

# Cone Theorem

## 1 Preliminary

**Theorem 1** (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27]). Let  $X$  be a projective variety and  $\mathcal{L}$  an semiample line bundle on  $X$ . Then there exists a fibration  $\varphi : X \rightarrow Y$  of projective varieties such that for any  $m \gg 0$  with  $\mathcal{L}^m$  base point free, we have that the morphism  $\varphi_{\mathcal{L}^m}$  induced by  $\mathcal{L}^m$  is isomorphic to  $\varphi$ . Such a fibration is called the *Iitaka fibration* associated to  $\mathcal{L}$ .

**Theorem 2** (Rigidity Lemma, ref. [Deb01, Lemma 1.15]). Let  $\pi_i : X \rightarrow Y_i$  be proper morphisms of varieties over a field  $k$  for  $i = 1, 2$ . Suppose that  $\pi_1$  is a fibration and  $\pi_2$  contracts  $\pi_1^{-1}(y_0)$ . Then there exists a rational map  $\varphi : Y_1 \dashrightarrow Y_2$  such that  $\pi_2 \circ \varphi = \pi_1$  and  $\varphi$  is well-defined near  $Y_1 \setminus \{y_0\}$ .

**Theorem 3.** Let  $A, B \subset \mathbb{R}^n$  be disjoint convex sets. Then there exists a linear functional  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f|_A \leq c$  and  $f|_B \geq c$  for some  $c \in \mathbb{R}$ .

**Proposition 4.** Let  $X$  be a normal projective variety of dimension  $n$  and  $H$  an ample divisor on  $X$ . Suppose that  $K_X \cdot H^{n-1} < 0$ . Then for a general point  $x \in X$ , there exists a rational curve  $\Gamma$  passing through  $x$  such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

*Schetch of proof.* Take a resolution  $f : Y \rightarrow X$ , then  $f^*H$  is nef on  $Y$  and  $K_Y \cdot f^*H^{n-1} < 0$  since  $E \cdot f^*H^{n-1} = 0$ . Choose an ample divisor  $H_Y$  on  $Y$  closed enough to  $f^*H$  such that  $K_Y \cdot H_Y^{n-1} < 0$ . By [MM86, Theorem 5] and take limit for  $H_Y$ .  $\square$

**Lemma 5** (ref. [Kaw91, Lemma]). Let  $(X, B)$  be a projective klt pair and  $f : X \rightarrow Y$  a birational projective morphism. Let  $E$  be an irreducible component of dimension  $d$  of the exceptional locus of  $f$  and  $\nu : E^\nu \rightarrow X$  the normalization of  $E$ . Suppose that  $f(E)$  is a point. Then for any ample divisor  $H$  on  $X$ , we have

$$K_{E^\nu} \cdot \nu^* H^{d-1} \leq K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

**Lemma 6.** Let  $f : Y \rightarrow X$  be a birational morphism of projective varieties with  $Y$  smooth and  $X$  has only rational singularities. Let  $E$  be an effective exceptional divisor on  $Y$  and  $D$  a divisor on  $X$ . Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^i f_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

*Proof.* Yang: I am unable to proof this lemma.  $\square$

## 2 Non-vanishing Theorem

**Theorem 7** (Non-vanishing Theorem). Let  $(X, B)$  be a projective klt pair and  $D$  a Cartier divisor on  $X$ . Suppose that  $D$  is nef and  $aD - K_{(X,B)}$  is nef and big for some  $a > 0$ . Then for  $m \gg 0$ , we have

$$H^0(X, mD) \neq 0.$$

## 3 Base Point Free Theorem

**Theorem 8** (Base Point Free Theorem). Let  $(X, B)$  be a projective klt pair and  $D$  a Cartier divisor on  $X$ . Suppose that  $D$  is nef and  $aD - K_{(X,B)}$  is nef and big for some  $a > 0$ . Then for  $m \gg 0$ ,  $mD$  is base point free.

