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# *Schemes and Varieties*

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# 1 Definition and First Properties of Schemes

## 1.1 Locally Ringed Space

## 1.2 Schemes

**Example 1.1** (Glue open subschemes). We construct a scheme by gluing open subschemes. Let  $X_i$  be schemes for  $i \in I$  and  $U_{ij} \subset X_i$  be open subschemes for  $i, j \in I$ . Suppose we have isomorphisms  $\varphi_{ij} : U_{ij} \rightarrow U_{ji}$  such that

- (a)  $\varphi_{ii} = \text{id}_{X_i}$  for all  $i \in I$ ;
- (b)  $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$  for all  $i, j \in I$ ;
- (c)  $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$  on  $U_{ij} \cap U_{ik}$  for all  $i, j, k \in I$ .

Then there exists a scheme  $X$  and open immersions  $\psi_i : X_i \rightarrow X$  such that the following diagram

$$\begin{array}{ccc} U_{ij} & \xrightarrow{\varphi_{ij}} & U_{ji} \\ \downarrow \iota_{ij} & & \downarrow \iota_{ji} \\ X_i & \xrightarrow{\psi_i} & X_j \end{array}$$

commutes for all  $i, j \in I$  and  $X = \bigcup_{i \in I} \psi_i(X_i)$ . Moreover,  $X$  is unique up to isomorphism. Such  $X$  is called *the scheme obtained by gluing the  $X_i$  along the  $\varphi_{ij}$* .

Yang: To be continued...

### 1.3 Integral, reduced and irreducible

### 1.4 Fiber product

### 1.5 Dimension

### 1.6 Noetherian and finite type

### 1.7 Separated and proper

## 2 Category of sheaves of modules

### 2.1 Sheaves of modules, quasi-coherent and coherent sheaves

**Definition 2.1.** Let  $X$  be a ringed space with structure sheaf  $\mathcal{O}_X$ . A **sheaf of (left)  $\mathcal{O}_X$ -modules** is a sheaf  $\mathcal{F}$  on  $X$  such that for every open set  $U \subseteq X$ ,  $\mathcal{F}(U)$  is an  $\mathcal{O}_X(U)$ -module, and for every inclusion of open sets  $V \subseteq U$ , the restriction map  $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$  is compatible with the restriction map  $\rho_{UV} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$  in the sense that for every  $s \in \mathcal{O}_X(U)$  and  $m \in \mathcal{F}(U)$ , we have

$$\rho_{UV}(s \cdot m) = \rho_{UV}(s) \cdot \rho_{UV}(m).$$

Yang: To be continued...

**Example 2.2.** Let  $X$  be a scheme. The structure sheaf  $\mathcal{O}_X$  is a sheaf of  $\mathcal{O}_X$ -modules. More generally, any quasi-coherent sheaf (to be defined later) is a sheaf of  $\mathcal{O}_X$ -modules. In particular, if  $X = \operatorname{Spec} A$  is an affine scheme, then for any  $A$ -module  $M$ , the associated sheaf  $\tilde{M}$  is a sheaf of  $\mathcal{O}_X$ -modules.

Yang: To be continued...

**Definition 2.3.** Let  $X$  be a scheme. A sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  is called **quasi-coherent** if for every point  $x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $\mathcal{F}|_U$  is isomorphic to the cokernel of a morphism of free  $\mathcal{O}_U$ -modules, i.e., there exists an exact sequence of sheaves of  $\mathcal{O}_U$ -modules

$$\mathcal{O}_U^{(I)} \rightarrow \mathcal{O}_U^{(J)} \rightarrow \mathcal{F}|_U \rightarrow 0,$$

where  $I, J$  are (possibly infinite) index sets. Yang: To be continued...

**Definition 2.4.** Let  $X$  be a scheme. A sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  is called **coherent** if it is quasi-coherent and for every point  $x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $\mathcal{F}|_U$  is isomorphic to the cokernel of a morphism of finite free  $\mathcal{O}_U$ -modules, i.e., there exists an exact sequence of sheaves of  $\mathcal{O}_U$ -modules

$$\mathcal{O}_U^m \rightarrow \mathcal{O}_U^n \rightarrow \mathcal{F}|_U \rightarrow 0,$$

where  $m, n$  are finite integers. Yang: To be continued...

