X, \mathbf{C})$ (resp. $\mathbf{PSh}(X, \mathbf{C})$), where the objects are sheaves (resp. presheaves) on X with values in $\mathbf{C}$ and the morphisms are morphisms of sheaves (resp. presheaves).

Definition 0.5. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in a category $\mathbf{C}$. For a point $x \in X$, the *stalk* of $\mathcal{F}$ at x, denoted by $\mathcal{F}_x$, is defined as the colimit

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the colimit is taken over all open neighborhoods U of x. An element of $\mathcal{F}_x$ is called a *germ* of sections of $\mathcal{F}$ at x.

More concretely, we have

$$\mathcal{F}_x = \{(U, s) : U \in \mathbf{Open}(X), U \ni x, s \in \mathcal{F}(U)\} / \sim,$$

where $(U, s) \sim (V, t)$ if there exists an open neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$.

Definition 0.6. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in **Set** (resp. **Ab**, **Ring**, etc.). A *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}^\dagger$ on X together with a morphism of presheaves $\eta : \mathcal{F} \to \mathcal{F}^+$ such that for every sheaf $\mathcal{G}$ on X and every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \eta$.
In other words, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & \mathcal{F}^\dagger \arrow[d, "\varphi^\dagger"] \\
& \mathcal{G}.
\end{tikzcd}$$

Yang: To be checked.

Yang: The concrete describe of sheafification.

Definition 0.7. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The morphism φ is called *injective* (resp. *surjective*) if for every $x \in X$, the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).

Proposition 0.8. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is injective if and only if for every open set $U \subset X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. Yang: To be checked.

Remark 0.9. The surjectivity on stalks cannot imply the surjectivity on sections. A counterexample is given by the exponential map $\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^*$ defined by $\exp(f) = e^f$, where $\mathcal{O}_{\mathbb{C}}$ is the sheaf of holomorphic functions on $\mathbb{C}$ and $\mathcal{O}_{\mathbb{C}}^*$ is the sheaf of non-vanishing holomorphic functions on $\mathbb{C}$. The induced map on stalks $\exp_z : \mathcal{O}_{\mathbb{C},z} \to \mathcal{O}_{\mathbb{C},z}^*$ is surjective for every $z \in \mathbb{C}$ by the existence of logarithm locally. However, the map on global sections $\exp(\mathbb{C}) : \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}^*(\mathbb{C})$ is not surjective since there is no entire function f such that $e^{f(z)} = z$ for all $z \in \mathbb{C}^*$. Yang: To be continued.

Proposition 0.10. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is an isomorphism if and only if it is injective and surjective.

Yang: Now we consider sheaves with values in an abelian category.

Definition 0.11. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The *kernel* of φ, denoted by $\ker\varphi$, is the sheaf defined by

$$(\ker\varphi)(U) := \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$.
The *cokernel* of φ, denoted by $\operatorname{coker}\varphi$, is the sheafification of the presheaf defined by

$$(\operatorname{coker}\varphi)_{\mathrm{pre}}(U) := \operatorname{coker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$. Yang: To be continued.

Theorem 0.12. Let X be a topological space and $\mathbf{C}$ be an abelian category (e.g., $\mathbf{Ab}$). Then the category of sheaves on X with values in $\mathbf{C}$ is an abelian category.

Proof. Yang: To be continued. □

Yang: To be checked and continuous.

0.2 Locally ringed space

Definition 0.13. Let $f : X \to Y$ be a continuous map between topological spaces. The *push-forward* functor $f_* : \mathbf{Sh}(X, \mathbf{C}) \to \mathbf{Sh}(Y, \mathbf{C})$ is defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

for every open set $V \subset Y$ and sheaf $\mathcal{F} \in \mathbf{Sh}(X, \mathbf{C})$.

Definition 0.14. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$

and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 0.15. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Example 0.16 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Example 0.17 (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i\in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i\in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^\sharp(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ'_i satisfying the same properties, then by gluing $\psi'_i \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

0.3 Manifolds as locally ringed spaces

0.4 Vector bundles and $\mathcal{O}_X$-modules

Let $(X, \mathcal{O}_X)$ be a manifold (real or complex) and $(\mathcal{E}, \pi, X)$ a vector bundle over X.

