Birational Geometry

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1 Kodaira Vanishing Theorem

1.1 Preliminary

Theorem 1.1 (Serre Duality)**.** Let X be a Cohen-Macaulay projective variety of dimension n over k and D a divisor on X. Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \ldots, n.$$

Theorem 1.2 (Log Resolution of Singularities)**.** Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there is a smooth variety (or analytic space) Y and a projective morphism $f : Y \to X$ such that

(a) f is an isomorphism over $X - (\mathrm{Sing}(X) \cup \mathrm{Supp}\, Z)$,

(b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(c) $\mathrm{Exc}(f) \cup D$ is an snc divisor.

Theorem 1.3 (Lefschetz Hyperplane Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 1.4 (Hodge Decomposition)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Then for any k, there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 1.3 and Theorem 1.4, we have the following lemma.

Lemma 1.5. Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map $r_k : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \to H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \to H^p(Y, \mathcal{O}_Y)$$

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is an isomorphism for $p < n-1$ and an injection for $p = n-1$.

Theorem 1.6 (Leray spectral sequence)**.** Let $f : Y \to X$ be a morphism of varieties and $\mathcal{F}$ a coherent sheaf on Y. Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_*\mathcal{F}) \implies H^{p+q}(Y, \mathcal{F}).$$

1.2 Kodaira Vanishing Theorem

Lemma 1.7. Let X be a smooth projective variety over k and $\mathcal{L}$ a line bundle on X. Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \to X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D. It induces a homomorphism $\mathcal{L}^{-m} \to \mathcal{O}_X$. Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}\right) \Big/ (\mathcal{L}^{-m} \to \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then $\mathcal{A}$ is a finite $\mathcal{O}_X$-algebra. Let $Y := \operatorname{Spec}_X \mathcal{A}$. Then Y is a finite $\mathcal{O}_X$-scheme and the natural morphism $f : Y \to X$ is finite and surjective.

For every $x \in X$, let $\mathcal{L}$ locally generated by t near x. Then $\mathcal{O}_Y$ locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y. Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. □

Remark 1.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D_i' = f^*(D_i)$. Then $D' + \sum_{i=1}^k D_i'$ is snc on Y by considering the local regular system of parameters.

Lemma 1.9. Let $f : Y \to X$ be a finite surjective morphism of projective varieties and $\mathcal{L}$ a line bundle on X. Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits by the trace map $(1/n)\operatorname{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. □

Theorem 1.10 (Kodaira Vanishing Theorem)**.** Let X be a smooth projective variety of dimension n over k of characteristic 0 and A an ample divisor on X. Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 1.7 and 1.9, after taking a multiple of A, we can assume that A is effective. Then we have an exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \to H^{i-1}(X, \mathcal{O}_A) \to H^i(X, \mathcal{O}_X(-A)) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 1.5 and Serre duality (Theorem 1.1). □

1.3 Vanishing theorem for nef and big divisors

Lemma 1.11. Let X be a smooth projective variety of dimension n over k of characteristic 0, A an ample divisor and E an snc divisor on X. Then

$$H^i(X, K_X + A + E) = 0, \quad \forall i > 0.$$

Proof. Let $E = \sum_{i=1}^k E_i$. We induct on k. Consider the exact sequence

$$0 \to \mathcal{O}_X(-A - \sum_{i=1}^{k} E_i) \to \mathcal{O}_X(-A - \sum_{i=1}^{k-1} E_i) \to \mathcal{O}_{E_k}(-A - \sum_{i=1}^{k-1} E_i) \to 0.$$

Yang: To be completed. □

Theorem 1.12 (Kawamata-Viehweg Vanishing Theorem for nef and big divisors)**.** Let X be a smooth projective variety of dimension n over k of characteristic 0 and D a nef and big $\mathbb{R}$-divisor on X. Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Proof. Yang: To be completed. □

1.4 Kawamata-Viehweg Vanishing Theorem for klt pairs

Lemma 1.13. Let X be a projective variety, $\mathcal{L}$ a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \to X$ and a line bundle $\mathcal{L}'$ on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X, then we can take f such that f^*D is an snc divisor on Y.

Proof. We can assume that $\mathcal{L}$ is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "g"] \\ X \arrow[r, "\varphi_{\mathcal{L}}"] & \mathbb{P}^N \end{tikzcd},$$

where $g : [x_0 : \ldots : x_N] \mapsto [x_0^m : \ldots : x_N^m]$. The morphism f is finite and surjective since so is g. Let $\mathcal{L}' := \psi^* \mathcal{L}\mathcal{O}(1)$.

