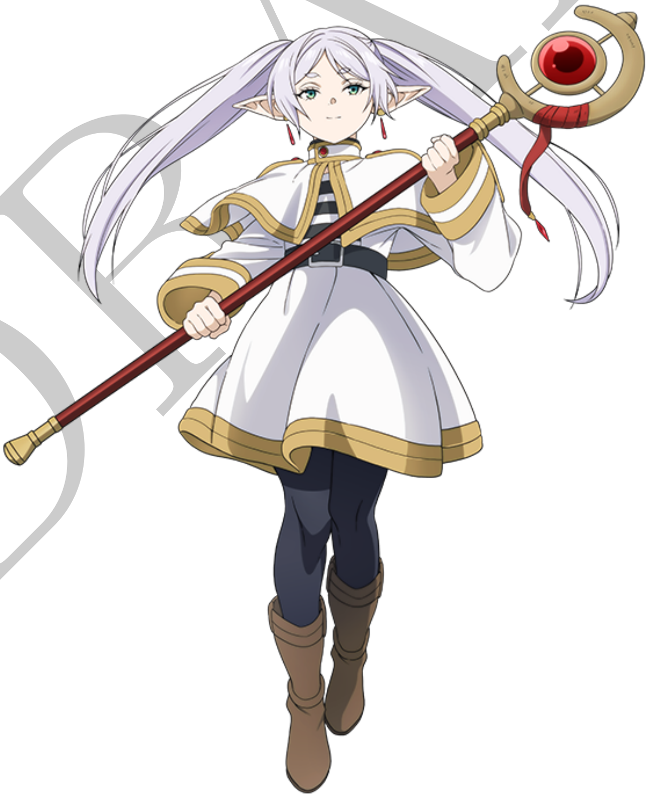
Abelian Varieties



“如果是勇者辛美尔，他一定会这么做的！”

Contents

1 The First Properties of Abelian Varieties

1.1 Definition and examples of Abelian Varieties

Definition 1.1. Let $\mathbf{k}$ be a field. An *abelian variety over* $\mathbf{k}$ is a proper variety A over $\mathbf{k}$ together with morphisms *identity* $e : \operatorname{Spec}\mathbf{k} \to A$, *multiplication* $m : A \times A \to A$ and *inversion* $i : A \to A$ such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd}
& A \times A \times A \arrow[dl, "\mathrm{id}_A \times m"'] \arrow[dr, "m \times \mathrm{id}_A"] & \\
A \times A \arrow[dr, "m"] & & A \times A \arrow[dl, "m"'] \\
& A &
\end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd}
A \times \operatorname{Spec}\mathbf{k} \arrow[r, "\mathrm{id}_A \times e"] \arrow[dr, "\cong"'] & A \times A \arrow[d, "m"] & \operatorname{Spec}\mathbf{k} \times A \arrow[l, "e \times \mathrm{id}_A"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd}
& A \arrow[dl, "\mathrm{id}_A \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_A"] & \\
A \times A \arrow[dr, "m"'] & \operatorname{Spec}\mathbf{k} \arrow[d, "e"] & A \times A \arrow[dl, "m"] \\
& A &
\end{tikzcd}.$$

In other words, an abelian variety is a group object in the category of proper varieties over $\mathbf{k}$.

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Example 1.2. Let E be an elliptic curve over a field $\mathbf{k}$. Then E is an abelian variety of dimension 1. Yang: To be completed.

In the following, we will always assume that A is an abelian variety over a field $\mathbf{k}$ of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The *left translation* by $a \in A(\mathbf{k})$ is defined as

$$l_a : A \xrightarrow{\cong} \operatorname{Spec}\mathbf{k} \times A \xrightarrow{a \times \mathrm{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a.

Proposition 1.3. Let A be an abelian variety. Then A is smooth.

Proof. By base changing to the algebraic closure of $\mathbf{k}$, we may assume that $\mathbf{k}$ is algebraically closed. Note that there is a non-empty open subset $U \subset A$ which is smooth. Then apply the left translation morphism l_a. □

Proposition 1.4. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. Yang: But how to show it is a morphism of varieties? Yang: To be completed. □

Theorem 1.5. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism, i.e., for every $\mathbf{k}$-scheme T, the induced map $f_T : A(T) \to B(T)$ is a group homomorphism.