Yang: It can regard as a sheaf on X.

Definition 0.18. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F}$ of abelian groups on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\mathrm{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear, where the $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(V)$ is induced by the restriction map $\mathrm{res}_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.
A *morphism of $\mathcal{O}_X$-modules* is a morphism of sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ such that for every open set $U \subseteq X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. Yang: To be checked...

Definition 0.19. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is said to be *locally free of rank r* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_U^r$, where $\mathcal{O}_U^r$ is the direct sum of r copies of $\mathcal{O}_U$. Yang: To be continued.

1 The First Properties of Schemes

If you learn the following content for the first time, it is recommended to skip all the proofs in this section and focus on the examples, remarks and the statements of propositions and theorems.

1.1 Schemes

Let R be a ring. Recall that the *spectrum* of R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set.*

Proposition 1.1. Let R be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

Proof. Yang: To be continued □

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 1.2. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i \in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.

These data form a category, denoted by **Sch**. If we fix a base scheme S, then an *S-scheme* is a scheme X together with a morphism $X \to S$. The category of S-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

Theorem 1.3. The functor $\mathsf{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. Yang: To be continued

Definition 1.4. A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if f induces an isomorphism of X onto an open (resp. closed) subscheme of Y. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. Yang: To be continued

Example 1.5. Let R be a graded ring. The *projective scheme* $\mathsf{Proj}\, R$ is defined as the scheme associated to the sheaf of rings

$$\mathcal{O}_{\mathsf{Proj}\, R} = \bigoplus_{d \geq 0} R_d.$$

It can be covered by open affine subschemes of the form $\mathsf{Spec}\, R_f$ for homogeneous elements $f \in R$. Yang: To be checked.

Example 1.6 (Glue open subschemes)**.** The construction in Example 0.17 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Example 0.17. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

Definition 1.7. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme Z of Y such that f factors through Z. More precisely, if $Y = \mathsf{Spec}\, A$ is affine, then the scheme theoretic image of f is $\mathsf{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover Y by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of f. Yang: To be checked.

1.2 Fiber product

Definition 1.8. Let $\mathcal{C}$ be a category and $X, Y, S \in \mathsf{Obj}(\mathcal{C})$ with morphisms $f : X \to S$ and $g : Y \to S$. A *fiber product* of X and Y over S is an object $Z \in \mathsf{Obj}(\mathcal{C})$ together with morphisms $p : Z \to X$ and $q : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

and satisfies the universal property that for any object $W \in \mathsf{Obj}(\mathcal{C})$ with morphisms $u : W \to X$ and $v : W \to Y$ such that $f \circ u = g \circ v$, there exists a unique morphism $h : W \to Z$ such that $p \circ h = u$ and $q \circ h = v$.

If a fiber product exists, it is unique up to a unique isomorphism. We denote the fiber product by

$X \times_S Y$. Yang: To be checked.

Example 1.9. In the category of sets, the fiber product $X \times_S Y$ is given by

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\},$$

with the projections $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ being the restrictions of the natural projections. Yang: To be checked.

Remark 1.10. If one reverses the arrows in Definition 1.8, one gets the notion of *fiber coproduct*. It is also called the *pushout* or *amalgamated sum* in some literature. We denote the fiber coproduct of X and Y over S by $X \amalg_S Y$. Note that in the category of rings, the fiber coproduct $A \amalg_R B$ of R-algebras A and B over R is given by the tensor product $A \otimes_R B$. Dually, one can expect that fiber products of affine schemes correspond to tensor products of rings.

Theorem 1.11. The category of schemes admits fiber products. More precisely, given morphisms of schemes $f : X \to S$ and $g : Y \to S$, there exists a scheme Z together with morphisms $p : Z \to X$ and $q : Z \to Y$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

commutes and satisfies the universal property of the fiber product. We denote this scheme by $X \times_S Y$. Yang: To be continued

Definition 1.12. Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a point. The *scheme theoretic fiber* of f over y is the fiber product $X_y = X \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y)$ is the residue field of the local ring $\mathcal{O}_{Y,y}$. Yang: To be checked.

