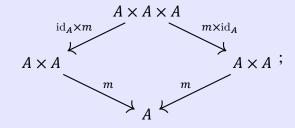
## The First Properties of Abelian Varieties

## 1 Definition and examples of Abelian Varieties

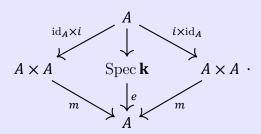
**Definition 1.** Let **k** be a field. An *abelian variety over* **k** is a proper variety A over **k** together with morphisms *identity* e: Spec  $\mathbf{k} \to A$ , *multiplication*  $m: A \times A \to A$  and *inversion*  $i: A \to A$  such that the following diagrams commute:

(a) (Associativity)



(b) (Identity)

(c) (Inversion)



In other words, an abelian variety is a group object in the category of proper varieties over  $\mathbf{k}$ .

**Example 2.** Let E be an elliptic curve over a field  $\mathbf{k}$ . Then E is an abelian variety of dimension 1. Yang: To be completed.

In the following, we will always assume that A is an abelian variety over a field  $\mathbf{k}$  of dimension d. Temporarily, we will use the notation  $e_A, m_A, i_A$  to denote the identity section, multiplication morphism and inversion morphism of an abelian variety A. The *left translation* by  $a \in A(\mathbf{k})$  is defined as

$$l_a: A \xrightarrow{\cong} \operatorname{Spec} \mathbf{k} \times A \xrightarrow{a \times \operatorname{id}_A} A \times A \xrightarrow{m_A} A.$$

Similar definition applies to the right translation  $r_a$ .

**Proposition 3.** Let A be an abelian variety. Then A is smooth.

*Proof.* By base changing to the algebraic closure of  $\mathbf{k}$ , we may assume that  $\mathbf{k}$  is algebraically closed. Note that there is a non-empty open subset  $U \subset A$  which is smooth. Then apply the left translation

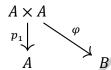
Date: October 1, 2025, Author: Tianle Yang, My Website

**Proposition 4.** Let A be an abelian variety. Then the cotangent bundle  $\Omega_A$  is trivial, i.e.,  $\Omega_A \cong \mathcal{O}_A^{\oplus d}$  where  $d = \dim A$ .

*Proof.* Consider  $\Omega_A$  as a geometric vector bundle of rank d. Then the conclusion follows from the fact that the left translation morphism  $l_a$  induces a morphism of varieties  $\Omega_A \to \Omega_A$  for every  $a \in A(\mathbf{k})$ . Yang: But how to show it is a morphism of varieties? Yang: To be completed.

**Theorem 5.** Let A and B be abelian varieties. Then any morphism  $f: A \to B$  with  $f(e_A) = e_B$  is a group homomorphism, i.e., for every **k**-scheme T, the induced map  $f_T: A(T) \to B(T)$  is a group homomorphism.

*Proof.* Consider the diagram



with  $\varphi$  be given by

$$A \times A \xrightarrow{\Delta \times \Delta} A \times A \times A \times A \xrightarrow{\cong} A \times A \times A \times A \xrightarrow{(f \circ m_A) \times (i_B \circ f) \times (i_B \circ f)} B \times B \times B \xrightarrow{m_B} B,$$
  
$$(x, y) \mapsto (x, x, y, y) \mapsto (x, y, y, x) \mapsto (f(xy), f(y)^{-1}, f(x)^{-1}) \mapsto f(xy)f(y)^{-1}f(x)^{-1}.$$

We have  $\varphi(p_1^{-1}(e_A)) = \varphi(\{e_A\} \times A) = \{e_B\}$ . Then by Rigidity Lemma (Theorem 9), there exists a unique rational map  $\psi : A \dashrightarrow B$  such that  $\varphi = \psi \circ p_1$ . Note that  $A \to A \times \{e_A\} \to A \times A$  gives a section of  $p_1$ . On this section, we have that  $\varphi$  is constant equal to  $e_B$ . Thus  $\psi$  is well-defined and  $\psi(A) = e_B$ . It follows that  $\varphi$  factors through the constant map  $A \times A \to \{e_B\} \to B$ . Then for every  $(x,y) \in A(\mathbb{k}) \times A(\mathbb{k})$ , we have

$$f(xy) = f(x)f(y).$$

Yang: Since  $A(\mathbb{k})$  is dense in A, the conclusion follows.

**Proposition 6.** Let A be an abelian variety. Then  $A(\mathbf{k})$  is an abelian group.

*Proof.* Note that a group is abelian if and only if the inversion map is a homomorphism of groups. Then the conclusion follows from Theorem 5.

From now on, we will use the notation  $0, +, [-1]_A, t_a$  to denote the identity section, addition morphism, inversion morphism and translation by a of an abelian variety A. For every  $n \in \mathbb{Z}_{>0}$ , the homomorphism of multiplication by n is defined as

$$[n]_A: A \xrightarrow{\Delta} A \times A \xrightarrow{[n-1]_A \times \mathrm{id}_A} A \times A \xrightarrow{+} A,$$

where  $\Delta$  is the diagonal morphism.

## 2 Complex abelian varieties

**Theorem 7.** Let A be a complex abelian variety. Then A is a complex torus, i.e., there exists a lattice  $\Lambda \subset \mathbb{C}^d$  such that  $A \cong \mathbb{C}^d/\Lambda$ . Conversely, let  $A = \mathbb{C}^n/\Lambda$  be a complex torus for some lattice  $\Lambda$ . Then A is a complex abelian variety if and only if there exists a positive definite Hermitian form H on  $\mathbb{C}^n$  such that  $\mathfrak{I}(H)(\Lambda,\Lambda) \subset \mathbb{Z}$ . Yang: To be completed.

## Requirements

**Proposition 8.** Let  $f: X \to Y$  be a morphism of varieties over a field **k**. Then the function  $y \mapsto \dim f^{-1}(y)$  is upper semicontinuous, i.e., for every integer m, the set  $\{y \in Y : \dim f^{-1}(y) \ge m\}$  is closed in Y. Yang: To be check.

**Theorem 9** (Rigidity Lemma). Let  $\pi_i: X \to Y_i$  be proper morphisms of varieties over a field **k** for i = 1, 2. Suppose that  $\pi_1$  is a fibration and  $\pi_2$  contracts  $\pi_1^{-1}(y_0)$ . Then there exists a rational map  $\varphi: Y_1 \dashrightarrow Y_2$  such that  $\pi_2 \circ \varphi = \pi_1$  and  $\varphi$  is well-defined near  $Y_1 \setminus \{y_0\}$ .

