Regularity and Smoothness

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Regularity and Smoothness

1 Modules of differentials and derivations

In this subsection, let R be a ring and A an R-algebra.

Definition 1 (Derivation)**.** A *derivation* of A over R is an R-linear map $\partial : A \to M$ with an A-module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module M, the set of all derivations of A over R into M forms an A-module, denoted by $\mathrm{Der}_R(A, M)$.

Given a module homomorphism $f : M \to N$ of A-modules and a derivation $\partial \in \mathrm{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of A over R into N.

Proposition 2. The functor $\mathrm{Der}_R(A, -)$ is representable. The representing object is denoted by $\Omega_{A/R}$, which is called the *module of differentials* of A over R.

Proof. Yang: To be completed. □

Proposition 3. Let A, R' be R-algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

Proof. Yang: To be completed. □

Proposition 4. Suppose A is of finite type over R. Then the module of differentials $\Omega_{A/R}$ is a finitely generated A-module.

Proof. Yang: To be completed. □

Theorem 5. Let A be an R-algebra and B an A-algebra. Then there is a short exact sequence

$$\Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to \Omega_{B/A} \to 0.$$

Proof. Yang: To be completed. □

Theorem 6. Let A be an R-algebra and I an ideal of A. Then there is a short exact sequence

$$I/I^2 \to \Omega_{A/R} \otimes_A A/I \to \Omega_{(A/I)/R} \to 0.$$

Proof. Yang: To be completed. □

2 Zariski's tangent space

Definition 7. Let A be a noetherian ring. For every $\mathfrak{p} \in \mathrm{Spec}\, A$, $\mathfrak{p}/\mathfrak{p}^2$ is a vector space over $\kappa(\mathfrak{p})$. The *Zariski's tangent space* $T_{A,\mathfrak{p}}$ of A at $\mathfrak{p}$ is defined as the dual $\kappa(\mathfrak{p})$-vector space of $\mathfrak{p}/\mathfrak{p}^2$.

Definition 8. A noetherian ring A is said to be *regular* if for every prime ideal $\mathfrak{p} \in \mathrm{Spec}\, A$, we have

$$\dim_{\kappa(\mathfrak{p})} T_{A,\mathfrak{p}} = \dim A_{\mathfrak{p}},$$

where $\dim A_{\mathfrak{p}}$ is the Krull dimension of the local ring $A_{\mathfrak{p}}$.

Proposition 9. Regularity is a local property, i.e., TFAE:

(a) A is regular;

(b) for every prime ideal $\mathfrak{p} \in \mathrm{Spec}\, A$, the local ring $A_{\mathfrak{p}}$ is regular;

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(c) for every maximal ideal $\mathfrak{m} \in \operatorname{mSpec} A$, the local ring $A_{\mathfrak{m}}$ is regular.

Proof. Yang: To be completed. □

Proposition 10.

Example 11.

3 Jacobiian criterion