Introduction to Moduli Problems

Let C be a smooth projective curve of genus g over an algebraically closed field $\Bbbk$ of characteristic 0.

We are interested in the moduli space of vector bundles on C.

1 Moduli functors

Let S be a noetherian scheme and T is a scheme of finite type over S. Recall the Yoneda lemma: there is a full and faithful functor

$$h : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathrm{Fun}((\mathbf{Sch}_S)^{\mathrm{op}}, \mathbf{Set}), \quad T \mapsto h_T(S) := \mathrm{Hom}_{\mathbf{Sch}_S}(T, S).$$

A functor $F : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$ is *representable* if there exists a scheme M over S such that $F \cong h_M$. We say that M is *the fine moduli space* of F.

Remark 1. If F is representable by M, then there is a universal object $\mathcal{U} \in F(M)$ given by $\mathrm{id}_M \in h_M(M)$ satisfying the following universal property: for any $T \in \mathbf{Sch}_S$ and any $\xi \in F(T)$, there exists a unique morphism $f : T \to M$ such that $F(f)(\mathcal{U}) = \xi$.

The most famous example of representable functor is the Quot functor. Let S be a noetherian scheme, $\pi : X \to S$ a projective morphism, $\mathcal{L}$ a relatively ample line bundle on X, $\mathcal{F}$ a coherent sheaf on X, and $P \in \mathbb{Q}[t]$ a polynomial. We define a functor

$$\begin{aligned} \mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{F}/X/S} :& (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set} \\ & T \mapsto \{\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q} \mid \mathcal{Q} \text{ is flat over } T, \forall t \in T, \mathcal{Q}_t \text{ has Hilbert polynomial } P\}/\sim, \end{aligned}$$

where $\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}$ and $\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}'$ are equivalent if $\mathrm{Ker}(\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}) = \mathrm{Ker}(\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}')$.

By Grothendieck, $\mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{F}/X/S}$ is representable by a projective S-scheme $\mathrm{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/S}$. Yang: Reference...

If we take $S = \mathrm{Spec}\,\Bbbk$, X a projective variety and $\mathcal{F} = \mathcal{O}_X$. Then the Quot functor $\mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{O}_X/X/\Bbbk}$ becomes the Hilbert functor $\mathcal{H}ilb^{P,\mathcal{L}}_{X/\Bbbk}$, which is representable by a projective $\Bbbk$-scheme called the *Hilbert scheme* $\mathrm{Hilb}^{P,\mathcal{L}}_X$.

2 Moduli functor of vector bundles

Consider the functor

$$\begin{aligned} \tilde{\mathcal{M}}_{r,d} :& (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set} \\ & T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a vector bundle on } X \times T \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\}/\sim, \end{aligned}$$

where $\mathcal{E} \sim \mathcal{E}'$ if there exists a line bundle $\mathcal{L}$ on T such that $\mathcal{E}' \cong \mathcal{E} \otimes \pi_T^*\mathcal{L}$, where $\pi_T : X \times T \to T$ is the projection.

Unfortunately, $\tilde{\mathcal{M}}_{r,d}$ is not representable. There are two main reasons:

- unboundedness and
- jumping phenomenon.

Date: October 2, 2025, **Author**: Tianle Yang, My Homepage

Definition 2. A family of vector bundles on a variety X is *bounded* if there exists a scheme S of finite type over $\Bbbk$ and a vector bundle $\mathcal{E}$ on $X \times S$ such that every vector bundle in the family is isomorphic to $\mathcal{E}_s$ for some $s \in S$.

If $\tilde{\mathcal{M}}_{r,d}$ is representable by a scheme M of finite type over $\Bbbk$, then the family of vector bundles parametrized by M is bounded. This is impossible since if so, $\{h^0(X, \mathcal{E}) \mid \mathcal{E} \in \tilde{\mathcal{M}}_{r,d}(\Bbbk)\}$ is bounded by semicontinuity theorem, which is not true. For example, consider the family $\mathcal{E}_n = \mathcal{O}_X(nP) \oplus \mathcal{O}_X(-nP) \in \tilde{\mathcal{M}}_{2,0}(\Bbbk)$ for $n \geq 0$, where $P \in X(\Bbbk)$ is a fixed point. By Riemann-Roch theorem, we have $h^0(X, \mathcal{E}_n) = n + 1 - g$ for n sufficiently large.

Example 3. Let us see a jumping phenomenon example due to Ress. Let $\mathcal{E}$ be a vector bundle on X of rank r and degree d with a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}.$$

On $X \times \mathbb{A}^1$, we can construct a vector bundle $\mathcal{F}$ by "deforming" $\mathcal{E}$ to $\bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}$ as follows: let t be the coordinate of $\mathbb{A}^1$, and define $\mathcal{F}$ to be the subsheaf of $\pi_X^*\mathcal{E}$ generated by $t^{-i} \cdot \pi_X^*\mathcal{E}_i$ for $1 \leq i \leq r$. Then $\mathcal{F}$ is a vector bundle on $X \times \mathbb{A}^1$ of rank r and degree d. We have

$$\mathcal{F}_t \cong \begin{cases} \mathcal{E}, & t \neq 0, \\ \bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}, & t = 0. \end{cases}$$

This is called the *jumping phenomenon*. Yang: To be checked...

