# Line bundles induce morphisms

## 1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

**Theorem 1.** Let $A$ be a ring and $X$ an $A$-scheme. Let $\mathcal{L}$ be a line bundle on $X$ and $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate $\mathcal{L}$, i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \to \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \to \mathbb{P}_A^n$ such that $\mathcal{L} \cong f^*\mathcal{O}(1)$ and $s_i = f^*x_i$, where $x_i$ are the standard coordinates on $\mathbb{P}_A^n$.

*Proof.* Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where $s_i$ does not vanish. Since $\{s_i\}$ generate $\mathcal{L}$, we have $X = \bigcup_i U_i$. Let $V_i$ be given by $x_i \neq 0$ in $\mathbb{P}_A^n$. On $U_i$, let $f_i : U_i \to V_i \subseteq \mathbb{P}_A^n$ be the morphism induced by the ring homomorphism

$$A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \to \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, $f_i$ and $f_j$ agree. Thus we can glue them to get a morphism $f : X \to \mathbb{P}_A^n$. By construction, we have $s_i = f^*x_i$ and $\mathcal{L} \cong f^*\mathcal{O}(1)$. If there is another morphism $g : X \to \mathbb{P}_A^n$ satisfying the same properties, then on each $U_i$, $g$ must agree with $f_i$ by the same construction. Thus $g = f$. $\square$

**Proposition 2.** Let $X$ be a **k**-scheme for some field **k** and $\mathcal{L}$ is a line bundle on $X$. Suppose that $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate $\mathcal{L}$. Let $f : X \to \mathbb{P}_\mathbf{k}^n$ and $g : X \to \mathbb{P}_\mathbf{k}^m$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}_\mathbf{k}^n \dashrightarrow \mathbb{P}_\mathbf{k}^m$ which is well defined near image of $f$ and satisfies $g = \phi \circ f$.

*Proof.* $\square$

**Example 3.** Let $X = \mathbb{P}_\mathbf{k}^n$ with **k** a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0, \ldots, i_n} = T_0^{i_0} T_1^{i_1} \cdots T_n^{i_n}$ for all $(i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = d$, where $T_i$ are the standard coordinates on $\mathbb{P}^n$. The they induce a morphism $f : X \to \mathbb{P}_\mathbf{k}^N$ where $N = \binom{n+d}{d} - 1$. On **k**-point level, it is given by

$$[x_0 : \cdots : x_n] \mapsto [\ldots : x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} : \ldots],$$

where the coordinates on the right-hand side are indexed by all $(i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = d$. This is called the *d-uple embedding* or *Veronese embedding* of $\mathbb{P}^n$ into $\mathbb{P}^N$.

**Example 4.** Let $X = \mathbb{P}_\mathbf{k}^m \times \mathbb{P}_\mathbf{k}^n$ and $\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1)$, where $\pi_1$ and $\pi_2$ are the projections. Let $T_0, \ldots, T_m$ and $S_0, \ldots, S_n$ be the standard coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^*T_i \otimes \pi_2^*S_j$ for $0 \leq i \leq m$ and $0 \leq j \leq n$. They induce a

morphism $f : X \to \mathbb{P}^{(m+1)(n+1)-1}_{\mathbf{k}}$. On $\mathbf{k}$-point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\ldots : x_i y_j : \ldots],$$

where the coordinates on the right-hand side are indexed by all $(i, j)$ with $0 \leq i \leq m$ and $0 \leq j \leq n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

**Example 5.** Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ over $\mathbb{P}^1$. Yang: To be continued.

**Definition 6.** A line bundle $\mathcal{L}$ on a scheme $X$ is *ample* if for every coherent sheaf $\mathcal{F}$ on $X$, there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Yang: To be continued.

**Definition 7.** A line bundle $\mathcal{L}$ on a scheme $X$ is *very ample* if there exists a closed embedding $i : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. Yang: To be continued.

**Definition 8.** Let $\mathcal{L}$ be a line bundle on a scheme $X$ and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace. The *base locus* of $V$ is the closed subset

$$\mathrm{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If $\mathrm{Bs}(V) = \emptyset$, we say that $V$ is *base-point free*. Yang: To be continued.

**Definition 9.** A line bundle $\mathcal{L}$ on a scheme $X$ is *globally generated* if $\Gamma(X, \mathcal{L})$ generates $\mathcal{L}$, i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$ is surjective. Yang: To be continued.

**Definition 10.** Let $\mathcal{L}$ be a line bundle on a scheme $X$. Yang: To be continued.

**Theorem 11.** Let $X$ be a scheme over a ring $A$ and $\mathcal{L}$ a line bundle on $X$. Then the following are equivalent:

(a) $\mathcal{L}$ is ample;

(b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample.

If furthermore $X$ is of finite type over noetherian ring $A$, then the above are also equivalent to:

(c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

**Proposition 12.**

# 2　Linear systems

In this subsection, we give a more geometric interpretation of last subsection using the language of linear systems.

**Definition 13.** Let $X$ be a normal proper variety over a field $\mathbf{k}$, $D$ a (Cartier) divisor on $X$ and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to $D$ is the set

$$|D| = \{D' \in \mathrm{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

# 3 Asymptotic behavior

**Definition 14.** Let $X$ be a scheme and $\mathcal{L}$ a line bundle on $X$. The *section ring* of $\mathcal{L}$ is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

**Definition 15.** A line bundle $\mathcal{L}$ on a scheme $X$ is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. Yang: To be continued.

**Theorem 16.** Let $X$ be a scheme over a ring $A$ and $\mathcal{L}$ a semiample line bundle on $X$. Then there exists a morphism $f : X \to Y$ over $A$ such that $\mathcal{L} \cong f^*\mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on $Y$. Moreover, $Y = \mathrm{Proj}\, R(X, \mathcal{L})$ and $f$ is induced by the natural map $R(X, \mathcal{L}) \to \Gamma(X, \mathcal{L}^{\otimes n})$. Yang: To be continued.

**Definition 17.** A line bundle $\mathcal{L}$ on a scheme $X$ is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq C n^{\dim X}$$

for all sufficiently large $n$. Yang: To be continued.

**Example 18.** Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $\pi : X \to \mathbb{P}^1$ be the projection and $E$ the unique section of $\pi$ with self-intersection $-2$. Then the anticanonical bundle $-K_X$ is big but not base-point free. In fact, we have

$$K_X = -2E - 4F,$$

where $F$ is a fiber of $\pi$. Thus

$$-K_X = 2E + 4F.$$

Since $E^2 = -2 < 0$, any section of $-K_X$ must vanish on $E$. Thus $E \subseteq \mathrm{Bs}(-K_X)$, so $-K_X$ is not base-point free. On the other hand, we have

$$H^0(X, \mathcal{O}_X(-nK_X)) = H^0(X, \mathcal{O}_X(2nE + 4nF)).$$

Using the projection formula and the fact that $\pi_* \mathcal{O}_X(E) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, we can compute that

$$H^0(X, \mathcal{O}_X(2nE + 4nF)) = H^0(\mathbb{P}^1, S^{2n}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \otimes \mathcal{O}_{\mathbb{P}^1}(4n)).$$

The right-hand side can be computed to be a direct sum of line bundles on $\mathbb{P}^1$, and for large $n$, it contains many global sections. In fact, one can show that

$$\dim H^0(X, \mathcal{O}_X(-nK_X)) \sim 4n^2 \text{ for large } n,$$

Yang: To be continued.