Notes in Algebraic Geometry

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Contents

Chapter 1

Schemes and Varieties

1.0 Locally Ringed Space

1.0.1 Sheaves

Definition 1.0.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

For two open sets $V \subset U \subset X$, the morphism $\mathcal{F}(U) \to \mathcal{F}(V)$, often denoted by res^U_V, is called the *restriction map*.

Example 1.0.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 1.0.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0, 1) \cup (2, 3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0, 1), (2, 3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0, 1))$ and $s_2 = 2 \in \mathcal{C}((2, 3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 1.0.4. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be presheaves on X with values in the same category (e.g., $\mathbf{Set}$, $\mathbf{Ab}$, $\mathbf{Ring}$, etc.). A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$. In other words, for every open set $U \subset X$, there is a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every inclusion of open sets $V \subset U$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}^U_V"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V). \end{tikzcd}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then φ is called a *morphism of sheaves.*

Fix a topological space X and a category $\mathbf{C}$. The sheaves (resp. presheaves) on X with values in $\mathbf{C}$ form a category, denoted by $\mathbf{Sh}(X, \mathbf{C})$ (resp. $\mathbf{PSh}(X, \mathbf{C})$), where the objects are sheaves (resp. presheaves) on X with values in $\mathbf{C}$ and the morphisms are morphisms of sheaves (resp. presheaves).

Definition 1.0.5. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in a category $\mathbf{C}$. For a point $x \in X$, the *stalk* of $\mathcal{F}$ at x, denoted by $\mathcal{F}_x$, is defined as the colimit

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the colimit is taken over all open neighborhoods U of x. An element of $\mathcal{F}_x$ is called a *germ* of sections of $\mathcal{F}$ at x.

More concretely, we have

$$\mathcal{F}_x = \{(U, s) : U \in \mathbf{Open}(X), U \ni x, s \in \mathcal{F}(U)\}/\sim,$$

where $(U, s) \sim (V, t)$ if there exists an open neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$.

Definition 1.0.6. Let X be a topological space and $\mathcal{F}$ a presheaf on X with values in $\mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.). A *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}^\dagger$ on X together with a morphism of presheaves $\eta : \mathcal{F} \to \mathcal{F}^+$ such that for every sheaf $\mathcal{G}$ on X and every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \eta$.
In other words, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & \mathcal{F}^\dagger \arrow[d, "\varphi^\dagger"] \\ & \mathcal{G}. \end{tikzcd}$$

To be checked.

The concrete describe of sheafification.

Definition 1.0.7. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The morphism φ is called *injective* (resp. *surjective*) if for every $x \in X$, the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective).

Proposition 1.0.8. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is injective if and only if for every open set $U \subset X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. To be checked.

Remark 1.0.9. The surjectivity on stalks cannot imply the surjectivity on sections. A counterexample is given by the exponential map $\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^*$ defined by $\exp(f) = e^f$, where $\mathcal{O}_{\mathbb{C}}$ is the sheaf of holomorphic functions on $\mathbb{C}$ and $\mathcal{O}_{\mathbb{C}}^*$ is the sheaf of non-vanishing holomorphic functions on $\mathbb{C}$. The induced map on stalks $\exp_z : \mathcal{O}_{\mathbb{C},z} \to \mathcal{O}_{\mathbb{C},z}^*$ is surjective for every $z \in \mathbb{C}$ by the existence of logarithm locally. However, the map on global sections $\exp(\mathbb{C}) : \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}^*(\mathbb{C})$ is not surjective since there is no entire function f such that $e^{f(z)} = z$ for all $z \in \mathbb{C}^*$. To be continued.

Proposition 1.0.10. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. Then φ is an isomorphism if and only if it is injective and surjective.

Now we consider sheaves with values in an abelian category.

Definition 1.0.11. Let X be a topological space and $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups on X. The *kernel* of φ, denoted by $\ker \varphi$, is the sheaf defined by

$$(\ker \varphi)(U) := \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$.
The *cokernel* of φ, denoted by $\mathsf{coker}\, \varphi$, is the sheafification of the presheaf defined by

$$(\mathsf{coker}\, \varphi)_{\mathrm{pre}}(U) := \mathsf{coker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$

for every open set $U \subset X$. To be continued.

Theorem 1.0.12. Let X be a topological space and $\mathbf{C}$ be an abelian category (e.g., $\mathbf{Ab}$). Then the category of sheaves on X with values in $\mathbf{C}$ is an abelian category.

Proof. To be continued. □

To be checked and continuous.

1.0.2 Locally ringed space

Definition 1.0.13. Let $f : X \to Y$ be a continuous map between topological spaces. The *push-forward* functor $f_* : \mathbf{Sh}(X, \mathbf{C}) \to \mathbf{Sh}(Y, \mathbf{C})$ is defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

for every open set $V \subset Y$ and sheaf $\mathcal{F} \in \mathbf{Sh}(X, \mathbf{C})$.

Definition 1.0.14. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 1.0.15. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Example 1.0.16 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Example 1.0.17 (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i\in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i\in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^{\sharp}(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ_i' satisfying the same properties, then by gluing $\psi_i' \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

1.0.3 Manifolds as locally ringed spaces

1.0.4 Vector bundles and $\mathcal{O}_X$-modules

Let $(X, \mathcal{O}_X)$ be a manifold (real or complex) and $(\mathcal{E}, \pi, X)$ a vector bundle over X.

It can regard as a sheaf on X.

Definition 1.0.18. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F}$ of abelian groups on X such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\mathsf{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear, where the $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(V)$ is induced by the restriction map $\mathsf{res}_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.
A *morphism of $\mathcal{O}_X$-modules* is a morphism of sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ such that for every open set $U \subseteq X$, the map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. To be checked...

Definition 1.0.19. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is said to be *locally free of rank r* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_U^r$, where $\mathcal{O}_U^r$ is the direct sum of r copies of $\mathcal{O}_U$. To be continued.

1.1 The First Properties of Schemes

If you learn the following content for the first time, it is recommended to skip all the proofs in this section and focus on the examples, remarks and the statements of propositions and theorems.

1.1.1 Schemes

Let R be a ring. Recall that the *spectrum* of R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set*.

Proposition 1.1.1. Let R be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

Proof. To be continued □

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 1.1.2. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i \in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.

These data form a category, denoted by **Sch**. If we fix a base scheme S, then an *S-scheme* is a scheme X together with a morphism $X \to S$. The category of S-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

Theorem 1.1.3. The functor $\mathsf{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. To be continued

Definition 1.1.4. A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if f induces an isomorphism of X onto an open (resp. closed) subscheme of Y. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. To be continued

Example 1.1.5. Let R be a graded ring. The *projective scheme* $\mathsf{Proj}\, R$ is defined as the scheme associated to the sheaf of rings

$$\mathcal{O}_{\mathsf{Proj}\, R} = \bigoplus_{d \geq 0} R_d.$$

It can be covered by open affine subschemes of the form $\mathsf{Spec}\, R_f$ for homogeneous elements $f \in R$. To be checked.

Example 1.1.6 (Glue open subschemes)**.** The construction in Example 1.0.17 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Example 1.0.17. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

Definition 1.1.7. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme Z of Y such that f factors through Z. More precisely, if $Y = \mathsf{Spec}\, A$ is affine, then the scheme theoretic image of f is $\mathsf{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover Y by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of f. To be checked.

1.1.2 Fiber product

Definition 1.1.8. Let $\mathcal{C}$ be a category and $X, Y, S \in \mathsf{Obj}(\mathcal{C})$ with morphisms $f : X \to S$ and $g : Y \to S$. A *fiber product* of X and Y over S is an object $Z \in \mathsf{Obj}(\mathcal{C})$ together with morphisms $p : Z \to X$ and $q : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

and satisfies the universal property that for any object $W \in \mathsf{Obj}(\mathcal{C})$ with morphisms $u : W \to X$ and $v : W \to Y$ such that $f \circ u = g \circ v$, there exists a unique morphism $h : W \to Z$ such that $p \circ h = u$ and $q \circ h = v$.

If a fiber product exists, it is unique up to a unique isomorphism. We denote the fiber product by $X \times_S Y$. To be checked.

Example 1.1.9. In the category of sets, the fiber product $X \times_S Y$ is given by

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\},$$

with the projections $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ being the restrictions of the natural projections. To be checked.

Remark 1.1.10. If one reverses the arrows in Definition 1.1.8, one gets the notion of *fiber coproduct*. It is also called the *pushout* or *amalgamated sum* in some literature. We denote the fiber coproduct of X and Y over S by $X \amalg_S Y$. Note that in the category of rings, the fiber coproduct $A \amalg_R B$ of R-algebras A and B over R is given by the tensor product $A \otimes_R B$. Dually, one can expect that fiber products of affine schemes correspond to tensor products of rings.

Theorem 1.1.11. The category of schemes admits fiber products. More precisely, given morphisms of schemes $f : X \to S$ and $g : Y \to S$, there exists a scheme Z together with morphisms $p : Z \to X$ and $q : Z \to Y$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

commutes and satisfies the universal property of the fiber product. We denote this scheme by $X \times_S Y$. To be continued

Definition 1.1.12. Let $f : X \to Y$ be a morphism of schemes and $y \in Y$ a point. The *scheme theoretic fiber* of f over y is the fiber product $X_y = X \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y)$ is the residue field of the local ring $\mathcal{O}_{Y,y}$. To be checked.

Definition 1.1.13. Let X be a scheme and $Z_1, Z_2 \subset X$ be closed subschemes defined by quasi-coherent sheaves of ideals $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{O}_X$, respectively. The *scheme theoretic intersection* of Z_1 and Z_2 is the closed subscheme $Z_1 \cap Z_2$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}_1 + \mathcal{I}_2$. To be checked.

1.1.3 Noetherian schemes and morphisms of finite type

Definition 1.1.14. A scheme X is *noetherian* if it admits a finite open cover $\{U_i\}_{i=1}^n$ such that each U_i is an affine scheme $\operatorname{Spec} A_i$ with A_i a noetherian ring. To be checked.

Proposition 1.1.15. A noetherian scheme is quasi-compact. To be checked.

Definition 1.1.16. Let S be a scheme. A scheme X is *of finite type* over S if there exists a finite open cover $\{U_i\}_{i=1}^n$ of S such that for each i, $f^{-1}(U_i)$ can be covered by finitely many affine open subsets $\{V_{ij}\}_{j=1}^{m_i}$ with $f(V_{ij}) \subseteq U_i$ and the induced morphism $f|_{V_{ij}} : V_{ij} \to U_i$ corresponds to a finitely generated algebra over the ring of global sections of U_i.
To be checked.

1.1.4 Integral, reduced and irreducible schemes

Definition 1.1.17. A topological space X is *irreducible* if it is non-empty and cannot be expressed as the union of two proper closed subsets. Equivalently, every non-empty open subset of X is dense in X. To be checked.

Proposition 1.1.18. Let X be a topological space satisfying the descending chain condition on closed subsets. Then X can be written as a finite union of irreducible closed subsets, called the *irreducible components* of X. Moreover, this decomposition is unique up to permutation of the components. To be checked.

Definition 1.1.19. A scheme X is *reduced* if its structure sheaf $\mathcal{O}_X$ has no nilpotent elements. To be checked.

Proposition 1.1.20. A scheme X is reduced if and only if for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a reduced ring. To be checked.

Proposition 1.1.21. Let X be a scheme. There exists a unique closed subscheme X of X such that X is reduced and has the same underlying topological space as X. Moreover, for any morphism of schemes $f : Y \to X$ with Y reduced, f factors uniquely through the inclusion $X \to X$. To be checked.

Definition 1.1.22. A scheme X is *integral* if it is both reduced and irreducible. To be checked.

Proposition 1.1.23. A scheme X is integral if and only if for every open affine subset $U = \operatorname{Spec} A \subset X$, the ring A is an integral domain. To be checked.

1.1.5 Dimension

Definition 1.1.24. The *Krull dimension* of a topological space X, denoted by $\dim X$, is the supremum of the lengths n of chains of distinct irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X. If no such finite supremum exists, we say that X has infinite dimension. To be checked.

1.1.6 Separated, proper and projective morphisms

Definition 1.1.25. A morphism of schemes $f : X \to Y$ is *separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. A scheme X is *separated* if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. To be checked.

Proposition 1.1.26. Any affine scheme is separated. More generally, any morphism between affine schemes is separated. To be checked.

Proposition 1.1.27. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any scheme T and any pair of morphisms $g_1, g_2 : T \to X$ such that $f \circ g_1 = f \circ g_2$, the equalizer of g_1 and g_2 is a closed subscheme of T. To be checked.

Proposition 1.1.28. A scheme X is separated if and only if for any pair of affine open subschemes $U, V \subset X$, the intersection $U \cap V$ is also an affine open subscheme. To be checked.

Proposition 1.1.29. The composition of separated morphisms is separated. Moreover, separatedness is stable under base change, i.e., if $f : X \to Y$ is a separated morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also separated. To be checked.

Proposition 1.1.30. A morphism of schemes $f : X \to Y$ is separated if and only if for every commutative diagram

$$\begin{tikzcd} \mathsf{Spec}\, K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathsf{Spec}\, R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists at most one morphism $\mathsf{Spec}\, R \to X$ making the entire diagram commute. To be checked.

