
Surfaces



“仿造的又如何，当不成真正的勇者也无妨，即便如此，我也是勇者！”

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1 The first properties of surfaces

Let k be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over k .

1.1 Basic concepts

1.2 Riemann-Roch Theorem for surfaces

1.3 Hodge Index Theorem

2 Birational geometry on surfaces

Let k be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over k .

2.1 Castelnuovo's Theorem and Run the MMP

Theorem 2.1 (Castelnuovo's contractibility criterion). Let X be a smooth projective surface over an algebraically closed field \mathbb{k} . Let $C \subseteq X$ be an irreducible curve. Then there exists a birational morphism $f : X \rightarrow Y$ contracting C to a smooth point if and only if $C \cong \mathbb{P}^1$ and $C^2 = -1$.

3 Coarse classification of surfaces

Let \mathbb{k} be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over \mathbb{k} .

Let X be a smooth projective surface over an algebraically closed field \mathbb{k} . We want to classify X up to birational equivalence. Let K_X be the canonical divisor of X .

Theorem 3.1. Let X be a smooth projective surface over an algebraically closed field \mathbb{k} . Suppose that the Kodaira dimension $\kappa(X) \geq 0$. Then the linear system $|12K_X|$ is base point free. Yang: To be checked.

3.1 Classification

Theorem 3.2 (Enriques-Kodaira classification). Let X be a smooth projective surface over \mathbb{k} . Then X is birational to a unique minimal model X' , unless X is birational to a ruled surface. Moreover, the minimal model X' falls into one of the following classes:

- (a) $\kappa(X') = -\infty$: $X' \cong \mathbb{P}^2$ or X' is a ruled surface;
- (b) $\kappa(X') = 0$: X' is a K3 surface, an abelian surface or their quotients;
- (c) $\kappa(X') = 1$: X' is an elliptic surface;
- (d) $\kappa(X') = 2$: X' is a surface of general type.

Yang: To be checked.

4 Ruled Surface

In this section, fix an algebraically closed field \mathbb{k} . This section is mainly based on [Har77, Chapter V.2].

4.1 Minimal Section and Classification

Definition 4.1 (Ruled surface). A *ruled surface* is a smooth projective surface X together with a surjective morphism $\pi : X \rightarrow C$ to a smooth curve C such that all geometric fibers of π are isomorphic to \mathbb{P}^1 .

Let $\pi : X \rightarrow C$ be a ruled surface over a smooth curve C of genus g .

Lemma 4.2. There exists a section of π .

Proof. Yang: To be continued... □

Proposition 4.3. Then there exists a vector bundle \mathcal{E} of rank 2 on \mathcal{C} such that $X \cong \mathbb{P}_{\mathcal{C}}(\mathcal{E})$ over \mathcal{C} .

Proof. Let $\sigma : \mathcal{C} \rightarrow X$ be a section of π and D be its image. Let $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{E} = \pi_*\mathcal{L}$. Since D is a section of π , $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in \mathcal{C}$, whence $h^0(X_t, \mathcal{L}|_{X_t}) = 2$ for any $t \in \mathcal{C}$. By Miracle Flatness (??), f is flat. By Grauert's Theorem (??), \mathcal{E} is a vector bundle of rank 2 on \mathcal{C} and we have a natural isomorphism $\mathcal{E} \otimes \kappa(t) \cong H^0(X_t, \mathcal{L}|_{X_t})$ for any $t \in \mathcal{C}$.

This gives a surjective homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{C}}} \kappa(t) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \cong H^0(X_t, \mathcal{L}|_{X_t}) \otimes_{\kappa(t)} \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{L}|_{X_t}.$$

For every $x \in X$, we have

$$\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{C}}} \kappa(\pi(x)) \otimes_{\kappa(\pi(x))} \mathcal{O}_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x) \twoheadrightarrow \mathcal{L}|_{X_{\pi(x)}} \otimes_{\mathcal{O}_{X_{\pi(x)}}} \kappa(x).$$

Yang: The left side coincides with $\pi^*\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ naturally. Hence by Nakayama's Lemma, the natural homomorphism $\pi^*\mathcal{E} \rightarrow \mathcal{L}$ is surjective.

