Notes in Algebraic Geometry

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Chapter 1

Schemes and Varieties

1.1 Definition and First Properties of Schemes

1.1.1 Locally Ringed Space

Definition 1.1.1. Let X be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of X with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Example 1.1.2. Let X be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U, \mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on X.

Example 1.1.3. Let X be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U\}$$

defines a presheaf $\mathcal{C}$ of rings on X but not a sheaf.

For a concrete example, let $X = (0, 1) \cup (2, 3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0, 1), (2, 3)\}$ of X. The sections $s_1 = 1 \in \mathcal{C}((0, 1))$ and $s_2 = 2 \in \mathcal{C}((2, 3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

Definition 1.1.4. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

Example 1.1.5. Let p be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p).

Example 1.1.6 (Glue morphisms)**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of X and for each $i \in I$, we have a morphism $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Example 1.1.7 (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i, \mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i\in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* φ_{ij}.

First φ_{ij} induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i\in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space X. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^\sharp(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with ψ_i' satisfying the same properties, then by gluing $\psi_i' \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

1.1.2 Schemes

Let R be a ring. Recall that the *spectrum* of R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set.*

Proposition 1.1.8. Let R be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

Proof. To be continued □

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Definition 1.1.9. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i \in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.
These data form a category, denoted by **Sch**. If we fix a base scheme S, then an S*-scheme* is a scheme X together with a morphism $X \to S$. The category of S-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

Theorem 1.1.10. The functor $\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. To be continued

Definition 1.1.11. A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if f induces an isomorphism of X onto an open (resp. closed) subscheme of Y. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. To be continued

Definition 1.1.12. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme Z of Y such that f factors through Z. More precisely, if $Y = \operatorname{Spec} A$ is affine, then the scheme theoretic image of f is $\operatorname{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover Y by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of f. To be continued

Example 1.1.13 (Glue open subschemes)**.** The construction in Example 1.1.7 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Example 1.1.7. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the φ_{ij} is a scheme.

1.1.3 Integral, reduced and irreducible

1.1.4 Fiber product

1.1.5 Dimension

1.1.6 Noetherian and finite type

1.1.7 Separated and proper

1.2 Line Bundles and Divisors

1.2.1 Cartier Divisors

1.2.2 Line Bundles and Picard Group

Definition 1.2.1. Let X be a scheme. The *Picard group* of X is defined to be $\mathsf{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. The group operation is given by the tensor product of line bundles.

Definition 1.2.2. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X \times \{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X \times \{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\text{alg}} \mathcal{L}'$. To be checked.

1.2.3 Weil Divisors and Reflexive Sheaves

1.3 Line bundles induce morphisms

1.3.1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 1.3.1. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^* x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On U_i, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^* x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each U_i, g must agree with f_i by the same construction. Thus $g = f$. □

Proposition 1.3.2. Let X be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on X. Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. To be continued. □

Example 1.3.3. Let $X = \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\dots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$, where T_i are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \dots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the d*-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

Example 1.3.4. Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with A a ring and $\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}^{(m+1)(n+1)-1}_A$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 1.3.5. A line bundle $\mathcal{L}$ on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. To be continued.

Example 1.3.6. Let

Example 1.3.7.

Definition 1.3.8. Let $\mathcal{L}$ be a line bundle on a scheme X. To be continued.

Definition 1.3.9. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. To be continued.

Definition 1.3.10. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. To be continued.

Theorem 1.3.11. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

To be continued.

Proposition 1.3.12. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$, $\mathcal{M}$ line bundles on X. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

To be continued.

Proof. To be continued. □

Proposition 1.3.13. Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ a line bundle on X. Then $\mathcal{L}$ is very ample if and only if the following two conditions hold:

(a) (separate points) for any two distinct points $x, y \in X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $s(y) \neq 0$;

(b) (separate tangent vectors) for any point $x \in X$ and non-zero tangent vector $v \in T_x X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $v(s) \neq 0$.

To be continued.

1.3.2 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 1.3.14. Let X be a normal proper variety over a field $\mathbf{k}$, D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

Definition 1.3.15. A *linear system* on a scheme X is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on X and $V \subseteq \Gamma(X, \mathcal{L})$ is a subspace. The dimension of the linear system is $\dim V - 1$. A linear system is *base-point free* if V is base-point free. A linear system is *complete* if $V = \Gamma(X, \mathcal{L})$. To be continued.

Definition 1.3.16. Let $\mathcal{L}$ be a line bundle on a scheme X and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace. The *base locus* of V is the closed subset

$$\mathrm{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If $\mathrm{Bs}(V) = \varnothing$, we say that V is *base-point free*. To be continued.

1.3.3 Asymptotic behavior

Definition 1.3.17. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. To be continued.

Definition 1.3.18. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. To be continued.

Theorem 1.3.19. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \mathrm{Proj}\, R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. To be continued.

Definition 1.3.20. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large n. To be continued.

Example 1.3.21. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2. To be continued.

Chapter 2

More Scattered Topics

Chapter 3

Surfaces

3.1 Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$. This section is mainly based on [Har77, Chapter V.2].

3.1.1 Preliminaries

Let S be a variety over $\Bbbk$ and $\mathcal{E}$ a vector bundle of rank $r+1$ on S.

Proposition 3.1.1. The S-varieties $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on S.

Theorem 3.1.2. Let $\pi : X = \mathbb{P}_S(\mathcal{E}) \to S$ be the projective bundle associated to a vector bundle $\mathcal{E}$ of rank $r+1$ on S. Then there is an exact sequence of vector bundles on $\mathbb{P}_S(\mathcal{E})$

$$0 \to \Omega_{\mathbb{P}_S(\mathcal{E})/S} \to \pi^*(\mathcal{E})(-1) \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \to 0.$$

In particular, $K_X \sim \pi^*(K_S + \det \mathcal{E}) - (r+1)\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. To be continued...

Theorem 3.1.3 (Tsen's Theorem, [Stacks, Tag 03RD])**.** Let C be a smooth curve over an algebraically closed field $\Bbbk$. Then $\mathbf{K} = \Bbbk(C)$ is a C_1 field, i.e., every degree d hypersurface in $\mathbb{P}^n_{\mathbf{K}}$ has a $\mathbf{K}$-rational point provided $d \leq n$.

Theorem 3.1.4 (Grauert's Theorem, [Har77, Corollary 12.9])**.** Let $f : X \to S$ be a projective morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf on X which is flat over S. Suppose that S is integral and the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$ is constant on S for some $i \geq 0$. Then $\mathrm{R}^i f_* \mathcal{F}$ is locally free and the base change homomorphism

$$\varphi^i_s : \mathrm{R}^i f_* \mathcal{F} \otimes_{\mathcal{O}_S} \kappa(s) \to H^i(X_s, \mathcal{F}_s)$$

is an isomorphism for all $s \in S$.

Theorem 3.1.5 (Miracle Flatness, [Mat89, Theorem 23.1])**.** Let $f : X \to Y$ be a morphism of noetherian schemes. Assume that Y is regular and X is Cohen-Macaulay. If all fibers of f have the same dimension $d = \dim X - \dim Y$, then f is flat.

Proposition 3.1.6 (Geometric form of Nakayama's Lemma)**.** Let X be a variety, $x \in X$ a closed point and $\mathcal{F}$ a coherent sheaf on X. If $a_1, \cdots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proposition 3.1.7. Let S be a noetherian scheme and $\mathcal{E}$ a vector bundle of rank $r+1$ on S. Denote by $\pi : \mathbb{P}_S(\mathcal{E}) \to S$ the projection. Let X be an S-scheme via a morphism $g : X \to S$. Then there is a bijection

$$\left\{ \begin{array}{l} S\text{-morphisms} \\ X \to \mathbb{P}_S(\mathcal{E}) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \mathcal{L} \in \mathsf{Pic}(X) \text{ and surjective} \\ \text{homomorphisms } g^*\mathcal{E} \to \mathcal{L} \end{array} \right\}.$$

Proof. We have a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ by the definition of $\mathbb{P}_S(\mathcal{E})$. If we have a morphism $f : X \to \mathbb{P}_S(\mathcal{E})$ over S, then we have a surjective homomorphism $f^*\pi^*\mathcal{E} \to f^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$.

Suppose we have a surjective homomorphism $g^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ where $\mathcal{L}$ is a line bundle on X. Take an affine cover $\{U_i\}$ of S such that $\mathcal{E}|_{U_i}$ is trivial. On U_i, choose a basis $e_0^{(i)}, \ldots, e_r^{(i)}$ of $\mathcal{E}|_{U_i}$. Suppose $\mathbb{P}_S(\mathcal{E})$ is given by gluing $\mathbb{P}^r_{U_i}$ via φ_{ij} induced by the transition functions of $\mathcal{E}$.

The surjection $g^*\mathcal{E}|_{U_i} \twoheadrightarrow \mathcal{L}|_{X_{U_i}}$ gives a unique morphism $f_i : X_{U_i} \to \mathbb{P}^r_{U_i}$ by Theorem 1.3.1. On $X_{U_i \cap U_j}$, f_i and f_j agree since we have

$$\begin{tikzcd} X_{U_i \cap U_j} \arrow[r, "="] \arrow[d, "f_i"'] & X_{U_i \cap U_j} \arrow[d, "f_j"] \\ \mathbb{P}_{U_i \cap U_j}(\bigoplus \mathcal{O}_{U_i \cap U_j} e_k^{(i)}) \arrow[r, "\varphi_{ij}"] & \mathbb{P}_{U_i \cap U_j}(\bigoplus \mathcal{O}_{U_i \cap U_j} e_k^{(j)}) \end{tikzcd}$$

and the bottom arrow is identical to the identity map on $\mathbb{P}_S(\mathcal{E})_{U_i \cap U_j}$. Gluing f_i gives a morphism $f : X \to \mathbb{P}_S(\mathcal{E})$ over S. In particular, we have $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. □

Definition 3.1.8. An *extension* of a coherent sheaf $\mathcal{F}$ by a coherent sheaf $\mathcal{G}$ on a scheme X is an exact sequence of coherent sheaves

$$S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0).$$

Two extensions S and S' are *equivalent* if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{G}}"] & \mathcal{E} \arrow[r] \arrow[d, "\cong"] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{F}}"] & 0 \\ 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0. \end{tikzcd}$$

Proposition 3.1.9. Let X be a scheme and $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. Then there is a one-to-one correspondence between equivalence classes of extensions

$$S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0)$$

and elements of $\mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})$ given by
$$S \mapsto \delta(\mathrm{id}_{\mathcal{F}})$$
where $\delta : \mathrm{Hom}_X(\mathcal{F},\mathcal{F}) \to \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})$ is the connecting homomorphism.

Proof. Take an exact sequence
$$0 \to \mathcal{G} \to \mathcal{I} \xrightarrow{\varphi} \mathcal{C} \to 0$$
with $\mathcal{I}$ injective. Applying $\mathrm{Hom}_X(\mathcal{F},-)$ gives a long exact sequence
$$0 \to \mathrm{Hom}_X(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_X(\mathcal{F},\mathcal{I}) \to \mathrm{Hom}_X(\mathcal{F},\mathcal{C}) \xrightarrow{\delta} \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}) \to 0.$$
For $a \in \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})$, choose a lifting $\alpha \in \mathrm{Hom}_X(\mathcal{F},\mathcal{C})$ of a. Let $\mathcal{E} := \mathrm{Ker}(\mathcal{I} \oplus \mathcal{F} \to \mathcal{C}, (i,f) \mapsto \varphi(i) - \alpha(f))$.

Let $\mathcal{E} \to \mathcal{F}$ be the projection to the second factor. It is surjective since φ is surjective. Consider the inclusion $\mathcal{G} \to \mathcal{I} \to \mathcal{I} \oplus \mathcal{F}$, which factors through $\mathcal{E}$. On the other hand, if $e \in \mathcal{E}$ maps to 0 in $\mathcal{F}$, then $e \in \mathcal{I}$ and $\varphi(e) = 0$, whence $e \in \mathcal{G}$. Hence we have an extension $S = (0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0)$.

To be continued... □

3.1.2 Minimal Section and Classification

Definition 3.1.10 (Ruled surface)**.** A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all geometric fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 3.1.11. There exists a section of π.

Proof. To be continued... □

Proposition 3.1.12. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Proof. Let $\sigma : C \to X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π, $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in C$. By Miracle Flatness (Theorem 3.1.5), f is flat. By Grauert's Theorem (Theorem 3.1.4), $\mathcal{E}$ is a vector bundle of rank 2 on C and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in C$.

This gives a surjective homomorphism
$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$
For every $x \in X$, we have
$$\mathcal{E} \otimes_{\mathcal{O}_C} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$
The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \to \mathcal{L}$ is surjective.

