

Line bundles induce morphisms

1 Ample and basepoint free line bundles

The story begins with the following theorem, which uses global sections of a line bundle to construct a morphism to projective space.

Theorem 1. Let A be a ring and X an A -scheme. Let \mathcal{L} be a line bundle on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Suppose that $\{s_i\}$ generate \mathcal{L} , i.e., $\bigoplus_i \mathcal{O}_X \cdot s_i \rightarrow \mathcal{L}$ is surjective. Then there is a unique morphism $f : X \rightarrow \mathbb{P}_A^n$ such that $\mathcal{L} \cong f^* \mathcal{O}(1)$ and $s_i = f^* x_i$, where x_i are the standard coordinates on \mathbb{P}_A^n .

Proof. Let $U_i := \{\xi \in X : s_i(\xi) \notin \mathfrak{m}_\xi \mathcal{L}_\xi\}$ be the open subset where s_i does not vanish. Since $\{s_i\}$ generate \mathcal{L} , we have $X = \bigcup_i U_i$. Let V_i be given by $x_i \neq 0$ in \mathbb{P}_A^n . On U_i , let $f_i : U_i \rightarrow V_i \subseteq \mathbb{P}_A^n$ be the morphism induced by the ring homomorphism

$$A \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] \rightarrow \Gamma(U_i, \mathcal{O}_X), \quad \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

Easy to check that on $U_i \cap U_j$, f_i and f_j agree. Thus we can glue them to get a morphism $f : X \rightarrow \mathbb{P}_A^n$. By construction, we have $s_i = f^* x_i$ and $\mathcal{L} \cong f^* \mathcal{O}(1)$. If there is another morphism $g : X \rightarrow \mathbb{P}_A^n$ satisfying the same properties, then on each U_i , g must agree with f_i by the same construction. Thus $g = f$. \square

Proposition 2. Let X be a \mathbf{k} -scheme for some field \mathbf{k} and \mathcal{L} is a line bundle on X . Suppose that $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and both generate \mathcal{L} . Let $f : X \rightarrow \mathbb{P}_{\mathbf{k}}^n$ and $g : X \rightarrow \mathbb{P}_{\mathbf{k}}^m$ be the morphisms induced by $\{s_i\}$ and $\{t_j\}$ respectively. Then there exists a linear transformation $\phi : \mathbb{P}_{\mathbf{k}}^n \dashrightarrow \mathbb{P}_{\mathbf{k}}^m$ which is well defined near image of f and satisfies $g = \phi \circ f$.

Proof. **Yang: To be continued.** \square

Example 3. Let $X = \mathbb{P}_A^n$ with A a ring and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d > 0$. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $S_{i_0, \dots, i_n} = T_0^{i_0} T_1^{i_1} \dots T_n^{i_n}$ for all (i_0, \dots, i_n) with $i_0 + \dots + i_n = d$, where T_i are the standard coordinates on \mathbb{P}^n . They induce a morphism $f : X \rightarrow \mathbb{P}_A^N$ where $N = \binom{n+d}{d} - 1$. If $A = \mathbf{k}$ is a field, on \mathbf{k} -point level, it is given by

$$[x_0 : \dots : x_n] \mapsto [\dots : x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} : \dots],$$

where the coordinates on the right-hand side are indexed by all (i_0, \dots, i_n) with $i_0 + \dots + i_n = d$. This is called the d -uple embedding or Veronese embedding of \mathbb{P}^n into \mathbb{P}^N .

Example 4. Let $X = \mathbb{P}_A^m \times_A \mathbb{P}_A^n$ with A a ring and $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$, where π_1 and π_2 are the projections. Let T_0, \dots, T_m and S_0, \dots, S_n be the standard coordinates on \mathbb{P}^m and \mathbb{P}^n respectively. Then $\Gamma(X, \mathcal{L})$ is generated by the global sections $T_i S_j = \pi_1^* T_i \otimes \pi_2^* S_j$ for $0 \leq i \leq m$ and $0 \leq j \leq n$.

