First properties of algebraic groups

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. All varieties are defined over $\mathbf{k}$ unless otherwise specified.

1 Basic concepts

Definition 1. A *group scheme* over S is an S-scheme G together with morphisms *multiplication* $\mu : G \times G \to G$, *identity* $\varepsilon : S \to G$ and *inversion* $\iota : G \to G$ over S such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd}
 & G \times G \times G \arrow[dl, "\mathrm{id}_G \times \mu"'] \arrow[dr, "\mu \times \mathrm{id}_G"] & \\
G \times G \arrow[dr, "\mu"] & & G \times G \arrow[dl, "\mu"'] \\
 & G &
\end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd}
G \times S \arrow[r, "\mathrm{id}_G \times \varepsilon"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] & S \times G \arrow[l, "\varepsilon \times \mathrm{id}_G"'] \arrow[dl, "\cong"] \\
 & G &
\end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd}
 & G \arrow[dl, "\mathrm{id}_G \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_G"] & \\
G \times G \arrow[dr, "\mu"'] & S \arrow[d, "\varepsilon"] & G \times G \arrow[dl, "\mu"] \\
 & G &
\end{tikzcd}.$$

In other words, a group scheme is a group object in the category of schemes.

Definition 2. An *algebraic group* is a $\mathbf{k}$-group scheme G which is reduced, separated and of finite type over a field $\mathbf{k}$.

Remark 3. Even if we work over $\Bbbk$ and just consider the closed points $G(\Bbbk)$ of an algebraic group G, $G(\Bbbk)$ is not a topological group with respect to the Zariski topology in general. The reason is that the topology on $G(\Bbbk) \times G(\Bbbk)$ is not the product topology of the topologies on $G(\Bbbk)$.

Definition 4. Let G be an algebraic group and $x \in G(\mathbf{k})$ a $\mathbf{k}$-point. The *left translation* by x is the morphism

$$l_x : G \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{\mu} G.$$

Similar definition applies to the right translation r_x.

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Remark 5. In the context of algebraic groups, we often use multiplicative notation for the group law. That is, for $g, h \in G(\mathbf{k})$, we write gh instead of $\mu(g,h)$ and g^{-1} instead of $\iota(g)$.

Sometimes we also abuse the notation by $\mu : G \times \cdots \times G \to G$ to denote the multiplication of multiple elements, i.e. $\mu(g_1, \dots, g_n) = g_1 \cdots g_n$ for $g_1, \dots, g_n \in G(\mathbf{k})$.

Proposition 6. Let G be an algebraic group. Then G is smooth over $\mathbf{k}$.

Proof. Since G is reduced and of finite type over a field, it is generically regular. Let $g \in G(\Bbbk)$ be a regular point. Then the left translation $l_{gh^{-1}} : G \to G$ is an isomorphism, hence G is regular at $h \in G(\Bbbk)$. It follows that G is regular at every $\Bbbk$-point, hence G is smooth over $\mathbf{k}$. □

Remark 7. Let G be an algebraic group. Then the irreducible components of G coincide with the connected components of G. We will use the term "connected" to refer to both concepts since "irreducible" has other meanings in the theory of representations.

Example 8. The *additive group* $\mathbb{G}_a$ is defined to be the affine line $\mathbb{A}^1$ with the group law given by addition. Concretely, we can write $\mathbb{G}_a = \operatorname{Spec} \mathbf{k}[T]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a, &\quad (x,y) \mapsto x+y, \\ \iota : \mathbb{G}_a \to \mathbb{G}_a, &\quad x \mapsto -x, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_a, &\quad * \mapsto 0. \end{aligned}$$

Example 9. The *multiplicative group* $\mathbb{G}_m$ is defined to be the affine variety $\mathbb{A}^1 \setminus \{0\}$ with the group law given by multiplication. Concretely, we can write $\mathbb{G}_m = \operatorname{Spec} \mathbf{k}[T, T^{-1}]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m, &\quad (x,y) \mapsto xy, \\ \iota : \mathbb{G}_m \to \mathbb{G}_m, &\quad x \mapsto x^{-1}, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_m, &\quad * \mapsto 1. \end{aligned}$$