Definition 1.13. Let X be a scheme and $Z_1, Z_2 \subset X$ be closed subschemes defined by quasi-coherent sheaves of ideals $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{O}_X$, respectively. The *scheme theoretic intersection* of Z_1 and Z_2 is the closed subscheme $Z_1 \cap Z_2$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}_1 + \mathcal{I}_2$. Yang: To be checked.

1.3 Noetherian schemes and morphisms of finite type

Definition 1.14. A scheme X is *noetherian* if it admits a finite open cover $\{U_i\}_{i=1}^n$ such that each U_i is an affine scheme $\operatorname{Spec} A_i$ with A_i a noetherian ring. Yang: To be checked.

Proposition 1.15. A noetherian scheme is quasi-compact. Yang: To be checked.

Definition 1.16. Let S be a scheme. A scheme X is *of finite type* over S if there exists a finite open cover $\{U_i\}_{i=1}^n$ of S such that for each i, $f^{-1}(U_i)$ can be covered by finitely many affine open subsets $\{V_{ij}\}_{j=1}^{m_i}$ with $f(V_{ij}) \subseteq U_i$ and the induced morphism $f|_{V_{ij}} : V_{ij} \to U_i$ corresponds to a finitely generated algebra over the ring of global sections of U_i.
Yang: To be checked.

1.4 Integral, reduced and irreducible schemes

Definition 1.17. A topological space X is *irreducible* if it is non-empty and cannot be expressed as the union of two proper closed subsets. Equivalently, every non-empty open subset of X is dense in X. Yang: To be checked.

Proposition 1.18. Let X be a topological space satisfying the descending chain condition on closed subsets. Then X can be written as a finite union of irreducible closed subsets, called the *irreducible components* of X. Moreover, this decomposition is unique up to permutation of the components. Yang: To be checked.

Definition 1.19. A scheme X is *reduced* if its structure sheaf $\mathcal{O}_X$ has no nilpotent elements. Yang: To be checked.

Proposition 1.20. A scheme X is reduced if and only if for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a reduced ring. Yang: To be checked.

Proposition 1.21. Let X be a scheme. There exists a unique closed subscheme X of X such that X is reduced and has the same underlying topological space as X. Moreover, for any morphism of schemes $f : Y \to X$ with Y reduced, f factors uniquely through the inclusion $X \to X$. Yang: To be checked.

Definition 1.22. A scheme X is *integral* if it is both reduced and irreducible. Yang: To be checked.

Proposition 1.23. A scheme X is integral if and only if for every open affine subset $U = \operatorname{Spec} A \subset X$, the ring A is an integral domain. Yang: To be checked.

1.5 Dimension

Definition 1.24. The *Krull dimension* of a topological space X, denoted by $\dim X$, is the supremum of the lengths n of chains of distinct irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X. If no such finite supremum exists, we say that X has infinite dimension. Yang: To be checked.

1.6 Separated, proper and projective morphisms

Definition 1.25. A morphism of schemes $f : X \to Y$ is *separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. A scheme X is *separated* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. Yang: To be checked.

Proposition 1.26. Any affine scheme is separated. More generally, any morphism between affine schemes is separated. Yang: To be checked.

Proposition 1.27. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any scheme T and any pair of morphisms $g_1, g_2 : T \to X$ such that $f \circ g_1 = f \circ g_2$, the equalizer of g_1 and g_2 is a closed subscheme of T. Yang: To be checked.

Proposition 1.28. A scheme X is separated if and only if for any pair of affine open subschemes $U, V \subset X$, the intersection $U \cap V$ is also an affine open subscheme. Yang: To be checked.

Proposition 1.29. The composition of separated morphisms is separated. Moreover, separatedness is stable under base change, i.e., if $f : X \to Y$ is a separated morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also separated. Yang: To be checked.