By Proposition 3.1.7, we have a morphism $\varphi : X \to \mathbb{P}_C(\mathcal{E})$ over C such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Since $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in C$, $\varphi|_{X_t} : X_t \to \mathbb{P}_C(\mathcal{E})_t$ is an isomorphism for any $t \in C$. Hence

φ is bijection on the underlying sets. Here is a serious gap. Why fiberwise isomorphism implies isomorphism? □

Lemma 3.1.13. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*. In particular, if $\mathcal{E}$ is normalized, then $e = -\deg c_1(\mathcal{E})$ is an invariant of the ruled surface X.

Proof. We can suppose that $\mathcal{E}$ is globally generated since we can always twist $\mathcal{E}$ by a sufficiently ample line bundle on C. Then for all line bundle $\mathcal{L}$ of degree sufficiently large, $\mathcal{L}$ is very ample and hence $H^0(C, \mathcal{E} \otimes \mathcal{L}) \neq 0$. By Lemma 3.1.11 and Proposition 3.1.7, $\mathcal{E}$ is an extension of line bundles. Then for all line bundle $\mathcal{L}$ of degree sufficiently negative, $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ since line bundles of negative degree have no global sections. Hence we can find a line bundle $\mathcal{M}$ on C of lowest degree such that $H^0(C, \mathcal{E} \otimes \mathcal{M}) \neq 0$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{M}$, we are done. □

Remark 3.1.14. The invariant e is unique but the normalization of $\mathcal{E}$ is not unique. For example, if $\mathcal{E}$ is normalized, then so is $\mathcal{E} \otimes \mathcal{L}$ for any line bundle $\mathcal{L}$ on C of degree 0. To be continued...

Suppose that $X \cong \mathbb{P}_C(\mathcal{E})$ where $\mathcal{E}$ is a normalized vector bundle of rank 2 on C. Since $H^0(C, \mathcal{E}) \neq 0$, choosing a non-zero section s, we get an exact sequence

$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{E} \to \mathcal{E}/\mathcal{O}_C \to 0.$$

We claim that $\mathcal{E}/\mathcal{O}_C$ is a line bundle on C. Since C is a curve, we only need to check that $\mathcal{E}/\mathcal{O}_C$ is torsion-free.

To be continued...

Definition 3.1.15. A section C_0 of π is called a *minimal section* if to be continued...

Lemma 3.1.16. Let $X = \mathbb{P}_C(\mathcal{E}) \to C$ be a ruled surface over a smooth curve C of genus g with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on C with $\deg \mathcal{L} = -e$.

(b) If $\mathcal{E}$ is indecomposable, then $-2g \leq e \leq 2g - 2$.

Proof. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable, we can assume that $H^0(C, \mathcal{L}_1) \neq 0$. If $\deg \mathcal{L}_1 > 0$, then $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$, contradicting the normalization of $\mathcal{E}$. Similarly $\deg \mathcal{L}_2 \leq 0$. Then $\mathcal{L}_1 \cong \mathcal{O}_C$. And hence $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$.

If $\mathcal{E}$ is indecomposable, we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

which is a non-trivial extension, with $\mathcal{L}$ a line bundle on C of degree $-e$. Hence by Proposition 3.1.9, we have $0 \neq \mathrm{Ext}^1_C(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$. By Serre duality, we have $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$. Hence $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$.

On the other hand, let $\mathcal{M}$ be a line bundle on C of degree -1. Twist the above exact sequence by $\mathcal{M}$ and take global sections, we have an equation

$$h^0(\mathcal{M}) - h^0(\mathcal{E} \otimes \mathcal{M}) + h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{M}) + h^1(\mathcal{E} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = 0.$$

Since $\deg \mathcal{M} < 0$ and $\mathcal{E}$ is normalized, we have $h^0(\mathcal{M}) = h^0(\mathcal{E} \otimes \mathcal{M}) = 0$. By Riemann-Roch, we have $h^1(\mathcal{M}) = g$ and $h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = -e - 1 + 1 - g$. Hence

$$h^1(\mathcal{E} \otimes \mathcal{M}) = e + 2g \geq 0.$$

This gives $e \geq -2g$. □

Theorem 3.1.17. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Proof. This is a direct consequence of Lemma 3.1.16. □

Example 3.1.18. Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$.

Let C be covered by two standard affine charts U_0, U_1 with coordinate u on U_0 and v on U_1 such that $u = 1/v$ on $U_0 \cap U_1$. On U_i, let $\mathcal{O}(-e)|_{U_i}$ be generated by s_i for $i = 0, 1$. We have $s_0 = u^e s_1$ on $U_0 \cap U_1$.

On $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$, let $[x_0 : x_1]$ and $[y_0 : y_1]$ be the homogeneous coordinates of $\mathbb{P}^1$ on X_0 and X_1 respectively. Then the transition function on $X_0 \cap X_1$ is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

Remark 3.1.19. The surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ is also called the *Hirzebruch surface*.

Theorem 3.1.20. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with invariant e and normalized $\mathcal{E}$.

(a) If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and for each e there exists a unique such ruled surface up to isomorphism.

(b) If $\mathcal{E}$ is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on E with $\deg \mathcal{L} = -e$.

Proof. Only the indecomposable case needs a proof. By Lemma 3.1.16, we have $-2 \leq e \leq 0$ and a non-trivial extension

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle on E of degree $-e$.

Case 1. $e = 0$.

In this case, $\mathcal{L}$ is of degree 0 and $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \neq 0$. Hence $\mathcal{L} \cong \mathcal{O}_E$. To be continued...

Case 2. $e = -1$.

In this case, $\mathcal{L}$ is of degree 1 and $H^1(E, \mathcal{L}) \cong H^0(E, \mathcal{L}^{-1}) = 0$. By Riemann-Roch, we have $h^0(E, \mathcal{L}) = 1$.

Case 3. $e = -2$.

To be continued...

□

Example 3.1.21. To be continued...

3.1.3 The Néron-Severi Group of Ruled Surfaces

Proposition 3.1.22. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and F a fiber of π. Then $\mathrm{Pic}(X) \cong \mathbb{Z}[C_0] \oplus \pi^* \mathrm{Pic}(C)$.

Proof. Let D be any divisor on X with $D.F = a \in \mathbb{Z}$. Then $D - aC_0$ is numerically trivial on the fibers of π. Let $\mathcal{L} = \mathcal{O}_X(D - aC_0)$. Then $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$ for any $t \in C$. By Grauert's Theorem (Theorem 3.1.4), $\pi_* \mathcal{L}$ is a line bundle on C and the natural map $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$ is an isomorphism. □

Proposition 3.1.23. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let F be a fiber of π. Then $K_X \sim -2C_0 + \pi^*(K_C - c_1(\mathcal{E}))$. Numerically, we have $K_X \equiv -2C_0 + (2g - 2 - e)F$ where e is the invariant of X. Check this carefully.

Proof. To be continued. □

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

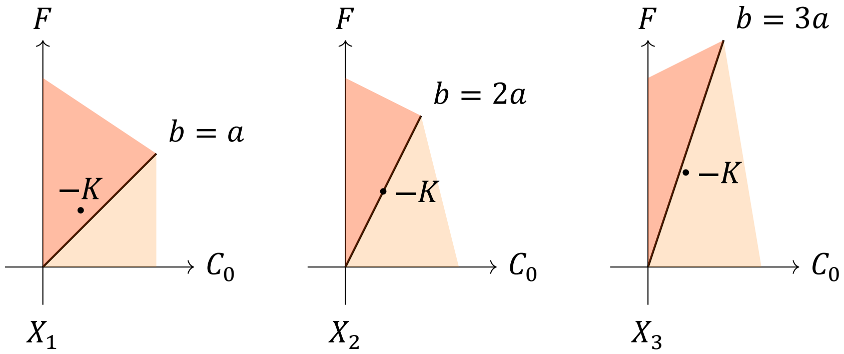
Theorem 3.1.24. Let $\pi : X \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with invariant e. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \sim aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a, b \geq 0$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$.

Proof. To be continued... □

Example 3.1.25. Here we draw the Néron-Severi group of the rational ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 1, 2, 3$.



We have $-K_{X_e} \equiv 2C_0 + (2 + e)F$. For $e = 1$, $-K$ is ample and hence X_1 is a del Pezzo surface. For $e = 2$, $-K$ is nef and big but not ample. For $e \geq 3$, $-K$ is big but not nef.

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.

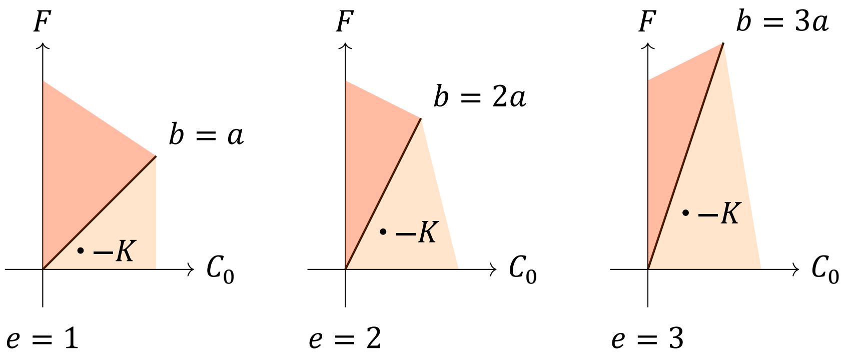
Theorem 3.1.26. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is decomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > ae$.

Proof. To be continued... □

Example 3.1.27. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with decomposable normalized $\mathcal{E}$ for $e = 1, 2, 3$.



In this case, $-K \equiv 2C_0 + eF$ is always big but not nef.

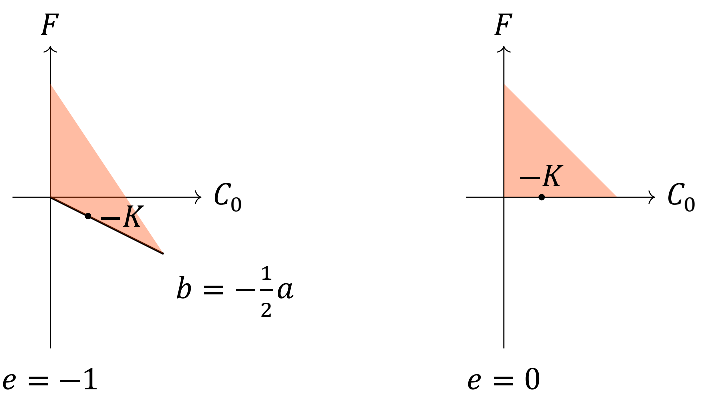
Theorem 3.1.28. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E with invariant e. Assume that $\mathcal{E}$ is indecomposable. Let C_0 be a minimal section of π and let F be a fiber of π. Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff$ $a \geq 0$ and $b \geq \frac{1}{2}ae$;

(b) D is ample $\iff$ D is very ample $\iff$ $a > 0$ and $b > \frac{1}{2}ae$.

Proof. To be continued... □

Example 3.1.29. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with indecomposable normalized $\mathcal{E}$ for $e = -1, 0$.



In this case, $-K \equiv 2C_0 + eF$ is always nef but not big.

Proposition 3.1.30. Let $\pi : X \to C$ be a ruled surface over a smooth curve C. Then every nef divisor on X is semi-ample. Check this carefully.

3.2 Some Singular Surfaces

In this section, fix an algebraically closed field $\Bbbk$. Everything is over $\Bbbk$ unless otherwise specified.

3.2.1 Projective cone over smooth projective curve

Let $C \subset \mathbb{P}^n$ be a smooth projective curve. The *projective cone* over C is the projective variety $X \subset \mathbb{P}^{n+1}$ defined by the same homogeneous equations as C. The variety X is singular at the vertex of the cone, which corresponds to the point $[0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}$.

Chapter 4

Moduli of vector bundles on curves

4.1 Introduction to Moduli Problems

Let C be a smooth projective curve of genus g over an algebraically closed field $\Bbbk$ of characteristic 0.

We are interested in the moduli space of vector bundles on C.

4.1.1 Moduli functors

Let S be a noetherian scheme and T is a scheme of finite type over S. Recall the Yoneda lemma: there is a full and faithful functor

$$h : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathrm{Fun}((\mathbf{Sch}_S)^{\mathrm{op}}, \mathbf{Set}), \quad T \mapsto h_T(S) \coloneqq \mathrm{Hom}_{\mathbf{Sch}_S}(T, S).$$

A functor $F : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$ is *representable* if there exists a scheme M over S such that $F \cong h_M$. We say that M is *the fine moduli space* of F.