## 2.2 As abelian categories

**Theorem 2.5.** Let  $X$  be a ringed space. The category of sheaves of  $\mathcal{O}_X$ -modules is an abelian category. Yang: To be continued...

**Theorem 2.6.** Let  $X$  be a scheme. The category of quasi-coherent sheaves on  $X$  is an abelian category. Yang: To be continued...

**Theorem 2.7.** Let  $X$  be a noetherian scheme. The category of coherent sheaves on  $X$  is an abelian category. Yang: To be continued...

## 2.3 Relevant functors

**Theorem 2.8.** Let  $X$  be a ringed space. The global sections functor

$$\Gamma(X, -) : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \rightarrow (\mathcal{O}_X(X)\text{-modules})$$

is left exact. Yang: To be continued...

**Theorem 2.9.** Let  $f : X \rightarrow Y$  be a morphism of ringed spaces. The direct image functor

$$f_* : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \rightarrow (\text{Sheaves of } \mathcal{O}_Y\text{-modules})$$

is left exact. Yang: To be continued...

**Theorem 2.10.** Let  $f : X \rightarrow Y$  be a morphism of ringed spaces. The inverse image functor

$$f^* : (\text{Sheaves of } \mathcal{O}_Y\text{-modules}) \rightarrow (\text{Sheaves of } \mathcal{O}_X\text{-modules})$$

is right exact. Yang: To be continued...

## 2.4 Locally free sheaves and vector bundles

**Definition 2.11.** Let  $X$  be a scheme. A sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  is called **locally free** if for every point  $x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $\mathcal{F}|_U$  is isomorphic to a finite free  $\mathcal{O}_U$ -module, i.e., there exists an isomorphism of sheaves of  $\mathcal{O}_U$ -modules

$$\mathcal{F}|_U \cong \mathcal{O}_U^n,$$

where  $n$  is a finite integer called the **rank** of  $\mathcal{F}$  at  $x$ . Yang: To be continued...

**Example 2.12.** A **line bundle** on a scheme  $X$  is a locally free sheaf of rank 1. The sheaf of differentials  $\Omega_{X/k}$  on a smooth variety  $X$  over a field  $k$  is a locally free sheaf of rank equal to the dimension of  $X$ . **Yang:** To be continued...

**Theorem 2.13.** Let  $X$  be a scheme. There is an equivalence of categories between the category of locally free sheaves of finite rank on  $X$  and the category of vector bundles on  $X$ . **Yang:** To be continued...

## 2.5 Cohomological theory

**Theorem 2.14.** Let  $X$  be a ringed space and  $\mathcal{F}$  a sheaf of  $\mathcal{O}_X$ -modules. Then the cohomology groups  $H^i(X, \mathcal{F})$  are  $\mathcal{O}_X(X)$ -modules for all  $i \geq 0$ . **Yang:** To be continued...

**Theorem 2.15.** Let  $X$  be a scheme and  $\mathcal{F}$  a quasi-coherent sheaf on  $X$ . Then the cohomology groups  $H^i(X, \mathcal{F})$  are  $\mathcal{O}_X(X)$ -modules for all  $i \geq 0$ . **Yang:** To be continued...

**Theorem 2.16.** Let  $X$  be a noetherian scheme and  $\mathcal{F}$  a coherent sheaf on  $X$ . Then the cohomology groups  $H^i(X, \mathcal{F})$  are  $\mathcal{O}_X(X)$ -modules for all  $i \geq 0$ . **Yang:** To be continued...

## 3 Normal, Cohen-Macaulay, and regular schemes

## 4 Line Bundles and Divisors

### 4.1 Cartier Divisors

### 4.2 Line Bundles and Picard Group

### 4.3 Weil Divisors and Reflexive Sheaves

## 5 Line bundles induce morphisms

### 5.1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

**Theorem 5.1.** Let  $A$  be a ring and  $X$  an  $A$ -scheme. Let  $\mathcal{L}$  be a line bundle on  $X$  and  $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ . Suppose that  $\{s_i\}$  generate  $\mathcal{L}$ , i.e.,  $\bigoplus_i \mathcal{O}_X \cdot s_i \rightarrow \mathcal{L}$  is surjective. Then there is a unique morphism  $f : X \rightarrow \mathbb{P}_A^n$  such that  $\mathcal{L} \cong f^* \mathcal{O}(1)$  and  $s_i = f^* x_i$ , where  $x_i$  are the standard coordinates on  $\mathbb{P}_A^n$ .

