
Notes in Algebraic Geometry



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Chapter 1

The First Properties

1.1 Setup and the first examples

1.1.1 Notations

All schemes are assumed to be separated. For a “scheme” which is not separated, we will use the term “prescheme”.

Let A be a ring. We denote by $\operatorname{Spec} A$ the spectrum of A . For an ideal $I \subset A$, we use $V(I)$ to denote the closed subscheme of $\operatorname{Spec} A$ defined by I .

Let S be $\operatorname{Spec} k$, $\operatorname{Spec} \mathcal{O}_K$ or an algebraic variety. An S -variety is an integral scheme X which is of finite type and flat over S . For an algebraic variety, we mean a k -variety.

We will use k, K to denote fields, and \mathbf{k}, \mathbf{K} to denote their algebraic closure relatively.

Let X be an integral scheme. We denote by $\mathcal{K}(X)$ the function field of X . For a closed point $x \in X$, we denote by $\kappa(x)$ the residue field of x .

We denote the category of S -varieties by \mathbf{Var}_S . We denote by $X(T)$ the set of T -points of X , that is, the set of morphisms $T \rightarrow X$.

Let X be an algebraic variety over k . A geometrical point is referred a morphism $\operatorname{Spec} \mathbf{k} \rightarrow X$.

When refer a point (may not be closed) in a scheme, we will use the notation $\xi \in X$. We use Z_ξ to denote the Zariski closure of $\{\xi\}$ in X . When we talk about a closed point on an algebraic variety, we will use the notation $x \in X(\mathbf{k})$.

Separated and proper morphisms

1.1.2 Examples

Appendix A

Commutative Algebra

A.1 Elementary Results Yang: To be completed

A.1.1 Notations

Proposition A.1.1. Let A be a ring, $\mathfrak{p}, \mathfrak{p}_i$ prime ideals of A and $\mathfrak{a}, \mathfrak{a}_i$ ideals of A .

- (a) Suppose $\mathfrak{a} \subset \bigcup_{i=1}^n \mathfrak{p}_i$. Then there exists i such that $\mathfrak{a} \subset \mathfrak{p}_i$.
- (b) Suppose $\bigcap_{i=1}^n \mathfrak{a}_i \subset \mathfrak{p}$. Then there exists i such that $\mathfrak{a}_i \subset \mathfrak{p}$.

Definition A.1.2. Let A be a ring and M an A -module. The *support* of M is defined as

$$\text{Supp } M := \{\mathfrak{p} \in \text{Spec } A : M_{\mathfrak{p}} \neq 0\}.$$

Proposition A.1.3. Let A be a ring and M a finite A -module. Then $\text{Supp } M = V(\text{Ann } M)$. In particular, $\text{Supp } M$ is a closed subset of $\text{Spec } A$.

Proof. Yang: To be completed. □

A.1.2 Nakayama's Lemma

Theorem A.1.4 (Nakayama's Lemma). Let A be a ring and \mathfrak{M} be its Jacobi radical. Suppose M is a finitely generated A -module. If $\mathfrak{a}M = M$ for $\mathfrak{a} \subset \mathfrak{M}$, then $M = 0$.

Proof. Suppose M is generated by x_1, \dots, x_n . Since $M = \mathfrak{a}M$, formally we have $(x_1, \dots, x_n)^T = \Phi(x_1, \dots, x_n)^T$ for $\Phi \in M_n(\mathfrak{a})$. Then $(\Phi - \text{id})(x_1, \dots, x_n)^T = 0$. Note that $\det(\Phi - \text{id}) = 1 + a$ for $a \in \mathfrak{a} \subset \mathfrak{M}$. Then $\Phi - \text{id}$ is invertible and then $M = 0$. □

Remark A.1.5. The finiteness of M is crucial in Nakayama's Lemma. For example, let $\overline{\mathbb{Z}}$ be the ring of algebraic integers in $\overline{\mathbb{Q}}$. Choose a non-zero prime ideal \mathfrak{p} of $\overline{\mathbb{Z}}$. Then we have that $\mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}} = \mathfrak{p}^2\overline{\mathbb{Z}}_{\mathfrak{p}}$. Indeed, if $a \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$, let $b = \sqrt{a} \in \overline{\mathbb{Z}}_{\mathfrak{p}}$. Then $b^2 = a \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$ and whence $b \in \mathfrak{p}\overline{\mathbb{Z}}_{\mathfrak{p}}$ since \mathfrak{p} is prime. It follows that $a = b^2 \in \mathfrak{p}^2\overline{\mathbb{Z}}_{\mathfrak{p}}$.

Proposition A.1.6 (Geometric form of Nakayama's Lemma). Let $X = \text{Spec } A$ be an affine scheme, $x \in X$ a closed point and \mathcal{F} a coherent sheaf on X . If $a_1, \dots, a_k \in \mathcal{F}(X)$ generate $\mathcal{F}|_x = \mathcal{F} \otimes \kappa(x)$, then there is an open subset $U \subset X$ such that $a_i|_U$ generate $\mathcal{F}(U)$.

Proof. Yang: To be completed. □

Corollary A.1.7.

Proof. Yang: To be completed. □

A.1.3 Nullstellensatz

Theorem A.1.8 (Noether's Normalization Lemma). Let A be a \mathbf{k} -algebra of finite type. Then there is an injection $\mathbf{k}[T_1, \dots, T_d] \hookrightarrow A$ such that A is finite over $\mathbf{k}[T_1, \dots, T_d]$.

Remark A.1.9. Here A does not need to be integral. For example,

Theorem A.1.10 (Hilbert's Nullstellensatz). Let A be a

A.2 Associated prime ideals

A.2.1 Associated prime ideals

Definition A.2.1 (Associated prime ideals). Let A be a noetherian ring and M an A -module. The *associated prime ideals* of M are the prime ideals \mathfrak{p} of form $\text{Ann}(x)$ for some $x \in M$. The set of associated prime ideals of M is denoted by $\text{Ass}(M)$.

Example A.2.2. Let $A = \mathbf{k}[x, y]/(xy)$ and $M = A$. First we see that $(x) = \text{Ann } y, (y) = \text{Ann } x \in \text{Ass } M$. Then we check other prime ideals. For (x, y) , if $xf = yf = 0$, then $f \in (x) \cap (y) = (0)$. If $(x - a) = \text{Ann } f$ for some f , note that $y \in (x - a)$ for $a \in \mathbf{k}^*$, then $f \in (x)$. Hence $f = 0$. Therefore $\text{Ass } M = \{(x), (y)\}$.

Example A.2.3. Let $A = \mathbf{k}[x, y]/(x^2, xy)$ and $M = A$. The underlying space of $\text{Spec } A$ is the y -axis since $\sqrt{(x^2, xy)} = (x)$. First note that $(x) = \text{Ann } y, (x, y) = \text{Ann } x \in \text{Ass } M$. For $(x, y - a)$ with $a \in \mathbf{k}^*$, easily see that $xf = (y - a)f = 0$ implies $f = 0$ since $A = \mathbf{k} \cdot x \oplus \mathbf{k}[y]$ as \mathbf{k} -vector space. Hence $\text{Ass } M = \{(x), (x, y)\}$.

Lemma A.2.4. Let A be a noetherian ring and M an A -module. Then the maximal element of the set

$$\{\text{Ann } x : x \in M_{\mathfrak{p}}, x \neq 0\}$$

belongs to $\text{Ass } M$.

Proof. We just need to show that such $\text{Ann } x$ is prime. Otherwise, there exist $a, b \in A$ such that $ab \in \text{Ann } x$ but $a, b \notin \text{Ann } x$. It follows that $\text{Ann } x \subsetneq \text{Ann } ax$ since $b \in \text{Ann } ax \setminus \text{Ann } x$. This contradicts the maximality of $\text{Ann } x$. \square

An element $a \in A$ is called a zero divisor for M if $M \rightarrow aM, m \mapsto am$ is not injective.

Corollary A.2.5. Let A be a noetherian ring and M an A -module. Then

$$\{\text{zero divisors for } M\} = \bigcup_{\mathfrak{p} \in \text{Ass } M} \mathfrak{p}.$$

Lemma A.2.6. Let A be a noetherian ring and M an A -module. Then $\mathfrak{p} \in \text{Ass}_A M$ iff $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Proof. Suppose $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $\mathfrak{p}A_{\mathfrak{p}} = \text{Ann } y_0/c$ with $y_0 \in M$ and $c \in A \setminus \mathfrak{p}$. For $a \in \text{Ann } y_0$, $ay_0 = 0$. Then $a/1 \in \mathfrak{p}A_{\mathfrak{p}}$. It follows that $a \in \mathfrak{p}$. Hence $\text{Ann } y_0 \subset \mathfrak{p}$.

Inductively, if $\text{Ann } y_n \subsetneq \mathfrak{p}$, then there exists $b_n \in A \setminus \mathfrak{p}$ such that $y_{n+1} := b_n y_n$, $\text{Ann } y_{n+1} \subset \mathfrak{p}$ and $\text{Ann } y_n \subsetneq \text{Ann } y_{n+1}$. To see this, choose $a_n \in \mathfrak{p} \setminus \text{Ann } y_n$. Then $(a_n/1)y_n = 0$ since $a_n/1 \in \mathfrak{p}A_{\mathfrak{p}}$. By definition, there exist $b_n \in A \setminus \mathfrak{p}$ such that $a_n b_n y_n = 0$. This process must terminate since A is noetherian. Thus $\text{Ann } y_n = \mathfrak{p}$ for some n . Hence $\mathfrak{p} \in \text{Ass}_A M$.