Remark 4.1.1. If F is representable by M, then there is a universal object $\mathcal{U} \in F(M)$ given by $\mathrm{id}_M \in h_M(M)$ satisfying the following universal property: for any $T \in \mathbf{Sch}_S$ and any $\xi \in F(T)$, there exists a unique morphism $f : T \to M$ such that $F(f)(\mathcal{U}) = \xi$.

The most famous example of representable functor is the Quot functor. Let S be a noetherian scheme, $\pi : X \to S$ a projective morphism, $\mathcal{L}$ a relatively ample line bundle on X, $\mathcal{F}$ a coherent sheaf on X, and $P \in \mathbb{Q}[t]$ a polynomial. We define a functor

$$\begin{aligned}\mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{F}/X/S} :&(\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}\\ &T \mapsto \{\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q} \mid \mathcal{Q} \text{ is flat over } T, \forall t \in T, \mathcal{Q}_t \text{ has Hilbert polynomial } P\}/\sim,\end{aligned}$$

where $\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}$ and $\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}'$ are equivalent if $\mathrm{Ker}(\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}) = \mathrm{Ker}(\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}')$.

By Grothendieck, $\mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{F}/X/S}$ is representable by a projective S-scheme $\mathrm{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/S}$. Reference...

If we take $S = \mathrm{Spec}\,\Bbbk$, X a projective variety and $\mathcal{F} = \mathcal{O}_X$. Then the Quot functor $\mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{O}_X/X/\Bbbk}$ becomes the Hilbert functor $\mathcal{H}ilb^{P,\mathcal{L}}_{X/\Bbbk}$, which is representable by a projective $\Bbbk$-scheme called the *Hilbert scheme* $\mathrm{Hilb}^{P,\mathcal{L}}_X$.

4.1.2 Moduli functor of vector bundles

Consider the functor

$$\begin{aligned}\tilde{\mathcal{M}}_{r,d} :& (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set} \\ & T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a vector bundle on } X \times T \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\} / \sim,\end{aligned}$$

where $\mathcal{E} \sim \mathcal{E}'$ if there exists a line bundle $\mathcal{L}$ on T such that $\mathcal{E}' \cong \mathcal{E} \otimes \pi_T^* \mathcal{L}$, where $\pi_T : X \times T \to T$ is the projection.

Unfortunately, $\tilde{\mathcal{M}}_{r,d}$ is not representable. There are two main reasons:

- unboundedness and
- jumping phenomenon.

Definition 4.1.2. A family of vector bundles on a variety X is *bounded* if there exists a scheme S of finite type over $\Bbbk$ and a vector bundle $\mathcal{E}$ on $X \times S$ such that every vector bundle in the family is isomorphic to $\mathcal{E}_s$ for some $s \in S$.

If $\tilde{\mathcal{M}}_{r,d}$ is representable by a scheme M of finite type over $\Bbbk$, then the family of vector bundles parametrized by M is bounded. This is impossible since if so, $\{h^0(X, \mathcal{E}) \mid \mathcal{E} \in \tilde{\mathcal{M}}_{r,d}(\Bbbk)\}$ is bounded by semicontinuity theorem, which is not true. For example, consider the family $\mathcal{E}_n = \mathcal{O}_X(nP) \oplus \mathcal{O}_X(-nP) \in \tilde{\mathcal{M}}_{2,0}(\Bbbk)$ for $n \geq 0$, where $P \in X(\Bbbk)$ is a fixed point. By Riemann-Roch theorem, we have $h^0(X, \mathcal{E}_n) = n + 1 - g$ for n sufficiently large.

Let us see a jumping phenomenon example due to Rees. Let $\mathcal{E}$ be a vector bundle on X of rank r and degree d with a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}.$$

On $X \times \mathbb{A}^1$, we can construct a vector bundle $\mathcal{F}$ by "deforming" $\mathcal{E}$ to $\bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}$ as follows: let t be the coordinate of $\mathbb{A}^1$, and define $\mathcal{F}$ to be

$$\bigoplus_{i=1}^r \pi_X^* \mathcal{E}_i \cdot e_i \big/ \mathcal{K},$$

where $\mathcal{K}$ is the subsheaf generated by $\{s(e_{i-1} - te_i) \mid s \in \pi_X^*\mathcal{E}_{i-1} \subset \pi_X^*\mathcal{E}_i, 1 \leq i \leq r\}$ and $e_1, \ldots, e_r$ are formal symbols. Suppose that $\mathcal{E}_i/\mathcal{E}_{i-1}$ are vector bundles for all $1 \leq i \leq r$. Then by computing locally, we can see that $\mathcal{F}$ is a vector bundle of rank r on $X \times \mathbb{A}^1$. We have

$$\mathcal{F}_t \cong \begin{cases} \mathcal{E}, & t \neq 0, \\ \bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}, & t = 0. \end{cases}$$

We see that all $\mathcal{F}_t$ is of rank r and degree d, but it jumps from $\mathcal{E}$ to $\bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}$ at $t = 0$. This is called the *jumping phenomenon*.

Example 4.1.3. For a concrete example, let $X = \mathbb{P}^1$, we have an exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O} \to 0$$

by ??. Fix the standard coordinate $\mathbb{P}^1 = \operatorname{Proj} \Bbbk[X_0, X_1]$ and let $e_0 = (1, 0)$, $e_1 = (0, 1)$ be the standard basis of $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. On the open subset $U_i = \{X_i \neq 0\}$, fix a trivialization

$\mathcal{O}(-1) \cong \mathcal{O}_{U_i} \cdot \frac{1}{X_i}$. Recall that $\mathcal{O}(-2) \subset \mathcal{E}$ is generated by $(X_1 e_0 - X_0 e_1)/X_i^2$ on U_i for $i = 0, 1$ and $\mathcal{E} \to \mathcal{O}$ is given by $e_0 \mapsto X_0$, $e_1 \mapsto X_1$.

Consider the filtration $0 \subset \mathcal{O}(-2) \subset \mathcal{E}$. Then on $U_i \times \mathbb{A}^1$, $\mathcal{F}$ is given by the quotient

$$\left(\mathcal{O} \cdot \frac{e_0 X_1 - X_0 e_1}{X_i^2} f_1 \oplus \mathcal{O} \cdot \frac{e_1}{X_i} f_2 \oplus \mathcal{O} \cdot \frac{e_2}{X_i} f_2\right) \Big/ \mathcal{O} \cdot \frac{X_1 e_0 - X_0 e_1}{X_i^2}(t f_1 - f_2),$$

where f_1, f_2 are formal symbols. When $t \neq 0$, the quotient makes f_1 and f_2 identified up to a scalar, thus $\mathcal{F}_t \cong \mathcal{E}$. When $t = 0$, the quotient kills $\frac{X_1 e_0 - X_0 e_1}{X_i^2} f_2$, thus $\mathcal{F}_0 \cong \mathcal{O}(-2) f_1 \oplus \mathcal{E} f_2 / \mathcal{O}(-2) f_2 \cong \mathcal{O}(-2) \oplus \mathcal{O}$.

If $\tilde{\mathcal{M}}_{r,d}$ is representable by a scheme M, then the family of vector bundles parametrized by M does not have jumping phenomenon. Indeed, if $\mathcal{F}$ is an vector bundle on $X \times \mathbb{A}^1$ such that $\mathcal{F}_t \cong \mathcal{E}$ for $t \neq 0$, then by the universal property of M, there exists a unique morphism $f : \mathbb{A}^1 \to M$ such that $\mathcal{F} \cong (\mathrm{id}_X \times f)^* \mathcal{U}$, where $\mathcal{U}$ is the universal vector bundle on $X \times M$. Since f is constant on the open subset $\mathbb{A}^1 \setminus \{0\}$, it is constant on $\mathbb{A}^1$. Thus, $\mathcal{F}_0 \cong \mathcal{E}$.

To fix the above problems, we need to

- restrict to a smaller family of vector bundles,
- kill jumping phenomenon, and
- weaken the notion of representability.

Definition 4.1.4. Let $F : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$ be a functor, M a scheme over S, and $\eta : F \to h_M$ a natural transformation. We say that (M, η) *corepresents* F if for any scheme N over S and any natural transformation $\eta' : F \to h_N$, there exists a unique morphism $f : M \to N$ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & h_M \arrow[d, "h_f"] \\ & h_N. \end{tikzcd}$$

Definition 4.1.5. A scheme M over S is called the *coarse moduli space* of F if

(a) there exists a natural transformation $\eta : F \to h_M$ such that (M, η) corepresents F;

(b) $\eta_{\Bbbk} : F(\Bbbk) \to M(\Bbbk)$ is a bijection.

To be continued...

4.1.3 Semistable vector bundles

Definition 4.1.6. Let C be a smooth projective curve over $\Bbbk$. For a vector bundle $\mathcal{E}$ of rank r and degree d on C, we define its slope to be $\mu(\mathcal{E}) := d/r$.

Proposition 4.1.7. Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ be an exact sequence of non-zero vector bundles on C. Then $\mu(\mathcal{E}_2) \geq \mu(\mathcal{E}_1)$ (resp. $\mu(\mathcal{E}_2) > \mu(\mathcal{E}_1)$) if and only if $\mu(\mathcal{E}_2) \leq \mu(\mathcal{E}_3)$ (resp. $\mu(\mathcal{E}_2) < \mu(\mathcal{E}_3)$).

Proof. We have
$$\mu(\mathcal{E}_2) = \frac{\deg \mathcal{E}_2}{\operatorname{rank} \mathcal{E}_2} = \frac{\deg \mathcal{E}_1 + \deg \mathcal{E}_3}{\operatorname{rank} \mathcal{E}_1 + \operatorname{rank} \mathcal{E}_3}.$$
Note that for any $a, b, c, d \in \mathbb{R}_{>0}$, we have
$$\frac{a+c}{b+d} \geq \frac{a}{b} \iff bc \geq ad \iff \frac{a+c}{b+d} \leq \frac{c}{d}.$$
The strict inequality case is similar. Then the proposition follows. □

Definition 4.1.8. Let C be a smooth projective curve over $\Bbbk$ and $\mathcal{E}$ a vector bundle on C. We say that $\mathcal{E}$ is *stable* (resp. *semistable*) if for any proper sub-bundle $\mathcal{F} \subset \mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$).

Proposition 4.1.9. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on C. Suppose that they are semistable and $\mu(\mathcal{E}) > \mu(\mathcal{F})$. Then any homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ is zero.
Suppose that they are stable and $\mu(\mathcal{E}) = \mu(\mathcal{F})$. Then any non-zero homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ is an isomorphism.

Proof. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a non-zero homomorphism of vector bundles on C. We have an exact sequence
$$0 \to \operatorname{Ker} \varphi \to \mathcal{E} \to \operatorname{Im} \varphi \to 0.$$
Since $\mathcal{F}$ is vector bundle, hence torsion-free, $\operatorname{Im} \varphi$ is also torsion-free, thus a vector bundle.

If $\mathcal{E}$ and $\mathcal{F}$ are semistable with $\mu(\mathcal{E}) > \mu(\mathcal{F})$, clearly $\operatorname{Ker} \varphi \neq 0$, then by Proposition 4.1.7, we have
$$\mu(\mathcal{E}) \leq \mu(\operatorname{Im} \varphi) \leq \mu(\mathcal{F}).$$
This is a contradiction, thus $\varphi = 0$.

Suppose that $\mathcal{E}$ and $\mathcal{F}$ are stable with $\mu(\mathcal{E}) = \mu(\mathcal{F})$. If $\operatorname{Ker} \varphi \neq 0$, then by Proposition 4.1.7, we have
$$\mu(\mathcal{E}) < \mu(\operatorname{Im} \varphi) \leq \mu(\mathcal{F}).$$
This is a contradiction, thus φ is injective. Since $\mathcal{F}$ is stable and $\operatorname{Im} \varphi \subset \mathcal{F}$ has the same slope as $\mathcal{F}$, we have $\operatorname{Im} \varphi = \mathcal{F}$. □

Corollary 4.1.10. A stable vector bundle is simple as a coherent sheaf, i.e., $\operatorname{End}(\mathcal{E}) \cong \Bbbk$.

Proof. Let $\varphi \in \operatorname{End}(\mathcal{E})$ be a non-zero endomorphism. Then there exists $P \in C(\Bbbk)$ such that $\varphi_P : \mathcal{E}_P \to \mathcal{E}_P$ is non-zero. Let $a \in \Bbbk$ be an eigenvalue of φ_P and consider the endomorphism $\varphi - a \cdot \mathrm{id}_{\mathcal{E}}$. Then $(\varphi - a \cdot \mathrm{id}_{\mathcal{E}})_P : \mathcal{E}_P \to \mathcal{E}_P$ is not an isomorphism, so is $\varphi - a \cdot \mathrm{id}_{\mathcal{E}}$. By Proposition 4.1.9, $\varphi - a \cdot \mathrm{id}_{\mathcal{E}} = 0$, thus $\varphi = a \cdot \mathrm{id}_{\mathcal{E}}$. □

Lemma 4.1.11. Let $\mathcal{E}$ be a semistable vector bundle on X.