Proposition 1.30. A morphism of schemes $f : X \to Y$ is separated if and only if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists at most one morphism $\operatorname{Spec} R \to X$ making the entire diagram commute. Yang: To be checked.

Definition 1.31. A morphism of schemes $f : X \to Y$ is *universally closed* if for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map. Yang: To be checked.

Definition 1.32. A morphism of schemes $f : X \to Y$ is *proper* if it is of finite type, separated, and universally closed (i.e., for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map). A scheme X is *proper* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is proper. Yang: To be checked.

Theorem 1.33. Any projective morphism is proper. In particular, any projective scheme is proper. Yang: To be checked.

Proposition 1.34. The composition of proper morphisms is proper. Moreover, properness is stable under base change, i.e., if $f : X \to Y$ is a proper morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also proper. Yang: To be checked.

Proposition 1.35. A morphism of schemes $f : X \to Y$ is proper if and only if for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists a unique morphism $\operatorname{Spec} R \to X$

making the entire diagram commute. Yang: To be checked.

1.7 Varieties

2 Category of sheaves of modules

Mostly results in this section fits into the context of ringed spaces.

2.1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets.

Definition 2.2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that there exists a surjective morphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

Remark 2.3. There are many versions of "local" properties for sheaves of $\mathcal{O}_X$-modules. Yang: To be continued.

Definition 2.4. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it is finitely generated, and for every open set $U \subseteq X$ and every morphism of sheaves of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of φ is finitely generated.

2.2 As abelian categories

Theorem 2.5. The categories of sheaves of abelian groups, quasi-coherent sheaves, and coherent sheaves on a ringed space $(X, \mathcal{O}_X)$ are all abelian categories. Yang: To be checked.

Theorem 2.6. Let $(X, \mathcal{O}_X)$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules has enough injectives. Yang: To be checked.

2.3 Relevant functors

Definition 2.7. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *sheaf Hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf of abelian groups defined as follows: for an open set $U \subseteq X$, we define

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of morphisms of sheaves of $\mathcal{O}_U$-modules from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. For an inclusion of open sets $V \subseteq U$, the restriction map

$$\mathrm{res}_{UV} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(V)$$

is defined by sending a morphism $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ to its restriction $\varphi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. Yang: To be continued.

Definition 2.8. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules defined as follows: for an open set $U \subseteq X$, we define

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

where $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is the tensor product of $\mathcal{O}_X(U)$-modules. For an inclusion of open sets $V \subseteq U$, the restriction map
Yang: To be continued.

Definition 2.9. Let $f : X \to Y$ be a morphism of ringed spaces. The *pull-back functor* $f^* : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$ is defined as follows: for an $\mathcal{O}_Y$-module $\mathcal{F}$, we define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

where $f^{-1}\mathcal{F}$ is the inverse image sheaf of $\mathcal{F}$. For a morphism of $\mathcal{O}_Y$-modules $\varphi : \mathcal{F} \to \mathcal{G}$, we define

$$f^*\varphi : f^*\mathcal{F} \to f^*\mathcal{G}$$

to be the morphism induced by the morphism of sheaves of abelian groups $f^{-1}\varphi : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$.
Yang: To be continued.

2.4 Cohomological theory

Definition 2.10. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *sheaf cohomology* $H^i(X, \mathcal{F})$ is defined as the i-th right derived functor of the global section functor $\Gamma(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ applied to $\mathcal{F}$, i.e.,

$$H^i(X, \mathcal{F}) := \mathrm{R}^i\Gamma(X, \mathcal{F}).$$

Yang: To be checked.

Definition 2.11. Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *i-th higher direct image* $\mathrm{R}^i f_* \mathcal{F}$ is defined as the i-th right derived functor of the direct image functor $f_* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ applied to $\mathcal{F}$, i.e.,

$$\mathrm{R}^i f_* \mathcal{F} := \mathrm{R}^i (f_* \mathcal{F}).$$

Yang: To be checked.

