

# Picard Groups of Abelian Varieties

Let  $\mathbf{k}$  be a field and  $\mathbb{k}$  its algebraic closure. Let  $A$  be an abelian variety over  $\mathbf{k}$ .

## 1 Pullback along group operations

**Theorem 1** (Theorem of the cube). Let  $X, Y, Z$  be proper varieties over  $\mathbf{k}$  and  $\mathcal{L}$  a line bundle on  $X \times Y \times Z$ . Suppose that there exist  $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$  such that the restriction  $\mathcal{L}|_{\{x\} \times Y \times Z}$ ,  $\mathcal{L}|_{X \times \{y\} \times Z}$  and  $\mathcal{L}|_{X \times Y \times \{z\}}$  are trivial. Then  $\mathcal{L}$  is trivial.

*Proof.* **Yang:** To be completed. □

**Remark 2.** If we assume the existence of the Picard scheme, then the [Theorem 1](#) can be deduced from the Rigidity Lemma. Consider the morphism

$$\varphi : X \times Y \rightarrow \text{Pic}(Z), \quad (x, y) \mapsto \mathcal{L}|_{\{x\} \times \{y\} \times Z}.$$

Since  $\varphi(x, y) = \mathcal{O}_Z$ ,  $\varphi$  factors through  $\text{Pic}^0(Z)$ . Then the assumption implies that  $\varphi$  contracts  $\{x\} \times Y$ ,  $X \times \{y\}$  and hence it maps  $X \times Y$  to a point. Thus  $\varphi(x', y') = \mathcal{O}_Z$  for every  $(x', y') \in X \times Y$ . Then by Grauert's theorem, we have  $\mathcal{L} \cong p^* p_* \mathcal{L}$  where  $p : X \times Y \times Z \rightarrow X \times Y$  is the projection. Note that  $p_* \mathcal{L} \cong \mathcal{L}|_{X \times Y \times \{z\}} \cong \mathcal{O}_{X \times Y}$ . Hence  $\mathcal{L}$  is trivial.

**Lemma 3.** Let  $A$  be an abelian variety over  $\mathbf{k}$ ,  $f, g, h : X \rightarrow A$  morphisms from a variety  $X$  to  $A$  and  $\mathcal{L}$  a line bundle on  $A$ . Then we have

$$(f + g + h)^* \mathcal{L} \cong (f + g)^* \mathcal{L} \otimes (f + h)^* \mathcal{L} \otimes (g + h)^* \mathcal{L} \otimes f^* \mathcal{L}^{-1} \otimes g^* \mathcal{L}^{-1} \otimes h^* \mathcal{L}^{-1}.$$

*Proof.* First consider  $X = A \times A \times A$ ,  $p : X \rightarrow A, (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$ ,  $p_{ij} : X \rightarrow A, (x_1, x_2, x_3) \mapsto x_i + x_j$  for  $1 \leq i < j \leq 3$  and  $p_i : X \rightarrow A, (x_1, x_2, x_3) \mapsto x_i$  for  $1 \leq i \leq 3$ . Then the conclusion follows from the theorem of the cube by taking  $\mathcal{L}' = p^* \mathcal{L}^{-1} \otimes p_{12}^* \mathcal{L} \otimes p_{13}^* \mathcal{L} \otimes p_{23}^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1} \otimes p_3^* \mathcal{L}^{-1}$  and considering the restriction to  $\{0\} \times A \times A$ ,  $A \times \{0\} \times A$  and  $A \times A \times \{0\}$ .

In general, consider the morphism  $\varphi = (f, g, h) : X \rightarrow A \times A \times A$  and pull back the above isomorphism along  $\varphi$ . □

**Proposition 4.** Let  $A$  be an abelian variety over  $\mathbf{k}$ ,  $n \in \mathbb{Z}$  and  $\mathcal{L}$  a line bundle on  $A$ . Then we have

$$[n]_A^* \mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}(n^2+n)} \otimes [-1]_A^* \mathcal{L}^{\otimes \frac{1}{2}(n^2-n)}.$$

*Proof.* For  $n = 0, 1$ , the conclusion is trivial. For  $n \geq 2$ , we can use the previous lemma on  $[n-2]_A, [1]_A, [1]_A$  and induct on  $n$ . Hence we have

$$[n]_A^* \mathcal{L} \cong [n-1]_A^* \mathcal{L} \otimes [n-1]_A^* \mathcal{L} \otimes [2]_A^* \mathcal{L} \otimes [1]_A^* \mathcal{L}^{-1} \otimes [1]_A^* \mathcal{L}^{-1} \otimes [n-2]_A^* \mathcal{L}^{-1}.$$

Then the conclusion follows from induction. **Yang:** To be completed. □

**Definition 5.** Let  $A$  be an abelian variety over  $\mathbf{k}$  and  $\mathcal{L}$  a line bundle on  $A$ . We say that  $\mathcal{L}$  is *symmetric* if  $[-1]_A^* \mathcal{L} \cong \mathcal{L}$  and *antisymmetric* if  $[-1]_A^* \mathcal{L} \cong \mathcal{L}^{-1}$ .