(a) if $\mu(\mathcal{E}) > 2g - 2$, then $H^1(X, \mathcal{E}) = 0$;

(b) if $\mu(\mathcal{E}) > 2g - 1$, then $\mathcal{E}$ is globally generated.

Proof. To be continued... □

Let $S_{r,d}$ be set of isomorphism classes of semistable vector bundles on X of rank r and degree d.

Proposition 4.1.12. The family $S_{r,d}$ is bounded.

Proof. To be continued... □

Definition 4.1.13 (Jordan-Hölder filtration)**.** Let $\mathcal{E}$ be a semistable vector bundle on C. A *Jordan-Hölder filtration* of $\mathcal{E}$ is a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ are stable with $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E})$ for all $1 \leq i \leq n$.

Proposition 4.1.14. Any semistable vector bundle on C admits a Jordan-Hölder filtration. Moreover, the associated graded object

$$\mathrm{gr}(\mathcal{E}) := \bigoplus_{i=1}^{n} \mathcal{E}_i/\mathcal{E}_{i-1}$$

is independent of the choice of Jordan-Hölder filtration up to isomorphism.

Proof. To be continued... □

Definition 4.1.15 (S-equivalence)**.** Two semistable vector bundles $\mathcal{E}$ and $\mathcal{F}$ of the same rank and degree on C are called *S-equivalent* if their associated graded objects $\mathrm{gr}(\mathcal{E})$ and $\mathrm{gr}(\mathcal{F})$ (from their Jordan-Hölder filtrations) are isomorphic.

Definition 4.1.16. We define a functor

$$\begin{aligned} \mathcal{M}^{ss}_{r,d} : (\mathbf{Sch}_{\Bbbk})^{\mathrm{op}} &\to \mathbf{Set} \\ T &\mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a family of semistable vector bundles on } X \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\}/\sim, \end{aligned}$$

where $\mathcal{E} \sim \mathcal{E}'$ if for any $t \in T$, the vector bundles $\mathcal{E}_t$ and $\mathcal{E}'_t$ are S-equivalent or .... To be continued...

Chapter 5

Birational Geometry

5.1 Kodaira Vanishing Theorem

5.1.1 Preliminary

Theorem 5.1.1 (Serre Duality)**.** Let X be a Cohen-Macaulay projective variety of dimension n over $\mathbf{k}$ and D a divisor on X. Then there is an isomorphism

$$H^i(X,D) \cong H^{n-i}(X,K_X-D)^\vee, \quad \forall i=0,1,\dots,n.$$

Theorem 5.1.2 (Log Resolution of Singularities)**.** Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there is a smooth variety (or analytic space) Y and a projective morphism $f:Y\to X$ such that

(a) f is an isomorphism over $X-(\mathrm{Sing}(X)\cup\mathrm{Supp}\,Z)$,

(b) $f^*I\subset\mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(c) $\mathrm{Exc}(f)\cup D$ is an snc divisor.

Theorem 5.1.3 (Lefschetz Hyperplane Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map

$$H^k(X,\mathbb{C})\to H^k(Y,\mathbb{C})$$

is an isomorphism for $k<n-1$ and an injection for $k=n-1$.

Theorem 5.1.4 (Hodge Decomposition)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Then for any k, there is a functorial decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^p(X,\Omega_X^q).$$

Combine Theorem 5.1.3 and Theorem 5.1.4, we have the following lemma.

Lemma 5.1.5. Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map $r_k : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \to H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \to H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n - 1$ and an injection for $p = n - 1$.

Theorem 5.1.6 (Leray spectral sequence)**.** Let $f : Y \to X$ be a morphism of varieties and $\mathcal{F}$ a coherent sheaf on Y. Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{F}) \implies H^{p+q}(Y, \mathcal{F}).$$

5.1.2 Kodaira Vanishing Theorem

Lemma 5.1.7. Let X be a smooth projective variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on X. Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \to X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D. It induces a homomorphism $\mathcal{L}^{-m} \to \mathcal{O}_X$. Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} \right) \bigg/ (\mathcal{L}^{-m} \to \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then $\mathcal{A}$ is a finite $\mathcal{O}_X$-algebra. Let $Y \coloneqq \mathrm{Spec}_X \mathcal{A}$. Then Y is a finite $\mathcal{O}_X$-scheme and the natural morphism $f : Y \to X$ is finite and surjective.

For every $x \in X$, let $\mathcal{L}$ locally generated by t near x. Then $\mathcal{O}_Y$ locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y. Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. □

Remark 5.1.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D_i' = f^*(D_i)$. Then $D' + \sum_{i=1}^k D_i'$ is snc on Y by considering the local regular system of parameters.

Lemma 5.1.9. Let $f : Y \to X$ be a finite surjective morphism of projective varieties and $\mathcal{L}$ a line bundle on X. Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits by the trace map $(1/n)\mathrm{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. □

Theorem 5.1.10 (Kodaira Vanishing Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and A an ample divisor on X. Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 5.1.7 and 5.1.9, after taking a multiple of A, we can assume that A is effective. Then we have an exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \to H^{i-1}(X, \mathcal{O}_A) \to H^i(X, \mathcal{O}_X(-A)) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 5.1.5 and Serre duality (Theorem 5.1.1). □

5.1.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

Theorem 5.1.11 (Kawamata-Viehweg Vanishing Theorem I)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{R}$-divisor on X. Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Theorem 5.1.12 (Kawamata-Viehweg Vanishing Theorem II)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{Q}$-divisor on X. Suppose that $\lceil D \rceil - D$ has snc support. Then

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall i > 0.$$

Theorem 5.1.13 (Kawamata-Viehweg Vanishing Theorem III)**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Let D be a nef $\mathbb{Q}$-divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

If we replace the assumption ”nef and big” of D by ”ample” in II and III, we denote them as “II(ample)” and “III(ample)” respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \implies \text{II(ample)} \implies \text{III(ample)} \implies \text{I} \implies \text{II} \implies \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

Proof of II (Theorem 5.1.12). Set $M := \lceil D \rceil$. Let

$$B := \sum_{i=1}^{k} b_i B_i := \lceil D \rceil - D = M - A, \quad b_i \in (0,1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k. When $k = 0$, the conclusion follows from Theorem 5.1.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 5.1.10.)) Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 5.1.15 and 5.1.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 5.1.7 on B_k, we can find a finite surjective morphism $f : X' \to X$ such that $f^*B_k = cB'_k, B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^{k} B'_i$ is an snc divisor on X'. Let $B' = \sum_{i=1}^{k-1} B'_i, A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$ vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^*\mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 5.1.9. □

Proof of III (Theorem 5.1.13). Let $f : \tilde{X} \to X$ be a resolution such that $\operatorname{Supp} f^*B \cup \operatorname{Exc} f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X},\tilde{B})} + f^*D,$$

where $\tilde{B} \in (0,1)$ has snc support and E is an effective exceptional divisor.

By Lemma 5.1.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X},\tilde{B})} + f^*D) = H^i(X, f_*\mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 5.1.12 in either case relative to the assumption of D. □

Proof of I (Theorem 5.1.11). By Lemma 5.1.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor A. Then the theorem comes down to Theorem 5.1.13. □

Lemma 5.1.14. Let $f : Y \to X$ be a birational morphism of projective varieties with Y smooth and X has only rational singularities. Let E be an effective exceptional divisor on Y and D a divisor on X. Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

Proof. I am unable to proof this lemma. □

Lemma 5.1.15. Let X be a projective variety, $\mathcal{L}$ a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \to X$ and a line bundle $\mathcal{L}'$ on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X, then we can take f such that f^*D is an snc divisor on Y.

Proof. We can assume that $\mathcal{L}$ is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & \mathbb{P}^N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\varphi_{\mathcal{L}}} & \mathbb{P}^N \end{array},$$

where $g : [x_0 : \ldots : x_N] \mapsto [x_0^m : \ldots : x_N^m]$. The morphism f is finite and surjective since so is g.

Let $\mathcal{L}' := \psi^*\mathcal{L}$.

For smoothness, we can compose g with a general automorphism of $\mathbb{P}^N$. Then the conclusion follows from [Har77, Chapter III, Theorem 10.8]. □

Lemma 5.1.16 (ref. [KM98, Theorem 5.10, 5.22])**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Lemma 5.1.17. Let X be a projective variety of dimension n and D a nef and big divisor on X. Then there exists an effective divisor B such that for every k, there is an ample divisor A_k such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k}B.$$

Proof. By definition of big divisor, there exists an ample divisor A_1 and effective divisor B such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k}B.$$

Since A is ample and D is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. □

5.2 Cone Theorem

5.2.1 Preliminary

Theorem 5.2.1 (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27])**.** Let X be a projective variety and $\mathcal{L}$ an semiample line bundle on X. Then there exists a fibration $\varphi : X \to Y$ of projective varieties such that for any $m \gg 0$ with $\mathcal{L}^m$ base point free, we have that the morphism $\varphi_{\mathcal{L}^m}$ induced by $\mathcal{L}^m$ is isomorphic to φ. Such a fibration is called the *Iitaka fibration* associated to $\mathcal{L}$.

Theorem 5.2.2 (Rigidity Lemma, ref. [Deb01, Lemma 1.15])**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Theorem 5.2.3. Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

Proposition 5.2.4. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

Schetch of proof. Take a resolution $f : Y \to X$, then f^*H is nef on Y and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor H_Y on Y closed enough to f^*H such that $K_Y \cdot H_Y^{n-1} < 0$. By [MM86, Theorem 5] and take limit for H_Y. □

Lemma 5.2.5 (ref. [Kaw91, Lemma]). Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Let E be an irreducible component of dimension d of the exceptional locus of f and $\nu : E^\nu \to X$ the normalization of E. Suppose that $f(E)$ is a point. Then for any ample divisor H on X, we have

$$K_{E^\nu} \cdot \nu^* H^{d-1} \le K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

5.2.2 Non-vanishing Theorem

Theorem 5.2.6 (Non-vanishing Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

Proof. To be completed. □

5.2.3 Base Point Free Theorem

Theorem 5.2.7 (Base Point Free Theorem). Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, mD is base point free.

Proof. To be completed. □

Remark 5.2.8. In general, we say that a Cartier divisor D is *semiample* if there exists a positive integer m such that mD is base point free. The statement in Base Point Free Theorem (Theorem 5.2.7) is strictly stronger than the semiample condition. For example, let $\mathcal{L}$ be a torsion line bundle, then $\mathcal{L}$ is semiample, but there exists no positive integer M such that $m\mathcal{L}$ is base point free for all $m > M$.

5.2.4 Rationality Theorem

Lemma 5.2.9 (ref. [KM98, Theorem 1.36]). Let X be a proper variety of dimension n and $D_1, \dots, D_m$ Cartier divisors on X. Then the Euler characteristic $\chi(n_1 D_1, \dots, n_m D_m)$ is a polynomial in $(n_1, \cdots, n_m)$ of degree at most n.

Theorem 5.2.10 (Rationality Theorem). Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let

$$t := \inf\{s \ge 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H. Then $t = v/u \in \mathbb{Q}$ and

$$0 \le v \le a(X) \cdot (\dim X + 1).$$

Proof. For every $r \in \mathbb{R}_{>0}$, let

$$v(r) := \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We need to show that $v(t) \leq a(\dim X + 1)$. For every $(p,q) \in \mathbb{Z}^2_{>0}$, set $D(p,q) := paK_{(X,B)} + qH$. If $(p,q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, then we have $D(p,q)$ is not nef and $D(p,q) - K_{(X,B)}$ is ample.

Step 1. We show that a polynomial $P(x,y) \neq 0 \in \mathbb{Q}[x,y]$ of degree at most n is not identically zero on the set

$$\{(p,q) \in \mathbb{Z}^2 : p,q > M, 0 < atp - q < t\varepsilon\}, \quad \forall M > 0,$$

if $v(t)\varepsilon > a(n+1)$.

If $v(t) = \infty$, for any n, we show that we can find infinitely many lines L such that $\#L \cap \Lambda \geq n+1$. If so, Λ is Zariski dense in $\mathbb{Q}^2$. Since $1/at \in \mathbb{R} \setminus \mathbb{Q}$, there exist $p_0, q_0 > M$ such that

$$0 < \frac{p_0}{q_0} - \frac{1}{at} < \frac{\varepsilon}{(n+1)a} \cdot \frac{1}{q_0}, \text{ i.e. } 0 < atp_0 - q_0 < \frac{\varepsilon t}{n+1}.$$

Then $(ip_0, iq_0) \in \Lambda \cap \{p_0 y = q_0 x\}$ for $i = 1, \cdots, n+1$. Since M is arbitrary, there are infinitely many such lines L.