Definition 1.1.31. A morphism of schemes $f : X \to Y$ is *universally closed* if for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map. To be checked.

Definition 1.1.32. A morphism of schemes $f : X \to Y$ is *proper* if it is of finite type, separated, and universally closed (i.e., for any morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a closed map). A scheme X is *proper* if the structure morphism $X \to \mathsf{Spec}\, \mathbb{Z}$ is proper. To be checked.

Theorem 1.1.33. Any projective morphism is proper. In particular, any projective scheme is proper. To be checked.

Proposition 1.1.34. The composition of proper morphisms is proper. Moreover, properness is stable under base change, i.e., if $f : X \to Y$ is a proper morphism and $Y' \to Y$ is any morphism, then the base change $X \times_Y Y' \to Y'$ is also proper. To be checked.

Proposition 1.1.35. A morphism of schemes $f : X \to Y$ is proper if and only if for every commutative diagram

$$\begin{tikzcd} \mathsf{Spec}\, K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathsf{Spec}\, R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where R is a valuation ring with field of fractions K, there exists a unique morphism $\mathsf{Spec}\, R \to X$ making the entire diagram commute. To be checked.

1.1.7 Varieties

1.2 Category of sheaves of modules

Mostly results in this section fits into the context of ringed spaces.

1.2.1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 1.2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets.

Definition 1.2.2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that there exists a surjective morphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

Remark 1.2.3. There are many versions of "local" properties for sheaves of $\mathcal{O}_X$-modules. To be continued.

Definition 1.2.4. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it is finitely generated, and for every open set $U \subseteq X$ and every morphism of sheaves of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of φ is finitely generated.

1.2.2 As abelian categories

Theorem 1.2.5. The categories of sheaves of abelian groups, quasi-coherent sheaves, and coherent sheaves on a ringed space $(X, \mathcal{O}_X)$ are all abelian categories. To be checked.

Theorem 1.2.6. Let $(X, \mathcal{O}_X)$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules has enough injectives. To be checked.

1.2.3 Relevant functors

Definition 1.2.7. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *sheaf Hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf of abelian groups defined as follows: for an open set $U \subseteq X$, we define

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of morphisms of sheaves of $\mathcal{O}_U$-modules from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. For an inclusion of open sets $V \subseteq U$, the restriction map

$$\mathrm{res}_{UV} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(V)$$

is defined by sending a morphism $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ to its restriction $\varphi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. To be continued.

Definition 1.2.8. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules defined as follows: for an open set $U \subseteq X$, we define

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

where $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is the tensor product of $\mathcal{O}_X(U)$-modules. For an inclusion of open sets $V \subseteq U$, the restriction map
To be continued.

Definition 1.2.9. Let $f : X \to Y$ be a morphism of ringed spaces. The *pull-back functor* $f^* : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$ is defined as follows: for an $\mathcal{O}_Y$-module $\mathcal{F}$, we define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

where $f^{-1}\mathcal{F}$ is the inverse image sheaf of $\mathcal{F}$. For a morphism of $\mathcal{O}_Y$-modules $\varphi : \mathcal{F} \to \mathcal{G}$, we define

$$f^*\varphi : f^*\mathcal{F} \to f^*\mathcal{G}$$

to be the morphism induced by the morphism of sheaves of abelian groups $f^{-1}\varphi : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$.
To be continued.

1.2.4 Cohomological theory

Definition 1.2.10. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *sheaf cohomology* $H^i(X, \mathcal{F})$ is defined as the i-th right derived functor of the global section functor $\Gamma(X, -) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$ applied to $\mathcal{F}$, i.e.,

$$H^i(X, \mathcal{F}) := \mathsf{R}^i\Gamma(X, \mathcal{F}).$$

To be checked.

Definition 1.2.11. Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The *i-th higher direct image* $\mathsf{R}^i f_*\mathcal{F}$ is defined as the i-th right derived functor of the direct image functor $f_* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ applied to $\mathcal{F}$, i.e.,

$$\mathsf{R}^i f_*\mathcal{F} := \mathsf{R}^i(f_*\mathcal{F}).$$

To be checked.

Proposition 1.2.12. Let $f : X \to Y$ be a morphism of ringed spaces, and let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. Then there are long exact sequences of $\mathcal{O}_Y$-modules

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to \mathrm{R}^1 f_*\mathcal{F} \to \mathrm{R}^1 f_*\mathcal{G} \to \mathrm{R}^1 f_*\mathcal{H} \to \mathrm{R}^2 f_*\mathcal{F} \to \cdots$$

To be checked.

Theorem 1.2.13 (Affine criterion by Serre)**.** Let X be a scheme. Then X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every $i > 0$. To be checked.

Theorem 1.2.14 (Leray spectral sequence)**.** Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there exists a spectral sequence

$$E_2^{p,q} = H^p(Y, \mathrm{R}^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

To be checked.

1.3 Line Bundles and Divisors

1.3.1 Cartier Divisors

Definition 1.3.1. Let X be a scheme. A *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of X. Equivalently, a Cartier divisor D can be represented by an open covering $\{U_i\}$ of X and a collection of rational functions $f_i \in \mathcal{K}_X^*(U_i)$ such that for any i, j, the function $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. We denote a Cartier divisor by $D = \{(U_i, f_i)\}$.

1.3.2 Line Bundles and Picard Group

Definition 1.3.2. Let X be a scheme. The *Picard group* of X is defined to be $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. The group operation is given by the tensor product of line bundles.

Definition 1.3.3. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X\times\{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X\times\{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\mathrm{alg}} \mathcal{L}'$. To be checked.

1.3.3 Weil Divisors and Reflexive Sheaves

To talk about Weil divisors, we need to work with normal schemes.

Definition 1.3.4. Let X be a normal integral scheme. A *Weil divisor* on X is a formal sum

$$D = \sum_{Z} n_Z Z,$$

where the sum runs over all prime divisors Z of X (i.e., integral closed subschemes of codimension 1) and $n_Z \in \mathbb{Z}$, such that for any affine open subset $U = \operatorname{Spec} A \subseteq X$, only finitely many Z intersecting U have nonzero coefficients n_Z. The group of Weil divisors on X is denoted by $\mathrm{WDiv}(X)$.

Definition 1.3.5. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. The *dual sheaf* of $\mathcal{F}$ is defined as $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. The sheaf $\mathcal{F}$ is called *reflexive* if the natural map $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is an isomorphism.

1.4 Projective morphisms and "positive" line bundles

1.4.1 Ample line bundles

Definition 1.4.1. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. To be continued.

Theorem 1.4.2 (Serre Vanishing)**.** Let X be a projective scheme over a field k and $\mathcal{L}$ an ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$ on X, there exists an integer N such that for all $n \geq N$, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

1.4.2 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 1.4.3. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^* x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^* x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$

satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Proposition 1.4.4. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. To be continued. □

Example 1.4.5. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 1.4.6. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}_A^{(m+1)(n+1)-1}$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 1.4.7. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. To be continued.

Example 1.4.8. Let

Example 1.4.9.

Definition 1.4.10. Let $\mathcal{L}$ be a line bundle on a scheme X. To be continued.

Definition 1.4.11. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. To be continued.

Theorem 1.4.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

To be continued.

Proposition 1.4.13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

To be continued.

Proof. To be continued. □

1.4.3 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 1.4.14. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

1.4.4 Asymptotic behavior

Definition 1.4.15. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. To be continued.

Definition 1.4.16. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. To be continued.

Theorem 1.4.17. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \mathrm{Proj}\, R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. To be continued.

Definition 1.4.18. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large n. To be continued.

Example 1.4.19. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. To be continued.

1.5 Flat, smooth and étale morphisms

1.5.1 Flat families

Definition 1.5.1. Let $f : X \to Y$ be a morphism of schemes. For a point $\xi \in X$, we say that f is *flat at* ξ if the local ring $\mathcal{O}_{X,\xi}$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that f is *flat* if it is flat at every point $\xi \in X$.

Definition 1.5.2. Let X be Y-scheme via a morphism $f : X \to Y$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *flat over* Y *at* $\xi \in X$ if the stalk $\mathcal{F}_\xi$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via the induced map $f_\xi^\sharp : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}$. We say that $\mathcal{F}$ is *flat over* Y if it is flat over Y at every point $\xi \in X$.

Proposition 1.5.3. We have the following fundamental properties of flat morphisms:

(a) open immersions are flat;

(b) the composition of flat morphisms is flat;

(c) flatness is preserved under base change;

(d) a coherent sheaf $\mathcal{F}$ on a noetherian scheme X is flat over X iff it is locally free.

To be checked.

Proposition 1.5.4. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. Then the set of points $\xi \in X$ at which f is flat is open in X. To be checked.

Proposition 1.5.5. Let X be a regular integral scheme of dimension 1 and $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is flat over X iff it is torsion-free, i.e., for every non-zero-divisor $s \in \mathcal{O}_{X,x}$, the multiplication map

$$s : \mathcal{F} \to \mathcal{F}$$

is injective. To be checked.

Proposition 1.5.6. Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field $\mathbf{k}$. Then for every point $\xi \in X$, we have

$$\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}.$$

To be checked.

Theorem 1.5.7 (Miracle flatness)**.** Let $f : X \to Y$ be a morphism between noetherian schemes. Suppose that X is Cohen–Macaulay and that Y is regular. Then f is flat at $\xi \in X$ iff $\dim_\xi X = \dim_{f(\xi)} Y + \dim_\xi X_{f(\xi)}$. To be checked.

Theorem 1.5.8. Let T be a integral noetherian scheme and $f : X \to T$ be a projective morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix a relatively ample line bundle H on X over T. Then $\mathcal{F}$ is flat over T iff the Hilbert polynomials

$$P(X_t, \mathcal{F}_t, H_t)(n) = \chi(X_t, \mathcal{F}_t \otimes H_t^{\otimes n})$$

are independent of $t \in T$. To be checked.

To be added: deformation, algebraic families...

1.5.2 Base change and semicontinuity

Theorem 1.5.9 (Grauert's theorem)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the sheaf $R^i f_* \mathcal{F}$ is coherent on Y, and there exists an open subset $U \subseteq Y$ such that for every point $y \in U$, the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism. To be checked.

Theorem 1.5.10 (Cohomology and base change)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. For each integer $i \geq 0$, the following are equivalent:

(a) the base change map

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

is an isomorphism for all points $y \in Y$;

(b) the sheaf $R^i f_* \mathcal{F}$ is locally free on Y.

To be checked.

Theorem 1.5.11 (Semicontinuity of cohomology)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the function

$$h^i : Y \to \mathbb{Z}, \quad y \mapsto \dim_{\kappa(y)} H^i(X_y, \mathcal{F}_y)$$

is upper semicontinuous on Y.

To be checked.

1.5.3 Smooth morphisms

Definition 1.5.12. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. For $\xi \in X$ with image $\zeta = f(\xi) \in Y$, set $\overline{\zeta} : \operatorname{Spec} \overline{\kappa(\zeta)} \to Y$ to be the geometric point over ζ and $X_{\overline{\zeta}}$ be the geometric fiber over ζ. We say that f is *smooth at* ξ if f is flat at ξ and the geometric fiber $X_{\overline{\zeta}}$ is regular over $\overline{\kappa(\zeta)}$ at every point lying over ξ. We say that f is *smooth* if it is smooth at every point $\xi \in X$.

To be checked.

1.5.4 Étale morphisms

Definition 1.5.13. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. We say that f is *étale at* ξ if f is smooth and finite at ξ. We say that f is *étale* if it is étale at every point $\xi \in X$.

To be checked.

Chapter 2

More Scattered Topics

Chapter 3

Surfaces

3.1 The first properties of surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

3.1.1 Basic concepts

3.1.2 Riemann-Roch Theorem for surfaces

3.1.3 Hodge Index Theorem

3.2 Birational geometry on surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

3.2.1 Castelnuovo's Theorem and Run the MMP

Theorem 3.2.1 (Castelnuovo's contractibility criterion)**.** Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. Let $C \subseteq X$ be an irreducible curve. Then there exists a birational morphism $f : X \to Y$ contracting C to a smooth point if and only if $C \cong \mathbb{P}^1$ and $C^2 = -1$.

3.3 Coarse classification of surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. We want to classify X up to birational equivalence. Let K_X be the canonical divisor of X.

Theorem 3.3.1. Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. Suppose that the Kodaira dimension $\kappa(X) \geq 0$. Then the linear system $|12K_X|$ is base point free. To be checked.

3.3.1 Classification

Theorem 3.3.2 (Enriques-Kodaira classification)**.** Let X be a smooth projective surface over $\Bbbk$. Then X is birational to a unique minimal model X', unless X is birational to a ruled surface. Moreover, the minimal model X' falls into one of the following classes:

(a) $\kappa(X') = -\infty$: $X' \cong \mathbb{P}^2$ or X' is a ruled surface;

(b) $\kappa(X') = 0$: X' is a K3 surface, an abelian surface or their quotients;

(c) $\kappa(X') = 1$: X' is an elliptic surface;

(d) $\kappa(X') = 2$: X' is a surface of general type.

