Dimension theory in commutative algebra

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Dimension and Depth

1 Artinian Rings and Length of Modules

Definition 1. Let A be a ring and M an A module. *A simple module filtration of M* is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i-1} is a simple module, i.e. it has no submodule except 0 and itself. If M has a simple module filtration as above, we define the *length of M* as n and say that M has *finite length*.

The following proposition guarantees the length is well-defined.

Proposition 2. Suppose M has a simple module filtration $M = M_{0,0} \supsetneq M_{1,0} \supsetneq \cdots \supsetneq M_{n,0} = 0$. Then for any other filtration $M = M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,m} = 0$ with $m > n$, there exist $k < m$ such that $M_{0,k} = M_{0,k+1}$.

Proof. We claim that there are at least $0 \leq k_1 < \cdots < k_{m-n} < m$ satisfies that $M_{0,k_i} = M_{0,k_i+1}$. Let $M_{i,j} := M_{i,0} \cap M_{0,j}$. Inductively on n, we can assume that there exist $k_1, \cdots, k_{n-m+1}$ such that $M_{1,k} = M_{1,k+1}$. Consider the sequence

$$M_{0,0}/M_{1,0} \supset (M_{0,1} + M_{1,0})/M_{1,0} \supset \cdots \supset (M_{0,m} + M_{1,0})/M_{1,0} = 0$$

in $M_{0,0}/M_{1,0}$. Since $M_{0,0}/M_{1,0}$ is simple, there is at most one k_i with $M_{0,k_i} + M_{1,0} \neq M_{0,k_i+1} + M_{1,0}$. And note that if $M_{0,k_i} + M_{1,0} = M_{0,k_i+1} + M_{1,0}$ and $M_{0,k_i} \cap M_{1,0} = M_{0,k_i} \cap M_{1,0}$, then $M_{0,k_i} = M_{0,k_i+1}$ by the Five Lemma. □

Example 3. Let A be a ring and $\mathfrak{m} \in \operatorname{mSpec} A$. Then $A/\mathfrak{m}$ is a simple module.

Proposition 4. Let A be a ring and M an A-module. Then M is of finite length iff it satisfies both a.c.c and d.c.c.

Proof. Note that if M has either a strictly ascending chain or a strictly descending chain, M is of infinite length. Conversely, d.c.c guarantee M has a simple submodule and a.c.c guarantee the sequence terminates. □

Proposition 5. The length $l(-)$ is an additive function for modules of finite length. That is, if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ with M_i of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

Proof. The simple module filtrations of M_1 and M_3 will give a simple module filtration of M_2. □

Proposition 6. Let $(A, \mathfrak{m})$ be a local ring. Then A is artinian iff $\mathfrak{m}^n = 0$ for some $n \geq 0$.

Proof. Suppose A is artinian. Then the sequence $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ will stable. It follows that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n. By the Nakayama's Lemma **??**, $\mathfrak{m}^n = 0$.
Conversely, we have

$$\mathfrak{m} \subset \mathfrak{N} \subset \bigcap_{\text{minimal prime ideal}} \mathfrak{p},$$

whence $\mathfrak{m}$ is minimal. □

Proposition 7. Let A be a ring. Then A is artinian iff A is of finite length.

Proof. First we show that A has only finite maximal ideal. Otherwise, consider the set $\{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k\}$. It has a minimal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ and for any maximal ideal $\mathfrak{m}$, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}$. It follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i. Let $\mathfrak{M} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ be the Jacobi radical of A. Consider the sequence $\mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots$ and by Nakayama's Lemma, we have $\mathfrak{M}^k = 0$ for some k. Consider the filtration

$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1^k \supset \mathfrak{m}_1^k \mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0).$$

We have $\mathfrak{m}_1^k \cdots \mathfrak{m}_i^j / \mathfrak{m}_1^k \cdots \mathfrak{m}_i^{j+1}$ is an $A/\mathfrak{m}_i$-vector space. It is artinian and then of finite length. Hence A is of finite length. □

Date: June 18, 2025, **Author**: Tianle Yang, My Website

Proposition 8. Let A be a ring. Then A is artinian iff A is noetherian and of dimension 0. For definition of dimension, see **??**.

