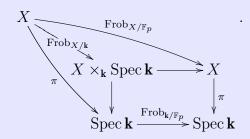
Bend and Break

1 Preliminary

Definition 1 (Frobinius morphism). Let X be a variety over a field \mathbf{k} of characteristic p > 0. Denote the structure morphism by $\pi : X \to \operatorname{Spec} \mathbf{k}$. The absolute Frobenius morphism is the morphism given by $\mathcal{O}_X \to \mathcal{O}_X$, $f \mapsto f^p$, denoted by $\operatorname{Frob}_{X/\mathbb{F}_p}$. The relative Frobenius morphism is the morphism $\operatorname{Frob}_{X/\mathbf{k}}$ given by the following commutative diagram:



We usually denote $X \times_{\mathbf{k}} \operatorname{Spec} \mathbf{k}$ appearing above by $X^{(p)}$.

Proposition 2. Let X be a variety of dimension d over a field \mathbf{k} of characteristic p > 0. Then the relative Frobenius morphism $\operatorname{Frob}_{X/\mathbf{k}} : X \to X^{(p)}$ is a finite morphism of degree p^d over \mathbf{k} .

2 Deformation of curves

Theorem 3 (ref. [Kol96, Chapter II, Theorem 1.2]). Let C be a smooth projective curve of genus g and X a smooth projective variety of dimension n. Let $f: C \to X$ be a non-constant morphism. Then every irreducible component of Mor(C, X) containing f has dimension at least

$$-K_Y \cdot f(C) + (1-g)n.$$

Proposition 4. Let X be a projective variety and $f: C \to X$ a non-constant morphism from a pointed smooth projective curve $p_0 \in C$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t: C \to X$ for $t \in T$ such that $f_0 = f$ and $f_t(p_0) = x_0$ for some point $x_0 \in X$ and all t. Then there exist some rational curves $\Gamma_1, \ldots, \Gamma_m \subset X$ such that

- (a) $x_0 \in \bigcup_{i=1}^m \Gamma_i;$
- (b) there is a morphism $g: C \to X$ such that $f(C) \equiv_{alg} g(C) + \sum_{i=1}^{m} a_i \Gamma_i$ with $a_i > 0$ for all i.

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Proposition 5. Let X be a projective variety and $f: \mathbb{P}^1 \to X$ a non-constant morphism with $f(0) = x_0, f(\infty) = x_\infty$. Let $0 \in T$ be a pointed smooth curve (may not be projective). Suppose that we have a non-trivial family of morphisms $f_t: \mathbb{P}^1 \to X$ for $t \in T$ such that $f_0 = f$ and $f_t(0) = x_0, f_t(\infty) = x_\infty$ for all t. Then there exists a curve $C \subset X$ such that $f(\mathbb{P}^1) \equiv_{alg} aC$ with a > 1.

3 Find rational curves

Theorem 6. Let X be a smooth Fano variety. Then for any $x \in X(\mathbf{k})$, there is a rational curve C passing through x with

$$0 < -C \cdot K_X \le \dim X + 1.$$

Proof. Yang: To be completed.

Theorem 7. Let X be a smooth projective variety such that $K_X \cdot C < 0$ for some irreducible curve $C \subset X$. Let H be an ample divisor on X. Then there exists a rational curve Γ such that

$$-(K_X \cdot C) \cdot \frac{H \cdot \Gamma}{H \cdot C} \le -K_X \cdot \Gamma \le \dim X + 1.$$

Proof. Yang: To be completed.

Theorem 8. Let (X, B) be a projective klt pair and $f: X \to Y$ a birational projective morphism. Suppose that $K_{(X,B)}$ is f-ample. Then the exceptional locus of f is covered by rational curves Γ with

$$0 < -K_{(X,B)} \cdot \Gamma \le 2 \dim X.$$

Theorem 9. Let X be a smooth projective variety of dimension n and H, H_1, \dots, H_{n-1} ample divisors on X. Suppose that $K_X \cdot H_1 \cdot \dots \cdot H_{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \le -2n \cdot \frac{H \cdot H_1 \cdots H_{n-1}}{K_X \cdot H_1 \cdots H_{n-1}}.$$

Proposition 10. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \le -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

References

[Kol96] János Kollár. Rational Curves on Algebraic Varieties. Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Berlin, Heidelberg:

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