
Birational Geometry



“要知道你为什么出枪，你的心里有闷烧的火，那是大地上燃烧的煤矿，它的火焰终有一天烧破地面去点燃天空。你会吼叫，因为你若是不吐出那火焰，它会烧穿你的胸膛，它像是愤怒，又像是高亢的歌，龙虎的吼声让时间停止。”

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Yang: This note is full of errors. Do not believe anything it says.

1 Kodaira Vanishing Theorem

1.1 Preliminary

Theorem 1.1 (Serre Duality). Let X be a Cohen-Macaulay projective variety of dimension n over k and D a divisor on X . Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \dots, n.$$

Theorem 1.2 (Log Resolution of Singularities). Let X be an irreducible reduced algebraic variety over \mathbb{C} (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z . Then there is a

smooth variety (or analytic space) Y and a projective morphism $f : Y \rightarrow X$ such that

- (a) f is an isomorphism over $X - (\text{Sing}(X) \cup \text{Supp } Z)$,
- (b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and
- (c) $\text{Exc}(f) \cup D$ is an snc divisor.

Theorem 1.3 (Lefschetz Hyperplane Theorem). Let X be a smooth projective variety of dimension n over \mathbb{C} and Y a hyperplane section of X . Then the restriction map

$$H^k(X, \mathbb{C}) \rightarrow H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 1.4 (Hodge Decomposition). Let X be a smooth projective variety of dimension n over \mathbb{C} . Then for any k , there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 1.3 and Theorem 1.4, we have the following lemma.

Lemma 1.5. Let X be a smooth projective variety of dimension n over \mathbb{C} and Y a hyperplane section of X . Then the restriction map $r_k : H^k(X, \mathbb{C}) \rightarrow H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \rightarrow H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \rightarrow H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n - 1$ and an injection for $p = n - 1$.

Theorem 1.6 (Leray spectral sequence). Let $f : Y \rightarrow X$ be a morphism of varieties and \mathcal{F} a coherent sheaf on Y . Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(Y, \mathcal{F}).$$

1.2 Kodaira Vanishing Theorem

Lemma 1.7. Let X be a smooth projective variety over \mathbf{k} and \mathcal{L} a line bundle on X . Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \rightarrow X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D . It induces a homomorphism $\mathcal{L}^{-m} \rightarrow \mathcal{O}_X$. Consider the \mathcal{O}_X -algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} \right) / (\mathcal{L}^{-m} \rightarrow \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then \mathcal{A} is a finite \mathcal{O}_X -algebra. Let $Y := \operatorname{Spec}_X \mathcal{A}$. Then Y is a finite \mathcal{O}_X -scheme and the natural morphism $f : Y \rightarrow X$ is finite and surjective.

For every $x \in X$, let \mathcal{L} locally generated by t near x . Then \mathcal{O}_Y locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y . Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. \square

Remark 1.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D'_i = f^*(D_i)$. Then $D' + \sum_{i=1}^k D'_i$ is snc on Y by considering the local regular system of parameters.

Lemma 1.9. Let $f : Y \rightarrow X$ be a finite surjective morphism of projective varieties and \mathcal{L} a line bundle on X . Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ splits by the trace map $(1/n) \operatorname{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. \square

Theorem 1.10 (Kodaira Vanishing Theorem). Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0 and A an ample divisor on X . Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 1.7 and 1.9, after taking a multiple of A , we can assume that A is effective. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(-A) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_A \rightarrow 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \rightarrow H^{i-1}(X, \mathcal{O}_A) \rightarrow H^i(X, \mathcal{O}_X(-A)) \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 1.5 and Serre duality (Theorem 1.1). \square

1.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

Theorem 1.11 (Kawamata-Viehweg Vanishing Theorem I). Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0 and D a nef and big \mathbb{R} -divisor on X . Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Theorem 1.12 (Kawamata-Viehweg Vanishing Theorem II). Let X be a smooth projective variety of dimension n over \mathbf{k} of characteristic 0 and D a nef and big \mathbb{Q} -divisor on X . Suppose that $[D] - D$ has snc support. Then

$$H^i(X, K_X + [D]) = 0, \quad \forall i > 0.$$

Theorem 1.13 (Kawamata-Viehweg Vanishing Theorem III). Let (X, B) be a klt pair over \mathbf{k} of characteristic 0. Let D be a nef \mathbb{Q} -divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

If we replace the assumption "nef and big" of D by "ample" in II and III, we denote them as "II(ample)" and "III(ample)" respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \Rightarrow \text{II(ample)} \Rightarrow \text{III(ample)} \Rightarrow \text{I} \Rightarrow \text{II} \Rightarrow \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

Proof of II (Theorem 1.12). Set $M := [D]$. Let

$$B := \sum_{i=1}^k b_i B_i := [D] - D = M - A, \quad b_i \in (0, 1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k . When $k = 0$, the conclusion follows from Theorem 1.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 1.10.)) Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 1.15 and 1.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 1.7 on B_k , we can find a finite surjective morphism $f : X' \rightarrow X$ such that $f^*B_k = cB'_k, B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^k B'_i$ is an snc divisor on X' . Let $B' = \sum_{i=1}^{k-1} B'_i, A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$

vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^* \mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 1.9. \square

Proof of III (Theorem 1.13). Let $f : \tilde{X} \rightarrow X$ be a resolution such that $\text{Supp } f^*B \cup \text{Exc } f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X}, \tilde{B})} + f^*D,$$

where $\tilde{B} \in (0, 1)$ has snc support and E is an effective exceptional divisor.

By Lemma 1.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X}, \tilde{B})} + f^*D) = H^i(X, f_* \mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 1.12 in either case relative to the assumption of D . \square

Proof of I (Theorem 1.11). By Lemma 1.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor A . Then the theorem comes down to Theorem 1.13. \square

Lemma 1.14. Let $f : Y \rightarrow X$ be a birational morphism of projective varieties with Y smooth and X has only rational singularities. Let E be an effective exceptional divisor on Y and D a divisor on X . Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

Proof. Yang: I am unable to proof this lemma. \square

Lemma 1.15. Let X be a projective variety, \mathcal{L} a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \rightarrow X$ and a line bundle \mathcal{L}' on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X , then we can take f such that f^*D is an snc divisor on Y .

Proof. We can assume that \mathcal{L} is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & \mathbb{P}^N \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{\varphi_{\mathcal{L}}} & \mathbb{P}^N \end{array}$$

where $g : [x_0 : \dots : x_N] \mapsto [x_0^m : \dots : x_N^m]$. The morphism f is finite and surjective since so is g . Let $\mathcal{L}' := \psi^*\mathcal{L}$.

For smoothness, we can compose g with a general automorphism of \mathbb{P}^N . Then the conclusion follows from [Har77, Chapter III, Theorem 10.8]. \square

Lemma 1.16 (ref. [KM98, Theorem 5.10, 5.22]). Let (X, B) be a klt pair over \mathbf{k} of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Lemma 1.17. Let X be a projective variety of dimension n and D a nef and big divisor on X . Then there exists an effective divisor B such that for every k , there is an ample divisor A_k such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k}B.$$

Proof. By **Yang: definition** of big divisor, there exists an ample divisor A_1 and effective divisor B such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k}B.$$

Since A is ample and D is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. \square

2 Cone Theorem

2.1 Preliminary

Theorem 2.1 (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27]). Let X be a projective variety and ℓ an semiample line bundle on X . Then there exists a fibration $\varphi : X \rightarrow Y$ of projective varieties such that for any $m \gg 0$ with ℓ^m base point free, we have that the morphism φ_{ℓ^m} induced by ℓ^m is isomorphic to φ . Such a fibration is called the *Iitaka fibration* associated to ℓ .

Theorem 2.2 (Rigidity Lemma, ref. [Deb01, Lemma 1.15]). Let $\pi_i : X \rightarrow Y_i$ be proper morphisms of varieties over a field \mathbf{k} for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Theorem 2.3. Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

Proposition 2.4. Let X be a normal projective variety of dimension n and H an ample divisor on X . Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

Sketch of proof. Take a resolution $f : Y \rightarrow X$, then f^*H is nef on Y and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor H_Y on Y closed enough to f^*H such that $K_Y \cdot H_Y^{n-1} < 0$. By [MM86, Theorem 5] and take limit for H_Y . \square

Lemma 2.5 (ref. [Kaw91, Lemma]). Let (X, B) be a projective klt pair and $f : X \rightarrow Y$ a birational projective morphism. Let E be an irreducible component of dimension d of the exceptional locus of f and $\nu : E^\nu \rightarrow X$ the normalization of E . Suppose that $f(E)$ is a point. Then for any ample divisor H on X , we have

$$K_{E^\nu} \cdot \nu^* H^{d-1} \leq K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

2.2 Non-vanishing Theorem

Theorem 2.6 (Non-vanishing Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X . Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

2.3 Base Point Free Theorem

Theorem 2.7 (Base Point Free Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X . Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, mD is base point free.