To be checked.

3.4 Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$. This section is mainly based on [Har77, Chapter V.2].

3.4.1 Minimal Section and Classification

Definition 3.4.1 (Ruled surface)**.** A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all geometric fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 3.4.2. There exists a section of π.

Proof. To be continued... □

Proposition 3.4.3. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Proof. Let $\sigma : C \to X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π, $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in C$. By Miracle Flatness (??), f is flat. By Grauert's Theorem (Theorem 1.5.9), $\mathcal{E}$ is a vector bundle of rank 2 on C and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in C$.

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every $x \in X$, we have

$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \to \mathcal{L}$ is surjective.

By ??, we have a morphism $\varphi : X \to \mathbb{P}_C(\mathcal{E})$ over C such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Since $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, $\varphi|_{X_t} : X_t \to \mathbb{P}_C(\mathcal{E})_t$ is an isomorphism for any $t \in C$. Hence φ is bijection on the underlying sets. Here is a serious gap. Why fiberwise isomorphism implies isomorphism? □

Lemma 3.4.4. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*. In particular, if $\mathcal{E}$ is normalized, then $e = -\deg c_1(\mathcal{E})$ is an invariant of the ruled surface X.

Proof. We can suppose that $\mathcal{E}$ is globally generated since we can always twist $\mathcal{E}$ by a sufficiently ample line bundle on C. Then for all line bundle $\mathcal{L}$ of degree sufficiently large, $\mathcal{L}$ is very ample and hence $H^0(C, \mathcal{E} \otimes \mathcal{L}) \neq 0$. By Lemma 3.4.2 and ??, $\mathcal{E}$ is an extension of line bundles. Then for all line bundle $\mathcal{L}$ of degree sufficiently negative, $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ since line bundles of negative degree have no global sections. Hence we can find a line bundle $\mathcal{M}$ on C of lowest degree such that $H^0(C, \mathcal{E} \otimes \mathcal{M}) \neq 0$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{M}$, we are done. □

Remark 3.4.5. The invariant e is unique but the normalization of $\mathcal{E}$ is not unique. For example, if $\mathcal{E}$ is normalized, then so is $\mathcal{E} \otimes \mathcal{L}$ for any line bundle $\mathcal{L}$ on C of degree 0. To be continued...

Suppose that $X \cong \mathbb{P}_C(\mathcal{E})$ where $\mathcal{E}$ is a normalized vector bundle of rank 2 on C. Since $H^0(C, \mathcal{E}) \neq 0$, choosing a non-zero section s, we get an exact sequence

$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{E} \to \mathcal{E}/\mathcal{O}_C \to 0.$$

We claim that $\mathcal{E}/\mathcal{O}_C$ is a line bundle on C. Since C is a curve, we only need to check that $\mathcal{E}/\mathcal{O}_C$ is torsion-free.

To be continued...

Definition 3.4.6. A section C_0 of π is called a *minimal section* if to be continued...

Lemma 3.4.7. Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface over a smooth curve C of genus g with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on C with $\deg \mathcal{L} = -e$.

(b) If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.

Proof. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable, we can assume that $H^0(C, \mathcal{L}_1) \neq 0$. If $\deg \mathcal{L}_1 > 0$, then $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$, contradicting the normalization of $\mathcal{E}$. Similarly $\deg \mathcal{L}_2 \leq 0$. Then $\mathcal{L}_1 \cong \mathcal{O}_C$. And hence $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$.

If $\mathcal{E}$ is indecomposable, we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

which is a non-trivial extension, with $\mathcal{L}$ a line bundle on C of degree $-e$. Hence by ??, we have

$0 \neq \mathrm{Ext}^1_C(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$. By Serre duality, we have $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$. Hence $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$.

On the other hand, let $\mathcal{M}$ be a line bundle on C of degree -1. Twist the above exact sequence by $\mathcal{M}$ and take global sections, we have an equation

$$h^0(\mathcal{M}) - h^0(\mathcal{E} \otimes \mathcal{M}) + h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{M}) + h^1(\mathcal{E} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = 0.$$

Since $\deg \mathcal{M} < 0$ and $\mathcal{E}$ is normalized, we have $h^0(\mathcal{M}) = h^0(\mathcal{E} \otimes \mathcal{M}) = 0$. By Riemann-Roch, we have $h^1(\mathcal{M}) = g$ and $h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = -e - 1 + 1 - g$. Hence

$$h^1(\mathcal{E} \otimes \mathcal{M}) = e + 2g \geq 0.$$

This gives $e \geq -2g$. □

Theorem 3.4.8. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Proof. This is a direct consequence of Lemma 3.4.7. □

Example 3.4.9. Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$.

Let C be covered by two standard affine charts U_0, U_1 with coordinate u on U_0 and v on U_1 such that $u = 1/v$ on $U_0 \cap U_1$. On U_i, let $\mathcal{O}(-e)|_{U_i}$ be generated by s_i for $i = 0, 1$. We have $s_0 = u^e s_1$ on $U_0 \cap U_1$.

On $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$, let $[x_0 : x_1]$ and $[y_0 : y_1]$ be the homogeneous coordinates of $\mathbb{P}^1$ on X_0 and X_1 respectively. Then the transition function on $X_0 \cap X_1$ is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

Remark 3.4.10. The surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ is also called the *Hirzebruch surface*.

Theorem 3.4.11. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and for each e there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on E with $\deg \mathcal{L} = -e$.

Proof. Only the indecomposable case needs a proof. By Lemma 3.4.7, we have $-2 \leq e \leq 0$ and a non-trivial extension

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle on E of degree $-e$.

Case 1. $e = 0$.

In this case, $\mathcal{L}$ is of degree 0 and $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \neq 0$. Hence $\mathcal{L} \cong \mathcal{O}_E$. To be continued...

Case 2. $e = -1$.

In this case, $\mathcal{L}$ is of degree 1 and $H^1(E,\mathcal{L}) \cong H^0(E,\mathcal{L}^{-1}) = 0$. By Riemann-Roch, we have $h^0(E,\mathcal{L}) = 1$.

Case 3. $e = -2$.

To be continued...

□

Example 3.4.12. To be continued...

3.4.2 The Néron-Severi Group of Ruled Surfaces

Proposition 3.4.13. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and F a fiber of π. Then $\mathrm{Pic}(X) \cong \mathbb{Z}[C_0] \oplus \pi^* \mathrm{Pic}(C)$.

Proof. Let D be any divisor on X with $D.F = a \in \mathbb{Z}$. Then $D - aC_0$ is numerically trivial on the fibers of π. Let $\mathcal{L} = \mathcal{O}_X(D - aC_0)$. Then $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$ for any $t \in C$. By Grauert's Theorem (Theorem 1.5.9), $\pi_*\mathcal{L}$ is a line bundle on C and the natural map $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. □

Proposition 3.4.14. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let F be a fiber of π. Then $K_X \sim -2C_0 + \pi^*(K_C - c_1(\mathcal{E}))$. Numerically, we have $K_X \equiv -2C_0 + (2g - 2 - e)F$ where e is the invariant of X. Check this carefully.

Proof. To be continued. □

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

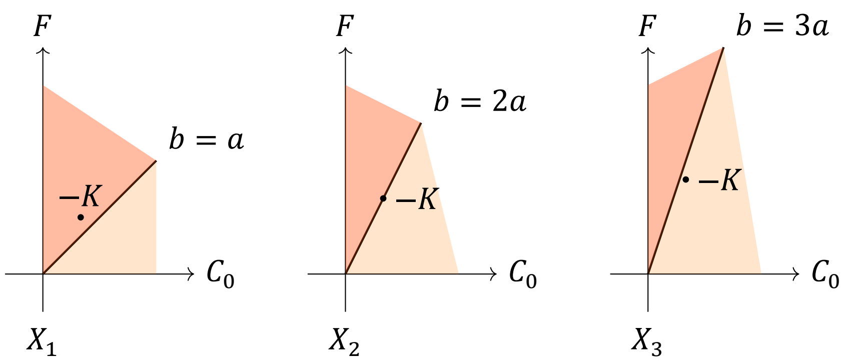
Theorem 3.4.15. Let $\pi : X \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with invariant e. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \sim aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a, b \geq 0$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$.

Proof. To be continued... □

Example 3.4.16. Here we draw the Néron-Severi group of the rational ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 1, 2, 3$.



We have $-K_{X_e} \equiv 2C_0 + (2+e)F$. For $e = 1$, $-K$ is ample and hence X_1 is a del Pezzo surface. For $e = 2$, $-K$ is nef and big but not ample. For $e \geq 3$, $-K$ is big but not nef.

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

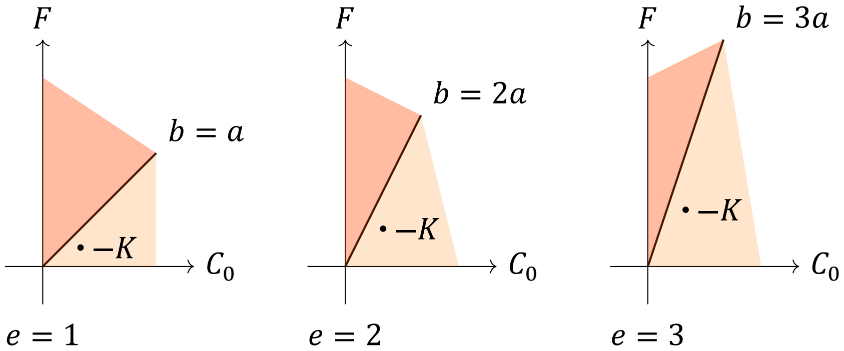
Theorem 3.4.17. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is decomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$.

Proof. To be continued... □

Example 3.4.18. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with decomposable normalized $\mathcal{E}$ for $e = 1, 2, 3$.



In this case, $-K \equiv 2C_0 + eF$ is always big but not nef.

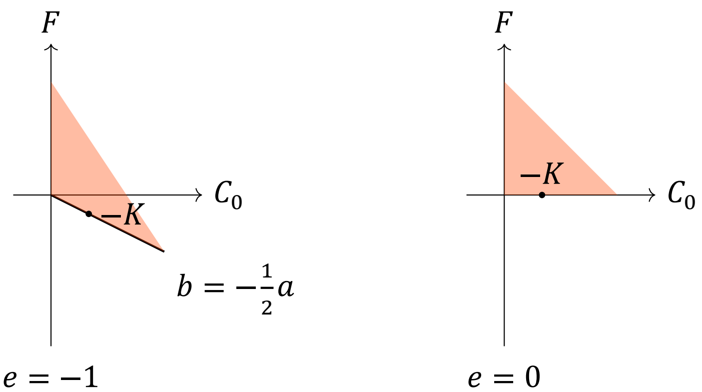
Theorem 3.4.19. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is indecomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq \frac{1}{2}ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > \frac{1}{2}ae$.

Proof. To be continued... □

Example 3.4.20. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with indecomposable normalized $\mathcal{E}$ for $e = -1, 0$.



In this case, $-K \equiv 2C_0 + eF$ is always nef but not big.

Proposition 3.4.21. Let $\pi : X \to C$ be a ruled surface over a smooth curve C. Then every nef divisor on X is semi-ample. Check this carefully.

3.5 K3 surface

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

3.5.1

Definition 3.5.1. A *K3 surface* is a smooth, projective surface X with trivial canonical bundle $K_X \cong \mathcal{O}_X$ and irregularity $q(X) = h^1(X, \mathcal{O}_X) = 0$.

Example 3.5.2. A smooth quartic surface $X \subseteq \mathbb{P}^3$ is a K3 surface. Indeed, by the adjunction formula, we have

$$K_X = (K_{\mathbb{P}^3} + X)|_X = (-4H + 4H)|_X = 0,$$

where H is a hyperplane in $\mathbb{P}^3$. Moreover, by the Lefschetz hyperplane theorem, we have $h^1(X, \mathcal{O}_X) = h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$. To be checked.

3.5.2 Hodge Structure and Moduli of K3 surfaces

3.6 Elliptic surfaces

3.6.1 The first properties

Definition 3.6.1. An *elliptic surface* is a smooth projective surface S together with a surjective morphism $\pi : S \to C$ to a smooth projective curve C such that the generic fiber of π is a smooth curve of genus 1, and π has a section $s : C \to S$. To be continued...

3.7 Some Singular Surfaces

In this section, fix an algebraically closed field $\Bbbk$. Everything is over $\Bbbk$ unless otherwise specified.

3.7.1 Projective cone over smooth projective curve

Let $C \subset \mathbb{P}^n$ be a smooth projective curve. The *projective cone* over C is the projective variety $X \subset \mathbb{P}^{n+1}$ defined by the same homogeneous equations as C. The variety X is singular at the vertex of the cone, which corresponds to the point $[0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}$.

Chapter 4

Birational Geometry

4.1 Technical Preparation

4.1.1 Resolution of singularities

Theorem 4.1.1. Let X be a normal variety over an algebraically closed field $\Bbbk$ of characteristic zero. Then there exists a log resolution of singularities $f : Y \to X$ such that Y is smooth and the exceptional divisor E is a simple normal crossing divisor.