Proof. Let $\bar{\mathbb{k}}$ be the algebraical closure of $\mathbf{k}$. For every $\mathbf{k}$-scheme T, we have the inclusion $A(T) \subset A_{\bar{\mathbb{k}}}(T_{\bar{\mathbb{k}}})$ and $B(T) \subset B_{\bar{\mathbb{k}}}(T_{\bar{\mathbb{k}}})$ which is compatible with the group structure and the morphism f. Thus we may assume that $\mathbf{k}$ is algebraically closed.

For every $a \in A(\mathbf{k})$, the fiber $m_A^{-1}(a)$ is isomorphic to A via the projection to the first factor. In particular, $m_A^{-1}(a)$ is connected.

Consider the composition

$$A \times A \xrightarrow{\varphi} A \times A \xrightarrow{m_A} A, \quad (x, y) \mapsto (x, m_A(i_A(x), y)) \mapsto m_A(x, m_A(i_A(x), y)) = y.$$

Hence we have $(m_A \circ \varphi)_* \mathcal{O}_{A\times A} \cong \mathcal{O}_A \cong m_{A*}\mathcal{O}_{A\times A}$ since φ is an isomorphism. Then consider the diagram

$$\begin{tikzcd} A \times A \arrow[r, "f\times f"] \arrow[d, "m_A"'] & B \times B \arrow[d, "m_B"] \\ A & B. \end{tikzcd}$$

For every closed point $a \in A$, the fiber $m_A^{-1}(a) = \{(x, m_A(i_A(x), a)) | x \in A\}$ is contrac Yang: To be completed. □

Proposition 1.6. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 1.5. □

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

1.2 Complex abelian varieties

Theorem 1.7. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if Λ Yang: To be completed.

2 Picard Groups of Abelian Varieties

Let $\mathbf{k}$ be a field and $\bar{\mathbf{k}}$ its algebraic closure. Let A be an abelian variety over $\mathbf{k}$.

2.1 Pullback along group operations

Theorem 2.1 (Theorem of the cube)**.** Let X, Y, Z be proper varieties over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\}\times Y\times Z}$, $\mathcal{L}|_{X\times\{y\}\times Z}$ and $\mathcal{L}|_{X\times Y\times\{z\}}$ are trivial. Then $\mathcal{L}$ is trivial.

Proof. Yang: To be completed. □

Remark 2.2. If we assume the existence of the Picard scheme, then the Theorem 2.1 can be deduced from the Rigidity Lemma. Consider the morphism

$$\varphi : X \times Y \to \mathrm{Pic}(Z), \quad (x, y) \mapsto \mathcal{L}|_{\{x\}\times\{y\}\times Z}.$$

Since $\varphi(x, y) = \mathcal{O}_Z$, φ factors through $\mathrm{Pic}^0(Z)$. Then the assumption implies that φ contracts $\{x\}\times Y$, $X \times \{y\}$ and hence it maps $X \times Y$ to a point. Thus $\varphi(x', y') = \mathcal{O}_Z$ for every $(x', y') \in X \times Y$. Then by Grauert's theorem, we have $\mathcal{L} \cong p^* p_* \mathcal{L}$ where $p : X \times Y \times Z \to X \times Y$ is the projection. Note that $p_*\mathcal{L} \cong \mathcal{L}|_{X\times Y\times\{z\}} \cong \mathcal{O}_{X\times Y}$. Hence $\mathcal{L}$ is trivial.

Lemma 2.3. Let A be an abelian variety over $\mathbf{k}$, $f, g, h : X \to A$ morphisms from a variety X to A and $\mathcal{L}$ a line bundle on A. Then we have

$$(f + g + h)^*\mathcal{L} \cong (f + g)^*\mathcal{L} \otimes (f + h)^*\mathcal{L} \otimes (g + h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof. First consider $X = A\times A\times A$, $p : X \to A, (x_1, x_2, x_3) \mapsto x_1+x_2+x_3$, $p_{ij} : X \to A, (x_1, x_2, x_3) \mapsto x_i + x_j$ for $1 \le i < j \le 3$ and $p_i : X \to A, (x_1, x_2, x_3) \mapsto x_i$ for $1 \le i \le 3$. Then the conclusion follows from the theorem of the cube by taking $\mathcal{L}' = p^*\mathcal{L}^{-1} \otimes p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \otimes p_3^*\mathcal{L}^{-1}$

and considering the restriction to $\{0\} \times A \times A$, $A \times \{0\} \times A$ and $A \times A \times \{0\}$.

