

# Differentials and duality

Let  $S$  be a base noetherian scheme,  $\mathbb{k}$  be an algebraically closed field. Unless otherwise specified, all schemes are assumed to be defined and of finite type over  $S$  and all varieties are assumed to be defined over  $\mathbb{k}$ .

## 1 The sheaves of differentials

**Definition 1.** Let  $f : X \rightarrow S$  be an  $S$ -scheme. The *sheaf of differentials* of  $X$  over  $S$ , denoted by  $\Omega_{X/S}$ , is the  $\mathcal{O}_X$ -module locally given by

$$\Omega_{X/S}(U) = \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$$

for any affine open subsets  $U \subseteq X$  and  $V \subseteq S$  with  $f(U) \subseteq V$ .

**Example 2.**

**Theorem 3** (Euler sequence for projective bundle). Let  $X$  be a normal variety over  $\mathbb{k}$  and  $\mathcal{E}$  be a locally free sheaf of rank  $r + 1$  on  $X$ . Let  $\pi : \mathbb{P}_X(\mathcal{E}) \rightarrow X$  be the projective bundle associated to  $\mathcal{E}$ . Then there is a natural exact sequence of  $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}$ -modules

$$0 \rightarrow \Omega_{\mathbb{P}_X(\mathcal{E})/X} \rightarrow \pi^*\mathcal{E}(-1) \rightarrow \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \rightarrow 0.$$

Yang: To be checked.

| *Proof.*

□

**Definition 4.** Let  $X$  be a normal variety over  $\mathbb{k}$  of dimension  $n$ . If  $X$  is smooth, then the *canonical divisor*  $K_X$  is defined to be  $c_1(\omega_X)$ . In general, let  $U \subseteq X$  be the smooth locus of  $X$  and  $i : U \hookrightarrow X$  be the inclusion map. Then the *canonical divisor*  $K_X$  is defined to be any Weil divisor on  $X$  such that  $\mathcal{O}_X(K_X) \cong i_*\omega_U$ . Note that  $U$  is big in  $X$  since  $X$  is normal, so such a Weil divisor always exists and is unique up to linear equivalence.

**Example 5.** Let  $\mathbb{P}_{\mathbb{k}}^n$  be the projective space of dimension  $n$  over  $\mathbb{k}$ . Then the canonical divisor  $K_{\mathbb{P}_{\mathbb{k}}^n} \sim -(n+1)H$ , where  $H$  is a hyperplane in  $\mathbb{P}_{\mathbb{k}}^n$ . Yang: To be checked.

## 2 Fundamental sequence

**Theorem 6** (The first fundamental sequence of differentials). Let  $f : X \rightarrow Y$  be a morphism of schemes. Then there is a natural exact sequence of  $\mathcal{O}_X$ -modules

$$f^*\Omega_{Y/S} \rightarrow \Omega_{X/S} \rightarrow \Omega_{X/Y} \rightarrow 0.$$

Suppose further that  $f$  is smooth. Then the above sequence is also exact on the left. Yang: ... it will be exact.

| *Proof.* Yang: To be completed. □

**Theorem 7** (The second fundamental sequence of differentials). Let  $Z \subseteq X$  be a closed subscheme defined by the sheaf of ideals  $\mathcal{I} \subseteq \mathcal{O}_X$ . Then there is a natural exact sequence of  $\mathcal{O}_X$ -modules

$$\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/S}|_Z \rightarrow \Omega_{Z/S} \rightarrow 0.$$

Suppose further that  $Z$  is a local complete intersection in  $X$ . Then the above sequence is also exact on the left. Yang: ... it will be exact.

| *Proof.* Yang: To be completed. □

### 3 Application of fundamental sequences

**Theorem 8** (Ramification formula). Let  $f : X \rightarrow Y$  be a finite morphism of normal varieties. Then

$$K_X = f^*K_Y + R_f,$$

where

$$R_f := \sum_{D \subseteq X \text{ prime divisor}} (\text{Mult}_D f^*(f(D)) - 1) D$$

is the ramification divisor of  $f$ .

| *Proof.* Yang: To be completed. □

**Theorem 9** (Adjunction formula). Let  $X$  be a smooth variety and  $Z \subseteq X$  be a smooth subvariety of codimension 1. Then

$$K_Z = (K_X + Z)|_Z.$$

| *Proof.* Yang: To be completed. □

### 4 Serre duality

**Definition 10** (Dualizing sheaf). Let  $X$  be a proper scheme of dimension  $n$  over  $\mathbb{k}$ . A *dualizing sheaf* on  $X$  is a coherent sheaf  $\omega_X^\circ$  together with a trace map  $\text{tr}_X : H^n(X, \omega_X^\circ) \rightarrow \mathbb{k}$  such that for every coherent sheaf  $\mathcal{F}$  on  $X$ , the natural pairing

$$\text{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \rightarrow H^n(X, \omega_X^\circ) \xrightarrow{\text{tr}_X} \mathbb{k}$$

induces an isomorphism

$$\text{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee.$$

**Theorem 11.** Let  $X$  be a projective scheme of dimension  $n$  over  $\mathbb{k}$ . Then there exists a dualizing sheaf  $\omega_X^\circ$  on  $X$  up to isomorphism. Moreover, if  $X$  is smooth,  $\omega_X^\circ \cong \omega_X = \bigwedge^n \Omega_{X/\mathbb{k}}$ .

| *Proof.* Yang: To be completed. □

**Theorem 12** (Serre duality). Let  $X$  be a projective, Cohen-Macaulay variety of dimension  $n$  over  $\mathbb{k}$  with dualizing sheaf  $\omega_X^\circ$ . Then for every coherent sheaf  $\mathcal{F}$  on  $X$ , there is a natural isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

| *Proof.* Yang: To be completed. □

**Corollary 13.** Let  $X$  be a projective, normal variety of dimension  $n$  over  $\mathbb{k}$ . Then for every integer  $m$  and  $0 \leq i \leq n$ , there is a natural isomorphism *Yang: To be completed.*

## 5 Logarithm version