# Schemes and Varieties



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# Contents





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# 1 Definition and First Properties of Schemes

## 1.1 Locally Ringed Space

**Definition 1.1.** Let $X$ be a topological space. A *presheaf* of sets (resp. abelian groups, rings, etc.) on $X$ is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Set}$ (resp. $\mathbf{Ab}$, $\mathbf{Ring}$, etc.), where $\mathbf{Open}(X)$ is the category of open subsets of $X$ with inclusions as morphisms.
A presheaf $\mathcal{F}$ is a *sheaf* if sections can be glued uniquely. More precisely, for every open covering $\{U_i\}_{i\in I}$ of an open set $U \subset X$ and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

**Example 1.2.** Let $X$ be a real (resp. complex) manifold. The assignment $U \mapsto C^\infty(U,\mathbb{R})$ (resp. $U \mapsto \{\text{holomorphic functions on } U\}$) defines a sheaf of rings on $X$.

**Example 1.3.** Let $X$ be a non-connected topological space. The assignment

$$U \mapsto \{\text{constant functions on } U\}$$

defines a presheaf $\mathcal{C}$ of rings on $X$ but not a sheaf.

For a concrete example, let $X = (0,1)\cup(2,3)$ with the subspace topology from $\mathbb{R}$. Consider the open covering $\{(0,1),(2,3)\}$ of $X$. The sections $s_1 = 1 \in \mathcal{C}((0,1))$ and $s_2 = 2 \in \mathcal{C}((2,3))$ agree on the intersection (which is empty), but there is no global section $s \in \mathcal{C}(X)$ such that $s|_{(0,1)} = s_1$ and $s|_{(2,3)} = s_2$.

**Definition 1.4.** A *locally ringed space* is a pair $(X,\mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of rings on $X$ such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
A *morphism of locally ringed spaces* $f : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the induced map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism, i.e., it maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$.

**Example 1.5.** Let $p$ be a prime number. Then the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is a homomorphism of local rings but not a local homomorphism. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal $(p)$.

**Example 1.6** (Glue morphisms)**.** Let $f : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a morphism of locally ringed spaces. If $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$, then the restriction $f|_U : (U,\mathcal{O}_X|_U) \to (V,\mathcal{O}_Y|_V)$ is a morphism of locally ringed spaces. Conversely, if $\{U_i\}_{i\in I}$ is an open covering of $X$ and for each $i \in I$, we have a morphism $f_i : (U_i,\mathcal{O}_X|_{U_i}) \to (Y,\mathcal{O}_Y)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a unique morphism $f : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

**Example 1.7** (Glue locally ringed space)**.** We construct a locally ringed space by gluing open subspaces. Let $(X_i,\mathcal{O}_{X_i})$ be locally ringed spaces for $i \in I$ and $(U_{ij},\mathcal{O}_{X_i}|_{U_{ij}})$ be open subspaces for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij},\mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji},\mathcal{O}_{X_j}|_{U_{ji}})$ such that

(a) $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;

(b) $\varphi_{ij}(U_{ij}\cap U_{ik}) = U_{ji}\cap U_{jk}$ for all $i, j \in I$;

(c) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$ for all $i, j, k \in I$.

Then there exists a locally ringed space $(X, \mathcal{O}_X)$ and open immersions $\psi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ uniquely up to isomorphism such that

(a) $\varphi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ for all $i, j \in I$;

(b) the following diagram

$$\begin{tikzcd}
(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \arrow[r, hook] \arrow[d, "\varphi_{ij}"'] & (X_i, \mathcal{O}_{X_i}) \arrow[r, hook, "\psi_i"] & (X, \mathcal{O}_X) \arrow[d, "="] \\
(U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}}) \arrow[r, hook] & (X_j, \mathcal{O}_{X_j}) \arrow[r, hook, "\psi_j"] & (X, \mathcal{O}_X)
\end{tikzcd}$$

commutes for all $i, j \in I$;

(c) $X = \bigcup_{i \in I} \psi_i(X_i)$.

Such $(X, \mathcal{O}_X)$ is called *the locally ringed space obtained by gluing the* $(X_i, \mathcal{O}_{X_i})$ *along the* $\varphi_{ij}$.