Conversely, suppose $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$. If $(a/s)(x/1) = 0 \in M_{\mathfrak{p}}$, there exist $t \in A \setminus \mathfrak{p}$ such that $tax = 0$. It follows that $ta \in \mathfrak{p}$ and then $(a/s) \in \mathfrak{p}A_{\mathfrak{p}}$. Hence $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. \square

Proposition A.2.7. We have $\text{Ass } M \subset \text{Supp } M$. Moreover, if $\mathfrak{p} \in \text{Supp } M$ satisfies $V(\mathfrak{p})$ is an irreducible component of $\text{Supp } M$, then $\mathfrak{p} \in \text{Ass } M$.

Proof. For any $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$, we have $A/\mathfrak{p} \cong A \cdot x \subset M$. Tensoring with $A_{\mathfrak{p}}$ gives $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$ since $A_{\mathfrak{p}}$ is flat. Hence $M_{\mathfrak{p}} \neq 0$ and $\mathfrak{p} \in \text{Supp } M$.

Now suppose $\mathfrak{p} \in \text{Supp } M$ and $V(\mathfrak{p})$ is an irreducible component of $\text{Supp } M$. First we show that $\mathfrak{p} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Let $x \in M_{\mathfrak{p}}$ such that $\text{Ann } x$ is maximal in the set

$$\{\text{Ann } x : x \in M_{\mathfrak{p}}, x \neq 0\}.$$

Then we claim that $\text{Ann } x = \mathfrak{p}A_{\mathfrak{p}}$. First, $\text{Ann } x$ is prime by Lemma A.2.4. If $\text{Ann } x \neq \mathfrak{p}$, then $V(\text{Ann } x) \supset V(\mathfrak{p})$. This implies that $\text{Ann } x \notin \text{Supp } M_{\mathfrak{p}}$ since $\text{Supp } M_{\mathfrak{p}} = \text{Supp } M \cap \text{Spec } A_{\mathfrak{p}}$. This is a contradiction. Thus $\mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. By Lemma A.2.6, we have $\mathfrak{p} \in \text{Ass } M$. \square

Remark A.2.8. The existence of irreducible component is guaranteed by Zorn's Lemma.

Definition A.2.9. A prime ideal $\mathfrak{p} \in \text{Ass } M$ is called *embedded* if $V(\mathfrak{p})$ is not an irreducible component of $\text{Supp } M$.

Example A.2.10. For $M = A = \mathbf{k}[x, y]/(x^2, xy)$, the origin (x, y) is an embedded point.

Proposition A.2.11. If we have exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$, then $\text{Ass } M_2 \subset \text{Ass } M_1 \cup \text{Ass } M_3$.

Proof. Let $\mathfrak{p} = \text{Ann } x \in \text{Ass } M_2 \setminus \text{Ass } M_1$. Then the image $[x]$ of x in M_3 is not equal to 0. We have that $\text{Ann } x \subset \text{Ann}[x]$. If $a \in \text{Ann}[x] \setminus \text{Ann } x$, then $ax \in M_1$. Since $\text{Ann } x \subsetneq \text{Ann } ax$, there is $b \in \text{Ann } ax \setminus \text{Ann } x$. However, it implies $ba \in \text{Ann } x$, and then $a \in \text{Ann } x$ since $\text{Ann } x$ is prime, which is a contradiction. \square

Corollary A.2.12. If M is finitely generated, then the set $\text{Ass } M$ is finite.

Proof. For $\mathfrak{p} = \text{Ann } x \in \text{Ass } M$, we know that the submodule M_1 generated by x is isomorphic to A/\mathfrak{p} . Inductively, we can choose M_n be the preimage of a submodule of M/M_{n-1} which is isomorphic to A/\mathfrak{q} for some $\mathfrak{q} \in \text{Ass } M/M_{n-1}$. We can take an ascending sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime \mathfrak{p}_i . Since M is finitely generated, this is a finite sequence. Then the conclusion follows by Proposition A.2.11. \square

A.2.2 Primary decomposition

Definition A.2.13. An A -module is called *co-primary* if $\text{Ass } M$ has a single element. Let M be an A -module and $N \subset M$ a submodule. Then N is called *primary* if M/N is co-primary. If $\text{Ass } M/N = \{\mathfrak{p}\}$, then N is called \mathfrak{p} -primary.

Remark A.2.14. This definition coincide with primary ideals in the case $M = A$. Recall an ideal $\mathfrak{q} \subset A$ is called *primary* if $\forall ab \in \mathfrak{p}, a \notin \mathfrak{q}$ implies $b^n \in \mathfrak{q}$ for some n .

Let \mathfrak{q} be a \mathfrak{q} -primary ideal. Since $\text{Supp } A/\mathfrak{q} = \{\mathfrak{p}\}$, $\mathfrak{p} \in \text{Ass } A/\mathfrak{q}$. Suppose $\text{Ann}[a] \in \text{Ass } A/\mathfrak{q}$. Then $\mathfrak{p} \subset \text{Ann}[a]$ since $V(\mathfrak{p}) = \text{Supp } A/\mathfrak{q}$. If $b \in \text{Ann}[a]$, then $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Hence $b^n \in \mathfrak{q}$, and then $b \in \mathfrak{p}$. This shows that $\text{Ass } A/\mathfrak{q} = \{\mathfrak{p}\}$ and \mathfrak{q} is \mathfrak{p} -primary as an A -submodule.

Let $\mathfrak{q} \subset A$ be a \mathfrak{p} -primary A -submodule. First we have $\mathfrak{p} = \sqrt{\mathfrak{q}}$ since $V(\mathfrak{p})$ is the unique irreducible component of $\text{Supp } A/\mathfrak{q}$. Suppose $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$. Then $b \in \text{Ann}[a] \subset \mathfrak{p}$ since \mathfrak{p} is the unique maximal element in $\{\text{Ann}[c] : c \in A \setminus \mathfrak{q}\}$. This implies that $b^n \in \mathfrak{q}$.

Definition A.2.15. Let A be a noetherian ring, M an A -module and $N \subset M$ a submodule. A *minimal primary decomposition* of N in M is a finite set of primary submodules $\{Q_i\}_{i=1}^n$ such that

$$N = \bigcap_{i=1}^n Q_i,$$

no Q_i can be omitted and $\text{Ass } M/Q_i$ are pairwise distinct. For $\text{Ass } M/Q_i = \{\mathfrak{p}\}$, Q_i is called belonging to \mathfrak{p} .

Indeed, if $N \subset M$ admits a minimal primary decomposition $N = \bigcap Q_i$ with Q_i belonging to \mathfrak{p} , then $\text{Ass}(M/N) = \{\mathfrak{p}\}$. For given i , consider $N_i := \bigcap_{j \neq i} Q_j$, then $N_i/N \cong (N_i + Q_i)/Q_i$. Since $N_i \neq N$, $\text{Ass } N_i/N \neq \emptyset$. On the other hand, $\text{Ass } N_i/N \subset \text{Ass } M/Q_i = \{\mathfrak{p}\}$. It follows that $\text{Ass } N_i/N = \{\mathfrak{p}\}$, whence $\mathfrak{p}_i \in \text{Ass } M/N$. Conversely, we have an injection $M/N \hookrightarrow \bigoplus M/Q_i$, so $\text{Ass } M/N \subset \bigcup \text{Ass } M/Q_i$. Due to this, if Q_i belongs to \mathfrak{p} , we also say that Q_i is the \mathfrak{p} -component of N .

Proposition A.2.16. Suppose $N \subset M$ has a minimal primary decomposition. If $\mathfrak{p} \in \text{Ass } M/N$ is not embedded, then the \mathfrak{p} component of N is unique. Explicitly, we have $Q = \nu^{-1}(N_{\mathfrak{p}})$, where $\nu : M \rightarrow M_{\mathfrak{p}}$.

Proof. First we show that $Q = \nu^{-1}(Q_{\mathfrak{p}})$. Clearly $Q \subset \nu^{-1}(Q_{\mathfrak{p}})$. Suppose $x \in \nu^{-1}(Q_{\mathfrak{p}})$. Then there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in Q$. That is, $[sx] = 0 \in M/Q$. If $[x] \neq 0$, we have $s \in \text{Ann}[x] \subset \mathfrak{p}$. This contradiction enforces $Q = \nu^{-1}(Q_{\mathfrak{p}})$.

Then we show that $N_{\mathfrak{p}} = Q_{\mathfrak{p}}$. Just need to show that for $\mathfrak{p}' \neq \mathfrak{p}$ and the \mathfrak{p}' component Q' of N , $Q'_{\mathfrak{p}} = M_{\mathfrak{p}}$. Since \mathfrak{p} is not embedded, $\mathfrak{p}' \not\subset \mathfrak{p}$. Then $\mathfrak{p} \notin V(\mathfrak{p}') = \text{Supp } M/Q'$. So $M_{\mathfrak{p}}/Q'_{\mathfrak{p}} = 0$. \square

Example A.2.17. If \mathfrak{p} is embedded, then its components may not be unique. For example, let $M = A = \mathbb{k}[x, y]/(x^2, xy)$. Then for every $n \in \mathbb{Z}_{\geq 1}$, $(x) \cap (x^2, xy, y^n)$ is a minimal primary decomposition of $(0) \subset M$.

Let A be a noetherian ring and $\mathfrak{p} \subset A$ a prime ideal. We consider the \mathfrak{p} component of \mathfrak{p}^n , which is called n -th symbolic power of \mathfrak{p} , denoted by $\mathfrak{p}^{(n)}$. We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A$. In general, $\mathfrak{p}^{(n)}$ is not equal to \mathfrak{p}^n ; see below example.