4.1.2 Negativity Lemma

Theorem 4.1.2. Let $f : Y \to X$ be a proper birational morphism between normal varieties. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y such that $-D$ is f-nef. Then D is effective if and only if f_*D is.

Proof. To be completed. □

4.1.3 General adjunction formula

Theorem 4.1.3 (Adjuction formula)**.** Let X be a normal variety and S be a reduced divisor on X. Need to check the statement.

Proof. To be completed. □

4.1.4 Exceptional divisors

Proposition 4.1.4. Let $f : Y \to X$ be a proper birational morphism between normal varieties. Let E be an effective f-exceptional divisor on Y. Then we have $f_*\mathcal{O}_Y(E) \cong \mathcal{O}_X$.

4.2 Kodaira Vanishing Theorem

4.2.1 Preliminary

Theorem 4.2.1 (Serre Duality)**.** Let X be a Cohen-Macaulay projective variety of dimension n over $\mathbf{k}$ and D a divisor on X. Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \dots, n.$$

Theorem 4.2.2 (Log Resolution of Singularities)**.** Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there is a smooth variety (or analytic space) Y and a projective morphism $f : Y \to X$ such that

(a) f is an isomorphism over $X - (\mathrm{Sing}(X) \cup \mathrm{Supp}\, Z)$,

(b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(c) $\mathrm{Exc}(f) \cup D$ is an snc divisor.

Theorem 4.2.3 (Lefschetz Hyperplane Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 4.2.4 (Hodge Decomposition)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Then for any k, there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 4.2.3 and Theorem 4.2.4, we have the following lemma.

Lemma 4.2.5. Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map $r_k : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \to H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \to H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n - 1$ and an injection for $p = n - 1$.

Theorem 4.2.6 (Leray spectral sequence)**.** Let $f : Y \to X$ be a morphism of varieties and $\mathcal{F}$ a coherent sheaf on Y. Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_*\mathcal{F}) \implies H^{p+q}(Y, \mathcal{F}).$$

4.2.2 Kodaira Vanishing Theorem

Lemma 4.2.7. Let X be a smooth projective variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on X. Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \to X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D. It induces a homomorphism $\mathcal{L}^{-m} \to \mathcal{O}_X$. Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}\right) \Big/ (\mathcal{L}^{-m} \to \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then $\mathcal{A}$ is a finite $\mathcal{O}_X$-algebra. Let $Y \coloneqq \mathsf{Spec}_X \mathcal{A}$. Then Y is a finite $\mathcal{O}_X$-scheme and the natural morphism $f : Y \to X$ is finite and surjective.

For every $x \in X$, let $\mathcal{L}$ locally generated by t near x. Then $\mathcal{O}_Y$ locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y. Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. □

Remark 4.2.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D_i' = f^*(D_i)$. Then $D' + \sum_{i=1}^k D_i'$ is snc on Y by considering the local regular system of parameters.

Lemma 4.2.9. Let $f : Y \to X$ be a finite surjective morphism of projective varieties and $\mathcal{L}$ a line bundle on X. Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits by the trace map $(1/n)\,\mathrm{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. □

Theorem 4.2.10 (Kodaira Vanishing Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and A an ample divisor on X. Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 4.2.7 and 4.2.9, after taking a multiple of A, we can assume that A is effective.

Then we have an exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \to H^{i-1}(X, \mathcal{O}_A) \to H^i(X, \mathcal{O}_X(-A)) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 4.2.5 and Serre duality (Theorem 4.2.1). □

4.2.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

Theorem 4.2.11 (Kawamata-Viehweg Vanishing Theorem I)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{R}$-divisor on X. Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Theorem 4.2.12 (Kawamata-Viehweg Vanishing Theorem II)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{Q}$-divisor on X. Suppose that $\lceil D \rceil - D$ has snc support. Then

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall i > 0.$$

Theorem 4.2.13 (Kawamata-Viehweg Vanishing Theorem III)**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Let D be a nef $\mathbb{Q}$-divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

If we replace the assumption ”nef and big” of D by ”ample” in II and III, we denote them as “II(ample)” and “III(ample)” respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \implies \text{II(ample)} \implies \text{III(ample)} \implies \text{I} \implies \text{II} \implies \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

Proof of II (Theorem 4.2.12). Set $M := \lceil D \rceil$. Let

$$B := \sum_{i=1}^{k} b_i B_i := \lceil D \rceil - D = M - A, \quad b_i \in (0,1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k. When $k = 0$, the conclusion follows from Theorem 4.2.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 4.2.10.)) Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 4.2.15 and 4.2.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 4.2.7 on B_k, we can find a finite surjective morphism $f : X' \to X$ such that $f^*B_k = cB'_k, B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^k B'_i$ is an snc divisor on X'. Let $B' = \sum_{i=1}^{k-1} B'_i, A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$ vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^*\mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 4.2.9. □

Proof of III (Theorem 4.2.13). Let $f : \tilde{X} \to X$ be a resolution such that $\operatorname{Supp} f^*B \cup \operatorname{Exc} f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X},\tilde{B})} + f^*D,$$

where $\tilde{B} \in (0,1)$ has snc support and E is an effective exceptional divisor.

By Lemma 4.2.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X},\tilde{B})} + f^*D) = H^i(X, f_*\mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 4.2.12 in either case relative to the assumption of D. □

Proof of I (Theorem 4.2.11). By Lemma 4.2.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor A. Then the theorem comes down to Theorem 4.2.13. □

Lemma 4.2.14. Let $f : Y \to X$ be a birational morphism of projective varieties with Y smooth and X has only rational singularities. Let E be an effective exceptional divisor on Y and D a divisor on X. Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

Proof. I am unable to proof this lemma. □

Lemma 4.2.15. Let X be a projective variety, $\mathcal{L}$ a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \to X$ and a line bundle $\mathcal{L}'$ on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X, then we can take f such that f^*D is an snc divisor on Y.

Proof. We can assume that $\mathcal{L}$ is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "g"] \\ X \arrow[r, "\varphi_{\mathcal{L}}"] & \mathbb{P}^N \end{tikzcd},$$

where $g : [x_0 : \ldots : x_N] \mapsto [x_0^m : \ldots : x_N^m]$. The morphism f is finite and surjective since so is g. Let $\mathcal{L}' := \psi^*\mathcal{L}$.

For smoothness, we can compose g with a general automorphism of $\mathbb{P}^N$. Then the conclusion follows from [Har77, Chapter III, Theorem 10.8]. □

Lemma 4.2.16 (ref. [KM98, Theorem 5.10, 5.22])**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Lemma 4.2.17. Let X be a projective variety of dimension n and D a nef and big divisor on X. Then there exists an effective divisor B such that for every k, there is an ample divisor A_k such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k}B.$$

Proof. By definition of big divisor, there exists an ample divisor A_1 and effective divisor B such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k}B.$$

Since A is ample and D is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. □

4.3 Cone Theorem

4.3.1 Preliminary

Theorem 4.3.1 (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27])**.** Let X be a projective variety and $\mathcal{L}$ an semiample line bundle on X. Then there exists a fibration $\varphi : X \to Y$ of projective varieties such that for any $m \gg 0$ with $\mathcal{L}^m$ base point free, we have that the morphism $\varphi_{\mathcal{L}^m}$ induced by $\mathcal{L}^m$ is isomorphic to φ. Such a fibration is called the *Iitaka fibration* associated to $\mathcal{L}$.

Theorem 4.3.2 (Rigidity Lemma, ref. [Deb01, Lemma 1.15])**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Theorem 4.3.3. Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

Proposition 4.3.4. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

Schetch of proof. Take a resolution $f : Y \to X$, then f^*H is nef on Y and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor H_Y on Y closed enough to f^*H such that $K_Y \cdot H_Y^{n-1} < 0$. By [MM86, Theorem 5] and take limit for H_Y. □

Lemma 4.3.5 (ref. [Kaw91, Lemma])**.** Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Let E be an irreducible component of dimension d of the exceptional locus of f and $\nu : E^\nu \to X$ the normalization of E. Suppose that $f(E)$ is a point. Then for any ample divisor

H on X, we have
$$K_{E^\nu} \cdot \nu^* H^{d-1} \leq K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

4.3.2 Non-vanishing Theorem

Theorem 4.3.6 (Non-vanishing Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have
$$H^0(X, mD) \neq 0.$$

Proof. To be completed. □

4.3.3 Base Point Free Theorem

Theorem 4.3.7 (Base Point Free Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, mD is base point free.

Proof. To be completed. □

Remark 4.3.8. In general, we say that a Cartier divisor D is *semiample* if there exists a positive integer m such that mD is base point free. The statement in Base Point Free Theorem (Theorem 4.3.7) is strictly stronger than the semiample condition. For example, let $\mathcal{L}$ be a torsion line bundle, then $\mathcal{L}$ is semiample, but there exists no positive integer M such that $m\mathcal{L}$ is base point free for all $m > M$.

4.3.4 Rationality Theorem

Lemma 4.3.9 (ref. [KM98, Theorem 1.36]). Let X be a proper variety of dimension n and $D_1, \ldots, D_m$ Cartier divisors on X. Then the Euler characteristic $\chi(n_1 D_1, \ldots, n_m D_m)$ is a polynomial in $(n_1, \cdots, n_m)$ of degree at most n.

Theorem 4.3.10 (Rationality Theorem). Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let
$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$
be the nef threshold of (X, B) with respect to H. Then $t = v/u \in \mathbb{Q}$ and
$$0 \leq v \leq a(X) \cdot (\dim X + 1).$$

Proof. For every $r \in \mathbb{R}_{>0}$, let
$$v(r) := \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
We need to show that $v(t) \leq a(\dim X + 1)$. For every $(p, q) \in \mathbb{Z}^2_{>0}$, set $D(p, q) := paK_{(X,B)} + qH$. If $(p, q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, then we have $D(p, q)$ is not nef and $D(p, q) - K_{(X,B)}$ is ample.

Step 1. We show that a polynomial $P(x,y)\neq 0\in\mathbb{Q}[x,y]$ of degree at most n is not identically zero on the set

$$\{(p,q)\in\mathbb{Z}^2 : p,q>M, 0<atp-q<t\varepsilon\},\quad \forall M>0,$$

if $v(t)\varepsilon>a(n+1)$.

If $v(t)=\infty$, for any n, we show that we can find infinitely many lines L such that $\#L\cap\Lambda\geq n+1$. If so, Λ is Zariski dense in $\mathbb{Q}^2$. Since $1/at\in\mathbb{R}\setminus\mathbb{Q}$, there exist $p_0,q_0>M$ such that

$$0<\frac{p_0}{q_0}-\frac{1}{at}<\frac{\varepsilon}{(n+1)a}\cdot\frac{1}{q_0},\ \text{i.e.}\ 0<atp_0-q_0<\frac{\varepsilon t}{n+1}.$$

Then $(ip_0,iq_0)\in\Lambda\cap\{p_0y=q_0x\}$ for $i=1,\cdots,n+1$. Since M is arbitrary, there are infinitely many such lines L.

Suppose $v(t)=v<\infty$ and $t=v/u$. Then the inequality is equivalent to $0<aup-vq<\varepsilon v$. Note that $\gcd(au,v)|a$, then $aup-vq=ai$ has integer solutions for $i=1,\cdots,n+1$. Since $v(t)\varepsilon>a(n+1)$, there are at least $n+1$ lines which intersect Λ in infinitely many points. This enforce any polynomial which vanishes on Λ has degree at least $n+1$.

Step 2. There exists an index set $\Lambda\subset\mathbb{Z}^2$ such that Λ contains all sufficiently large (p,q) with $0\leq atp-q\leq t$ and

$$Z:=\operatorname{Bs}|D(p,q)|=\operatorname{Bs}|D(p',q')|\neq\emptyset,\quad\forall (p,q),(p',q')\in\Lambda.$$

For every $(p,q)\in\mathbb{Z}^2_{>0}$ with $0<atp-q<t$, there exists $M>0$ such that

$$D(\alpha,\beta)=\alpha aK_{(X,B)}+\beta H$$

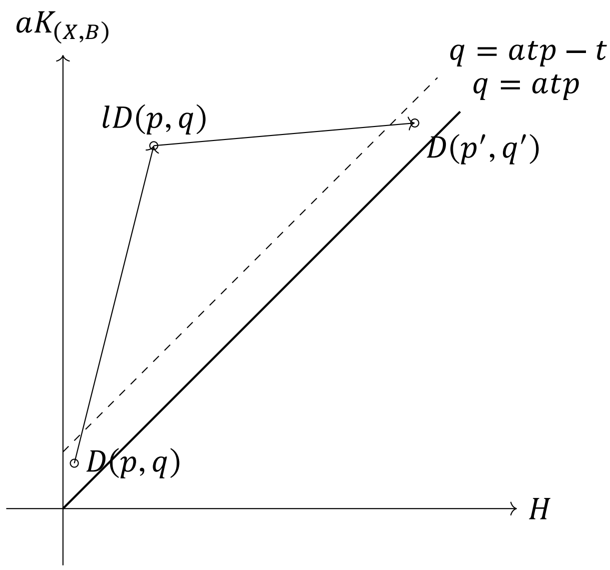
is base point free for all $\alpha=0,\cdots,p$ and $\beta>M$. Choose M' large enough such that for all $(p',q')\in\mathbb{Z}^2_{>0}$ with $p',q'>M'$ and $0<atp'-q'<t$, write

$$p'=lp+p_0,\quad q'=lq+q_0$$

for some $l\in\mathbb{Z}_{\geq 0}$ and $0\leq p_0<p$, we have $q_0>M$. The existence of such M' follows from the estimate

$$q_0=q'-lq=q'-\frac{p'-p_0}{p}q>q'-(p'-p_0)(at-\delta)>p'\delta,$$

where $\delta>0$ is a small enough number such that $at-\delta>q/p$.