**Remark 9.** In general, we say that a Cartier divisor  $D$  is *semiample* if there exists a positive integer  $m$  such that  $mD$  is base point free. The statement in Base Point Free Theorem (Theorem 8) is strictly stronger than the semiample condition. For example, let  $\mathcal{L}$  be a torsion line bundle, then  $\mathcal{L}$  is semiample but there exists no positive integer  $M$  such that  $m\mathcal{L}$  is base point free for all  $m > M$ .

## 4 Rationality Theorem

**Lemma 10** (ref. [KM98, Theorem 1.36]). Let  $X$  be a proper variety of dimension  $n$  and  $D_1, \dots, D_m$  Cartier divisors on  $X$ . Then the Euler characteristic  $\chi(n_1 D_1, \dots, n_m D_m)$  is a polynomial in  $(n_1, \dots, n_m)$  of degree at most  $n$ .

**Theorem 11** (Rationality Theorem). Let  $(X, B)$  be a projective klt pair,  $a = a(X) \in \mathbb{Z}$  with  $aK_{(X,B)}$  Cartier and  $H$  an ample divisor on  $X$ . Let

$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of  $(X, B)$  with respect to  $H$ . Then  $t = v/u \in \mathbb{Q}$  and

$$0 \leq v \leq a(X) \cdot (\dim X + 1).$$

*Proof.* For every  $r \in \mathbb{R}_{>0}$ , let

$$v(r) := \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We need to show that  $v(t) \leq a(\dim X + 1)$ . For every  $(p, q) \in \mathbb{Z}_{>0}^2$ , set  $D(p, q) := paK_{(X,B)} + qH$ . If  $(p, q) \in \mathbb{Z}_{>0}^2$  with  $0 < atp - q < t$ , then we have  $D(p, q)$  is not nef and  $D(p, q) - K_{(X,B)}$  is ample.

**Step 1.** We show that a polynomial  $P(x, y) \neq 0 \in \mathbb{Q}[x, y]$  of degree at most  $n$  is not identically zero on the set

$$\{(p, q) \in \mathbb{Z}^2 : p, q > M, 0 < atp - q < t\varepsilon\}, \quad \forall M > 0,$$

if  $v(t)\varepsilon > a(n+1)$ .

If  $v(t) = \infty$ , for any  $n$ , we show that we can find infinitely many lines  $L$  such that  $\#L \cap \Lambda \geq n+1$ . If so,  $\Lambda$  is Zariski dense in  $\mathbb{Q}^2$ . Since  $1/at \in \mathbb{R} \setminus \mathbb{Q}$ , there exist  $p_0, q_0 > M$  such that

$$0 < \frac{p_0}{q_0} - \frac{1}{at} < \frac{\varepsilon}{(n+1)a} \cdot \frac{1}{q_0}, \text{ i.e. } 0 < atp_0 - q_0 < \frac{\varepsilon t}{n+1}.$$

Then  $(ip_0, iq_0) \in \Lambda \cap \{p_0y = q_0x\}$  for  $i = 1, \dots, n+1$ . Since  $M$  is arbitrary, there are infinitely many such lines  $L$ .

Suppose  $v(t) = v < \infty$  and  $t = v/u$ . Then the inequality is equivalent to  $0 < aup - vq < \varepsilon v$ . Note that  $\gcd(au, v) \mid a$ , then  $aup - vq = ai$  has integer solutions for  $i = 1, \dots, n+1$ . Since  $v(t)\varepsilon > a(n+1)$ , there are at least  $n+1$  lines which intersect  $\Lambda$  in infinitely many points. This enforces any polynomial which vanishes on  $\Lambda$  has degree at least  $n+1$ .