Example A.2.18. Let $A = \mathbb{k}[x, y, z, w]/(y^2 - zx^2, yz - xw)$ and $\mathfrak{p} = (y, z, w)$. We have $z = y^2/x^2, w = yz/x \in \mathfrak{p}^2 A_{\mathfrak{p}}$, whence $\mathfrak{p}^2 A_{\mathfrak{p}} = (z, w) \neq \mathfrak{p}^2$.

Theorem A.2.19. Let A be a noetherian ring and M an A -module. Then for every $\mathfrak{p} \in \text{Ass } M$, there is a \mathfrak{p} -primary submodule $Q(\mathfrak{p})$ such that

$$(0) = \bigcap_{\mathfrak{p} \in \text{Ass } M} Q(\mathfrak{p}).$$

Proof. Consider the set

$$\mathcal{N} := \{N \subset M : \mathfrak{p} \notin \text{Ass } N\}.$$

Note that $\text{Ass} \bigcup N_i = \bigcup \text{Ass } N_i$ by definition of associated prime ideals. Then it is easy to check that \mathcal{N} satisfies the conditions of Zorn's Lemma. Hence \mathcal{N} has a maximal element $Q(\mathfrak{p})$. We claim that $Q(\mathfrak{p})$ is \mathfrak{p} -primary. If there is $\mathfrak{p}' \neq \mathfrak{p} \in \text{Ass } M/Q(\mathfrak{p})$, then there is a submodule $N' \cong A/\mathfrak{p}'$. Let N'' be the preimage of N' in M . We have $Q(\mathfrak{p}) \subsetneq N''$ and $N'' \in \mathcal{N}$. This is a contradiction. By the fact $\text{Ass} \bigcap N_i = \bigcap \text{Ass } N_i$, we get the conclusion. \square

Corollary A.2.20. Let A be a noetherian ring and M a finitely generated A -module. Then every submodule of M has a minimal primary decomposition.

A.3 Dimension and Depth

There are three numbers measuring the “size” of a local ring (A, \mathfrak{m}) :

- $\dim A$: the Krull dimension of A .
- $\text{depth } A$: the depth of A .
- $\dim_{\kappa(\mathfrak{m})} T_{A, \mathfrak{m}}$: the dimension of Zariski tangent space $T_{A, \mathfrak{m}} := (\mathfrak{m}/\mathfrak{m}^2)^{\vee}$ as a $\kappa(\mathfrak{m})$ -vector space.

Somehow the Krull dimension is “homological” and the depth is “cohomological”.

Definition A.3.1. Let A be a noetherian ring. The *height of a prime ideal* \mathfrak{p} in A is defined as the maximum length of chains of prime ideals contained in \mathfrak{p} , that is,

$$\text{ht}(\mathfrak{p}) := \sup\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

The *Krull dimension* of A is defined as

$$\dim A := \max_{\mathfrak{p} \in \text{Spec } A} \text{ht}(\mathfrak{p}).$$

Example A.3.2. Let A be a PID. For every two non-zero prime ideals \mathfrak{p}_1 and \mathfrak{p}_2 , if $\mathfrak{p}_1 = t_1 A \subset \mathfrak{p}_2 = t_2 A$, then $t_2 \mid t_1$ and hence $\mathfrak{p}_1 = \mathfrak{p}_2$. It follows that $\dim A = 1$. Consequently, the ring of integers \mathbb{Z} and the polynomial ring $\mathbb{k}[T]$ in one variable over a field have Krull dimension 1.

Definition A.3.3. Let A be a noetherian ring, $I \subset A$ an ideal and M a finitely generated A -module. A sequence $t_1, \dots, t_n \in I$ is called an *M -regular sequence in I* if t_i is not a zero divisor on $M/(t_1, \dots, t_{i-1})M$ for all i .

Example A.3.4. Let $A = \mathbb{k}[x, y]/(x^2, xy)$ and $I = (x, y)$. Then $\text{depth}_I A = 0$.

Definition A.3.5. Let A be a noetherian ring. For every $\mathfrak{p} \in \text{Spec } A$, $\mathfrak{p}/\mathfrak{p}^2$ is a vector space over $\kappa(\mathfrak{p})$. The *Zariski's tangent space* $T_{A,\mathfrak{p}}$ of A at \mathfrak{p} is defined as $(\mathfrak{p}/\mathfrak{p}^2)^\vee$, the dual $\kappa(\mathfrak{p})$ -vector space of $\mathfrak{p}/\mathfrak{p}^2$.

A.3.1 Artinian Rings and Length of Modules

Definition A.3.6. Let A be a ring and M an A module. A *simple module filtration* of M is a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i-1} is a simple module, i.e. it has no submodule except 0 and itself. If M has a simple module filtration as above, we define the *length* of M as n and say that M has *finite length*.

The following proposition guarantees the length is well-defined.

Proposition A.3.7. Suppose M has a simple module filtration $M = M_{0,0} \supsetneq M_{1,0} \supsetneq \cdots \supsetneq M_{n,0} = 0$. Then for any other filtration $M = M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,m} = 0$ with $m > n$, there exist $k < m$ such that $M_{0,k} = M_{0,k+1}$.

Proof. We claim that there are at least $0 \leq k_1 < \cdots < k_{m-n} < m$ satisfies that $M_{0,k_i} = M_{0,k_i+1}$. Let $M_{i,j} := M_{i,0} \cap M_{0,j}$. Inductively on n , we can assume that there exist k_1, \dots, k_{n-m+1} such that $M_{1,k} = M_{1,k+1}$. Consider the sequence

$$M_{0,0}/M_{1,0} \supset (M_{0,1} + M_{1,0})/M_{1,0} \supset \cdots \supset (M_{0,m} + M_{1,0})/M_{1,0} = 0$$

in $M_{0,0}/M_{1,0}$. Since $M_{0,0}/M_{1,0}$ is simple, there is at most one k_i with $M_{0,k_i} + M_{1,0} \neq M_{0,k_i+1} + M_{1,0}$. And note that if $M_{0,k_i} + M_{1,0} = M_{0,k_i+1} + M_{1,0}$ and $M_{0,k_i} \cap M_{1,0} = M_{0,k_i} \cap M_{1,0}$, then $M_{0,k_i} = M_{0,k_i+1}$ by the Five Lemma. \square

Example A.3.8. Let A be a ring and $\mathfrak{m} \in \text{mSpec } A$. Then A/\mathfrak{m} is a simple module. **Yang:** To be completed.

Proposition A.3.9. Let A be a ring and M an A -module. Then M is of finite length iff it satisfies both a.c.c and d.c.c.

Proof. Note that if M has either a strictly ascending chain or a strictly descending chain, M is of infinite length. Conversely, d.c.c guarantee M has a simple submodule and a.c.c guarantee the sequence terminates. \square

Proposition A.3.10. The length $l(-)$ is an additive function for modules of finite length. That is, if we have an exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ with M_i of finite length, then $l(M_2) = l(M_1) + l(M_3)$.

Proof. The simple module filtrations of M_1 and M_3 will give a simple module filtration of M_2 . \square

Proposition A.3.11. Let (A, \mathfrak{m}) be a local ring. Then A is artinian iff $\mathfrak{m}^n = 0$ for some $n \geq 0$.

Proof. Suppose A is artinian. Then the sequence $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ is stable. It follows that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n . By the Nakayama's Lemma A.1.4, $\mathfrak{m}^n = 0$.

Conversely, we have

$$\mathfrak{m} \subset \mathfrak{N} \subset \bigcap_{\text{minimal prime ideal}} \mathfrak{p},$$

whence \mathfrak{m} is minimal. \square

Proposition A.3.12. Let A be a ring. Then A is artinian iff A is of finite length.

Proof. First we show that A has only finite maximal ideal. Otherwise, consider the set $\{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k\}$. It has a minimal element $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ and for any maximal ideal \mathfrak{m} , $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}$. It follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i . Let $\mathfrak{M} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ be the Jacobi radical of A . Consider the sequence $\mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots$ and by Nakayama's Lemma, we have $\mathfrak{M}^k = 0$ for some k . Consider the filtration

$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1^k \supset \mathfrak{m}_1^k \mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0).$$

We have $\mathfrak{m}_1^k \cdots \mathfrak{m}_i^j / \mathfrak{m}_1^k \cdots \mathfrak{m}_i^{j+1}$ is an A/\mathfrak{m}_i -vector space. It is artinian and then of finite length. Hence A is of finite length. \square

Theorem A.3.13. Let A be a ring. Then A is artinian iff A is noetherian and of dimension 0.

Proof. Suppose A is artinian. Then A is noetherian by Proposition A.3.12. Let $\mathfrak{p} \in \text{Spec } A$. Then A/\mathfrak{p} is an artinian integral domain. If there is $a \in A/\mathfrak{p}$ is not invertible, consider $(a) \supset (a^2) \supset \dots$, we see $a = 0$. Hence \mathfrak{p} is maximal and $\dim A = 0$.

Suppose that A is noetherian and of dimension 0. Then every maximal ideal is minimal. In particular, A has only finite maximal ideal $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Let \mathfrak{q}_i be the \mathfrak{p}_i -component of (0) . Then we have $A \hookrightarrow \bigoplus_i A/\mathfrak{q}_i$. We just need to show that A/\mathfrak{q}_i is of finite length as A -module. If $\mathfrak{q}_i \subset \mathfrak{p}_j$, take radical we get $\mathfrak{p}_i \subset \mathfrak{p}_j$ and hence $i = j$. So A/\mathfrak{q}_i is a local ring with maximal ideal $\mathfrak{p}_i A/\mathfrak{q}_i$. Then every element in $\mathfrak{p}_i A/\mathfrak{q}_i$ is nilpotent. Since \mathfrak{p}_i is finitely generated, $(\mathfrak{p}_i A/\mathfrak{q}_i)^k = 0$ for some k . Then A/\mathfrak{q}_i is artinian and then of finite length as A/\mathfrak{q}_i -module. Then the conclusion follows. \square

A.3.2 Dedekind Domains Yang: To be completed

A.3.3 Krull's Principal Ideal Theorem

Theorem A.3.14 (Krull's Principal Ideal Theorem). Let A be a noetherian ring. Suppose $f \in A$ is not a unit. Let \mathfrak{p} be a minimal prime ideal among those containing f . Then $\text{ht}(\mathfrak{p}) \leq 1$.

