

Picard Groups of Abelian Varieties

Let \mathbf{k} be a field and \mathbb{k} its algebraic closure. Let A be an abelian variety over \mathbf{k} .

1 Pullback along group operations

Theorem 1 (Theorem of the cube). Let X, Y, Z be proper varieties over \mathbf{k} and \mathcal{L} a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\} \times Y \times Z}$, $\mathcal{L}|_{X \times \{y\} \times Z}$ and $\mathcal{L}|_{X \times Y \times \{z\}}$ are trivial. Then \mathcal{L} is trivial.

Proof. **Yang:** To be completed. □

Remark 2. If we assume the existence of the Picard scheme, then the [Theorem 1](#) can be deduced from the Rigidity Lemma. Consider the morphism

$$\varphi : X \times Y \rightarrow \text{Pic}(Z), \quad (x, y) \mapsto \mathcal{L}|_{\{x\} \times \{y\} \times Z}.$$

Since $\varphi(x, y) = \mathcal{O}_Z$, φ factors through $\text{Pic}^0(Z)$. Then the assumption implies that φ contracts $\{x\} \times Y$, $X \times \{y\}$ and hence it maps $X \times Y$ to a point. Thus $\varphi(x', y') = \mathcal{O}_Z$ for every $(x', y') \in X \times Y$. Then by Grauert's theorem, we have $\mathcal{L} \cong p^* p_* \mathcal{L}$ where $p : X \times Y \times Z \rightarrow X \times Y$ is the projection. Note that $p_* \mathcal{L} \cong \mathcal{L}|_{X \times Y \times \{z\}} \cong \mathcal{O}_{X \times Y}$. Hence \mathcal{L} is trivial.

Lemma 3. Let A be an abelian variety over \mathbf{k} , $f, g, h : X \rightarrow A$ morphisms from a variety X to A and \mathcal{L} a line bundle on A . Then we have

$$(f + g + h)^* \mathcal{L} \cong (f + g)^* \mathcal{L} \otimes (f + h)^* \mathcal{L} \otimes (g + h)^* \mathcal{L} \otimes f^* \mathcal{L}^{-1} \otimes g^* \mathcal{L}^{-1} \otimes h^* \mathcal{L}^{-1}.$$

Proof. First consider $X = A \times A \times A$, $p : X \rightarrow A, (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$, $p_{ij} : X \rightarrow A, (x_1, x_2, x_3) \mapsto x_i + x_j$ for $1 \leq i < j \leq 3$ and $p_i : X \rightarrow A, (x_1, x_2, x_3) \mapsto x_i$ for $1 \leq i \leq 3$. Then the conclusion follows from the theorem of the cube by taking $\mathcal{L}' = p^* \mathcal{L}^{-1} \otimes p_{12}^* \mathcal{L} \otimes p_{13}^* \mathcal{L} \otimes p_{23}^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1} \otimes p_3^* \mathcal{L}^{-1}$ and considering the restriction to $\{0\} \times A \times A$, $A \times \{0\} \times A$ and $A \times A \times \{0\}$.

In general, consider the morphism $\varphi = (f, g, h) : X \rightarrow A \times A \times A$ and pull back the above isomorphism along φ . □

Proposition 4. Let A be an abelian variety over \mathbf{k} , $n \in \mathbb{Z}$ and \mathcal{L} a line bundle on A . Then we have

$$[n]_A^* \mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}(n^2+n)} \otimes [-1]_A^* \mathcal{L}^{\otimes \frac{1}{2}(n^2-n)}.$$

Proof. For $n = 0, 1$, the conclusion is trivial. For $n \geq 2$, we can use the previous lemma on $[n-2]_A, [1]_A, [1]_A$ and induct on n . Hence we have

$$[n]_A^* \mathcal{L} \cong [n-1]_A^* \mathcal{L} \otimes [n-1]_A^* \mathcal{L} \otimes [2]_A^* \mathcal{L} \otimes [1]_A^* \mathcal{L}^{-1} \otimes [1]_A^* \mathcal{L}^{-1} \otimes [n-2]_A^* \mathcal{L}^{-1}.$$

Then the conclusion follows from induction. **Yang:** To be completed. □

Definition 5. Let A be an abelian variety over \mathbf{k} and \mathcal{L} a line bundle on A . We say that \mathcal{L} is *symmetric* if $[-1]_A^* \mathcal{L} \cong \mathcal{L}$ and *antisymmetric* if $[-1]_A^* \mathcal{L} \cong \mathcal{L}^{-1}$.