Proposition 2.12. Let $f : X \to Y$ be a morphism of ringed spaces, and let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. Then there are long exact sequences of $\mathcal{O}_Y$-modules

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to \mathrm{R}^1 f_*\mathcal{F} \to \mathrm{R}^1 f_*\mathcal{G} \to \mathrm{R}^1 f_*\mathcal{H} \to \mathrm{R}^2 f_*\mathcal{F} \to \cdots$$

Yang: To be checked.

Theorem 2.13 (Affine criterion by Serre)**.** Let X be a scheme. Then X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i > 0$. Yang: To be checked.

Theorem 2.14 (Leray spectral sequence)**.** Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there exists a spectral sequence

$$E_2^{p,q} = H^p(Y, \mathrm{R}^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Yang: To be checked.

3 Normal, Cohen-Macaulay, and regular schemes

3.1 Normal scheme

Definition 3.1. A scheme X is called *normal* if for every open affine subset $U = \operatorname{Spec} A$ of X, the ring A is an integrally closed domain. Yang: To be checked.

Definition 3.2. The *normalization* of a scheme X is a normal scheme $\widetilde{X}$ together with a finite birational morphism $\pi : \widetilde{X} \to X$ such that for every normal scheme Y and every birational morphism $f : Y \to X$, there exists a unique morphism $g : Y \to \widetilde{X}$ such that $f = \pi \circ g$. Yang: To be checked.

Theorem 3.3. Let X be a scheme. Then there exists a normalization $\widetilde{X}$ of X.

Theorem 3.4 (Hartog's phenomenon)**.** Let X be a normal integral scheme, and let $U \subseteq X$ be an open subset whose complement has codimension at least 2. Then every regular function on U extends uniquely to a regular function on X, i.e., the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is an isomorphism.

Yang: To be checked.

Proposition 3.5. Let X be a normal scheme, and let $x \in X$ be a point with codimension 1. Then X is regular at x.

3.2 Cohen-Macaulay scheme

Definition 3.6. Let X be a scheme, and let $Z \subseteq X$ be a closed subset. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the *local cohomology* $H^i_Z(X, \mathcal{F})$ is defined as the i-th right derived functor of the functor $\Gamma_Z(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ that sends a sheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ to the abelian group of sections of $\mathcal{G}$ with support in Z, i.e.,

$$H^i_Z(X, \mathcal{F}) := \mathrm{R}^i\Gamma_Z(X, \mathcal{F}).$$

Yang: To be checked.

Definition 3.7. A scheme X is called *Cohen-Macaulay* if for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a Cohen-Macaulay ring. Yang: To be checked.

Theorem 3.8. Let X be a Cohen-Macaulay scheme, and let $Z \subseteq X$ be a closed subset of codimension at least 2. Then for every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the local cohomology $H^i_Z(X, \mathcal{F}) = 0$ for every $i < 2$. Yang: To be checked.

3.3 Regular scheme

We first define the tangent space of a scheme at a point.

There are many descriptions of the tangent space of a scheme at a point. Here we give one of them.

Let X be a scheme over a field $\mathbf{k}$, and let $x \in X(\mathbf{k})$.

Proposition 3.9. Let $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$ be the spectrum of the ring of dual numbers over $\mathbf{k}$ with point $* : \operatorname{Spec} \mathbf{k} \to \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$. The tangent space T_xX is naturally isomorphic to the set of morphisms $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X$ that send $*$ to x, i.e.

$$T_xX \cong \{f : \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X \mid f(*) = x\}.$$

Proof. Yang: To be filled. □

4 Line Bundles and Divisors

4.1 Cartier Divisors

Definition 4.1. Let X be a scheme. A *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of X. Equivalently, a Cartier divisor D can be represented by an open covering $\{U_i\}$ of X and a collection of rational functions $f_i \in \mathcal{K}_X^*(U_i)$ such that for any i, j, the function $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. We denote a Cartier divisor by $D = \{(U_i, f_i)\}$.

4.2 Line Bundles and Picard Group

Definition 4.2. Let X be a scheme. The *Picard group* of X is defined to be $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. The group operation is given by the tensor product of line bundles.

Definition 4.3. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a Yang: non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X\times\{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X\times\{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\mathrm{alg}} \mathcal{L}'$. Yang: To be checked.

