Preliminaries in Category Theory

1 Sites

Definition 1. Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of sets of arrows $\{U_i \to U\}_{i\in I}$, called *covering*, for each object U in $\mathbf{C}$ such that:

(a) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering;

(b) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a arrow, then the fiber product $U_i \times_U V \to V$ exists and $\{U_i \times_U V \to V\}$ is a covering of V;

(c) if $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ are coverings, then the collection of composition $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \jmath)$ where $\mathbf{C}$ is a category and $\jmath$ is a Grothendieck topology on $\mathbf{C}$.

Note that sheaf is indeed defined on a site.

Definition 2. Let $(\mathbf{C}, \jmath)$ be a site. A *sheaf* on $(\mathbf{C}, \jmath)$ is a functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ satisfying the following condition: for every object U in $\mathbf{C}$ and every covering $\{U_i \to U\}_{i\in I}$ of U, if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every i, j, the pullback $s_i|_{U_i\times_U U_j}$ and $s_j|_{U_i\times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every i, the pullback $s|_{U_i} = s_i$.

Definition 3. Let X be a scheme. The *big étale site* of X, denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over X with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if and only if each U_i is étale over U and the union of their images is the whole U.

Let X be a scheme over S. By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \to \mathbf{Set}$ such that for any S-scheme T, $h_X(T) = \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$. Yang: Easy to check that h_X is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

2 Fibered categories and descent conditions

Definition 4. Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. A morphism $f : b \to a$ in $\mathbf{X}$ is called *strongly Cartesian* if for every object $c \in \mathrm{Obj}(\mathbf{X})$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{X}}(c,b) \arrow[r, "f\circ-"] \arrow[d, "\mathbf{p}"'] & \mathrm{Hom}_{\mathbf{X}}(c,a) \arrow[d, "\mathbf{p}"] \\ \mathrm{Hom}_{\mathbf{S}}(w,v) \arrow[r, "\mathbf{p}(f)\circ-"] & \mathrm{Hom}_{\mathbf{S}}(w,u) \end{tikzcd}$$

is a pullback of sets, where $u = \mathbf{p}(a), v = \mathbf{p}(b), w = \mathbf{p}(c)$.

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The condition in Definition 4 can be interpreted as follows: for any diagram as below black part with $\mathbf{p}(g) = \mathbf{p}(f) \circ \alpha$,

$$\begin{tikzcd}
c \arrow[drr, bend left, "g"] \arrow[dr, "h"] \arrow[dd, mapsto] & & \\
& b \arrow[r, "f"] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
w \arrow[r, "\alpha"] & v \arrow[r, "\mathbf{p}(f)"] & u
\end{tikzcd}$$

there exists a unique gray morphism $h : c \to a$ such that $\mathbf{p}(h) = \alpha$ and $f \circ h = g$.

Notation 5. Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a functor. For $a, b \in \mathsf{Obj}(\mathbf{X})$ and $f \in \mathrm{Hom}_{\mathbf{X}}(a, b)$, we say that a is *over* $\mathbf{p}(a)$ and f is *over* $\mathbf{p}(f)$. In a diagram, we have

$$\begin{tikzcd}
\mathbf{X} \arrow[d, "\mathbf{p}"'] \\
\mathbf{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\
\mathbf{p}(a) \arrow[r, "\mathbf{p}(f)"] & \mathbf{p}(b)
\end{tikzcd}$$

Definition 6. Let $\mathbf{S}$ be a category. A category $\mathbf{X}$ over $\mathbf{S}$ via $\mathbf{p}$ is called a *category fibred* over the site $\mathbf{S}$ if for every morphism $\iota : v \to u$ in $\mathbf{S}$ and every object $a \in \mathsf{Obj}(\mathbf{X})$ over u, there exists an object $b \in \mathsf{Obj}(\mathbf{X})$ over v and a strongly Cartesian morphism $f : b \to a$ over ι. Such an object b is called a *pullback* of a along ι, and is often denoted by $\iota^* a$.

Definition 7. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a category fibred over $\mathbf{S}$. For every object $u \in \mathsf{Obj}(\mathbf{S})$, the *fiber* of $\mathbf{X}$ over u is the category $\mathbf{X}_u$ given by

$$\mathsf{Obj}(\mathbf{X}_u) = \{a \in \mathsf{Obj}(\mathbf{X}) \mid \mathbf{p}(a) = u\}, \quad \mathrm{Hom}_{\mathbf{X}_u}(a, b) = \{f \in \mathrm{Hom}_{\mathbf{X}}(a, b) \mid \mathbf{p}(f) = \mathrm{id}_u\}.$$

Remark 8. Note that in Definition 6, the pullback $r^* b$ of an object b along a morphism r is not necessarily unique. Yang: To be continued.