**Step 2.** There exists an index set  $\Lambda \subset \mathbb{Z}^2$  such that  $\Lambda$  contains all sufficiently large  $(p, q)$  with  $0 \leq atp - q \leq t$  and

$$Z := \text{Bs } |D(p, q)| = \text{Bs } |D(p', q')| \neq \emptyset, \quad \forall (p, q), (p', q') \in \Lambda.$$

For every  $(p, q) \in \mathbb{Z}_{>0}^2$  with  $0 < atp - q < t$ , choose  $k \in \mathbb{Z}_{>0}$  such that  $k(atp - q) > t$ . Then for all  $p', q' > kp$  with  $0 < atp' - q' < t$ , we have

$$p' - kp \geq 0, \quad q' - kp > t(p' - kp).$$

It follows that

Yang: To be completed.

**Step 3.** Suppose the contradiction that  $v(t) > a(\dim X + 1)$ . Then we show that  $H^0(X, D(p, q)) \neq 0$  for all  $(p, q) \in \Lambda$ . This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem ([Theorem 8](#)).

Let  $P(x, y) := \chi(D(x, y))$  be the Hilbert polynomial of  $D(x, y)$ . Note that  $P(0, n) = \chi(nH) \neq 0$  since  $H$  is ample. Then  $P(x, y) \neq 0$  and  $\deg P \leq \dim X$ . By ??,  $P$  is not identically zero on  $\Lambda$ . Note that  $D(p, q) - K_{(X, B)}$  is ample for all  $(p, q) \in \Lambda$ , then  $h^i(X, D(p, q)) = 0$  for all  $i > 0$  by Kawamata-Viehweg vanishing theorem (??). Then

$$P(p, q) = \chi(D(p, q)) = h^0(X, D(p, q)) \neq 0$$

for some  $(p, q) \in \Lambda$ . This is equivalent to that  $Z \neq X$  and hence  $H^0(X, D(p, q)) \neq 0$  for all  $(p, q) \in \Lambda$ .

**Step 4.** We follow the same line of the proof of Base Point Free Theorem ([Theorem 8](#)) to show that there is a section which does not vanish on  $Z$ .

Fix  $(p, q) \in \Lambda$ . If  $v(t) < \infty$ , we assume that  $t = v/u$  and  $atp - q = a(n+1)/u$ . Let  $f : Y \rightarrow X$  be a resolution such that

- (a)  $K_{Y,B_Y} = f^*K_{(X,B)} + E_Y$  for some effective exceptional divisor  $E_Y$ , and  $Y, B_Y$  is a klt pair;
- (b)  $f^*|D(p, q)| = |L| + F$  for some effective divisor  $F$  and a base point free divisor  $L$ , and  $f(\text{Supp } F) = Z$ ;
- (c)  $f^*D(p, q) - f^*K_{(X,B)} - E_0$  is ample for some effective  $\mathbb{Q}$ -divisor  $E_0 \in (0, 1)$ , and coefficients of  $E_0$  are sufficiently small;
- (d)  $B_Y + E_Y + F + E_0$  has snc support.

**Yang:** Such resolution exists by [KM98].

Let  $c := \inf\{\lfloor B_Y + E_0 + tF \rfloor \neq 0\}$ . Adjust the coefficients of  $E_0$  slightly such that  $\lfloor B_Y + E_0 + cF \rfloor = F_0$  for unique prime divisor  $F_0$  with  $F_0 \subset \text{Supp } F$ . Set  $\Delta_Y := B_Y + cF + E_0 - F_0$ . Then  $(Y, \Delta_Y)$  is a klt pair.

Let

$$\begin{aligned} N(p', q') &:= f^*D(p', q') + E_Y - F_0 - K_{(Y, \Delta_Y)} \\ &= \left( f^*D(p', q') - (1+c)f^*D(p, q) \right) + \left( f^*D(p, q) - f^*K_{(X,B)} - E_0 \right) + c \left( f^*D(p, q) - F \right). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p', q') \in \Lambda : 0 < atp' - q' < atp - q, \ p', q' > (1+c) \max\{p, q\}\},$$

the divisor  $f^*D(p', q') - (1+c)f^*D(p, q) = f^*D(p' - (1+c)p, q' - (1+c)q)$  is ample, and hence  $N(p', q')$  is ample.

