Picard Groups of Abelian Varieties

1 Pullback along group operations

Theorem 1 (Seesaw Theorem)**.** Let A be an abelian variety over $\Bbbk$.

Theorem 2 (Theorem of the cube)**.** Let X, Y, Z be completed varieties over $\Bbbk$ and $\mathcal{L}$ a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\Bbbk), y \in Y(\Bbbk), z \in Z(\Bbbk)$ such that the restriction $\mathcal{L}|_{\{x\}\times Y\times Z}$, $\mathcal{L}|_{X\times\{y\}\times Z}$ and $\mathcal{L}|_{X\times Y\times\{z\}}$ are trivial. Then $\mathcal{L}$ is trivial.

Proof. Yang: To be completed. □

Remark 3. If we assume the existence of the Picard scheme, then the theorem of the cube can be deduced from the Rigidity Lemma. Yang: To be completed.

Proposition 4. Let A be an abelian variety over $\Bbbk$, $f, g, h : X \to A$ morphisms from a variety X to A and $\mathcal{L}$ a line bundle on A. Then

$$(f+g+h)^*\mathcal{L} \cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof. Yang: To be completed. □

Proposition 5. Let A be an abelian variety over $\Bbbk$, $n \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle on A. Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes \frac{1}{2}(n^2-n)}.$$

Proof. Yang: To be completed. □

Theorem 6 (Theorem of the square)**.** Let A be an abelian variety over $\Bbbk$, $x, y \in A(\Bbbk)$ two points and $\mathcal{L}$ a line bundle on A. Then

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Remark 7. We can define a map

$$\Phi_{\mathcal{L}} : A(\Bbbk) \to \mathrm{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary $\mathcal{L}$, the map

$$\Phi_{\square} : \mathrm{Pic}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A(\Bbbk), \mathrm{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is a group homomorphism. For any $x \in A(\Bbbk)$, we have

$$\Phi_{t_x^*\mathcal{L}} = \Phi_{\mathcal{L}}.$$

In the other words,

$$\Phi_{\mathcal{L}}(x) \in \mathrm{Ker}\,\Phi_{\square}, \quad \forall \mathcal{L} \in \mathrm{Pic}(A), x \in A(\Bbbk).$$

Yang: To be completed.

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If we assume the scheme structure on $\mathrm{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\mathrm{Pic}^0(A)$. Let $K(\mathcal{L}) \coloneqq \mathrm{Ker}\,\Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A. We give another description of $K(\mathcal{L})$. From this point, we can recover the dual abelian variety $A^\vee = \mathrm{Pic}^0(A)$ as the quotient $A/K(\mathcal{L})$. Yang: To be completed.

2 Projectivity

Proposition 8. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. Then $|2D|$ is base point free.

Theorem 9. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. TFAE:

(a) the stabilizer $\mathrm{Stab}(D)$ of D is finite;

(b) the morphism $\Phi_{|2D|}$ induced by the complete linear system $|2D|$ is finite;

(c) D is ample;

(d) $K(\mathscr{O}_A(D))$ is finite.

Theorem 10. Let A be an abelian variety over $\mathbf{k}$. Then A is projective.

Proof. Yang: To be completed. □

3 Isogenies and finite subgroups

Theorem 11. Let A be an abelian variety of dimension d over $\Bbbk$. Then the subgroup $A[n]$ of n torsion points is finite and we have

(a) if n is coprime to $\mathrm{char}(\mathbf{k})$, then $A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2d}$;

(b) if $n = p^k$ for $p = \mathrm{char}(\mathbf{k}) > 0$

Proof. Yang: To be completed. □

Theorem 12. Let A be an abelian variety over $\Bbbk$. There is a bijection between the isogenies from A over $\Bbbk$ and the finite subgroup schemes of A.

4 Dual abelian varieties

Theorem 13. Let A be an abelian variety over $\mathbf{k}$. Then $\mathrm{Pic}^0(A)$ has a natural structure of an abelian variety, called the *dual abelian variety* of A, denoted by $A^\vee$.

Proposition 14. There exists a unique line bundle $\mathcal{P}$ on $A \times A^\vee$ such that for every $y = \mathcal{L} \in A^\vee = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A \times \{y\}} \cong \mathcal{L}$.