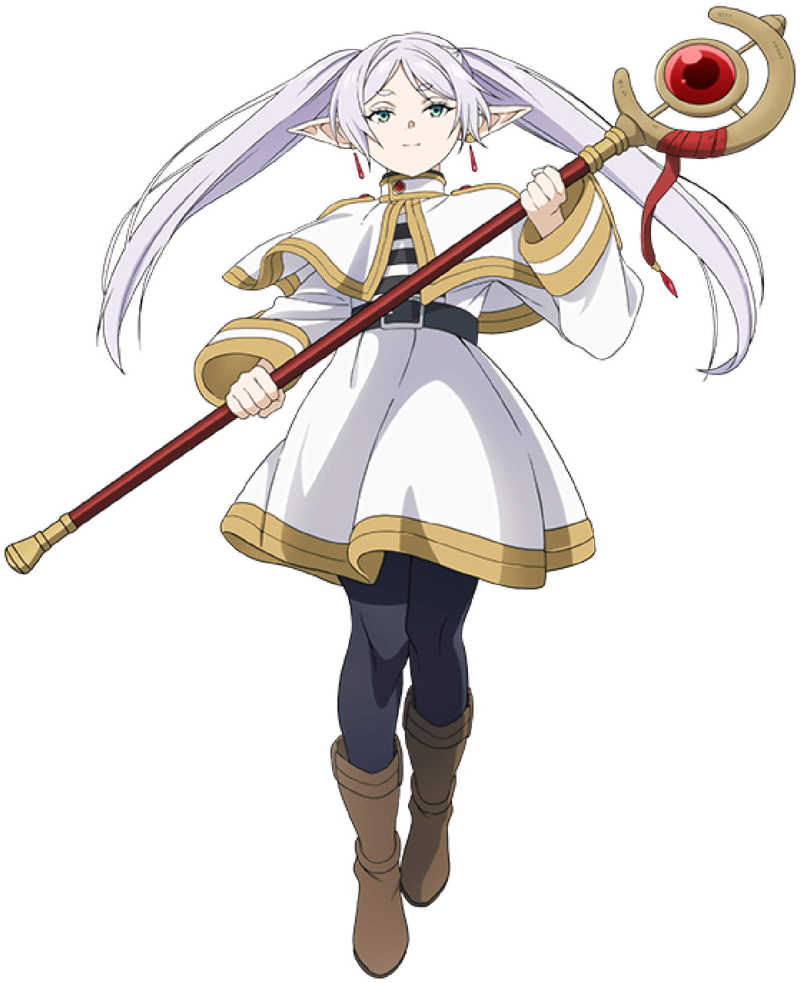
Normal, Cohen-Macaulay and regular schemes



如果是勇者辛美尔，他一定会这么做的！

Normal, Cohen-Macaulay and regular schemes

1 Height, Depth and Dimension Yang: To be completed

Krull dimension and height of prime ideals Algebraically, we have the following definitions.

Definition 1. Let A be a noetherian ring. The *height of a prime ideal* $\mathfrak{p}$ in A is defined as the maximum length of chains of prime ideals contained in $\mathfrak{p}$, that is,

$$\mathrm{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as

$$\dim A := \max_{\mathfrak{p} \in \operatorname{Spec} A} \mathrm{ht}(\mathfrak{p}).$$

Geometrically, we have the corresponding definition.

Definition 2. Let X be a noetherian scheme. The *codimension of an irreducible subscheme* Y in X is defined as the length of the longest chain of irreducible closed subsets containing Y, that is,

$$\mathrm{codim}_X(Y) := \sup\{n \mid \exists \text{ a chain of irreducible closed subsets } Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n\}.$$

The *dimension* of X is defined as

$$\dim X := \max_{\xi \in X} \mathrm{codim}_X Z_\xi.$$

For an affine scheme $X = \operatorname{Spec} A$, above two definitions coincide by the correspondence of prime ideals and irreducible closed subsets.

Proposition 3. Let A be a noetherian ring and $\mathfrak{p} \in \operatorname{Spec} A$. Then

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{codim}_{\operatorname{Spec} A} V(\mathfrak{p}) = \dim A_{\mathfrak{p}}.$$

Lemma 4. Let $A \subset B$ be noetherian rings such that B is finite over A. Then the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Proof. For $\mathfrak{p} \in \operatorname{Spec} A$, let $S := A - \mathfrak{p}$ and denote $S^{-1}B$ by $B_{\mathfrak{p}}$. Then we have $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$ is finite over $A_{\mathfrak{p}}$. Let $\mathfrak{P}B_{\mathfrak{p}}$ be a maximal ideal of $B_{\mathfrak{p}}$. We claim that $\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$ is maximal. Indeed, consider $A_{\mathfrak{p}}/(\mathfrak{P} \cap A_{\mathfrak{p}}) \hookrightarrow B_{\mathfrak{p}}/\mathfrak{P}B_{\mathfrak{p}}$, the latter is finite over the former. This enforces $A_{\mathfrak{p}}/(\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}})$ be a field. Hence $\mathfrak{P}B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, and then $\mathfrak{P} \cap A = \mathfrak{p}$. ☐

Proposition 5. Let $A \subset B$ be noetherian rings such that B is finite over A. Then $\dim A = \dim B$.