By the exact sequence

$$0 \rightarrow \mathcal{O}_Y(f^*D(p', q') + E_Y - F_0) \rightarrow \mathcal{O}_Y(f^*D(p', q') + E_Y) \rightarrow \mathcal{O}_{F_0}((f^*D(p', q') + E_Y)|_{F_0}) \rightarrow 0$$

and Kawamata-Viehweg Vanishing Theorem (??), we get a surjective map

$$H^0(Y, f^*D(p', q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}).$$

On  $F_0$ , consider the polynomial  $\chi((f^*D(p', q') + E_Y)|_{F_0})$ . Note that  $\dim F_0 = n - 1$  and by the construction of  $(p, q), \Lambda_0$ , similar to Step 3, we can show that  $\chi((f^*D(p', q') + E_Y)|_{F_0})$  is not identically zero on  $\Lambda_0$ . By adjunction, we have  $(f^*D(p', q') + E_Y)|_{F_0} = N(p', q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$  with  $N(p', q')|_{F_0}$  ample and  $(F_0, \Delta_Y|_{F_0})$  klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (??) to get

$$h^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}) = \chi(F_0, (D(p', q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that  $f(F_0) \subset Z = \text{Bs } |D(p', q')|$ .

□

## 5 Cone Theorem and Contraction Theorem

**Theorem 12** (Cone Theorem). Let  $(X, B)$  be a projective klt pair. Then there exist countably many rational curves  $C_i \subset X$  with

$$0 < -K_{(X,B)} \cdot C_i \leq 2 \dim X$$

such that

(a) we have a decomposition of cones

$$\text{Psef}_1(X) = \text{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any  $\varepsilon > 0$  and an ample divisor  $H$  on  $X$ , we have

$$\text{Psef}_1(X) = \text{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

*Proof.* Let  $F_D := \text{Psef}_1(X) \cap D^\perp$  for a nef divisor  $D$  on  $X$ . If  $\dim F_D = 1$ , we also write  $R_D := F_D$ . Let  $H_1, \dots, H_{\rho-1}$  be ample divisors on  $X$  such that they together with  $K_{(X,B)}$  form a basis of  $N^1(X)_\mathbb{Q}$ . Fix a norm  $\|\cdot\|$  on  $N_1(X)_\mathbb{R}$  and let  $S^{\rho-1} := S(N_1(X)_\mathbb{R})$  be the unit sphere in  $N_1(X)_\mathbb{R}$ .

**Step 1.** There exists an integer  $N$  such that for every  $K_{(X,B)}$ -negative extremal face  $F_D$  and for every ample divisor  $H$ , there exists  $n_0, r \in \mathbb{Z}_{>0}$  such that for all  $n > n_0$ ,  $\{0\} \neq F_{nD+rK_{(X,B)}+NH} \subset F_D$ .

Let  $N := (a(X)(\dim X + 1))!$ , where  $a(X)$  is the number in Theorem 11. For every  $n$ ,  $nD + H$  is an ample divisor and by Theorem 11, the nef threshold of  $K_{(X,B)}$  with respect to  $nD + H$  is of form

$$\inf\{s \geq 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since  $K_{(X,B)} + (N/r_n)((n+1)D + H)$  is nef, we have  $r_n \leq r_{n+1}$ . On the other hand, let  $\xi \in F_D \setminus \{0\}$ . Then  $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$  implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence  $r_n \rightarrow r \in \mathbb{Z}_{\geq 0}$ . It follows that  $rK_{(X,B)} + nND + NH$  is a nef but not ample divisor for all  $n \gg 0$ . Note that for every nef divisors  $N_1, N_2$ , we have  $F_{N_1+N_2} = F_{N_1} \cap F_{N_2}$ . Then for all  $n \gg 0$ , there exists  $m$  large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

**Step 2.** Let  $\Phi : N_1(X)_{K_{(X,B)} < 0} \rightarrow \mathbb{R}^{\rho-1}$  be the map defined by

$$\alpha \mapsto \left( \frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha} \right).$$

We show that the image of  $R_D$  under  $\Phi$  lies in a  $\mathbb{Z}$ -lattice in  $\mathbb{R}^{\rho-1}$ .