Proof. By replacing A by $A_{\mathfrak{p}}$, we may assume A is local with maximal ideal \mathfrak{p} . Note that $A/(f)$ is artinian since it has only one prime ideal $\mathfrak{p}/(f)$.

Let $\mathfrak{q} \subsetneq \mathfrak{p}$. Consider the sequence $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \dots$, its image in $A/(f)$ is stationary. Then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{q}^{(n)} + (f) = \mathfrak{q}^{(n+1)} + (f)$. For $x \in \mathfrak{q}^{(n)}$, we may write $x = y + af$ for $y \in \mathfrak{q}^{(n+1)}$. Then $af \in \mathfrak{q}^{(n)}$. Since $\mathfrak{q}^{(n)}$ is \mathfrak{q} -primary and $f \notin \mathfrak{q}$, $a \in \mathfrak{q}^{(n)}$. Then we get $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$. That is, $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = f\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$. Note that $f \in \mathfrak{p}$, by Nakayama's Lemma, $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. That is, $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1} A_{\mathfrak{q}}$. By Nakayama's Lemma again, $\mathfrak{q}^n A_{\mathfrak{q}} = 0$. It follows that $\mathfrak{q} A_{\mathfrak{q}}$ is minimal, whence $A_{\mathfrak{q}}$ is artinian. Therefore, \mathfrak{q} is minimal in A . \square

Corollary A.3.15. Let A be a noetherian local ring. Suppose $f \in A$ is not a unit. Then $\dim A/(f) \geq \dim A - 1$. If f is not contained in a minimal prime ideal, the equality holds.

Proof. Let $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$ be a sequence of prime ideals. By assumption, $f \in \mathfrak{p}_n$. If $f \in \mathfrak{p}_0$, we get a sequence of prime ideals in $A/(f)$ of length n . Now we suppose $f \notin \mathfrak{p}_0$. Then there exists $k \geq 0$ such that $f \in \mathfrak{p}_{k+1} \setminus \mathfrak{p}_k$.

Choose \mathfrak{q} be a minimal prime ideal among those containing (\mathfrak{p}_{k-1}, f) and contained in \mathfrak{p}_{k+1} . Then by Krull's Principal Ideal Theorem A.3.14, $\mathfrak{q}_k \subsetneq \mathfrak{p}_{k+1}$. Replace \mathfrak{p}_k by \mathfrak{q}_k , we have $f \in \mathfrak{q}_k \setminus \mathfrak{p}_{k-1}$.

Repeat this process, we get a sequence $\mathfrak{p}'_0 \subsetneq \dots \subsetneq \mathfrak{p}'_n$ such that $f \in \mathfrak{p}'_1$. This gives a sequence $\mathfrak{p}'_1 \subsetneq \dots \subsetneq \mathfrak{p}'_n$ in $A/(f)$. Hence we get $\dim A/(f) \geq \dim A - 1$.

Since f is not contained in minimal prime ideal, preimage of a minimal prime ideal in $A/(f)$ has height 1. Hence a sequence of prime ideals in A/fA can be extended by a minimal prime ideal in A . It follows that $\dim A/(f) + 1 \leq \dim A$. \square

Proposition A.3.16. Let (A, \mathfrak{m}) be a local noetherian ring with residue field k . Then the following inequalities hold:

$$\text{depth } A \leq \dim A \leq \dim_k T_{A, \mathfrak{m}}.$$

Proof. The first inequality is a direct corollary of Corollary A.3.15.

Let t_1, \dots, t_n be a $\kappa(\mathfrak{m})$ -basis of $\mathfrak{m}/\mathfrak{m}^2$. Then we have $\mathfrak{m}/(t_1, \dots, t_n) + \mathfrak{m}^2 = 0$, whence $\mathfrak{m}/(t_1, \dots, t_n) = \mathfrak{m}(\mathfrak{m}/(t_1, \dots, t_n))$. It follows that $\mathfrak{m} = (t_1, \dots, t_n)$ by Nakayama's Lemma. By Corollary A.3.15,

$$n + \dim A/(t_1, \dots, t_n) \geq n - 1 + \dim A/(t_1, \dots, t_{n-1}) \geq \dots \geq 1 + \dim A/(t_1) \geq \dim A.$$

We conclude the result. \square

Definition A.3.17. Let X be a locally noetherian scheme and $k \in \mathbb{Z}_{\geq 0}$. We say that X *verifies property (R_k)* or is *regular in codimension k* if $\forall \xi \in X$ with $\text{codim } Z_{\xi} \leq k$,

$$\dim_{\kappa(\xi)} T_{X, \xi} = \dim \mathcal{O}_{X, \xi}.$$

We say that X *verifies property (S_k)* if $\forall \xi \in X$ with $\text{depth } \mathcal{O}_{X, \xi} < k$,

$$\text{depth } \mathcal{O}_{X, \xi} = \dim \mathcal{O}_{X, \xi}.$$

Example A.3.18. Let A be a noetherian ring. Then A verifies (S_1) iff A has no embedded point.

Suppose A verifies (S_1) . If $\mathfrak{p} \in \text{Ass } A$, every element in \mathfrak{p} is a zero divisor. Then $\text{depth } A_{\mathfrak{p}} = 0$. It follows that $\dim A_{\mathfrak{p}} = 0$ and then \mathfrak{p} is minimal.

Suppose A has no embedded point. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{depth} A_{\mathfrak{p}} = 0$. This means every element in $\mathfrak{p}A_{\mathfrak{p}}$ is a zero divisor. Then

$$\mathfrak{p} \subset \{\text{zero divisors in } A\} = \bigcup_{\text{minimal prime ideals}} \mathfrak{q}.$$

By Proposition A.1.1, $\mathfrak{p} = \mathfrak{q}$ for some minimal \mathfrak{q} , whence $\operatorname{dim} A_{\mathfrak{p}} = 0$.

Example A.3.19. Let A be a noetherian ring. Then A is reduced iff it verifies (R_0) and (S_1) .

Suppose A is reduced. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all minimal prime ideals of A . We have $\bigcap \mathfrak{p}_i = \mathfrak{N} = (0)$, where \mathfrak{N} is the nilradical of A . Hence A has no embedded point. Since $A_{\mathfrak{p}}$ is artinian, local and reduced, $A_{\mathfrak{p}}$ is a field and hence regular.

Conversely, let $\operatorname{Ass} A$ be equal to $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Then every \mathfrak{p}_i is minimal by (S_1) . Let f be in \mathfrak{N} . Then the image of f in $A_{\mathfrak{p}_i}$ is 0 since by (R_0) , $A_{\mathfrak{p}_i}$ is a field. It follows that $f \in \mathfrak{q}_i$, where \mathfrak{q}_i is the \mathfrak{p}_i component of (0) in A . Hence $f \in \bigcap \mathfrak{q}_i = (0)$. That is, A is reduced.

A.3.4 Cohen-Macaulay rings

Definition A.3.20 (Cohen-Macaulay). A noetherian local ring (A, \mathfrak{m}) is called *Cohen-Macaulay* if $\operatorname{dim} A = \operatorname{depth} A$. A noetherian ring A is called *Cohen-Macaulay* if for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is Cohen-Macaulay. This is equivalent to that A verifies (S_k) for all $k \geq 0$.

Example A.3.21 (Non Cohen-Macaulay rings). **Yang:** To be completed.

Proposition A.3.22. Let $(A, \mathfrak{m}, \mathfrak{k})$ be a noetherian local ring and M a finite A -module. Then

$$\operatorname{depth} M := \inf\{i : \operatorname{Ext}_A^i(\mathfrak{k}, M) \neq 0\}.$$

Proof. Let $a \in \mathfrak{m}$ be M -regular and $N = M/aM$. Then we claim that

$$\inf\{i : \operatorname{Ext}_A^i(\mathfrak{k}, N) \neq 0\} = \inf\{i : \operatorname{Ext}_A^i(\mathfrak{k}, M) \neq 0\} - 1.$$

Indeed, we have an exact sequence

$$0 \rightarrow M \xrightarrow{a} M \rightarrow N \rightarrow 0.$$

It induces a long exact sequence

$$\dots \rightarrow \operatorname{Ext}_A^{i-1}(\mathfrak{k}, M) \rightarrow \operatorname{Ext}_A^{i-1}(\mathfrak{k}, N) \rightarrow \operatorname{Ext}_A^i(\mathfrak{k}, M) \xrightarrow{\operatorname{Ext}_A^i(\mathfrak{k}, \operatorname{Mult}_a)} \operatorname{Ext}_A^i(\mathfrak{k}, M) \rightarrow \dots$$

Note that $a \in \mathfrak{m}$, then $\operatorname{Ext}_A^i(\mathfrak{k}, \operatorname{Mult}_a) = 0$. It follows that when $\operatorname{Ext}_A^{i-1}(\mathfrak{k}, M) = 0$, we have $\operatorname{Ext}_A^{i-1}(\mathfrak{k}, N) = 0$ iff $\operatorname{Ext}_A^i(\mathfrak{k}, M) = 0$, whence the claim.