Remark 2.8. In general, we say that a Cartier divisor D is *semiample* if there exists a positive integer m such that mD is base point free. The statement in Base Point Free Theorem (Theorem 2.7) is strictly stronger than the semiample condition. For example, let ℓ be a torsion line bundle, then ℓ is semiample but there exists no positive integer M such that $m\ell$ is base point free for all $m > M$.

2.4 Rationality Theorem

Lemma 2.9 (ref. [KM98, Theorem 1.36]). Let X be a proper variety of dimension n and D_1, \dots, D_m Cartier divisors on X . Then the Euler characteristic $\chi(n_1 D_1, \dots, n_m D_m)$ is a polynomial in (n_1, \dots, n_m) of degree at most n .

Theorem 2.10 (Rationality Theorem). Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X . Let

$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H . Then $t = v/u \in \mathbb{Q}$ and

$$0 \leq v \leq a(X) \cdot (\dim X + 1).$$

Proof. For every $r \in \mathbb{r}_{>0}$, let

$$v(r) := \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{r} \setminus \mathbb{Q}. \end{cases}$$

We need to show that $v(t) \leq a(\dim X + 1)$. For every $(p, q) \in \mathbb{Z}_{>0}^2$, set $D(p, q) := paK_{(X,B)} + qH$. If $(p, q) \in \mathbb{Z}_{>0}^2$ with $0 < atp - q < t$, then we have $D(p, q)$ is not nef and $D(p, q) - K_{(X,B)}$ is ample.

Step 1. We show that a polynomial $P(x, y) \neq 0 \in \mathbb{Q}[x, y]$ of degree at most n is not identically zero on the set

$$\{(p, q) \in \mathbb{Z}^2 : p, q > M, 0 < atp - q < t\varepsilon\}, \quad \forall M > 0,$$

if $v(t)\varepsilon > a(n + 1)$.

If $v(t) = \infty$, for any n , we show that we can find infinitely many lines L such that $\#L \cap \Lambda \geq n + 1$. If so, Λ is Zariski dense in \mathbb{Q}^2 . Since $1/at \in \mathbb{r} \setminus \mathbb{Q}$, there exist $p_0, q_0 > M$ such that

$$0 < \frac{p_0}{q_0} - \frac{1}{at} < \frac{\varepsilon}{(n+1)a} \cdot \frac{1}{q_0}, \text{ i.e. } 0 < atp_0 - q_0 < \frac{\varepsilon t}{n+1}.$$

Then $(ip_0, iq_0) \in \Lambda \cap \{p_0y = q_0x\}$ for $i = 1, \dots, n+1$. Since M is arbitrary, there are infinitely many such lines L .

Suppose $v(t) = v < \infty$ and $t = v/u$. Then the inequality is equivalent to $0 < aup - vq < \varepsilon v$. Note that $\gcd(au, v) | a$, then $aup - vq = ai$ has integer solutions for $i = 1, \dots, n+1$. Since $v(t)\varepsilon > a(n+1)$, there are at least $n+1$ lines which intersect Λ in infinitely many points. This enforces any polynomial which vanishes on Λ has degree at least $n+1$.

Step 2. There exists an index set $\Lambda \subset \mathbb{Z}^2$ such that Λ contains all sufficiently large (p, q) with $0 \leq atp - q \leq t$ and

$$Z := \text{Bs } |D(p, q)| = \text{Bs } |D(p', q')| \neq \emptyset, \quad \forall (p, q), (p', q') \in \Lambda.$$

For every $(p, q) \in \mathbb{Z}_{>0}^2$ with $0 < atp - q < t$, choose $k \in \mathbb{Z}_{>0}$ such that $k(atp - q) > t$. Then for all $p', q' > kp$ with $0 < atp' - q' < t$, we have

$$p' - kp \geq 0, \quad q' - kp > t(p' - kp).$$

It follows that

Yang: To be completed.

Step 3. Suppose the contradiction that $v(t) > a(\dim X + 1)$. Then we show that $H^0(X, D(p, q)) \neq 0$ for all $(p, q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem ([Theorem 2.7](#)).

Let $P(x, y) := \chi(D(x, y))$ be the Hilbert polynomial of $D(x, y)$. Note that $P(0, n) = \chi(nH) \neq 0$ since H is ample. Then $P(x, y) \neq 0$ and $\deg P \leq \dim X$. By [Step 1](#), P is not identically zero on Λ . Note that $D(p, q) - K_{(X,B)}$ is ample for all $(p, q) \in \Lambda$, then $h^i(X, D(p, q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem ([Theorem 1.13](#)). Then

$$P(p, q) = \chi(D(p, q)) = h^0(X, D(p, q)) \neq 0$$

for some $(p, q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p, q)) \neq 0$ for all $(p, q) \in \Lambda$.

Step 4. We follow the same line of the proof of Base Point Free Theorem (Theorem 2.7) to show that there is a section which does not vanish on Z .

Fix $(p, q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n+1)/u$. Let $f : Y \rightarrow X$ be a resolution such that

- (a) $K_{Y, B_Y} = f^*K_{(X, B)} + E_Y$ for some effective exceptional divisor E_Y , and Y, B_Y is a klt pair;
- (b) $f^*[D(p, q)] = [L] + F$ for some effective divisor F and a base point free divisor L , and $f(\text{Supp } F) = Z$;
- (c) $f^*D(p, q) - f^*K_{(X, B)} - E_0$ is ample for some effective \mathbb{Q} -divisor $E_0 \in (0, 1)$, and coefficients of E_0 are sufficiently small;
- (d) $B_Y + E_Y + F + E_0$ has snc support.

Yang: Such resolution exists by [KM98].

Let $c := \inf\{[B_Y + E_0 + tF] \neq 0\}$. Adjust the coefficients of E_0 slightly such that $[B_Y + E_0 + cF] = F_0$ for unique prime divisor F_0 with $F_0 \subset \text{Supp } F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then (Y, Δ_Y) is a klt pair.

Let

$$\begin{aligned} N(p', q') &:= f^*D(p', q') + E_Y - F_0 - K_{(Y, \Delta_Y)} \\ &= \left(f^*D(p', q') - (1+c)f^*D(p, q)\right) + \left(f^*D(p, q) - f^*K_{(X, B)} - E_0\right) + c\left(f^*D(p, q) - F\right). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p', q') \in \Lambda : 0 < atp' - q' < atp - q, p', q' > (1+c) \max\{p, q\}\},$$

the divisor $f^*D(p', q') - (1+c)f^*D(p, q) = f^*D(p' - (1+c)p, q' - (1+c)q)$ is ample, and hence $N(p', q')$ is ample.

By the exact sequence

$$0 \rightarrow \mathcal{O}_Y(f^*D(p', q') + E_Y - F_0) \rightarrow \mathcal{O}_Y(f^*D(p', q') + E_Y) \rightarrow \mathcal{O}_{F_0}((f^*D(p', q') + E_Y)|_{F_0}) \rightarrow 0$$

and Kawamata-Viehweg Vanishing Theorem (Theorem 1.13), we get a surjective map

$$H^0(Y, f^*D(p', q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}).$$

On F_0 , consider the polynomial $\chi((f^*D(p', q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n-1$ and by the construction of $(p, q), \Lambda_0$, similar to Step 3, we can show that $\chi((f^*D(p', q') + E_Y)|_{F_0})$ is not identically zero on Λ_0 . By adjunction, we have $(f^*D(p', q') + E_Y)|_{F_0} = N(p', q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p', q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (Theorem 1.13) to get

$$h^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}) = \chi(F_0, (D(p', q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \text{Bs } |D(p', q')|$. □

2.5 Cone Theorem and Contraction Theorem

Theorem 2.11 (Cone Theorem). Let (X, B) be a projective klt pair. Then there exist countably many rational curves $C_i \subset X$ with

$$0 < -K_{(X,B)} \cdot C_i \leq 2 \dim X$$

such that

(a) we have a decomposition of cones

$$\text{Psef}_1(X) = \text{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum_{\mathbb{R}_{\geq 0}} \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X , we have

$$\text{Psef}_1(X) = \text{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. Let $F_D := \text{Psef}_1(X) \cap D^\perp$ for a nef divisor D on X . If $\dim F_D = 1$, we also write $R_D := F_D$. Let $H_1, \dots, H_{\rho-1}$ be ample divisors on X such that they together with $K_{(X,B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} := S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

Step 1. There exists an integer N such that for every $K_{(X,B)}$ -negative extremal face F_D and for every ample divisor H , there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \neq F_{nD+rK_{(X,B)}+NH} \subset F_D$.

