Structure of linear algebraic groups I: commutative and solvable groups

Yang: In this section, everything is defined on an algebraically closed field $\Bbbk$.

1 Commutative algebraic groups and character groups

Definition 1. Let G be a linear algebraic group over a field $\Bbbk$. The *character group* of G, denoted by $\chi(G)$, is defined to be the group of all homomorphisms of linear algebraic groups from G to the multiplicative group $\mathbb{G}_m$:

$$\chi(G) = \mathrm{Hom}_{\mathbf{AlgGrp}_{\Bbbk}}(G, \mathbb{G}_m).$$

The group operation on $\chi(G)$ is given by pointwise multiplication of characters.

Example 2. Let us compute the character group of the multiplicative group $\mathbb{G}_m$. Let $\varphi : \mathbb{G}_m \to \mathbb{G}_m$ be a character of $\mathbb{G}_m$. Since φ is a morphism of algebraic varieties, it induces a homomorphism of coordinate rings $\varphi^\sharp : \Bbbk[T, T^{-1}] \to \Bbbk[T, T^{-1}]$. Note that $(\Bbbk[T, T^{-1}])^\times = \{cT^n \mid c \in \Bbbk^\times, n \in \mathbb{Z}\}$. Thus, we have $\varphi^\sharp(T) = cT^n$ for some $c \in \Bbbk^\times$ and $n \in \mathbb{Z}$. That is, $\varphi(a) = ca^n$ for all $a \in \mathbb{G}_m(\Bbbk) = \Bbbk^\times$. However, since φ is a group homomorphism, we must have $\varphi(1) = c = 1$. Therefore, $\varphi(a) = a^n$ for some integer n. Conversely, for each integer n, the map $\chi_n : \mathbb{G}_m \to \mathbb{G}_m$ defined by $\chi_n(a) = a^n$ is indeed a character of $\mathbb{G}_m$. Hence, we have shown that $\chi(\mathbb{G}_m) \cong \mathbb{Z}$.

Example 3. The character group of the additive group $\mathbb{G}_a$ is trivial, i.e., $\chi(\mathbb{G}_a) = \{0\}$. Indeed, every morphism from $\mathbb{A}^1$ to $\mathbb{A}^1 \setminus \{0\}$ is constant since every regular function on $\mathbb{A}^1$ is a polynomial, and there is no non-constant polynomial function without zeros.

Definition 4. A linear algebraic group T over a field $\Bbbk$ is called a *torus* if T is isomorphic to a finite product of copies of the multiplicative group $\mathbb{G}_m$, i.e., $T \cong \mathbb{G}_m^n$ for some non-negative integer n.

Lemma 5. Let T be a torus and H be a connected closed algebraic subgroup of T. Then H is also a torus.

Proof. Yang: To be continued. □

Recall that for a linear operator $T : V \to V$ of finite-dimensional $\Bbbk$-vector space V is called *semisimple* if it is diagonalizable, and *unipotent* if $T - \mathrm{id}_V$ is nilpotent.

Proposition 6. Let $T \subset \mathrm{GL}_n(\Bbbk)$ be a torus. Then every element of $T(\Bbbk)$ is semisimple. Conversely, if $g \in \mathrm{GL}_n(\Bbbk)$ is semisimple and of infinite order, then the neutral component of the algebraic subgroup generated by g is a torus.

Proposition 7. Let $g \subset \mathrm{GL}_n(\Bbbk)$. Then g is unipotent if and only if the algebraic subgroup generated by g is isomorphic to $\mathbb{G}_a$. Yang: To be revised.

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Theorem 8. Let G be a connected commutative linear algebraic group over an algebraically closed field $\Bbbk$ of characteristic zero. Then G is isomorphic to $\mathbb{G}_m^r \times \mathbb{G}_a^s$ for some non-negative integers r and s.

Proof. Yang: To be continued. □

2 Jordan-Chevalley Decomposition of elements

Definition 9. Let G be a linear algebraic group and $g \in G(\Bbbk)$. We say that g is *semisimple* (resp. *unipotent*) if its image under some (equivalently, any) faithful linear representation of G is a semisimple (resp. unipotent) linear operator.

Lemma 10. The notion of semisimple and unipotent elements in Definition 9 does not depend on the choice of faithful linear representation.

Proof. Yang: To be added. □

Theorem 11 (Jordan-Chevalley Decomposition)**.** Let G be a linear algebraic group and $g \in G(\Bbbk)$. Then there exist unique commuting elements $g_s, g_u \in G(\Bbbk)$ such that $g = g_s g_u$, where g_s is semisimple and g_u is unipotent.
Moreover, this decomposition is functorial in the sense that for any homomorphism of linear algebraic groups $\varphi : G \to H$, we have $\varphi(g)_s = \varphi(g_s)$ and $\varphi(g)_u = \varphi(g_u)$. Yang: To be checked

Proof. Yang: To be continued. □

3 Solvable groups and Borel subgroups

Definition 12. A group G is said to be *solvable* if there exists a finite sequence of algebraic subgroups

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = \{e\}$$

such that each G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is commutative for all $0 \le i < n$. Yang: to be checked.

Theorem 13. Let G be a solvable linear algebraic group acting on a proper variety X. Then there exists a fixed point $x \in X(\Bbbk)$ such that $g \cdot x = x$ for all $g \in G(\Bbbk)$.

Corollary 14 (Lie-Kolchin Theorem)**.** Let $G < \mathrm{GL}_n(\Bbbk)$ be a solvable linear algebraic group over an algebraically closed field $\Bbbk$. Then there exists a basis of $\Bbbk^n$ such that G is contained in the group of upper triangular matrices with respect to this basis.

Theorem 15. Let G be a linear algebraic group of dimension 1 over an algebraically closed field $\Bbbk$. Then G is isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$.

4 Decomposition of linear algebraic groups

Definition 16. Let G be a linear algebraic group over a field $\Bbbk$. The *radical* of G, denoted by $\operatorname{rad}(G)$, is defined to be the unique maximal connected normal solvable subgroup of G.

Yang: Well-defined?

Definition 17. Let G be a linear algebraic group. The *unipotent radical* of G, denoted by $\operatorname{rad}_u(G)$, is defined to be the subgroup of $\operatorname{rad}(G)$ consisting of all unipotent elements.

Yang: Why a group?

Definition 18. Let G be a linear algebraic group over a field $\Bbbk$. We say that G is *semisimple* if $\operatorname{rad}(G)$ is trivial.

Definition 19. Let G be a linear algebraic group over a field $\Bbbk$. We say that G is *reductive* if the unipotent radical of G is trivial.

Slogan

$$\begin{array}{ccc} \textit{“unipotent radical”} & \rightarrowtail\!\leftarrowtail & \textit{“reductive”} \\ \Downarrow & & \Uparrow \\ \textit{“solvable radical”} & \rightarrowtail\!\leftarrowtail & \textit{“semisimple”} \end{array}$$

Theorem 20 (Levi Decomposition)**.** Let G be a linear algebraic group over an algebraically closed field $\Bbbk$. Then there exists a reductive subgroup H of G such that the multiplication map $\operatorname{rad}_u(G) \rtimes H \to G$ is an isomorphism of algebraic groups. Such a subgroup H is called a *Levi subgroup* of G. Yang: To be checked.

Proof. Yang: To be continued. $\square$