Let $n = \inf\{i : \operatorname{Ext}_A^i(\mathfrak{k}, M) \neq 0\}$. Induct on n . Suppose first $n = 0$. Since \mathfrak{k} is a simple A -module, there is an injective homomorphism $\mathfrak{k} \rightarrow M$. Then $\mathfrak{m} \in \operatorname{Ass} M$ and hence $\operatorname{depth} M = 0$.

Suppose $n > 0$, let $a_1, \dots, a_m \in \mathfrak{m}$ be any M -regular sequence. Using the claim inductively on $M/(a_1, \dots, a_m)M$, we have $n \geq \operatorname{depth} M$. If M has no regular element, then $\mathfrak{m} \subset \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$. Then $\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} M$. This shows that we can find $x \neq 0 \in M$ such that $\mathfrak{p} = \operatorname{Ann} x$. It gives a homomorphism $\mathfrak{k} = A/\mathfrak{m} \rightarrow M$. That is a contradiction and hence M has a regular element. Let a be M -regular and $N = M/aM$. Then $\operatorname{depth} N = n - 1$ by the claim and induction hypothesis. Hence we have $\operatorname{depth} M \geq n$. \square

Corollary A.3.23. Let A be a noetherian ring, M a finite A -module and $a \in A$ an M -regular element. Then $\operatorname{depth} M = \operatorname{depth} M/aM + 1$.

Corollary A.3.24. Let A be a noetherian ring $a \in A$ a nonzero divisor. Then A verifies (S_d) iff A/aA verifies (S_{d-1}) .

Definition A.3.25. An ideal I of a noetherian ring A is called *unmixed* if

$$\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p}), \quad \forall \mathfrak{p} \in \operatorname{Ass}(A/I).$$

Here $\operatorname{ht}(I)$ is defined as

$$\operatorname{ht}(I) := \inf\{\operatorname{ht}(\mathfrak{p}) : I \subset \mathfrak{p}\}.$$

We say that *the unmixedness theorem holds for a noetherian ring A* if any ideal $I \subset A$ generated by $\operatorname{ht}(I)$ elements is

unmixed. We say that *the unmixedness theorem holds for a locally noetherian scheme X* if $\mathcal{O}_{X,\xi}$ is unmixed for any point $\xi \in X$.

Theorem A.3.26. Let X be a locally noetherian scheme. Then the unmixedness theorem holds for X if and only if X is Cohen-Macaulay.

Proof. We can assume that $X = \operatorname{Spec} A$ is affine.

Suppose X is Cohen-Macaulay. Let $I \subset A$ be an ideal generated by a_1, \dots, a_r with $r = \operatorname{ht}(I)$. We claim that a_1, \dots, a_r is an A -regular sequence. If so, we get that the unmixedness theorem holds for A by applying Example A.3.18 on A/I . Since $\operatorname{ht}(a_1, \dots, a_{r-1}) \leq r-1$ by Krull's Principal Ideal Theorem A.3.14 and $\operatorname{ht}(a_1, \dots, a_r) = r \leq \operatorname{ht}(a_1, \dots, a_{r-1}) + 1$, we have $\operatorname{ht}(a_1, \dots, a_{r-1}) = r-1$. By induction on r , we can assume that a_1, \dots, a_{r-1} is an A -regular sequence. Hence any prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1, \dots, a_{r-1})$ has height $r-1$. Now suppose a_r is a zero divisor in $A/(a_1, \dots, a_{r-1})$. Then there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass} A/(a_1, \dots, a_{r-1})$ such that $a_r \in \mathfrak{p}$. Then $I \subset \mathfrak{p}$ and $\operatorname{ht}(I) \leq r-1$. This contradicts that $\operatorname{ht}(I) = r$.

Suppose the unmixedness theorem holds for A . Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal with $\operatorname{ht}(\mathfrak{p}) = r$. Then $\mathfrak{p} \in \operatorname{Ass} A$ if and only if $\operatorname{ht}(\mathfrak{p}) = 0$. If $r > 0$, there is a nonzero divisor $a \in \mathfrak{p}$. By Krull's Principal Ideal Theorem A.3.14, $\operatorname{ht}(\mathfrak{p}A/aA) = r-1$. Inductively, we can find a regular sequence a_1, \dots, a_r in \mathfrak{p} . Then $\operatorname{depth} A_{\mathfrak{p}} = r$. \square

Theorem A.3.27. Let X be a locally noetherian scheme. Suppose that X is Cohen-Macaulay. Let $F \subset X$ be a closed subset of codimension $\geq k$. Then the restriction $H^i(X, \mathcal{O}_X) \rightarrow H^i(X \setminus F, \mathcal{O}_X)$ is an isomorphism.

Proof. Yang: To be completed. \square

A.3.5 Regular rings

Definition A.3.28. A noetherian ring A is said to be *regular at $\mathfrak{p} \in \operatorname{Spec} A$* if we have

$$\dim_{\kappa(\mathfrak{p})} T_{A,\mathfrak{p}} = \dim A_{\mathfrak{p}},$$

where $\dim A_{\mathfrak{p}}$ is the Krull dimension of the local ring $A_{\mathfrak{p}}$.

A noetherian ring A is said to be *regular* if it is regular at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. This is equivalent to the condition that A verifies (R_k) for all $k \geq 0$.

Definition A.3.29. Let A be a noetherian ring that is regular at $\mathfrak{p} \in \operatorname{Spec} A$. A sequence $t_1, \dots, t_n \in \mathfrak{p}$ is called a *regular system of parameters at \mathfrak{p}* if their images form a basis of the $\kappa(\mathfrak{p})$ -vector space $\mathfrak{p}/\mathfrak{p}^2$.

Proposition A.3.30. Let (A, \mathfrak{m}) be a noetherian local ring that is regular at \mathfrak{m} . Let t_1, \dots, t_n be a regular system of parameters at \mathfrak{m} , $\mathfrak{p}_i = (t_1, \dots, t_i)$ and $\mathfrak{p}_0 = (0)$. Then \mathfrak{p}_i is a prime ideal of height i , and A/\mathfrak{p}_i is a regular local ring for all i . In particular, regular local ring is integral, and the regular system of parameters t_1, \dots, t_n is a regular sequence in A .

Proof. By the Krull's Principal Ideal Theorem A.3.14, we have

$$n-1 = \dim A - 1 \leq \dim A/(t_1) \leq \dim_{\kappa(\mathfrak{m}/(t_1))} T_{A/(t_1), \mathfrak{m}/(t_1)} \leq n-1.$$

Hence $\dim A/(t_1) = n-1$ and $\operatorname{ht}(t_1) = 1$. Since t_2, \dots, t_n generate $\mathfrak{m}/(t_1)$, we have that $A/(t_1)$ is regular at $\mathfrak{m}/(t_1)$ and the images of t_2, \dots, t_n form a regular system of parameters.

For integrality, we induct on the dimension of A . If $\dim A = 0$, then A is a field and hence integral. Suppose $\dim A > 0$, let \mathfrak{q} be a minimal prime ideal of A . Then $t_1 \notin \mathfrak{q}$. We have

$$n-1 = \dim A - 1 \leq \dim A/(\mathfrak{q} + t_1 A) \leq \dim_{\kappa(\mathfrak{q}/(t_1))} T_{A/(\mathfrak{q} + t_1 A), \mathfrak{q}/(t_1)} \leq n-1.$$

By similar arguments, we have $A/(\mathfrak{q} + t_1 A)$ is regular at $\mathfrak{m}/(\mathfrak{q} + t_1 A)$. By induction hypothesis, both of $A/t_1 A$ and $A/(\mathfrak{q} + t_1 A)$ are integral and of dimension $n-1$. Hence $t_1 A = t_1 A + \mathfrak{q}$, i.e. $\mathfrak{q} \subset t_1 A$. For every $a = bt_1 \in \mathfrak{q}$, we have $b \in \mathfrak{q}$ since $t_1 \notin \mathfrak{q}$. Then $\mathfrak{q} \subset t_1 \mathfrak{q} \subset \mathfrak{m} \mathfrak{q}$. By Nakayama's Lemma, $\mathfrak{q} = 0$, whence A is integral. \square

Corollary A.3.31. A regular ring is Cohen-Macaulay.

Corollary A.3.32. A regular ring is normal.

Proposition A.3.33. A noetherian ring A is regular if and only if it is regular at every maximal ideal $\mathfrak{m} \in \text{mSpec } A$.

Proof. Suppose $\mathfrak{p} \subset \mathfrak{m}$ and A is regular at \mathfrak{m} .

Yang: To be completed. □

Remark A.3.34. Let k be arbitrary field, $A = k[T_1, \dots, T_n]$ and g_i irreducible polynomials in one variable T_i over k . Then for every $f \in A$, we can write

$$f = \sum_{I=(i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n} a_I g_1^{i_1} \cdots g_n^{i_n}, \quad a_I \in A, \quad \deg_{T_i} a_I \leq \deg g_i.$$

This is called the *Taylor expansion of f with respect to g_1, \dots, g_n* .

A.4 Finite Algebra and Normality **Yang:** To be completed

Definition A.4.1. An integral domain A is called *normal* if it is integrally closed in its field of fractions $\text{Frac}(A)$.

Lemma A.4.2. Let $A \subset C$ be rings and B the integral closure of A in C , S a multiplicatively closed subset of A . Then the integral closure of $S^{-1}A$ in $S^{-1}C$ is $S^{-1}B$.

Proof. For every $b \in B$ and $\forall s \in S$, there exists $a_i \in A$ s.t.

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0.$$

Then

$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s^1} \left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0.$$

Hence b/s is integral over $S^{-1}A$, $S^{-1}B$ is integral over $S^{-1}A$.