Proof. If we have a sequence $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ of prime ideals in B, then there exists $f \in \mathfrak{P}_2 \setminus \mathfrak{P}_1$. Since B is finite over A, there exist $a_1, \cdots, a_n \in A$ such that

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0.$$

Then $a_n \in \mathfrak{P}_2 \cap A$. If $a_n \in \mathfrak{P}_1$, $f^{n-1} + \cdots + a_{n_1} \in \mathfrak{P}_1$ since $f \notin \mathfrak{P}_1$. Then $a_{n-1} \in \mathfrak{P}_2$. Repeat the process, it will terminate, whence $\mathfrak{P}_1 \cap A \subsetneq \mathfrak{P}_2 \cap A$. Otherwise, we have $f^n \in a_1 B + \cdots + a_n B \subset \mathfrak{P}_1$.
Conversely, suppose we have $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} A$ with $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Choose $\mathfrak{P}_1 \in \operatorname{Spec} B$ such that $\mathfrak{P}_1 \cap A = \mathfrak{p}_1$, then we have $A/\mathfrak{p}_1 \subset B/\mathfrak{P}_1$. Let $\mathfrak{P}_2$ be the preimage of the prime ideal in $B/\mathfrak{P}_1$ which is over image of $\mathfrak{p}_2$ in $A/\mathfrak{p}_1$. Proposition 4 guarantees that such $\mathfrak{P}_2$ exists. Then we get $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$. Repeat this progress, we get $\dim B \geq \dim A$. ☐

Theorem 6 (Krull's Principal Ideal Theorem)**.** Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let $\mathfrak{p}$ be a minimal prime ideal among those containing f. Then $\mathrm{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_{\mathfrak{p}}$, we may assume A is local with maximal ideal $\mathfrak{p}$. Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.
Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is

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$\mathfrak{q}$-primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_\mathfrak{q} = \mathfrak{q}^{n+1} A_\mathfrak{q}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_\mathfrak{q} = 0$. It follows that $\mathfrak{q}A_\mathfrak{q}$ is minimal, whence $A_\mathfrak{q}$ is artinian. Therefore, $\mathfrak{q}$ is minimal in A. □

Corollary 7. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n. Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.
Choose $\mathfrak{q}$ be a minimal prime ideal among those containing $(\mathfrak{p}_{k-1}, f)$ and contained in $\mathfrak{p}_{k+1}$. Then by Krull's Principal Ideal Theorem 6, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace $\mathfrak{p}_k$ by $\mathfrak{q}_k$, we have $f \in \mathfrak{q}_k \setminus \mathfrak{p}_{k-1}$
Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.
Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$ has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A. It follows that $\dim A/(f) + 1 \leq \dim A$. □

For varieties, the Krull dimension behaves well by follows.

Lemma 8. Let X be an algebraic variety over k. Then for every closed point $x \in X(\mathbf{k})$, we have

$$\dim X = \dim \mathcal{O}_{X,x} = \operatorname{trdeg}(\mathscr{K}(X)/\mathsf{k}).$$

Proof. Since X is irreducible, we may assume that $X = \operatorname{Spec} A$ is affine. Let $d = \operatorname{trdeg}(\mathscr{K}(X)/\mathsf{k})$.
By Noether's Normalization Lemma **??**, there is an injective and finite homomorphism $A_0 = \mathsf{k}[T_1, \cdots, T_d] \hookrightarrow A$. Let $\mathfrak{M}$ be the corresponding maximal ideal of x in A and $\mathfrak{m} = \mathfrak{M} \cap \mathsf{k}[T_1, \cdots, T_d]$. Denote the image of T_i in $\mathsf{l} := A_0/\mathfrak{m}$ by t_i. The extension l/k is finite by Nullstellensatz **??**. Let $f_i \in \mathsf{k}[T]$ be the minimal polynomial of t_i and $g_i := f_i(T_i) \in A_0$. Then $g_i \in \mathfrak{m}$ and $\mathfrak{m} = g_1 A_0 + \cdots, g_d A_0$. In particular, $g_1, \cdots, g_d \in \mathfrak{M}$.
We have $A/g_1 A + \cdots + g_d A$ is finite over $A_0/\mathfrak{m}$, whence it is artinian. This implies that $A_\mathfrak{M}/g_1 A_\mathfrak{M} + \cdots + g_d A_\mathfrak{M}$ is also artinian. Since g_{k+1} is not a zero divisor in $A_0/g_1 A_0 + \cdots + g_k A_0$, g_{k+1} is not contained in any minimal prime ideal of $A_0/g_1 A_0 + \cdots + g_k A_0$. Then g_{k+1} is also not contained in any minimal prime ideal of $A/g_1 A + \cdots + g_k A$. By Corollary 7, $\dim A_\mathfrak{M} = \dim(A_\mathfrak{M}/g_1 A_\mathfrak{M} + \cdots + g_d A_\mathfrak{M}) + d = d$. □