Let $N := (a(X)(\dim X + 1))!$, where $a(X)$ is the number in Theorem 2.10. For every n , $nD + H$ is an ample divisor and by Theorem 2.10, the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \geq 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since $K_{(X,B)} + (N/r_n)((n+1)D + H)$ is nef, we have $r_n \leq r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$ implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \rightarrow r \in \mathbb{Z}_{\geq 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors N_1, N_2 , we have $F_{N_1+N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists m large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

Step 2. Let $\Phi : N_1(X)_{K_{(X,B)} < 0} \rightarrow \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left(\frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha} \right).$$

We show that the image of R_D under Φ lies in a \mathbb{Z} -lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class ξ . By [Step 1](#), we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers n, r . Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of R_D under Φ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

Step 3. We show that every $K_{(X,B)}$ -negative extremal ray of $\text{Psef}_1(X)$ is of the form R_D for some nef divisor D on X .

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$ -negative extremal ray. **Yang:** Then R is of form $D^\perp \cap \text{Psef}_1(X)$ for some nef \mathbb{R} -divisor D on X by [Theorem 2.3](#). We need to show that D can be choose as a nef \mathbb{Q} -divisor. There is a sequence of nef but not ample \mathbb{Q} -divisors D_m such that $D_m \rightarrow D$ as $m \rightarrow \infty$. We adjust D_m such that $\dim F_{D_m} = 1$ for all n .

By re-choosing H_i , we can assume that $D = a_1H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is D_m . Then F_{D_m} is $K_{(X,B)}$ -negative. Note that $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by [Step 1](#). If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{F_{D_m}}$. We can assume that $r_1K_{(X,B)}+NH_1$ is not identically zero on F_{D_m} . Then we can choose n large enough such that $\|r_1K_{(X,B)} + NH_1\|/n < 1/m$. Replace D_m by $D_m + (r_1K_{(X,B)} + NH_1)/n$. Inductively we construct D_m nef \mathbb{Q} -divisor with $D_m \rightarrow D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that ξ_m converges. Then $\xi_m \rightarrow \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, Φ is well-defined at ξ and the image of ξ_m under Φ is discrete. Hence $\xi = \xi_m$ for all m large enough. It follows that $R = R_{D_m}$ for a nef \mathbb{Q} -divisor D_m .

Step 4. We show that any $K_{(X,B)}$ -negative extremal ray R_D contains the class of a rational curve C with $0 < -K_{(X,B)} \cdot C \leq 2 \dim X$.

By [Theorem 2.13](#), let $\varphi_D : X \rightarrow Y$ be the contraction associated to R_D (note that we do not need the step to proof [Theorem 2.13](#)). If $\dim Y < \dim X$, let F be a general fiber of φ_D . **Yang:** By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that H is ample on F . By [Proposition 2.4](#). **Yang:** In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By [Lemma 2.5](#), we can use [Proposition 2.4](#) to get the result.

Yang: To be completed.

Step 5. Proof of the theorem.

Given an ample divisor H on X , note that εH has positive minimum δ on $\text{Psef}_1(X) \cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \text{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and Φ is well-defined on it. By [Steps 2](#) and [3](#), there are only finitely many extremal rays on $\text{Psef}_1(X)_{K_{(X,B)}+\varepsilon H \leq 0}$. By [Step 4](#), we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$\mathcal{C} := \text{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\} \subset \mathcal{C}$ such that $\alpha_n \rightarrow \alpha \in N_1(X)_{\mathbb{R}}$. Note that $\text{Psef}_1(X)$ is closed, hence $\alpha \in \text{Psef}_1(X)$. We only need to consider the case $\alpha \cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$. Then $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$ for all n large enough. Note that $\mathcal{C} \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$ is a polyhedral cone by [Step 2](#) and hence is closed. Then $\alpha \in \mathcal{C}$ and the conclusion follows. \square

Remark 2.12. Yang: Thanks for my friend Qin for pointing out that the extremal ray in [Theorem 2.11](#) may not be exposed.

Theorem 2.13 (Contraction Theorem). Let (X, B) be a projective klt pair and $F \subset \text{Psef}_1(X)$ a $K_{(X,B)}$ -negative extremal face of $\text{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \rightarrow Y$ of projective varieties such that

- (a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;
- (b) up to linearly equivalence, any Cartier divisor G with $F \subset G^\perp = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$ comes from a Cartier divisor on Y , i.e., there exists a Cartier divisor G_Y on Y such that $G \sim \varphi_F^* G_Y$.

Proof. We follow the following steps to prove the theorem.

Step 1. We show that there exists a nef divisor D on X such that $F = D^\perp \cap \text{Psef}_1(X)$. In other words, F is defined on $N_1(X)_{\mathbb{Q}}$.

We can choose an ample divisor H and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on F since $F \cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in $N_1(X)_{\mathbb{R}}$. Then by Cone Theorem ([Theorem 2.11](#)), F is an extremal face of a rational polyhedral cone, namely $\text{Psef}_1(X)_{K_{(X,B)} + (1/n)H \leq 0}$. It follows that $F^\perp \subset N^1(X)_{\mathbb{R}}$ is defined on \mathbb{Q} . Since F is extremal and $K_{(X,B)} + (1/n)H$ -negative, the set $\{L \in F^\perp : L|_{\text{Psef}_1(X) \setminus F} > 0\}$ has non-empty interior in F^\perp by [Theorems 2.3](#) and [2.11](#). Then there exists a Cartier divisor D such that $D \in F^\perp$ and $D|_{\text{Psef}_1(X) \setminus F} > 0$. It follows that D is nef and $F = D^\perp \cap \text{Psef}_1(X)$.

Step 2. Let $\varphi : X \rightarrow Y$ be the Iitaka fibration associated to D by [Theorem 2.1](#). We show that φ is the desired fibration.

Note that $\text{Psef}_1(X)_{K_{(X,B)} \geq 0} \cap S^{\rho-1}$ is compact and D is strictly positive on it. Then there exist $a \geq 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\text{Psef}_1(X)_{K_{(X,B)} \geq 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since F is $K_{(X,B)}$ -negative. Then by Base Point Free Theorem ([Theorem 2.7](#)), we know that mD is base point free for all $m \gg 0$. Hence we can apply [Theorem 2.1](#) to get a fibration $\varphi_D : X \rightarrow Y$.

First we show that D comes from Y . Note that mD and $(m+1)D$ induces the same fibration φ_D for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m)D_{Y,m}$ and $D_{Y,m}$ is ample. Hence D_Y is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that C is contracted by φ_D if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let G be arbitrary Cartier divisor on X such that $F \subset G^\perp$. Since D is strictly positive on $\text{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' := mD + G$, we have $D'^\perp \cap \text{Psef}_1(X) = F$. Then by the same argument as above, we get an other fibration $\varphi_{D'} : X \rightarrow Y'$ such that a curve C is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 2.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on Y . In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor D'_Y on Y . Then $G = D' - mD$ also comes from Y . \square

Remark 2.14. The Step 1 is amazing. If F is not $K_{(X,B)}$ -negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E . By [Laz04, Lemma 1.5.4], we know that $\text{Psef}_1(X)$ is a circular cone. Then we see there indeed exist some irrational extremal faces of $\text{Psef}_1(X)$.

Definition 2.15. Let (X, B) be a projective klt pair and R a $K_{(X,B)}$ -negative extremal ray of $\text{Psef}_1(X)$ with contraction $\varphi_R : X \rightarrow Y$. There are three types of contractions:

- (a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension one;
- (b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension at least two;
- (c) *Mori fiber space*: if $\dim X > \dim Y$.

Proposition 2.16. Let (X, B) be a \mathbb{Q} -factorial projective klt pair and R a $K_{(X,B)}$ -negative extremal ray of $\text{Psef}_1(X)$. Suppose that the contraction $\varphi : X \rightarrow Y$ associated to R is either divisorial or a Mori fiber space. Then Y is \mathbb{Q} -factorial.

Proof. Let D be a prime Weil divisor on Y and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\geq 0}[C]$ for an irreducible curve C contracted by φ . Set $D_X := \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$. Then D_X is a prime Weil divisor on X and hence is \mathbb{Q} -Cartier.

If φ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber F of φ . Then by Contraction Theorem (Theorem 2.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor D' on Y . We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence D is \mathbb{Q} -Cartier.