*Proof.* Let  $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$  be the open subset where  $s_i$  does not vanish. Since  $\{s_i\}$  generate  $\mathcal{L}$ , we have  $X = \bigcup_i U_i$ . Let  $V_i$  be given by  $x_i \neq 0$  in  $\mathbb{P}_A^n$ . On  $U_i$ , let  $f_i : U_i \rightarrow V_i \subseteq \mathbb{P}_A^n$  be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \rightarrow \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on  $U_i \cap U_j$ ,  $f_i$  and  $f_j$  agree. Thus we can glue them to get a morphism  $f : X \rightarrow \mathbb{P}_A^n$ . By construction, we have  $s_i = f^*x_i$  and  $\mathcal{L} \cong f^*\mathcal{O}(1)$ . If there is another morphism  $g : X \rightarrow \mathbb{P}_A^n$  satisfying the same properties, then on each  $U_i$ ,  $g$  must agree with  $f_i$  by the same construction. Thus  $g = f$ .  $\square$

**Proposition 5.2.** Let  $X$  be a  $\mathbf{k}$ -scheme for some field  $\mathbf{k}$  and  $\mathcal{L}$  is a line bundle on  $X$ . Suppose that  $\{s_0, \dots, s_n\}$  and  $\{t_0, \dots, t_m\}$  span the same subspace  $V \subseteq \Gamma(X, \mathcal{L})$  and both generate  $\mathcal{L}$ . Let  $f : X \rightarrow \mathbb{P}_{\mathbf{k}}^n$  and  $g : X \rightarrow \mathbb{P}_{\mathbf{k}}^m$  be the morphisms induced by  $\{s_i\}$  and  $\{t_j\}$  respectively. Then there exists a linear transformation  $\phi : \mathbb{P}_{\mathbf{k}}^n \dashrightarrow \mathbb{P}_{\mathbf{k}}^m$  which is well defined near image of  $f$  and satisfies  $g = \phi \circ f$ .

*Proof.* **Yang:** To be continued.  $\square$

**Example 5.3.** Let  $X = \mathbb{P}_{\mathbf{k}}^n$  with  $\mathbf{k}$  a field and  $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$  for some  $d > 0$ . Then  $\Gamma(X, \mathcal{L})$  is generated by the global sections  $S_{i_0, \dots, i_n} = T_0^{i_0} T_1^{i_1} \dots T_n^{i_n}$  for all  $(i_0, \dots, i_n)$  with  $i_0 + \dots + i_n = d$ , where  $T_i$  are the standard coordinates on  $\mathbb{P}^n$ . They induce a morphism  $f : X \rightarrow \mathbb{P}_{\mathbf{k}}^N$  where  $N = \binom{n+d}{d} - 1$ . On  $\mathbf{k}$ -point level, it is given by

$$[x_0 : \dots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all  $(i_0, \dots, i_n)$  with  $i_0 + \dots + i_n = d$ . This is called the *d-uple embedding* or *Veronese embedding* of  $\mathbb{P}^n$  into  $\mathbb{P}^N$ .

**Example 5.4.** Let  $X = \mathbb{P}_{\mathbf{k}}^m \times \mathbb{P}_{\mathbf{k}}^n$  with  $\mathbf{k}$  a field and  $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$ , where  $\pi_1$  and  $\pi_2$  are the projections. Let  $T_0, \dots, T_m$  and  $S_0, \dots, S_n$  be the standard coordinates on  $\mathbb{P}^m$  and  $\mathbb{P}^n$  respectively. Then  $\Gamma(X, \mathcal{L})$  is generated by the global sections  $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$  for  $0 \leq i \leq m$  and  $0 \leq j \leq n$ . They induce a morphism  $f : X \rightarrow \mathbb{P}_{\mathbf{k}}^{(m+1)(n+1)-1}$ . On  $\mathbf{k}$ -point level, it is given by

$$([x_0 : \dots : x_m], [y_0 : \dots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all  $(i, j)$  with  $0 \leq i \leq m$  and  $0 \leq j \leq n$ . This is called the *Segre embedding* of  $\mathbb{P}^m \times \mathbb{P}^n$  into  $\mathbb{P}^{(m+1)(n+1)-1}$ .

