# Introduction to Moduli Problems

Let $X$ be a smooth projective curve of genus $g$ over an algebraically closed field $\Bbbk$ of characteristic $0$.

We are interested in the moduli space of vector bundles on $X$.

## 1 Moduli functors

Let $S$ be a noetherian scheme and $T$ is a scheme of finite type over $S$. Recall the Yoneda lemma: there is a full and faithful functor

$$h : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathrm{Fun}((\mathbf{Sch}_S)^{\mathrm{op}}, \mathbf{Set}), \quad T \mapsto h_T(S) := \mathrm{Hom}_{\mathbf{Sch}_S}(T, S).$$

A functor $F : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$ is *representable* if there exists a scheme $M$ over $S$ such that $F \cong h_M$. We say that $M$ is *the fine moduli space* of $F$.

**Remark 1.** If $F$ is representable by $M$, then there is a universal object $\mathcal{U} \in F(M)$ given by $\mathrm{id}_M \in h_M(M)$ satisfying the following universal property: for any $T \in \mathbf{Sch}_S$ and any $\xi \in F(T)$, there exists a unique morphism $f : T \to M$ such that $F(f)(\mathcal{U}) = \xi$.

The most famous example of representable functor is the Quot functor. Let $S$ be a noetherian scheme, $\pi : X \to S$ a projective morphism, $\mathcal{L}$ a relatively ample line bundle on $X$, $\mathcal{F}$ a coherent sheaf on $X$, and $P \in \mathbb{Q}[t]$ a polynomial. We define a functor

$$\begin{aligned} \mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{F}/X/S} :& (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set} \\ & T \mapsto \{\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q} \mid \mathcal{Q} \text{ is flat over } T, \forall t \in T, \mathcal{Q}_t \text{ has Hilbert polynomial } P\}/\sim, \end{aligned}$$

where $\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}$ and $\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}'$ are equivalent if $\mathrm{Ker}(\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}) = \mathrm{Ker}(\pi_T^*\mathcal{F} \twoheadrightarrow \mathcal{Q}')$.

By Grothendieck, $\mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{F}/X/S}$ is representable by a projective $S$-scheme $\mathrm{Quot}^{P,\mathcal{L}}_{\mathcal{F}/X/S}$. Yang: Reference...

If we take $S = \operatorname{Spec}\Bbbk$, $X$ a projective variety and $\mathcal{F} = \mathcal{O}_X$. Then the Quot functor $\mathcal{Q}uot^{P,\mathcal{L}}_{\mathcal{O}_X/X/\Bbbk}$ becomes the Hilbert functor $\mathcal{H}ilb^{P,\mathcal{L}}_{X/\operatorname{Spec}\Bbbk}$, which is representable by a projective $\Bbbk$-scheme called the *Hilbert scheme* $\mathrm{Hilb}^{P,\mathcal{L}}_X$.

## 2 Moduli functor of vector bundles

Consider the functor

$$\begin{aligned} \tilde{\mathcal{M}}_{r,d} :& (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set} \\ & T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a vector bundle on } X \times T \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\}/\sim, \end{aligned}$$

where $\mathcal{E} \sim \mathcal{E}'$ if there exists a line bundle $\mathcal{L}$ on $T$ such that $\mathcal{E}' \cong \mathcal{E} \otimes \pi_T^*\mathcal{L}$, where $\pi_T : X \times T \to T$ is the projection.

Unfortunately, $\tilde{\mathcal{M}}_{r,d}$ is not representable. There are two main reasons:

- unboundedness and
- jumping phenomenon.

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**Definition 2.** A family of vector bundles on $X$ is *bounded* if there exists a scheme $S$ of finite type over $\Bbbk$ and a vector bundle $\mathcal{E}$ on $X \times S$ such that every vector bundle in the family is isomorphic to $\mathcal{E}_s$ for some $s \in S$.

If $\tilde{\mathcal{M}}_{r,d}$ is representable by a scheme $M$ of finite type over $\Bbbk$, then the family of vector bundles parametrized by $M$ is bounded. This is impossible since if so, $\{h^0(X,\mathcal{E}) \mid \mathcal{E} \in \tilde{\mathcal{M}}_{r,d}(\Bbbk)\}$ is bounded by semicontinuity theorem, which is not true. For example, consider the family $\mathcal{E}_n = \mathcal{O}_X(np) \oplus \mathcal{O}_X(-np) \in \tilde{\mathcal{M}}_{2,0}(\Bbbk)$ for $n \geq 0$, where $p \in X$ is a fixed point. By Riemann-Roch theorem, we have $h^0(X,\mathcal{E}_n) = n + 1 - g$ for $n$ sufficiently large.