First $\varphi_{ij}$ induces an equivalence relation $\sim$ on the disjoint union $\coprod_{i \in I} X_i$. By taking the quotient space, we can glue the underlying topological spaces to get a topological space $X$. The structure sheaf $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) := \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \;\middle|\; s_i|_{U_{ij}} = \varphi_{ij}^{\sharp}(s_j|_{U_{ji}}) \text{ for all } i, j \in I \right\}.$$

Easy to check that $(X, \mathcal{O}_X)$ is a locally ringed space and satisfies the required properties. If there is another locally ringed space $(X', \mathcal{O}_{X'})$ with $\psi'_i$ satisfying the same properties, then by gluing $\psi'_i \circ \psi_i^{-1}$ we get an isomorphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$.

## 1.2 Schemes

Let $R$ be a ring. Recall that the *spectrum* of $R$, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of $R$ equipped with the Zariski topology, where the closed sets are of the form $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$ for some ideal $I \subset R$.

For each $f \in R$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Such $D(f)$ is open in $\operatorname{Spec} R$ and called a *principal open set*.

**Proposition 1.8.** Let $R$ be a ring. The collection of principal open sets $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$.

*Proof.* Yang: To be continued □

Define a sheaf of rings on $\operatorname{Spec} R$ by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R[1/f].$$

Then $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

**Definition 1.9.** An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring $R$. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}_{i\in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for each $i \in I$.
A *morphism of schemes* is a morphism of locally ringed spaces.
These data form a category, denoted by $\mathbf{Sch}$. If we fix a base scheme $S$, then an $S$*-scheme* is a scheme $X$ together with a morphism $X \to S$. The category of $S$-schemes is denoted by $\mathbf{Sch}/S$ or $\mathbf{Sch}_S$.

**Theorem 1.10.** The functor $\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is fully faithful and induces an equivalence of categories between the category of rings and the category of affine schemes. Yang: To be continued

**Definition 1.11.** A morphism of schemes $f : X \to Y$ is an *open immersion* (resp. *closed immersion*) if $f$ induces an isomorphism of $X$ onto an open (resp. closed) subscheme of $Y$. An *immersion* is a morphism which factors as a closed immersion followed by an open immersion. Yang: To be continued

**Definition 1.12.** Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of $f$ is the smallest closed subscheme $Z$ of $Y$ such that $f$ factors through $Z$. More precisely, if $Y = \operatorname{Spec} A$ is affine, then the scheme theoretic image of $f$ is $\operatorname{Spec}(A/\ker(f^\sharp))$, where $f^\sharp : A \to \Gamma(X, \mathcal{O}_X)$ is the induced map on global sections. In general, we can cover $Y$ by affine open subsets and glue the scheme theoretic images on each affine open subset to get the scheme theoretic image of $f$. Yang: To be continued

**Example 1.13** (Glue open subschemes)**.** The construction in Example 1.7 allows us to glue open subschemes to get a scheme. More precisely, let $(X_i, \mathcal{O}_{X_i})$ be schemes for $i \in I$ and $(U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}})$ be open subschemes for $i, j \in I$. Suppose we have isomorphisms $\varphi_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$ satisfying the cocycle condition as in Example 1.7. Then the locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing the $(X_i, \mathcal{O}_{X_i})$ along the $\varphi_{ij}$ is a scheme.

## 1.3 Integral, reduced and irreducible

## 1.4 Fiber product

## 1.5 Dimension

## 1.6 Noetherian and finite type

## 1.7 Separated and proper

# 2 Category of sheaves of modules

## 2.1 Sheaves of modules, quasi-coherent and coherent sheaves

**Definition 2.1.** Let $X$ be a ringed space with structure sheaf $\mathcal{O}_X$. A **sheaf of (left) $\mathcal{O}_X$-modules** is a sheaf $\mathcal{F}$ on $X$ such that for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ in the sense that for every $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$, we have

$$\rho_{UV}(s \cdot m) = \rho_{UV}(s) \cdot \rho_{UV}(m).$$

Yang: To be continued...

**Example 2.2.** Let $X$ be a scheme. The structure sheaf $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-modules. More generally, any quasi-coherent sheaf (to be defined later) is a sheaf of $\mathcal{O}_X$-modules. In particular, if $X = \operatorname{Spec} A$ is an affine scheme, then for any $A$-module $M$, the associated sheaf $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules. Yang: To be continued...