If φ is a divisorial contraction, let E be the exceptional divisor of φ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier \mathbb{Q} -divisor E_Y on Y). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 2.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor D' on Y . Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence D is \mathbb{Q} -Cartier. \square

Remark 2.17. If φ is a small contraction, then Y is never \mathbb{Q} -factorial. Otherwise, let B_Y be the strict transform of B on Y . Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset U . Suppose $K_{(Y,B_Y)}$ is \mathbb{Q} -Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

3 Basepoint Free Theorem on Positive Characteristic

This section refers to [Kee99]. For site and algebraic space, we refer to [Knu71], [Art70], [Stacks] and [FGA05]. Throughout this section, all schemes (or algebraic space) are of finite type over a base scheme S with S noetherian.

3.1 Preliminaries

Theorem 3.1 (Serre vanishing in relative setting, ref. [Laz04, Theorem 1.7.6]). Let $f : X \rightarrow S$ be a proper morphism of schemes, ℓ a line bundle and \mathcal{F} a coherent sheaf on X . Suppose that ℓ is relatively ample. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the higher direct image sheaves $R^i f_* \mathcal{F} \otimes \ell^{\otimes n}$ are zero for all $i > 0$.

Theorem 3.2 (ref. [Laz04, Proposition 1.4.37]). Let X be a projective scheme over a field \mathbb{k} . Then there exists a scheme T of finite type over \mathbb{k} and a line bundle ℓ on $X \times T$ such that every numerically trivial line bundle on X arises as the restriction $\ell|_{X \times \{t\}}$ for some $t \in T$.

Theorem 3.3 (Theorem on Formal Functions, ref. [Har77, Chapter III, Theorem 11.1]). Let $f : X \rightarrow Y$ be a projective morphism of noetherian schemes, let \mathcal{F} be a coherent sheaf on X , and let $y \in Y$. Then the natural map

$$(R^i f_* \mathcal{F})_y^\wedge \rightarrow \varprojlim H^i(X_n, \mathcal{F}_n)$$

is an isomorphism for all $i \geq 0$, where $X_n = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n$ and $\mathcal{F}_n = \mathcal{F}|_{X_n}$.

Definition 3.4. Let X be a proper variety and ℓ a nef line bundle on X . A closed subvariety $Z \subseteq X$ is called the *exceptional* for ℓ if $\ell^{\dim Z} \cdot Z = 0$. The *exceptional locus* of ℓ , denoted by $\operatorname{Exc} \ell$, is defined as the closure of the union of all exceptional subvarieties of ℓ .

If ℓ is semiample, then $\operatorname{Exc} \ell = \operatorname{Exc} \varphi$ for the fibration $\varphi : X \rightarrow Y$ induced by ℓ .

Definition 3.5. Let X be a proper scheme and ℓ a nef line bundle on X . We say that ℓ is *endowed with a map (EWM)* if there is a proper morphism $\varphi : X \rightarrow Y$ to a proper algebraic space such that

$\dim Z > \dim f(Z)$ if and only if Z is an exceptional subvariety of ℓ . If such a morphism is a fibration, then it is unique, called the *fibration associated to ℓ* .

Proposition 3.6. Let X be a proper variety and ℓ a nef line bundle on X endowed with a map. Let $\varphi : X \rightarrow Y$ be the associated fibration. Then TFAE:

- (a) ℓ is semiample;
- (b) $\ell^{\otimes m}$ is pulled back from an ample line bundle on Y for some $m \in \mathbb{Z}_{>0}$;
- (c) $\ell^{\otimes m}$ is pulled back from a line bundle on Y for some $m \in \mathbb{Z}_{>0}$;

Proof. (a) \Leftrightarrow (b) \Rightarrow (c) is clear. Replacing ℓ by $\ell^{\otimes m}$ for some $m \in \mathbb{Z}_{>0}$, suppose that $\ell = \varphi^* \ell_Y$ for some line bundle ℓ_Y on Y . We show that ℓ_Y is ample. Indeed, for all closed subvarieties $Z \subset Y$, we can find $Z' \subset X$ such that $Z' \twoheadrightarrow Z$ and $\dim Z' = \dim Z$. Then

$$\ell_Y^{\dim Z} \cdot Z = d \ell^{\dim Z'} \cdot Z' > 0$$

where $d = \deg(Z' \rightarrow Z)$. Hence ℓ_Y is ample. \square

Definition 3.7. A morphism $f : X \rightarrow Y$ of schemes is called a *universal homeomorphism* if for every Y -scheme Y' , the base change $X \times_Y Y' \rightarrow Y'$ is a homeomorphism between the underlying topological spaces.

Example 3.8. Let X be a scheme of finite type over \mathbb{k} . Then the natural morphism $X_{\text{red}} \rightarrow X$ is a universal homeomorphism.

Let X be a scheme over S of characteristic p . Then the absolute and relative Frobenius morphisms are universal homeomorphisms. **Yang: To be completed.**

The morphism $\text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{R}$ is not a universal homeomorphism.

Lemma 3.9. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms of schemes with g finite. Let \mathcal{f} be a coherent sheaf on X . Then the we have

$$R^i(g \circ f)_* \mathcal{f} = g_*(R^i f_* \mathcal{f}).$$

Proof. **Yang: This is a simple application of the Grothendieck spectral sequence. However, I do not know anything about it.** \square

3.2 Algebraic space

Definition 3.10. Let \mathbf{C} be a category. A *Grothendieck topology* on \mathbf{C} is a collection of sets of arrows $\{U_i \rightarrow U\}_{i \in I}$, called *covering*, for each object U in \mathbf{C} such that:

- (a) if $V \rightarrow U$ is an isomorphism, then $\{V \rightarrow U\}$ is a covering;
- (b) if $\{U_i \rightarrow U\}_{i \in I}$ is a covering and $V \rightarrow U$ is a arrow, then the fiber product $U_i \times_U V \rightarrow V$ exists

and $\{U_i \times_U V \rightarrow V\}$ is a covering of V ;

- (c) if $\{U_i \rightarrow U\}_{i \in I}$ and $\{U_{ij} \rightarrow U_i\}_{j \in J_i}$ are coverings, then the collection of composition $\{U_{ij} \rightarrow U_i \rightarrow U\}_{i \in I, j \in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{J})$ where \mathbf{C} is a category and \mathcal{J} is a Grothendieck topology on \mathbf{C} .

Note that sheaf is indeed defined on a site.

Definition 3.11. Let $(\mathbf{C}, \mathcal{J})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{J})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \rightarrow \mathbf{Set}$ satisfying the following condition: for every object U in \mathbf{C} and every covering $\{U_i \rightarrow U\}_{i \in I}$ of U , if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every i, j , the pullback $s_i|_{U_i \times_U U_j}$ and $s_j|_{U_i \times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every i , the pullback $s|_{U_i} = s_i$.

Definition 3.12. Let X be a scheme. The *big étale site* of X , denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over X with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \rightarrow U\}_{i \in I}$ is a covering if and only if each U_i is étale over U and the union of their images is the whole U .

Let X be a scheme over S . By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \rightarrow \mathbf{Set}$ such that for any S -scheme T , $h_X(T) = \text{Hom}_{\mathbf{Sch}_S}(T, X)$. **Yang:** Easy to check that h_X is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

Definition 3.13. Let U be a scheme over a base scheme S . An *étale equivalence relation* on U is a morphism $R \rightarrow U \times_S U$ between schemes over S such that:

- (a) the projections in two factors $R \rightarrow U$ are étale and surjective;
- (b) for every S -scheme T , $h_R(T) \rightarrow h_U(T) \times h_U(T)$ gives an equivalence relation on $h_U(T)$ set-theoretically.

Definition 3.14. An *algebraic space* X over a base scheme S is an S -scheme U together with an étale equivalence relation $R \rightarrow U \times_S U$.

Let $X = (U, R)$ be an algebraic space over S . We explain X as a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. For any scheme T over S , $h_R(T)$ is an equivalence relation on $h_U(T)$. The rule sending T to the set of equivalence classes of $h_R(T)$ gives a presheaf on the site $(\mathbf{Sch}/S)_{\text{ét}}$. The sheafification of this presheaf is the sheaf associated to the algebraic space X . Explicitly, we have

$$X(T) := \left\{ f = (f_i) \left| \begin{array}{l} \{T_i \rightarrow T\} \text{ a covering, } f_i \in h_U(T_i) \text{ such} \\ \text{that } (f_i|_{T_i \times_T T_j}, f_j|_{T_i \times_T T_j}) \in h_R(T_i \times_T T_j) \end{array} \right. \right\} / \sim,$$

where

$$\alpha \sim \beta \quad \text{if } \exists \{S_i \rightarrow T\} \text{ such that } (\alpha|_{S_i}, \beta|_{S_i}) \in h_R(S_i).$$

Definition 3.15. An *algebraic space* over a base scheme S is a sheaf F on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$ such that

- (a) the diagonal morphism $F \rightarrow F \times_S F$ is representable;
- (b) there exists a scheme U over S and a map $h_U \rightarrow F$ which is surjective and étale.