Then $D(p',q') - lD(p,q) = D(p_0,q_0)$ is base point free. It follows that $\mathrm{Bs}\,|D(p',q')| \subseteq \mathrm{Bs}\,|D(p,q)|$. By noetherian induction, there exists an index set Λ such that $\mathrm{Bs}\,|D(p,q)| = Z$ for all $(p,q) \in \Lambda$.

Step 3. Suppose the contradiction that $v(t) > a(\dim X+1)$. Then we show that $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem (Theorem 4.3.7).

Let $P(x,y) := \chi(D(x,y))$ be the Hilbert polynomial of $D(x,y)$. Note that $P(0,n) = \chi(nH) \neq 0$ since H is ample. Then $P(x,y) \neq 0$ and $\deg P \leq \dim X$. By Step 1, P is not identically zero on Λ. Note that $D(p,q) - K_{(X,B)}$ is ample for all $(p,q) \in \Lambda$, then $h^i(X, D(p,q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem (Theorem 4.2.13). Then

$$P(p,q) = \chi(D(p,q)) = h^0(X, D(p,q)) \neq 0$$

for some $(p,q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$.

Step 4. We follow the same line of the proof of Base Point Free Theorem (Theorem 4.3.7) to show that there is a section which does not vanish on Z.

Fix $(p,q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n+1)/u$. Let $f : Y \to X$ be a resolution such that

(a) $K_{Y,B_Y} = f^*K_{(X,B)} + E_Y$ for some effective exceptional divisor E_Y, and Y, B_Y is a klt pair;

(b) $f^*|D(p,q)| = |L| + F$ for some effective divisor F and a base point free divisor L, and $f(\mathrm{Supp}\,F) = Z$;

(c) $f^*D(p,q) - f^*K_{(X,B)} - E_0$ is ample for some effective $\mathbb{Q}$-divisor $E_0 \in (0,1)$, and coefficients of E_0 are sufficiently small;

(d) $B_Y + E_Y + F + E_0$ has snc support.

Such resolution exists by [KM98].

Let $c := \inf\{\lfloor B_Y + E_0 + tF \rfloor \neq 0\}$. Adjust the coefficients of E_0 slightly such that $\lfloor B_Y + E_0 + cF \rfloor = F_0$ for unique prime divisor F_0 with $F_0 \subset \mathrm{Supp}\,F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then (Y, Δ_Y) is a klt pair.

Let

$$\begin{aligned} N(p',q') &:= f^*D(p',q') + E_Y - F_0 - K_{(Y,\Delta_Y)} \\ &= \Big(f^*D(p',q') - (1+c)f^*D(p,q)\Big) + \Big(f^*D(p,q) - f^*K_{(X,B)} - E_0\Big) + c\Big(f^*D(p,q) - F\Big). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p',q') \in \Lambda : 0 < atp' - q' < atp - q,\ p', q' > (1+c)\max\{p,q\}\},$$

the divisor $f^*D(p',q') - (1+c)f^*D(p,q) = f^*D(p' - (1+c)p, q' - (1+c)q)$ is ample, and hence $N(p',q')$ is ample.

By the exact sequence

$$0 \to \mathscr{O}_Y(f^*D(p',q') + E_Y - F_0) \to \mathscr{O}_Y(f^*D(p',q') + E_Y) \to \mathscr{O}_{F_0}((f^*D(p',q') + E_Y)|_{F_0}) \to 0$$

and Kawamata-Viehweg Vanishing Theorem (Theorem 4.2.13), we get a surjective map

$$H^0(Y, f^*D(p', q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}).$$

On F_0, consider the polynomial $\chi((f^*D(p', q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n - 1$ and by the construction of $(p, q), \Lambda_0$, similar to Step 3, we can show that $\chi((f^*D(p', q') + E_Y)|_{F_0})$ is not identically zero on Λ_0. By adjunction, we have $(f^*D(p', q') + E_Y)|_{F_0} = N(p', q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p', q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (Theorem 4.2.13) to get

$$h^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}) = \chi(F_0, (D(p', q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \mathrm{Bs}\,|D(p', q')|$.

□

4.3.5 Cone Theorem and Contraction Theorem

Theorem 4.3.11 (Cone Theorem)**.** Let (X, B) be a projective klt pair. Then there exist countably many curves $C_i \subset X$ such that

(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. Let $F_D := \mathrm{Psef}_1(X) \cap D^{\perp}$ for a nef divisor D on X. If $\dim F_D = 1$, we also write $R_D := F_D$. Let $H_1, \cdots, H_{\rho-1}$ be ample divisors on X such that they together with $K_{(X,B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} := S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

Step 1. There exists an integer N such that for every $K_{(X,B)}$-negative extremal face F_D and for every ample divisor H, there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \neq F_{nD + rK_{(X,B)} + NH} \subset F_D$.

Let $N := (a(X)(\dim X + 1))!$, where $a(X)$ is the number in Theorem 4.3.10. For every n, $nD + H$ is an ample divisor and by Theorem 4.3.10, the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \geq 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since $K_{(X,B)} + (N/r_n)((n+1)D + H)$ is nef, we have $r_n \leq r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$ implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \to r \in \mathbb{Z}_{\geq 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors N_1, N_2, we have $F_{N_1+N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists m large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

Step 2. Let $\Phi : N_1(X)_{K_{(X,B)}<0} \to \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left(\frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha}\right).$$

We show that the image of R_D under Φ lies in a $\mathbb{Z}$-lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class ξ. By Step 1, we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers n, r. Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of R_D under Φ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

Step 3. We show that every $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ is of the form R_D for some nef divisor D on X.

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$-negative exposed ray. Then R is of form $D^{\perp} \cap \mathrm{Psef}_1(X)$ for some nef $\mathbb{R}$-divisor D on X. We need to show that D can be choose as a nef $\mathbb{Q}$-divisor. There is a sequence of nef but not ample $\mathbb{Q}$-divisors D_m such that $D_m \to D$ as $m \to \infty$. We adjust D_m such that $\dim F_{D_m} = 1$ for all n.

By re-choosing H_i, we can assume that $D = a_1H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is D_m. Then F_{D_m} is $K_{(X,B)}$-negative. Note that $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by Step 1. If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{D_m}$. We can assume that$r_1K_{(X,B)}+NH_1$ is not identically zero on F_{D_m}. Then we can choose n large enough such that $\|r_1K_{(X,B)}+NH_1\|/n < 1/m$. Replace D_m by $D_m + (r_1K_{(X,B)} + NH_1)/n$. Inductively we construct D_m nef $\mathbb{Q}$-divisor with $D_m \to D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that ξ_m converges. Then $\xi_m \to \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, Φ is well-defined at ξ and the image of ξ_m under Φ is discrete. Hence $\xi = \xi_m$ for all m large enough. It follows that $R = R_{D_m}$ for a nef $\mathbb{Q}$-divisor D_m.

By Step 2, the $K_{(X,B)}$-negative extremal rays form a discrete set in $\{\alpha \in \mathrm{Psef}_1(X) : K_{(X,B)} \cdot \alpha < 0\}$. Hence every $K_{(X,B)}$-negative extremal ray is an exposed ray by Straszewicz's Theorem.

Step 4. Proof of the theorem.

Given an ample divisor H on X, note that εH has positive minimum δ on $\mathrm{Psef}_1(X) \cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \mathrm{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and Φ is well-defined on it. By Steps 2 and 3, there are only finitely many extremal rays on $\mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H \leq 0}$. Hence we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$\mathcal{C} := \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\} \subset \mathcal{C}$ such that $\alpha_n \to \alpha \in N_1(X)_{\mathbb{R}}$. Note that $\mathrm{Psef}_1(X)$ is closed, hence $\alpha \in \mathrm{Psef}_1(X)$. We only need to consider the case $\alpha \cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$. Then $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$ for all n large enough. Note that $\mathcal{C} \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$ is a polyhedral cone by Step 2 and hence is closed. Then $\alpha \in \mathcal{C}$ and the conclusion follows. □

Remark 4.3.12. Thanks for my friend Qin for pointing out that the extremal ray may not be exposed.

Theorem 4.3.13 (Contraction Theorem)**.** Let (X, B) be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;

(b) up to linearly equivalence, any Cartier divisor G with $F \subset G^{\perp} = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$ comes from a Cartier divisor on Y, i.e., there exists a Cartier divisor G_Y on Y such that $G \sim \varphi_F^* G_Y$.

Proof. We follow the following steps to prove the theorem.

Step 1. We show that there exists a nef divisor D on X such that $F = D^{\perp} \cap \mathrm{Psef}_1(X)$. In other words, F is defined on $N_1(X)_{\mathbb{Q}}$.

We can choose an ample divisor H and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on F since $F \cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in $N_1(X)_{\mathbb{R}}$. Then by Cone Theorem (Theorem 4.3.11), F is an extremal face of a rational polyhedral cone, namely $\mathrm{Psef}_1(X)_{K_{(X,B)}+(1/n)H\leq 0}$. It follows that $F^{\perp} \subset N^1(X)_{\mathbb{R}}$ is defined on $\mathbb{Q}$. Since F is extremal and $K_{(X,B)} + (1/n)H$-negative, the set $\{L \in F^{\perp} : L|_{\mathrm{Psef}_1(X)\setminus F} > 0\}$ has non-empty interior in $F^{\perp}$ by Theorems 4.3.3 and 4.3.11. Then there exists a Cartier divisor D such that $D \in F^{\perp}$ and $D|_{\mathrm{Psef}_1(X)\setminus F} > 0$. It follows that D is nef and $F = D^{\perp} \cap \mathrm{Psef}_1(X)$.

Step 2. Let $\varphi : X \to Y$ be the Iitaka fibration associated to D by Theorem 4.3.1. We show that φ is the desired fibration.

Note that $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$ is compact and D is strictly positive on it. Then there exist $a \geq 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since F is $K_{(X,B)}$-negative. Then by Base Point Free Theorem (Theorem 4.3.7), we know that mD is base point free for all $m \gg 0$. Hence we can apply Theorem 4.3.1 to get a fibration $\varphi_D : X \to Y$.

First we show that D comes from Y. Note that mD and $(m+1)D$ induces the same fibration φ_D for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m)D_{Y,m}$ and $D_{Y,m}$ is ample. Hence D_Y is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that C is contracted by φ_D if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let G be arbitrary Cartier divisor on X such that $F \subset G^{\perp}$. Since D is strictly positive on $\mathrm{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' \coloneqq mD + G$, we have $D'^{\perp} \cap \mathrm{Psef}_1(X) = F$. Then by the same argument as above, we get an other fibration $\varphi_{D'} : X \to Y'$ such that a curve C is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 4.3.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on Y. In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor D'_Y on Y. Then $G = D' - mD$ also comes from Y. ☐

Remark 4.3.14. The Step 1 is amazing. If F is not $K_{(X,B)}$-negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E. By [Laz04, Lemma 1.5.4], we know that $\mathrm{Psef}_1(X)$ is a circular cone. The we see there indeed exist some irrational extremal faces of $\mathrm{Psef}_1(X)$.

Theorem 4.3.15 (Length of extremal rays)**.** Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Then there exists a rational curve $C \subset X$ such that $[C] \in R$ and

$$0 < -K_{(X,B)} \cdot C \leq 2 \dim X.$$

Proof. By Theorem 4.3.13, let $\varphi_D : X \to Y$ be the contraction associated to R_D (note that we do not need the step to proof Theorem 4.3.13). If $\dim Y < \dim X$, let F be a general fiber of φ_D. By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that H is ample on F. By Proposition 4.3.4. In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By Lemma 4.3.5, we can use Proposition 4.3.4 to get the result. To be completed. ☐

Definition 4.3.16. Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ with contraction $\varphi_R : X \to Y$. There are three types of contractions:

(a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension one;

(b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension at least two;

(c) *Mori fiber space*: if $\dim X > \dim Y$.

Proposition 4.3.17. Let (X, B) be a $\mathbb{Q}$-factorial projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Suppose that the contraction $\varphi : X \to Y$ associated to R is either divisorial or a Mori fiber space. Then Y is $\mathbb{Q}$-factorial.

Proof. Let D be a prime Weil divisor on Y and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\geq 0}[C]$ for an irreducible curve C contracted by φ. Set $D_X := \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$. Then D_X is a prime Weil divisor on X and hence is $\mathbb{Q}$-Cartier.

If φ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber F of φ. Then by Contraction Theorem (Theorem 4.3.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor D' on Y. We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence D is $\mathbb{Q}$-Cartier.