**Example 5.5.** Let  $X = \mathbb{F}_2$  be the second Hirzebruch surface, i.e., the projective bundle  $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$  over  $\mathbb{P}^1$ . **Yang:** To be continued.

**Definition 5.6.** A *linear system* on a scheme  $X$  is a pair  $(\mathcal{L}, V)$  where  $\mathcal{L}$  is a line bundle on  $X$  and  $V \subseteq \Gamma(X, \mathcal{L})$  is a subspace. The dimension of the linear system is  $\dim V - 1$ . A linear system is *base-point free* if  $V$  is base-point free. A linear system is *complete* if  $V = \Gamma(X, \mathcal{L})$ . **Yang:** To be continued.

**Definition 5.7.** A line bundle  $\mathcal{L}$  on a scheme  $X$  is *ample* if for every coherent sheaf  $\mathcal{F}$  on  $X$ , there exists  $n_0 > 0$  such that for all  $n \geq n_0$ ,  $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$  is globally generated. **Yang: To be continued.**

**Definition 5.8.** A line bundle  $\mathcal{L}$  on a scheme  $X$  is *very ample* if there exists a closed embedding  $i : X \rightarrow \mathbb{P}_A^n$  such that  $\mathcal{L} \cong i^* \mathcal{O}(1)$ . **Yang: To be continued.**

**Definition 5.9.** Let  $\mathcal{L}$  be a line bundle on a scheme  $X$  and  $V \subseteq \Gamma(X, \mathcal{L})$  a subspace. The *base locus* of  $V$  is the closed subset

$$\text{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If  $\text{Bs}(V) = \emptyset$ , we say that  $V$  is *base-point free*. **Yang: To be continued.**

**Definition 5.10.** A line bundle  $\mathcal{L}$  on a scheme  $X$  is *globally generated* if  $\Gamma(X, \mathcal{L})$  generates  $\mathcal{L}$ , i.e., the natural map  $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \rightarrow \mathcal{L}$  is surjective. **Yang: To be continued.**

**Definition 5.11.** Let  $\mathcal{L}$  be a line bundle on a scheme  $X$ . **Yang: To be continued.**

**Theorem 5.12.** Let  $X$  be a scheme of finite type over a noetherian ring  $A$  and  $\mathcal{L}$  a line bundle on  $X$ . Then the following are equivalent:

- (a)  $\mathcal{L}$  is ample;
- (b) for some  $n > 0$ ,  $\mathcal{L}^{\otimes n}$  is very ample;
- (c) for all  $n \gg 0$ ,  $\mathcal{L}^{\otimes n}$  is very ample.

**Yang: To be continued.**

**Proposition 5.13.** Let  $X$  be a scheme of finite type over a noetherian ring  $A$  and  $\mathcal{L}, \mathcal{M}$  line bundles on  $X$ . Then we have the following:

- (a) if  $\mathcal{L}$  is ample and  $\mathcal{M}$  is globally generated, then  $\mathcal{L} \otimes \mathcal{M}$  is ample;
- (b) if  $\mathcal{L}$  is very ample and  $\mathcal{M}$  is globally generated, then  $\mathcal{L} \otimes \mathcal{M}$  is very ample;
- (c) if both  $\mathcal{L}$  and  $\mathcal{M}$  are ample, then so is  $\mathcal{L} \otimes \mathcal{M}$ ;
- (d) if both  $\mathcal{L}$  and  $\mathcal{M}$  are globally generated, then so  $\mathcal{L} \otimes \mathcal{M}$ ;
- (e) if  $\mathcal{L}$  is ample and  $\mathcal{M}$  is arbitrary, then for some  $n > 0$ ,  $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$  is ample;