The *morphism between algebraic spaces* F_1, F_2 is defined as a natural transformation of functors F_1, F_2 .

Remark 3.16. By Yoneda's Lemma, given a morphism $h_U \rightarrow F$ between sheaves is the same as giving an element of $F(U)$. We may abuse the notation.

Definition 3.17. Let \mathcal{P} be a property of morphisms of schemes satisfying the following conditions:

- (a) is preserved under any base change;
- (b) is étale local on the base. Yang: In [Stacks], this requires that “fppf local”.

Let $\alpha : F \rightarrow G$ be a representable morphism of sheaves on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. We say that α has property \mathcal{P} if for every $h_T \rightarrow G$, the base change $h_T \times_G F \rightarrow F$ has property \mathcal{P} .

Remark 3.18. The fiber product $F_1 \times_F F_2$ is just defined as $F_1 \times_F F_2(T) := F_1(T) \times_{F(T)} F_2(T)$ for any object $T \in \text{Obj}(\mathbf{Sch}_S)$. We say that a morphism $f : F_1 \rightarrow F_2$ of sheaves is *representable* if for every $T \in \text{Obj}(\mathbf{Sch}/S)$ and every $\xi \in F_2(T)$, the sheaf $F_1 \times_{F_2} h_T$ is representable as a functor. Here $h_T \rightarrow F_2$ is given by

$$h_T(U) \rightarrow F_2(U), \quad f \in \text{Hom}(U, T) \mapsto F_2(f)(\xi) \in F_2(U).$$

In our case, given an arbitrary $h_U \rightarrow F \times F$ is equivalent to giving morphisms $h_{U_i} \rightarrow F$ for $i = 1, 2$. And the fiber product $F \times_{F \times F} (h_{U_1} \times h_{U_2})$ is just the fiber product $h_{U_1} \times_F h_{U_2}$. Hence the first condition in Definition 3.15 is equivalent to that $h_{U_1} \times_F h_{U_2}$ is representable for any U_1, U_2 over F . This implies that $h_U \rightarrow F$ is representable, whence the second condition in Definition 3.15 makes sense.

Definition 3.19. Let X be an algebraic space over a base scheme S . Two morphisms from field $\text{Spec } k_i \rightarrow X$ is called equivalent if there is a common extension $K \supset k_1, k_2$ such that we have $\text{Spec } K \rightarrow \text{Spec } k_i \rightarrow X$ are the same for $i = 1, 2$. The *underlying point set* of X , denote by $|X|$, is defined as the set of equivalence classes of morphisms $\text{Spec } k \rightarrow X$ for all field k over the base field \mathbb{k} .

This definition coincides with the underlying set of a scheme. Let $\alpha : X \rightarrow Y$ be a morphism of algebraic spaces. It induces a map $|\alpha| : |X| \rightarrow |Y|$ by $x \mapsto \alpha \circ x$ (vertical composition).

Proposition 3.20 (ref. [Stacks, Lemma 66.4.6]). There is a unique topology on $|X|$ such that

- (a) if X is a scheme, then the topology coincides with the usual topology.
- (b) every morphism of algebraic spaces $f : X \rightarrow Y$ induces a continuous map $|f| : |X| \rightarrow |Y|$.
- (c) if U is a scheme and $U \rightarrow X$ is étale, then the induced map $|U| \rightarrow |X|$ is open.

This topology is called the *Zariski topology* on $|X|$.

Definition 3.21. Let X be an algebraic space over a base scheme S . All étale morphisms $U \rightarrow X$ with U scheme form a small site $X_{\text{ét}}$. All étale morphisms $U \rightarrow X$ with U algebraic space form a small site $X_{\text{sp}, \text{ét}}$. The *structure sheaf* \mathcal{O}_X of X is given by $U \mapsto \Gamma(U, \mathcal{O}_U)$ for every étale morphism $U \rightarrow X$ from a scheme. It extends to a sheaf on the site $X_{\text{sp}, \text{ét}}$ uniquely.

Example 3.22. Let $U = \mathbb{A}_{\mathbb{C}}^1$ and $R \subset U \times U$ given by $y = x + n, n \in \mathbb{Z}$. Then R is a disjoint union of lines in $U \times U$. Write $R = \coprod_{n \in \mathbb{Z}} R_n$ with $R_n = \{(x, x + n) : x \in \mathbb{C}\}$. Then the projection is given by

$$\begin{aligned} \pi_1|_{R_n} : R_n &\rightarrow U, & (x, x + n) &\mapsto x, \\ \pi_2|_{R_n} : R_n &\rightarrow U, & (x, x + n) &\mapsto x + n. \end{aligned}$$

Easily see that the projection $\pi_i : R \rightarrow U$ is étale and surjective for $i = 1, 2$. Let $r_{ij} : R \times U \rightarrow U \times U \times U$ be the morphism which maps $((x, y), u)$ to (a_1, a_2, a_3) where $a_i = x$, $a_j = y$ and $a_k = u$ for $k \neq i, j$. Since $\Delta_U \rightarrow U \times U$ factors through R , $(\pi_1, \pi_2) = (\pi_2, \pi_1)$ and $r_{12} \times_{(U \times U \times U)} r_{23}$ factors through r_{13} , we have that $h_R(T)$ is an equivalence relation on $h_U(T)$ for all T over S . Then $X := (U, R)$ is an algebraic space.

We do not check the representability here but give an example. Let $U \rightarrow X$ be the natural morphism given by $\text{id}_U \in X(U)$. For any scheme T over \mathbb{C} , we have

$$(U \times_X U)(T) = \{(f, g) \in h_{U \times U}(T) : \exists \{T_i \rightarrow T\} \text{ s.t. } (f_i, g_i) \in h_R(T_i)\} = h_R(T).$$

Hence the fiber product $h_U \times_X h_U$ is represented by R .

We show that $X \not\cong \mathbb{C}^\times$ by computing the the global sections. Consider the covering $U \rightarrow X$, a section $s \in \mathcal{O}_X(X)$ is given by a section $s \in \Gamma(U, \mathcal{O}_U) = \mathbb{C}[t]$ such that $\pi_1^*s = \pi_2^*s$ in $\Gamma(R, \mathcal{O}_R)$. This means that $s(x + n) = s(x)$ for all $n \in \mathbb{Z}$. Hence s is a constant function. In particular, $\mathcal{O}_X(X) = \mathbb{C} \neq \mathbb{C}[t, t^{-1}]$.

The underlying set $|X|$ is union of the quotient set \mathbb{C}/\mathbb{Z} and a generic point. The Zariski topology on $|X|$ is the trivial topology.

In following, we will use the technique of *local construction* to construct many scheme-like objects on algebraic spaces. For local construction, see [Knu71]. Roughly speaking, for every étale morphism $U \rightarrow X$ with U a scheme, we construct a scheme-theoretic object on U which is compatible under base change. Then we glue these objects together to get a global object on X .

Definition 3.23. Let X be an algebraic space over a base scheme S . A *coherent sheaf* on X is a sheaf \mathcal{F} on $X_{\text{ét}}$ such that for every covering $\{U_i \rightarrow X\}$ with U_i schemes, the sheaf $\mathcal{F}|_{U_i}$ is coherent for every i . It extends to a sheaf on the site $X_{\text{sp}, \text{ét}}$ uniquely.

An *ideal sheaf* on X is a coherent sheaf $\mathcal{I} \subset \mathcal{O}_X$. It defines a closed subspace $V(\mathcal{I}) \subset X$ by **Yang: to be completed**. And every closed subspace $Y \subset X$ is defined by an ideal sheaf \mathcal{I}_Y such that $V(\mathcal{I}_Y) = Y$.

Definition 3.24. Let X be an algebraic space over a base scheme S . A *line bundle* on X is a coherent sheaf \mathcal{L} on X such that for every covering $\{U_i \rightarrow X\}$ with U_i schemes, the sheaf $\mathcal{L}|_{U_i}$ is a line bundle on U_i . It extends to a sheaf on the site $X_{\text{sp}, \text{ét}}$ uniquely.

