Line bundles induce morphisms

1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 1. Let A be a ring and X an A-scheme. Let $\mathcal{L}$ be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where x_i are the standard coordinates on $\mathbb{P}^n_A$.

Proof. Yang: To be continued. □

Definition 2. A line bundle $\mathcal{L}$ on a scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

Definition 3. A line bundle $\mathcal{L}$ on a scheme X is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. Yang: To be continued.

Definition 4. Let $\mathcal{L}$ be a line bundle on a scheme X and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace. The *base locus* of V is the closed subset

$$\mathrm{Bs}(V) = \{x \in X : s(x) = 0 \text{ for all } s \in V\}.$$

If $\mathrm{Bs}(V) = \emptyset$, we say that V is *base-point free*. Yang: To be continued.

Definition 5. A *linear system* on a scheme X is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on X and $V \subseteq \Gamma(X, \mathcal{L})$ is a subspace. The dimension of the linear system is $\dim V - 1$. A linear system is *base-point free* if V is base-point free. A linear system is *complete* if $V = \Gamma(X, \mathcal{L})$. Yang: To be continued.

Theorem 6. Let X be a scheme over a ring A and $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(a) $\mathcal{L}$ is ample.

(b) For some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample.

(c) For some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free.

(d) For every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

Yang: To be continued.

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2 Asymptotic behavior

Definition 7. Let X be a scheme and $\mathcal{L}$ a line bundle on X. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

Definition 8. A line bundle $\mathcal{L}$ on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. Yang: To be continued.

Theorem 9. Let X be a scheme over a ring A and $\mathcal{L}$ a semiample line bundle on X. Then there exists a morphism $f : X \to Y$ over A such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y. Moreover, $Y = \operatorname{Proj} R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. Yang: To be continued.

Definition 10. A line bundle $\mathcal{L}$ on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large n. Yang: To be continued.

3 Iitaka fibration

Theorem 11. Let X be a projective variety over a field k and $\mathcal{L}$ a line bundle on X. Then there exists a unique rational map $f : X \dashrightarrow Y$ to a projective variety Y such that:

(a) The general fiber of f is connected.

(b) The dimension of Y is equal to the Iitaka dimension of $\mathcal{L}$, i.e., the transcendence degree of the section ring $R(X, \mathcal{L})$ minus one.

(c) For some $n > 0$, the linear system associated to $\mathcal{L}^{\otimes n}$ defines the map f.

The map f is called the *Iitaka fibration* associated to $\mathcal{L}$. Yang: To be continued.