Example 10. The *general linear group* GL_n is defined to be the open subvariety of $\mathbb{A}^{n^2}$ consisting of invertible matrices, with the group law given by matrix multiplication. Concretely, we can write $\mathrm{GL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}, \det(T_{ij})^{-1}]$ where $1 \le i, j \le n$ and the group law is given by the morphism

$$\begin{aligned} \mu : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, &\quad (A,B) \mapsto AB, \\ \iota : \mathrm{GL}_n \to \mathrm{GL}_n, &\quad A \mapsto A^{-1}, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathrm{GL}_n, &\quad * \mapsto I_n. \end{aligned}$$

Example 11. An abelian variety is an algebraic group that is also a proper variety.

Example 12. Let G and H be algebraic groups. The *product* $G \times H$ is an algebraic group with the group law defined by

$$\begin{aligned} \mu_{G\times H} = \mu_G \times \mu_H &: (G \times H) \times (G \times H) \cong (G \times G) \times (H \times H) \to G \times H, \\ \varepsilon_{G\times H} = \varepsilon_G \times \varepsilon_H &: \operatorname{Spec} \mathbf{k} \cong \operatorname{Spec} \mathbf{k} \times \operatorname{Spec} \mathbf{k} \to G \times H, \\ \iota_{G\times H} = \iota_G \times \iota_H &: G \times H \to G \times H. \end{aligned}$$

Example 13. Let G be an algebraic group over $\mathbf{k}$ and $\mathbf{K}/\mathbf{k}$ a field extension. The base change $G_{\mathbf{K}} = G \times_{\operatorname{Spec}\mathbf{k}} \operatorname{Spec}\mathbf{K}$ is an algebraic group over $\mathbf{K}$ with the group law defined by the base change of the original group law of G to $\mathbf{K}$.

Definition 14. A *homomorphism* of algebraic groups is a morphism of schemes that is also a group homomorphism. Explicitly, a morphism $f : G \to H$ between algebraic groups G and H is a homomorphism if the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

where μ_G and μ_H are the group laws of G and H, respectively.

Definition 15. An *algebraic subgroup* of an algebraic group G is a closed subscheme $H \subseteq G$ that is also a subgroup of G. More precisely, H is an algebraic subgroup and the inclusion morphism $H \hookrightarrow G$ is compatible with the group laws.
An algebraic subgroup H of G is called *normal* if for any $\mathbf{k}$-scheme S, the subgroup $H(S)$ is a normal subgroup of the abstract group $G(S)$.

Example 16. The *special linear group* SL_n is defined to be the closed subvariety of GL_n defined by the equation $\det = 1$. It is an algebraic subgroup of GL_n.

Proposition 17. Let G be an algebraic group and S is a closed subgroup of $G(\Bbbk)$. Then there exists a unique algebraic subgroup H of G such that $H(\Bbbk) = S$.

Proof. Yang: To be continued... □

Remark 18. By Proposition 17, we often identify an algebraic group G with its set of closed points $G(\Bbbk)$ when there is no confusion.

Remark 19. If one replaces $\Bbbk$ by $\mathbf{k}$ in Proposition 17, the statement may not hold. For example, let $\mathbf{k} = \mathbb{Q}$ and G be the elliptic curve defined by $X^3+Y^3 = Z^3$ in $\mathbb{P}^2$. It is well-known that $\#G(\mathbb{Q}) = 3$. Let S be the disjoint union of the three $\mathbb{Q}$-points of G endowed with the reduced subscheme structure and the group structure induced from G. Then S is a proper closed subgroup of G and we have $S(\mathbb{Q}) = G(\mathbb{Q})$. This contradicts the uniqueness in Proposition 17.

Indeed, in this chapter, despite working over an arbitrary field $\mathbf{k}$, we mostly consider the closed points of algebraic groups over $\Bbbk$.