**Yang: To be continued.**

*Proof.* **Yang: To be continued.** □

**Proposition 5.14.** Let  $X$  be a scheme of finite type over a noetherian ring  $A$  and  $\mathcal{L}$  a line bundle on  $X$ . Then  $\mathcal{L}$  is very ample if and only if the following two conditions hold:



- (a) (separate points) for any two distinct points  $x, y \in X$ , there exists  $s \in \Gamma(X, \mathcal{L})$  such that  $s(x) = 0$  but  $s(y) \neq 0$ ;
- (b) (separate tangent vectors) for any point  $x \in X$  and non-zero tangent vector  $v \in T_x X$ , there exists  $s \in \Gamma(X, \mathcal{L})$  such that  $s(x) = 0$  but  $v(s) \neq 0$ .

Yang: To be continued.

## 5.2 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

**Definition 5.15.** Let  $X$  be a normal proper variety over a field  $\mathbf{k}$ ,  $D$  a (Cartier) divisor on  $X$  and  $\mathcal{L} = \mathcal{O}_X(D)$  the associated line bundle. The *complete linear system* associated to  $D$  is the set

$$|D| = \{D' \in \text{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system  $|D|$  and the projective space  $\mathbb{P}(\Gamma(X, \mathcal{L}))$ . Here the elements in  $\mathbb{P}(\Gamma(X, \mathcal{L}))$  are one-dimensional subspaces of  $\Gamma(X, \mathcal{L})$ . Consider the vector subspace  $V \subseteq \Gamma(X, \mathcal{L})$ , we can define the generate linear system  $|V|$  as the image of  $V \setminus \{0\}$  in  $\mathbb{P}(\Gamma(X, \mathcal{L}))$ .

## 5.3 Asymptotic behavior

**Definition 5.16.** Let  $X$  be a scheme and  $\mathcal{L}$  a line bundle on  $X$ . The *section ring* of  $\mathcal{L}$  is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

**Definition 5.17.** A line bundle  $\mathcal{L}$  on a scheme  $X$  is *semiample* if for some  $n > 0$ ,  $\mathcal{L}^{\otimes n}$  is base-point free. Yang: To be continued.

**Theorem 5.18.** Let  $X$  be a scheme over a ring  $A$  and  $\mathcal{L}$  a semiample line bundle on  $X$ . Then there exists a morphism  $f : X \rightarrow Y$  over  $A$  such that  $\mathcal{L} \cong f^* \mathcal{O}_Y(1)$  for some very ample line bundle  $\mathcal{O}_Y(1)$  on  $Y$ . Moreover,  $Y = \text{Proj } R(X, \mathcal{L})$  and  $f$  is induced by the natural map  $R(X, \mathcal{L}) \rightarrow \Gamma(X, \mathcal{L}^{\otimes n})$ . Yang: To be continued.

**Definition 5.19.** A line bundle  $\mathcal{L}$  on a scheme  $X$  is *big* if the section ring  $R(X, \mathcal{L})$  has maximal growth, i.e., there exists  $C > 0$  such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq Cn^{\dim X}$$

for all sufficiently large  $n$ . Yang: To be continued.

**Example 5.20.** Let  $X = \mathbb{F}_2$  be the second Hirzebruch surface, i.e., the projective bundle  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$  over  $\mathbb{P}^1$ . Let  $\pi : X \rightarrow \mathbb{P}^1$  be the projection and  $E$  the unique section of  $\pi$  with self-intersection  $-2$ . Yang: To be continued.

## 6 Differentials and duality

## 7 Flat, smooth and étale morphisms

## 8 Relative objects

### 8.1 Relative schemes

**Definition 8.1.** Let  $X$  be a scheme. An  $\mathcal{O}_X$ -algebra is a sheaf . Yang: To be continued...

**Definition 8.2.** Let  $X$  be a scheme and  $\mathcal{A}$  be a quasi-coherent  $\mathcal{O}_X$ -algebra. The relative Spec of  $\mathcal{A}$ , denoted by  $\text{Spec}_X \mathcal{A}$ , is the scheme obtained by gluing the affine schemes  $\text{Spec} \mathcal{A}(U)$  for all affine open subsets  $U \subset X$ . Yang: To be continued...