4.3 Weil Divisors and Reflexive Sheaves

To talk about Weil divisors, we need to work with normal schemes.

Definition 4.4. Let X be a normal integral scheme. A *Weil divisor* on X is a formal sum

$$D = \sum_Z n_Z Z,$$

where the sum runs over all prime divisors Z of X (i.e., integral closed subschemes of codimension 1) and $n_Z \in \mathbb{Z}$, such that for any affine open subset $U = \operatorname{Spec} A \subseteq X$, only finitely many Z intersecting U have nonzero coefficients n_Z. The group of Weil divisors on X is denoted by $\mathrm{WDiv}(X)$.

Definition 4.5. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. The *dual sheaf* of $\mathcal{F}$ is defined as $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. The sheaf $\mathcal{F}$ is called *reflexive* if the natural map $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is an isomorphism.

5 Projective morphisms and "positive" line bundles

5.1 Ample line bundles

Definition 5.1. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. Yang: To be continued.

Theorem 5.2 (Serre Vanishing). Let X be a projective scheme over a field k and $\mathcal{L}$ an ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that for all $n \geq N$, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

5.2 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 5.3. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Proposition 5.4. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. Yang: To be continued. □

Example 5.5. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 5.6. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}^{(m+1)(n+1)-1}_A$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 5.7. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. Yang: To be continued.

Example 5.8. Let

Example 5.9.

Definition 5.10. Let $\mathcal{L}$ be a line bundle on a scheme X. Yang: To be continued.

Definition 5.11. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

Theorem 5.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

Proposition 5.13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}, \mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

Yang: To be continued.

Proof. Yang: To be continued. □

5.3 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 5.14. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space

$\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

5.4 Asymptotic behavior

Definition 5.15. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

Definition 5.16. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. Yang: To be continued.

Theorem 5.17. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \operatorname{Proj} R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. Yang: To be continued.

Definition 5.18. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large n. Yang: To be continued.

Example 5.19. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. Yang: To be continued.

6 Relative objects

6.1 Relative schemes

Definition 6.1. Let X be a scheme. An $\mathcal{O}_X$-algebra is a sheaf . Yang: To be continued...

Definition 6.2. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. The relative Spec of $\mathcal{A}$, denoted by $\mathcal{Spec}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Spec} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. Yang: To be continued...

Proposition 6.3. Let X be a scheme and $\mathcal{E}$ be a locally free sheaf of finite rank on X. Then the relative Spec of the symmetric algebra of $\mathcal{E}$, denoted by $\mathbb{V}(\mathcal{E}) = \operatorname{Spec}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}$, is called the geometric vector bundle associated to $\mathcal{E}$. The projection morphism $\pi : \mathbb{V}(\mathcal{E}) \to X$ is affine and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Spec} \operatorname{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$. Yang: To be continued... Yang: To be revised, need to take dual.

Definition 6.4. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_0 = \mathcal{O}_X$ and $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{O}_X$-algebra. The relative Proj of $\mathcal{A}$, denoted by $\mathcal{Proj}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Proj} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. The projection morphism $\pi : \mathcal{Proj}_X \mathcal{A} \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{A}(U)$. Yang: To be continued...

6.2 Blowing up

Definition 6.5. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blowing up of X along $\mathcal{I}$, denoted by $\mathrm{Bl}_{\mathcal{I}} X$, is defined to be the relative Proj of the Rees algebra of $\mathcal{I}$:

$$\mathrm{Bl}_{\mathcal{I}} X = \mathcal{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{I}^n.$$

The projection morphism $\pi : \mathrm{Bl}_{\mathcal{I}} X \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \mathrm{Bl}_{\mathcal{I}(U)} U$. The exceptional divisor of the blowing up is defined to be the closed subscheme $E = \pi^{-1}(V(\mathcal{I}))$ of $\mathrm{Bl}_{\mathcal{I}} X$. Yang: To be continued...