**Example 3.** Let us see a jumping phenomenon example due to Ress. Let $\mathcal{E}$ be a vector bundle on $X$ of rank $r$ and degree $d$ with a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}.$$

On $X \times \mathbb{A}^1$, we can construct a vector bundle $\mathcal{F}$ by "deforming" $\mathcal{E}$ to $\bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}$ as follows: let $t$ be the coordinate of $\mathbb{A}^1$, and define $\mathcal{F}$ to be the subsheaf of $\pi_X^*\mathcal{E}$ generated by $t^{-i} \cdot \pi_X^*\mathcal{E}_i$ for $1 \leq i \leq r$, where $\pi_X : X \times \mathbb{A}^1 \to X$ is the projection. Then $\mathcal{F}$ is a vector bundle on $X \times \mathbb{A}^1$ of rank $r$ and degree $d$. We have

$$\mathcal{F}_t \cong \begin{cases} \mathcal{E}, & t \neq 0, \\ \bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}, & t = 0. \end{cases}$$

This is called the *jumping phenomenon.*

For a concrete example, let $X = \mathbb{P}^1$, we have an exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O} \to 0.$$

Let $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and consider the filtration $F : 0 \subset \mathcal{O}(-2) \subset \mathcal{E}$. Yang: To be continued...

If $\tilde{\mathcal{M}}_{r,d}$ is representable by a scheme $M$, then the family of vector bundles parametrized by $M$ does not have jumping phenomenon. Yang: To be continued...

To fix the above problems, we need to

- restrict to a smaller family of vector bundles,
- kill jumping phenomenon, and
- weaken the notion of representability.

## 3 Coarse moduli space

Let $F : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$ be a functor, $M$ a scheme over $S$, and $\eta : F \to h_M$ a natural transformation. We say that $(M, \eta)$ *corepresents* $F$ if it satisfies the following universal property: for any scheme $N$ over $S$ and any natural transformation $\eta' : F \to h_N$, there exists a unique morphism $f : M \to N$ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\eta'"'] & h_M \arrow[d, "h_f"] \\ h_N \arrow[r, "\sim"] & N. \end{tikzcd}$$

**Definition 4.** A scheme $M$ over $S$ is called the *coarse moduli space* of $F$ if

(a) there exists a natural transformation $\eta : F \to h_M$ such that $(M, \eta)$ corepresents $F$;

(b) $\eta_{\Bbbk} : F(\Bbbk) \to M(\Bbbk)$ is a bijection.

For a vector bundle $\mathcal{E}$ of rank $r$ and degree $d$ on $X$, we define its slope to be $\mu(\mathcal{E}) := d/r$. We say that $\mathcal{E}$ is *stable* (resp. *semistable*) if for any proper sub-bundle $\mathcal{F} \subset \mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$).

**Proposition 5.** If $\mathcal{E}$ is a stable vector bundle on $X$ and $\mathcal{F}$ is a semistable vector bundle on $X$ with $\mu(\mathcal{E}) \geq \mu(\mathcal{F})$, then any non-zero homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ is an isomorphism.
If $\mathcal{E}$ and $\mathcal{F}$ are stable vector bundles on $X$ with $\mu(\mathcal{E}) \geq \mu(\mathcal{F})$, then any non-zero homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ is an isomorphism. In particular, if $\mathcal{E}$ is a stable vector bundle, then $\mathsf{End}(\mathcal{E}) \cong \Bbbk$. Yang: To be continued...

**Corollary 6.** A stable vector bundle is simple.

**Lemma 7.** Let $\mathcal{E}$ be a semistable vector bundle on $X$.

(a) if $\mu(\mathcal{E}) > 2g - 2$, then $H^1(X, \mathcal{E}) = 0$;

(b) if $\mu(\mathcal{E}) > 2g - 1$, then $\mathcal{E}$ is globally generated.

Let $S_{r,d}$ be set of isomorphism classes of semistable vector bundles on $X$ of rank $r$ and degree $d$.

**Proposition 8.** The family $S_{r,d}$ is bounded.

**Definition 9** (Jordan-Hölder filtration)**.** Let $\mathcal{E}$ be a semistable vector bundle on $X$. A *Jordan-Hölder filtration* of $\mathcal{E}$ is a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

such that each successive quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a stable vector bundle with slope $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E})$. The associated graded object of $F$ is defined to be

$$\mathrm{gr}(\mathcal{E}) := \bigoplus_{i=1}^{n} \mathcal{E}_i/\mathcal{E}_{i-1}.$$

Any semistable vector bundle on $X$ admits a Jordan-Hölder filtration, and the associated graded object is unique up to isomorphism. Yang: To be continued...

**Definition 10** (S-equivalence)**.** Two semistable vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $X$ are S-equivalent if their associated graded objects $\mathsf{gr}(\mathcal{E})$ and $\mathsf{gr}(\mathcal{F})$ (from their Jordan-Hölder filtrations) are isomorphic.

**Definition 11.** We define a functor

$$\begin{aligned} \mathcal{M}_{r,d} : &(\mathbf{Sch}_{\Bbbk})^{\mathrm{op}} \to \mathbf{Set} \\ &T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a family of semistable vector bundles on } X \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\}/\sim, \end{aligned}$$

where $\mathcal{E} \sim \mathcal{E}'$ if for any $t \in T$, the vector bundles $\mathcal{E}_t$ and $\mathcal{E}'_t$ are S-equivalent. Yang: To be continued...