**Definition 2.3.** Let $X$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **quasi-coherent** if for every point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0,$$

where $I, J$ are (possibly infinite) index sets. Yang: To be continued...

**Definition 2.4.** Let $X$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **coherent** if it is quasi-coherent and for every point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of finite free $\mathcal{O}_U$-modules, i.e., there exists an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0,$$

where $m, n$ are finite integers. Yang: To be continued...

## 2.2 As abelian categories

**Theorem 2.5.** Let $X$ be a ringed space. The category of sheaves of $\mathcal{O}_X$-modules is an abelian category. Yang: To be continued...

**Theorem 2.6.** Let $X$ be a scheme. The category of quasi-coherent sheaves on $X$ is an abelian category. Yang: To be continued...

**Theorem 2.7.** Let $X$ be a noetherian scheme. The category of coherent sheaves on $X$ is an abelian category. Yang: To be continued...

## 2.3 Relevant functors

**Theorem 2.8.** Let $X$ be a ringed space. The global sections functor

$$\Gamma(X, -) : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\mathcal{O}_X(X)\text{-modules})$$

is left exact. Yang: To be continued...

**Theorem 2.9.** Let $f : X \to Y$ be a morphism of ringed spaces. The direct image functor

$$f_* : (\text{Sheaves of } \mathcal{O}_X\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_Y\text{-modules})$$

is left exact. Yang: To be continued...

**Theorem 2.10.** Let $f : X \to Y$ be a morphism of ringed spaces. The inverse image functor

$$f^* : (\text{Sheaves of } \mathcal{O}_Y\text{-modules}) \to (\text{Sheaves of } \mathcal{O}_X\text{-modules})$$

is right exact. Yang: To be continued...

## 2.4 Locally free sheaves and vector bundles

**Definition 2.11.** Let $X$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **locally free** if for every point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|_U$ is isomorphic to a finite free $\mathcal{O}_U$-module, i.e., there exists an isomorphism of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{F}|_U \cong \mathcal{O}_U^n,$$

where $n$ is a finite integer called the **rank** of $\mathcal{F}$ at $x$. Yang: To be continued...

**Example 2.12.** A **line bundle** on a scheme $X$ is a locally free sheaf of rank 1. The sheaf of differentials $\Omega_{X/k}$ on a smooth variety $X$ over a field $k$ is a locally free sheaf of rank equal to the dimension of $X$. Yang: To be continued...

**Theorem 2.13.** Let $X$ be a scheme. There is an equivalence of categories between the category of locally free sheaves of finite rank on $X$ and the category of vector bundles on $X$. Yang: To be continued...

## 2.5 Cohomological theory

**Theorem 2.14.** Let $X$ be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

**Theorem 2.15.** Let $X$ be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on $X$. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

**Theorem 2.16.** Let $X$ be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on $X$. Then the cohomology groups $H^i(X, \mathcal{F})$ are $\mathcal{O}_X(X)$-modules for all $i \geq 0$. Yang: To be continued...

# 3 Normal, Cohen-Macaulay, and regular schemes

## 3.1 Tangent spaces

There are many description of the tangent space of a scheme at a point. Here we give one of them.

Let $X$ be a scheme over a field $\mathbf{k}$, and let $x \in X(\mathbf{k})$.

**Proposition 3.1.** Let $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$ be the spectrum of the ring of dual numbers over $\mathbf{k}$ with point $* : \operatorname{Spec} \mathbf{k} \to \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2)$. The tangent space $T_x X$ is naturally isomorphic to the set of morphisms $\operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X$ that send $*$ to $x$, i.e.

$$T_x X \cong \{f : \operatorname{Spec} \mathbf{k}[\epsilon]/(\epsilon^2) \to X \mid f(*) = x\}.$$

*Proof.* Yang: To be filled. □

# 4 Line Bundles and Divisors

## 4.1 Cartier Divisors

## 4.2 Line Bundles and Picard Group

**Definition 4.1.** Let $X$ be a scheme. The *Picard group* of $X$ is defined to be $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. The group operation is given by the tensor product of line bundles.