For a concrete example, let $X = \mathbb{P}^1$, we have an exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O} \to 0$$

by Proposition 17.

Fix the standard coordinate $\mathbb{P}^1 = \operatorname{Proj} \Bbbk[X_0, X_1]$ and let $e_0 = (1, 0)$, $e_1 = (0, 1)$ be the standard basis of $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. On the open subset $U_i = \{X_i \neq 0\}$, fix a trivialization $\mathcal{O}(-1) \cong \mathcal{O}_{U_i} \cdot \frac{1}{X_i}$. Recall that $\mathcal{O}(-2) \subset \mathcal{E}$ is generated by $(X_1 e_0 - X_0 e_1)/X_i^2$ on U_i for $i = 0, 1$ and $\mathcal{E} \to \mathcal{O}$ is given by $e_0 \mapsto X_0$, $e_1 \mapsto X_1$.

Let $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and consider the filtration $F : 0 \subset \mathcal{O}(-2) \subset \mathcal{E}$. Yang: To be continued...

If $\tilde{\mathcal{M}}_{r,d}$ is representable by a scheme M, then the family of vector bundles parametrized by M does not have jumping phenomenon. Indeed, if $\mathcal{F}$ is an vector bundle on $X \times \mathbb{A}^1$ such that $\mathcal{F}_t \cong \mathcal{E}$ for $t \neq 0$, then by the universal property of M, there exists a unique morphism $f : \mathbb{A}^1 \to M$ such that $\mathcal{F} \cong (\mathrm{id}_X \times f)^*\mathcal{U}$, where $\mathcal{U}$ is the universal vector bundle on $X \times M$. Since f is constant on the open subset $\mathbb{A}^1 \setminus \{0\}$, it is constant on $\mathbb{A}^1$. Thus, $\mathcal{F}_0 \cong \mathcal{E}$.

To fix the above problems, we need to

- restrict to a smaller family of vector bundles,
- kill jumping phenomenon, and
- weaken the notion of representability.

Definition 4. Let $F : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$ be a functor, M a scheme over S, and $\eta : F \to h_M$ a natural transformation. We say that (M, η) *corepresents* F if for any scheme N over S and any natural transformation $\eta' : F \to h_N$, there exists a unique morphism $f : M \to N$ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & h_M \arrow[d, "h_f"] \\ & h_N. \end{tikzcd}$$

Definition 5. A scheme M over S is called the *coarse moduli space* of F if

(a) there exists a natural transformation $\eta : F \to h_M$ such that (M, η) corepresents F;

(b) $\eta_{\mathbb{k}} : F(\mathbb{k}) \to M(\mathbb{k})$ is a bijection.

Yang: To be continued...

3 Semistable vector bundles

Definition 6. Let C be a smooth projective curve over $\mathbb{k}$. For a vector bundle $\mathcal{E}$ of rank r and degree d on C, we define its slope to be $\mu(\mathcal{E}) \coloneqq d/r$.

Proposition 7. Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ be an exact sequence of non-zero vector bundles on C. Then $\mu(\mathcal{E}_2) \geq \mu(\mathcal{E}_1)$ (resp. $\mu(\mathcal{E}_2) > \mu(\mathcal{E}_1)$) if and only if $\mu(\mathcal{E}_2) \leq \mu(\mathcal{E}_3)$ (resp. $\mu(\mathcal{E}_2) < \mu(\mathcal{E}_3)$).

Proof. We have

$$\mu(\mathcal{E}_2) = \frac{\deg \mathcal{E}_2}{\operatorname{rank} \mathcal{E}_2} = \frac{\deg \mathcal{E}_1 + \deg \mathcal{E}_3}{\operatorname{rank} \mathcal{E}_1 + \operatorname{rank} \mathcal{E}_3}.$$

Note that for any $a, b, c, d \in \mathbb{R}_{>0}$, we have

$$\frac{a+c}{b+d} \geq \frac{a}{b} \iff bc \geq ad \iff \frac{a+c}{b+d} \leq \frac{c}{d}.$$

The strict inequality case is similar. Then the proposition follows. □

Definition 8. Let C be a smooth projective curve over $\mathbb{k}$ and $\mathcal{E}$ a vector bundle on C. We say that $\mathcal{E}$ is *stable* (resp. *semistable*) if for any proper sub-bundle $\mathcal{F} \subset \mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$).

Proposition 9. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on C. Suppose that they are semistable and $\mu(\mathcal{E}) > \mu(\mathcal{F})$. Then any homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ is zero.
Suppose that they are stable and $\mu(\mathcal{E}) = \mu(\mathcal{F})$. Then any non-zero homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ is an isomorphism.

Proof. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a non-zero homomorphism of vector bundles on C. We have an exact sequence

$$0 \to \operatorname{Ker} \varphi \to \mathcal{E} \to \operatorname{Im} \varphi \to 0.$$

Since $\mathcal{F}$ is vector bundle, hence torsion-free, $\operatorname{Im}\varphi$ is also torsion-free, thus a vector bundle.

