Line bundles and divisors

1 Cartier divisors

Definition 1. Let X be a scheme. A *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of X. Equivalently, a Cartier divisor D can be represented by an open covering $\{U_i\}$ of X and a collection of rational functions $f_i \in \mathcal{K}_X^*(U_i)$ such that for any i, j, the function $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. We denote a Cartier divisor by $D = \{(U_i, f_i)\}$.

2 Line bundles and Picard group

Definition 2. Let X be a scheme. A *line bundle* on X is a locally free sheaf of $\mathcal{O}_X$-modules of rank 1.

Example 3. Let $X = \mathbb{P}_A^n = \operatorname{Proj} A[T_0, T_1, \dots, T_n] = \operatorname{Proj} B$ be the projective n-space over a ring A. For each integer $d \in \mathbb{Z}$, the sheaf $\mathcal{O}_X(d)$, defined by

$$\{f \neq 0\} \mapsto B(d)_{(f)},$$

is a line bundle on X, called the Yang: *twisted line bundle* of degree d. Recall that here $B(d)_{(f)}$ is the degree-zero part of the localization of the shifted graded ring $B(d)$ at the multiplicative set generated by f, and $B(d)$ is defined by $B(d)_m = B_{m+d}$ for all $m \in \mathbb{Z}$.

Let us verify this by direct computation. On the standard open subset $U_i = D_+(T_i) = \operatorname{Spec} B_i$, where $B_i = A\left[T_0/T_i, \dots, T_n/T_i\right]$, write $t_{j,i} = T_j/T_i$. We have

$$\mathcal{O}_X(d)(U_i) = B(d)^0_{(T_i)} = \left\{ \frac{f}{T_i^k} \middle| f \in B, \deg f = k + d \right\} = B_i \cdot T_i^d =: B_i \cdot e_i,$$

where we denote $e_i = T_i^d$. Hence $\mathcal{O}_X(d)(U_i)$ is a free B_i-module of rank 1 and thus $\mathcal{O}_X(d)$ is locally free of rank 1.

In the language of bundles, on $U_{ij} = U_i \cap U_j$, we have

$$e_i = t_{i,j}^d \cdot e_j.$$

Thus the transition functions of $\mathcal{O}_X(d)$ are given by $\{(U_{ij},\ t_{i,j}^d : U_{ij} \to \mathbb{G}_m)\}$.

Proposition 4. Let X be a scheme and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. Then

(a) the tensor product $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is also a line bundle on X;

(b) the dual $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also a line bundle on X;

(c) there is a natural isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$.

Proof. □

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Definition 5. Let X be a scheme. The *Picard group* of X is defined to be the group of isomorphism classes of line bundles on X with the group operation given by the tensor product. It is denoted by $\mathsf{Pic}(X)$.

Definition 6. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a Yang: non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X\times\{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X\times\{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\mathrm{alg}} \mathcal{L}'$. Yang: To be checked.

3 Weil divisors and reflexive sheaves

To talk about Weil divisors, we need to work with normal schemes.

Definition 7. Let X be a normal integral scheme. A *Weil divisor* on X is a formal sum

$$D = \sum_Z n_Z Z,$$

where the sum runs over all prime divisors Z of X (i.e., integral closed subschemes of codimension 1) and $n_Z \in \mathbb{Z}$, such that for any affine open subset $U = \operatorname{Spec} A \subseteq X$, only finitely many Z intersecting U have nonzero coefficients n_Z. The group of Weil divisors on X is denoted by $\mathrm{WDiv}(X)$.

Definition 8. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. The sheaf $\mathcal{F}$ is called *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

Proposition 9. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. If $\mathcal{F}$ is reflexive, then it is determined by its restriction to any open subset $U \subseteq X$ whose complement has codimension at least 2, i.e., $\mathcal{F} \cong i_*(\mathcal{F}|_U)$, where $i : U \hookrightarrow X$ is the inclusion map. Yang: To be checked.

Proof. Yang: To be continued. □

Theorem 10. Let X be a normal integral scheme. There is a one-to-one correspondence between the set of isomorphism classes of reflexive sheaves of rank 1 on X and the Yang: Weil divisor class group $\mathrm{WDiv}(X)$ of X. Under this correspondence, a Weil divisor D corresponds to the reflexive sheaf $\mathcal{O}_X(D)$. Yang: To be checked.

Proof. Yang: To be continued. □

4 The first Chern class

Definition 11. Let X be a normal scheme and $\mathcal{L}$ a vector bundle on X. The *first Chern class* of $\mathcal{L}$, denoted by $c_1(\mathcal{L})$, is a Weil divisor class defined as follows:
Yang: To be completed.

Definition 12. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. On X_{reg}, the regular locus of X, $\mathcal{F}|_{X_{\mathrm{reg}}}$ admits a finite resolution by vector bundles

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F}|_{X_{\mathrm{reg}}} \to 0.$$

The *first Chern class* of $\mathcal{F}$, denoted by $c_1(\mathcal{F})$, is defined to be

$$c_1(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i c_1(\mathcal{E}_i).$$

Yang: To be revised.

Proposition 13. Let X be a normal scheme and $\mathcal{F}$ a torsion sheaf on X. Then

$$c_1(\mathcal{F}) = \sum_Z \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{F}_Z) \cdot Z,$$

where the sum runs over all prime divisors Z of X and $\mathcal{F}_Z$ is the stalk of $\mathcal{F}$ at the generic point of Z. Yang: To be checked.