Suppose  $R = \mathbb{R}_{\geq 0}\xi$  for a class  $\xi$ . By [Step 1](#), we have  $R_{nD+rK_{(X,B)}+NH_i} = R_D$  for some integers  $n, r$ . Then  $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$  implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of  $R_D$  under  $\Phi$  lies in  $\frac{1}{N}\mathbb{Z}^{\rho-1}$ .

**Step 3.** We show that every  $K_{(X,B)}$ -negative extremal ray of  $\text{Psef}_1(X)$  is of the form  $R_D$  for some nef divisor  $D$  on  $X$ .

Let  $R = \mathbb{R}_{\geq 0}\xi$  be a  $K_{(X,B)}$ -negative extremal ray. **Yang:** Then  $R$  is of form  $D^\perp \cap \text{Psef}_1(X)$  for some nef  $\mathbb{R}$ -divisor  $D$  on  $X$  by [Theorem 3](#). We need to show that  $D$  can be choose as a nef  $\mathbb{Q}$ -divisor. There is a sequence of nef but not ample  $\mathbb{Q}$ -divisors  $D_m$  such that  $D_m \rightarrow D$  as  $m \rightarrow \infty$ . We adjust  $D_m$  such that  $\dim F_{D_m} = 1$  for all  $n$ .

By re-choosing  $H_i$ , we can assume that  $D = a_1H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$  for  $a_i > 0$  since  $aD - K$  is ample for  $a \gg 0$ . After truncation, we can assume that so is  $D_m$ . Then  $F_{D_m}$  is  $K_{(X,B)}$ -negative. Note that  $F_{nD_m+r_iK_{(X,B)}+NH_i} \subset F_{D_m}$  for some  $r_i > 0$  and  $n \gg 0$  by [Step 1](#). If  $\dim F_{D_m} > 1$ , then not all  $H_i|_{F_{D_m}}$  are proportional to  $K_{(X,B)}|_{D_m}$ . We can assume that  $r_1K_{(X,B)} + NH_1$  is not identically zero on  $F_{D_m}$ . Then we can choose  $n$  large enough such that  $\|r_1K_{(X,B)} + NH_1\|/n < 1/m$ . Replace  $D_m$  by  $D_m + (r_1K_{(X,B)} + NH_1)/n$ . Inductively we construct  $D_m$  nef  $\mathbb{Q}$ -divisor with  $D_m \rightarrow D$  and  $\dim F_{D_m} = 1$ .

Let  $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$ . Suppose that  $\|\xi_m\| = \|\xi\| = 1$ . By passing to a subsequence, we can assume that  $\xi_m$  converges. Then  $\xi_m \rightarrow \xi$  since  $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$ . However,  $\Phi$  is well-defined at  $\xi$  and the image of  $\xi_m$  under  $\Phi$  is discrete. Hence  $\xi = \xi_m$  for all  $m$  large enough. It follows that  $R = R_{D_m}$  for a nef  $\mathbb{Q}$ -divisor  $D_m$ .

**Step 4.** We show that any  $K_{(X,B)}$ -negative extremal ray  $R_D$  contains the class of a rational curve  $C$  with  $0 < -K_{(X,B)} \cdot C \leq 2 \dim X$ .