**Proposition 8.3.** Let  $X$  be a scheme and  $\mathcal{E}$  be a locally free sheaf of finite rank on  $X$ . Then the relative Spec of the symmetric algebra of  $\mathcal{E}$ , denoted by  $\mathbb{V}(\mathcal{E}) = \text{Spec}_X \text{Sym}_{\mathcal{O}_X} \mathcal{E}$ , is called the geometric vector bundle associated to  $\mathcal{E}$ . The projection morphism  $\pi : \mathbb{V}(\mathcal{E}) \rightarrow X$  is affine and for any open subset  $U \subset X$ , we have  $\pi^{-1}(U) \cong \text{Spec} \text{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$ . Yang: To be continued...

**Definition 8.4.** Let  $X$  be a scheme and  $\mathcal{A}$  be a quasi-coherent graded  $\mathcal{O}_X$ -algebra such that  $\mathcal{A}_0 = \mathcal{O}_X$  and  $\mathcal{A}$  is generated by  $\mathcal{A}_1$  as an  $\mathcal{O}_X$ -algebra. The relative Proj of  $\mathcal{A}$ , denoted by  $\text{Proj}_X \mathcal{A}$ , is the scheme obtained by gluing the affine schemes  $\text{Proj} \mathcal{A}(U)$  for all affine open subsets  $U \subset X$ . The projection morphism  $\pi : \text{Proj}_X \mathcal{A} \rightarrow X$  is projective and for any open subset  $U \subset X$ , we have  $\pi^{-1}(U) \cong \text{Proj} \mathcal{A}(U)$ . Yang: To be continued...

### 8.2 Blowing up

**Definition 8.5.** Let  $X$  be a scheme and  $\mathcal{I} \subset \mathcal{O}_X$  be a quasi-coherent sheaf of ideals. The blowing up of  $X$  along  $\mathcal{I}$ , denoted by  $\text{Bl}_{\mathcal{I}} X$ , is defined to be the relative Proj of the Rees algebra of  $\mathcal{I}$ :

$$\text{Bl}_{\mathcal{I}} X = \text{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{I}^n.$$

The projection morphism  $\pi : \text{Bl}_{\mathcal{I}} X \rightarrow X$  is projective and for any open subset  $U \subset X$ , we have  $\pi^{-1}(U) \cong \text{Bl}_{\mathcal{I}(U)} U$ . The exceptional divisor of the blowing up is defined to be the closed subscheme

$E = \pi^{-1}(V(\mathcal{I}))$  of  $\mathrm{Bl}_{\mathcal{I}} X$ . Yang: To be continued...

## 8.3 Relative ampleness and relative morphisms

# 9 Finite morphisms and fibrations

# 10 Varieties in more general settings

## 10.1 Varieties

**Definition 10.1.** A *variety* over an algebraically closed field  $\mathbb{k}$  is an integral separated scheme of finite type over  $\mathrm{Spec} \mathbb{k}$ .

Yang: Suppose that  $\mathbf{k}$  is not algebraically closed, let  $\mathbf{k}'$  be an algebraic extension of  $\mathbf{k}$ . What is the relation between  $X$ ,  $X_{\mathbf{k}'}$ ,  $X(\mathbf{k}')$  and  $X_{\mathbf{k}'}(\mathbf{k}')$ ?

## 10.2 Geometric properties

## 10.3 Points in varieties

**Proposition 10.2.** Let  $\mathcal{K}$  be a field and  $\ell$  an extension of  $\mathcal{K}$ . Let  $X$  be a variety over  $\mathcal{K}$ . Then we have the following:

- (a) there is a natural bijection between  $X(\ell)$  and  $X_{\ell}(\ell)$ ;
- (b) let  $m/\ell$  be an extension, then there is a natural inclusion  $X(\ell) \subseteq X(m)$ ;
- (c) suppose that  $X = \mathrm{Spec} \mathcal{K}[T_1, \dots, T_n]/I$  is an affine variety, then there is a natural bijection between  $X(\ell)$  and the set  $\{(x_1, \dots, x_n) \in \ell^n \mid f(x_1, \dots, x_n) = 0, \forall f \in I\}$ .