Example 9. Let $\mathbf{S}$ be a category and $\mathcal{F} : \mathbf{S}^{op} \to \mathbf{Set}$ be a presheaf on $\mathbf{S}$ taking values in $\mathbf{Set}$. We can construct a category $\mathbf{F}$ fibred over $\mathbf{S}$ as follows:

- The objects of $\mathbf{F}$ are pairs (U, x) where $U \in \mathsf{Obj}(\mathbf{S})$ and $x \in \mathcal{F}(U)$;
- morphisms from (V, y) to (U, x) in $\mathbf{F}$ are morphisms $\iota : V \to U$ in $\mathbf{S}$ such that $\mathcal{F}(\iota)(x) = y$, denoted by res_ι.

The functor $\mathbf{p} : \mathbf{F} \to \mathbf{S}$ is defined by $\mathbf{p}(U, x) = U$ on objects and $\mathbf{p}(\iota) = \iota$ on morphisms. If one has the diagram

$$\begin{tikzcd}
(W, z) \arrow[drr, bend left, "\mathsf{res}_\tau"] \arrow[dd, mapsto] & & \\
& (V, y) \arrow[r, "\mathsf{res}_\iota"] \arrow[d, mapsto] & (U, x) \arrow[d, mapsto] \\
W \arrow[r, "\sigma"] & V \arrow[r, "\iota"] & U
\end{tikzcd}$$

with $\mathbf{p}(\mathsf{res}_\tau) = \iota \circ \sigma$. By definition, we have $\tau = \iota \circ \sigma$ and $\mathcal{F}(\tau)(x) = z, \mathcal{F}(\iota)(x) = y$. Thus, we have $\mathcal{F}(\sigma)(y) = z$. This verifies that res_σ is a strongly Cartesian morphism. Note that the fiber of $\mathbf{F}$ over an $U \in \mathrm{Obj}(\mathbf{S})$ is the discrete category associated to the set $\mathcal{F}(U)$. Therefore, presheaves of sets can be viewed as categories fibred in sets.

Conversely, given a category $\mathbf{F}$ fibred in sets over $\mathbf{S}$ via $\mathbf{p} : \mathbf{F} \to \mathbf{S}$, one can construct a presheaf of sets $\mathcal{F} : \mathbf{S}^{op} \to \mathbf{Set}$ by defining $\mathcal{F}(U) = \mathrm{Obj}(\mathbf{F}_U)$ for each $U \in \mathrm{Obj}(\mathbf{S})$, and for each morphism $\iota : V \to U$ in $\mathbf{S}$, defining $\mathcal{F}(\iota) : \mathcal{F}(U) \to \mathcal{F}(V)$ by sending an object $x \in \mathcal{F}(U)$ to its pullback $\iota^* x \in \mathcal{F}(V)$ along ι. This establishes an equivalence between presheaves of sets on $\mathbf{S}$ and categories fibred in sets over $\mathbf{S}$.

Example 10. Yang: case $\mathbf{S} = set, group$. To be added.

Slogan *Presheaves of sets are categories fibered in sets.*

In following, we describe categories fibered in groupoids.

Definition 11. Let $\mathbf{X}$ be a category fibred over a category $\mathbf{S}$ via $\mathbf{p} : \mathbf{X} \to \mathbf{S}$. For every $u \in \mathrm{Obj}(\mathbf{S})$ and every pair of objects a, b over u, we define the *presheaf of morphisms* $\mathrm{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ by

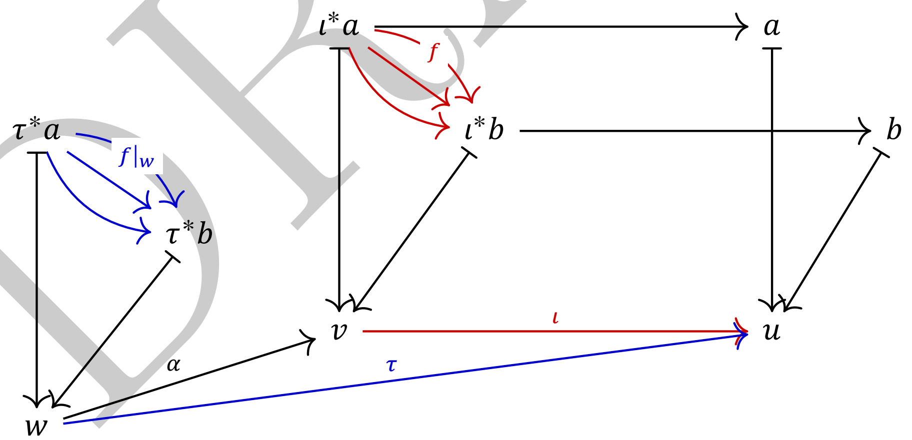
$$\mathrm{Hom}_{\mathbf{X}}(a, b)(\iota : v \to u) = \mathrm{Hom}_{\mathbf{X}_v}(\iota^* a, \iota^* b)$$

