The Quot functor

1 Definitions and examples

Definition 1. Let S be a noetherian scheme, $(X, \mathcal{O}(1))$ a projective scheme over S, and $\mathcal{E}$ a vector bundle on X. For a polynomial $P \in \mathbb{Q}[\lambda]$, The *Quot functor* $\mathfrak{Quot}_{\mathcal{E}/X/S,P} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ is defined as

$$T \mapsto \left\{ \mathcal{E}_{X_T} \to \mathcal{Q} \to 0 \text{ on } X_T \;\middle|\; \mathcal{Q} \text{ is flat over } T, \text{ and } P_{\mathcal{Q}|_{X_\xi}} = P,\ \forall \xi \in T \right\} / \sim,$$

where two quotients $\mathcal{E}_{X_T} \to \mathcal{Q} \to 0$ and $\mathcal{E}_{X_T} \to \mathcal{Q}' \to 0$ are equivalent if there is an isomorphism $\mathcal{Q} \cong \mathcal{Q}'$ making the diagram

$$\begin{tikzcd} \mathcal{E}_{X_T} \arrow[r] \arrow[dr] & \mathcal{Q} \arrow[d, "\cong"] \\ & \mathcal{Q}' \end{tikzcd}$$

commute.

The main goal of this section is to prove the following representability theorem.

Theorem 2. The Quot functor $\mathfrak{Quot}_{\mathcal{E}/X/S,P}$ is representable by a projective S-scheme $\mathrm{Quot}_{\mathcal{E}/X/S,P}$ and a universal quotient $p_X^*\mathcal{E} \to \mathcal{Q} \to 0$ on $X \times_S \mathrm{Quot}_{\mathcal{E}/X/S,P}$. Yang: To be checked.

Many important moduli spaces can be realized as special cases of the Quot scheme.

Grassmannian scheme The first example is the Grassmannian scheme.

Definition 3. Let S be a noetherian scheme and $\mathcal{E}$ a vector bundle of rank n on S. The *Grassmannian functor* $\mathfrak{Grass}_{\mathcal{E},r} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ is defined as

$$T \mapsto \{ \mathcal{E}_T \to \mathcal{Q} \to 0 \mid \mathcal{Q} \text{ locally free of rank } r \text{ on } T \} / \sim,$$

where two quotients $\mathcal{E}_T \to \mathcal{Q} \to 0$ and $\mathcal{E}_T \to \mathcal{Q}' \to 0$ are equivalent if there is an isomorphism $\mathcal{Q} \cong \mathcal{Q}'$ making the diagram

$$\begin{tikzcd} \mathcal{E}_{T} \arrow[r] \arrow[dr] & \mathcal{Q} \arrow[d, "\cong"] \\ & \mathcal{Q}' \end{tikzcd}$$

commute.

Let $i : \xi \to S$ be a point of S. Then the fiber $\mathcal{Q}|_\xi = i^*\mathcal{Q}$ is a vector space over the residue field $\kappa(\xi)$ of dimension r. By taking $(X, \mathcal{O}(1)) = (S, \mathcal{O}_S)$ and $P(\lambda) = r$, the Grassmannian functor $\mathfrak{Grass}_{\mathcal{E},r}$ is a special case of the Quot functor $\mathfrak{Quot}_{\mathcal{E}/X/S,P}$.

Let us further specialize to the case where $S = \operatorname{Spec} \mathbf{k}$ for a field $\mathbf{k}$ and $\mathcal{E} = V = \mathbf{k}^{\oplus n}$ is a finite-dimensional $\mathbf{k}$-vector space. Note that $V \to W \to 0$ is equivalent to $V \to W' \to 0$ if and only if $\ker(V \to W) = \ker(V \to W')$. Hence in this case, the Grassmannian functor $\mathfrak{Grass}_{\mathcal{E},r}$ becomes the

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classical Grassmannian variety $\mathbf{Gr}(n-r, V)$ parameterizing $n-r$-dimensional subspaces of $\mathbf{k}^{\oplus n}$.

