Coarse classification of surfaces

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Unless otherwise specified, all varieties are defined over $\Bbbk$.

Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. We want to classify X up to birational equivalence. Let K_X be the canonical divisor of X.

1 Tools

Theorem 1. Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. Suppose that the Kodaira dimension $\kappa(X) \geq 0$. Then the linear system $|12K_X|$ is base point free. Yang: To be checked.

2 Classification

Theorem 2 (Enriques-Kodaira classification)**.** Let X be a smooth projective surface over an algebraically closed field $\Bbbk$. Then X is birational to a unique minimal model X', unless X is birational to a ruled surface. Moreover, the minimal model X' falls into one of the following classes:

(a) $\kappa(X') = -\infty$: $X' \cong \mathbb{P}^2$ or X' is a ruled surface;

(b) $\kappa(X') = 0$: X' is a K3 surface, an abelian surface or their quotients;

(c) $\kappa(X') = 1$: X' is an elliptic surface;

(d) $\kappa(X') = 2$: X' is a surface of general type.

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