If $c/s \in S^{-1}C$ is integral over $S^{-1}A$, then $\exists a_i \in S^{-1}A$ s.t.

$$\left(\frac{c}{s}\right)^n + a_1 \left(\frac{c}{s}\right)^{n-1} + \cdots + a_n = 0.$$

Then

$$c^n + a_1 s c^{n-1} + \cdots + a_n s^n = 0 \in S^{-1}C$$

Then $\exists t \in S$ s.t.

$$t(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0 \in C.$$

Then

$$(ct)^n + a_1 st(ct)^{n-1} + \cdots + a_n s^n t^n = t^n(c^n + a_1 s c^{n-1} + \cdots + a_n s^n) = 0.$$

Hence ct is integral over A , then $ct \in B$. Then $c/s = (ct)/(st) \in S^{-1}B$. This completes the proof. □

Proposition A.4.3. Normality is a local property. That is, for an integral domain A , TFAE:

- (i) A is normal.
- (ii) For any prime ideal $\mathfrak{p} \in \text{Spec } A$, the localization $A_{\mathfrak{p}}$ is normal.
- (iii) For any maximal ideal $\mathfrak{m} \in \text{mSpec } A$, the localization $A_{\mathfrak{m}}$ is normal.

Proof. When A is normal, $A_{\mathfrak{p}}$ is normal by Lemma A.4.2.

Assume that $A_{\mathfrak{m}}$ is normal for every $\mathfrak{m} \in \text{mSpec } A$. If A is not normal, let \tilde{A} be the integral closure of A in $\text{Frac } A$, \tilde{A}/A is a nonzero A -module. Suppose $\mathfrak{p} \in \text{Supp } \tilde{A}/A$ and $\mathfrak{p} \subset \mathfrak{m}$. We have $\tilde{A}_{\mathfrak{m}}/A_{\mathfrak{m}} = 0$ and $\tilde{A}_{\mathfrak{p}}/A_{\mathfrak{p}} = (\tilde{A}_{\mathfrak{m}}/A_{\mathfrak{m}})_{\mathfrak{p}} \neq 0$. This is a contradiction. □

Proposition A.4.4. Let A be a normal ring. Then $A[X]$ is also normal.

Definition A.4.5. A scheme X is called *normal* if the local ring $\mathcal{O}_{X,\xi}$ is normal for any point $\xi \in X$. A ring A is called *normal* if $\text{Spec } A$ is normal.

Remark A.4.6. For a general ring A , let $S := A \setminus (\bigcup_{\mathfrak{p} \in \text{Ass } A} \mathfrak{p}) = \bigcap_{\mathfrak{p} \in \text{Ass } A} A \setminus \mathfrak{p}$. Then S is a multiplicative set. The localization $S^{-1}A$ is called *the total ring of fractions* of A .

Suppose A is reduced and $\text{Ass } A = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Denote its total ring of fractions by Q . Note that elements in Q are either unit or zero divisor. Hence any maximal ideal \mathfrak{m} is contained in $\bigcup \mathfrak{p}_i Q$, whence contained in some $\mathfrak{p}_i Q$. Thus $\mathfrak{p}_i Q$ are maximal ideals. And we have $\bigcap \mathfrak{p}_i Q = 0$. By the Chinese Remainder Theorem, we have $Q = \prod Q/\mathfrak{p}_i Q = \prod A_{\mathfrak{p}_i}$. Let A be a reduced ring with total ring of fractions Q . Then A is normal iff A is integral closed in Q . If A is normal, then for every $\mathfrak{p} \in \text{Spec } A$, $A_{\mathfrak{p}}$ is integral. Then there is unique minimal prime ideal $\mathfrak{p}_i \subset \mathfrak{p}$. In particular, any two minimal prime ideal are relatively prime. By the Chinese Remainder Theorem, $A = \prod A/\mathfrak{p}_i$. Just need to check A/\mathfrak{p}_i is integral closed in $A_{\mathfrak{p}_i}$. This is clear by check pointwise.

Conversely, suppose A is integral closed in Q . Let e_i be the unit element of $A_{\mathfrak{p}_i}$. It belongs to A since $e_i^2 - e_i = 0$. Since $1 = e_1 + \dots + e_n$ and $e_i e_j = \delta_{ij}$, we have $A = \prod A e_i$. Since $A e_i$ is integral closed in $A_{\mathfrak{p}_i}$, it is normal. Hence A is normal.

Lemma A.4.7. Let A be a normal ring. Then A verifies (R_1) and (S_2) .

Proof. Since all properties are local, we can assume A is integral and local.

For (S_2) , by Example ??, we only need to show that $\text{Ass}_A A/f$ has no embedded point. Let $\mathfrak{p} = (f : g) \in \text{Ass}_A A/fA$ and $t := f/g \in \text{Frac } A$. After Replacing A by $A_{\mathfrak{p}}$, we can assume that \mathfrak{p} is maximal. By definition, $t^{-1}\mathfrak{p} \subset A$. If $t^{-1}\mathfrak{p} \subset \mathfrak{p}$, suppose \mathfrak{p} is generated by (x_1, \dots, x_n) and $t^{-1}(x_1, \dots, x_n)^T = \Phi(x_1, \dots, x_n)^T$ for $\Phi \in M_n(A)$. There is a monic polynomial $\chi(T) \in A[T]$ vanishing Φ . Then $\chi(t^{-1}) = 0$ and $t^{-1} \in A$. This is impossible by definition of t . Then $t^{-1}\mathfrak{p} = A$, and $\mathfrak{p} = (t)$ is principal. By Krull's Principal Ideal Theorem A.3.14, $\text{ht}(\mathfrak{p}) = 1$.

Now we show that A verifies (R_1) . Suppose (A, \mathfrak{m}) is local of dimension 1. Choosing $a \in \mathfrak{m}$, A/a is of dimension 0. Then by A.3.11, $\mathfrak{m}^n \subset aA$ for some $n \geq 1$. Suppose $\mathfrak{m}^{n-1} \not\subset aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$ and let $t = a/b$. By construction, $t^{-1} \notin A$ and $t^{-1}\mathfrak{m} \subset A$. After similar argument, we see that $\mathfrak{m} = tA$, whence A is regular. \square

Lemma A.4.8. Let (A, \mathfrak{m}) be a noetherian local ring of dimension 1. Then A is normal iff A is regular.

Proof. By lemma A.4.7, we just need to show that regularity implies normality.

Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since A is regular, $\mathfrak{m} = (t)$. Let $I \subset \mathfrak{m}$ be an ideal. If $I \subset \bigcap_n \mathfrak{m}^n$, then for every $a \in I$, there exists a_n such that $a = a_n t^n$. Then we get an ascending chain of ideals $(a_1) \subset (a_2) \subset \dots$. Hence $a = 0$ by Nakayama's Lemma. Suppose I is not zero. Then there is some n such that $I \subset \mathfrak{m}^n$ and $I \not\subset \mathfrak{m}^{n+1}$. For every $at^n \in I \setminus \mathfrak{m}^{n+1}$, $a \notin \mathfrak{m}$, whence a is a unit in A . Then $I = (t^n)$. Hence A is PID and hence normal. \square

Proposition A.4.9. Let A be a noetherian integral domain of dimension ≥ 1 verifying (S_2) . Then

$$A = \bigcap_{\mathfrak{p} \in \text{Spec } A, \text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

Proof. Clearly $A \subset \bigcap A_{\mathfrak{p}}$. Let $t = f/g \in \bigcap A_{\mathfrak{p}}$. Since $f \in gA_{\mathfrak{p}}$ and we have $gA = \bigcap (gA_{\mathfrak{p}} \cap A)$, $f \in gA$. It follows that $t \in A$. \square

Theorem A.4.10 (Serre's criterion for normality). Let X be a locally noetherian scheme. Then X is normal if and only if it verifies (R_1) and (S_2) .

Proof. One direction has been proved in Lemma A.4.7. Suppose X verifies (R_1) and (S_2) . Again we can assume $X = \text{Spec } A$ is affine and A is local. By Remark A.4.6, we just need to show that A is integral closed in its total ring of fractions Q . Suppose we have

$$\left(\frac{a}{b}\right)^n + c_1 \left(\frac{a}{b}\right)^{n-1} + \dots + c_n = 0 \in Q.$$

Since A verifies (S_2) , $bA = \bigcap \nu_{\mathfrak{p}}^{-1}(b_{\mathfrak{p}}A_{\mathfrak{p}})$. So it is sufficient to show that $a_{\mathfrak{p}} \in b_{\mathfrak{p}}A_{\mathfrak{p}}$ with $\text{ht}(\mathfrak{p}) = 1$. Note that $A_{\mathfrak{p}}$ is regular and hence normal by Lemma A.4.8. Then above equation gives us desired result. \square

A.5 Smoothness

A.5.1 Modules of differentials and derivations

In this subsection, let R be a ring and A an R -algebra.

Definition A.5.1 (Derivation). A *derivation* of A over R is an R -linear map $\partial : A \rightarrow M$ with an A -module such that for all $a, b \in A$, we have

$$\partial(ab) = a\partial(b) + b\partial(a).$$

Given the module M , the set of all derivations of A over R into M forms an A -module, denoted by $\text{Der}_R(A, M)$.

Given a module homomorphism $f : M \rightarrow N$ of A -modules and a derivation $\partial \in \text{Der}_R(A, M)$, the map $f \circ \partial$ is a derivation of A over R into N .

Proposition A.5.2. The functor $\text{Der}_R(A, -)$ is representable. The representing object is denoted by $\Omega_{A/R}$, which is called the *module of differentials* of A over R .

