First properties of algebraic groups

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. All varieties are defined over $\mathbf{k}$ unless otherwise specified.

1 Basic concepts

Definition 1. A *group scheme* over S is an S-scheme G together with morphisms *multiplication* $\mu : G \times G \to G$, *identity* $\varepsilon : S \to G$ and *inversion* $\iota : G \to G$ over S such that the following diagrams commute:

(a) (Associativity)

$$\begin{tikzcd}
& G \times G \times G \arrow[dl, "\mathrm{id}_G \times \mu"'] \arrow[dr, "\mu \times \mathrm{id}_G"] & \\
G \times G \arrow[dr, "\mu"] & & G \times G \arrow[dl, "\mu"'] \\
& G &
\end{tikzcd};$$

(b) (Identity)

$$\begin{tikzcd}
G \times S \arrow[r, "\mathrm{id}_G \times \varepsilon"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] & S \times G \arrow[l, "\varepsilon \times \mathrm{id}_G"'] \arrow[dl, "\cong"] \\
& G &
\end{tikzcd};$$

(c) (Inversion)

$$\begin{tikzcd}
& G \arrow[dl, "\mathrm{id}_G \times i"'] \arrow[d] \arrow[dr, "i \times \mathrm{id}_G"] & \\
G \times G \arrow[dr, "\mu"'] & S \arrow[d, "\varepsilon"] & G \times G \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}.$$

In other words, a group scheme is a group object in the category of schemes.

Definition 2. An *algebraic group* is a $\mathbf{k}$-group scheme G which is reduced, separated and of finite type over a field $\mathbf{k}$.

Definition 3. Let G be an algebraic group and $x \in G(\mathbf{k})$ a $\mathbf{k}$-point. The *left translation* by x is the morphism

$$l_x : G \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{\mu} G.$$

Similar definition applies to the right translation r_x.

Proposition 4. Let G be an algebraic group. Then G is a smooth over $\mathbf{k}$.

Proof. Yang: To be continued... ☐

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Remark 5. Let G be an algebraic group. Then the irreducible components of G coincide with the connected components of G. We will use the term "connected" to refer to both concepts since "irreducible" has other meanings in the theory of representations.

Example 6. The *additive group* $\mathbb{G}_a$ is defined to be the affine line $\mathbb{A}^1$ with the group law given by addition. Concretely, we can write $\mathbb{G}_a = \operatorname{Spec} \mathbf{k}[T]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a &\quad \mathbf{k}[T] \to \mathbf{k}[T] \otimes_{\mathbf{k}} \mathbf{k}[T], \quad T \mapsto T \otimes 1 + 1 \otimes T. \\ \iota : \mathbb{G}_a \to \mathbb{G}_a &\quad \mathbf{k}[T] \to \mathbf{k}[T], \quad T \mapsto -T. \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_a &\quad \mathbf{k}[T] \to \mathbf{k}, \quad T \mapsto 0. \end{aligned}$$

Example 7. The *multiplicative group* $\mathbb{G}_m$ is defined to be the affine variety $\mathbb{A}^1 \setminus \{0\}$ with the group law given by multiplication. Concretely, we can write $\mathbb{G}_m = \operatorname{Spec} \mathbf{k}[T, T^{-1}]$ with the group law given by the morphism

$$\begin{aligned} \mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m &\rightsquigarrow \mathbf{k}[T, T^{-1}] \to \mathbf{k}[T, T^{-1}] \otimes_{\mathbf{k}} \mathbf{k}[T, T^{-1}], \quad T \mapsto T \otimes T. \\ \iota : \mathbb{G}_m \to \mathbb{G}_m &\rightsquigarrow \mathbf{k}[T, T^{-1}] \to \mathbf{k}[T, T^{-1}], \quad T \mapsto T^{-1}. \\ \varepsilon : \operatorname{Spec} \mathbf{k} \to \mathbb{G}_m &\rightsquigarrow \mathbf{k}[T, T^{-1}] \to \mathbf{k}, \quad T \mapsto 1. \end{aligned}$$

Example 8. The *general linear group* GL_n is defined to be the open subvariety of $\mathbb{A}^{n^2}$ consisting of invertible matrices, with the group law given by matrix multiplication. Concretely, we can write $\mathrm{GL}_n = \operatorname{Spec} \mathbf{k}[T_{ij}, \det(T_{ij})^{-1}]$ where $1 \le i, j \le n$ and the group law is given by the morphism

$$\begin{aligned} \mu &: \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n, \quad (A, B) \mapsto AB, \\ \iota &: \mathrm{GL}_n \to \mathrm{GL}_n, \quad A \mapsto A^{-1}, \\ \varepsilon &: \operatorname{Spec} \mathbf{k} \to \mathrm{GL}_n, \quad 1 \mapsto I_n. \end{aligned}$$

Example 9. An abelian variety is an algebraic group that is also a proper variety.