By ??, we have a morphism $\varphi : X \rightarrow \mathbb{P}_{\mathcal{C}}(\mathcal{E})$ over \mathcal{C} such that $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}_{\mathcal{C}}(\mathcal{E})}(1)$. Since $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ for any $t \in \mathcal{C}$, $\varphi|_{X_t} : X_t \rightarrow \mathbb{P}_{\mathcal{C}}(\mathcal{E})_t$ is an isomorphism for any $t \in \mathcal{C}$. Hence φ is bijection on the underlying sets. Yang: Here is a serious gap. Why fiberwise isomorphism implies isomorphism? □

Lemma 4.4. It is possible to write $X \cong \mathbb{P}_{\mathcal{C}}(\mathcal{E})$ such that $H^0(\mathcal{C}, \mathcal{E}) \neq 0$ but $H^0(\mathcal{C}, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle \mathcal{L} on \mathcal{C} with $\deg \mathcal{L} < 0$. Such a vector bundle \mathcal{E} is called a *normalized vector bundle*. In particular, if \mathcal{E} is normalized, then $e = -\deg c_1(\mathcal{E})$ is an invariant of the ruled surface X .

Proof. We can suppose that \mathcal{E} is globally generated since we can always twist \mathcal{E} by a sufficiently ample line bundle on \mathcal{C} . Then for all line bundle \mathcal{L} of degree sufficiently large, \mathcal{L} is very ample and hence $H^0(\mathcal{C}, \mathcal{E} \otimes \mathcal{L}) \neq 0$. By Lemma 4.2 and ??, \mathcal{E} is an extension of line bundles. Then for all line bundle \mathcal{L} of degree sufficiently negative, $H^0(\mathcal{C}, \mathcal{E} \otimes \mathcal{L}) = 0$ since line bundles of negative degree have no global sections. Hence we can find a line bundle \mathcal{M} on \mathcal{C} of lowest degree such that $H^0(\mathcal{C}, \mathcal{E} \otimes \mathcal{M}) \neq 0$. Replacing \mathcal{E} by $\mathcal{E} \otimes \mathcal{M}$, we are done. □

Remark 4.5. The invariant e is unique but the normalization of \mathcal{E} is not unique. For example, if \mathcal{E} is normalized, then so is $\mathcal{E} \otimes \mathcal{L}$ for any line bundle \mathcal{L} on \mathcal{C} of degree 0. Yang: To be continued...

Suppose that $X \cong \mathbb{P}_{\mathcal{C}}(\mathcal{E})$ where \mathcal{E} is a normalized vector bundle of rank 2 on \mathcal{C} . Since $H^0(\mathcal{C}, \mathcal{E}) \neq 0$, choosing a non-zero section s , we get an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{C}} \xrightarrow{s} \mathcal{E} \rightarrow \mathcal{E}/\mathcal{O}_{\mathcal{C}} \rightarrow 0.$$

We claim that $\mathcal{E}/\mathcal{O}_{\mathcal{C}}$ is a line bundle on \mathcal{C} . Since \mathcal{C} is a curve, we only need to check that $\mathcal{E}/\mathcal{O}_{\mathcal{C}}$ is torsion-free.

Yang: To be continued...

Definition 4.6. A section \mathcal{C}_0 of π is called a *minimal section* if Yang: to be continued...

Lemma 4.7. Let $X = \mathbb{P}_C(\mathcal{E}) \rightarrow C$ be a ruled surface over a smooth curve C of genus g with invariant e and normalized \mathcal{E} .

- (a) If \mathcal{E} is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ where \mathcal{L} is a line bundle on C with $\deg \mathcal{L} = -e$.
- (b) If \mathcal{E} is indecomposable, then $-2g \leq e \leq 2g - 2$.

Proof. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable, we can assume that $H^0(C, \mathcal{L}_1) \neq 0$. If $\deg \mathcal{L}_1 > 0$, then $H^0(C, \mathcal{E} \otimes \mathcal{L}_1^{-1}) \neq 0$, contradicting the normalization of \mathcal{E} . Similarly $\deg \mathcal{L}_2 \leq 0$. Then $\mathcal{L}_1 \cong \mathcal{O}_C$. And hence $e = -\deg c_1(\mathcal{E}) = -\deg \mathcal{L}_2 \geq 0$.

