
Picard groups of abelian varieties



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1 Pullback along group operations

Theorem 1 (Seesaw Theorem). Let A be an abelian variety over \mathbf{k} .

Theorem 2 (Theorem of the cube). Let X, Y, Z be completed varieties over \mathbf{k} and \mathcal{L} a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\} \times Y \times Z}$, $\mathcal{L}|_{X \times \{y\} \times Z}$ and $\mathcal{L}|_{X \times Y \times \{z\}}$ are trivial. Then \mathcal{L} is trivial.

Proof. Yang: To be completed. □

Remark 3. If we assume the existence of the Picard scheme, then the theorem of the cube can be deduced from the Rigidity Lemma. Yang: To be completed.

Proposition 4. Let A be an abelian variety over \mathbf{k} , $f, g, h : X \rightarrow A$ morphisms from a variety X to A and \mathcal{L} a line bundle on A . Then

$$(f + g + h)^* \mathcal{L} \cong (f + g)^* \mathcal{L} \otimes (f + h)^* \mathcal{L} \otimes (g + h)^* \mathcal{L} \otimes f^* \mathcal{L}^{-1} \otimes g^* \mathcal{L}^{-1} \otimes h^* \mathcal{L}^{-1}.$$

Proof. Yang: To be completed. □

Proposition 5. Let A be an abelian variety over \mathbf{k} , $n \in \mathbb{Z}$ and \mathcal{L} a line bundle on A . Then we have

$$[n]_A^* \mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}(n^2+n)} \otimes [-1]_A^* \mathcal{L}^{\otimes \frac{1}{2}(n^2-n)}.$$

Proof. Yang: To be completed. □

Theorem 6 (Theorem of the square). Let A be an abelian variety over \mathbf{k} , $x, y \in A(\mathbf{k})$ two points and \mathcal{L} a line bundle on A . Then

$$t_{x+y}^* \mathcal{L} \otimes \mathcal{L} \cong t_x^* \mathcal{L} \otimes t_y^* \mathcal{L}.$$

Remark 7. We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \rightarrow \text{Pic}(A), \quad x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary \mathcal{L} , this gives an action of $A(\mathbf{k})$ on $\text{Pic}(A)$. Yang: To be completed.

2 Positivity

Theorem 8. Let A be an abelian variety over \mathbf{k} . Then A is projective.

Proof. Yang: To be completed. □

3 Isogenies and finite subgroups

Theorem 9. Let A be an abelian variety of dimension d over \mathbf{k} . Then the subgroup $A[n]$ of n torsion points is finite and we have

- (a) if n is coprime to $\text{char}(\mathbf{k})$, then $A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2d}$;
- (b) if $n = p^k$ for $p = \text{char}(\mathbf{k}) > 0$

Proof. Yang: To be completed. □

4 Dual abelian varieties

Theorem 10. Let A be an abelian variety over \mathbf{k} . Then $\text{Pic}^0(A)$ has a natural structure of an abelian variety, called the *dual abelian variety* of A , denoted by A^\vee .

Proposition 11.