**Theorem 6** (Theorem of the square). Let  $A$  be an abelian variety over  $\mathbf{k}$ ,  $x, y \in A(\mathbf{k})$  two points and  $\mathcal{L}$  a line bundle on  $A$ . Then

$$t_{x+y}^* \mathcal{L} \otimes \mathcal{L} \cong t_x^* \mathcal{L} \otimes t_y^* \mathcal{L}.$$

*Proof.* Yang: To be completed. □

**Remark 7.** We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \rightarrow \text{Pic}(A), \quad x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that  $\Phi_{\mathcal{L}}$  is a homomorphism of groups. When we vary  $\mathcal{L}$ , the map

$$\Phi_{\square} : \text{Pic}(A) \rightarrow \text{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \text{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is also a group homomorphism. For any  $x \in A(\mathbf{k})$ , we have

$$\Phi_{t_x^* \mathcal{L}} = \Phi_{\mathcal{L}}$$

by Theorem 6. In the other words,

$$\Phi_{\mathcal{L}}(x) \in \text{Ker } \Phi_{\square}, \quad \forall \mathcal{L} \in \text{Pic}(A), x \in A(\mathbf{k}).$$

If we assume the scheme structure on  $\text{Pic}(A)$ , then  $\Phi_{\mathcal{L}}$  is a morphism of scheme and factors through  $\text{Pic}^0(A)$ . Let  $K(\mathcal{L}) := \text{Ker } \Phi_{\mathcal{L}}$ , then  $K(\mathcal{L})$  is a subgroup scheme of  $A$ . We give another description of  $K(\mathcal{L})$ . From this point, when  $K(\mathcal{L})$  is finite, we can recover the dual abelian variety  $A^{\vee} = \text{Pic}_{A/\mathbf{k}}^0$  as the quotient  $A/K(\mathcal{L})$ .

## 2 Projectivity

In this subsection, we work over the algebraically closed field  $\mathbf{k}$ .

**Proposition 8.** Let  $A$  be an abelian variety over  $\mathbf{k}$  and  $D$  an effective divisor on  $A$ . Then  $|2D|$  is base point free.

*Proof.* Yang: To be completed. □

**Theorem 9.** Let  $A$  be an abelian variety over  $\mathbf{k}$  and  $D$  an effective divisor on  $A$ . TFAE:

- (a) the stabilizer  $\text{Stab}(D)$  of  $D$  is finite;
- (b) the morphism  $\phi_{|2D|}$  induced by the complete linear system  $|2D|$  is finite;
- (c)  $D$  is ample;
- (d)  $K(\mathcal{O}_A(D))$  is finite.

*Proof.* Yang: To be completed. □

**Theorem 10.** Let  $A$  be an abelian variety over  $\mathbb{k}$ . Then  $A$  is projective.

*Proof.* Yang: To be completed. □

**Corollary 11.** Let  $A$  be an abelian variety over  $\mathbb{k}$  and  $D$  a divisor on  $A$ . Then  $D$  is pseudo-effective if and only if it is nef, i.e.  $\text{Psef}^1(A) = \text{Nef}^1(A)$ .

*Proof.* Yang: To be completed. □

### 3 Dual abelian varieties

In this subsection, we work over the algebraically closed field  $\mathbb{k}$ .

**Proposition 12.** Let  $A$  be an abelian variety over  $\mathbf{k}$  and  $\mathcal{L}$  an ample line bundle on  $A$ . Then the homomorphism  $\Phi_{\mathcal{L}} : A(\mathbb{k}) \rightarrow \text{Pic}(A)$  factors through  $\text{Pic}^0(A)$  and  $A(\mathbb{k}) \rightarrow \text{Pic}^0(A)$  is surjective.

*Proof.* Yang: To be completed. □

**Definition 13.** Let  $A$  be an abelian variety over  $\mathbb{k}$ . We define the *dual abelian variety* of  $A$  to be  $A/K(\mathcal{L})$  for some ample line bundle  $\mathcal{L}$  on  $A$ . We denote it by  $A^{\vee}$ .

**Theorem 14.** Let  $A$  be an abelian variety over  $\mathbf{k}$ . Then the dual abelian variety  $A^{\vee}$  does not depend on the choice of the ample line bundle  $\mathcal{L}$ . Moreover, there is a natural bijection  $A^{\vee}(\mathbf{k}) \rightarrow \text{Pic}^0(A)$ .

*Proof.* Yang: To be completed. □

**Proposition 15.** Let  $A$  be an abelian variety over  $\mathbf{k}$ . Then the dual abelian variety  $A^{\vee}$  is also an abelian variety and the natural map  $A \rightarrow A^{\vee\vee}$  is an isomorphism.

*Proof.* Yang: To be completed. □

**Proposition 16.** There exists a unique line bundle  $\mathcal{P}$  on  $A \times A^{\vee}$  such that for every  $y = \mathcal{L} \in A^{\vee} = \text{Pic}^0(A)$ , we have  $\mathcal{P}|_{A \times \{y\}} \cong \mathcal{L}$ .

*Proof.* Yang: To be completed. □

### 4 The Néron-Severi group

**Theorem 17.** Let  $A$  be an abelian variety over  $\mathbb{k}$ . Then we have an inclusion  $\text{NS}(A) \hookrightarrow \text{Hom}_{\mathbf{Grp}}(A, A^{\vee})$  given by Yang: To be completed.