Theorem 9. Let S be spectrum of a field k or an algebraic integer ring $\mathcal{O}_K$ and X an integral S-variety. Then we have the follows:

(i) For every point $\xi \in X$, $\dim X = \dim \mathcal{O}_{X,\xi} + \operatorname{codim} Z_\xi$.

(ii) For every non-empty open subset $U \subset X$, $\dim U = \dim X$.

(iii) $\dim X = \operatorname{trdeg}(\mathscr{K}(X)/\mathscr{K}(S)) + \dim S$.

Proof. Yang: To be continued. □

Example 10. For general noetherian schemes, Theorem 9 may not hold. Let $A = \mathsf{k}[t]$, $\mathfrak{m} = (t)$, $B = A_\mathfrak{m}[x]$ and $X = \operatorname{Spec} B$. Then we have $\dim X = 2$ since Yang: To be added.

Depth For a noetherian local ring $(A, \mathfrak{m})$, we can define the depth of an A-module M. Somehow the Krull dimension is "homological" and the depth is "cohomological".

Definition 11. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A-module. A sequence $t_1, \cdots, t_n \in \mathfrak{m}$ is called an *M-regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \cdots, t_{i-1})M$ for all i.

Example 12. Let $A = \mathsf{k}[x, y]/(x^2, xy)$ and $I = (x, y)$. Then $\operatorname{depth}_I A = 0$.

Definition 13. The *I-depth* of M is defined as the maximum length of M-regular sequences in I, denoted by $\operatorname{depth}_I M$. When A is a local ring with maximal ideal $\mathfrak{m}$, we write $\operatorname{depth} M$ for $\operatorname{depth}_\mathfrak{m} M$.

Regular and Serre's conditions Up to now, there are three numbers measuring the "size" of a local ring $(A, \mathfrak{m})$:

- $\dim A$: the Krull dimension of A.

- depth A: the depth of A.
- $\dim_{\kappa(\mathfrak{m})} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$-vector space.

These three numbers are related by the following inequalities.

Proposition 14. Let $(A, \mathfrak{m})$ be a local noetherian ring with residue field k. Then the following inequalities hold:

$$\operatorname{depth} A \le \dim A \le \dim_{\mathsf{k}} T_{A,\mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary 7.
Let $t_1, \cdots, t_n$ be a $\kappa(\mathfrak{m})$-basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \cdots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \cdots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \cdots, t_n))$. It follows that $\mathfrak{m} = (t_1, \cdots, t_n)$ by Nakayama's Lemma. By Corollary 7,

$$n + \dim A/(t_1, \cdots, t_n) \ge n - 1 + \dim A/(t_1, \cdots, t_{n-1}) \ge \cdots \ge 1 + \dim A/(t_1) \ge \dim A.$$

We conclude the result. □

Definition 15. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\ge 0}$. We say that X *verifies property* (R_k) or *is regular in codimension* k if $\forall \xi \in X$ with $\operatorname{codim} Z_\xi \le k$,

$$\dim_{\kappa(\xi)} T_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

We say that X *verifies property* (S_k) if $\forall \xi \in X$ with $\operatorname{depth} \mathcal{O}_{X,\xi} < k$,

$$\operatorname{depth} \mathcal{O}_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

Lemma 16. Let A be a ring and $\mathfrak{a} \subset \bigcup_i \mathfrak{p}_i$. Then $\mathfrak{a} \subset \mathfrak{p}_i$ for some i.

Proof. Yang: To be completed. □

Example 17. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.
Suppose A verifies (S_1). If $\mathfrak{p} \in AssA$, every element in $\mathfrak{p}$ is a zero divisor. Then $\operatorname{depth} A_\mathfrak{p} = 0$. It follows that $\dim A_\mathfrak{p} = 0$ and then $\mathfrak{p}$ is minimal.
Suppose A has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_\mathfrak{p} = 0$. This means every element in $\mathfrak{p}A_\mathfrak{p}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Lemma 16, $\mathfrak{p} = \mathfrak{q}$ for some minimal $\mathfrak{q}$, whence $\dim A_\mathfrak{p} = 0$.