Proof. Suppose A is artinian. Then A is noetherian by Proposition 7. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A/\mathfrak{p}$ is an artinian integral domain. If there is $a \in A/\mathfrak{p}$ is not invertible, consider $(a) \supset (a^2) \supset \cdots$, we see $a = 0$. Hence $\mathfrak{p}$ is maximal and $\dim A = 0$.
Suppose that A is noetherian and of dimension 0. Then every maximal ideal is minimal. In particular, A has only finite maximal ideal $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$. Let $\mathfrak{q}_i$ be the $\mathfrak{p}_i$-component of (0). Then we have $A \hookrightarrow \bigoplus_i A/\mathfrak{q}_i$. We just need to show that $A/\mathfrak{q}_i$ is of finite length as A-module. If $\mathfrak{q}_i \subset \mathfrak{p}_j$, take radical we get $\mathfrak{p}_i \subset \mathfrak{q}_j$ and hence $i = j$. So $A/\mathfrak{q}_i$ is a local ring with maximal ideal $\mathfrak{p}_i A/\mathfrak{q}_i$. Then every element in $\mathfrak{p}_i A/\mathfrak{q}_i$ is nilpotent. Since $\mathfrak{p}_i$ is finitely generated, $(\mathfrak{p}_i A/\mathfrak{q}_i)^k = 0$ for some k. Then $A/\mathfrak{q}_i$ is artinian and then of finite length as $A/\mathfrak{q}_i$-module. Then the conclusion follows. □

2 Dedekind Domains

3 Dimension and Serre's conditions

Proposition 9. Let $A \subset B$ be noetherian rings such that B is finite over A. Then $\dim A = \dim B$.

Proof. If we have a sequence $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ of prime ideals in B, then there exists $f \in \mathfrak{P}_2 \setminus \mathfrak{P}_1$. Since B is finite over A, there exist $a_1, \cdots, a_n \in A$ such that

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0.$$

Then $a_n \in \mathfrak{P}_2 \cap A$. If $a_n \in \mathfrak{P}_1$, $f^{n-1} + \cdots + a_{n_1} \in \mathfrak{P}_1$ since $f \notin \mathfrak{P}_1$. Then $a_{n-1} \in \mathfrak{P}_2$. Repeat the process, it will terminate, whence $\mathfrak{P}_1 \cap A \subsetneq \mathfrak{P}_2 \cap A$. Otherwise, we have $f^n \in a_1 B + \cdots + a_n B \subset \mathfrak{P}_1$.
Conversely, suppose we have $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} A$ with $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Choose $\mathfrak{P}_1 \in \operatorname{Spec} B$ such that $\mathfrak{P}_1 \cap A = \mathfrak{p}_1$, then we have $A/\mathfrak{p}_1 \subset B/\mathfrak{P}_1$. Let $\mathfrak{P}_2$ be the preimage of the prime ideal in $B/\mathfrak{P}_1$ which is over image of $\mathfrak{p}_2$ in $A/\mathfrak{p}_1$. Proposition **??** guarantees that such $\mathfrak{P}_2$ exists. Then we get $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$. Repeat this progress, we get $\dim B \geq \dim A$. □

Theorem 10 (Krull's Principal Ideal Theorem)**.** Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let $\mathfrak{p}$ be a minimal prime ideal among those containing f. Then $\operatorname{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_\mathfrak{p}$, we may assume A is local with maximal ideal $\mathfrak{p}$. Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.
Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is $\mathfrak{q}$-primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_\mathfrak{q} = \mathfrak{q}^{n+1} A_\mathfrak{q}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_\mathfrak{q} = 0$. It follows that $\mathfrak{q} A_\mathfrak{q}$ is minimal, whence $A_\mathfrak{q}$ is artinian. Therefore, $\mathfrak{q}$ is minimal in A. □

Corollary 11. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n. Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.
Choose $\mathfrak{q}$ be a minimal prime ideal among those containing $(\mathfrak{p}_{k-1}, f)$ and contained in $\mathfrak{p}_{k+1}$. Then by Krull's Principal Ideal Theorem 10, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace $\mathfrak{p}_k$ by $\mathfrak{q}_k$, we have $f \in \mathfrak{q}_k \setminus \mathfrak{p}_{k-1}$
Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \cdots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.
Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$ has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A. It follows that $\dim A/(f) + 1 \leq \dim A$. □

Depth For a noetherian local ring $(A, \mathfrak{m})$, we can define the depth of an A-module M. Somehow the Krull dimension is "homological" and the depth is "cohomological".

Definition 12. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A-module. A sequence $t_1, \cdots, t_n \in \mathfrak{m}$ is called an *M-regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \cdots, t_{i-1})M$ for all i.

