Category of sheaves of modules

1 Sheaves of modules, quasi-coherent and coherent sheaves

Definition 1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where I, J are (possibly infinite) index sets.

Definition 2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that there exists a surjective morphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

Remark 3. There are many versions of "local" properties for sheaves of $\mathcal{O}_X$-modules. Yang: To be continued.

Definition 4. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if it is finitely generated, and for every open set $U \subseteq X$ and every morphism of sheaves of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of φ is finitely generated.

2 As abelian categories

Theorem 5. The categories of sheaves of abelian groups, quasi-coherent sheaves, and coherent sheaves on a ringed space $(X, \mathcal{O}_X)$ are all abelian categories. Yang: To be checked.

Theorem 6. Let $(X, \mathcal{O}_X)$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules has enough injectives. Yang: To be checked.

3 Relevant functors

Definition 7. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *sheaf Hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf of abelian groups defined as follows: for an open set $U \subseteq X$, we define

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of morphisms of sheaves of $\mathcal{O}_U$-modules from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. For an inclusion of open sets $V \subseteq U$, the restriction map

$$\mathrm{res}_{UV} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(V)$$

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is defined by sending a morphism $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ to its restriction $\varphi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. Yang: To be continued.

Definition 8. Let $f : X \to Y$ be a morphism of ringed spaces. The *pull-back functor* $f^* : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$ is defined as follows: for an $\mathcal{O}_Y$-module $\mathcal{F}$, we define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

where $f^{-1}\mathcal{F}$ is the inverse image sheaf of $\mathcal{F}$. For a morphism of $\mathcal{O}_Y$-modules $\varphi : \mathcal{F} \to \mathcal{G}$, we define

$$f^*\varphi : f^*\mathcal{F} \to f^*\mathcal{G}$$

to be the morphism induced by the morphism of sheaves of abelian groups $f^{-1}\varphi : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$. Yang: To be continued.

Definition 9. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules defined as follows: for an open set $U \subseteq X$, we define

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

where $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is the tensor product of $\mathcal{O}_X(U)$-modules. For an inclusion of open sets $V \subseteq U$, the restriction map
Yang: To be continued.

4 Locally free sheaves and vector bundles

5 Cohomological theory