Example 18. Let A be a noetherian ring verifies (S_1). Then A verifies (S_2) iff for any nonzero divisor $f \in A$, $\operatorname{Ass}_A A/fA$ has no embedded point.
Suppose A verifies (S_2). Let $f \in A$ be a nonzero divisor and $\mathfrak{p} \in \operatorname{Ass}_A A/fA$. There exist $g \in A \setminus fA$ such that $\mathfrak{p} = (f : g)$. For any $t_1, t_2 \in \mathfrak{p}$, there exist s_1, s_2 with $s_i \notin (t_i)$ and $t_i g = f s_i$. Then $t_1 t_2 g = f s_1 t_2 = f s_2 t_1$. Since f is not a zero divisor, $s_1 t_2 = s_2 t_1$. Then t_2 is a zero divisor in $A_\mathfrak{p}/t_1 A_\mathfrak{p}$ since $s_1 \notin (t_1)$. Since $f \in \mathfrak{p}$, $\operatorname{depth} A_\mathfrak{p} = 1$ and then $\operatorname{ht} \mathfrak{p} = 1$. This show that $\mathfrak{p}$ is not embedded in $\operatorname{Ass}_A A/fA$.
Conversely, suppose $\operatorname{Ass}_A A/fA$ has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_\mathfrak{p} = 1$. Then there exists $f \in A_\mathfrak{p}$ which is not a zero divisor. We have $\operatorname{depth} A_\mathfrak{p}/fA_\mathfrak{p} = 0$ and $\operatorname{Ass}_A A/fA$ has no embedded point, whence $\mathfrak{p}$ is minimal in A/fA. Then $\operatorname{ht} \mathfrak{p} = 1$ by Krull's Principal Ideal Theorem 6 and the fact f is not a zero divisor.

Example 19. Let X be a locally noetherian scheme. Then X is reduced iff it verifies (R_0) and (S_1).
The properties are local, whence we can assume $X = \operatorname{Spec} A$. Suppose A is reduced. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be all minimal prime ideals of A. We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where $\mathfrak{N}$ is the nilradical of A. Hence A has no embedded point. Since $A_\mathfrak{p}$ is artinian, local and reduced, $A_\mathfrak{p}$ is a field and hence regular.
Conversely, let $\operatorname{Ass} A$ be equal to $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Then every $\mathfrak{p}_i$ is minimal by (S_1). Let f be in $\mathfrak{N}$. Then the image of f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0), $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where $\mathfrak{q}_i$ is the $\mathfrak{p}_i$ component of (0) in A. Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

2 Normal schemes Yang: To be completed

Definition 20. An integral domain A is called *normal* if it is integrally closed in its field of fractions $\mathrm{Frac}(A)$.

Lemma 21. Let $A \subset C$ be rings and B the integral closure of A in C, S a multiplicatively closed subset of A. Then the integral closure of $S^{-1}A$ in $S^{-1}C$ is $S^{-1}B$.

Proof. For every $b \in B$ and $\forall s \in S$, there exists $a_i \in A$ s.t.

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then

$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s^1}\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0.$$

Hence b/s is integral over $S^{-1}A$, $S^{-1}B$ is integral over $S^{-1}A$.
If $c/s \in S^{-1}C$ is integral over $S^{-1}A$, then $\exists a_i \in S^{-1}A$ s.t.

$$\left(\frac{c}{s}\right)^n + a_1\left(\frac{c}{s}\right)^{n-1} + \cdots + a_n = 0.$$

Then

$$c^n + a_1 s c^{n-1} + \cdots + a_n s^n = 0 \in S^{-1}C$$

Then $\exists t \in S$ s.t.

$$t(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0 \in C.$$

Then

$$(ct)^n + a_1 st(ct)^{n-1} + \cdots + a_n s^n t^n = t^n(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0.$$

Hence ct is integral over A, then $ct \in B$. Then $c/s = (ct)/(st) \in S^{-1}B$. This completes the proof. □

Proposition 22. Normality is a local property. That is, for an integral domain A, TFAE:

(i) A is normal.

(ii) For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_\mathfrak{p}$ is normal.

(iii) For any maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$, the localization $A_\mathfrak{m}$ is normal.

