Definition and First Properties of Varieties

1 Varieties

Definition 1. A *variety* over an algebraically closed field $\mathbb{k}$ is an integral separated scheme of finite type over $\operatorname{Spec}\mathbf{k}$.

Yang: Suppose that $\mathbf{k}$ is not algebraically closed, let $\mathbf{k}'$ be an algebraic extension of $\mathbf{k}$. What is the relation between X, $X_{\mathbf{k}'}$, $X(\mathbf{k}')$ and $X_{\mathbf{k}'}(\mathbf{k}')$?

2 Geometric properties

3 Points in varieties

Proposition 2. Let $\mathscr{k}$ be a field and ℓ an extension of $\mathscr{k}$. Let X be a variety over $\mathscr{k}$. Then we have the following:

(a) there is a natural bijection between $X(\ell)$ and $X_\ell(\ell)$;

(b) let m/ℓ be an extension, then there is a natural inclusion $X(\ell) \subseteq X(m)$;

(c) suppose that $X = \operatorname{Spec}\mathscr{k}[T_1, \dots, T_n]/I$ is an affine variety, then there is a natural bijection between $X(\ell)$ and the set $\{(x_1, \dots, x_n) \in \ell^n | f(x_1, \dots, x_n) = 0, \forall f \in I\}$.

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