Suppose $v(t) = v < \infty$ and $t = v/u$. Then the inequality is equivalent to $0 < aup - vq < \varepsilon v$. Note that $\gcd(au, v) | a$, then $aup - vq = ai$ has integer solutions for $i = 1, \cdots, n+1$. Since $v(t)\varepsilon > a(n+1)$, there are at least $n+1$ lines which intersect Λ in infinitely many points. This enforce any polynomial which vanishes on Λ has degree at least $n+1$.

Step 2. There exists an index set $\Lambda \subset \mathbb{Z}^2$ such that Λ contains all sufficiently large (p,q) with $0 \leq atp - q \leq t$ and

$$Z := \operatorname{Bs}|D(p,q)| = \operatorname{Bs}|D(p',q')| \neq \varnothing, \quad \forall (p,q), (p',q') \in \Lambda.$$

For every $(p,q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, there exists $M > 0$ such that

$$D(\alpha, \beta) = \alpha a K_{(X,B)} + \beta H$$

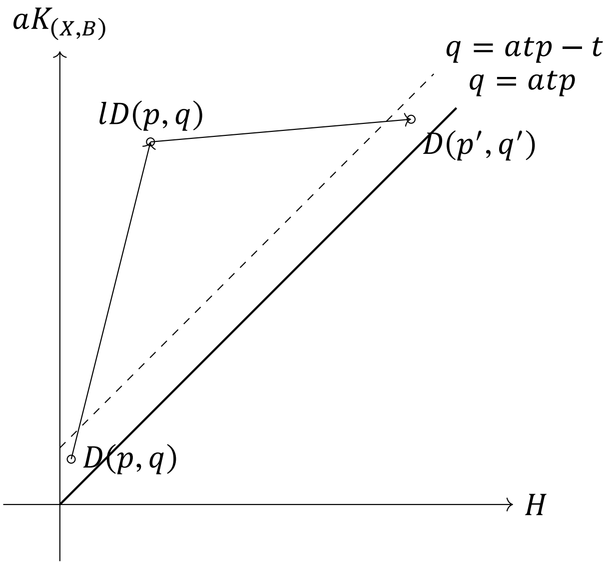
is base point free for all $\alpha = 0, \cdots, p$ and $\beta > M$. Choose M' large enough such that for all $(p',q') \in \mathbb{Z}^2_{>0}$ with $p', q' > M'$ and $0 < atp' - q' < t$, write

$$p' = lp + p_0, \quad q' = lq + q_0$$

for some $l \in \mathbb{Z}_{\geq 0}$ and $0 \leq p_0 < p$, we have $q_0 > M$. The existence of such M' follows from the estimate

$$q_0 = q' - lq = q' - \frac{p' - p_0}{p} q > q' - (p' - p_0)(at - \delta) > p'\delta,$$

where $\delta > 0$ is a small enough number such that $at - \delta > q/p$.



Then $D(p', q') - lD(p, q) = D(p_0, q_0)$ is base point free. It follows that $\operatorname{Bs}|D(p',q')| \subseteq \operatorname{Bs}|D(p,q)|$. By noetherian induction, there exists an index set Λ such that $\operatorname{Bs}|D(p,q)| = Z$ for all $(p,q) \in \Lambda$.

Step 3. Suppose the contradiction that $v(t) > a(\dim X+1)$. Then we show that $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem (Theorem 5.2.7).

Let $P(x,y) := \chi(D(x,y))$ be the Hilbert polynomial of $D(x,y)$. Note that $P(0,n) = \chi(nH) \neq 0$ since H is ample. Then $P(x,y) \neq 0$ and $\deg P \leq \dim X$. By Step 1, P is not identically zero on Λ. Note that $D(p,q) - K_{(X,B)}$ is ample for all $(p,q) \in \Lambda$, then $h^i(X, D(p,q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem (Theorem 5.1.13). Then

$$P(p,q) = \chi(D(p,q)) = h^0(X, D(p,q)) \neq 0$$

for some $(p,q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$.

Step 4. We follow the same line of the proof of Base Point Free Theorem (Theorem 5.2.7) to show that there is a section which does not vanish on Z.

Fix $(p,q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n+1)/u$. Let $f : Y \to X$ be a resolution such that

(a) $K_{Y,B_Y} = f^*K_{(X,B)} + E_Y$ for some effective exceptional divisor E_Y, and Y, B_Y is a klt pair;

(b) $f^*|D(p,q)| = |L| + F$ for some effective divisor F and a base point free divisor L, and $f(\operatorname{Supp} F) = Z$;

(c) $f^*D(p,q) - f^*K_{(X,B)} - E_0$ is ample for some effective $\mathbb{Q}$-divisor $E_0 \in (0,1)$, and coefficients of E_0 are sufficiently small;

(d) $B_Y + E_Y + F + E_0$ has snc support.

Such resolution exists by [KM98].

Let $c := \inf\{\lfloor B_Y + E_0 + tF \rfloor \neq 0\}$. Adjust the coefficients of E_0 slightly such that $\lfloor B_Y + E_0 + cF \rfloor = F_0$ for unique prime divisor F_0 with $F_0 \subset \operatorname{Supp} F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then (Y, Δ_Y) is a klt pair.

Let

$$\begin{aligned} N(p',q') &:= f^*D(p',q') + E_Y - F_0 - K_{(Y,\Delta_Y)} \\ &= \Big(f^*D(p',q') - (1+c)f^*D(p,q)\Big) + \Big(f^*D(p,q) - f^*K_{(X,B)} - E_0\Big) + c\Big(f^*D(p,q) - F\Big). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p',q') \in \Lambda : 0 < atp' - q' < atp - q,\ p',q' > (1+c)\max\{p,q\}\},$$

the divisor $f^*D(p',q') - (1+c)f^*D(p,q) = f^*D(p'-(1+c)p, q'-(1+c)q)$ is ample, and hence $N(p',q')$ is ample.

By the exact sequence

$$0 \to \mathcal{O}_Y(f^*D(p',q') + E_Y - F_0) \to \mathcal{O}_Y(f^*D(p',q') + E_Y) \to \mathcal{O}_{F_0}((f^*D(p',q') + E_Y)|_{F_0}) \to 0$$

and Kawamata-Viehweg Vanishing Theorem (Theorem 5.1.13), we get a surjective map

$$H^0(Y, f^*D(p',q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p',q') + E_Y)|_{F_0}).$$

On F_0, consider the polynomial $\chi((f^*D(p',q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n-1$ and by the construction of $(p,q), \Lambda_0$, similar to Step 3, we can show that $\chi((f^*D(p',q') + E_Y)|_{F_0})$ is not identically zero on Λ_0. By adjunction, we have $(f^*D(p',q') + E_Y)|_{F_0} = N(p',q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p',q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (Theorem 5.1.13) to get

$$h^0(F_0, (f^*D(p',q') + E_Y)|_{F_0}) = \chi(F_0, (D(p',q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \operatorname{Bs}|D(p',q')|$.

□

5.2.5 Cone Theorem and Contraction Theorem

Theorem 5.2.11 (Cone Theorem)**.** Let (X,B) be a projective klt pair. Then there exist countably many curves $C_i \subset X$ such that

(a) we have a decomposition of cones

$$\operatorname{Psef}_1(X) = \operatorname{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\operatorname{Psef}_1(X) = \operatorname{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. Let $F_D := \operatorname{Psef}_1(X) \cap D^\perp$ for a nef divisor D on X. If $\dim F_D = 1$, we also write $R_D := F_D$. Let $H_1, \cdots, H_{\rho-1}$ be ample divisors on X such that they together with $K_{(X,B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} := S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

Step 1. There exists an integer N such that for every $K_{(X,B)}$-negative extremal face F_D and for every ample divisor H, there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \neq F_{nD+rK_{(X,B)}+NH} \subset F_D$.

Let $N := (a(X)(\dim X + 1))!$, where $a(X)$ is the number in Theorem 5.2.10. For every n, $nD + H$ is an ample divisor and by Theorem 5.2.10, the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \geq 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since $K_{(X,B)} + (N/r_n)((n + 1)D + H)$ is nef, we have $r_n \leq r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$ implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \to r \in \mathbb{Z}_{\geq 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors N_1, N_2, we have $F_{N_1+N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists m large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

Step 2. Let $\Phi : N_1(X)_{K_{(X,B)}<0} \to \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left(\frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha}\right).$$

We show that the image of R_D under Φ lies in a $\mathbb{Z}$-lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class ξ. By Step 1, we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers n, r. Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of R_D under Φ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

Step 3. We show that every $K_{(X,B)}$-negative extremal ray of $\mathsf{Psef}_1(X)$ is of the form R_D for some nef divisor D on X.

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$-negative exposed ray. Then R is of form $D^\perp \cap \mathsf{Psef}_1(X)$ for some nef $\mathbb{R}$-divisor D on X. We need to show that D can be choose as a nef $\mathbb{Q}$-divisor. There is a sequence of nef but not ample $\mathbb{Q}$-divisors D_m such that $D_m \to D$ as $m \to \infty$. We adjust D_m such that $\dim F_{D_m} = 1$ for all n.

By re-choosing H_i, we can assume that $D = a_1H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is D_m. Then F_{D_m} is $K_{(X,B)}$-negative. Note that $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by Step 1. If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{D_m}$. We can assume that$r_1K_{(X,B)} + NH_1$ is not identically zero on F_{D_m}. Then we can choose n large enough such that $\|r_1K_{(X,B)} + NH_1\|/n < 1/m$. Replace D_m by $D_m + (r_1K_{(X,B)} + NH_1)/n$. Inductively we construct D_m nef $\mathbb{Q}$-divisor with $D_m \to D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that ξ_m converges. Then $\xi_m \to \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, Φ is well-defined at

ξ and the image of ξ_m under Φ is discrete. Hence $\xi = \xi_m$ for all m large enough. It follows that $R = R_{D_m}$ for a nef $\mathbb{Q}$-divisor D_m.

By Step 2, the $K_{(X,B)}$-negative extremal rays form a discrete set in $\{\alpha \in \mathrm{Psef}_1(X) : K_{(X,B)}\cdot\alpha < 0\}$. Hence every $K_{(X,B)}$-negative extremal ray is an exposed ray by Straszewicz's Theorem.

Step 4. Proof of the theorem.

Given an ample divisor H on X, note that εH has positive minimum δ on $\mathrm{Psef}_1(X)\cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \mathrm{Psef}_1(X)\cap S^{\rho-1} : K_{(X,B)}\cdot\alpha \leq -\varepsilon H\cdot\alpha\} \subset \{\alpha : K_{(X,B)}\cdot\alpha \leq -\delta\}$$

is compact, and Φ is well-defined on it. By Steps 2 and 3, there are only finitely many extremal rays on $\mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H\leq 0}$. Hence we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$\mathcal{C} := \mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\}\subset\mathcal{C}$ such that $\alpha_n \to \alpha \in N_1(X)_{\mathbb{R}}$. Note that $\mathrm{Psef}_1(X)$ is closed, hence $\alpha \in \mathrm{Psef}_1(X)$. We only need to consider the case $\alpha\cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha\cdot(K_{(X,B)}+\varepsilon H) < 0$. Then $\alpha_n\cdot(K_{(X,B)}+\varepsilon H) < 0$ for all n large enough. Note that $\mathcal{C}\cap\{K_{(X,B)}+\varepsilon H \leq 0\}$ is a polyhedral cone by Step 2 and hence is closed. Then $\alpha\in\mathcal{C}$ and the conclusion follows. □

Remark 5.2.12. Thanks for my friend Qin for pointing out that the extremal ray may not be exposed.

Theorem 5.2.13 (Contraction Theorem)**.** Let (X,B) be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C\subset X$ is contracted by φ_F if and only if $[C]\in F$;

(b) up to linearly equivalence, any Cartier divisor G with $F\subset G^{\perp} = \{\alpha\in N_1(X) : \alpha\cdot G = 0\}$ comes from a Cartier divisor on Y, i.e., there exists a Cartier divisor G_Y on Y such that $G\sim\varphi_F^*G_Y$.

Proof. We follow the following steps to prove the theorem.

Step 1. We show that there exists a nef divisor D on X such that $F = D^{\perp}\cap\mathrm{Psef}_1(X)$. In other words, F is defined on $N_1(X)_{\mathbb{Q}}$.