Definition 20. Let G be an algebraic group. The *neutral component* G^0 is the connected component of G containing the identity element ε.

Proposition 21. The neutral component G^0 is a closed, normal algebraic subgroup of G.

Proof. Yang: To be continued... □

Proposition 22. Let G be an algebraic group and $H \subseteq G(\Bbbk)$ a subgroup (not necessarily closed). Then the Zariski closure $\overline{H}$ of H in G is an algebraic subgroup of G. If $H \subset G(\Bbbk)$ is constructible, then $H = \overline{H}(\Bbbk)$.

Proof. Yang: To be continued... □

Example 23. Let $G = \mathrm{SL}_2$ over $\Bbbk$, $T = \{\mathrm{diag}(t, t^{-1}) | t \in \Bbbk^\times\}$ and $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Set $S = gTg^{-1}$. Then both T and S are closed algebraic subgroups of $G(\Bbbk)$, but the product TS is not closed in $G(\Bbbk)$. By direct computation, we have

$$S = \left\{ \begin{pmatrix} s & s^{-1} - s \\ 0 & s^{-1} \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

Then

$$TS = \left\{ \begin{pmatrix} ts & t(s^{-1} - s) \\ 0 & (ts)^{-1} \end{pmatrix} \middle| t, s \in \Bbbk^\times \right\}.$$

We have

$$TS \cap \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \Bbbk \right\} = \left\{ \begin{pmatrix} 1 & s^{-2} - 1 \\ 0 & 1 \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

The right hand side is not closed in $\mathrm{SL}_2(\Bbbk)$ since it does not contain the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Hence TS is not closed in $G(\Bbbk)$.

Proposition 24. Let G be an algebraic group, X_i varieties over $\mathbf{k}$ and $f_i : X_i \to G$ morphisms for $i = 1, \dots, n$ with images $Y_i = f_i(X_i)$. Suppose that Y_i pass through the identity element of G. Let H be the closed subgroup of G generated by $Y_1, \dots, Y_n$, i.e. the smallest closed subgroup of G containing $Y_1, \dots, Y_n$. Then H is connected and $H = Y_{a_1}^{e_1} \cdots Y_{a_m}^{e_m}$ for some $a_1, \dots, a_m \in \{1, \dots, n\}$ and $e_1, \dots, e_m \in \{\pm 1\}$.

Proof. Yang: To be continued... □

Remark 25. We can take $m \le 2 \dim G$ in Proposition 24.

2 Action and representations

Definition 26. An *action* of an algebraic group G on a variety X is a morphism

$$\sigma : G \times X \to X$$

such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \sigma"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} \mathbf{k} \times X \arrow[r, "\varepsilon \times \mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

where μ is the group law of G and ε is the identity element of G. In other words, for any $\mathbf{k}$-scheme S, the induced map $G(S) \times X(S) \to X(S)$ defines a group action of the abstract group $G(S)$ on the set $X(S)$.

For simplicity, we often write $g.x$ instead of $\sigma(g, x)$ for $g \in G(\mathbf{k})$ and $x \in X(\mathbf{k})$.

Example 27. There are three natural actions of an algebraic group G on itself:

(a) Left translation: $g.h = l_g(h) = gh$;

(b) Right translation: $g.h = r_g(h) = hg^{-1}$;

(c) Conjugation: $g.h = \mathrm{Ad}_g(h) = ghg^{-1}$.

All of them are morphisms of varieties since they are defined by the group law and inversion of G.

Example 28. The general linear group GL_n acts on the affine space $\mathbb{A}^n$ by matrix multiplication. It is given by polynomials, hence is a morphism of varieties.