Proof. When A is normal, $A_\mathfrak{p}$ is normal by Lemma 21.
Assume that $A_\mathfrak{m}$ is normal for every $\mathfrak{m} \in \operatorname{mSpec} A$. If A is not normal, let $\tilde{A}$ be the integral closure of A in $\mathrm{Frac}\, A$, $\tilde{A}/A$ is a nonzero A-module. Suppose $\mathfrak{p} \in \operatorname{Supp} \tilde{A}/A$ and $\mathfrak{p} \subset \mathfrak{m}$. We have $\tilde{A}_\mathfrak{m}/A_\mathfrak{m} = 0$ and $\tilde{A}_\mathfrak{p}/A_\mathfrak{p} = (\tilde{A}_\mathfrak{m}/A_\mathfrak{m})_\mathfrak{p} \neq 0$. This is a contradiction. □

Definition 23. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,\xi}$ is normal for any point $\xi \in X$. A ring A is called *normal* if $\operatorname{Spec} A$ is normal.

Remark 24. For a general ring A, let $S := A \setminus (\bigcup_{\mathfrak{p} \in \operatorname{Ass} A} \mathfrak{p}) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} A} A \setminus \mathfrak{p}$. Then S is a multiplicative set. The localization $S^{-1}A$ is called *the total ring of fractions* of A.
Suppose A is reduced and $\operatorname{Ass} A = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Denote its total ring of fractions by Q. Note that elements in Q are either unit or zero divisor. Hence any maximal ideal $\mathfrak{m}$ is contained in $\bigcup \mathfrak{p}_i Q$, whence contained in some $\mathfrak{p}_i Q$. Thus $\mathfrak{p}_i Q$ are maximal ideals. And we have $\bigcap \mathfrak{p}_i Q = 0$. By the Chinese Remainder Theorem, we have $Q = \prod Q/\mathfrak{p}_i Q = \prod A_{\mathfrak{p}_i}$.
Let A be a reduced ring with total ring of fractions Q. Then A is normal iff A is integral closed in Q. If A is normal, then for every $\mathfrak{p} \in \operatorname{Spec} A$, $A_\mathfrak{p}$ is integral. Then there is unique minimal prime ideal $\mathfrak{p}_i \subset \mathfrak{p}$. In particular, any two minimal prime ideal are relatively prime. By the Chinese Remainder Theorem, $A = \prod A/\mathfrak{p}_i$. Just need to check $A/\mathfrak{p}_i$ is integral closed in $A_{\mathfrak{p}_i}$. This is clear by check pointwise.
Conversely, suppose A is integral closed in Q. Let e_i be the unit element of $A_{\mathfrak{p}_i}$. It belongs to A since $e_i^2 - e_i = 0$. Since $1 = e_1 + \cdots + e_n$ and $e_i e_j = \delta_{ij}$, we have $A = \prod A e_i$. Since Ae_i is integral closed in $A_{\mathfrak{p}_i}$, it is normal. Hence A is normal.

Definition 25. Let X be a scheme. The *normalization* of X is an X-scheme X^ν with the following universal property: for any normal X-scheme Y with dominant structure morphism, its structure morphism $Y \to X$ factors through X^ν.

Proposition 26. The normalization X^ν of X exists. Moreover, if X is reduced, $X^\nu \to X$ is birational.

Proof. Suppose there is a dominant morphism $Y \to X$ with Y normal. Since Y is normal, it is reduced. Then it factors through X_{red}. Hence we can assume that X is reduced by replacing X by X_{red}.
Suppose $X = \operatorname{Spec} A$ is affine. Let A^ν be the integral closure of A in it total ring of fractions and $X^\nu := \operatorname{Spec} A^\nu$. It gives a homomorphism $A \to \mathcal{O}_Y(Y)$. We claim that it is injective. Otherwise, it factors through $A \to A/I$ and then $Y \to \operatorname{Spec} A$ factors through $\operatorname{Spec} A/I \to \operatorname{Spec} A$. It contradicts that $Y \to X$ is dominant. Since Y is normal, $\mathcal{O}_Y(Y)$ is integral closed in its total ring of fraction. Then $\mathcal{O}_Y(Y)$ contains A^ν. This shows that X^ν is the normalization of X.
In general case, take an affine cover $\{U_i\}$ of X and clue these U_i^ν by universal property. □

Lemma 27. Let A be a normal ring. Then A verifies (R_1) and (S_2).

