Relative objects

1 Relative schemes

Definition 1. Let X be a scheme. An $\mathcal{O}_X$-algebra is a sheaf . Yang: To be continued...

Definition 2. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. The relative Spec of $\mathcal{A}$, denoted by $\mathcal{Spec}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Spec} \mathcal{A}(U) \to U$ for all affine open subsets $U \subset X$. Yang: To be continued...

Proposition 3. Let X be a scheme and $\mathcal{E}$ be a locally free sheaf of finite rank on X. Then the relative Spec of the symmetric algebra of $\mathcal{E}$, denoted by $\mathbb{V}(\mathcal{E}) = \operatorname{Spec}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}$, is called the geometric vector bundle associated to $\mathcal{E}$. The projection morphism $\pi : \mathbb{V}(\mathcal{E}) \to X$ is affine and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Spec} \operatorname{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$. Yang: To be continued... Yang: To be revised, need to take dual.

Definition 4. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_0 = \mathcal{O}_X$ and $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{O}_X$-algebra. The relative Proj of $\mathcal{A}$, denoted by $\mathcal{Proj}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Proj} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. The projection morphism $\pi : \mathcal{Proj}_X \mathcal{A} \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{A}(U)$. Yang: To be continued...

Let X be a scheme and $\mathcal{F}$ be a quasi-coherent sheaf on X. The *projective bundle* associated to $\mathcal{F}$ is defined to be $\mathbb{P}(\mathcal{F}) = \mathcal{Proj}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{F}$. The *vector bundle* associated to $\mathcal{F}$ is defined to be $\mathbb{V}(\mathcal{F}) = \mathcal{Spec}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{F}$.

2 Relative ampleness and projective morphisms

Definition 5. Let X be an S-scheme via a morphism $f : X \to S$. A line bundle $\mathcal{L}$ on X is called *relatively ample* or *f-ample* if for every affine open subset $U \subset S$, the restriction $\mathcal{L}|_{f^{-1}(U)}$ is an ample line bundle on the scheme $f^{-1}(U)$. Yang: To be revised.

Definition 6. Let $f : X \to S$ be a morphism of schemes. The morphism f is called *projective* if there exists a coherent sheaf $\mathcal{F}$ on S and a closed immersion $i : X \to \mathbb{P}(\mathcal{F})$ over S.

Remark 7. There may be three different definitions of projective morphisms in the literature:

- (Grothendieck) this is the definition in the sense of Definition 6;
- (Altman and Kleiman) this need the coherent sheaf $\mathcal{F}$ to be locally free of finite rank in Definition 6;
- (Hartshorne) this need the coherent sheaf $\mathcal{F}$ to be free of finite rank in Definition 6.

For more details, see [FGA05, Section 5.5].

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Definition 8. Let $f : X \to S$ be a proper morphism of schemes. A line bundle $\mathcal{L}$ on X is called *relatively very ample* or *f-very ample* if there exists a closed immersion $i : X \to \mathbb{P}(\mathcal{F})$ over S for some coherent sheaf $\mathcal{F}$ on S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.

Theorem 9. Let $f : X \to S$ be a proper morphism of schemes and $\mathcal{L}$ be a line bundle on X. The following are equivalent:

(a) $\mathcal{L}$ is f-ample;

(b) for every coherent sheaf $\mathcal{F}$ on X, there exists an integer $n_0 > 0$ such that for all integers $n \geq n_0$,

$$f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

is surjective;

(c) $\mathcal{L}^{\otimes n}$ is f-very ample for some integer $n > 0$;

Moreover, the morphism f is projective if and only if there exists an f-ample line bundle on X. Yang: To be continued...

Theorem 10 (Relative Serre Vanishing)**.** Let $f : X \to S$ be a projective morphism of Noetherian schemes and $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is f-ample if and only if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer $n_0 > 0$ such that for all integers $n \geq n_0$ and all $i > 0$, we have

$$R^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0.$$

Yang: To be continued...

Theorem 11 (Openness of ample locus)**.** Let $f : X \to S$ be a projective morphism of varieties over $\mathbf{k}$ and $\mathcal{L}$ be a line bundle on X. Then the set

$$U = \{s \in S(\Bbbk) : \mathcal{L}|_{X_s} \text{ is an ample line bundle on the fibre } X_s = f^{-1}(s)\}$$

is an open subset of $S(\Bbbk)$. Yang: To be continued...

Theorem 12 (Fibrewise ampleness)**.** Let $f : X \to S$ be a projective morphism of varieties over $\mathbf{k}$ and $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is f-ample if and only if for every point $s \in S(\Bbbk)$, the restriction $\mathcal{L}|_{X_s}$ is an ample line bundle on the fibre $X_s = f^{-1}(s)$. Yang: To be continued...

3 Blowing up

Definition 13. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blow up of X along $\mathcal{I}$, denoted by $\mathrm{Bl}_{\mathcal{I}} X$, is defined to be the relative Proj of the Rees algebra of $\mathcal{I}$:

$$\mathrm{Bl}_{\mathcal{I}} X = \mathcal{P}roj_X \bigoplus_{n=0}^{\infty} \mathcal{I}^n.$$

Yang: To be continued...

Proposition 14. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blow up morphism $\pi : \mathrm{Bl}_{\mathcal{I}} X \to X$ is projective. Moreover, if the support of $\mathcal{O}_X/\mathcal{I}$ is nowhere dense in X, then π is birational. Yang: To be continued...

Proposition 15. Let X be a regular scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then the blow up $\mathrm{Bl}_{\mathcal{I}} X$ is also a regular scheme. Yang: To be continued...

Proposition 16. Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The exceptional locus of the blow up morphism $\pi : \mathrm{Bl}_{\mathcal{I}} X \to X$ is equal to the inverse image of the support of $\mathcal{O}_X/\mathcal{I}$:

$$\mathrm{Exc}(\pi) = \pi^{-1}(\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})).$$

Yang: To be continued...

References

[FGA05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*. Vol. 123. Mathematical Surveys and Monographs. Grothendieck's FGA explained. American Mathematical Society, Providence, RI, 2005, pp. x+339. ISBN: 0-8218-3541-6. DOI: 10.1090/surv/123. URL: https://doi.org/10.1090/surv/123 (cit. on p. 1).