Theorem 3.25 (ref. [Stacks, Theorem 76.36.4]). Let $f : X \rightarrow Y$ be a proper morphism of algebraic spaces over a base scheme S . Then there exists a factorization

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y,$$

where f_1 has geometrically connected fibers and $(f_1)_*\mathcal{O}_X = \mathcal{O}_Z$ and f_2 is finite.

Definition 3.26. Let X be an algebraic space over a base scheme S and Y a closed subset of $|X|$. The *formal completion* of X along Y , denoted by \mathfrak{X} , is

Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined as $\varprojlim_n \mathcal{O}_X / \mathcal{I}^n$ where \mathcal{I} is the ideal sheaf of Y in \mathcal{O}_X . **Yang: to be completed**.

Definition 3.27. Let X be an algebraic space and Y a closed subset of X . A *modification* of X along Y is a proper morphism $f : X' \rightarrow X$ and a closed subset $Y' \subset X'$ such that $X' \setminus Y' \rightarrow X \setminus Y$ is an isomorphism and $f^{-1}(Y) = Y'$.

Theorem 3.28 (ref. [Art70, Theorem 3.1]). Let Y' be a closed subset of an algebraic space X' of finite type over \mathbb{k} . Let \mathfrak{X}' be the formal completion of X' along Y' . Suppose that there is a formal modification $\mathfrak{f} : \mathfrak{X}' \rightarrow \mathfrak{X}$. Then there is a unique modification

$$f : X' \rightarrow X, \quad Y' \subset X'$$

such that the formal completion of X along Y is isomorphic to \mathfrak{X} and the induced morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ is isomorphic to \mathfrak{f} .

Theorem 3.29 (ref. [Art70, Theorem 6.2]). Let \mathfrak{X}' be a formal algebraic space and $Y' = V(\mathcal{I}')$ with \mathcal{I}' the defining ideal sheaf of \mathfrak{X}' . Let $f : Y' \rightarrow Y$ be a proper morphism. Suppose that

(a) for every coherent sheaf \mathcal{F} on \mathfrak{X}' , we have

$$R^1 f_* \mathcal{I}'^n \mathcal{F} / \mathcal{I}'^{n+1} \mathcal{F} = 0, \quad \forall n \gg 0;$$

(b) for every n , the homomorphism

$$f_*(\mathcal{O}_{\mathfrak{X}'} / i'^n) \otimes_{f_*\mathcal{O}_{Y'}} \mathcal{O}_Y \rightarrow \mathcal{O}_Y$$

is surjective.

Then there exists a modification $\mathfrak{f} : \mathfrak{X}' \rightarrow \mathfrak{X}$ and a defining ideal sheaf i of \mathfrak{X} such that $V(i) = Y$ and \mathfrak{f} induces f on Y .

Theorem 3.30 (ref. [Art70, Theorem 6.1]). Let Y' be a closed algebraic subspace of an algebraic space X' and $f_0 : Y' \rightarrow Y$ a finite morphism. Then there exists a modification $f : X' \rightarrow X$ whose restriction to Y' is f_0 . It is the amalgamated sum $X = X' \amalg_{Y'} Y$ in the category of algebraic spaces \mathbf{AlgSp} .

Example 3.31. Let $X = \mathbb{A}^2 = \text{Spec } \mathbb{k}[x, y]$ and $Y = V(y)$ be the x -axis. Let $f_0 : Y' = \mathbb{A}^1 \rightarrow Y, x \mapsto x^2$. Then there exists a modification $f : X' \rightarrow X$ such that the restriction $f|_{Y'} : Y' \rightarrow Y$ is f_0 . **Yang:** To be completed.

3.3 A sufficient and necessary condition for EWM

In this and next subsection, we assume that all schemes (algebraic spaces) are of finite type over a field \mathbb{k} with characteristic $p > 0$.

Lemma 3.32. Let $f : X \rightarrow Y$ be a finite morphism of algebraic space which is of finite type over \mathbb{k} . Suppose that f is a universal homeomorphism. Then there exists $q = p^n$ such that the relative Frobenius morphism $\text{Frob}_{X/\mathbb{k}}^n$ factors as

$$\text{Frob}_{X/\mathbb{k}}^n : X \xrightarrow{f} Y \rightarrow X^{(q)}.$$

Proof. **Yang: I can only prove this for schemes.** Suppose that X, Y are affine. Factor it as $A \twoheadrightarrow B \hookrightarrow C$ with A, B, C \mathbb{k} -algebras.

For $A \twoheadrightarrow B$, let I be the kernel of the surjection. Since $\text{Spec } B \rightarrow \text{Spec } A$ is finite universal homeomorphism, we have that I is a nilpotent ideal. Hence there exists q such that $I^q = 0$. Let $a, a' \in A$ with the same image b in B . Then we have $a^q - a'^q \in I^q = 0$. Hence $a^q = a'^q$ in A . This gives a map $B^q \rightarrow A, b^q \mapsto a^q$.

For $B \hookrightarrow C$, we induct on the dimension. If C is artinian, then $0 = C^q \subset B \subset C$. In general case, this shows that $B \cdot C^{q_1} \subset C$ is an isomorphism at generic points. Let $I := \text{Ann}(B \cdot C^q / B) \subset B$. This is the conductor of extension $B \cdot C^{q_1} \subset C$, whence also an ideal of $B \cdot C^{q_1}$. To see this, for every $x \in B \cdot C^{q_1}$, $b \in I$, we have $xb \in B \cdot C^{q_1} = bB \cdot C^{q_1} \subset B$. By induction hypothesis, we have $(BC^{q_1}/I)^{q_2} \subset B/I$. For $x \in BC^{q_1}$, there exists $b \in B$ and $\delta \in I \subset B$ such that $x^{q_2} = b + \delta \in B$. Hence we have $(BC^{q_1})^{q_2} \subset B$. In particular, we have $C^{q_1 q_2} \subset (B \cdot C^{q_1})^{q_2} \subset B$.

In general case, we have

$$\begin{array}{ccccc} C^{q_1 q_2} & \longrightarrow & A' & \twoheadrightarrow & C^{q_1} \\ & & \downarrow & & \downarrow \\ & & A & \twoheadrightarrow & B \hookrightarrow C \end{array},$$

where A' is the preimage of C^{q_1} in A . One we have $C^q \rightarrow A \rightarrow C$, note that $A \rightarrow C$ is over \mathbb{k} , then it gives

$$C^q \rightarrow C^{(q)} \rightarrow A \rightarrow C.$$

□

Corollary 3.33. Let $Z \rightarrow X$ be a finite universal homeomorphism of algebraic spaces and $Z \rightarrow Y$ any finite morphism of algebraic spaces. Suppose that X, Y, Z are all of finite type over \mathbb{k} . Then the amalgamated sum $X \amalg_Z Y$ exists in the category of algebraic spaces. Moreover, $Y \rightarrow X \amalg_Z Y$ is a finite universal homeomorphism.

Proof. By Lemma 3.32, we have a diagram

$$\begin{array}{ccc} Y^{(q)} & \longleftarrow & Y \\ \uparrow & & \uparrow \\ Z^{(q)} & & g \\ \uparrow & & \uparrow \\ X & \xleftarrow{f} & Z \end{array}.$$

Denote $X \rightarrow Y^{(q)}$ by f . Let

$$\mathcal{a} := \text{Ker}(\sigma_X \times \sigma_Y \rightarrow \sigma_Z, \quad (s, t) \mapsto f^*s - g^*t).$$

Then \mathcal{a} is an $\sigma_{Y^{(q)}}$ -algebra. Set $W := \text{Spec}_{Y^{(q)}} \mathcal{a}$. Then $W = X \amalg_Z Y$ is the amalgamated sum in the category of algebraic spaces. **Yang: The most important point is that $Z \rightarrow W$ is finite. Yang: At least in the cat of schemes.** □

Proposition 3.34. Let $g : X' \rightarrow X$ be a proper, finite universal homeomorphism between algebraic spaces. Then a line bundle ℓ on X is endowed with a map if and only if $g^*\ell$ is endowed with a map.

Proof. Let $f : X' \rightarrow Z$ be the map endowed on $g^*\ell$. By Lemma 3.32, we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow f & & \downarrow \\ Z & \longrightarrow & Z^{(q)} \end{array}.$$

Easy to check that $X \rightarrow Z^{(q)}$ is a map associated to ℓ . □

Proposition 3.35. Let X be a projective scheme and ℓ a nef line bundle on X . Assume that $X = X_1 \cup X_2$ for closed subsets X_1 and X_2 . Suppose that $\ell|_{X_i}$ is endowed with a fibration $g_i : X_i \rightarrow Z_i$ for $i = 1, 2$. Then ℓ is endowed with a map $g : X \rightarrow Z$.