Proof. Since all properties are local, we can assume A is integral and local.
For (S_2), by Example 18, we only need to show that $\operatorname{Ass}_A A/f$ has no embedded point. Let $\mathfrak{p} = (f : g) =\in \operatorname{Ass}_A A/fA$ and $t := f/g \in \operatorname{Frac} A$. After Replacing A by $A_\mathfrak{p}$, we can assume that $\mathfrak{p}$ is maximal. By definition, $t^{-1}\mathfrak{p} \subset A$. If $t^{-1}\mathfrak{p} \subset \mathfrak{p}$, suppose $\mathfrak{p}$ is generated by $(x_1, \cdots, x_n)$ and $t^{-1}(x_1, \cdots, x_n)^T = \Phi(x_1, \cdots, x_n)^T$ for $\Phi \in M_n(A)$. There is a monic polynomial $\chi(T) \in A[T]$ vanishing Φ. Then $\chi(t^{-1}) = 0$ and $t^{-1} \in A$. This is impossible by definition of t. Then $t^{-1}\mathfrak{p} = A$, and $\mathfrak{p} = (t)$ is principal. By Krull's Principal Ideal Theorem 6, $\operatorname{ht}(\mathfrak{p}) = 1$.
Now we show that A verifies (R_1). Suppose $(A, \mathfrak{m})$ is local of dimension 1. Choosing $a \in \mathfrak{m}$, A/a is of dimension 0. Then by **??**, $\mathfrak{m}^n \subset aA$ for some $n \geq 1$. Suppose $\mathfrak{m}^{n-1} \not\subset aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$ and let $t = a/b$. By construction, $t^{-1} \notin A$ and $t^{-1}\mathfrak{m} \subset A$. After similar argument, we see that $\mathfrak{m} = tA$, whence A is regular. □

Lemma 28. Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension 1. Then A is normal iff A is regular.

Proof. By lemma 27, we just need to show that regularity implies normality.
Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since A is regular, $\mathfrak{m} = (t)$. Let $I \subset \mathfrak{m}$ be an ideal. If $I \subset \bigcap_n \mathfrak{m}^n$, then for every $a \in I$, there exists a_n such that $a = a_n t^n$. Then we get an ascending chain of ideals $(a_1) \subset (a_2) \subset \cdots$. Hence $a = 0$ by Nakayama's Lemma. Suppose I is not zero. Then there is some n such that $I \subset \mathfrak{m}^n$ and $I \not\subset \mathfrak{m}^{n+1}$. For every $at^n \in I \setminus \mathfrak{m}^{n+1}$, $a \notin \mathfrak{m}$, whence a is a unit in A. Then $I = (t^n)$. Hence A is PID and hence normal. □

Proposition 29. Let A be a noetherian integral domain of dimension ≥ 1 verifying (S_2). Then

$$A = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A, \operatorname{ht}(\mathfrak{p})=1} A_\mathfrak{p}.$$

Proof. Clearly $A \subset \bigcap A_\mathfrak{p}$. Let $t = f/g \in \bigcap A_\mathfrak{p}$. Since $f \in gA_\mathfrak{p}$ and we have $gA = \bigcap (gA_\mathfrak{p} \cap A)$, $f \in gA$. It follows that $t \in A$. □

Theorem 30 (Serre's criterion for normality)**.** Let X be a locally noetherian scheme. Then X is normal if and only if it verifies (R_1) and (S_2).

Proof. One direction has been proved in Lemma 27. Suppose X verifies (R_1) and (S_2). Again we can assume $X = \operatorname{Spec} A$ is affine and A is local. By Remark 24, we just need to show that A is integral closed in its total ring of fractions Q. Suppose we have

$$\left(\frac{a}{b}\right)^n + c_1 \left(\frac{a}{b}\right)^{n-1} + \cdots + c_n = 0 \in Q.$$

Since A verifies (S_2), $bA = \bigcap \nu_\mathfrak{p}^{-1}(b_\mathfrak{p} A_\mathfrak{p})$. So it is sufficient to show that $a_\mathfrak{p} \in b_\mathfrak{p} A_\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) = 1$. Note that $A_\mathfrak{p}$ is regular and hence normal by Lemma 28. Then above equation gives us desired result. □

Theorem 31. Let X be a normal and locally noetherian scheme. Let $F \subset X$ be a closed subset of codimension ≥ 2. Then the restriction $H^0(X, \mathcal{O}_X) \to H^0(X \setminus F, \mathcal{O}_X)$ is an isomorphism.

Proof. By the exact sequences

$$0 \to \mathcal{F}(X) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j),$$

where $\{U_i\}$ is an affine open cover of X, we can reduce to the case that X is affine. Then $X = \operatorname{Spec} A$ for some normal noetherian ring A. For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{ht}(\mathfrak{p}) = 1$, we have $\mathfrak{p} \in X \setminus F$. By Proposition 29, the conclusion follows. □

Theorem 32 (Valuation criterion for properness)**.** Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. Then f is proper iff for any valuation ring A, $\mathsf{K} = \operatorname{Frac} A$ and commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathsf{K} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd},$$

the morphism $\operatorname{Spec} A \to Y$ factors through f uniquely.