Example 10. Let G and H be algebraic groups. The *product* $G \times H$ is an algebraic group with the group law defined by

$$\begin{aligned} \mu_{G \times H} = \mu_G \times \mu_H &: (G \times H) \times (G \times H) \cong (G \times G) \times (H \times H) \to G \times H, \\ \varepsilon_{G \times H} = \varepsilon_G \times \varepsilon_H &: \operatorname{Spec} \mathbf{k} \cong \operatorname{Spec} \mathbf{k} \times \operatorname{Spec} \mathbf{k} \to G \times H, \\ \iota_{G \times H} = \iota_G \times \iota_H &: G \times H \to G \times H. \end{aligned}$$

Definition 11. A *homomorphism* of algebraic groups is a morphism of schemes that is also a group homomorphism. Explicitly, a morphism $f : G \to H$ between algebraic groups G and H is a homomorphism if the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f \times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

where μ_G and μ_H are the group laws of G and H, respectively.

Definition 12. An *algebraic subgroup* of an algebraic group G is a closed subscheme $H \subseteq G$ that is also a subgroup of G. More precisely, H is an algebraic subgroup and the inclusion morphism $H \hookrightarrow G$ is a morphism of algebraic groups.

Example 13. The *special linear group* SL_n is defined to be the closed subvariety of GL_n defined by the equation $\det = 1$. It is an algebraic subgroup of GL_n. Yang: To be continued...

Definition 14. Let G be an algebraic group. The *neutral component* G^0 is the connected component of G containing the identity element ε.

Proposition 15. The neutral component G^0 is a closed, normal algebraic subgroup of G of finite index. Moreover, each closed subgroup H of finite index contains G^0.

Proof. Yang: To be continued... □

Proposition 16. Let G be an algebraic group and $H \subseteq G$ a subgroup (not necessarily closed). Then the Zariski closure $\overline{H}$ of H in G is an algebraic subgroup of G. If H is constructible, then $H = \overline{H}$. Yang: To be continued...

Proof. Yang: To be continued... □

Example 17. Let $G = \mathrm{SL}_2$ over $\Bbbk$, $T = \{\mathrm{diag}(t, t^{-1}) | t \in \Bbbk^\times\}$ and $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Set $S = gTg^{-1}$. Then both T and S are closed algebraic subgroups of G, but the product TS is not closed in G. By direct computation, we have

$$S = \left\{ \begin{pmatrix} s & s^{-1} - s \\ 0 & s^{-1} \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

Then

$$TS = \left\{ \begin{pmatrix} ts & t(s^{-1} - s) \\ 0 & (ts)^{-1} \end{pmatrix} \middle| t, s \in \Bbbk^\times \right\}.$$

We have

$$TS \cap \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \Bbbk \right\} = \left\{ \begin{pmatrix} 1 & s^{-2} - 1 \\ 0 & 1 \end{pmatrix} \middle| s \in \Bbbk^\times \right\}.$$

The right hand side is not closed in SL_2 since it does not contain the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Hence TS is not closed in G.

Proposition 18. Let G be an algebraic group, X_i varieties over $\mathbf{k}$ and $f_i : X_i \to G$ morphisms for $i = 1, \dots, n$ with images $Y_i = f_i(X_i)$. Suppose that Y_i pass through the identity element of G. Let H be the closed subgroup of G generated by $Y_1, \dots, Y_n$, i.e. the smallest closed subgroup of G containing $Y_1, \dots, Y_n$. Then H is connected and $H = Y_{a_1}^{e_1} \cdots Y_{a_m}^{e_m}$ for some $a_1, \dots, a_m \in \{1, \dots, n\}$ and $e_1, \dots, e_m \in \{\pm 1\}$.

Proof. Yang: To be continued... □

Remark 19. We can take $m \leq 2 \dim G$ in Proposition 18. Yang: To be continued...

2 Action and representations

Definition 20. An *action* of an algebraic group G on a variety X is a morphism

$$\sigma : G \times X \to X$$

such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \sigma"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} \mathbf{k} \times X \arrow[r, "\varepsilon \times \mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

where μ is the group law of G and ε is the identity element of G. In other words, for any $\mathbf{k}$-scheme S, the induced map $G(S) \times X(S) \to X(S)$ defines a group action of the abstract group $G(S)$ on the set $X(S)$. We say that X is a *G-variety*. Yang: To be checked.