**Definition 4.2.** Let $X$ be a scheme over a field $\mathbf{k}$ and $\mathcal{L}, \mathcal{L}'$ two line bundles on $X$. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *algebraically equivalent* if there exists a Yang: non-singular variety $T$ over $\mathbf{k}$, two points

$t_0, t_1 \in T(\Bbbk)$ and a line bundle $\mathcal{M}$ on $X \times T$ such that

$$\mathcal{M}|_{X\times\{t_0\}} \cong \mathcal{L}, \quad \mathcal{M}|_{X\times\{t_1\}} \cong \mathcal{L}'.$$

We denote it by $\mathcal{L} \sim_{\mathrm{alg}} \mathcal{L}'$. Yang: To be checked.

### 4.3 Weil Divisors and Reflexive Sheaves

## 5 Line bundles induce morphisms

### 5.1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

**Theorem 5.1.** Let $A$ be a ring and $X$ an $A$-scheme. Let $\mathcal{L}$ be a line bundle on $X$ and $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where $x_i$ are the standard coordinates on $\mathbb{P}^n_A$.

*Proof.* Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where $s_i$ does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let $V_i$ be given by $x_i \neq 0$ in $\mathbb{P}^n_A$. On $U_i$, let $f_i : U_i \to V_i \subseteq \mathbb{P}^n_A$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, $f_i$ and $f_j$ agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}^n_A$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}^n_A$ satisfying the same properties, then on each $U_i$, $g$ must agree with $f_i$ by the same construction. Thus $g = f$. □

**Proposition 5.2.** Let $X$ be a $\mathbf{k}$-scheme for some field $\mathbf{k}$ and $\mathcal{L}$ is a line bundle on $X$. Suppose that $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}^n_{\mathbf{k}}$ and $g : X \to \mathbb{P}^m_{\mathbf{k}}$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^m_{\mathbf{k}}$ which is well defined near image of $f$ and satisfies $g = \phi \circ f$.

*Proof.* Yang: To be continued. □

**Example 5.3.** Let $X = \mathbb{P}^n_A$ with $A$ a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0,\ldots,i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = d$, where $T_i$ are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}^N_A$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\ldots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \ldots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *$d$-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

**Example 5.4.** Let $X = \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ with $A$ a ring and $\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$, where $\pi_1$ and $\pi_2$ are the projections. Let $T_0, \dots, T_m$ and $S_0, \dots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_iS_j = \pi_1^*T_i \otimes \pi_2^*S_j$ for $0 \le i \le m$ and $0 \le j \le n$. They induce a morphism $f : X \to \mathbb{P}_A^{(m+1)(n+1)-1}$. If $A = \mathbf{k}$ is a field, on $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_iy_j : \dots],$$

where the coordinates on the right-hand side are indexed by all $(i, j)$ with $0 \le i \le m$ and $0 \le j \le n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

**Definition 5.5.** A line bundle $\mathcal{L}$ on a scheme $X$ is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. Yang: To be continued.

**Example 5.6.** Let

**Example 5.7.**

**Definition 5.8.** Let $\mathcal{L}$ be a line bundle on a scheme $X$. Yang: To be continued.

**Definition 5.9.** A line bundle $\mathcal{L}$ on a scheme $X$ is *ample* if for every coherent sheaf $\mathcal{F}$ on $X$, there exists $n_0 > 0$ such that for all $n \ge n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

**Definition 5.10.** A line bundle $\mathcal{L}$ on a scheme $X$ is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. Yang: To be continued.

**Theorem 5.11.** Let $X$ be a scheme of finite type over a noetherian ring $A$ and $\mathcal{L}$ a line bundle on $X$. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

**Proposition 5.12.** Let $X$ be a scheme of finite type over a noetherian ring $A$ and $\mathcal{L}$, $\mathcal{M}$ line bundles on $X$. Then we have the following:

(a) if $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;

(b) if $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;

(c) if both $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;

(d) if both $\mathcal{L}$ and $\mathcal{M}$ are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;

(e) if $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

Yang: To be continued.

*Proof.* Yang: To be continued. □

**Proposition 5.13.** Let $X$ be a scheme of finite type over a noetherian ring $A$ and $\mathcal{L}$ a line bundle on $X$. Then $\mathcal{L}$ is very ample if and only if the following two conditions hold:

(a) (separate points) for any two distinct points $x, y \in X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $s(y) \neq 0$;

(b) (separate tangent vectors) for any point $x \in X$ and non-zero tangent vector $v \in T_x X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $v(s) \neq 0$.