By [Theorem 14](#), let  $\varphi_D : X \rightarrow Y$  be the contraction associated to  $R_D$  (note that we do not need the step to proof [Theorem 14](#)). If  $\dim Y < \dim X$ , let  $F$  be a general fiber of  $\varphi_D$ . **Yang:** By adjunction,  $(F, B|_F)$  is a klt pair and  $K_{(F,B|_F)} = K_{(X,B)}|_F$ . Take  $H = aD - K_{(X,B)}$  for some  $a > 0$  such that  $H$  is ample on  $F$ . By [Proposition 4](#). **Yang:** In birational case, by adjunction, suppose  $\varphi_D(E)$  is a point. By [Lemma 5](#), we can use [Proposition 4](#) to get the result.

**Yang:** To be completed.

**Step 5.** Proof of the theorem.

Given an ample divisor  $H$  on  $X$ , note that  $\varepsilon H$  has positive minimum  $\delta$  on  $\text{Psef}_1(X) \cap S^{\rho-1}$ . Note that the set

$$\{\alpha \in \text{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and  $\Phi$  is well-defined on it. By [Steps 2](#) and [3](#), there are only finitely many extremal rays on  $\text{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \leq 0}$ . By [Step 4](#), we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal

ray. We only need to show that the cone

$$\mathcal{C} := \text{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence  $\{\alpha_n\} \subset \mathcal{C}$  such that  $\alpha_n \rightarrow \alpha \in N_1(X)_{\mathbb{R}}$ . Note that  $\text{Psef}_1(X)$  is closed, hence  $\alpha \in \text{Psef}_1(X)$ . We only need to consider the case  $\alpha \cdot K_{(X,B)} < 0$ . We can choose an ample divisor and  $\varepsilon > 0$  such that  $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$ . Then  $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$  for all  $n$  large enough. Note that  $\mathcal{C} \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$  is a polyhedral cone by [Step 2](#) and hence is closed. Then  $\alpha \in \mathcal{C}$  and the conclusion follows.  $\square$

**Remark 13.** Yang: Thanks for my friend Qin for pointing out that the extremal ray in [Theorem 12](#) may not be exposed.

**Theorem 14** (Contraction Theorem). Let  $(X, B)$  be a projective klt pair and  $F \subset \text{Psef}_1(X)$  a  $K_{(X,B)}$ -negative extremal face of  $\text{Psef}_1(X)$ . Then there exists a fibration  $\varphi_F : X \rightarrow Y$  of projective varieties such that

- (a) an irreducible curve  $C \subset X$  is contracted by  $\varphi_F$  if and only if  $[C] \in F$ ;
- (b) up to linearly equivalence, any Cartier divisor  $G$  with  $F \subset G^\perp = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$  comes from a Cartier divisor on  $Y$ , i.e., there exists a Cartier divisor  $G_Y$  on  $Y$  such that  $G \sim \varphi_F^* G_Y$ .

*Proof.* We follow the following steps to prove the theorem.

**Step 1.** We show that there exists a nef divisor  $D$  on  $X$  such that  $F = D^\perp \cap \text{Psef}_1(X)$ . In other words,  $F$  is defined on  $N_1(X)_{\mathbb{Q}}$ .

We can choose an ample divisor  $H$  and  $n > 0$  such that  $K_{(X,B)} + (1/n)H$  is negative on  $F$  since  $F \cap S^{\rho-1}$  is compact and  $K_{(X,B)}$  is strictly negative on it, where  $S^{\rho-1}$  is the unit sphere in  $N_1(X)_{\mathbb{R}}$ . Then by Cone Theorem ([Theorem 12](#)),  $F$  is an extremal face of a rational polyhedral cone, namely  $\text{Psef}_1(X)_{K_{(X,B)} + (1/n)H \leq 0}$ . It follows that  $F^\perp \subset N^1(X)_{\mathbb{R}}$  is defined on  $\mathbb{Q}$ . Since  $F$  is extremal and  $K_{(X,B)} + (1/n)H$ -negative, the set  $\{L \in F^\perp : L|_{\text{Psef}_1(X) \setminus F} > 0\}$  has non-empty interior in  $F^\perp$  by [Theorems 3](#) and [12](#). Then there exists a Cartier divisor  $D$  such that  $D \in F^\perp$  and  $D|_{\text{Psef}_1(X) \setminus F} > 0$ . It follows that  $D$  is nef and  $F = D^\perp \cap \text{Psef}_1(X)$ .