Theorem 6 (Theorem of the square). Let A be an abelian variety over \mathbf{k} , $x, y \in A(\mathbf{k})$ two points and \mathcal{L} a line bundle on A . Then

$$t_{x+y}^* \mathcal{L} \otimes \mathcal{L} \cong t_x^* \mathcal{L} \otimes t_y^* \mathcal{L}.$$

Proof. Yang: To be completed. □

Remark 7. We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \rightarrow \text{Pic}(A), \quad x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary \mathcal{L} , the map

$$\Phi_{\square} : \text{Pic}(A) \rightarrow \text{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \text{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is also a group homomorphism. For any $x \in A(\mathbf{k})$, we have

$$\Phi_{t_x^* \mathcal{L}} = \Phi_{\mathcal{L}}$$

by Theorem 6. In the other words,

$$\Phi_{\mathcal{L}}(x) \in \text{Ker } \Phi_{\square}, \quad \forall \mathcal{L} \in \text{Pic}(A), x \in A(\mathbf{k}).$$

If we assume the scheme structure on $\text{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\text{Pic}^0(A)$. Let $K(\mathcal{L}) := \text{Ker } \Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A . We give another description of $K(\mathcal{L})$. From this point, when $K(\mathcal{L})$ is finite, we can recover the dual abelian variety $A^{\vee} = \text{Pic}_{A/\mathbf{k}}^0$ as the quotient $A/K(\mathcal{L})$.

2 Projectivity

In this subsection, we work over the algebraically closed field \mathbf{k} .

Proposition 8. Let A be an abelian variety over \mathbf{k} and D an effective divisor on A . Then $|2D|$ is base point free.

Proof. Yang: To be completed. □

Theorem 9. Let A be an abelian variety over \mathbf{k} and D an effective divisor on A . TFAE:

- (a) the stabilizer $\text{Stab}(D)$ of D is finite;
- (b) the morphism $\phi_{|2D|}$ induced by the complete linear system $|2D|$ is finite;
- (c) D is ample;
- (d) $K(\mathcal{O}_A(D))$ is finite.

Proof. Yang: To be completed. □

Theorem 10. Let A be an abelian variety over \mathbb{k} . Then A is projective.

Proof. Yang: To be completed. □

Corollary 11. Let A be an abelian variety over \mathbb{k} and D a divisor on A . Then D is pseudo-effective if and only if it is nef, i.e. $\text{Psef}^1(A) = \text{Nef}^1(A)$.

Proof. Yang: To be completed. □

3 Dual abelian varieties

In this subsection, we work over the algebraically closed field \mathbb{k} .

Definition 12. Let A be an abelian variety over \mathbb{k} . We define the *dual abelian variety* of A to be $A/K(\mathcal{L})$ for some ample line bundle \mathcal{L} on A . We denote it by A^\vee .

Yang: We have a natural map $A^\vee(\mathbb{k}) \rightarrow \text{Pic}^0(A)$ by sending $x + K(\mathcal{L}) \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$. We will show that this map is an isomorphism.

Lemma 13. There exists a unique line bundle \mathcal{P} on $A \times A^\vee$ such that for every $y = \mathcal{L} \in A^\vee = \text{Pic}^0(A)$, we have $\mathcal{P}|_{A \times \{y\}} \cong \mathcal{L}$.

Proof. Yang: To be completed. □

Lemma 14. Let A be an abelian variety over \mathbb{k} and B a group variety over \mathbb{k} . Then there is a natural bijection between the morphisms $f : B \rightarrow A^\vee$ and the line bundles \mathcal{L} on $A \times B$ such that for every $b \in B(\mathbb{k})$, we have $\mathcal{L}|_{A \times \{b\}} \in \text{Pic}^0(A)$. The bijection is given by $f \mapsto (1_A \times f)^* \mathcal{P}$ where \mathcal{P} is the Poincaré line bundle on $A \times A^\vee$. Yang: To be completed.

Proof. Yang: To be completed. □

Theorem 15. Let A be an abelian variety over \mathbb{k} . Then the dual abelian variety A^\vee and the Poincaré line bundle \mathcal{P} on $A \times A^\vee$ do not depend on the choice of the ample line bundle \mathcal{L} . Moreover, there is a natural bijection $A^\vee(\mathbb{k}) \rightarrow \text{Pic}^0(A)$ of groups. Under this bijection, for every $x = \mathcal{L} \in A^\vee(\mathbb{k}) = \text{Pic}^0(A)$, we have $\mathcal{P}|_{A \times \{x\}} \cong \mathcal{L}$.

Proof. Yang: To be completed. □

Proposition 16. Let A be an abelian variety over \mathbb{k} . Then the dual abelian variety A^\vee is also an abelian variety and the natural morphism $A \rightarrow A^{\vee\vee}$ is an isomorphism.

Proof. Yang: To be completed. □

4 The Néron-Severi group

Theorem 17. Let A be an abelian variety over \mathbb{k} . Then we have an inclusion $\text{NS}(A) \hookrightarrow \text{Hom}_{\text{Grp}}(A, A^\vee)$ given by Yang: To be completed.