Example 13. Let $A = \mathsf{k}[x, y]/(x^2, xy)$ and $I = (x, y)$. Then $\operatorname{depth}_I A = 0$.

Definition 14. The *I-depth* of M is defined as the maximum length of M-regular sequences in I, denoted by $\operatorname{depth}_I M$. When A is a local ring with maximal ideal $\mathfrak{m}$, we write $\operatorname{depth} M$ for $\operatorname{depth}_{\mathfrak{m}} M$.

Regular and Serre's conditions Up to now, there are three numbers measuring the "size" of a local ring $(A, \mathfrak{m})$:

- $\dim A$: the Krull dimension of A.
- $\operatorname{depth} A$: the depth of A.
- $\dim_{\kappa(\mathfrak{m})} T_{A,\mathfrak{m}}$: the dimension of Zariski tangent space $T_{A,\mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as a $\kappa(\mathfrak{m})$-vector space.

These three numbers are related by the following inequalities.

Proposition 15. Let $(A, \mathfrak{m})$ be a local noetherian ring with residue field k. Then the following inequalities hold:

$$\operatorname{depth} A \leq \dim A \leq \dim_{\mathsf{k}} T_{A,\mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary 11.
Let $t_1, \cdots, t_n$ be a $\kappa(\mathfrak{m})$-basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \cdots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \cdots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \cdots, t_n))$. It follows that $\mathfrak{m} = (t_1, \cdots, t_n)$ by Nakayama's Lemma. By Corollary 11,

$$n + \dim A/(t_1, \cdots, t_n) \geq n - 1 + \dim A/(t_1, \cdots, t_{n-1}) \geq \cdots \geq 1 + \dim A/(t_1) \geq \dim A.$$

We conclude the result. □

Definition 16. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property* (R_k) or *is regular in codimension* k if $\forall \xi \in X$ with $\operatorname{codim} Z_\xi \leq k$,

$$\dim_{\kappa(\xi)} T_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

We say that X *verifies property* (S_k) if $\forall \xi \in X$ with $\operatorname{depth} \mathcal{O}_{X,\xi} < k$,

$$\operatorname{depth} \mathcal{O}_{X,\xi} = \dim \mathcal{O}_{X,\xi}.$$

Example 17. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.
Suppose A verifies (S_1). If $\mathfrak{p} \in AssA$, every element in $\mathfrak{p}$ is a zero divisor. Then $\operatorname{depth} A_{\mathfrak{p}} = 0$. It follows that $\dim A_{\mathfrak{p}} = 0$ and then $\mathfrak{p}$ is minimal.
Suppose A has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_{\mathfrak{p}} = 0$. This means every element in $\mathfrak{p}A_{\mathfrak{p}}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Lemma **??**, $\mathfrak{p} = \mathfrak{q}$ for some minimal $\mathfrak{q}$, whence $\dim A_{\mathfrak{p}} = 0$.

Example 18. Let A be a noetherian ring verifies (S_1). Then A verifies (S_2) iff for any nonzero divisor $f \in A$, $\operatorname{Ass}_A A/fA$ has no embedded point.
Suppose A verifies (S_2). Let $f \in A$ be a nonzero divisor and $\mathfrak{p} \in \operatorname{Ass}_A A/fA$. There exist $g \in A \setminus fA$ such that $\mathfrak{p} = (f : g)$. For any $t_1, t_2 \in \mathfrak{p}$, there exist s_1, s_2 with $s_i \notin (t_i)$ and $t_i g = f s_i$. Then $t_1 t_2 g = f s_1 t_2 = f s_2 t_1$. Since f is not a zero divisor, $s_1 t_2 = s_2 t_1$. Then t_2 is a zero divisor in $A_{\mathfrak{p}}/t_1 A_{\mathfrak{p}}$ since $s_1 \notin (t_1)$. Since $f \in \mathfrak{p}$, $\operatorname{depth} A_{\mathfrak{p}} = 1$ and then $\operatorname{ht} \mathfrak{p} = 1$. This show that $\mathfrak{p}$ is not embedded in $\operatorname{Ass}_A A/fA$.
Conversely, suppose $\operatorname{Ass}_A A/fA$ has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_{\mathfrak{p}} = 1$. Then there exists $f \in A_{\mathfrak{p}}$ which is not a zero divisor. We have $\operatorname{depth} A_{\mathfrak{p}}/fA_{\mathfrak{p}} = 0$ and $\operatorname{Ass}_A A/fA$ has no embedded point, whence $\mathfrak{p}$ is minimal in A/fA. Then $\operatorname{ht} \mathfrak{p} = 1$ by Krull's Principal Ideal Theorem 10 and the fact f is not a zero divisor.

