Cone Theorem

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Cone Theorem

1 Preliminary

Theorem 1 (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27])**.** Let X be a projective variety and $\mathcal{L}$ an semiample line bundle on X. Then there exists a fibration $\varphi : X \to Y$ of projective varieties such that for any $m \gg 0$ with $\mathcal{L}^m$ base point free, we have that the morphism $\varphi_{\mathcal{L}^m}$ induced by $\mathcal{L}^m$ is isomorphic to φ. Such a fibration is called the *Iitaka fibration* associated to $\mathcal{L}$.

2 Non-vanishing Theorem

Theorem 2 (Non-vanishing Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

3 Base Point Free Theorem

Theorem 3 (Base Point Free Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then D is semiample.

4 Rationality Theorem

Theorem 4 (Rationality Theorem)**.** Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let

$$t := \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H. Then $t = u/v \in \mathbb{Q}$ and

$$0 \leq u \leq a(X) \cdot (\dim X + 1).$$

5 Cone Theorem and Contraction Theorem

Theorem 5 (Cone Theorem)**.** Let (X, B) be a projective klt pair. Then there exist countably many rational curves $C_i \subset X$ with

$$0 < -K_{(X,B)} \cdot C_i \leq 2 \dim X$$

Date: July 19, 2025, **Author**: Tianle Yang, My Website such that

(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. We follow the following steps to prove the theorem.

Step 1. Let $F_D := \mathrm{Psef}_1(X) \cap D^\perp$ for a nef divisor D on X. We show that if $\dim F_D > 1$ and $F_D \not\subset \mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0}$, then we can choose D' nef such that $F_{D'} \subset F_D$ and $\dim F_{D'} < \dim F_D$.

Yang: To be completed.

Step 2. If $\dim F_D = 1$, we also write $R_D := F_D$. We show that

$$\mathrm{Psef}_1(X) = \overline{\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} + \sum R_D}.$$

Yang: To be completed.

Step 3. For any $\varepsilon > 0$ and an ample divisor H on X, we show that

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H\geq 0} + \sum_{\text{finite}} R_D.$$

Yang: To be completed.

Step 4. We show that any $K_{(X,B)}$-negative extremal ray R_D contains the class of a rational curve C with $0 < -K_{(X,B)} \cdot C \leq 2\dim X$.

Yang: To be completed. □

Theorem 6 (Contraction Theorem)**.** Let (X, B) be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;

(b) any line bundle $\mathcal{L}$ with $F \subset \mathcal{L}^\perp = \{\alpha \in N_1(X) : \alpha \cdot \mathcal{L} = 0\}$ comes from a line bundle on Y, i.e., there exists a line bundle $\mathcal{L}_Y$ on Y such that $\mathcal{L} \cong \varphi_F^* \mathcal{L}_Y$.

Proof. We follow the following steps to prove the theorem.

Step 1. We show that there exists a nef divisor D on X such that $F = D^\perp \cap \mathrm{Psef}_1(X)$. In other words, F is defined on $N_1(X)_{\mathbb{Q}}$.

Yang: To be completed.

Step 2. We show that D is semiample.

Yang: To be completed.

Step 3. Let $\varphi : X \to Y$ be the Iitaka fibration associated to D by **??**. We show that φ is the desired fibration.

Yang: To be completed. □

Remark 7. The **??** is amazing. If F is not $K_{(X,B)}$-negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E. By [Laz04, Lemma 1.5.4], we know that $\mathrm{Psef}_1(X)$ is a circular cone. The we see there indeed exist some irrational extremal faces of $\mathrm{Psef}_1(X)$.

References

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*. Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: 10.1007/978-3-642-18808-4. URL: https://doi.org/10.1007/978-3-642-18808-4.