Example 29. The general linear group GL_{n+1} acts on the projective space $\mathbb{P}^n$ by

$$A \cdot [x_0 : \cdots : x_n] = [y_0 : \cdots : y_n], \quad \text{where } (y_0, \dots, y_n)^T = A(x_0, \dots, x_n)^T.$$

Let U_i be the standard affine open subset of $\mathbb{P}^n$ defined by $x_i \neq 0$. The map is given by polynomials on the principal open subset of $\mathrm{GL}_{n+1} \times U_i$ defined by $y_j \neq 0$ for any j. Hence it is a morphism of varieties.

Definition 30. A *linear representation* of an algebraic group G on a finite-dimensional vector space V over $\Bbbk$ is an abstract group representation $\rho : G(\Bbbk) \to \mathrm{GL}(V)$ such that if we identify V with $\mathbb{A}^n$ for some n, then the map $G(\Bbbk) \times \mathbb{A}^n(\Bbbk) \to \mathbb{A}^n(\Bbbk)$ is is a morphism of varieties.

Definition 31. Let G be an algebraic group acting on a variety X. For any $x \in X(\mathbf{k})$, the *orbit* of x is the locally closed subvariety $G \cdot x = \sigma(G \times \{x\})$ of X.

Proposition 32. Let G be an algebraic group acting on a variety X. Then for any $x \in X(\mathbf{k})$, the orbit $G \cdot x$ is a locally closed subvariety of X, and $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of strictly smaller dimension.

Proof. Yang: To be continued... □

Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For $x \in G(\mathbf{k})$, we have the left translation of functions $\tau_x : A \to A$ defined by $\tau_x(f)(y) = f(x^{-1}y)$ for $y \in X(\mathbf{k})$.

Lemma 33. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For any finite-dimensional subspace $V \subseteq A$, there exists a finite-dimensional G-invariant subspace $W \subseteq A$ containing V.

Proof. Yang: To be continued... □

Theorem 34. Any affine algebraic group is isomorphic to a closed algebraic subgroup of some GL_n.

Proof. Yang: To be continued... □

3 Lie algebra of an algebraic group

Let G be an algebraic group. The *Lie algebra* of G is defined to be the tangent space of G at the identity element ε:

$$\mathrm{Lie}(G) = T_\varepsilon G.$$

It is a finite-dimensional vector space over $\mathbf{k}$.

Proposition 35. The group law $\mu : G \times G \to G$ induces the plus map on $\mathrm{Lie}(G)$:

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)} : T_{(\varepsilon,\varepsilon)}(G \times G) \cong T_\varepsilon G \oplus T_\varepsilon G \to T_\varepsilon G, \quad (v, w) \mapsto v + w.$$

Proof. We have

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v,w) = \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v,0) + \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(0,w) = (\mathrm{d}\mu \circ (\mathrm{id}_G \times \varepsilon))_\varepsilon(v) + (\mathrm{d}\mu \circ (\varepsilon \times \mathrm{id}_G))_\varepsilon(w) = v + w.$$

□

Preliminaries

Definition 36. Let X be a scheme with underlying topological space $|X|$. The family $\mathfrak{C}$ of constructible sets in $|X|$ is the smallest family of subsets of $|X|$ that contains all open subsets and is closed under finite intersections, finite unions, and complements. A subset $E \subseteq |X|$ is called a *constructible set* if $E \in \mathfrak{C}$.

Theorem 37. Let $f : X \to Y$ be a morphism of varieties. Then the image of f is a constructible set in Y.

Lemma 38. Let X and Y be varieties over a field $\mathbf{k}$. For any point $x \in X(\mathbf{k})$ and $y \in Y(\mathbf{k})$, there is a natural isomorphism of $\mathbf{k}$-vector spaces

$$T_{(x,y)}(X \times Y) \cong T_x X \oplus T_y Y$$

given by $v \mapsto (d\pi_1(v), d\pi_2(v))$, where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projection morphisms.

Proof. The inverse map is given by $(u, w) \mapsto d(\iota_1)(u) + d(\iota_2)(w)$, where $\iota_1 : X \cong X \times \{y\} \to X \times Y$ and $\iota_2 : Y \cong \{x\} \times Y \to X \times Y$ are the natural inclusions. □