Example 19. Let X be a locally noetherian scheme. Then X is reduced iff it verifies (R_0) and (S_1).
The properties are local, whence we can assume $X = \operatorname{Spec} A$. Suppose A is reduced. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be all minimal prime ideals of A. We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where $\mathfrak{N}$ is the nilradical of A. Hence A has no embedded point. Since $A_{\mathfrak{p}}$ is artinian, local and reduced, $A_{\mathfrak{p}}$ is a field and hence regular.
Conversely, let $\operatorname{Ass} A$ be equal to $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. Then every $\mathfrak{p}_i$ is minimal by (S_1). Let f be in $\mathfrak{N}$. Then the image of

f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0), $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where $\mathfrak{q}_i$ is the $\mathfrak{p}_i$ component of (0) in A. Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

4 Cohen-Macaulay rings

Definition 20 (Cohen-Macaulay)**.** A noetherian local ring $(A, \mathfrak{m})$ is called *Cohen-Macaulay* if $\dim A = \operatorname{depth} A$. A locally noetherian scheme X is called *Cohen-Macaulay* if $\mathcal{O}_{X,\xi}$ is Cohen-Macaulay for any point $\xi \in X$.

By definition, it is easy to see that X is Cohen-Macaulay if and only if it verifies (S_k) for all $k \geq 0$.

Example 21 (Non Cohen-Macaulay rings)**.**

Proposition 22. Let $(A, \mathfrak{m}, \mathsf{k})$ be a noetherian local ring and M a finite A-module. Then

$$\operatorname{depth} M := \inf\{i : \operatorname{Ext}^i_A(\mathsf{k}, M) \neq 0\}.$$

Proof. Let $a \in \mathfrak{m}$ be M-regular and $N = M/aM$. Then we claim that

$$\inf\{i : \operatorname{Ext}^i_A(\mathsf{k}, N) \neq 0\} = \inf\{i : \operatorname{Ext}^i_A(\mathsf{k}, M) \neq 0\} - 1.$$

Indeed, we have an exact sequence

$$0 \to M \xrightarrow{a} M \to N \to 0.$$

It induces a long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}_A(\mathsf{k}, M) \to \operatorname{Ext}^{i-1}_A(\mathsf{k}, N) \to \operatorname{Ext}^i_A(\mathsf{k}, M) \xrightarrow{\operatorname{Ext}^i_A(\mathsf{k}, \mathrm{Mult}_a)} \operatorname{Ext}^i_A(\mathsf{k}, M) \to \cdots.$$

Note that $a \in \mathfrak{m}$, then $\operatorname{Ext}^i_A(\mathsf{k}, \mathrm{Mult}_a) = 0$. It follows that when $\operatorname{Ext}^{i-1}_A(\mathsf{k}, M) = 0$, we have $\operatorname{Ext}^{i-1}_A(\mathsf{k}, N) = 0$ iff $\operatorname{Ext}^i_A(\mathsf{k}, M) = 0$, whence the claim.
Let $n = \inf\{i : \operatorname{Ext}^i_A(\mathsf{k}, M) \neq 0\}$. Induct on n. Suppose first $n = 0$. Since k is a simple A-module, there is an injective homomorphism $\mathsf{k} \to M$. Then $\mathfrak{m} \in \operatorname{Ass} M$ and hence $\operatorname{depth} M = 0$.
Suppose $n > 0$., let $a_1, \cdots, a_m \in \mathfrak{m}$ be any M-regular sequence. Using the claim inductively on $M/(a_1, \cdots, a_m)M$, we have $n \geq$ depth. If M has no regular element, then $\mathfrak{m} \subset \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$. Then $\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} M$. This show that we can find $x \neq 0 \in M$ such that $\mathfrak{p} = \operatorname{Ann} x$. It gives a homomorphism $\mathsf{k} = A/\mathfrak{m} \to M$. That is a contradiction and hence M has a regular element. Let a be M-regular and $N = M/aM$. Then $\operatorname{depth} N = n - 1$ by the claim and induction hypothesis. Hence we have $\operatorname{depth} M \geq n$. □

Corollary 23. Let A be a noetherian ring, M a finite A-module and $a \in A$ an M-regular element. Then $\operatorname{depth} M = \operatorname{depth} M/aM + 1$.