We can choose an ample divisor H and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on F since $F\cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in $N_1(X)_{\mathbb{R}}$. Then by Cone Theorem (Theorem 5.2.11), F is an extremal face of a rational polyhedral cone, namely $\mathrm{Psef}_1(X)_{K_{(X,B)}+(1/n)H\leq 0}$. It follows that $F^{\perp}\subset N^1(X)_{\mathbb{R}}$ is defined on $\mathbb{Q}$. Since F is extremal and $K_{(X,B)}+(1/n)H$-negative, the set $\{L\in F^{\perp} : L|_{\mathrm{Psef}_1(X)\setminus F} > 0\}$ has non-empty interior in $F^{\perp}$ by Theorems 5.2.3 and 5.2.11. Then there exists a Cartier divisor D such that $D\in F^{\perp}$ and $D|_{\mathrm{Psef}_1(X)\setminus F} > 0$. It follows that D is nef and $F = D^{\perp}\cap\mathrm{Psef}_1(X)$.

Step 2. Let $\varphi : X \to Y$ be the Iitaka fibration associated to D by Theorem 5.2.1. We show that φ is the desired fibration.

Note that $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$ is compact and D is strictly positive on it. Then there exist $a \geq 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since F is $K_{(X,B)}$-negative. Then by Base Point Free Theorem (Theorem 5.2.7), we know that mD is base point free for all $m \gg 0$. Hence we can apply Theorem 5.2.1 to get a fibration $\varphi_D : X \to Y$.

First we show that D comes from Y. Note that mD and $(m+1)D$ induces the same fibration φ_D for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m)D_{Y,m}$ and $D_{Y,m}$ is ample. Hence D_Y is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that C is contracted by φ_D if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let G be arbitrary Cartier divisor on X such that $F \subset G^\perp$. Since D is strictly positive on $\mathrm{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' := mD + G$, we have $D'^\perp \cap \mathrm{Psef}_1(X) = F$. Then by the same argument as above, we get an other fibration $\varphi_{D'} : X \to Y'$ such that a curve C is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 5.2.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on Y. In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor D'_Y on Y. Then $G = D' - mD$ also comes from Y. □

Remark 5.2.14. The Step 1 is amazing. If F is not $K_{(X,B)}$-negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E. By [Laz04, Lemma 1.5.4], we know that $\mathrm{Psef}_1(X)$ is a circular cone. The we see there indeed exist some irrational extremal faces of $\mathrm{Psef}_1(X)$.

Theorem 5.2.15 (Length of extremal rays)**.** Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Then there exists a rational curve $C \subset X$ such that $[C] \in R$ and

$$0 < -K_{(X,B)} \cdot C \leq 2 \dim X.$$

Proof. By Theorem 5.2.13, let $\varphi_D : X \to Y$ be the contraction associated to R_D (note that we do not need the step to proof Theorem 5.2.13). If $\dim Y < \dim X$, let F be a general fiber of φ_D. By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that H is ample on F. By Proposition 5.2.4. In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By Lemma 5.2.5, we can use Proposition 5.2.4 to get the result. To be completed. □

Definition 5.2.16. Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ with contraction $\varphi_R : X \to Y$. There are three types of contractions:

(a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension one;

(b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension at least two;

(c) *Mori fiber space*: if $\dim X > \dim Y$.

Proposition 5.2.17. Let (X,B) be a $\mathbb{Q}$-factorial projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Suppose that the contraction $\varphi : X \to Y$ associated to R is either divisorial or a Mori fiber space. Then Y is $\mathbb{Q}$-factorial.

Proof. Let D be a prime Weil divisor on Y and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\geq 0}[C]$ for an irreducible curve C contracted by φ. Set $D_X := \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$. Then D_X is a prime Weil divisor on X and hence is $\mathbb{Q}$-Cartier.

If φ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber F of φ. Then by Contraction Theorem (Theorem 5.2.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor D' on Y. We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence D is $\mathbb{Q}$-Cartier.

If φ is a divisorial contraction, let E be the exceptional divisor of φ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier $\mathbb{Q}$-divisor E_Y on Y). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 5.2.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor D' on Y. Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence D is $\mathbb{Q}$-Cartier. □

Remark 5.2.18. If φ is a small contraction, then Y is never $\mathbb{Q}$-factorial. Otherwise, let B_Y be the strict transform of B on Y. Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset U. Suppose $K_{(Y,B_Y)}$ is $\mathbb{Q}$-Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

Example 5.2.19. Let $X = E \times E \times \mathbb{P}^1$. To be completed.

5.3 F-singularities

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let X be a projective variety over $\Bbbk$. Let F denote the relative Frobenius morphism on X.

Definition 5.3.1. We say that X is *F-finite* if $F : X \to X^{(p)}$ is finite.

Definition 5.3.2. We say that X is *globally F-split* if $\mathcal{O}_X \to F_*^e \mathcal{O}_X$ splits as $\mathcal{O}_X$-modules for some $e \geq 0$. This is equivalent to for every $e \in \mathbb{Z}_{>0}$, $\mathcal{O}_X \to F_*^e \mathcal{O}_X$ splits as $\mathcal{O}_X$-modules.

Definition 5.3.3. Fix $\phi : F_*^e L \to \mathcal{O}_X$ a splitting of $\mathcal{O}_X \to F_*^e \mathcal{O}_X$. Define $\phi^n : F_*^{ne} L^{1+p^e+\cdots+p^{(n-1)e}} \to \mathcal{O}_X$ by induction:

$$\phi^n := \phi \circ F_*^e(\phi^{n-1}).$$

Theorem 5.3.4. Above ϕ^n will be stable. That is, $\Im\phi^n = \Im\phi^{n+1}$ for all $n \gg 0$.

Definition 5.3.5. Let $\sigma(X,\phi) := \Im\phi^n$. We say that (X,ϕ) is *F-pure* if $\sigma(X,\phi) = \mathcal{O}_X$.

Proposition 5.3.6. There is a bijection between

$$\{\text{effective } \mathbb{Q}\text{-divisor } \Delta \text{ such that } (p^e-1)(K_X+\Delta) \text{ is Cartier}\}/\sim$$

and

$$\{\text{line bundles } \mathscr{L} \text{ and } \phi : F^e_*\mathscr{L} \to \mathscr{O}_X\}.$$

Proof. We have

$$F^e_X\mathscr{O}_X((1-p^e)K_X) \to \mathscr{O}_X$$

given by $F^e\mathscr{O}_X(K_X) \to \mathscr{O}_X(K_X)$ and reflexivity of $\mathscr{O}_X(K_X)$. Since Δ is effective, we have

$$F^e(\mathscr{O}_X((1-p^e)(K_X+\Delta))) \to F^e\mathscr{O}_X((1-p^e)(K_X)) \to \mathscr{O}_X.$$

The another direction is by Grothendieck's duality

$$\mathscr{H}om_{\mathscr{O}_X}(F^e\mathscr{L}, \mathscr{O}_X) \cong F^e_*(\mathscr{L}^{-1} \otimes \mathscr{O}_X((1-p^e)K_X)).$$

□

Definition 5.3.7. Let $\phi_{e,\Delta} : F^e_*(\mathscr{O}_X((1-p^e)(K_X+\Delta))) \to \mathscr{O}_X$ be the morphism corresponding to the effective $\mathbb{Q}$-divisor Δ.
We say that (X,Δ) is *F-pure* if $(X,\phi_{e,\Delta})$ is F-pure.
We say that (X,Δ) is *globally F-split* if for every Weil divisor $D \geq 0$, $\mathscr{O}_X \to F^e_*(\mathscr{O}_X(\lceil (p^e-1)\Delta \rceil + D))$ admits a splitting for some $e \geq 0$.
We say that (X,Δ) is *strongly F-split* if for every Weil divisor $D \geq 0$, $\mathscr{O}_X \to F^e_*(\mathscr{O}_X(\lceil (p^e-1)\Delta \rceil + D))$ admits a local splitting for some $e \geq 0$.

Definition 5.3.8.

Definition 5.3.9. $S^0(X, \sigma(X,\Delta) \otimes m)$

Proposition 5.3.10. Let X be a globally F-split projective variety. Then we have

(a) suppose that $H^i(X, \mathscr{L}^n) = 0$ for all $i > 0$ and all $n \gg 0$, then $H^i(X, \mathscr{L}) = 0$ for all $i > 0$;

(b) for every ample divisor A on X, we have $H^i(X, \mathscr{O}_X(A)) = 0$ for all $i > 0$;

(c) suppose that X is Cohen-Macaulay and A-ample, then $H^i(X, \mathscr{O}_X(-A)) = 0$ for all $i < \dim X$;

(d) suppose that X is normal and A-ample, then $H^i(X, \omega_X(A)) = 0$ for all $i > 0$.

Chapter 6

Abelian Varieties

6.1 The First Properties of Abelian Varieties

6.1.1 Definition and examples of Abelian Varieties

Definition 6.1.1. Let $\mathbf{k}$ be a field. An *abelian variety over* $\mathbf{k}$ is a proper variety A over $\mathbf{k}$ together with morphisms *identity* $e : \operatorname{Spec}\mathbf{k} \to A$, *multiplication* $m : A \times A \to A$ and *inversion* $i : A \to A$ such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd}
& A\times A\times A \arrow[dl, "\mathrm{id}_A\times m"'] \arrow[dr, "m\times\mathrm{id}_A"] & \\
A\times A \arrow[dr, "m"] & & A\times A \arrow[dl, "m"'] \\
& A &
\end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd}
A\times\operatorname{Spec}\mathbf{k} \arrow[r, "\mathrm{id}_A\times e"] \arrow[dr, "\cong"'] & A\times A \arrow[d, "m"] & \operatorname{Spec}\mathbf{k}\times A \arrow[l, "e\times\mathrm{id}_A"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd}
& A \arrow[dl, "\mathrm{id}_A\times i"'] \arrow[d] \arrow[dr, "i\times\mathrm{id}_A"] & \\
A\times A \arrow[dr, "m"'] & \operatorname{Spec}\mathbf{k} \arrow[d, "e"] & A\times A \arrow[dl, "m"] \\
& A &
\end{tikzcd}.$$

In other words, an abelian variety is a group object in the category of proper varieties over $\mathbf{k}$.

Example 6.1.2. Let E be an elliptic curve over a field $\mathbf{k}$. Then E is an abelian variety of dimension 1. To be completed.

In the following, we will always assume that A is an abelian variety over a field $\mathbf{k}$ of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The *left translation* by $a \in A(\mathbf{k})$ is defined as

$$l_a : A \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times A \xrightarrow{a \times \mathrm{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a.

Proposition 6.1.3. Let A be an abelian variety. Then A is smooth.

Proof. By base changing to the algebraic closure of $\mathbf{k}$, we may assume that $\mathbf{k}$ is algebraically closed. Note that there is a non-empty open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a. $\square$

Proposition 6.1.4. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. But how to show it is a morphism of varieties? To be completed. $\square$

Theorem 6.1.5. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism, i.e., for every $\mathbf{k}$-scheme T, the induced map $f_T : A(T) \to B(T)$ is a group homomorphism.

Proof. Consider the diagram

$$\begin{tikzcd} A \times A \arrow[d, "p_1"'] \arrow[dr, "\varphi"] & \\ A & B \end{tikzcd}$$

with φ be given by

$$\begin{aligned} A \times A \xrightarrow{\Delta\times\Delta} A \times A \times A \times A \xrightarrow{\cong} A \times A \times A \times A \xrightarrow{(f\circ m_A)\times(i_B\circ f)\times(i_B\circ f)} B \times B \times B \xrightarrow{m_B} B, \\ (x,y) \mapsto (x,x,y,y) \mapsto (x,y,y,x) \mapsto (f(xy), f(y)^{-1}, f(x)^{-1}) \mapsto f(xy)f(y)^{-1}f(x)^{-1}. \end{aligned}$$

We have $\varphi(p_1^{-1}(e_A)) = \varphi(\{e_A\} \times A) = \{e_B\}$. Then by Rigidity Lemma (??), there exists a unique rational map $\psi : A \dashrightarrow B$ such that $\varphi = \psi \circ p_1$. Note that $A \to A \times \{e_A\} \to A \times A$ gives a section of p_1. On this section, we have that φ is constant equal to e_B. Thus ψ is well-defined and $\psi(A) = e_B$. It follows that φ factors through the constant map $A \times A \to \{e_B\} \to B$. Then for every $(x, y) \in A(\mathbb{k}) \times A(\mathbb{k})$, we have

$$f(xy) = f(x)f(y).$$

Since $A(\mathbb{k})$ is dense in A, the conclusion follows. $\square$

Proposition 6.1.6. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 6.1.5. $\square$

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 6.1.7. Let A be an abelian variety over $\Bbbk$ and n a positive integer not divisible by $\mathrm{char}\,\Bbbk$. Then the multiplication by n morphism $[n]_A : A \to A$ is finite surjective and étale.