If φ is a divisorial contraction, let E be the exceptional divisor of φ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier $\mathbb{Q}$-divisor E_Y on Y). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 4.3.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor D' on Y. Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence D is $\mathbb{Q}$-Cartier. ☐

Remark 4.3.18. If φ is a small contraction, then Y is never $\mathbb{Q}$-factorial. Otherwise, let B_Y be the strict transform of B on Y. Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset U. Suppose $K_{(Y,B_Y)}$ is $\mathbb{Q}$-Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

Example 4.3.19. Let $X = E \times E \times \mathbb{P}^1$. To be completed.

Chapter 5

Abelian Varieties

5.1 The First Properties of Abelian Varieties

5.1.1 Definition and examples of Abelian Varieties

Definition 5.1.1. Let $\mathbf{k}$ be a field. An *abelian variety over* $\mathbf{k}$ is a proper variety A over $\mathbf{k}$ together with morphisms *identity* $e : \operatorname{Spec} \mathbf{k} \to A$, *multiplication* $m : A \times A \to A$ and *inversion* $i : A \to A$ such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd} & A \times A \times A \arrow[dl, "\mathrm{id}_A \times m"'] \arrow[dr, "m \times \mathrm{id}_A"] & \\ A \times A \arrow[dr, "m"'] & & A \times A \arrow[dl, "m"] \\ & A & \end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd} A \times \operatorname{Spec} \mathbf{k} \arrow[r, "\mathrm{id}_A \times e"] \arrow[dr, "\cong"'] & A \times A \arrow[d, "m"] & \operatorname{Spec} \mathbf{k} \times A \arrow[l, "e \times \mathrm{id}_A"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_A"] & \\ A \times A \arrow[dr, "m"'] & \operatorname{Spec} \mathbf{k} \arrow[d, "e"] & A \times A \arrow[dl, "m"] \\ & A & \end{tikzcd}.$$

In other words, an abelian variety is a group object in the category of proper varieties over $\mathbf{k}$.

Example 5.1.2. Let E be an elliptic curve over a field $\mathbf{k}$. Then E is an abelian variety of dimension 1. To be completed.

In the following, we will always assume that A is an abelian variety over a field $\mathbf{k}$ of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The *left translation* by $a \in A(\mathbf{k})$ is defined as

$$l_a : A \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times A \xrightarrow{a \times \mathrm{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a.

Proposition 5.1.3. Let A be an abelian variety. Then A is smooth.

Proof. By base changing to the algebraic closure of $\mathbf{k}$, we may assume that $\mathbf{k}$ is algebraically closed. Note that there is a non-empty open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a. □

Proposition 5.1.4. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. But how to show it is a morphism of varieties? To be completed. □

Theorem 5.1.5. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism, i.e., for every $\mathbf{k}$-scheme T, the induced map $f_T : A(T) \to B(T)$ is a group homomorphism.

Proof. Consider the diagram

$$\begin{tikzcd} A \times A \arrow[d, "p_1"'] \arrow[dr, "\varphi"] & \\ A & B \end{tikzcd}$$

with φ be given by

$$A \times A \xrightarrow{\Delta \times \Delta} A \times A \times A \times A \xrightarrow{\cong} A \times A \times A \times A \xrightarrow{(f \circ m_A) \times (i_B \circ f) \times (i_B \circ f)} B \times B \times B \xrightarrow{m_B} B,$$
$$(x, y) \mapsto (x, x, y, y) \mapsto (x, y, y, x) \mapsto (f(xy), f(y)^{-1}, f(x)^{-1}) \mapsto f(xy) f(y)^{-1} f(x)^{-1}.$$

We have $\varphi(p_1^{-1}(e_A)) = \varphi(\{e_A\} \times A) = \{e_B\}$. Then by Rigidity Lemma (??), there exists a unique rational map $\psi : A \dashrightarrow B$ such that $\varphi = \psi \circ p_1$. Note that $A \to A \times \{e_A\} \to A \times A$ gives a section of p_1. On this section, we have that φ is constant equal to e_B. Thus ψ is well-defined and $\psi(A) = e_B$. It follows that φ factors through the constant map $A \times A \to \{e_B\} \to B$. Then for every $(x, y) \in A(\mathbb{k}) \times A(\mathbb{k})$, we have

$$f(xy) = f(x) f(y).$$

Since $A(\mathbb{k})$ is dense in A, the conclusion follows. □

Proposition 5.1.6. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 5.1.5. □

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 5.1.7. Let A be an abelian variety over $\Bbbk$ and n a positive integer not divisible by $\operatorname{char} \Bbbk$. Then the multiplication by n morphism $[n]_A : A \to A$ is finite surjective and étale.

Proof. To be completed. □

5.1.2 Complex abelian varieties

Theorem 5.1.8. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if there exists a positive definite Hermitian form H on $\mathbb{C}^n$ such that $\Im(H)(\Lambda, \Lambda) \subset \mathbb{Z}$. To be completed.

5.2 Picard Groups of Abelian Varieties

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. Let A be an abelian variety over $\mathbf{k}$.

5.2.1 Pullback along group operations

Theorem 5.2.1 (Theorem of the cube)**.** Let X, Y, Z be proper varieties over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\} \times Y \times Z}$, $\mathcal{L}|_{X \times \{y\} \times Z}$ and $\mathcal{L}|_{X \times Y \times \{z\}}$ are trivial. Then $\mathcal{L}$ is trivial.

Proof. To be completed. □

Remark 5.2.2. If we assume the existence of the Picard scheme, then the Theorem 5.2.1 can be deduced from the Rigidity Lemma. Consider the morphism

$$\varphi : X \times Y \to \mathrm{Pic}(Z), \quad (x, y) \mapsto \mathcal{L}|_{\{x\} \times \{y\} \times Z}.$$

Since $\varphi(x, y) = \mathcal{O}_Z$, φ factors through $\mathrm{Pic}^0(Z)$. Then the assumption implies that φ contracts $\{x\} \times Y$, $X \times \{y\}$ and hence it maps $X \times Y$ to a point. Thus $\varphi(x', y') = \mathcal{O}_Z$ for every $(x', y') \in X \times Y$. Then by Grauert's theorem, we have $\mathcal{L} \cong p^* p_* \mathcal{L}$ where $p : X \times Y \times Z \to X \times Y$ is the projection. Note that $p_* \mathcal{L} \cong \mathcal{L}|_{X \times Y \times \{z\}} \cong \mathcal{O}_{X \times Y}$. Hence $\mathcal{L}$ is trivial.

Lemma 5.2.3. Let A be an abelian variety over $\mathbf{k}$, $f, g, h : X \to A$ morphisms from a variety X to A and $\mathcal{L}$ a line bundle on A. Then we have

$$(f + g + h)^* \mathcal{L} \cong (f + g)^* \mathcal{L} \otimes (f + h)^* \mathcal{L} \otimes (g + h)^* \mathcal{L} \otimes f^* \mathcal{L}^{-1} \otimes g^* \mathcal{L}^{-1} \otimes h^* \mathcal{L}^{-1}.$$

Proof. First consider $X = A\times A\times A$, $p : X \to A, (x_1, x_2, x_3) \mapsto x_1+x_2+x_3$, $p_{ij} : X \to A, (x_1, x_2, x_3) \mapsto x_i + x_j$ for $1 \le i < j \le 3$ and $p_i : X \to A, (x_1, x_2, x_3) \mapsto x_i$ for $1 \le i \le 3$. Then the conclusion follows from the theorem of the cube by taking $\mathcal{L}' = p^*\mathcal{L}^{-1}\otimes p_{12}^*\mathcal{L}\otimes p_{13}^*\mathcal{L}\otimes p_{23}^*\mathcal{L}\otimes p_1^*\mathcal{L}^{-1}\otimes p_2^*\mathcal{L}^{-1}\otimes p_3^*\mathcal{L}^{-1}$ and considering the restriction to $\{0\} \times A \times A$, $A \times \{0\} \times A$ and $A \times A \times \{0\}$.

In general, consider the morphism $\varphi = (f, g, h) : X \to A \times A \times A$ and pull back the above isomorphism along φ. $\square$

Proposition 5.2.4. Let A be an abelian variety over $\mathbf{k}$, $n \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle on A. Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes\frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes\frac{1}{2}(n^2-n)}.$$

Proof. For $n = 0, 1$, the conclusion is trivial. For $n \ge 2$, we can use the previous lemma on $[n-2]_A, [1]_A, [1]_A$ and induct on n. Hence we have

$$[n]_A^*\mathcal{L} \cong [n-1]_A^*\mathcal{L} \otimes [n-1]_A^*\mathcal{L} \otimes [2]_A^*\mathcal{L} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [n-2]_A^*\mathcal{L}^{-1}.$$

Then the conclusion follows from induction. To be completed. $\square$

Definition 5.2.5. Let A be an abelian variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on A. We say that $\mathcal{L}$ is *symmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}$ and *antisymmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}^{-1}$.

Theorem 5.2.6 (Theorem of the square)**.** Let A be an abelian variety over $\mathbf{k}$, $x, y \in A(\mathbf{k})$ two points and $\mathcal{L}$ a line bundle on A. Then

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Proof. To be completed. $\square$

Remark 5.2.7. We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \to \mathrm{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary $\mathcal{L}$, the map

$$\Phi_{\square} : \mathrm{Pic}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \mathrm{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is also a group homomorphism. For any $x \in A(\mathbf{k})$, we have

$$\Phi_{t_x^*\mathcal{L}} = \Phi_{\mathcal{L}}$$

by Theorem 5.2.6. In the other words,

$$\Phi_{\mathcal{L}}(x) \in \mathrm{Ker}\,\Phi_{\square}, \quad \forall \mathcal{L} \in \mathrm{Pic}(A), x \in A(\mathbf{k}).$$

If we assume the scheme structure on $\mathrm{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\mathrm{Pic}^0(A)$. Let $K(\mathcal{L}) \coloneqq \mathrm{Ker}\,\Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A. We give another description of $K(\mathcal{L})$. From this point, when $K(\mathcal{L})$ is finite, we can recover the dual abelian variety $A^\vee = \mathrm{Pic}^0_{A/\mathbf{k}}$ as the quotient $A/K(\mathcal{L})$.

5.2.2 Projectivity

In this subsection, we work over the algebraically closed field $\Bbbk$.

Proposition 5.2.8. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. Then $|2D|$ is base point free.

Proof. To be completed. □

Theorem 5.2.9. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. TFAE:

(a) the stabilizer $\mathsf{Stab}(D)$ of D is finite;

(b) the morphism $\phi_{|2D|}$ induced by the complete linear system $|2D|$ is finite;

(c) D is ample;

(d) $K(\mathcal{O}_A(D))$ is finite.

Proof. To be completed. □

Theorem 5.2.10. Let A be an abelian variety over $\Bbbk$. Then A is projective.

Proof. To be completed. □

Corollary 5.2.11. Let A be an abelian variety over $\Bbbk$ and D a divisor on A. Then D is pseudo-effective if and only if it is nef, i.e. $\mathsf{Psef}^1(A) = \mathsf{Nef}^1(A)$.

Proof. To be completed. □

5.2.3 Dual abelian varieties

In this subsection, we work over the algebraically closed field $\Bbbk$.

Definition 5.2.12. Let A be an abelian variety over $\Bbbk$. We define the *dual abelian variety* of A to be $A/K(\mathcal{L})$ for some ample line bundle $\mathcal{L}$ on A. We denote it by $A^\vee$.

We have a natural map $A^\vee(\Bbbk) \to \mathsf{Pic}^0(A)$ by sending $x + K(\mathcal{L}) \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$. We will show that this map is an isomorphism.

Lemma 5.2.13. There exists a unique line bundle $\mathcal{P}$ on $A \times A^\vee$ such that for every $y = \mathcal{L} \in A^\vee = \mathsf{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{y\}} \cong \mathcal{L}$.

Proof. To be completed. □

Lemma 5.2.14. Let A be an abelian variety over $\Bbbk$ and B a group variety over $\Bbbk$. Then there is a natural bijection between the morphisms $f : B \to A^\vee$ and the line bundles $\mathcal{L}$ on $A \times B$ such that for every $b \in B(\Bbbk)$, we have $\mathcal{L}|_{A\times\{b\}} \in \mathsf{Pic}^0(A)$. The bijection is given by $f \mapsto (1_A \times f)^*\mathcal{P}$ where $\mathcal{P}$ is the Poincaré line bundle on $A \times A^\vee$. To be completed.

Proof. To be completed. □

Theorem 5.2.15. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ and the Poincaré line bundle $\mathcal{P}$ on $A \times A^\vee$ do not depend on the choice of the ample line bundle $\mathcal{L}$. Moreover, there is a natural bijection $A^\vee(\mathbf{k}) \to \mathrm{Pic}^0(A)$ of groups. Under this bijection, for every $x = \mathcal{L} \in A^\vee(\mathbf{k}) = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{x\}} \cong \mathcal{L}$.

Proof. To be completed. □

Proposition 5.2.16. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ is also an abelian variety and the natural morphism $A \to A^{\vee\vee}$ is an isomorphism.