for every morphism $\iota : v \to u$ in $\mathbf{S}/u$. For a morphism $\alpha : w \to v$ in $\mathbf{S}/u$, the restriction map

$$\mathrm{Hom}_{\mathbf{X}}(a, b)(\iota) \to \mathrm{Hom}_{\mathbf{X}}(a, b)(\iota \circ \alpha)$$

is given by sending a morphism $f : \iota^* a \to \iota^* b$ in $\mathbf{X}_v$ to the pullback morphism Yang: $\alpha^* f : (\iota \circ \alpha)^* a \to (\iota \circ \alpha)^* b$ need to conjugate with a natural transformation. in $\mathbf{X}_w$. Yang: To be checked.

In a diagram, the presheaf of morphisms can be visualized as follows:



Proposition 12. Let $\mathbf{S}$ be a category and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a category fibred over $\mathbf{S}$. Then $\mathbf{X}$ is a category fibred in groupoids if and only if for every object $u \in \mathrm{Obj}(\mathbf{S})$ and every pair of objects a, b over u, the presheaf of morphisms $\mathrm{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf. Yang: To be checked.

Definition 13. Let $\mathbf{S}$ be a category. A category $\mathbf{X}$ fibred over $\mathbf{S}$ via $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *category fibred in groupoids* over $\mathbf{S}$ if for every object $u \in \mathrm{Obj}(\mathbf{S})$ and every pair of objects a, b over u, the presheaf of morphisms $\mathrm{Hom}_{\mathbf{X}}(a, b) : (\mathbf{S}/u)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf. Yang: To be checked.

Now let us discuss how sheaves fit into the framework of fibered categories. Of course, we need assume the base category $\mathbf{S}$ is a site. The glued condition for sheaves can be interpreted in terms of descent data in fibered categories.

Definition 14. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. Let $U \in \mathsf{Obj}(\mathbf{S})$ and $\{U_i \to U\}$ be a covering in $\mathbf{S}$. A *descent datum* for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ consists of

- a collection of objects $a_i \in \mathsf{Obj}(\mathbf{X}_{U_i})$ for each i,
- a collection of isomorphisms $\varphi_{ij} : a_j|_{U_{ij}} \to a_i|_{U_{ij}}$ in $\mathbf{X}_{U_{ij}}$ for each pair (i, j), where $U_{ij} = U_i \times_U U_j$,

such that the cocycle condition

$$\varphi_{ik}|_{U_{ijk}} = \varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}}$$

holds for all triples (i, j, k), where $U_{ijk} = U_i \times_U U_j \times_U U_k$. Yang: To be checked.

Example 15. Yang: To be added.

Definition 16. Let $\mathbf{S}$ be a site and $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ a fibered category over $\mathbf{S}$. A descent datum $(\{a_i\}, \{\varphi_{ij}\})$ for objects of $\mathbf{X}$ relative to a covering $\{U_i \to U\}$ in $\mathbf{S}$ is called *effective* if there exists an object $a \in \mathsf{Obj}(\mathbf{X}_U)$ and isomorphisms $\psi_i : a|_{U_i} \to a_i$ in $\mathbf{X}_{U_i}$ such that for all pairs (i, j), the diagram

$$\begin{tikzcd}
a|_{U_{ij}} \arrow[r, "\psi_j|_{U_{ij}}"] \arrow[d, "\psi_i|_{U_{ij}}"'] & a_j|_{U_{ij}} \arrow[d, "\varphi_{ij}"] \\
a_i|_{U_{ij}} \arrow[r, "\varphi_{ij}"'] & a_j|_{U_{ij}}
\end{tikzcd}$$

commutes. Yang: To be checked.

Slogan *Descent data are like gluing data for objects, and effectiveness means that the glued object exists.*

3 Prestacks and stacks

Definition 17. A *prestack* over the site $\mathbf{S}$ is a category $\mathbf{X}$ fibered in groupoids over $\mathbf{S}$.