In general, consider the morphism $\varphi = (f, g, h) : X \to A \times A \times A$ and pull back the above isomorphism along φ. □

Proposition 2.4. Let A be an abelian variety over $\mathbf{k}$, $n \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle on A. Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes \frac{1}{2}(n^2-n)}.$$

Proof. For $n = 0, 1$, the conclusion is trivial. For $n \geq 2$, we can use the previous lemma on $[n-2]_A, [1]_A, [1]_A$ and induct on n. Hence we have

$$[n]_A^*\mathcal{L} \cong [n-1]_A^*\mathcal{L} \otimes [n-1]_A^*\mathcal{L} \otimes [2]_A^*\mathcal{L} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [n-2]_A^*\mathcal{L}^{-1}.$$

Then the conclusion follows from induction. □

Definition 2.5. Let A be an abelian variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on A. We say that $\mathcal{L}$ is *symmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}$ and *antisymmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}^{-1}$.

Theorem 2.6 (Theorem of the square)**.** Let A be an abelian variety over $\mathbf{k}$, $x, y \in A(\mathbf{k})$ two points and $\mathcal{L}$ a line bundle on A. Then

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Proof. Yang: To be completed. □

Remark 2.7. We can define a map

$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \to \mathrm{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary $\mathcal{L}$, the map

$$\Phi_{\square} : \mathrm{Pic}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \mathrm{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$

is also a group homomorphism.

If we assume the scheme structure on $\mathrm{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\mathrm{Pic}^0(A)$. Let $K(\mathcal{L}) := \mathrm{Ker}\,\Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A. We give another description of $K(\mathcal{L})$. From this point, when $K(\mathcal{L})$ is finite, we can recover the dual abelian variety $A^\vee = \mathrm{Pic}^0_{A/\mathbf{k}}$ as the quotient $A/K(\mathcal{L})$.

2.2 Projectivity

In this subsection, we work over the algebraically closed field $\Bbbk$.

Proposition 2.8. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. Then $|2D|$ is base point free.

Proof. Yang: To be completed. □

Theorem 2.9. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. TFAE:

(a) the stabilizer $\mathrm{Stab}(D)$ of D is finite;

(b) the morphism $\phi_{|2D|}$ induced by the complete linear system $|2D|$ is finite;

(c) D is ample;

(d) $K(\mathcal{O}_A(D))$ is finite.

Proof. Yang: To be completed. □

Theorem 2.10. Let A be an abelian variety over $\Bbbk$. Then A is projective.

Proof. Yang: To be completed. □

Corollary 2.11. Let A be an abelian variety over $\Bbbk$ and D a divisor on A. Then D is pseudo-effective if and only if it is nef, i.e. $\mathrm{Psef}^1(A) = \mathrm{Nef}^1(A)$.

Proof. Yang: To be completed. □

2.3 Dual abelian varieties

In this subsection, we work over the algebraically closed field $\Bbbk$.

Proposition 2.12. Let A be an abelian variety over $\mathbf{k}$ and $\mathcal{L}$ an ample line bundle on A. Then the homomorphism $\Phi_{\mathcal{L}} : A(\Bbbk) \to \mathrm{Pic}(A)$ factors through $\mathrm{Pic}^0(A)$ and $A(\Bbbk) \to \mathrm{Pic}^0(A)$ is surjective.

Proof. Yang: To be completed. □

Definition 2.13. Let A be an abelian variety over $\Bbbk$. We define the *dual abelian variety* of A to be $A/K(\mathcal{L})$ for some ample line bundle $\mathcal{L}$ on A. We denote it by $A^\vee$.

Theorem 2.14. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ does not depend on the choice of the ample line bundle $\mathcal{L}$. Moreover, there is a natural bijection $A^\vee(\mathbf{k}) \to \mathrm{Pic}^0(A)$.

Proof. Yang: To be completed. □

Proposition 2.15. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ is also an abelian variety and the natural map $A \to A^{\vee\vee}$ is an isomorphism.

Proof. Yang: To be completed. □

Proposition 2.16. There exists a unique line bundle $\mathcal{P}$ on $A \times A^\vee$ such that for every $y = \mathcal{L} \in A^\vee = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{y\}} \cong \mathcal{L}$.

Proof. Yang: To be completed. □

2.4 The Néron-Severi group

Theorem 2.17. Let A be an abelian variety over $\Bbbk$. The we have an inclusion $\mathrm{NS}(A) \hookrightarrow \mathrm{Hom}_{\mathbf{Grp}}(A, A^\vee)$ given by Yang: To be completed.