Ruled Surface

In this section, fix an algebraically closed field $\Bbbk$.

1 Preliminaries on Projective Bundles

Let S be a variety over $\Bbbk$ and $\mathcal{E}$ a vector bundle of rank $r+1$ on S.

Proposition 1. The S-varieties $\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on S.

Theorem 2. Let $\pi : X = \mathbb{P}_S(\mathcal{E}) \to S$ be the projective bundle associated to a vector bundle $\mathcal{E}$ of rank $r+1$ on S. Then there is an exact sequence of vector bundles on $\mathbb{P}_S(\mathcal{E})$

$$0 \to \Omega_{\mathbb{P}_S(\mathcal{E})/S} \to \pi^*(\mathcal{E})(-1) \to \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \to 0.$$

In particular, $K_X \sim \pi^*(K_S + \det \mathcal{E}) - (r+1)\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$. Yang: To be continued...

Theorem 3 (Tsen's Theorem)**.** Let C be a smooth curve over an algebraically closed field $\Bbbk$. Then $\mathbf{K} = \Bbbk(C)$ is a C_1 field, i.e., every degree d hypersurface in $\mathbb{P}^n_{\mathbf{K}}$ has a $\mathbf{K}$-rational point provided $d \le n$. Yang: Need a reference.

2 Minimal Section and Classification

Definition 4 (Ruled surface)**.** A *ruled surface* is a projective surface X together with a surjective morphism $\pi : X \to C$ to a smooth curve C such that all fibers of π are isomorphic to $\mathbb{P}^1$.

Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g.

Lemma 5. There exists a section of π.

Proposition 6. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.

Lemma 7. There is a one-to-one correspondence between sections of π and quotient line bundles of $\mathcal{E}$.

Lemma 8. It is possible to write $X \cong \mathbb{P}_C(\mathcal{E})$ such that $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. Such a vector bundle $\mathcal{E}$ is called a *normalized vector bundle*.

Definition 9. A section C_0 of π is called a *minimal section* if $C_0^2 \le C_1^2$ for any other section C_1 of π.

Theorem 10. Let

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Theorem 11. Let $\pi : X \to C$ be a ruled surface over $C = \mathbb{P}^1$ with invariant e. Then $X \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-e))$.

Theorem 12. Let

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Proposition 13. Let $\pi : X \to C$ be a ruled surface over a smooth curve C of genus g. Let C_0 be a minimal section of π and let f be a fiber of π. Then $K_X \sim -2C_0 + (K_C -)f$ where $e = -C_0^2$. Yang: Check this carefully.

Rational case. Let $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ be a ruled surface over $\mathbb{P}^1$ with $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$.

Elliptic case. Let $\pi : X = \mathbb{P}_C(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve E with $\mathcal{E}$ a normalized vector bundle of rank 2 and degree $-e$.