6.3 Relative ampleness and relative morphisms

7 Finite morphisms and fibrations

Theorem 7.1 (Zariski's Main Theorem)**.** Let $f : Y \to X$ be a quasi-finite and separated morphism of schemes. Then there exists a factorization

Theorem 7.2 (Stein factorization)**.** Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then there exists a factorization

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where g is a proper morphism with connected fibers and h is a finite morphism. Moreover, this factorization is unique up to isomorphism. Yang: To be checked.

7.1 Finite morphisms

Theorem 7.3. Let $f : Y \to X$ be a finite morphism of schemes. If $\mathcal{L}$ is an ample line bundle on X, then $f^*\mathcal{L}$ is an ample line bundle on Y. If and only if.

7.2 Fibrations

Definition 7.4. Let $f : Y \to X$ be a proper morphism of noetherian schemes. We say that f is a fibration if for every point $x \in X$, the fiber $f^{-1}(x)$ is a geometrically connected scheme.

Proposition 7.5. Let $f : Y \to X$ be a proper morphism of noetherian schemes. Then f is a fibration if and only if the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is an isomorphism. In particular, if X is an algebraically closed field and f is a fibration, then the fibers $f^{-1}(x)$ are also algebraically closed in the function field $K(X)$. Yang: To be revised

Definition 7.6. Let $f : Y \dashrightarrow X$ be a rational map of noetherian schemes. We say that f is a fibration if there exists an open subset $U \subseteq Y$ such that the restriction $f|_U : U \to X$ is a fibration.

8 Differentials and duality

Let $\Bbbk$ be an algebraically closed field. Unless otherwise specified, all schemes and varieties are assumed to be defined over $\Bbbk$.

8.1 The sheaves of differentials

Definition 8.1. Let X be a normal variety over $\Bbbk$ of dimension n. If X is smooth, then the *canonical divisor* K_X is defined to be $c_1(\omega_X)$. In general, let $U \subseteq X$ be the smooth locus of X and $i : U \hookrightarrow X$ be the inclusion map. Then the *canonical divisor* K_X is defined to be any Weil divisor on X such that $\mathcal{O}_X(K_X) \cong i_*\omega_U$. Note that U is big in X since X is normal, so such a Weil divisor always exists and is unique up to linear equivalence.

8.2 Fundamental sequence

Theorem 8.2. Let $f : X \to Y$ be a morphism of schemes. Then there is a natural exact sequence of $\mathcal{O}_X$-modules

$$f^*\Omega_{Y/\Bbbk} \to \Omega_{X/\Bbbk} \to \Omega_{X/Y} \to 0.$$

Yang: ... it will be exact.

Theorem 8.3 (Ramification formula)**.** Let $f : X \to Y$ be a morphism of schemes and let $Z \subseteq Y$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_Y$. Then there is a natural isomorphism

$$\Omega_{X/Y} \cong \mathcal{J}/\mathcal{J}^2,$$

where $\mathcal{J} = f^*\mathcal{I} \cdot \mathcal{O}_X \subseteq \mathcal{O}_X$ is the sheaf of ideals defining the preimage $W = f^{-1}(Z)$ in X. Yang: It is wrong.

Theorem 8.4. Let $Z \subseteq Y$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_Y$ and let $W = f^{-1}(Z)$ be the preimage of Z in X, defined by the sheaf of ideals $\mathcal{J} = f^*\mathcal{I} \cdot \mathcal{O}_X \subseteq \mathcal{O}_X$. Then there is a natural exact sequence of $\mathcal{O}_W$-modules

$$\mathcal{J}/\mathcal{J}^2 \to \Omega_{X/\Bbbk}|_W \to \Omega_{W/\Bbbk} \to 0.$$

Yang: ... it will be exact.

Theorem 8.5 (Adjunction formula)**.** Let $f : X \to Y$ be a smooth morphism of schemes. Then there is a natural isomorphism

$$\Omega_{X/Y} \cong \Omega_{X/\Bbbk}|_W,$$

where $W = f^{-1}(Z)$ is the preimage of a closed subscheme $Z \subseteq Y$. Yang: It is wrong.