Proof. To be completed. □

6.1.2 Complex abelian varieties

Theorem 6.1.8. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if there exists a positive definite Hermitian form H on $\mathbb{C}^n$ such that $\Im(H)(\Lambda, \Lambda) \subset \mathbb{Z}$. To be completed.

6.2 Picard Groups of Abelian Varieties

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. Let A be an abelian variety over $\mathbf{k}$.

6.2.1 Pullback along group operations

Theorem 6.2.1 (Theorem of the cube)**.** Let X, Y, Z be proper varieties over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\}\times Y\times Z}$, $\mathcal{L}|_{X\times\{y\}\times Z}$ and $\mathcal{L}|_{X\times Y\times\{z\}}$ are trivial. Then $\mathcal{L}$ is trivial.

Proof. To be completed. □

Remark 6.2.2. If we assume the existence of the Picard scheme, then the Theorem 6.2.1 can be deduced from the Rigidity Lemma. Consider the morphism

$$\varphi : X \times Y \to \mathrm{Pic}(Z), \quad (x, y) \mapsto \mathcal{L}|_{\{x\}\times\{y\}\times Z}.$$

Since $\varphi(x, y) = \mathcal{O}_Z$, φ factors through $\mathrm{Pic}^0(Z)$. Then the assumption implies that φ contracts $\{x\}\times Y$, $X \times \{y\}$ and hence it maps $X \times Y$ to a point. Thus $\varphi(x', y') = \mathcal{O}_Z$ for every $(x', y') \in X \times Y$. Then by Grauert's theorem, we have $\mathcal{L} \cong p^* p_* \mathcal{L}$ where $p : X \times Y \times Z \to X \times Y$ is the projection. Note that $p_*\mathcal{L} \cong \mathcal{L}|_{X\times Y\times\{z\}} \cong \mathcal{O}_{X\times Y}$. Hence $\mathcal{L}$ is trivial.

Lemma 6.2.3. Let A be an abelian variety over $\mathbf{k}$, $f, g, h : X \to A$ morphisms from a variety X to A and $\mathcal{L}$ a line bundle on A. Then we have

$$(f + g + h)^*\mathcal{L} \cong (f + g)^*\mathcal{L} \otimes (f + h)^*\mathcal{L} \otimes (g + h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof. First consider $X = A \times A \times A$, $p : X \to A, (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$, $p_{ij} : X \to A, (x_1, x_2, x_3) \mapsto x_i + x_j$ for $1 \leq i < j \leq 3$ and $p_i : X \to A, (x_1, x_2, x_3) \mapsto x_i$ for $1 \leq i \leq 3$. Then the conclusion follows from the theorem of the cube by taking $\mathcal{L}' = p^*\mathcal{L}^{-1} \otimes p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \otimes p_3^*\mathcal{L}^{-1}$ and considering the restriction to $\{0\} \times A \times A$, $A \times \{0\} \times A$ and $A \times A \times \{0\}$.

In general, consider the morphism $\varphi = (f, g, h) : X \to A \times A \times A$ and pull back the above isomorphism along φ. □

Proposition 6.2.4. Let A be an abelian variety over $\mathbf{k}$, $n \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle on A. Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes \frac{1}{2}(n^2-n)}.$$

Proof. For $n = 0, 1$, the conclusion is trivial. For $n \geq 2$, we can use the previous lemma on $[n-2]_A, [1]_A, [1]_A$ and induct on n. Hence we have

$$[n]_A^*\mathcal{L} \cong [n-1]_A^*\mathcal{L} \otimes [n-1]_A^*\mathcal{L} \otimes [2]_A^*\mathcal{L} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [n-2]_A^*\mathcal{L}^{-1}.$$

Then the conclusion follows from induction. To be completed. □

Definition 6.2.5. Let A be an abelian variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on A. We say that $\mathcal{L}$ is *symmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}$ and *antisymmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}^{-1}$.

Theorem 6.2.6 (Theorem of the square)**.** Let A be an abelian variety over $\mathbf{k}$, $x, y \in A(\mathbf{k})$ two points and $\mathcal{L}$ a line bundle on A. Then

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Proof. To be completed. □

Remark 6.2.7. We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \to \mathrm{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary $\mathcal{L}$, the map

$$\Phi_{\square} : \mathrm{Pic}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \mathrm{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is also a group homomorphism. For any $x \in A(\mathbf{k})$, we have

$$\Phi_{t_x^*\mathcal{L}} = \Phi_{\mathcal{L}}$$

by Theorem 6.2.6. In the other words,

$$\Phi_{\mathcal{L}}(x) \in \mathrm{Ker}\,\Phi_{\square}, \quad \forall \mathcal{L} \in \mathrm{Pic}(A), x \in A(\mathbf{k}).$$

If we assume the scheme structure on $\mathrm{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\mathrm{Pic}^0(A)$. Let $K(\mathcal{L}) \coloneqq \mathrm{Ker}\,\Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A. We give another description of $K(\mathcal{L})$. From this point, when $K(\mathcal{L})$ is finite, we can recover the dual abelian variety $A^\vee = \mathrm{Pic}^0_{A/\mathbf{k}}$ as the quotient $A/K(\mathcal{L})$.

6.2.2 Projectivity

In this subsection, we work over the algebraically closed field $\Bbbk$.

Proposition 6.2.8. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. Then $|2D|$ is base point free.

Proof. To be completed. □

Theorem 6.2.9. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. TFAE:

(a) the stabilizer $\mathrm{Stab}(D)$ of D is finite;

(b) the morphism $\phi_{|2D|}$ induced by the complete linear system $|2D|$ is finite;

(c) D is ample;

(d) $K(\mathcal{O}_A(D))$ is finite.

Proof. To be completed. □

Theorem 6.2.10. Let A be an abelian variety over $\Bbbk$. Then A is projective.

Proof. To be completed. □

Corollary 6.2.11. Let A be an abelian variety over $\Bbbk$ and D a divisor on A. Then D is pseudo-effective if and only if it is nef, i.e. $\mathrm{Psef}^1(A) = \mathrm{Nef}^1(A)$.

Proof. To be completed. □

6.2.3 Dual abelian varieties

In this subsection, we work over the algebraically closed field $\Bbbk$.

Definition 6.2.12. Let A be an abelian variety over $\Bbbk$. We define the *dual abelian variety* of A to be $A/K(\mathcal{L})$ for some ample line bundle $\mathcal{L}$ on A. We denote it by $A^\vee$.

We have a natural map $A^\vee(\Bbbk) \to \mathrm{Pic}^0(A)$ by sending $x + K(\mathcal{L}) \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$. We will show that this map is an isomorphism.

Lemma 6.2.13. There exists a unique line bundle $\mathcal{P}$ on $A \times A^\vee$ such that for every $y = \mathcal{L} \in A^\vee = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{y\}} \cong \mathcal{L}$.

Proof. To be completed. □

Lemma 6.2.14. Let A be an abelian variety over $\Bbbk$ and B a group variety over $\Bbbk$. Then there is a natural bijection between the morphisms $f : B \to A^\vee$ and the line bundles $\mathcal{L}$ on $A \times B$ such that for every $b \in B(\Bbbk)$, we have $\mathcal{L}|_{A\times\{b\}} \in \mathrm{Pic}^0(A)$. The bijection is given by $f \mapsto (1_A \times f)^*\mathcal{P}$ where $\mathcal{P}$ is the Poincaré line bundle on $A \times A^\vee$. To be completed.

Proof. To be completed. □

Theorem 6.2.15. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ and the Poincaré line bundle $\mathcal{P}$ on $A \times A^\vee$ do not depend on the choice of the ample line bundle $\mathcal{L}$. Moreover, there is a natural bijection $A^\vee(\mathbf{k}) \to \mathrm{Pic}^0(A)$ of groups. Under this bijection, for every $x = \mathcal{L} \in A^\vee(\mathbf{k}) = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{x\}} \cong \mathcal{L}$.

Proof. To be completed. □

Proposition 6.2.16. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ is also an abelian variety and the natural morphism $A \to A^{\vee\vee}$ is an isomorphism.

Proof. To be completed. □

6.2.4 The Néron-Severi group

Theorem 6.2.17. Let A be an abelian variety over $\mathbb{k}$. The we have an inclusion $\mathrm{NS}(A) \hookrightarrow \mathrm{Hom}_{\mathbf{Grp}}(A, A^\vee)$ given by To be completed.

Chapter 7

Algebraic Groups

7.1 First properties of algebraic groups

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. All varieties are defined over $\mathbf{k}$ unless otherwise specified.

7.1.1 Basic concepts

Definition 7.1.1. A *group scheme* over S is an S-scheme G together with morphisms *multiplication* $\mu : G \times G \to G$, *identity* $\varepsilon : S \to G$ and *inversion* $\iota : G \to G$ over S such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd}
& G \times G \times G \arrow[dl, "\mathrm{id}_G\times\mu"'] \arrow[dr, "\mu\times\mathrm{id}_G"] & \\
G \times G \arrow[dr, "\mu"] & & G \times G \arrow[dl, "\mu"'] \\
& G &
\end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd}
G \times S \arrow[r, "\mathrm{id}_G\times\varepsilon"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] & S \times G \arrow[l, "\varepsilon\times\mathrm{id}_G"'] \arrow[dl, "\cong"] \\
& G &
\end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd}
& G \arrow[dl, "\mathrm{id}_G\times i"'] \arrow[d] \arrow[dr, "i\times\mathrm{id}_G"] & \\
G \times G \arrow[dr, "\mu"'] & S \arrow[d, "\varepsilon"] & G \times G \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}.$$

In other words, a group scheme is a group object in the category of schemes.

Definition 7.1.2. An *algebraic group* is a $\mathbf{k}$-group scheme G which is reduced, separated and of finite type over a field $\mathbf{k}$.

Remark 7.1.3. Even if we work over $\Bbbk$ and just consider the closed points $G(\Bbbk)$ of an algebraic group G, $G(\Bbbk)$ is not a topological group with respect to the Zariski topology in general. The reason is that the topology on $G(\Bbbk) \times G(\Bbbk)$ is not the product topology of the topologies on $G(\Bbbk)$.

Definition 7.1.4. Let G be an algebraic group and $x \in G(\mathbf{k})$ a $\mathbf{k}$-point. The *left translation* by x is the morphism

$$l_x : G \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{\mu} G.$$

Similar definition applies to the right translation r_x.

Remark 7.1.5. In the context of algebraic groups, we often use multiplicative notation for the group law. That is, for $g, h \in G(\mathbf{k})$, we write gh instead of $\mu(g,h)$ and g^{-1} instead of $\iota(g)$.

Sometimes we also abuse the notation by $\mu : G \times \cdots \times G \to G$ to denote the multiplication of multiple elements, i.e. $\mu(g_1, \dots, g_n) = g_1 \cdots g_n$ for $g_1, \dots, g_n \in G(\mathbf{k})$.

Proposition 7.1.6. Let G be an algebraic group. Then G is smooth over $\mathbf{k}$.

Proof. Since G is reduced and of finite type over a field, it is generically regular. Let $g \in G(\Bbbk)$ be a regular point. Then the left translation $l_{gh^{-1}} : G \to G$ is an isomorphism, hence G is regular at $h \in G(\Bbbk)$. It follows that G is regular at every $\Bbbk$-point, hence G is smooth over $\mathbf{k}$. □

Remark 7.1.7. Let G be an algebraic group. Then the irreducible components of G coincide with the connected components of G. We will use the term "connected" to refer to both concepts since "irreducible" has other meanings in the theory of representations.

Example 7.1.8. The *additive group* $\mathbb{G}_a$ is defined to be the affine line $\mathbb{A}^1$ with the group law given by addition. Concretely, we can write $\mathbb{G}_a = \operatorname{Spec} \mathbf{k}[T]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a, &\quad (x,y) \mapsto x + y, \\ \iota : \mathbb{G}_a \to \mathbb{G}_a, &\quad x \mapsto -x, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_a, &\quad * \mapsto 0. \end{aligned}$$

Example 7.1.9. The *multiplicative group* $\mathbb{G}_m$ is defined to be the affine variety $\mathbb{A}^1 \setminus \{0\}$ with the group law given by multiplication. Concretely, we can write $\mathbb{G}_m = \operatorname{Spec} \mathbf{k}[T, T^{-1}]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m, &\quad (x,y) \mapsto xy, \\ \iota : \mathbb{G}_m \to \mathbb{G}_m, &\quad x \mapsto x^{-1}, \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_m, &\quad * \mapsto 1. \end{aligned}$$

Example 7.1.10. The *general linear group* GL_n is defined to be the open subvariety of $\mathbb{A}^{n^2}$ consisting of invertible matrices, with the group law given by matrix multiplication. Concretely, we can write

$\mathrm{GL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}, \det(T_{ij})^{-1}]$ where $1 \le i, j \le n$ and the group law is given by the morphism

$$\begin{aligned} \mu &: \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, \quad (A, B) \mapsto AB, \\ \iota &: \mathrm{GL}_n \to \mathrm{GL}_n, \quad A \mapsto A^{-1}, \\ \varepsilon &: \operatorname{Spec} \mathbf{k} \to \mathrm{GL}_n, \quad * \mapsto I_n. \end{aligned}$$

Example 7.1.11. An abelian variety is an algebraic group that is also a proper variety.