**Step 2.** Let  $\varphi : X \rightarrow Y$  be the Iitaka fibration associated to  $D$  by [Theorem 1](#). We show that  $\varphi$  is the desired fibration.

Note that  $\text{Psef}_1(X)_{K_{(X,B)} \geq 0} \cap S^{\rho-1}$  is compact and  $D$  is strictly positive on it. Then there exist  $a \geq 0$  such that  $aD - K_{(X,B)}$  is strictly positive on  $\text{Psef}_1(X)_{K_{(X,B)} \geq 0} \cap S^{\rho-1}$ . And  $K_{(X,B)}$  is strictly negative on  $F \setminus \{0\}$  since  $F$  is  $K_{(X,B)}$ -negative. Then by Base Point Free Theorem ([Theorem 8](#)), we know that  $mD$  is base point free for all  $m \gg 0$ . Hence we can apply [Theorem 1](#) to get a fibration  $\varphi_D : X \rightarrow Y$ .

First we show that  $D$  comes from  $Y$ . Note that  $mD$  and  $(m+1)D$  induces the same fibration  $\varphi_D$  for  $m \gg 0$ . Then there exists  $D_{Y,m}$  and  $D_{Y,m+1}$  such that  $\varphi_D^* D_{Y,m} \sim mD$  and  $\varphi_D^* D_{Y,m+1} \sim (m+1)D$ . Then set  $D_Y = D_{Y,m+1} - D_{Y,m}$ , we have  $\varphi_D^* D_Y \sim D$ .



Note that  $D_Y \equiv (1/m)D_{Y,m}$  and  $D_{Y,m}$  is ample. Hence  $D_Y$  is ample. Then for any curve  $C \subset X$ , we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that  $C$  is contracted by  $\varphi_D$  if and only if  $D \cdot C = 0$ , which is equivalent to  $[C] \in F$ .

Let  $G$  be arbitrary Cartier divisor on  $X$  such that  $F \subset G^\perp$ . Since  $D$  is strictly positive on  $\text{Psef}_1(X) \setminus F$ , for  $m \gg 0$ , let  $D' := mD + G$ , we have  $D'^\perp \cap \text{Psef}_1(X) = F$ . Then by the same argument as above, we get an other fibration  $\varphi_{D'} : X \rightarrow Y'$  such that a curve  $C$  is contracted by  $\varphi_{D'}$  if and only if  $[C] \in F$ . Then by Rigidity Lemma (Theorem 2), we see that  $\varphi_D = \varphi_{D'}$  up to an isomorphism on  $Y$ . In particular,  $D' \sim \varphi_D^* D'_Y$  for some Cartier divisor  $D'_Y$  on  $Y$ . Then  $G = D' - mD$  also comes from  $Y$ .  $\square$

**Remark 15.** The Step 1 is amazing. If  $F$  is not  $K_{(X,B)}$ -negative, then it may not be rational. For example, let  $X = E \times E$  for a general elliptic curve  $E$ . By [Laz04, Lemma 1.5.4], we know that  $\text{Psef}_1(X)$  is a circular cone. Then we see there indeed exist some irrational extremal faces of  $\text{Psef}_1(X)$ .

**Definition 16.** Let  $(X, B)$  be a projective klt pair and  $R$  a  $K_{(X,B)}$ -negative extremal ray of  $\text{Psef}_1(X)$  with contraction  $\varphi_R : X \rightarrow Y$ . There are three types of contractions:

- (a) *Divisorial contraction*: if  $\dim X = \dim Y$  and the exceptional locus of  $\varphi_R$  is of codimension one;
- (b) *Small contraction*: if  $\dim X = \dim Y$  and the exceptional locus of  $\varphi_R$  is of codimension at least two;
- (c) *Mori fiber space*: if  $\dim X > \dim Y$ .

