

# Relative objects

## 1 Relative schemes

**Definition 1.** Let  $X$  be a scheme. An  $\mathcal{O}_X$ -algebra is a sheaf . Yang: To be continued...

**Definition 2.** Let  $X$  be a scheme and  $\mathcal{A}$  be a quasi-coherent  $\mathcal{O}_X$ -algebra. The relative Spec of  $\mathcal{A}$ , denoted by  $\text{Spec}_X \mathcal{A}$ , is the scheme obtained by gluing the affine schemes  $\text{Spec} \mathcal{A}(U) \rightarrow U$  for all affine open subsets  $U \subset X$ . Yang: To be continued...

**Proposition 3.** Let  $X$  be a scheme and  $\mathcal{E}$  be a locally free sheaf of finite rank on  $X$ . Then the relative Spec of the symmetric algebra of  $\mathcal{E}$ , denoted by  $\mathbf{V}(\mathcal{E}) = \text{Spec}_X \text{Sym}_{\mathcal{O}_X} \mathcal{E}$ , is called the geometric vector bundle associated to  $\mathcal{E}$ . The projection morphism  $\pi : \mathbf{V}(\mathcal{E}) \rightarrow X$  is affine and for any open subset  $U \subset X$ , we have  $\pi^{-1}(U) \cong \text{Spec} \text{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$ . Yang: To be continued... Yang: To be revised, need to take dual.

**Definition 4.** Let  $X$  be a scheme and  $\mathcal{A}$  be a quasi-coherent graded  $\mathcal{O}_X$ -algebra such that  $\mathcal{A}_0 = \mathcal{O}_X$  and  $\mathcal{A}$  is generated by  $\mathcal{A}_1$  as an  $\mathcal{O}_X$ -algebra. The relative Proj of  $\mathcal{A}$ , denoted by  $\text{Proj}_X \mathcal{A}$ , is the scheme obtained by gluing the affine schemes  $\text{Proj} \mathcal{A}(U)$  for all affine open subsets  $U \subset X$ . The projection morphism  $\pi : \text{Proj}_X \mathcal{A} \rightarrow X$  is projective and for any open subset  $U \subset X$ , we have  $\pi^{-1}(U) \cong \text{Proj} \mathcal{A}(U)$ . Yang: To be continued...

## 2 Relative ampleness and projective morphisms

**Definition 5.** Let  $X$  be an  $S$ -scheme via a morphism  $f : X \rightarrow S$ . A line bundle  $\mathcal{L}$  on  $X$  is called *relatively ample* (or  $f$ -ample) if for every affine open subset  $U \subset S$ , the restriction  $\mathcal{L}|_{f^{-1}(U)}$  is an ample line bundle on the scheme  $f^{-1}(U)$ . Yang: To be revised.

**Definition 6.** Let  $f : X \rightarrow S$  be a morphism of schemes. The morphism  $f$  is called *projective* if there exists a quasi-coherent

**Remark 7.** There are subtle differences between various definitions of projective morphisms in the literature. (see [FGA05, abaaba])

## 3 Blowing up

**Definition 8.** Let  $X$  be a scheme and  $\mathcal{I} \subset \mathcal{O}_X$  be a quasi-coherent sheaf of ideals. The blow up of  $X$  along  $\mathcal{I}$ , denoted by  $\mathrm{Bl}_{\mathcal{I}} X$ , is defined to be the relative Proj of the Rees algebra of  $\mathcal{I}$ :

$$\mathrm{Bl}_{\mathcal{I}} X = \mathrm{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{I}^n.$$

Yang: To be continued...

**Proposition 9.** Let  $X$  be a scheme and  $\mathcal{I} \subset \mathcal{O}_X$  be a quasi-coherent sheaf of ideals. The blow up morphism  $\pi : \mathrm{Bl}_{\mathcal{I}} X \rightarrow X$  is projective. Moreover, if the support of  $\mathcal{O}_X/\mathcal{I}$  is nowhere dense in  $X$ , then  $\pi$  is birational. Yang: To be continued...

**Proposition 10.** Let  $X$  be a regular scheme and  $\mathcal{I} \subset \mathcal{O}_X$  be a quasi-coherent sheaf of ideals. Then the blow up  $\mathrm{Bl}_{\mathcal{I}} X$  is also a regular scheme. Yang: To be continued...

**Proposition 11.** Let  $X$  be a scheme and  $\mathcal{I} \subset \mathcal{O}_X$  be a quasi-coherent sheaf of ideals. The exceptional locus of the blow up morphism  $\pi : \mathrm{Bl}_{\mathcal{I}} X \rightarrow X$  is equal to the inverse image of the support of  $\mathcal{O}_X/\mathcal{I}$ :

$$\mathrm{Exc}(\pi) = \pi^{-1}(\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})).$$

Yang: To be continued...

## References

- [FGA05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*. Vol. 123. Mathematical Surveys and Monographs. Grothendieck's FGA explained. American Mathematical Society, Providence, RI, 2005, pp. x+339. ISBN: 0-8218-3541-6. DOI: [10.1090/surv/123](https://doi.org/10.1090/surv/123). URL: <https://doi.org/10.1090/surv/123> (cit. on p. 1).