If $\mathcal{E}$ and $\mathcal{F}$ are semistable with $\mu(\mathcal{E}) > \mu(\mathcal{F})$, clearly $\operatorname{Ker}\varphi \neq 0$, then by Proposition 7, we have

$$\mu(\mathcal{E}) \leq \mu(\operatorname{Im}\varphi) \leq \mu(\mathcal{F}).$$

This is a contradiction, thus $\varphi = 0$.

Suppose that $\mathcal{E}$ and $\mathcal{F}$ are stable with $\mu(\mathcal{E}) = \mu(\mathcal{F})$. If $\operatorname{Ker}\varphi \neq 0$, then by Proposition 7, we have

$$\mu(\mathcal{E}) < \mu(\operatorname{Im}\varphi) \leq \mu(\mathcal{F}).$$

This is a contradiction, thus φ is injective. Since $\mathcal{F}$ is stable and $\operatorname{Im}\varphi \subset \mathcal{F}$ has the same slope as $\mathcal{F}$, we have $\operatorname{Im}\varphi = \mathcal{F}$. □

Corollary 10. A stable vector bundle is simple as a coherent sheaf, i.e., $\operatorname{End}(\mathcal{E}) \cong \Bbbk$.

Proof. Let $\varphi \in \operatorname{End}(\mathcal{E})$ be a non-zero endomorphism. Then there exists $P \in C(\Bbbk)$ such that $\varphi_P : \mathcal{E}_P \to \mathcal{E}_P$ is non-zero. Let $a \in \Bbbk$ be an eigenvalue of φ_P and consider the endomorphism $\varphi - a \cdot \mathrm{id}_{\mathcal{E}}$. Then $(\varphi - a \cdot \mathrm{id}_{\mathcal{E}})_P : \mathcal{E}_P \to \mathcal{E}_P$ is not an isomorphism, so is $\varphi - a \cdot \mathrm{id}_{\mathcal{E}}$. By Proposition 9, $\varphi - a \cdot \mathrm{id}_{\mathcal{E}} = 0$, thus $\varphi = a \cdot \mathrm{id}_{\mathcal{E}}$. □

Lemma 11. Let $\mathcal{E}$ be a semistable vector bundle on X.

(a) if $\mu(\mathcal{E}) > 2g - 2$, then $H^1(X, \mathcal{E}) = 0$;

(b) if $\mu(\mathcal{E}) > 2g - 1$, then $\mathcal{E}$ is globally generated.

Proof. Yang: To be continued... □

Let $S_{r,d}$ be set of isomorphism classes of semistable vector bundles on X of rank r and degree d.

Proposition 12. The family $S_{r,d}$ is bounded.

Proof. Yang: To be continued... □

Definition 13 (Jordan-Hölder filtration)**.** Let $\mathcal{E}$ be a semistable vector bundle on C. A *Jordan-Hölder filtration* of $\mathcal{E}$ is a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ are stable with $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E})$ for all $1 \leq i \leq n$.

Proposition 14. Any semistable vector bundle on C admits a Jordan-Hölder filtration. Moreover, the associated graded object

$$\operatorname{gr}(\mathcal{E}) \coloneqq \bigoplus_{i=1}^{n} \mathcal{E}_i/\mathcal{E}_{i-1}$$

is independent of the choice of Jordan-Hölder filtration up to isomorphism.

Proof. Yang: To be continued... □

Definition 15 (S-equivalence)**.** Two semistable vector bundles $\mathcal{E}$ and $\mathcal{F}$ of the same rank and degree on C are called *S-equivalent* if their associated graded objects $\mathrm{gr}(\mathcal{E})$ and $\mathrm{gr}(\mathcal{F})$ (from their Jordan-Hölder filtrations) are isomorphic.

Definition 16. We define a functor

$$\begin{aligned}\mathcal{M}^{ss}_{r,d} : &(\mathbf{Sch}_{\Bbbk})^{\mathrm{op}} \to \mathbf{Set} \\ &T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a family of semistable vector bundles on } X \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\}/\sim,\end{aligned}$$

where $\mathcal{E} \sim \mathcal{E}'$ if for any $t \in T$, the vector bundles $\mathcal{E}_t$ and $\mathcal{E}'_t$ are S-equivalent or Yang: .... Yang: To be continued...

Requirements

Proposition 17. Let $\mathbb{P}^n_R$ be the projective space of dimension n over a ring R. Then we have the following exact sequence of vector bundles on $\mathbb{P}^n_R$:

$$0 \to \Omega_{\mathbb{P}^n_R/R} \to \mathcal{O}_{\mathbb{P}^n_R}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_R} \to 0.$$

Proof. Fixing a non-zero element in $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(1))$, this gives a homomorphism $\mathcal{O}_{\mathbb{P}^n_R} \to \mathcal{O}_{\mathbb{P}^n_R}(1)$. Twisting by $\mathcal{O}_{\mathbb{P}^n_R}(-1)$, we get a homomorphism $\mathcal{O}_{\mathbb{P}^n_R}(-1) \to \mathcal{O}_{\mathbb{P}^n_R}$. Yang: To be continued... □