Proposition 33. Let X, Y be S-schemes with S locally noetherian. Suppose Y is of finite type over S. Let $\xi \in X$ and $f_x : \operatorname{Spec} \mathcal{O}_{X,\xi} \to Y$ be a morphism. Then there exists an open subset $U \subset X$ containing ξ such that the morphism extends to a morphism $U \to Y$.

Proof. Replacing S, X, Y by affine open neighborhoods of images of ξ, we can assume that $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A[T_1, \cdots, T_n]/I$ are affine. Then we get a homomorphism $A[T_1, \cdots, T_n]/I \to B_\xi$ of A-algebra. Denote the image of T_i by f_i/g_i in B_ξ, where $f_i, g_i \in B$. Then above homomorphism factors through $B[1/g_1, \cdots, 1/g_n] \to B_\xi$. Let U be the open subset of X defined by $g_1 \cdots g_n \neq 0$. Then the morphism f_x extends to a morphism $U \to Y$. $\square$

Theorem 34. Let X, Y be S-schemes of finite type with S noetherian. Suppose X is normal, and Y is proper over S. Let $f : X \dashrightarrow Y$ be a rational map. Then f is well-defined on an open subset $U \subset X$ whose complement has codimension ≥ 2.

Proof. We can assume that X is irreducible and hence integral. Suppose f is defined on $U \subset X$. For every $\xi \in X$ with codimension 1, we have following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathscr{K}(X) \arrow[r] \arrow[d] & U \arrow[r, "f"] & Y \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{X,\xi} \arrow[rr] & & S \end{tikzcd},$$

By Theorem 32 and Proposition 33, there exists an open subset $U_\xi \subset X$ containing ξ such that the morphism extends to a morphism $U_\xi \to Y$.
Yang: To be completed. $\square$

Remark 35. Theorem 31 and Theorem 34 are very similar. However, they are base on different properties. Theorem 31 is based on (S_2), while Theorem 34 is based on (R_1). Philosophically, the (S_k) conditions are used to control the "bad part of codimension larger than k". The (R_k) conditions are used to control the "bad part of codimension smaller than or equal to k". We will see more examples in the next section.Yang: To be completed.

3 Cohen-Macaulay schemes

Definition 36 (Cohen-Macaulay)**.** A noetherian local ring $(A, \mathfrak{m})$ is called *Cohen-Macaulay* if $\dim A = \operatorname{depth} A$. A locally noetherian scheme X is called *Cohen-Macaulay* if $\mathcal{O}_{X,\xi}$ is Cohen-Macaulay for any point $\xi \in X$.

By definition, it is easy to see that X is Cohen-Macaulay if and only if it verifies (S_k) for all $k \geq 0$.

Example 37 (Non Cohen-Macaulay rings)**.**

Proposition 38. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring and M a finite A-module. Then

$$\operatorname{depth} M := \inf\{i : \operatorname{Ext}^i_A(\mathsf{k}, M) \neq 0\}.$$

Proof. Let $a \in \mathfrak{m}$ be M-regular and $N = M/aM$. Then we claim that

$$\inf\{i : \operatorname{Ext}^i_A(\mathsf{k}, N) \neq 0\} = \inf\{i : \operatorname{Ext}^i_A(\mathsf{k}, M) \neq 0\} - 1.$$

Indeed, we have an exact sequence

$$0 \to M \xrightarrow{a} M \to N \to 0.$$

It induces a long exact sequence

$$\cdots \to \mathrm{Ext}_A^{i-1}(\mathsf{k}, M) \to \mathrm{Ext}_A^{i-1}(\mathsf{k}, N) \to \mathrm{Ext}_A^{i}(\mathsf{k}, M) \xrightarrow{\mathrm{Ext}_A^i(\mathsf{k},\mathrm{Mult}_a)} \mathrm{Ext}_A^{i}(\mathsf{k}, M) \to \cdots.$$