Slogan *Prestacks are "presheaf remembering automorphisms".*

Example 18. presheaf is a prestack. Yang: To be added.

Example 19. The moduli problem of classifying algebraic curves of a fixed genus g can be formulated as a prestack over the site of schemes. Consider the category $\mathbf{M}_g$ whose objects are families of smooth projective curves of genus g over schemes, and whose morphisms are isomorphisms of such families. The functor $\mathbf{p} : \mathbf{M}_g \to \mathbf{Sch}$ sending a family of curves to its base scheme makes $\mathbf{M}_g$ a category fibred in groupoids over $\mathbf{Sch}$. For each scheme S, the fiber category $\mathbf{M}_{g,S}$ consists of families of smooth projective curves of genus g over S and their isomorphisms. The descent data for objects in $\mathbf{M}_g$ relative to a covering of schemes correspond to gluing families of curves along isomorphisms on overlaps, which is effective due to the nature of algebraic curves. Thus, $\mathbf{M}_g$ is a prestack over the site of schemes. Yang: To be revised.

Proposition 20. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$, $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$, and $\mathbf{r} : \mathbf{Z} \to \mathbf{S}$ be prestacks over $\mathbf{S}$. Let $\Phi : \mathbf{X} \to \mathbf{Z}$ and $\Psi : \mathbf{Y} \to \mathbf{Z}$ be morphisms of prestacks over $\mathbf{S}$. Then the fiber product $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ exists in the category of prestacks over $\mathbf{S}$. Yang: To be checked.

Definition 21. Let $\mathbf{S}$ be a site. A prestack $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ is called a *stack* over the site $\mathbf{S}$ if for every object $U \in \mathrm{Obj}(\mathbf{S})$ and every covering $\{U_i \to U\}$ in $\mathbf{S}$, the descent data for objects of $\mathbf{X}$ relative to the covering $\{U_i \to U\}$ are effective. Yang: To be revised.

Definition 22. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ and $\mathbf{q} : \mathbf{Y} \to \mathbf{S}$ be stacks over $\mathbf{S}$. A *morphism of stacks* $F : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$ is a functor $F : \mathbf{X} \to \mathbf{Y}$ such that $\mathbf{q} \circ F = \mathbf{p}$. Yang: To be checked.

Slogan *Stacks are to prestacks as sheaves are to presheaves.*

Example 23. Let X be a scheme over a base noetherian scheme S. The functor of points $h_X : (\mathbf{Sch}/S)^{\mathrm{op}}_{\text{ét}} \to \mathbf{Set}$ is a sheaf, and thus a stack.

Construction 24. Let $\mathbf{S}$ be a site, and let $\mathbf{p} : \mathbf{X} \to \mathbf{S}$ be a prestack over $\mathbf{S}$. There exists a stack $\mathbf{p}^+ : \mathbf{X}^+ \to \mathbf{S}$ over $\mathbf{S}$ together with a morphism of prestacks $F : \mathbf{X} \to \mathbf{X}^+$ over $\mathbf{S}$ satisfying the following universal property: for every stack $\mathbf{p}' : \mathbf{Y} \to \mathbf{S}$ over $\mathbf{S}$ and every morphism of prestacks $G : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$, there exists a unique morphism of stacks $G^+ : \mathbf{X}^+ \to \mathbf{Y}$ over $\mathbf{S}$ such that $G = G^+ \circ F$. The stack $\mathbf{X}^+$ is called the *stackification* of the prestack $\mathbf{X}$. Yang: To be checked.

Notation 25. As Example 9, we can associate a prestack $\mathbf{X}$ over a $\mathbf{S}$ to a functor $\mathcal{X} : \mathbf{S}^{\mathrm{op}} \to \mathbf{Grpd}$ by setting $\mathbf{X}_u = \mathcal{X}(u)$ for each $u \in \mathrm{Obj}(\mathbf{S})$ and defining the pullback functors accordingly. In particular, we can talk about representability of such prestacks. Yang: To be revised. Yang: Why do not we just talk about sheaves of groupoid?

Definition 26. Let $\mathbf{S}$ be a site, and let $\mathbf{X}, \mathbf{Y}$ be prestacks over $\mathbf{S}$. A morphism of prestacks $F : \mathbf{X} \to \mathbf{Y}$ over $\mathbf{S}$ is called *representable* if for every $\mathbf{Z} \to \mathbf{Y}$ over $\mathbf{S}$ with $\mathbf{Z}$ representable in $\mathbf{S}$, the fiber product $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}$ is representable in $\mathbf{S}$.

Appendix