Proof. First suppose A is a free R -algebra with a set of generators $a_\lambda, \lambda \in \Lambda$. Then an R -derivation $\partial \in \text{Der}_R(A, M)$ is uniquely determined by its values on the generators a_λ . Let

$$\Omega_{A/R} := \bigoplus_{\lambda \in \Lambda} A \cdot da_\lambda$$

and $d : A \rightarrow \Omega_{A/R}$ be the R -derivation defined by $a_\lambda \mapsto da_\lambda$. For any R -derivation $\partial \in \text{Der}_R(A, M)$, we can define a unique A -module homomorphism $\Phi_\partial : \Omega_{A/R} \rightarrow M$ by sending da_λ to $\partial(a_\lambda)$ such that $\partial = \Phi_\partial \circ d$. This gives a bijection

$$\text{Der}_R(A, M) \cong \text{Hom}_A(\Omega_{A/R}, M), \quad \partial \mapsto \Phi_\partial.$$

Now suppose $A = F/I$ is an arbitrary R -algebra, where F is a free R -algebra and I is an ideal of F . Then we can define the module of differentials

$$\Omega_{A/R} := (\Omega_{F/R} \otimes_F A) / \sum_{f \in I} A \cdot df.$$

The R -linear map $d_A : F \otimes_F A \xrightarrow{d_F} \Omega_{F/R} \otimes_F A \rightarrow \Omega_{A/R}$ is a derivation of A over R .

For any R -derivation $\partial \in \text{Der}_R(A, M)$, note that $F \rightarrow A \xrightarrow{\partial} M$ is an R -derivation of F over R into M . Then we get an F -module homomorphism $\Omega_F \rightarrow M$. It gives an A -module homomorphism $\Omega_F \otimes_F A \rightarrow M, df \otimes 1 \mapsto \partial f$. This map factors into $\Omega_F \otimes_F A \rightarrow \Omega_{A/R}$ and $\Phi_\partial : \Omega_{A/R} \rightarrow M$. Since Φ_∂ is A -linear and $\Omega_{A/R}$ is generated by da_λ as A -module, such Φ_∂ is unique. \square

Corollary A.5.3. Suppose A is of finite type over R . Then the module of differentials $\Omega_{A/R}$ is a finitely generated A -module.

Remark A.5.4. Let B be an A -algebra, M an A -module and N a B -module. If there is a homomorphism of A -modules $M \rightarrow N$, then we can extend it to a homomorphism of B -modules $M \otimes_A B \rightarrow N$ by sending $m \otimes b$ to $m \cdot b$. And such extension is unique in the sense of following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\quad} & N \\ \downarrow & \nearrow \exists! & \\ M \otimes_A B & & \end{array}$$

Hence we get a natural bijection

$$\text{Hom}_A(M, N) \cong \text{Hom}_B(M \otimes_A B, N).$$

Proposition A.5.5. Let A, R' be R -algebras and $A' := A \otimes_R R'$. Then the module of differentials $\Omega_{A'/R'}$ is isomorphic to $\Omega_{A/R} \otimes_A A'$.

Proof. We check the universal property of $\Omega_{A/R} \otimes_A A'$. First, the map

$$d_{A'} : A \otimes_R R' \rightarrow \Omega_{A/R} \otimes_R R' \cong \Omega_{A/R} \otimes_A A', \quad a \otimes r \mapsto da \otimes r$$

is an R' -derivation of A' into $\Omega_{A/R} \otimes_A A'$. For any R' -derivation $\partial' : A' \rightarrow M$ into an A' -module M , we can compose it with the homomorphism $A' \rightarrow A$ and get an R -derivation $\partial : A \rightarrow M$. By the universal property of $\Omega_{A/R}$, there is a unique A -module homomorphism $\Phi : \Omega_{A/R} \rightarrow M$ such that $\partial = \Phi \circ d_A$. Then we can extend it to an A' -module homomorphism $\Phi' : \Omega_{A/R} \otimes_A A' \rightarrow M$ by Remark A.5.4. By the construction, we have $\Phi' \circ d_{A'} = \partial'$. \square

Proposition A.5.6. Let A be an R -algebra and S a multiplicative set of A . Then we have an isomorphism

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

Proof. Let

$$d_{S^{-1}A} : S^{-1}A \rightarrow S^{-1}\Omega_{A/R}, \quad \frac{a}{s} \mapsto \frac{sda - ads}{s^2}.$$

By direct computation, $d_{S^{-1}A}$ is an R -derivation of $S^{-1}A$ over R into $S^{-1}\Omega_{A/R}$. For any R -derivation $\partial : S^{-1}A \rightarrow M$ into an $S^{-1}A$ -module M , we can get an $S^{-1}A$ -module homomorphism $\Phi' : S^{-1}\Omega_{A/R} \rightarrow M$ as proof of Proposition A.5.5. We have

$$\partial(s \cdot \frac{a}{s}) = s\partial(\frac{a}{s}) + \frac{a}{s}\partial s.$$

It follows that

$$\partial(\frac{a}{s}) = \frac{s\partial a - a\partial s}{s^2} = \frac{s\Phi'(da) - a\Phi'(ds)}{s^2} = \Phi'(\frac{sda - ads}{s^2}).$$

Thus, $\Phi' \circ d_{S^{-1}A} = \partial$. □

Theorem A.5.7. Let A be an R -algebra and B an A -algebra. Then there is a natural short exact sequence

$$\Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow \Omega_{B/A} \rightarrow 0$$

of B -modules.

Proof. Let $d_{A/R} : A \rightarrow \Omega_{A/R}$ be the R -derivation of A over R . The map $A \rightarrow B \xrightarrow{d_{B/R}} \Omega_{B/R}$ induces a B -linear map

$$u : \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R}, \quad d_{A/R}(a) \otimes b \mapsto bd_{B/R}(a).$$

The map $d_{B/A}$ is an A -derivation and hence R -derivation. Then it induces a B -linear map

$$v : \Omega_{B/R} \rightarrow \Omega_{B/A}, \quad d_{B/R}(b) \mapsto d_{B/A}(b).$$

Since $\Omega_{B/A}$ is generated by elements of the form $d_{B/A}(b)$ for $b \in B$, the map v is surjective. And clearly $d_{B/A}(a) = ad_{B/A}(1) = 0$ for $a \in A$.

Consider the composition $B \xrightarrow{d_{B/R}} \Omega_{B/R} \rightarrow \Omega_{B/R} / \text{Im } u$. For every $a \in A, b \in B$, we have

$$[d_{B/R}(ab)] = [bd_{B/R}(a) + ad_{B/R}(b)] = [bd_{B/R}(a)] + [ad_{B/A}(b)] = [ad_{B/A}(b)].$$

Hence it is indeed an A -derivation of B . Then it induces a B -linear map

$$\varphi : \Omega_{B/A} \rightarrow \Omega_{B/R} / \text{Im } u, \quad d_{B/A}(b) \mapsto [d_{B/R}(b)].$$

The map φ is surjective since $\Omega_{B/R}$ is generated by elements of the form $d_{B/R}(b)$ for $b \in B$. Note that the composition

$$\Omega_{B/A} \xrightarrow{\varphi} \Omega_{B/R} / \text{Im } u \rightarrow \Omega_{B/A} / \text{Ker } v$$

is the identity map. Thus, φ is injective and hence an isomorphism. In particular, we have $\text{Ker } v = \text{Im } u$. □

Theorem A.5.8. Let A be an R -algebra and I an ideal of A . Set $B := A/I$. Then there is a natural short exact sequence

$$I/I^2 \rightarrow \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow 0$$

of B -modules.

Proof. Suppose $A = F/\mathfrak{b}$ for some free R -algebra F and an ideal \mathfrak{b} of F . Let \mathfrak{a} be the preimage of I in F . Let $d\mathfrak{b}$ (resp. $d\mathfrak{a}$) denote the image of \mathfrak{b} (resp. \mathfrak{a}) in $\Omega_{F/R}$. Then we have

$$\Omega_{A/R} \otimes_A B = \Omega_{F/R} \otimes_F B / (d\mathfrak{b} \otimes_F B), \quad \Omega_{B/R} = \Omega_{F/R} \otimes_F B / (d\mathfrak{a} \otimes_F B).$$

Clearly

$$I/I^2 \cong (\mathfrak{a}/\mathfrak{b}) \otimes_F B \rightarrow (d\mathfrak{a} \otimes_F B) / (d\mathfrak{b} \otimes_F B)$$

is surjective. Then the exact sequence follows. □

Definition A.5.9. Let k be a field and A an integral k -algebra of finite type of dimension n . We say A is *smooth at* $\mathfrak{p} \in \operatorname{Spec} A$ if the module of differentials $\Omega_{A,\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module of rank n .

A.5.2 Applications to affine varieties

Let k be arbitrary field, $A = k[T_1, \dots, T_n]$ and \mathfrak{m} a maximal ideal of A such that $\kappa(\mathfrak{m})$ is separable over k . We try to give an explanation of Zariski's tangent space at \mathfrak{m} using the language of derivation. We know that $\Omega_{A/k} = \bigoplus_{i=1}^n A dT_i$, thus $\Omega_{A_{\mathfrak{m}}/k} \cong \bigoplus_{i=1}^n A_{\mathfrak{m}} dT_i$. Then

$$\operatorname{Der}_k(A_{\mathfrak{m}}, A_{\mathfrak{m}}) \cong \operatorname{Hom}_k(\Omega_{A_{\mathfrak{m}}/k}, A_{\mathfrak{m}}) \cong \bigoplus_{i=1}^n A_{\mathfrak{m}} \partial_i,$$

where $\partial_i \in \operatorname{Der}_k(A_{\mathfrak{m}}, A_{\mathfrak{m}})$ is the derivation defined by $dT_i \mapsto 1$ and $dT_j \mapsto 0$ for $j \neq i$. It coincides with the usual derivation $f \mapsto \partial f / \partial T_i$. Consider the restriction of ∂_i to \mathfrak{m} and take values in the residue field $\kappa(\mathfrak{m})$, we get

$$\Phi : \mathfrak{m} \xrightarrow{(\partial_1, \dots, \partial_n)^T} A_{\mathfrak{m}}^n \rightarrow \kappa(\mathfrak{m})^n.$$

Since $\kappa(\mathfrak{m})$ is separable over k , the map $\operatorname{Ker} \Phi = \mathfrak{m}^2$. Hence Φ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa(\mathfrak{m})^n$ of $\kappa(\mathfrak{m})$ -vector spaces. Then we get

$$T_{A,\mathfrak{m}} = (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong \bigoplus_{i=1}^n \kappa(\mathfrak{m}) \cdot \partial_i|_x,$$

where $x \in \mathbb{A}_k^n$ is the point corresponding to \mathfrak{m} . This coincides with the usual tangent space at x in language of differential geometry.