Proof. Let $X_{12} := X_1 \cap X_2$. Let $X_{12} \rightarrow Z_{12}$ be the Stein factorization of the map $g_1|_{X_{12}}$. Then by [Yang: Rigidity Lemma](#), it is also the Stein factorization of the map $g_2|_{X_{12}}$. Denote Y_i be the image of Z_{12} in Z_i for $i = 1, 2$. Then we have a commutative diagram

$$\begin{array}{ccccc}
 & & Z_1 & & \\
 & & \uparrow & \swarrow h' & \\
 & X & \longleftarrow X_1 & & Y_1 \\
 & \uparrow & \uparrow & & \uparrow h \\
 Z_2 & \longleftarrow X_2 & \longleftarrow X_{12} & & Z_{12} \\
 & \searrow f & \searrow & \searrow & \\
 & Y_2 & & &
 \end{array}$$

Consider the sub-diagram

$$\begin{array}{ccc}
 & Z_1 & \\
 & \uparrow h' & \\
 & Y_1 & \\
 & \uparrow h & \\
 Z_2 & \xleftarrow{f} & Z_{12}
 \end{array}$$

Here f is finite, h is finite universal homeomorphism and h' is a closed immersion. By [Corollary 3.33](#), we have the amalgamated sum $Z' := Y_1 \amalg_{Z_{12}} Z_2$ exists in the category of algebraic spaces. Since f is finite, so is the induced morphism $Y_1 \rightarrow Z'$. Then by [Theorem 3.30](#), the amalgamated sum $Z := Z' \amalg_{Y_1} Z_1$ exists in the category of algebraic spaces.

Then we have a commutative diagram

$$\begin{array}{ccccc}
 Z & \longleftarrow & & Z_1 & \\
 \uparrow & \swarrow g & & \uparrow & \\
 & X & \longleftarrow & X_1 & \\
 \uparrow & \uparrow & & \uparrow & \\
 Z_2 & \longleftarrow X_2 & \longleftarrow & X_{12} &
 \end{array}$$

Directly check shows that g is a map associated to ℓ . □

Proposition 3.36. Let X be a projective scheme and D a nef and big divisor on X . Then we can write $D = A + E$ where A is an ample divisor and E is an effective divisor. Then D is endowed with a map iff $D|_{E_{red}}$ is endowed with a map.

Proof. By [Proposition 3.34](#), we may assume that $D|_E$ is endowed with a map $f : E \rightarrow Z$. Let $\ell = \mathcal{O}_X(-E)$ be the ideal sheaf of E . note that $-E = A - D$ and D is f -numerically trivial. Hence $\ell|_E$ is f -ample. By Serre's vanishing, for every coherent sheaf f on X , there exists $n_0 \in \mathbb{m}$ such that for all $n \geq n_0$, we have

$$R^i f_* f|_E \otimes \ell|_E^{\otimes n} = 0$$

for all $i > 0$. In particular, let $n \in \mathbb{z}$ such that $R^i f_* \mathcal{O}_X / \ell \otimes \ell^{\otimes m} = 0$ for all $i > 0, m \geq n$. Set $i := \ell^{\otimes n}$. Then by the exact sequence

$$0 \rightarrow \ell^{n-1} \otimes \mathcal{O}_X / \ell \rightarrow \mathcal{O}_X / \ell^n \rightarrow \mathcal{O}_X / \ell \rightarrow 0,$$

we have that $R^i f_*(\mathcal{O}_X / i \otimes i^t) = 0$ for all $i > 0, t \geq 1$. This implies that $f_* \mathcal{O}_X / i^t \rightarrow f_* \mathcal{O}_X / i$ is surjective for all $t \geq 1$.

Let

$$\begin{aligned} \mathfrak{a} &:= \mathcal{O}_X \oplus iT \oplus i^2 T^2 \oplus \dots, \\ \mathfrak{m} &:= f \oplus ifT \oplus i^2 fT^2 \oplus \dots, \end{aligned}$$

where T is a formal variable to denote the grading. Then \mathfrak{a} is a graded \mathcal{O}_X -algebra of finite type and \mathfrak{m} is a finite graded \mathfrak{a} -module. We have an exact sequence of graded \mathfrak{a} -modules

$$0 \rightarrow \mathfrak{k} \rightarrow \mathfrak{m} \otimes_{\mathfrak{a}} iT \rightarrow \mathfrak{m} \rightarrow 0,$$

where $\mathfrak{k} = \bigoplus \mathfrak{k}_r T^r$ is a finite graded \mathfrak{a} -module. Hence for $r \gg 1$, we have that $iT \cdot \mathfrak{k}_r T^r = \mathfrak{k}_{r+1} T^{r+1}$. It implies that the image of $\mathfrak{k}_{r+1} T^{r+1} \rightarrow \mathfrak{m}_r T^r \otimes_{\mathfrak{a}} iT$ is contained in $i\mathfrak{m}_r$ for all $r \gg 1$. Tensor with $\mathfrak{a} \otimes_{\mathcal{O}_X} \mathcal{O}_X / i$, we have that

$$\mathfrak{k}_{r+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X / i \rightarrow 0 \rightarrow \mathfrak{m}_r \otimes_{\mathcal{O}_X} i \otimes_{\mathcal{O}_X} \mathcal{O}_X / i \rightarrow \mathfrak{m}_{r+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X / i \rightarrow 0.$$

That is, $i^r f / i^{r+1} f \otimes_{\mathcal{O}_X / i} i / i^2 \cong i^{r+1} f / i^{r+2} f$ for all $r \gg 1$. Hence we have that

$$R^i f_*(i^{r-1} f / i^r f) = 0$$

for all $i > 0, r \gg 1$.

Let $E' := V(i)$, we have that $D|_{E'}$ is endowed with a map $f' : E' \rightarrow Z'$ by [Proposition 3.34](#). Moreover, we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & Z \\ \downarrow & & \downarrow g \\ E' & \xrightarrow{f'} & Z' \end{array}$$

with g finite. Then by Grothendieck Spectral Sequence, we have that

$$R^i f'_*(i^{r-1} f / i^r f) = 0$$

for all $i > 0, r \gg 1$.

Then we can apply [Theorems 3.28](#) and [3.29](#) to get a modification $X \rightarrow Y$. Note that $\text{Exc } D \subset \text{Supp } E$. It follows that $X \rightarrow Y$ is a map associated to D . \square

Theorem 3.37. Let X be a proper variety and ℓ a nef line bundle on X . Then ℓ is endowed with a map if and only if $\ell|_{\text{Exc}} \ell$ is endowed with a map.

Proof. By Proposition 3.35, we can assume that ℓ is big. Then the result follows from Proposition 3.36 and induction on dimension. \square

3.4 For semiample

Lemma 3.38. Let X be a projective scheme over $\mathbb{k} = \overline{\mathbb{F}_p}$. Then ℓ is numerically trivial if and only if ℓ is torsion in $\text{Pic}(X)$.

Proof. Let T be the scheme in Theorem 3.2. Then ℓ corresponds to a \mathbb{F}_q -point of T . Note that there are only finitely many \mathbb{F}_q -points in T . Hence ℓ is torsion in $\text{Pic}(X)$. \square

Proposition 3.39. Let $f : X \rightarrow Y$ be a finite universal homeomorphism between algebraic spaces of finite type over \mathbb{k} and ℓ a line bundle on Y . Then there exists $q = p^n$ such that

- (a) for every section $s \in H^0(X, f^*\ell)$, we have $s^q \in \mathfrak{I}(H^0(Y, \ell^{\otimes q}) \rightarrow H^0(X, f^*\ell^{\otimes q}))$;
- (b) ℓ is semiample if and only if $f^*\ell$ is semiample;
- (c) the map

$$f^* : \text{Pic}(Y) \otimes \mathbb{Z}[1/q] \rightarrow \text{Pic}(X) \otimes \mathbb{Z}[1/q]$$

is an isomorphism;

- (d) if $f^*s_1 = f^*s_2$ for two sections $s_1, s_2 \in H^0(Y, \ell)$, then $s_1^q = s_2^q$ in $H^0(X, \ell^{\otimes q})$.

Proof. Note that $\text{Frob}^* \ell \cong \ell^{\otimes p}$. Then all the properties follows from Lemma 3.32. \square

Proposition 3.40. Let X be a projective scheme and ℓ a nef line bundle on X . Assume that $X = X_1 \cup X_2$ for closed subsets X_1 and X_2 . Suppose that $\ell|_{X_i}$ is semiample for $i = 1, 2$. Then ℓ is semiample.