If \mathcal{E} is indecomposable, we have an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$$

which is a non-trivial extension, with \mathcal{L} a line bundle on C of degree $-e$. Hence by ??, we have $0 \neq \text{Ext}_C^1(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^{-1})$. By Serre duality, we have $H^1(C, \mathcal{L}^{-1}) \cong H^0(C, \mathcal{L} \otimes \omega_C)$. Hence $\deg(\mathcal{L} \otimes \omega_C) = 2g - 2 - e \geq 0$.

On the other hand, let \mathcal{M} be a line bundle on C of degree -1 . Twist the above exact sequence by \mathcal{M} and take global sections, we have an equation

$$h^0(\mathcal{M}) - h^0(\mathcal{E} \otimes \mathcal{M}) + h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{M}) + h^1(\mathcal{E} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = 0.$$

Since $\deg \mathcal{M} < 0$ and \mathcal{E} is normalized, we have $h^0(\mathcal{M}) = h^0(\mathcal{E} \otimes \mathcal{M}) = 0$. By Riemann-Roch, we have $h^1(\mathcal{M}) = g$ and $h^0(\mathcal{L} \otimes \mathcal{M}) - h^1(\mathcal{L} \otimes \mathcal{M}) = -e - 1 + 1 - g$. Hence

$$h^1(\mathcal{E} \otimes \mathcal{M}) = e + 2g \geq 0.$$

This gives $e \geq -2g$. □

Theorem 4.8. Let $\pi : X \rightarrow C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e . Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Proof. This is a direct consequence of Lemma 4.7. □

Example 4.9. Here we give an explicit description of the ruled surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e \geq 0$.

Let C be covered by two standard affine charts U_0, U_1 with coordinate u on U_0 and v on U_1 such that $u = 1/v$ on $U_0 \cap U_1$. On U_i , let $\mathcal{O}(-e)|_{U_i}$ be generated by s_i for $i = 0, 1$. We have $s_0 = u^e s_1$ on $U_0 \cap U_1$.

On $X_i = X_{U_i} \cong U_i \times \mathbb{P}^1$, let $[x_0 : x_1]$ and $[y_0 : y_1]$ be the homogeneous coordinates of \mathbb{P}^1 on X_0 and X_1 respectively. Then the transition function on $X_0 \cap X_1$ is given by

$$(u, [x_0 : x_1]) \mapsto (1/u, [x_0 : u^e x_1]).$$

Remark 4.10. The surface $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ is also called the *Hirzebruch surface*.

Theorem 4.11. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \rightarrow E$ be a ruled surface over an elliptic curve E with invariant e and normalized \mathcal{E} .

- (a) If \mathcal{E} is indecomposable, then $e = 0$ or -1 , and for each e there exists a unique such ruled surface up to isomorphism.
- (b) If \mathcal{E} is decomposable, then $e \geq 0$ and $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$ where \mathcal{L} is a line bundle on E with $\deg \mathcal{L} = -e$.

Proof. Only the indecomposable case needs a proof. By Lemma 4.7, we have $-2 \leq e \leq 0$ and a non-trivial extension

$$0 \rightarrow \mathcal{O}_E \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$$

where \mathcal{L} is a line bundle on E of degree $-e$.

Case 1. $e = 0$.

In this case, \mathcal{L} is of degree 0 and $H^1(E, \mathcal{L}^{-1}) \cong H^0(E, \mathcal{L} \otimes \omega_E) \cong H^0(E, \mathcal{L}) \neq 0$. Hence $\mathcal{L} \cong \mathcal{O}_E$.

Yang: To be continued...

Case 2. $e = -1$.

In this case, \mathcal{L} is of degree 1 and $H^1(E, \mathcal{L}) \cong H^0(E, \mathcal{L}^{-1}) = 0$. By Riemann-Roch, we have $h^0(E, \mathcal{L}) = 1$.

Case 3. $e = -2$.

Yang: To be continued...

□

Example 4.12. Yang: To be continued...

