Line bundles and divisors

1 Cartier divisors

Definition 1. Let X be a scheme. A *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of X. Equivalently, a Cartier divisor D can be represented by an open covering $\{U_i\}$ of X and a collection of rational functions $f_i \in \mathcal{K}_X^*(U_i)$ such that for any i, j, the function $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. We denote a Cartier divisor by $D = \{(U_i, f_i)\}$.

2 Line bundles and Picard group

Definition 2. Let X be a scheme. The *Picard group* of X is defined to be the group of isomorphism classes of line bundles on X with the group operation given by the tensor product. It is denoted by $\mathrm{Pic}(X)$.

Definition 3. Let X be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on X. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a Yang: non-singular variety T over $\mathbf{k}$, two points $t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X \times \{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X \times \{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\mathrm{alg}} \mathcal{L}'$. Yang: To be checked.

3 Weil divisors and reflexive sheaves

To talk about Weil divisors, we need to work with normal schemes.

Definition 4. Let X be a normal integral scheme. A *Weil divisor* on X is a formal sum

$$D = \sum_Z n_Z Z,$$

where the sum runs over all prime divisors Z of X (i.e., integral closed subschemes of codimension 1) and $n_Z \in \mathbb{Z}$, such that for any affine open subset $U = \mathrm{Spec}\, A \subseteq X$, only finitely many Z intersecting U have nonzero coefficients n_Z. The group of Weil divisors on X is denoted by $\mathrm{WDiv}(X)$.

Definition 5. Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. The sheaf $\mathcal{F}$ is called *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

Proposition 6. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. If $\mathcal{F}$ is reflexive, then it is determined by its restriction to any open subset $U \subseteq X$ whose complement has codimension at least 2, i.e., $\mathcal{F} \cong i_*(\mathcal{F}|_U)$, where $i : U \hookrightarrow X$ is the inclusion map. Yang: To be checked.

Proof. Yang: To be continued.   □

Theorem 7. Let X be a normal integral scheme. There is a one-to-one correspondence between the set of isomorphism classes of reflexive sheaves of rank 1 on X and the Yang: Weil divisor class group $\mathrm{WDiv}(X)$ of X. Under this correspondence, a Weil divisor D corresponds to the reflexive sheaf $\mathcal{O}_X(D)$. Yang: To be checked.

Proof. Yang: To be continued.   □

4   The first Chern class

Definition 8. Let X be a normal scheme and $\mathcal{L}$ a vector bundle on X. The *first Chern class* of $\mathcal{L}$, denoted by $c_1(\mathcal{L})$, is the image of $\mathcal{L}$ under the connecting homomorphism
Yang: To be completed.

Definition 9. Let X be a normal scheme and $\mathcal{F}$ a coherent sheaf on X. On X_{reg}, the regular locus of X, $\mathcal{F}|_{X_{\mathrm{reg}}}$ admits a finite resolution by vector bundles

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F}|_{X_{\mathrm{reg}}} \to 0.$$

The *first Chern class* of $\mathcal{F}$, denoted by $c_1(\mathcal{F})$, is defined to be

$$c_1(\mathcal{F}) = \sum_{i=0}^{n}(-1)^i c_1(\mathcal{E}_i).$$

Yang: To be revised.

Proposition 10. Let X be a normal scheme and $\mathcal{F}$ a torsion sheaf on X. Then

$$c_1(\mathcal{F}) = \sum_{Z} \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{F}_Z) \cdot Z,$$

where the sum runs over all prime divisors Z of X and $\mathcal{F}_Z$ is the stalk of $\mathcal{F}$ at the generic point of Z. Yang: To be checked.