**Proposition 17.** Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial projective klt pair and  $R$  a  $K_{(X,B)}$ -negative extremal ray of  $\text{Psef}_1(X)$ . Suppose that the contraction  $\varphi : X \rightarrow Y$  associated to  $R$  is either divisorial or a Mori fiber space. Then  $Y$  is  $\mathbb{Q}$ -factorial.

*Proof.* Let  $D$  be a prime Weil divisor on  $Y$  and  $U \subset Y$  a big open smooth subset. Let  $R = \mathbb{R}_{\geq 0}[C]$  for an irreducible curve  $C$  contracted by  $\varphi$ . Set  $D_X := \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$ . Then  $D_X$  is a prime Weil divisor on  $X$  and hence is  $\mathbb{Q}$ -Cartier.

If  $\varphi$  is a Mori fiber space, then  $D_X|_F \equiv 0$  for general fiber  $F$  of  $\varphi$ . Then by Contraction Theorem (Theorem 14), we see that  $mD_X \sim \varphi^* D'$  for some Cartier divisor  $D'$  on  $Y$ . We have  $mD|_U \sim D'|_U$  since  $\varphi|_{\varphi^{-1}(U)}$  is a fibration. Then  $mD \sim D'$  and hence  $D$  is  $\mathbb{Q}$ -Cartier.

If  $\varphi$  is a divisorial contraction, let  $E$  be the exceptional divisor of  $\varphi$  and assume that  $\varphi^{-1}|_U$  is an isomorphism. Then  $E \cdot C \neq 0$  (otherwise  $E \sim_{\mathbb{Q}} f^* E_Y$  for some Cartier  $\mathbb{Q}$ -divisor  $E_Y$  on  $Y$ ). Then we can choose  $a \in \mathbb{Q}$  such that  $(D_X + aE) \cdot C = 0$ . By Contraction Theorem (Theorem 14), we have  $mD_X + maE \sim \varphi^* D'$  for some Cartier divisor  $D'$  on  $Y$ . Then we also have  $D|_U \sim mD'|_U$  since  $\varphi|_{\varphi^{-1}(U)}$  is an isomorphism. Hence  $D$  is  $\mathbb{Q}$ -Cartier.  $\square$



**Remark 18.** If  $\varphi$  is a small contraction, then  $Y$  is never  $\mathbb{Q}$ -factorial. Otherwise, let  $B_Y$  be the strict transform of  $B$  on  $Y$ . Note that  $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$  on a big open subset  $U$ . Suppose  $K_{(Y,B_Y)}$  is  $\mathbb{Q}$ -Cartier. Then  $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$ . Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

## References

- [Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001, pp. xiv+233. ISBN: 0-387-95227-6. DOI: [10.1007/978-1-4757-5406-3](https://doi.org/10.1007/978-1-4757-5406-3). URL: <https://doi.org/10.1007/978-1-4757-5406-3> (cit. on p. 1).
- [Kaw91] Yujiro Kawamata. “On the length of an extremal rational curve”. In: *Inventiones mathematicae* 105.1 (1991), pp. 609–611 (cit. on p. 1).
- [KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*. Vol. 134. Cambridge Tracts in Mathematics. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. Cambridge University Press, Cambridge, 1998, pp. viii+254. ISBN: 0-521-63277-3. DOI: [10.1017/CB09780511662560](https://doi.org/10.1017/CB09780511662560). URL: <https://doi.org/10.1017/CB09780511662560> (cit. on pp. 2, 4).
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: [10.1007/978-3-642-18808-4](https://doi.org/10.1007/978-3-642-18808-4). URL: <https://doi.org/10.1007/978-3-642-18808-4> (cit. on pp. 1, 8).
- [MM86] Yoichi Miyaoka and Shigefumi Mori. “A numerical criterion for uniruledness”. In: *Annals of Mathematics* 124.1 (1986), pp. 65–69 (cit. on p. 1).