4.2 The Néron-Severi Group of Ruled Surfaces

Proposition 4.13. Let $\pi : X \rightarrow \mathcal{C}$ be a ruled surface over a smooth curve \mathcal{C} of genus g . Let C_0 be a minimal section of π and F a fiber of π . Then $\text{Pic}(X) \cong \mathbb{Z}[C_0] \oplus \pi^* \text{Pic}(\mathcal{C})$.

Proof. Let D be any divisor on X with $D.F = a \in \mathbb{Z}$. Then $D - aC_0$ is numerically trivial on the fibers of π . Let $\mathcal{L} = \mathcal{O}_X(D - aC_0)$. Then $\mathcal{L}|_{X_t} \cong \mathcal{O}_{X_t}$ for any $t \in \mathcal{C}$. By Grauert's Theorem (??), $\pi_* \mathcal{L}$ is a line bundle on \mathcal{C} . Yang: and the natural map $\pi^* \pi_* \mathcal{L} \rightarrow \mathcal{L}$ is an isomorphism. □

Proposition 4.14. Let $\pi : X \rightarrow \mathcal{C}$ be a ruled surface over a smooth curve \mathcal{C} of genus g . Let C_0 be a minimal section of π and let F be a fiber of π . Then $K_X \sim -2C_0 + \pi^*(K_{\mathcal{C}} - c_1(\mathcal{E}))$. Numerically, we have $K_X \equiv -2C_0 + (2g - 2 - e)F$ where e is the invariant of X . Yang: Check this carefully.

Proof. Yang: To be continued. □

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \rightarrow \mathbb{P}^1$ be a ruled surface over \mathbb{P}^1 with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

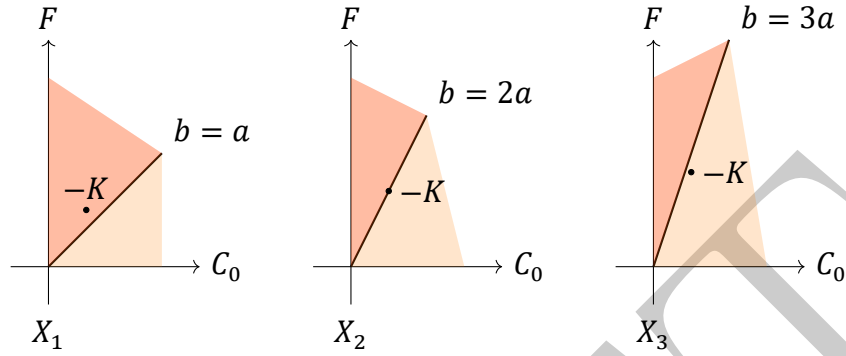
Theorem 4.15. Let $\pi : X \rightarrow \mathbb{P}^1$ be a ruled surface over \mathbb{P}^1 with invariant e . Let C_0 be a minimal section of π and let F be a fiber of π . Let $D \sim aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

- (a) D is effective $\iff a, b \geq 0$;

(b) D is ample $\iff D$ is very ample $\iff a > 0$ and $b > ae$.

Proof. Yang: To be continued... □

Example 4.16. Here we draw the Néron-Severi group of the rational ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ for $e = 1, 2, 3$.



We have $-K_{X_e} \equiv 2C_0 + (2+e)F$. For $e = 1$, $-K$ is ample and hence X_1 is a del Pezzo surface. For $e = 2$, $-K$ is nef and big but not ample. For $e \geq 3$, $-K$ is big but not nef.

Elliptic case. Let $\pi : X = \mathbb{P}_E(\mathcal{E}) \rightarrow E$ be a ruled surface over an elliptic curve E with \mathcal{E} a normalized vector bundle of rank 2 and degree $-e$.