Corollary 24. Let A be a noetherian ring $a \in A$ a nonzero divisor. Then A verifies (S_d) iff A/aA verifies (S_{d-1}).

Definition 25. An ideal I of a noetherian ring A is called *unmixed* if

$$\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \operatorname{Ass}(A/I).$$

Here $\operatorname{ht}(I)$ is defined as

$$\operatorname{ht}(I) := \inf\{\operatorname{ht}(\mathfrak{p}) : I \subset \mathfrak{p}\}.$$

We say that *the unmixedness theorem holds for a noetherian ring* A if any ideal $I \subset A$ generated by $\operatorname{ht}(I)$ elements is unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme* X if $\mathcal{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Theorem 26. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Proof. We can assume that $X = \operatorname{Spec} A$ is affine.
Suppose X is Cohen-Macaulay. Let $I \subset A$ be an ideal generated by $a_1, \cdots, a_r$ with $r = \operatorname{ht}(I)$. We claim that $a_1, \cdots, a_r$

is an A-regular sequence. If so, we get that the unmixedness theorem holds for A by applying Example 17 on A/I. Since $\mathrm{ht}(a_1,\cdots,a_{r-1}) \le r-1$ by Krull's Principal Ideal Theorem 10 and $\mathrm{ht}(a_1,\cdots,a_r) = r \le \mathrm{ht}(a_1,\cdots,a_{r-1})+1$, we have $\mathrm{ht}(a_1,\cdots,a_{r-1}) = r-1$. By induction on r, we can assume that $a_1,\cdots,a_{r-1}$ is an A-regular sequence. Hence any prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1,\cdots,a_{r-1})$ has height $r-1$. Now suppose a_r is a zero divisor in $A/(a_1,\cdots,a_{r-1})$. Then there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1,\cdots,a_{r-1})$ such that $a_r \in \mathfrak{p}$. Then $I \subset \mathfrak{p}$ and $\mathrm{ht}(I) \le r-1$. This contradicts that $\mathrm{ht}(I) = r$.
Suppose the unmixedness theorem holds for A. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal with $\mathrm{ht}(\mathfrak{p}) = r$. Then $\mathfrak{p} \in \operatorname{Ass} A$ if and only if $\mathrm{ht}(\mathfrak{p}) = 0$. If $r > 0$, there is a nonzero divisor $a \in \mathfrak{p}$. By Krull's Principal Ideal Theorem 10, $\mathrm{ht}(\mathfrak{p}A/aA) = r-1$. Inductively, we can find a regular sequence $a_1,\cdots,a_r$ in $\mathfrak{p}$. Then $\operatorname{depth} A_{\mathfrak{p}} = r$. □

5 Regular rings I

Definition 27. Let A be a noetherian ring. For every $\mathfrak{p} \in \operatorname{Spec} A$, $\mathfrak{p}/\mathfrak{p}^2$ is a vector space over $\kappa(\mathfrak{p})$. The *Zariski's tangent space* $T_{A,\mathfrak{p}}$ of A at $\mathfrak{p}$ is defined as the dual $\kappa(\mathfrak{p})$-vector space of $\mathfrak{p}/\mathfrak{p}^2$.

Definition 28. A noetherian ring A is said to be *regular at* $\mathfrak{p} \in \operatorname{Spec} A$ if we have

$$\dim_{\kappa(\mathfrak{p})} T_{A,\mathfrak{p}} = \dim A_{\mathfrak{p}},$$

where $\dim A_{\mathfrak{p}}$ is the Krull dimension of the local ring $A_{\mathfrak{p}}$. A noetherian ring A is said to be *regular* if it is regular at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$.

Definition 29. Let A be a noetherian ring that is regular at $\mathfrak{p} \in \operatorname{Spec} A$. A sequence $t_1,\cdots,t_n \in \mathfrak{p}$ is called a *regular system of parameters* at $\mathfrak{p}$ if their images form a basis of the $\kappa(\mathfrak{p})$-vector space $\mathfrak{p}/\mathfrak{p}^2$.