Let $B = A/I$ be a k of finite type, $I = (F_1, \dots, F_m) \subset \mathfrak{m}$ and \mathfrak{n} the image of \mathfrak{m} in B . We have an exact sequence of $\kappa(\mathfrak{m})$ -vector spaces

$$0 \rightarrow I/(I \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0.$$

It induces an isomorphism

$$T_{B,\mathfrak{n}} \cong \{\partial \in T_{A,\mathfrak{m}} : \partial(f) = 0, \forall f \in I\}.$$

The *Jacobian matrix* of F_1, \dots, F_m is the $m \times n$ matrix

$$J(F_1, \dots, F_m) := \left(\frac{\partial F_i}{\partial T_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

with entries in B .

Theorem A.5.10. Setting as above. Then B is regular at \mathfrak{n} if and only if the Jacobian matrix J has maximal rank $n - \dim B_{\mathfrak{n}}$ after taking values in the residue field $\kappa(\mathfrak{m})$.

Proof. We have an exact sequence

$$0 \rightarrow T_{B,\mathfrak{n}} \rightarrow T_{A,\mathfrak{m}} \xrightarrow{\Psi} \kappa(\mathfrak{m})^m \rightarrow 0,$$

where Ψ sends $\partial \in T_{A,\mathfrak{m}}$ to $(\partial(F_1), \dots, \partial(F_m))^T$. Note that the matrix of Ψ is just J^T , the transpose of the Jacobian matrix. Hence

$$\operatorname{rank} J = n - \dim_{\kappa} T_{B,\mathfrak{n}} \leq n - \dim B_{\mathfrak{n}}$$

and the equality holds if and only if B is regular at \mathfrak{n} . □

Remark A.5.11. If $\kappa(\mathfrak{m})$ is not separable over k , then we still have the inequality

$$\operatorname{rank} J \leq n - \dim B_{\mathfrak{n}}.$$

Indeed, in any case, we have an exact sequence

$$0 \rightarrow I/(I \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0.$$

Hence $\dim_{\kappa} I/(I \cap \mathfrak{m}^2) = n - \dim B_{\mathfrak{n}}$. There is a $\kappa(\mathfrak{m})$ -linear map

$$I/(I \cap \mathfrak{m}^2) \rightarrow \kappa(\mathfrak{m})^n, \quad [f] \mapsto (\partial_1(f), \dots, \partial_n(f))^T,$$

and every row of the Jacobian matrix J is in the image of this map. Thus, the rank of J is at most $n - \dim B_{\mathfrak{n}}$.

Hence if $\operatorname{rank} J = n - \dim B_{\mathfrak{n}}$, we can still see that B is regular at \mathfrak{n} . However, the converse does not hold in general.

Proposition A.5.12. Let k be a field, \mathbf{k} the algebraic closure of k , A a k -algebra of finite type and $A_{\mathbf{k}} := A \otimes_k \mathbf{k}$. **Yang:** Suppose $A_{\mathbf{k}}$ is integral. Let $\mathfrak{m} \in \operatorname{mSpec} A$ and \mathfrak{m}' be a maximal ideal of $A_{\mathbf{k}}$ lying over \mathfrak{m} . Then

- (a) If $A_{\mathbf{k}}$ is regular at \mathbf{m}' , then A is regular at \mathbf{m} ;
- (b) suppose $\kappa(\mathbf{m})$ is separable over \mathbf{k} , the converse holds.

Proof. Regarding $J_{\mathbf{m}}$ and $J_{\mathbf{m}'}$ as matrices with entries in \mathbf{k} , they are the same and hence have the same rank. If $A_{\mathbf{k}}$ is regular at \mathbf{m}' , since $\kappa(\mathbf{m}) = \mathbf{k}$, then $\text{rank } J_{\mathbf{m}'} = n - \dim A_{\mathbf{k}, \mathbf{m}'}$. Note that $\dim A_{\mathbf{k}, \mathbf{m}'} = \text{trdeg}(\mathcal{K}(A_{\mathbf{k}})/\mathbf{k}) = \text{trdeg}(\mathcal{K}(A)/\mathbf{k}) = \dim A_{\mathbf{m}}$, we have $\text{rank } J_{\mathbf{m}} = n - \dim A_{\mathbf{m}}$. Hence A is regular at \mathbf{m} .

Conversely, suppose A is regular at \mathbf{m} and $\kappa(\mathbf{m})$ is separable over \mathbf{k} . Then $\text{rank } J_{\mathbf{m}} = n - \dim A_{\mathbf{m}}$. Hence $A_{\mathbf{k}}$ is regular at \mathbf{m}' . \square

Proposition A.5.13. Let \mathbf{k} be a field and A an integral \mathbf{k} -algebra of finite type and of dimension n . Let $\bar{\mathbf{k}}$ be the algebraic closure of \mathbf{k} and $A_{\bar{\mathbf{k}}} := A \otimes_{\mathbf{k}} \bar{\mathbf{k}}$. Then A is smooth at $\mathbf{p} \in \text{Spec } A$ if and only if $A_{\bar{\mathbf{k}}}$ is regular at every \mathbf{m}' over \mathbf{m} .

Proof. Since $\Omega_{A_{\bar{\mathbf{k}}}/\bar{\mathbf{k}}} \cong \Omega_{A/\mathbf{k}} \otimes_A A_{\bar{\mathbf{k}}}$ is free of rank n if and only if $\Omega_{A/\mathbf{k}}$ is free of rank n , we can assume that $\mathbf{k} = \bar{\mathbf{k}}$. If A is smooth at \mathbf{p} , then $\Omega_{A_{\mathbf{p}}/\mathbf{k}} \cong \bigoplus A_{\mathbf{p}} df_i$ is free of rank n . Let $\mathfrak{P}_i \in \text{Der}_{\mathbf{k}}(A_{\mathbf{m}}, A_{\mathbf{m}})$ be the derivation defined by $df_i \mapsto 1$ and $dT_j \mapsto 0$ for $j \neq i$. Then we have $\partial_i f_j = \delta_{ij}$ for $1 \leq i, j \leq n$. Then similar to above argument, we have an isomorphism

$$\mathbf{m}/\mathbf{m}^2 \xrightarrow{(\partial_1, \dots, \partial_n)^T} \mathbf{k}^n.$$

This shows that $A_{\mathbf{k}}$ is regular at \mathbf{m} .

Conversely, suppose $A_{\bar{\mathbf{k}}}$ is regular at \mathbf{m} . Note that $\mathbf{m}/\mathbf{m}^2 \rightarrow \Omega_{A, \mathbf{k}} \otimes_A \bar{\mathbf{k}}$ is surjective since $\Omega_{A_{\bar{\mathbf{k}}}/\bar{\mathbf{k}}} = 0$. Then by Nakayama's lemma, $\Omega_{A_{\mathbf{m}}/\mathbf{k}}$ is generated by n elements as an $A_{\mathbf{m}}$ -module.

Note that $\dim_{\mathcal{K}(A)} \Omega_{\mathcal{K}(A)/\mathbf{k}} = \text{trdeg}(\mathcal{K}(A)/\mathbf{k}) = \dim A_{\mathbf{m}} = n$. **Yang: By induction on transcendental degree.**

Yang: By Nakayama's Lemma, $\Omega_{A_{\mathbf{m}}/\mathbf{k}}$ is free of rank n as an $A_{\mathbf{m}}$ -module.

Yang: To be completed. \square

Example A.5.14. Let \mathbf{k} be an imperfect field of characteristic $p > 2$. Suppose $\alpha = \beta^p \in \mathbf{k}$ and β is not in \mathbf{k} . Let $A = \mathbf{k}[x, y]/(x^2 - y^p - \alpha)$ and $\mathbf{m} = (x, y^p - \alpha) = (x)$. Note that \mathbf{m} is principal, so A is regular at \mathbf{m} . However,

$$J_{\mathbf{m}} = \left(\frac{\partial}{\partial x}(x^2 - y^p - \alpha), \frac{\partial}{\partial y}(y^p - \alpha) \right) = (2x, 0) = (0, 0) \in M_{1 \times 2}(\kappa(\mathbf{m})).$$

Thus, A is not smooth at \mathbf{m} . From the view of differentials, we have

$$\Omega_{A_{\mathbf{m}}/\mathbf{k}} = A_{\mathbf{m}} dx \oplus A_{\mathbf{m}} dy / A_{\mathbf{m}} \cdot x dx = \kappa(\mathbf{m}) dx \oplus A_{\mathbf{m}} dy,$$

which is not free as an $A_{\mathbf{m}}$ -module.

Appendix B

Homological Algebra

B.1 Complexes and Homology

B.2 Derived Functors

B.3 Applications to Commutative Algebra

B.3.1 Depth and Ext

B.3.2 Regularity and cohomological dimension

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