Proof. To be completed. □

5.2.4 The Néron-Severi group

Theorem 5.2.17. Let A be an abelian variety over $\Bbbk$. The we have an inclusion $\mathrm{NS}(A) \hookrightarrow \mathrm{Hom}_{\mathbf{Grp}}(A, A^\vee)$ given by To be completed.

Chapter 6

Algebraic Groups

6.1 First properties of algebraic groups

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. All varieties are defined over $\mathbf{k}$ unless otherwise specified.

6.1.1 Basic concepts

Definition 6.1.1. A *group scheme* over S is an S-scheme G together with morphisms *multiplication* $\mu : G \times G \to G$, *identity* $\varepsilon : S \to G$ and *inversion* $\iota : G \to G$ over S such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd}
 & G \times G \times G \arrow[dl, "\mathrm{id}_G \times \mu"'] \arrow[dr, "\mu \times \mathrm{id}_G"] & \\
G \times G \arrow[dr, "\mu"] & & G \times G \arrow[dl, "\mu"'] \\
 & G &
\end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd}
G \times S \arrow[r, "\mathrm{id}_G \times \varepsilon"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] & S \times G \arrow[l, "\varepsilon \times \mathrm{id}_G"'] \arrow[dl, "\cong"] \\
 & G &
\end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd}
 & G \arrow[dl, "\mathrm{id}_G \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_G"] & \\
G \times G \arrow[dr, "\mu"'] & S \arrow[d, "\varepsilon"] & G \times G \arrow[dl, "\mu"] \\
 & G &
\end{tikzcd}.$$

In other words, a group scheme is a group object in the category of schemes.

Definition 6.1.2. An *algebraic group* is a $\mathbf{k}$-group scheme G which is reduced, separated and of finite type over a field $\mathbf{k}$.

Remark 6.1.3. Even if we work over $\Bbbk$ and just consider the closed points $G(\Bbbk)$ of an algebraic group G, $G(\Bbbk)$ is not a topological group with respect to the Zariski topology in general. The reason is that the topology on $G(\Bbbk) \times G(\Bbbk)$ is not the product topology of the topologies on $G(\Bbbk)$.

Definition 6.1.4. Let G be an algebraic group and $x \in G(\mathbf{k})$ a $\mathbf{k}$-point. The *left translation* by x is the morphism

$$l_x : G \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{\mu} G.$$

Similar definition applies to the right translation r_x.

Remark 6.1.5. In the context of algebraic groups, we often use multiplicative notation for the group law. That is, for $g, h \in G(\mathbf{k})$, we write gh instead of $\mu(g,h)$ and g^{-1} instead of $\iota(g)$.

Sometimes we also abuse the notation by $\mu : G \times \cdots \times G \to G$ to denote the multiplication of multiple elements, i.e. $\mu(g_1, \ldots, g_n) = g_1 \cdots g_n$ for $g_1, \ldots, g_n \in G(\mathbf{k})$.

Proposition 6.1.6. Let G be an algebraic group. Then G is smooth over $\mathbf{k}$.

Proof. Since G is reduced and of finite type over a field, it is generically regular. Let $g \in G(\Bbbk)$ be a regular point. Then the left translation $l_{gh^{-1}} : G \to G$ is an isomorphism, hence G is regular at $h \in G(\Bbbk)$. It follows that G is regular at every $\Bbbk$-point, hence G is smooth over $\mathbf{k}$. □

Remark 6.1.7. Let G be an algebraic group. Then the irreducible components of G coincide with the connected components of G. We will use the term "connected" to refer to both concepts since "irreducible" has other meanings in the theory of representations.

Example 6.1.8. The *additive group* $\mathbb{G}_a$ is defined to be the affine line $\mathbb{A}^1$ with the group law given by addition. Concretely, we can write $\mathbb{G}_a = \operatorname{Spec} \mathbf{k}[T]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a, &\quad (x, y) \mapsto x + y, \\ \iota : \mathbb{G}_a \to \mathbb{G}_a, &\quad x \mapsto -x, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_a, &\quad * \mapsto 0. \end{aligned}$$

Example 6.1.9. The *multiplicative group* $\mathbb{G}_m$ is defined to be the affine variety $\mathbb{A}^1 \setminus \{0\}$ with the group law given by multiplication. Concretely, we can write $\mathbb{G}_m = \operatorname{Spec} \mathbf{k}[T, T^{-1}]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m, &\quad (x, y) \mapsto xy, \\ \iota : \mathbb{G}_m \to \mathbb{G}_m, &\quad x \mapsto x^{-1}, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_m, &\quad * \mapsto 1. \end{aligned}$$

Example 6.1.10. The *general linear group* GL_n is defined to be the open subvariety of $\mathbb{A}^{n^2}$ consisting of invertible matrices, with the group law given by matrix multiplication. Concretely, we can write

$\mathrm{GL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}, \det(T_{ij})^{-1}]$ where $1 \le i, j \le n$ and the group law is given by the morphism

$$\begin{aligned} \mu &: \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, \quad (A, B) \mapsto AB, \\ \iota &: \mathrm{GL}_n \to \mathrm{GL}_n, \quad A \mapsto A^{-1}, \\ \varepsilon &: \operatorname{Spec} \mathbf{k} \to \mathrm{GL}_n, \quad * \mapsto I_n. \end{aligned}$$

Example 6.1.11. An abelian variety is an algebraic group that is also a proper variety.

Example 6.1.12. Let G and H be algebraic groups. The *product* $G \times H$ is an algebraic group with the group law defined by

$$\begin{aligned} \mu_{G\times H} = \mu_G \times \mu_H &: (G \times H) \times (G \times H) \cong (G \times G) \times (H \times H) \to G \times H, \\ \varepsilon_{G\times H} = \varepsilon_G \times \varepsilon_H &: \operatorname{Spec} \mathbf{k} \cong \operatorname{Spec} \mathbf{k} \times \operatorname{Spec} \mathbf{k} \to G \times H, \\ \iota_{G\times H} = \iota_G \times \iota_H &: G \times H \to G \times H. \end{aligned}$$

Example 6.1.13. Let G be an algebraic group over $\mathbf{k}$ and $\mathbf{K}/\mathbf{k}$ a field extension. The base change $G_{\mathbf{K}} = G \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} \mathbf{K}$ is an algebraic group over $\mathbf{K}$ with the group law defined by the base change of the original group law of G to $\mathbf{K}$.

Definition 6.1.14. A *homomorphism* of algebraic groups is a morphism of schemes that is also a group homomorphism. Explicitly, a morphism $f : G \to H$ between algebraic groups G and H is a homomorphism if the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

where μ_G and μ_H are the group laws of G and H, respectively.

Definition 6.1.15. An *algebraic subgroup* of an algebraic group G is a closed subscheme $H \subseteq G$ that is also a subgroup of G. More precisely, H is an algebraic subgroup and the inclusion morphism $H \hookrightarrow G$ is compatible with the group laws.
An algebraic subgroup H of G is called *normal* if for any $\mathbf{k}$-scheme S, the subgroup $H(S)$ is a normal subgroup of the abstract group $G(S)$.

Example 6.1.16. The *special linear group* SL_n is defined to be the closed subvariety of GL_n defined by the equation $\det = 1$. It is an algebraic subgroup of GL_n.

Proposition 6.1.17. Let G be an algebraic group and S is a closed subgroup of $G(\mathbb{k})$. Then there exists a unique algebraic subgroup H of G such that $H(\mathbb{k}) = S$.

Proof. To be continued... □

Remark 6.1.18. By Proposition 6.1.17, we often identify an algebraic group G with its set of closed points $G(\mathbb{k})$ when there is no confusion.

Remark 6.1.19. If one replaces $\mathbb{k}$ by $\mathbf{k}$ in Proposition 6.1.17, the statement may not hold. For example, let $\mathbf{k} = \mathbb{Q}$ and G be the elliptic curve defined by $X^3 + Y^3 = Z^3$ in $\mathbb{P}^2$. It is well-known that $\#G(\mathbb{Q}) = 3$. Let S be the disjoint union of the three $\mathbb{Q}$-points of G endowed with the reduced subscheme structure and the group structure induced from G. Then S is a proper closed subgroup of G and we have $S(\mathbb{Q}) = G(\mathbb{Q})$. This contradicts the uniqueness in Proposition 6.1.17.

Indeed, in this chapter, despite working over an arbitrary field $\mathbf{k}$, we mostly consider the closed points of algebraic groups over $\Bbbk$.

Definition 6.1.20. Let G be an algebraic group. The *neutral component* G^0 is the connected component of G containing the identity element ε.

Proposition 6.1.21. The neutral component G^0 is a closed, normal algebraic subgroup of G.

Proof. To be continued... □

Proposition 6.1.22. Let G be an algebraic group and $H \subseteq G(\Bbbk)$ a subgroup (not necessarily closed). Then the Zariski closure $\overline{H}$ of H in G is an algebraic subgroup of G. If $H \subset G(\Bbbk)$ is constructible, then $H = \overline{H}(\Bbbk)$.

Proof. To be continued... □

Example 6.1.23. Let $G = \mathrm{SL}_2$ over $\Bbbk$, $T = \{\mathrm{diag}(t, t^{-1}) | t \in \Bbbk^\times\}$ and $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Set $S = gTg^{-1}$. Then both T and S are closed algebraic subgroups of $G(\Bbbk)$, but the product TS is not closed in $G(\Bbbk)$. By direct computation, we have

$$S = \left\{ \begin{pmatrix} s & s^{-1} - s \\ 0 & s^{-1} \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

Then

$$TS = \left\{ \begin{pmatrix} ts & t(s^{-1} - s) \\ 0 & (ts)^{-1} \end{pmatrix} \middle| t, s \in \Bbbk^\times \right\}.$$

We have

$$TS \cap \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \Bbbk \right\} = \left\{ \begin{pmatrix} 1 & s^{-2} - 1 \\ 0 & 1 \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

The right hand side is not closed in $\mathrm{SL}_2(\Bbbk)$ since it does not contain the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Hence TS is not closed in $G(\Bbbk)$.

Proposition 6.1.24. Let G be an algebraic group, X_i varieties over $\mathbf{k}$ and $f_i : X_i \to G$ morphisms for $i = 1, \dots, n$ with images $Y_i = f_i(X_i)$. Suppose that Y_i pass through the identity element of G. Let H be the closed subgroup of G generated by $Y_1, \dots, Y_n$, i.e. the smallest closed subgroup of G containing $Y_1, \dots, Y_n$. Then H is connected and $H = Y_{a_1}^{e_1} \cdots Y_{a_m}^{e_m}$ for some $a_1, \dots, a_m \in \{1, \dots, n\}$ and $e_1, \dots, e_m \in \{\pm 1\}$.

Proof. To be continued... □

Remark 6.1.25. We can take $m \leq 2 \dim G$ in Proposition 6.1.24.

6.1.2 Action and representations

Definition 6.1.26. An *action* of an algebraic group G on a variety X is a morphism

$$\sigma : G \times X \to X$$

such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu\times\mathrm{id}_X"] \arrow[d, "\mathrm{id}_G\times\sigma"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}\,\mathbf{k} \times X \arrow[r, "\varepsilon\times\mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

where μ is the group law of G and ε is the identity element of G. In other words, for any $\mathbf{k}$-scheme S, the induced map $G(S) \times X(S) \to X(S)$ defines a group action of the abstract group $G(S)$ on the set $X(S)$.
For simplicity, we often write $g.x$ instead of $\sigma(g, x)$ for $g \in G(\mathbf{k})$ and $x \in X(\mathbf{k})$.

Example 6.1.27. There are three natural actions of an algebraic group G on itself:

(a) Left translation: $g.h = l_g(h) = gh$;

(b) Right translation: $g.h = r_g(h) = hg^{-1}$;

(c) Conjugation: $g.h = \mathrm{Ad}_g(h) = ghg^{-1}$.

All of them are morphisms of varieties since they are defined by the group law and inversion of G.

Example 6.1.28. The general linear group GL_n acts on the affine space $\mathbb{A}^n$ by matrix multiplication. It is given by polynomials, hence is a morphism of varieties.

Example 6.1.29. The general linear group GL_{n+1} acts on the projective space $\mathbb{P}^n$ by

$$A \cdot [x_0 : \cdots : x_n] = [y_0 : \cdots : y_n], \quad \text{where } (y_0, \dots, y_n)^T = A(x_0, \dots, x_n)^T.$$

Let U_i be the standard affine open subset of $\mathbb{P}^n$ defined by $x_i \neq 0$. The map is given by polynomials on the principal open subset of $\mathrm{GL}_{n+1} \times U_i$ defined by $y_j \neq 0$ for any j. Hence it is a morphism of varieties.

Definition 6.1.30. A *linear representation* of an algebraic group G on a finite-dimensional vector space V over $\mathbb{k}$ is an abstract group representation $\rho : G(\mathbb{k}) \to \mathrm{GL}(V)$ such that if we identify V with $\mathbb{A}^n$ for some n, then the map $G(\mathbb{k}) \times \mathbb{A}^n(\mathbb{k}) \to \mathbb{A}^n(\mathbb{k})$ is is a morphism of varieties.

Definition 6.1.31. Let G be an algebraic group acting on a variety X. For any $x \in X(\mathbf{k})$, the *orbit* of x is the locally closed subvariety $G \cdot x = \sigma(G \times \{x\})$ of X.