Proposition 30. Let $(A,\mathfrak{m})$ be a noetherian local ring that is regular at $\mathfrak{m}$. Let $t_1,\cdots,t_n$ be a regular system of parameters at $\mathfrak{m}$, $\mathfrak{p}_i = (t_1,\cdots,t_i)$ and $\mathfrak{p}_0 = (0)$. Then $\mathfrak{p}_i$ is a prime ideal of height i, and $A/\mathfrak{p}_i$ is a regular local ring for all i. In particular, regular local ring is integral, and the regular system of parameters $t_1,\cdots,t_n$ is a regular sequence in A.

Proof. By the Krull's Principal Ideal Theorem 10, we have

$$n-1 = \dim A - 1 \le \dim A/(t_1) \le \dim_{\kappa(\mathfrak{m}/(t_1))} T_{A/(t_1),\mathfrak{m}/(t_1)} \le n-1.$$

Hence $\dim A/(t_1) = n-1$ and $\mathrm{ht}(t_1) = 1$. Since $t_2,\cdots,t_n$ generate $\mathfrak{m}/(t_1)$, we have that $A/(t_1)$ is regular at $\mathfrak{m}/(t_1)$ and the images of $t_2,\cdots,t_n$ form a regular system of parameters.
For integrality, we induct on the dimension of A. If $\dim A = 0$, then A is a field and hence integral. Suppose $\dim A > 0$, let $\mathfrak{q}$ be a minimal prime ideal of A. Then $t_1 \notin \mathfrak{q}$. We have

$$n-1 = \dim A - 1 \le \dim A/(\mathfrak{q}+t_1A) \le \dim_{\kappa(\mathfrak{q}/(t_1))} T_{A/(\mathfrak{q}+t_1A),\mathfrak{q}/(t_1)} \le n-1.$$

By similar arguments, we have $A/(\mathfrak{q}+t_1A)$ is regular at $\mathfrak{m}/(\mathfrak{q}+t_1A)$. By induction hypothesis, both of A/t_1A and $A/(\mathfrak{q}+t_1A)$ are integral and of dimension $n-1$. Hence $t_1A = t_1A+\mathfrak{q}$, i.e. $\mathfrak{q} \subset t_1A$. For every $a = bt_1 \in \mathfrak{q}$, we have $b \in \mathfrak{q}$ since $t_1 \notin \mathfrak{q}$. Then $\mathfrak{q} \subset t_1\mathfrak{q} \subset \mathfrak{m}\mathfrak{q}$. By Nakayama's Lemma, $\mathfrak{q} = 0$, whence A is integral. □

Corollary 31. A regular ring is Cohen-Macaulay.

Corollary 32. A regular ring is normal.

Proposition 33. A noetherian ring A is regular if and only if it is regular at every maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$.

Proof. Suppose $\mathfrak{p} \subset \mathfrak{m}$ and A is regular at $\mathfrak{m}$.
Yang: To be completed. □

Proposition 34. Let k be a field, k' an algebraic extension of k, A an integral k-algebra of finite type and $A' := A \otimes_{\mathsf{k}} \mathsf{k}'$. Let $\mathfrak{m} \in \operatorname{mSpec} A$ and $\mathfrak{m}'$ be a maximal ideal of A' lying over $\mathfrak{m}$. Then

(a) If A' is regular at $\mathfrak{m}'$, then A is regular at $\mathfrak{m}$;

(b) suppose $\kappa(\mathfrak{m})$ is separable over k, the converse holds.

Proof. We claim that $\mathfrak{m}'^2 \cap A = \mathfrak{m}^2$. Suppose $\mathfrak{m} = (g_1, \cdots, g_n)$. Let $f \in A \cap \mathfrak{m}'^2$. We can assume that f is in the form $f = \sum_{i=1}^n a_i g_i$ for some $a_i \in A$ satisfy that $\deg_{T_i} a_j \le \deg g_i, \forall i, j$.
The map $\mathfrak{m} \to \mathfrak{m}' \to \mathfrak{m}'/\mathfrak{m}'^2$ induces a map $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/\mathfrak{m}'^2$. This is a $\kappa(\mathfrak{m})$-linear map. Yang: To be completed. □

Remark 35. Let k be arbitrary field, $A = \mathsf{k}[T_1, \cdots, T_n]$ and g_i irreducible polynomials in one variable T_i over k. Then for every $f \in A$, we can write

$$f = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{Z}_{\ge 0}^n} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \le \deg g_i.$$

This is called the *Taylor expansion of f with respect to* $g_1, \cdots, g_n$.