More specially, when $r=1$ or $r=n-1$, the Grassmannian variety $\mathbf{Gr}(n-r, V)$ is the projective space $\mathbb{P}^{n-1}_{\mathbf{k}}$. However, although the space is the same, the universal object is different. When $r=1$, i.e., $\mathbb{P}^{n-1}_{\mathbf{k}}$ parameterizes quotients $V \to W \to 0$ where W is a one-dimensional vector space, the universal object is

$$\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}} \cdot e_i \to \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}}(1) \to 0, \quad e_i \mapsto x_i,$$

where $x_1, \ldots, x_n$ are the homogeneous coordinates on $\mathbb{P}^{n-1}_{\mathbf{k}}$. When $r=n-1$, i.e., $\mathbb{P}^{n-1}_{\mathbf{k}}$ parameterizes quotients $V \to W \to 0$ where W is an $(n-1)$-dimensional vector space, the universal object is

$$0 \to \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}}(-1) \xrightarrow{\varphi} \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}} \cdot e_i \to \mathcal{Q} \to 0, \quad \varphi(1/x_i) = e_i,$$

where we view $\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbf{k}}}(-1)$ locally generated by $1/x_i$ on the chart $x_i \neq 0$.

When we say $\mathbb{P}(V)$, we usually mean the case $r=1$. This is also called the projectivization of the vector space V of hyperplanes in V or in the sense of Grothendieck. Under this convention, the functor $\mathbb{P} : \mathbf{vect}_{\mathbf{k}} \to \mathbf{Sch}_{\mathbf{k}}$ sending a finite-dimensional vector space V to the projective space $\mathbb{P}(V)$ is contravariant. Hence one should view V as the space of linear functions rather than points.

Yang: To be continued, describe $\mathbb{P}^n_S$ for general S.

Hilbert scheme Another important example is the Hilbert scheme.

Definition 4. Let S be a noetherian scheme and X a projective scheme over S. For a polynomial $P \in \mathbb{Q}[\lambda]$, the *Hilbert functor* $\mathfrak{Hilb}_{X/S,P} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ is defined as

$$T \mapsto \left\{ Y \subseteq X_T \;\middle|\; Y \to T \text{ is flat and for all } \xi \in T,\ P_{\mathcal{O}_{Y_\xi}} = P \right\},$$

where Y_ξ is the fiber of Y over the point ξ and $P_{\mathcal{O}_{Y_\xi}}$ is the Hilbert polynomial of $\mathcal{O}_{Y_\xi}$ with respect to $\mathcal{O}_X(1)|_{X_\xi}$.

By taking $\mathcal{E} = \mathcal{O}_X$ and noting that a quotient $\mathcal{O}_{X_T} \to \mathcal{Q} \to 0$ corresponds to a closed subscheme $Y = \mathcal{S}pec_{X_T} \mathcal{Q} \subseteq X_T$, we see that the Hilbert functor $\mathfrak{Hilb}_{X/S,P}$ is a special case of the Quot functor $\mathfrak{Quot}_{\mathcal{E}/X/S,P}$.

Yang: To be continued

Morphisms space Yet another example is the morphisms space.

Definition 5. Let S be a noetherian scheme, X, Y projective schemes over S, and $f : X \to Y$ a morphism over S. The *functor of morphism space through* f $\mathfrak{Mor}_{(X,Y)/S,f} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ is defined as

$$T \mapsto \left\{ g_T : X_T \to Y_T \text{ over } T \;\middle|\; \Gamma_g \text{ is flat over } T \text{ and } P_{\Gamma_{g_\xi}} = P_{\Gamma_{f_\xi}} \text{ for all } \xi \in T \right\},$$

where $X_T = X \times_S T$ and $Y_T = Y \times_S T$. Yang: To be revised.

2 Castelnuovo-Mumford regularity

3 Construction of Quot scheme

Appendix