Proposition 6.1.32. Let G be an algebraic group acting on a variety X. Then for any $x \in X(\mathbf{k})$, the orbit $G \cdot x$ is a locally closed subvariety of X, and $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of strictly smaller dimension.

Proof. To be continued... $\square$

Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For $x \in G(\mathbf{k})$, we have the left translation of functions $\tau_x : A \to A$ defined by $\tau_x(f)(y) = f(x^{-1}y)$ for $y \in X(\mathbf{k})$.

Lemma 6.1.33. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For any finite-dimensional subspace $V \subseteq A$, there exists a finite-dimensional G-invariant subspace $W \subseteq A$ containing V.

Proof. To be continued... □

Theorem 6.1.34. Any affine algebraic group is isomorphic to a closed algebraic subgroup of some GL_n.

Proof. To be continued... □

6.1.3 Lie algebra of an algebraic group

Let G be an algebraic group. The *Lie algebra* of G is defined to be the tangent space of G at the identity element ε:

$$\operatorname{Lie}(G) = T_\varepsilon G.$$

It is a finite-dimensional vector space over $\mathbf{k}$.

Proposition 6.1.35. The group law $\mu : G \times G \to G$ induces the plus map on $\operatorname{Lie}(G)$:

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)} : T_{(\varepsilon,\varepsilon)}(G \times G) \cong T_\varepsilon G \oplus T_\varepsilon G \to T_\varepsilon G, \quad (v, w) \mapsto v + w.$$

Proof. We have

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v, w) = \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v, 0) + \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(0, w) = (\mathrm{d}\mu \circ (\mathrm{id}_G \times \varepsilon))_\varepsilon(v) + (\mathrm{d}\mu \circ (\varepsilon \times \mathrm{id}_G))_\varepsilon(w) = v + w.$$

□

6.2 Quotient by algebraic group

Everything in this section is over an arbitrary field $\mathbf{k}$ unless otherwise specified.

6.2.1 Quotient

Definition 6.2.1. Let G be an algebraic group acting on a variety X. A *quotient* of X by G is a variety Y together with a morphism $\pi : X \to Y$ such that

(a) π is G-invariant, i.e., $\pi(g \cdot x) = \pi(x)$ for all $g \in G$ and $x \in X$.

(b) For any variety Z and any G-invariant morphism $f : X \to Z$, there exists a unique morphism $\overline{f} : Y \to Z$ such that $f = \overline{f} \circ \pi$.

In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "\bar{f}"] \\ & Z \end{tikzcd}$$

If a quotient exists, it is unique up to a unique isomorphism. To be continued...

Such a quotient does not always exist.

Theorem 6.2.2. Let G be an affine algebraic group acting on a variety X. Then there exists a variety Y and a rational morphism $\pi : X \dashrightarrow Y$ with commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "\bar{f}"] \\ & Z \end{tikzcd}$$

satisfying the following universal property: If a quotient exists, it is unique up to a unique isomorphism.
Furthermore, if all orbits of G in X are closed, then π is a morphism (i.e., defined everywhere). To be continued... Ref?

6.2.2 Quotient of affine algebraic group by closed subgroup

Lemma 6.2.3. Let V be a finite-dimensional vector space over $\mathbf{k}$ and G an abstract group acting linearly on V. Let $W \subseteq V$ be a subspace of dimension m. Then $G.W = W$ if and only if $G. \wedge^m W = \wedge^m W$.

Proof. To be filled. □

Lemma 6.2.4. Let G be an affine algebraic group and H a closed subgroup. Then there exists a finite-dimensional linear representation V of G and a one-dimensional subspace $L \subseteq V$ such that H is the stabilizer of L.

Proof. To be filled. □

Theorem 6.2.5. Let G be an affine algebraic group and H a closed subgroup. Then the quotient G/H exists as a quasi-projective variety.

Proof. To be filled. □

6.3 Decomposition of algebraic groups

6.3.1

6.4 Application: birational group of varieties of general type

In this section, we apply the results from the previous sections to study the birational automorphism groups of varieties of general type.

Theorem 6.4.1. Let X be a projective variety of general type over an algebraically closed field $\Bbbk$ of characteristic zero. Then the group of birational automorphisms $\mathrm{Bir}(X)$ is finite.

Proof. We will prove this theorem in several steps. By replacing X with its resolution of singularities, we may assume that X is smooth.

Step 1. For every $m \geq 1$, $\mathrm{Bir}(X)$ linearly acts on $H^0(X, mK_X)$ via pull-back of functions (as abstract group).

Let $\mathscr{K}(X)$ be the function field of X. Then for every $g \in \mathrm{Bir}(X)$, g induces an automorphism of $\mathscr{K}(X)$ over $\Bbbk$, which we denote by g^*. In particular we know that g^* is injective and $\Bbbk$-linear. By definition, $H^0(X, mK_X) = \{s \in \mathscr{K}(X) \mid \mathrm{div}(s) + mK_X \geq 0\}$. We only need to show that for every $s \in H^0(X, mK_X)$, $g^*(s) \in H^0(X, mK_X)$ since $\dim_{\Bbbk} H^0(X, mK_X) < \infty$. Consider the commutative diagram

$$\begin{tikzcd} \Gamma \arrow[d, "p"'] \arrow[dr, "q"] & \\ X \arrow[r, dashed, "g"] & X \end{tikzcd}$$

with Γ smooth and p, q birational morphisms. Then we have

$$K_\Gamma = p^*K_X + E_p = q^*K_X + E_q,$$

where E_p and E_q are p- and q-exceptional divisors respectively. Moreover, E_p and E_q are effective since X is smooth. For every $s \in H^0(X, mK_X)$, we have

$$\mathrm{div}(q^*s) + mK_\Gamma = q^*(\mathrm{div}(s) + mK_X) + mE_q \geq 0.$$

Then

$$\begin{aligned} \mathrm{div}(g^*s) + mK_X &= p_*p^*(\mathrm{div}(g^*s) + mK_X) \\ &= p_*\left(\mathrm{div}(q^*s) + mK_\Gamma - mE_p\right) \\ &= p_*\left(\mathrm{div}(q^*s) + mK_\Gamma\right) \geq 0. \end{aligned}$$

It follows that $g^*(s) \in H^0(X, mK_X)$.

Note this action $g \mapsto g^*$ is contravariant, i.e., for every $g_1, g_2 \in \mathrm{Bir}(X)$, we have $(g_1 \circ g_2)^* = g_2^* \circ g_1^*$.

Step 2. The group $\mathrm{Bir}(X)$ is a linear algebraic group by identifying it with a closed subgroup of $\mathrm{Aut}(\mathbb{P}(V))$ for some finite-dimensional $\Bbbk$-vector space V (subspace of $H^0(X, mK_X)$ for some $m > 0$). Moreover, its rational action on X is algebraic.

By ??, there exists an integer $m > 0$ such that the map $\psi : X \dashrightarrow \mathbb{P}(H^0(X, mK_X))$ is birational onto its image Y. Let V be the subspace of $H^0(X, mK_X)$ spanned by the affine cone over Y. Since $\mathrm{Bir}(X)$ linearly acts on $H^0(X, mK_X)$ by Step 1, it also linearly acts on V. we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, dashed, "\psi"] & X \arrow[d, dashed, "\psi"] \\ Y \arrow[r, "\varphi_g|_Y"] \arrow[d, hook] & Y \arrow[d, hook] \\ \mathbb{P}(V) \arrow[r, "\varphi_g"] & \mathbb{P}(V) \end{tikzcd}$$

for every $g \in \mathrm{Bir}(X)$, where φ_g is the induced automorphism of $\mathbb{P}(V)$.

Since ψ is birational, the map $g \mapsto \varphi_g$ defines an injective group homomorphism from $\mathrm{Bir}(X)$ to $\mathrm{Aut}(\mathbb{P}(V))$. Consider the natural algebraic group structure on $\mathrm{Aut}(\mathbb{P}(V))$ and let G be the Zariski closure of the image of $\mathrm{Bir}(X)$ in $\mathrm{Aut}(\mathbb{P}(V))$. Note that $\mathrm{Bir}(X)$ fixes Y. Thus G also fixes Y. Since the affine cone over Y spans V, we conclude that any element $g \in G$ is uniquely determined by its restriction to Y. In particular, we have $G = \mathrm{Bir}(X)$. Note that $\mathrm{Aut}(\mathbb{P}(V))$ is a linear algebraic group and so is its closed subgroup $\mathrm{Bir}(X)$.

Step 3. If $\dim \mathrm{Bir}(X) > 0$, then it contains $\mathbb{G}_a$ or $\mathbb{G}_m$ as a subgroup. We show that the action of $\mathbb{G}_a$ or $\mathbb{G}_m$ on X leads to X being uniruled, which contradicts the assumption that X is of general type.

By ?? and ??, if $\dim \mathrm{Bir}(X) > 0$, then $\mathrm{Bir}(X)$ contains either $\mathbb{G}_a$ or $\mathbb{G}_m$ as a subgroup. Note that both $\mathbb{G}_a$ and $\mathbb{G}_m$ are rational varieties, without loss of generality, we may assume that $\mathrm{Bir}(X)$ contains $\mathbb{G}_m$ as a subgroup. Then we have a rational map

$$\Phi : \mathbb{G}_m \times X \dashrightarrow X.$$

Fix $x \in X$ such that $\Phi|_{\mathbb{G}_m \times \{x\}} : \mathbb{G}_m \to X$ is not constant. Choose $Z \subset X$ a closed subvariety of codimension 1 passing through x such that $\mathbb{G}_m.x \nsubseteq Z$. Then the closure of $\Phi(\mathbb{G}_m \times Z)$ in X has dimension at least $\dim Z + 1 = \dim X$. Hence we have a dominant rational map

$$\Phi : \mathbb{P}^1 \times Z \dashrightarrow X.$$

This contradicts ?? and the assumption that X is of general type. Therefore, we must have $\dim \mathrm{Bir}(X) = 0$, i.e., $\mathrm{Bir}(X)$ is finite. $\square$

Remark 6.4.2. In the proof of Theorem 6.4.1, by $\mathbb{P}(V)$ we mean the projective space associated to the vector space V in the sense of Grothendieck, i.e., $\mathbb{P}(V) = \mathrm{Proj}(\bigoplus_{k \geq 0} \mathrm{Sym}^k V)$. Hence if one have a linear map $f : V \to W$ between two finite-dimensional $\Bbbk$-vector spaces, then it induces a morphism $\mathbb{P}(W) \to \mathbb{P}(V)$ (not $\mathbb{P}(V) \to \mathbb{P}(W)$).

Corollary 6.4.3. Let X be a projective variety of general type over an algebraically closed field $\Bbbk$ of characteristic zero. Then there exists a projective variety Y birational to X such that $\mathrm{Bir}(Y) = \mathrm{Aut}(Y)$.

Corollary 6.4.4. Let X be a smooth projective Fano variety over an algebraically closed field $\mathbb{k}$ of characteristic zero. Then the group of automorphisms $\mathrm{Aut}(X)$ is a linear algebraic group.

Proof. Note that for every $g \in \mathrm{Aut}(X)$, g induces an automorphism of $H^0(X, -mK_X)$ for every integer $m \geq 1$ via pull-back of functions. Then the same argument as in Step 2 shows that $\mathrm{Aut}(X)$ is a linear algebraic group. $\square$

Chapter 7

Moduli Spaces

References

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001, pp. xiv+233. ISBN: 0-387-95227-6. DOI: 10.1007/978-1-4757-5406-3. URL: https://doi.org/10.1007/978-1-4757-5406-3 (cit. on p. 34).

[Har77] Robin Hartshorne. *Algebraic geometry*. Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9 (cit. on pp. 22, 33).

[Kaw91] Yujiro Kawamata. "On the length of an extremal rational curve". In: *Inventiones mathematicae* 105.1 (1991), pp. 609–611 (cit. on p. 34).

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*. Vol. 134. Cambridge Tracts in Mathematics. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. Cambridge University Press, Cambridge, 1998, pp. viii+254. ISBN: 0-521-63277-3. DOI: 10.1017/CBO9780511662560. URL: https://doi.org/10.1017/CBO9780511662560 (cit. on pp. 33, 35, 37).

[Kol96] János Kollár. *Rational Curves on Algebraic Varieties*. Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Berlin, Heidelberg: Springer-Verlag, 1996, p. 320. ISBN: 978-3-540-60168-5. DOI: 10.1007/978-3-662-03276-3. URL: https://doi.org/10.1007/978-3-662-03276-3.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: 10.1007/978-3-642-18808-4. URL: https://doi.org/10.1007/978-3-642-18808-4 (cit. on pp. 34, 41).

[Mat89] Hideyuki Matsumura. *Commutative ring theory*. 8. Cambridge university press, 1989.

[MM86] Yoichi Miyaoka and Shigefumi Mori. "A numerical criterion for uniruledness". In: *Annals of Mathematics* 124.1 (1986), pp. 65–69 (cit. on p. 34).

[Stacks] The Stacks Project Authors. *Stacks Project*. URL: https://stacks.math.columbia.edu/.