For smoothness, we can compose g with a general automorphism of $\mathbb{P}^N$. Then the conclusion follows from [Har13, Chapter III, Theorem 10.8]. □

Theorem 1.14. Let X be a smooth projective variety of dimension n over k of characteristic 0 and D a nef $\mathbb{R}$-divisor on X. Suppose that $\lceil D \rceil - D$ has snc support. Then

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall i > 0.$$

Proof. By the Bertini, we can assume that $A := D$ is ample and a $\mathbb{Q}$-divisor by adding a sufficiently small ample divisor and adjusting the coefficients slightly. Set $M := \lceil D \rceil$. Let

$$B := \sum_{i=1}^{k} b_i B_i := \lceil D \rceil - D = M - A, \quad b_i \in (0,1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k. Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 1.13 and 1.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 1.7 on B_k, we can find a finite surjective morphism $f : X' \to X$ such that $f^* B_k = cB'_k, B'_i = f^* B_i$ for $i < k$ and $\sum_{i=1}^{k} B'_i$ is an snc divisor on X'. Let $B' = \sum_{i=1}^{k-1} B'_i, A' = f^* A$ and $M' = f^* M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$ vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^* \mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 1.9. □

Lemma 1.15 (ref. [KM98, Theorem 5.10, 5.22])**.** Let (X, B) be a klt pair over k of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Theorem 1.16 (Kawamata-Viehweg Vanishing Theorem for klt pairs)**.** Let (X, B) be a klt pair over k of characteristic 0. Let D be a nef $\mathbb{R}$-divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

Proof. Let $f : \tilde{X} \to X$ be a resolution such that $\operatorname{Supp} f^*B \cup \operatorname{Exc} f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X},\tilde{B})} + f^*D,$$

where $\tilde{B} \in (0,1)$ has snc support and E is an effective exceptional divisor.

Claim 1.17. The higher direct image sheaves $R^i f_*(\mathcal{O}_{\tilde{X}}(f^*(K_{(X,B)} + D) + E))$ vanish for $i > 0$ and $f_*(\mathcal{O}_{\tilde{X}}(f^*(K_{(X,B)} + D) + E)) \cong \mathcal{O}_X(K_{(X,B)} + D)$.

By the Claim, we have

$$H^i(\tilde{X}, K_{(\tilde{X},\tilde{B})} + f^*D) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 1.14. □

Proof of Claim 1.17. Let $\mathcal{F} \coloneqq \mathcal{O}_{\tilde{X}}(f^*(K_{(X,B)} + D) + E)$. Yang: To be completed. □

2 Cone Theorem

2.1 Preliminary

Theorem 2.1 (Iitaka fibration)**.** Let X be a projective variety and $\mathcal{L}$ a line bundle on X. Let $\varphi_n : X \dashrightarrow Y_n$ be the dominant rational map associated to $\mathcal{L}^n$. Then for $n \gg 0$, the rational maps φ_n stable to a fibration $\varphi_\infty : X \dashrightarrow Y_\infty$ up to birational equivalence.

Proof. Here we test cref for the step environment. Test Step 2 for a step label. □

2.2 Non-vanishing Theorem

Theorem 2.2 (Non-vanishing Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

2.3 Base Point Free Theorem

Theorem 2.3 (Base Point Free Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then D is semiample.

2.4 Rationality Theorem

Theorem 2.4 (Rationality Theorem)**.** Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let

$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H. Then $t = u/v \in \mathbb{Q}$ and

$$0 \leq u \leq a(X) \cdot (\dim X + 1).$$

2.5 Cone Theorem and Contraction Theorem

Theorem 2.5 (Cone Theorem)**.** Let (X, B) be a projective klt pair. Then there exist countably many rational curves $C_i \subset X$ with

$$0 < -K_{(X,B)} \cdot C_i \leq 2 \dim X$$

such that

(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. We only need to prove (b) and (a) follows from (b) by taking $\varepsilon = 1/n$.

Step 1. We show that

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

why it is so long?

Step 2 (Test Name). This is a test.

Yang: To be completed. □

Proof. The follows are test steps for the step environment.

Step 1. test again. In this step, we refer to 2 for a test.

Step 2. This is a test. Test cref Theorem 2.3.

□

Theorem 2.6 (Contraction Theorem)**.** Let (X, B) be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;

(b) any line bundle $\mathcal{L}$ with $F \subset \mathcal{L}^{\perp} = \{\alpha \in N_1(X) : \alpha \cdot \mathcal{L} = 0\}$ comes from a line bundle on Y, i.e., there exists a line bundle $\mathcal{L}_Y$ on Y such that $\mathcal{L} \cong \varphi_F^* \mathcal{L}_Y$.

References

[Har13] Robin Hartshorne. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 2013.

[KM98] János Kollár and Shigefumi Mori. “Birational geometry of algebraic varieties”. In: *(No Title)* 134 (1998).