Yang: To be continued.

## 5.2 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

**Definition 5.14.** Let $X$ be a normal proper variety over a field $\mathbf{k}$, $D$ a (Cartier) divisor on $X$ and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to $D$ is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

**Definition 5.15.** A *linear system* on a scheme $X$ is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on $X$ and $V \subseteq \Gamma(X, \mathcal{L})$ is a subspace. The dimension of the linear system is $\dim V - 1$. A linear system is *base-point free* if $V$ is base-point free. A linear system is *complete* if $V = \Gamma(X, \mathcal{L})$. Yang: To be continued.

**Definition 5.16.** Let $\mathcal{L}$ be a line bundle on a scheme $X$ and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace. The *base locus* of $V$ is the closed subset

$$\mathrm{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If $\mathrm{Bs}(V) = \varnothing$, we say that $V$ is *base-point free*. Yang: To be continued.

## 5.3 Asymptotic behavior

**Definition 5.17.** Let $X$ be a scheme and $\mathcal{L}$ a line bundle on $X$. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

**Definition 5.18.** A line bundle $\mathcal{L}$ on a scheme $X$ is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. Yang: To be continued.

**Theorem 5.19.** Let $X$ be a scheme over a ring $A$ and $\mathcal{L}$ a semiample line bundle on $X$. Then there exists a morphism $f : X \to Y$ over $A$ such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on $Y$. Moreover, $Y = \operatorname{Proj} R(X, \mathcal{L})$ and $f$ is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. Yang: To be continued.

**Definition 5.20.** A line bundle $\mathcal{L}$ on a scheme $X$ is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large $n$. Yang: To be continued.

**Example 5.21.** Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and $E$ the unique section of $\pi$ with self-intersection $-2$. Yang: To be continued.

# 6 Differentials and duality

# 7 Flat, smooth and étale morphisms

# 8 Relative objects

## 8.1 Relative schemes

**Definition 8.1.** Let $X$ be a scheme. An $\mathcal{O}_X$-algebra is a sheaf . Yang: To be continued...

**Definition 8.2.** Let $X$ be a scheme and $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. The relative Spec of $\mathcal{A}$, denoted by $\operatorname{Spec}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Spec} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. Yang: To be continued...

**Proposition 8.3.** Let $X$ be a scheme and $\mathcal{E}$ be a locally free sheaf of finite rank on $X$. Then the relative Spec of the symmetric algebra of $\mathcal{E}$, denoted by $\mathbb{V}(\mathcal{E}) = \operatorname{Spec}_X \operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}$, is called the geometric vector bundle associated to $\mathcal{E}$. The projection morphism $\pi : \mathbb{V}(\mathcal{E}) \to X$ is affine and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Spec} \operatorname{Sym}_{\mathcal{O}_X(U)} \mathcal{E}(U)$. Yang: To be continued...

**Definition 8.4.** Let $X$ be a scheme and $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_0 = \mathcal{O}_X$ and $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{O}_X$-algebra. The relative Proj of $\mathcal{A}$, denoted by $\operatorname{Proj}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Proj} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. The projection morphism $\pi : \operatorname{Proj}_X \mathcal{A} \to X$ is projective and for any open subset $U \subset X$, we have

$\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{A}(U)$. Yang: To be continued...

## 8.2 Blowing up

**Definition 8.5.** Let $X$ be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blowing up of $X$ along $\mathcal{I}$, denoted by $\mathrm{Bl}_{\mathcal{I}} X$, is defined to be the relative Proj of the Rees algebra of $\mathcal{I}$:

$$\mathrm{Bl}_{\mathcal{I}} X = \mathrm{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{I}^n.$$

The projection morphism $\pi : \mathrm{Bl}_{\mathcal{I}} X \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \mathrm{Bl}_{\mathcal{I}(U)} U$. The exceptional divisor of the blowing up is defined to be the closed subscheme $E = \pi^{-1}(V(\mathcal{I}))$ of $\mathrm{Bl}_{\mathcal{I}} X$. Yang: To be continued...

## 8.3 Relative ampleness and relative morphisms

# 9 Finite morphisms and fibrations