8.3 Serre duality

Theorem 8.6 (Serre duality)**.** Let X be a proper variety over $\Bbbk$ and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a natural isomorphism

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee,$$

where ω_X is the canonical sheaf on X and $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual sheaf. Yang: there are some errors. Need to be revised

9 Flat, smooth and étale morphisms

9.1 Flat families

Definition 9.1. Let $f : X \to Y$ be a morphism of schemes. For a point $\xi \in X$, we say that f is *flat at* ξ if the local ring $\mathcal{O}_{X,\xi}$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that f is *flat* if it is flat at every point $\xi \in X$.

Definition 9.2. Let X be Y-scheme via a morphism $f : X \to Y$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *flat over* Y *at* $\xi \in X$ if the stalk $\mathcal{F}_\xi$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that $\mathcal{F}$ is *flat over* Y if it is flat over Y at every point $\xi \in X$.

Proposition 9.3. We have the following fundamental properties of flat morphisms:

(a) open immersions are flat;

(b) the composition of flat morphisms is flat;

(c) flatness is preserved under base change;

(d) a coherent sheaf $\mathcal{F}$ on a noetherian scheme X is flat over X iff it is locally free.

Yang: To be checked.

Proposition 9.4. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. Then the set of points $\xi \in X$ at which f is flat is open in X. Yang: To be checked.

Proposition 9.5. Let X be a regular integral scheme of dimension 1 and $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is flat over X iff it is torsion-free, i.e., for every non-zero-divisor $s \in \mathcal{O}_{X,x}$, the multiplication map

$$s : \mathcal{F} \to \mathcal{F}$$

is injective. Yang: To be checked.

Proposition 9.6. Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field $\mathbf{k}$. Then for every point $\xi \in X$, we have

$$\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}.$$

Yang: To be checked.

Theorem 9.7 (Miracle flatness)**.** Let $f : X \to Y$ be a morphism between noetherian schemes. Suppose that X is Cohen–Macaulay and that Y is regular. Then f is flat at $\xi \in X$ iff $\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}$. Yang: To be checked.

Theorem 9.8. Let T be a integral noetherian scheme and $f : X \to T$ be a projective morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix a relatively ample line bundle H on X over T. Then $\mathcal{F}$ is flat over T iff the Hilbert polynomials

$$P(X_t, \mathcal{F}_t, H_t)(n) = \chi(X_t, \mathcal{F}_t \otimes H_t^{\otimes n})$$

are independent of $t \in T$. Yang: To be checked.

Yang: To be added: deformation, algebraic families...

9.2 Base change and semicontinuity

Theorem 9.9 (Grauert's theorem)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the sheaf $R^i f_* \mathcal{F}$ is coherent on Y, and there exists an open subset $U \subseteq Y$ such that for every point $y \in U$, the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism. Yang: To be checked.

Theorem 9.10 (Cohomology and base change)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. For each integer $i \geq 0$, the following are equivalent:

(a) the base change map
$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$
is an isomorphism for all points $y \in Y$;

(b) the sheaf $R^i f_* \mathcal{F}$ is locally free on Y.

Yang: To be checked.

Theorem 9.11 (Semicontinuity of cohomology)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the function
$$h^i : Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$
is upper semicontinuous on Y.

Yang: To be checked.

9.3 Smooth morphisms

Definition 9.12. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. For $\xi \in X$ with image $\zeta = f(\xi) \in Y$, set $\overline{\zeta} : \operatorname{Spec} \overline{\kappa(\zeta)} \to Y$ to be the geometric point over ζ and $X_{\overline{\zeta}}$ be the geometric fiber over ζ. We say that f is *smooth at* ξ if f is flat at ξ and the geometric fiber $X_{\overline{\zeta}}$ is regular over $\overline{\kappa(\zeta)}$ at every point lying over ξ. We say that f is *smooth* if it is smooth at every point $\xi \in X$.

Yang: To be checked.

9.4 Étale morphisms

Definition 9.13. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. We say that f is *étale at* ξ if f is smooth and finite at ξ. We say that f is *étale* if it is étale at every point $\xi \in X$.

Yang: To be checked.