Definition 21. Let G be an algebraic group acting on a variety X. For any field extension $K/\mathbf{k}$ and $x \in X(K)$, the *orbit* of x is the subset

$$G(K) \cdot x = \{\sigma(g, x) | g \in G(K)\} \subseteq X(K).$$

The *stabilizer* of x is the subgroup

$$G_x(K) = \{g \in G(K) | \sigma(g, x) = x\} \subseteq G(K).$$

The action is called *transitive* if for any field extension $K/\mathbf{k}$, the induced map $G(K) \times X(K) \to X(K)$ is transitive. The action is called *faithful* if for any field extension $K/\mathbf{k}$, the induced map $G(K) \times X(K) \to X(K)$ is faithful. Yang: To be checked.

Example 22. A *linear representation* of an algebraic group G on a finite-dimensional vector space $V = \mathbb{A}^k_{\mathbf{k}}$ over $\mathbf{k}$ is an action of G on the affine space associated to V, i.e. a morphism

$$\rho : G \times V \to V$$

such that for any field extension $K/\mathbf{k}$, the induced map $G(K) \times V(K) \to V(K)$ defines a group homomorphism from the abstract group $G(K)$ to the general linear group of the vector space $V(K)$. In other words, for any $g \in G(K)$, the map $\rho_g : V(K) \to V(K)$ defined by $\rho_g(v) = \rho(g, v)$ is a linear automorphism of $V(K)$. We say that V is a G-module. Yang: To be checked.

Definition 23. An *rational action* of an algebraic group G on a variety X is a rational map

$$\sigma : G \times X \dashrightarrow X$$

such that the following diagrams commute wherever the maps are defined:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu\times\mathrm{id}_X"] \arrow[d, dashed, "\mathrm{id}_G\times\sigma"] & G \times X \arrow[d, dashed, "\sigma"] \\ G \times X \arrow[r, dashed, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}\mathbf{k} \times X \arrow[r, "\varepsilon\times\mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

where μ is the group law of G and ε is the identity element of G. In other words, for any field extension $K/\mathbf{k}$, the induced map $G(K)\times X(K) \dashrightarrow X(K)$ defines a group action of the abstract group $G(K)$ on the set $X(K)$. We say that X is a *rational G-variety*. Yang: To be checked.

Definition 24. Let G be an algebraic group acting on a variety X. For any $x \in X(\mathbf{k})$, the *orbit* of x is the locally closed subvariety $G \cdot x = \sigma(G \times \{x\})$ of X. Note that this definition is different from the one in Definition 21. Yang: To be checked.

Theorem 25 (Weil's Regularization Theorem)**.** Let G be an algebraic group acting on a variety X. Then there exists a variety Y with a regular action of G and a G-equivariant birational isomorphism $X \dashrightarrow Y$. Yang: To be checked.

Proposition 26. Let G be an algebraic group acting on a variety X. Then for any $x \in X(\mathbf{k})$, the orbit $G \cdot x$ is a locally closed subvariety of X, and $\overline{G\cdot x} \setminus G \cdot x$ is a union of orbits of strictly smaller dimension. Yang: To be checked.

Proof. Yang: To be continued... □

3 Affine algebraic groups

Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For $x \in G(\mathbf{k})$, we have the left translation of functions $\tau_x : A \to A$ defined by $\tau_x(f)(y) = f(x^{-1}y)$ for $y \in X(\mathbf{k})$.

Lemma 27. Let G be an algebraic group acting on an affine variety $X = \operatorname{Spec} A$. For any finite-dimensional subspace $V \subseteq A$, there exists a finite-dimensional G-invariant subspace $W \subseteq A$ containing V. Yang: To be continued...

Theorem 28. Any affine algebraic group is isomorphic to a closed algebraic subgroup of some GL_n.

Theorem 29. Let G be an algebraic group. Then there exists a unique maximal connected affine normal algebraic subgroup G_{aff} of G such that the quotient G/G_{aff} is an abelian variety. This subgroup is called the *affine part* of G. Yang: To be continued...

Theorem 30. Let G be an algebraic group. Then there exists a smallest normal connected algebraic subgroup G_{ant} of G such that the quotient G/G_{ant} is affine. This subgroup is called the *anti-affine part* of G. Moreover, G_{ant} is contained in the center of G^0 and is smooth and connected. Yang: To be continued...