Note that $a \in \mathfrak{m}$, then $\mathrm{Ext}_A^i(\mathsf{k}, \mathrm{Mult}_a) = 0$. It follows that when $\mathrm{Ext}_A^{i-1}(\mathsf{k}, M) = 0$, we have $\mathrm{Ext}_A^{i-1}(\mathsf{k}, N) = 0$ iff $\mathrm{Ext}_A^{i}(\mathsf{k}, M) = 0$, whence the claim.
Let $n = \inf\{i : \mathrm{Ext}_A^i(\mathsf{k}, M) \neq 0\}$. Induct on n. Suppose first $n = 0$. Since k is a simple A-module, there is an injective homomorphism $\mathsf{k} \to M$. Then $\mathfrak{m} \in \mathrm{Ass}\, M$ and hence $\operatorname{depth} M = 0$.
Suppose $n > 0$., let $a_1, \cdots, a_m \in \mathfrak{m}$ be any M-regular sequence. Using the claim inductively on $M/(a_1, \cdots, a_m)M$, we have $n \geq \operatorname{depth}$. If M has no regular element, then $\mathfrak{m} \subset \bigcup_{\mathfrak{p} \in \mathrm{Ass}\, M} \mathfrak{p}$. Then $\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}\, M$. This show that we can find $x \neq 0 \in M$ such that $\mathfrak{p} = \mathrm{Ann}\, x$. It gives a homomorphism $\mathsf{k} = A/\mathfrak{m} \to M$. That is a contradiction and hence M has a regular element. Let a be M-regular and $N = M/aM$. Then $\operatorname{depth} N = n - 1$ by the claim and induction hypothesis. Hence we have $\operatorname{depth} M \geq n$. □

Corollary 39. Let A be a noetherian ring, M a finite A-module and $a \in A$ an M-regular element. Then $\operatorname{depth} M = \operatorname{depth} M/aM + 1$.

Corollary 40. Let A be a noetherian ring $a \in A$ a nonzero divisor. Then A verifies (S_d) iff A/aA verifies (S_{d-1}).

Definition 41. An ideal I of a noetherian ring A is called *unmixed* if

$$\mathrm{ht}(I) = \mathrm{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \mathrm{Ass}(A/I).$$

Here $\mathrm{ht}(I)$ is defined as

$$\mathrm{ht}(I) := \inf\{\mathrm{ht}(\mathfrak{p}) : I \subset \mathfrak{p}\}.$$

We say that *the unmixedness theorem holds for a noetherian ring* A if any ideal $I \subset A$ generated by $\mathrm{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme* X if $\mathcal{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Theorem 42. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Proof. We can assume that $X = \operatorname{Spec} A$ is affine.
Suppose X is Cohen-Macaulay. Let $I \subset A$ be an ideal generated by $a_1, \cdots, a_r$ with $r = \mathrm{ht}(I)$. We claim that $a_1, \cdots, a_r$ is an A-regular sequence. If so, we get that the unmixedness theorem holds for A by applying Example 17 on A/I. Since $\mathrm{ht}(a_1, \cdots, a_{r-1}) \leq r-1$ by Krull's Principal Ideal Theorem 6 and $\mathrm{ht}(a_1, \cdots, a_r) = r \leq \mathrm{ht}(a_1, \cdots, a_{r-1}) + 1$, we have $\mathrm{ht}(a_1, \cdots, a_{r-1}) = r - 1$. By induction on r, we can assume that $a_1, \cdots, a_{r-1}$ is an A-regular sequence. Hence any prime ideal $\mathfrak{p} \in \mathrm{Ass}\, A/(a_1, \cdots, a_{r-1})$ has height $r - 1$. Now suppose a_r is a zero divisor in $A/(a_1, \cdots, a_{r-1})$. Then there exists a prime ideal $\mathfrak{p} \in \mathrm{Ass}\, A/(a_1, \cdots, a_{r-1})$ such that $a_r \in \mathfrak{p}$. Then $I \subset \mathfrak{p}$ and $\mathrm{ht}(I) \leq r - 1$. This contradicts that $\mathrm{ht}(I) = r$.
Suppose the unmixedness theorem holds for A. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal with $\mathrm{ht}(\mathfrak{p}) = r$. Then $\mathfrak{p} \in \mathrm{Ass}\, A$ if and only if $\mathrm{ht}(\mathfrak{p}) = 0$. If $r > 0$, there is a nonzero divisor $a \in \mathfrak{p}$. By Krull's Principal Ideal Theorem 6, $\mathrm{ht}(\mathfrak{p}A/aA) = r - 1$. Inductively, we can find a regular sequence $a_1, \cdots, a_r$ in $\mathfrak{p}$. Then $\operatorname{depth} A_{\mathfrak{p}} = r$. □

Theorem 43. Let X be a locally noetherian scheme. Suppose that X is Cohen-Macaulay. Let $F \subset X$ be a closed subset of codimension $\geq k$. Then the restriction $H^i(X, \mathcal{O}_X) \to H^i(X \setminus F, \mathcal{O}_X)$ induced by the is an isomorphism.

Proof. Yang: To be completed. □

4 Regular schemes

Proposition 44. If X verifies (R_k), then $\operatorname{codim}_X X_{\mathrm{sing}} \geq k + 1$.

Proposition 45. A regular scheme is Cohen-Macaulay.

Corollary 46. A regular scheme is normal.