Example 7.1.12. Let G and H be algebraic groups. The *product* $G \times H$ is an algebraic group with the group law defined by

$$\begin{aligned} \mu_{G\times H} = \mu_G \times \mu_H &: (G \times H) \times (G \times H) \cong (G \times G) \times (H \times H) \to G \times H, \\ \varepsilon_{G\times H} = \varepsilon_G \times \varepsilon_H &: \operatorname{Spec} \mathbf{k} \cong \operatorname{Spec} \mathbf{k} \times \operatorname{Spec} \mathbf{k} \to G \times H, \\ \iota_{G\times H} = \iota_G \times \iota_H &: G \times H \to G \times H. \end{aligned}$$

Example 7.1.13. Let G be an algebraic group over $\mathbf{k}$ and $\mathbf{K}/\mathbf{k}$ a field extension. The base change $G_{\mathbf{K}} = G \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} \mathbf{K}$ is an algebraic group over $\mathbf{K}$ with the group law defined by the base change of the original group law of G to $\mathbf{K}$.

Definition 7.1.14. A *homomorphism* of algebraic groups is a morphism of schemes that is also a group homomorphism. Explicitly, a morphism $f : G \to H$ between algebraic groups G and H is a homomorphism if the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

where μ_G and μ_H are the group laws of G and H, respectively.

Definition 7.1.15. An *algebraic subgroup* of an algebraic group G is a closed subscheme $H \subseteq G$ that is also a subgroup of G. More precisely, H is an algebraic subgroup and the inclusion morphism $H \hookrightarrow G$ is compatible with the group laws.
An algebraic subgroup H of G is called *normal* if for any $\mathbf{k}$-scheme S, the subgroup $H(S)$ is a normal subgroup of the abstract group $G(S)$.

Example 7.1.16. The *special linear group* SL_n is defined to be the closed subvariety of GL_n defined by the equation $\det = 1$. It is an algebraic subgroup of GL_n.

Proposition 7.1.17. Let G be an algebraic group and S is a closed subgroup of $G(\Bbbk)$. Then there exists a unique algebraic subgroup H of G such that $H(\Bbbk) = S$.

Proof. To be continued... □

Remark 7.1.18. By Proposition 7.1.17, we often identify an algebraic group G with its set of closed points $G(\Bbbk)$ when there is no confusion.

Remark 7.1.19. If one replaces $\Bbbk$ by $\mathbf{k}$ in Proposition 7.1.17, the statement may not hold. For example, let $\mathbf{k} = \mathbb{Q}$ and G be the elliptic curve defined by $X^3 + Y^3 = Z^3$ in $\mathbb{P}^2$. It is well-known that $\#G(\mathbb{Q}) = 3$. Let S be the disjoint union of the three $\mathbb{Q}$-points of G endowed with the reduced subscheme structure and the group structure induced from G. Then S is a proper closed subgroup of G and we have $S(\mathbb{Q}) = G(\mathbb{Q})$. This contradicts the uniqueness in Proposition 7.1.17.

Indeed, in this chapter, despite working over an arbitrary field $\mathbf{k}$, we mostly consider the closed points of algebraic groups over $\Bbbk$.

Definition 7.1.20. Let G be an algebraic group. The *neutral component* G^0 is the connected component of G containing the identity element ε.

Proposition 7.1.21. The neutral component G^0 is a closed, normal algebraic subgroup of G.

Proof. To be continued... □

Proposition 7.1.22. Let G be an algebraic group and $H \subseteq G(\Bbbk)$ a subgroup (not necessarily closed). Then the Zariski closure $\overline{H}$ of H in G is an algebraic subgroup of G. If $H \subset G(\Bbbk)$ is constructible, then $H = \overline{H}(\Bbbk)$.

Proof. To be continued... □

Example 7.1.23. Let $G = \mathrm{SL}_2$ over $\Bbbk$, $T = \{\mathrm{diag}(t, t^{-1}) | t \in \Bbbk^\times\}$ and $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Set $S = gTg^{-1}$. Then both T and S are closed algebraic subgroups of $G(\Bbbk)$, but the product TS is not closed in $G(\Bbbk)$. By direct computation, we have

$$S = \left\{ \begin{pmatrix} s & s^{-1} - s \\ 0 & s^{-1} \end{pmatrix} \middle| \, s \in \Bbbk^\times \right\}.$$

Then

$$TS = \left\{ \begin{pmatrix} ts & t(s^{-1} - s) \\ 0 & (ts)^{-1} \end{pmatrix} \middle| \, t, s \in \Bbbk^\times \right\}.$$

We have

$$TS \cap \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in \Bbbk \right\} = \left\{ \begin{pmatrix} 1 & s^{-2} - 1 \\ 0 & 1 \end{pmatrix} \middle| \, s \in \Bbbk^\times \right\}.$$

The right hand side is not closed in $\mathrm{SL}_2(\Bbbk)$ since it does not contain the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Hence TS is not closed in $G(\Bbbk)$.

Proposition 7.1.24. Let G be an algebraic group, X_i varieties over $\mathbf{k}$ and $f_i : X_i \to G$ morphisms for $i = 1, \dots, n$ with images $Y_i = f_i(X_i)$. Suppose that Y_i pass through the identity element of G. Let H be the closed subgroup of G generated by $Y_1, \dots, Y_n$, i.e. the smallest closed subgroup of G containing $Y_1, \dots, Y_n$. Then H is connected and $H = Y_{a_1}^{e_1} \cdots Y_{a_m}^{e_m}$ for some $a_1, \dots, a_m \in \{1, \dots, n\}$ and $e_1, \dots, e_m \in \{\pm 1\}$.

Proof. To be continued... □

Remark 7.1.25. We can take $m \leq 2 \dim G$ in Proposition 7.1.24.

7.1.2 Action and representations

Definition 7.1.26. An *action* of an algebraic group G on a variety X is a morphism

$$\sigma : G \times X \to X$$

such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu\times\mathrm{id}_X"] \arrow[d, "\mathrm{id}_G\times\sigma"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}\mathbf{k} \times X \arrow[r, "\varepsilon\times\mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

where μ is the group law of G and ε is the identity element of G. In other words, for any $\mathbf{k}$-scheme S, the induced map $G(S)\times X(S) \to X(S)$ defines a group action of the abstract group $G(S)$ on the set $X(S)$.

For simplicity, we often write $g.x$ instead of $\sigma(g,x)$ for $g \in G(\mathbf{k})$ and $x \in X(\mathbf{k})$.

Example 7.1.27. There are three natural actions of an algebraic group G on itself:

(a) Left translation: $g.h = l_g(h) = gh$;

(b) Right translation: $g.h = r_g(h) = hg^{-1}$;

(c) Conjugation: $g.h = \mathrm{Ad}_g(h) = ghg^{-1}$.

All of them are morphisms of varieties since they are defined by the group law and inversion of G.

Example 7.1.28. The general linear group GL_n acts on the affine space $\mathbb{A}^n$ by matrix multiplication. It is given by polynomials, hence is a morphism of varieties.

Example 7.1.29. The general linear group GL_{n+1} acts on the projective space $\mathbb{P}^n$ by

$$A \cdot [x_0 : \cdots : x_n] = [y_0 : \cdots : y_n], \quad \text{where } (y_0, \ldots, y_n)^T = A(x_0, \ldots, x_n)^T.$$

Let U_i be the standard affine open subset of $\mathbb{P}^n$ defined by $x_i \neq 0$. The map is given by polynomials on the principal open subset of $\mathrm{GL}_{n+1} \times U_i$ defined by $y_j \neq 0$ for any j. Hence it is a morphism of varieties.

Definition 7.1.30. A *linear representation* of an algebraic group G on a finite-dimensional vector space V over $\Bbbk$ is an abstract group representation $\rho : G(\Bbbk) \to \mathrm{GL}(V)$ such that if we identify V with $\mathbb{A}^n$ for some n, then the map $G(\Bbbk) \times \mathbb{A}^n(\Bbbk) \to \mathbb{A}^n(\Bbbk)$ is is a morphism of varieties.

Definition 7.1.31. Let G be an algebraic group acting on a variety X. For any $x \in X(\mathbf{k})$, the *orbit* of x is the locally closed subvariety $G \cdot x = \sigma(G \times \{x\})$ of X.

Proposition 7.1.32. Let G be an algebraic group acting on a variety X. Then for any $x \in X(\mathbf{k})$, the orbit $G \cdot x$ is a locally closed subvariety of X, and $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of strictly smaller dimension.

Proof. To be continued... ☐

Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For $x \in G(\mathbf{k})$, we have the left translation of functions $\tau_x : A \to A$ defined by $\tau_x(f)(y) = f(x^{-1}y)$ for $y \in X(\mathbf{k})$.

Lemma 7.1.33. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For any finite-dimensional subspace $V \subseteq A$, there exists a finite-dimensional G-invariant subspace $W \subseteq A$ containing V.

Proof. To be continued... □

Theorem 7.1.34. Any affine algebraic group is isomorphic to a closed algebraic subgroup of some GL_n.

Proof. To be continued... □

7.1.3 Lie algebra of an algebraic group

Let G be an algebraic group. The *Lie algebra* of G is defined to be the tangent space of G at the identity element ε:

$$\operatorname{Lie}(G) = T_\varepsilon G.$$

It is a finite-dimensional vector space over $\mathbf{k}$.

Proposition 7.1.35. The group law $\mu : G \times G \to G$ induces the plus map on $\operatorname{Lie}(G)$:

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)} : T_{(\varepsilon,\varepsilon)}(G \times G) \cong T_\varepsilon G \oplus T_\varepsilon G \to T_\varepsilon G, \quad (v, w) \mapsto v + w.$$

Proof. We have

$$\mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v, w) = \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(v, 0) + \mathrm{d}\mu_{(\varepsilon,\varepsilon)}(0, w) = (\mathrm{d}\mu \circ (\mathrm{id}_G \times \varepsilon))_\varepsilon(v) + (\mathrm{d}\mu \circ (\varepsilon \times \mathrm{id}_G))_\varepsilon(w) = v + w.$$

□

7.2 Quotient by algebraic group

Everything in this section is over an arbitrary field $\mathbf{k}$ unless otherwise specified.

7.2.1 Quotient

Definition 7.2.1. Let G be an algebraic group acting on a variety X. A *quotient* of X by G is a variety Y together with a morphism $\pi : X \to Y$ such that

(a) π is G-invariant, i.e., $\pi(g \cdot x) = \pi(x)$ for all $g \in G$ and $x \in X$.

(b) For any variety Z and any G-invariant morphism $f : X \to Z$, there exists a unique morphism $\overline{f} : Y \to Z$ such that $f = \overline{f} \circ \pi$.

In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "\bar{f}"] \\ & Z \end{tikzcd}$$

If a quotient exists, it is unique up to a unique isomorphism. To be continued...

Such a quotient does not always exist.

Theorem 7.2.2. Let G be an affine algebraic group acting on a variety X. Then there exists a variety Y and a rational morphism $\pi : X \dashrightarrow Y$ with commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "\bar{f}"] \\ & Z \end{tikzcd}$$

satisfying the following universal property: If a quotient exists, it is unique up to a unique isomorphism.
Furthermore, if all orbits of G in X are closed, then π is a morphism (i.e., defined everywhere). To be continued... Ref?

7.2.2 Quotient of affine algebraic group by closed subgroup

Lemma 7.2.3. Let V be a finite-dimensional vector space over $\mathbf{k}$ and G an abstract group acting linearly on V. Let $W \subseteq V$ be a subspace of dimension m. Then $G.W = W$ if and only if $G. \wedge^m W = \wedge^m W$.

Proof. To be filled. □

Lemma 7.2.4. Let G be an affine algebraic group and H a closed subgroup. Then there exists a finite-dimensional linear representation V of G and a one-dimensional subspace $L \subseteq V$ such that H is the stabilizer of L.

Proof. To be filled. □

Theorem 7.2.5. Let G be an affine algebraic group and H a closed subgroup. Then the quotient G/H exists as a quasi-projective variety.

Proof. To be filled. □

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