The First Properties of Abelian Varieties

1 Definition and examples of Abelian Varieties

Definition 1. Let $\mathbf{k}$ be a field. An *abelian variety over* $\mathbf{k}$ is a proper variety A over $\mathbf{k}$ together with morphisms *identity* $e : \operatorname{Spec}\mathbf{k} \to A$, *multiplication* $m : A \times A \to A$ and *inversion* $i : A \to A$ such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd} & A \times A \times A \arrow[dl, "\mathrm{id}_A \times m"'] \arrow[dr, "m \times \mathrm{id}_A"] & \\ A \times A \arrow[dr, "m"] & & A \times A \arrow[dl, "m"'] \\ & A & \end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd} A \times \operatorname{Spec}\mathbf{k} \arrow[r, "\mathrm{id}_A \times e"] \arrow[dr, "\cong"'] & A \times A \arrow[d, "m"] & \operatorname{Spec}\mathbf{k} \times A \arrow[l, "e \times \mathrm{id}_A"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_A"] & \\ A \times A \arrow[dr, "m"'] & \operatorname{Spec}\mathbf{k} \arrow[d, "e"] & A \times A \arrow[dl, "m"] \\ & A & \end{tikzcd}.$$

In other words, an abelian variety is a group object in the category of proper varieties over $\mathbf{k}$.

Example 2. Let E be an elliptic curve over a field $\mathbf{k}$. Then E is an abelian variety of dimension 1. Yang: To be completed.

In the following, we will always assume that A is an abelian variety over a field $\mathbf{k}$ of dimension d.

Temporarily, we will use the notation e_A, m_A, i_A to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The *left translation* by $a \in A(\mathbf{k})$ is defined as

$$l_a : A \xrightarrow{\cong} \operatorname{Spec}\mathbf{k} \times A \xrightarrow{a \times \mathrm{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation r_a.

Proposition 3. Let A be an abelian variety. Then A is smooth.

Proof. By base changing to the algebraic closure of $\mathbf{k}$, we may assume that $\mathbf{k}$ is algebraically closed. Note that there is a non-empty open subset $U \subset A$ which is smooth. Then apply the left translation

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Proposition 4. Let A be an abelian variety. Then the cotangent bundle Ω_A is trivial, i.e., $\Omega_A \cong \mathcal{O}_A^{\oplus d}$ where $d = \dim A$.

Proof. Consider Ω_A as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism l_a induces a morphism of varieties $\Omega_A \to \Omega_A$ for every $a \in A(\mathbf{k})$. Yang: But how to show it is a morphism of varieties? Yang: To be completed. □

Theorem 5. Let A and B be abelian varieties. Then any morphism $f : A \to B$ with $f(e_A) = e_B$ is a group homomorphism, i.e., for every $\mathbf{k}$-scheme T, the induced map $f_T : A(T) \to B(T)$ is a group homomorphism.

Proof. Consider the diagram

$$\begin{tikzcd} A \times A \arrow[d, "p_1"'] \arrow[dr, "\varphi"] & \\ A & B \end{tikzcd}$$

with φ be given by

$$\begin{aligned} &A \times A \xrightarrow{\Delta \times \Delta} A \times A \times A \times A \xrightarrow{\cong} A \times A \times A \times A \xrightarrow{(f \circ m_A) \times (i_B \circ f) \times (i_B \circ f)} B \times B \times B \xrightarrow{m_B} B, \\ &(x, y) \mapsto (x, x, y, y) \mapsto (x, y, y, x) \mapsto (f(xy), f(y)^{-1}, f(x)^{-1}) \mapsto f(xy) f(y)^{-1} f(x)^{-1}. \end{aligned}$$

We have $\varphi(p_1^{-1}(e_A)) = \varphi(\{e_A\} \times A) = \{e_B\}$. Then by Rigidity Lemma (Theorem 9), there exists a unique rational map $\psi : A \dashrightarrow B$ such that $\varphi = \psi \circ p_1$. Note that $A \to A \times \{e_A\} \to A \times A$ gives a section of p_1. On this section, we have that φ is constant equal to e_B. Thus ψ is well-defined and $\psi(A) = e_B$. It follows that φ factors through the constant map $A \times A \to \{e_B\} \to B$. Then for every $(x, y) \in A(\mathbb{k}) \times A(\mathbb{k})$, we have

$$f(xy) = f(x) f(y).$$

Yang: Since $A(\mathbb{k})$ is dense in A, the conclusion follows. □

Proposition 6. Let A be an abelian variety. Then $A(\mathbf{k})$ is an abelian group.

Proof. Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 5. □

From now on, we will use the notation $0, +, [-1]_A, t_a$ to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every $n \in \mathbb{Z}_{>0}$, the homomorphism of multiplication by n is defined as

$$[n]_A : A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where Δ is the diagonal morphism.

2 Complex abelian varieties

Theorem 7. Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice $\Lambda \subset \mathbb{C}^d$ such that $A \cong \mathbb{C}^d/\Lambda$. Conversely, let $A = \mathbb{C}^n/\Lambda$ be a complex torus for some lattice Λ. Then A is a complex abelian variety if and only if there exists a positive definite Hermitian form H on $\mathbb{C}^n$ such that $\Im(H)(\Lambda, \Lambda) \subset \mathbb{Z}$. Yang: To be completed.

Requirements

Proposition 8. Let $f : X \to Y$ be a morphism of varieties over a field $\mathbf{k}$. Then the function $y \mapsto \dim f^{-1}(y)$ is upper semicontinuous, i.e., for every integer m, the set $\{y \in Y : \dim f^{-1}(y) \geq m\}$ is closed in Y. Yang: To be check.

Theorem 9 (Rigidity Lemma)**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.