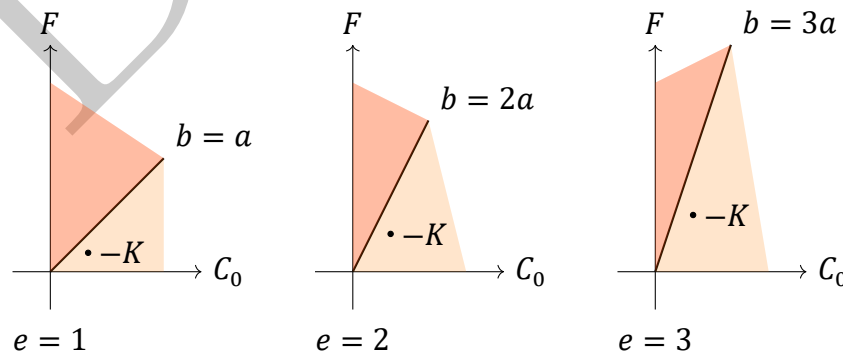
Theorem 4.17. Let $\pi : X \rightarrow E$ be a ruled surface over an elliptic curve E with invariant e . Assume that \mathcal{E} is decomposable. Let C_0 be a minimal section of π and let F be a fiber of π . Let $D \equiv aC_0 + bF$ be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff a \geq 0$ and $b \geq ae$;

(b) D is ample $\iff D$ is very ample $\iff a > 0$ and $b > ae$.

Proof. Yang: To be continued... □

Example 4.18. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with decomposable normalized \mathcal{E} for $e = 1, 2, 3$.



In this case, $-K \equiv 2C_0 + eF$ is always big but not nef.

Theorem 4.19. Let $\pi : X \rightarrow E$ be a ruled surface over an elliptic curve E with invariant e . Assume that \mathcal{E} is indecomposable. Let C_0 be a minimal section of π and let F be a fiber of π . Let $D \equiv aC_0 + bF$

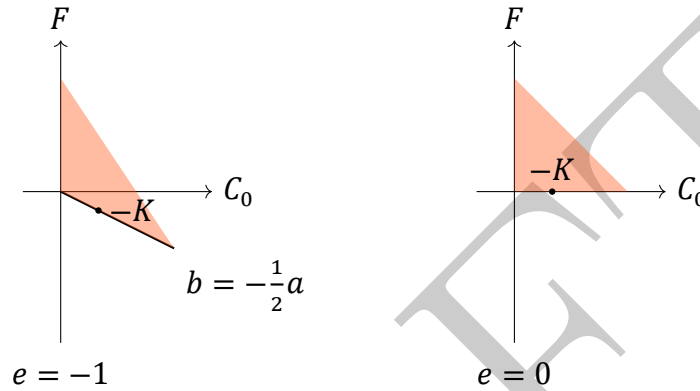
be a divisor on X with $a, b \in \mathbb{Z}$.

(a) D is effective $\iff a \geq 0$ and $b \geq \frac{1}{2}ae$;

(b) D is ample $\iff D$ is very ample $\iff a > 0$ and $b > \frac{1}{2}ae$.

Proof. Yang: To be continued... □

Example 4.20. Here we draw the Néron-Severi group of the ruled surface X over an elliptic curve E with indecomposable normalized \mathcal{E} for $e = -1, 0$.



In this case, $-K \equiv 2C_0 + eF$ is always nef but not big.

Proposition 4.21. Let $\pi : X \rightarrow \mathcal{C}$ be a ruled surface over a smooth curve \mathcal{C} . Then every nef divisor on X is semi-ample. Yang: Check this carefully.

5 K3 surface

Let \mathbb{k} be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over \mathbb{k} .

5.1

Definition 5.1. A *K3 surface* is a smooth, projective surface X with trivial canonical bundle $K_X \cong \mathcal{O}_X$ and irregularity $q(X) = h^1(X, \mathcal{O}_X) = 0$.

Example 5.2. A smooth quartic surface $X \subseteq \mathbb{P}^3$ is a K3 surface. Indeed, by the adjunction formula, we have

$$K_X = (K_{\mathbb{P}^3} + X)|_X = (-4H + 4H)|_X = 0,$$

where H is a hyperplane in \mathbb{P}^3 . Moreover, by the Lefschetz hyperplane theorem, we have $h^1(X, \mathcal{O}_X) = h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$. Yang: To be checked.

5.2 Hodge Structure and Moduli of K3 surfaces

6 Elliptic surfaces

6.1 The first properties

Definition 6.1. An *elliptic surface* is a smooth projective surface S together with a surjective morphism $\pi : S \rightarrow C$ to a smooth projective curve C such that the generic fiber of π is a smooth curve of genus 1, and π has a section $s : C \rightarrow S$. Yang: To be continued...