Preliminaries

Proposition 1. Let $f: X \to Y$ be a morphism of varieties over a field **k**. Then the function $y \mapsto \dim f^{-1}(y)$ is upper semicontinuous, i.e., for every integer m, the set $\{y \in Y : \dim f^{-1}(y) \ge m\}$ is closed in Y. Yang: To be check.

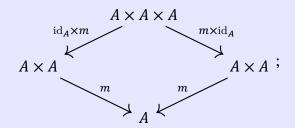
Theorem 2 (Rigidity Lemma). Let $\pi_i: X \to Y_i$ be proper morphisms of varieties over a field **k** for i = 1, 2. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi: Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

The First Properties of Abelian Varieties

1 Definition and examples of Abelian Varieties

Definition 3. Let **k** be a field. An *abelian variety over* **k** is a proper variety A over **k** together with morphisms $identity e : \operatorname{Spec} \mathbf{k} \to A$, $multiplication m : A \times A \to A$ and $inversion i : A \to A$ such that the following diagrams commute:

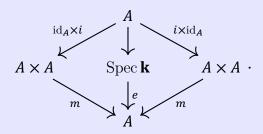
(a) (Associativity)



(b) (Identity)

$$A \times \operatorname{Spec} \mathbf{k} \xrightarrow{\operatorname{id}_A \times \varrho} A \times A \xrightarrow{\varrho \times \operatorname{id}_A} \operatorname{Spec} \mathbf{k} \times A$$

(c) (Inversion)



In other words, an abelian variety is a group object in the category of proper varieties over \mathbf{k} .

Example 4. Let E be an elliptic curve over a field \mathbf{k} . Then E is an abelian variety of dimension 1. Yang: To be completed.

Date: September 14, 2025, Author: Tianle Yang, My Website

In the following, we will always assume that A is an abelian variety over a field \mathbf{k} of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The *left translation* by $a \in A(\mathbf{k})$ is defined as

$$l_a: A \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times A \xrightarrow{a \times \operatorname{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a .

Proposition 5. Let A be an abelian variety. Then A is smooth.

Proof. By base changing to the algebraic closure of \mathbf{k} , we may assume that \mathbf{k} is algebraically closed. Note that there is a non-empty open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a .

Proposition 6. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\bigoplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. Yang: But how to show it is a morphism of varieties? Yang: To be completed.

Theorem 7. Let A and B be abelian varieties. Then any morphism $f: A \to B$ with $f(e_A) = e_B$ is a group homomorphism, i.e., for every **k**-scheme T, the induced map $f_T: A(T) \to B(T)$ is a group homomorphism.

Proof. Let \mathbb{k} be the algebraical closure of \mathbf{k} . For every \mathbf{k} -scheme T, we have the inclusion $A(T) \subset A_{\mathbb{k}}(T_{\mathbb{k}})$ and $B(T) \subset B_{\mathbb{k}}(T_{\mathbb{k}})$ which is compatible with the group structure and the morphism f. Thus we may assume that \mathbf{k} is algebraically closed.

For every $a \in A(\mathbf{k})$, the fiber $m_A^{-1}(a)$ is isomorphic to A via the projection to the first factor. In particular, $m_A^{-1}(a)$ is connected.

Consider the composition

$$A \times A \xrightarrow{\varphi} A \times A \xrightarrow{m_A} A$$
, $(x, y) \mapsto (x, m_A(i_A(x), y)) \mapsto m_A(x, m_A(i_A(x), y)) = y$.

Hence we have $(m_A \circ \varphi)_* \mathcal{O}_{A \times A} \cong \mathcal{O}_A \cong m_{A*} \mathcal{O}_{A \times A}$ since φ is an isomorphism. Then consider the diagram

$$\begin{array}{ccc}
A \times A & \xrightarrow{f \times f} & B \times B \\
\downarrow^{m_A} & & \downarrow^{m_B} \\
A & & B.
\end{array}$$

For every closed point $a \in A$, the fiber $m_A^{-1}(a) = \{(x, m_A(i_A(x), a)) | x \in A\}$ is contrac Yang: To be completed.

Proposition 8. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 7.

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition mor-

phism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A: A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

Proposition 9. Let A be an abelian variety over k and n a positive integer. Then the multiplication by n morphism $[n]_A: A \to A$ is finite surjective and étale.

Proof. Yang: To be completed.

2 Complex abelian varieties

Theorem 10. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ . Then A is a complex abelian variety if and only if Λ Yang: To be completed.