Proof. Yang: To be learned. \square

Lemma 3.41. Let $f : X \rightarrow Y$ be a proper map between algebraic spaces with $f_*\mathcal{O}_X = \mathcal{O}_Y$ and ℓ a line bundle on X . Let $D = V(\mathfrak{i}) \subset X$ be a closed subspace defined by an ideal sheaf \mathfrak{i} , $Z = f(D)$ and $D_k := V(\mathfrak{i}^k)$. Suppose that f is a modification with respect to D, Z and $R^1f_*\mathfrak{i}^k/\mathfrak{i}^{k+1} = 0$ for all $k \gg 0$. Suppose for every k , there exists $r > 0$ such that $\ell^{\otimes r}|_{D_k}$ is pulled back from $f(D_k)$. Then $\ell^{\otimes r}$ is pulled back from Y for some $r > 0$.

Proof. Replace D by D_k and ℓ by $\ell^{\otimes r}$ for some $k, r > 0$, we can assume that $R^1f_*\mathfrak{i}^k/\mathfrak{i}^{k+1} = 0$ for all k and $\ell|_D$ is pulled back from $f(D)$. Then we show that $f_*\ell$ is a line bundle and $f^*f_*\ell \cong \ell$. Both of them are local, so we can assume that $X = \text{Spec } B, Z = \text{Spec } A$ are spectrum of local rings. Hence $\ell|_{D_k}$ is trivial for all k . By vanishing of $R^1f_*\mathfrak{i}^k/\mathfrak{i}^{k+1}$, we have a surjection $H^0(D_{k+1}, \ell|_{D_{k+1}}) \twoheadrightarrow H^0(D_k, \ell|_{D_k})$

for all k . This allow us to choose a section $s_k \in H^0(D_k, \ell|_{D_k})$ such that $s_k = s_{k+1}|_{D_k}$ for all k . Then we have a section $s \in H^0(D, \ell|_D)$ such that $s|_{D_k} = s_k$ for all k . By Nakayama's Lemma, we can assume that s_k is nowhere vanishing. **Yang: To be completed.** \square

Proposition 3.42. Let X be a projective scheme and D a nef and big divisor on X . Then we can write $D = A + E$ where A is an ample divisor and E is an effective divisor. Then D is semiample iff $D|_{E_{\text{red}}}$ is semiample.

Proof. **Yang: To be completed.** \square

Theorem 3.43. Let X be a proper variety and ℓ a nef line bundle on X . Then ℓ is semiample if and only if $\ell|_{E_{\text{xc}} \ell}$ is semiample.

Proof. **Yang: To be completed.** \square

3.5 Basepoint free theorem on positive characteristic

Proposition 3.44 (ref. **Yang:**). Let $T \subset X$ be a reduced Weil divisor on a normal variety X . Let $T^\vee \rightarrow T$ be the normalization, $C \subset T^\vee$ the effective Weil divisor defined by the conductor and $p : T^\vee \rightarrow T \hookrightarrow X$ the composition. Suppose that $K_X + T$ is \mathbb{Q} -Cartier. Then there exists an effective \mathbb{Q} -Weil divisor D on T^\vee such that

$$K_{T^\vee} + C + D = p^*(K_X + T).$$

Theorem 3.45. Let X be a normal projective \mathbb{Q} -factorial threefold and $B \in (0, 1)$ a \mathbb{Q} -divisor. Let ℓ be a nef and big line bundle on X such that $\ell - K_{(X,B)}$ is nef and big. Then ℓ is endowed with a map. Moreover, if $\mathbb{k} = \overline{\mathbb{F}_p}$, ℓ is semiample.

Proof. Let $\ell = \sigma_X(A + E)$ with A an ample divisor and E an effective divisor. Write $E = E_0 + E_1 + E_2$ such that the restriction of ℓ to every irreducible component of E_i is of numerical dimension i . Let $S := \text{Supp } E_1$ and $S = \sum S_i$ with S_i irreducible components. Let $S^\vee \rightarrow S$ and $S_i^\vee \rightarrow S_i$ be the normalizations.

Step 1. Reduce to show that $\ell|_S$ is endowed with a map (semiample).

Yang: To be completed.

Step 2. Reduce to show that $\ell|_{S_i^\vee}$ is endowed with a map (semiample).

Yang: To be completed.

Step 3. Show that $\ell|_{S_i^\vee}$ is endowed with a map (semiample).

Yang: To be completed. \square

4 F-singularities

Let \mathbb{k} be an algebraically closed field of characteristic $p > 0$. Let X be a projective variety over \mathbb{k} . Let F denote the relative Frobenius morphism on X .

Definition 4.1. We say that X is *F-finite* if $F : X \rightarrow X^{(p)}$ is finite.

Definition 4.2. We say that X is *globally F-split* if $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ splits as \mathcal{O}_X -modules for some $e \geq 0$. This is equivalent to for every $e \in \mathbb{Z}_{>0}$, $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ splits as \mathcal{O}_X -modules.

Definition 4.3. Fix $\phi : F_*^e L \rightarrow \mathcal{O}_X$ a splitting of $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$. Define $\phi^n : F_*^{ne} L^{1+p^e+\dots+p^{(n-1)e}} \rightarrow \mathcal{O}_X$ by induction:

$$\phi^n := \phi \circ F_*^e(\phi^{n-1}).$$

Theorem 4.4. Above ϕ^n will be stable. That is, $\mathfrak{I}\phi^n = \mathfrak{I}\phi^{n+1}$ for all $n \gg 0$.

Definition 4.5. Let $\sigma(X, \phi) := \mathfrak{I}\phi^n$. We say that (X, ϕ) is *F-pure* if $\sigma(X, \phi) = \mathcal{O}_X$.

Proposition 4.6. There is a bijection between

$$\{\text{effective } \mathbb{Q}\text{-divisor } \Delta \text{ such that } (p^e - 1)(K_X + \Delta) \text{ is Cartier}\} / \sim$$

and

$$\{\text{line bundles } \ell \text{ and } \phi : F_*^e \ell \rightarrow \mathcal{O}_X\}.$$

Proof. We have

$$F_X^e \mathcal{O}_X((1 - p^e)K_X) \rightarrow \mathcal{O}_X$$

given by $F^e \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_X(K_X)$ and reflexivity of $\mathcal{O}_X(K_X)$. Since Δ is effective, we have

$$F^e(\mathcal{O}_X((1 - p^e)(K_X + \Delta))) \rightarrow F^e \mathcal{O}_X((1 - p^e)(K_X)) \rightarrow \mathcal{O}_X.$$

The another direction is by Grothendieck's duality

$$\mathcal{H}om_{\mathcal{O}_X}(F^e \ell, \mathcal{O}_X) \cong F_*^e(\ell^{-1} \otimes \mathcal{O}_X((1 - p^e)K_X)).$$

□

Definition 4.7. Let $\phi_{e,\Delta} : F_*^e(\mathcal{O}_X((1 - p^e)(K_X + \Delta))) \rightarrow \mathcal{O}_X$ be the morphism corresponding to the effective \mathbb{Q} -divisor Δ .

We say that (X, Δ) is *F-pure* if $(X, \phi_{e,\Delta})$ is *F-pure*.

We say that (X, Δ) is *globally F-split* if for every Weil divisor $D \geq 0$, $\mathcal{O}_X \rightarrow F_*^e(\mathcal{O}_X([(p^e - 1)\Delta] + D))$ admits a splitting for some $e \geq 0$.

We say that (X, Δ) is *strongly F-split* if for every Weil divisor $D \geq 0$, $\mathcal{O}_X \rightarrow F_*^e(\mathcal{O}_X([(p^e - 1)\Delta] + D))$

admits a local splitting for some $e \geq 0$.

Definition 4.8.

Definition 4.9. $S^0(X, \sigma(X, \Delta) \otimes m)$

Proposition 4.10. Let X be a globally F -split projective variety. Then we have

- (a) suppose that $H^i(X, \ell^n) = 0$ for all $i > 0$ and all $n \gg 0$, then $H^i(X, \ell) = 0$ for all $i > 0$;
- (b) for every ample divisor A on X , we have $H^i(X, \mathcal{O}_X(A)) = 0$ for all $i > 0$;
- (c) suppose that X is Cohen-Macaulay and A -ample, then $H^i(X, \mathcal{O}_X(-A)) = 0$ for all $i < \dim X$;
- (d) suppose that X is normal and A -ample, then $H^i(X, \omega_X(A)) = 0$ for all $i > 0$.

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