Flat, smooth and étale morphisms

1 Flat family and Hilbert polynomial

Theorem 1. Let $X \subseteq \mathbb{P}^n_T$ be a closed subscheme, where T is a noetherian scheme. Then the following are equivalent:

(a) $X \to T$ is flat.

(b) The Hilbert polynomial of the fiber $X_t \subseteq \mathbb{P}^n_{k(t)}$ is independent of the choice of $t \in T$.

Yang: To be checked.

2 Base change and semicontinuity

Theorem 2 (Base change of flat morphisms)**.** Let $f : X \to Y$ be a flat morphism of schemes. Then for any morphism $Y' \to Y$, the base change morphism $f' : X \times_Y Y' \to Y'$ is also flat.

Theorem 3 (Semicontinuity theorem)**.** Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Then for each integer $i \geq 0$, the function

$$h^i : Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is upper semicontinuous on Y.

Yang: To be checked.

3 Smooth morphisms

Definition 4 (Smooth morphism)**.** A morphism of schemes $f : X \to Y$ is called **smooth** at a point $x \in X$ if there exists an open neighborhood U of x such that the restriction $f|_U : U \to Y$ factors as

$$U \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{p} Y,$$

where g is étale and p is the projection morphism. The morphism f is called **smooth** if it is smooth at every point of X.

Yang: To be checked.

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4 Étale morphisms

Definition 5 (Étale morphism)**.** A morphism of schemes $f : X \to Y$ is called **étale** at a point $x \in X$ if there exists an open neighborhood U of x such that the restriction $f|_U : U \to Y$ factors as

$$U \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{p} Y,$$

where g is unramified and p is the projection morphism. The morphism f is called **étale** if it is étale at every point of X.

Yang: To be checked.