They induce a morphism $f : X \rightarrow \mathbb{P}_A^{(m+1)(n+1)-1}$. If $A = \mathbf{k}$ is a field, on \mathbf{k} -point level, it is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [\dots : x_i y_j : \dots],$$

where the coordinates on the right-hand side are indexed by all (i, j) with $0 \leq i \leq m$ and $0 \leq j \leq n$. This is called the *Segre embedding* of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{(m+1)(n+1)-1}$.

Definition 5. A line bundle \mathcal{L} on a scheme X is *globally generated* if $\Gamma(X, \mathcal{L})$ generates \mathcal{L} , i.e., the natural map $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_X \rightarrow \mathcal{L}$ is surjective. **Yang: To be continued.**

Example 6. Let

Example 7.

Definition 8. Let \mathcal{L} be a line bundle on a scheme X . **Yang: To be continued.**

Definition 9. A line bundle \mathcal{L} on a scheme X is *ample* if for every coherent sheaf \mathcal{F} on X , there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. **Yang: To be continued.**

Definition 10. A line bundle \mathcal{L} on a scheme X is *very ample* if there exists a closed embedding $i : X \rightarrow \mathbb{P}_A^n$ such that $\mathcal{L} \cong i^* \mathcal{O}(1)$. **Yang: To be continued.**

Theorem 11. Let X be a scheme of finite type over a noetherian ring A and \mathcal{L} a line bundle on X . Then the following are equivalent:

- (a) \mathcal{L} is ample;
- (b) for some $n > 0$, $\mathcal{L}^{\otimes n}$ is very ample;
- (c) for all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample.

Yang: To be continued.

Proposition 12. Let X be a scheme of finite type over a noetherian ring A and \mathcal{L}, \mathcal{M} line bundles on X . Then we have the following:

- (a) if \mathcal{L} is ample and \mathcal{M} is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample;
- (b) if \mathcal{L} is very ample and \mathcal{M} is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample;
- (c) if both \mathcal{L} and \mathcal{M} are ample, then so is $\mathcal{L} \otimes \mathcal{M}$;
- (d) if both \mathcal{L} and \mathcal{M} are globally generated, then so $\mathcal{L} \otimes \mathcal{M}$;
- (e) if \mathcal{L} is ample and \mathcal{M} is arbitrary, then for some $n > 0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample;

Yang: To be continued.

Proof. **Yang: To be continued.**

□

Proposition 13. Let X be a scheme of finite type over a noetherian ring A and \mathcal{L} a line bundle on X . Then \mathcal{L} is very ample if and only if the following two conditions hold:

- (a) (separate points) for any two distinct points $x, y \in X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $s(y) \neq 0$;
- (b) (separate tangent vectors) for any point $x \in X$ and non-zero tangent vector $v \in T_x X$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ but $v(s) \neq 0$.

Yang: To be continued.

2 Linear systems

In this subsection, when work over a field, we give a more geometric interpretation of last subsection using the language of linear systems.

Definition 14. Let X be a normal proper variety over a field \mathbf{k} , D a (Cartier) divisor on X and $\mathcal{L} = \mathcal{O}_X(D)$ the associated line bundle. The *complete linear system* associated to D is the set

$$|D| = \{D' \in \text{CaDiv}(X) : D' \sim D, D' \geq 0\}.$$

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}))$. Here the elements in $\mathbb{P}(\Gamma(X, \mathcal{L}))$ are one-dimensional subspaces of $\Gamma(X, \mathcal{L})$. Consider the vector subspace $V \subseteq \Gamma(X, \mathcal{L})$, we can define the generate linear system $|V|$ as the image of $V \setminus \{0\}$ in $\mathbb{P}(\Gamma(X, \mathcal{L}))$.

Definition 15. A *linear system* on a scheme X is a pair (\mathcal{L}, V) where \mathcal{L} is a line bundle on X and $V \subseteq \Gamma(X, \mathcal{L})$ is a subspace. The dimension of the linear system is $\dim V - 1$. A linear system is *base-point free* if V is base-point free. A linear system is *complete* if $V = \Gamma(X, \mathcal{L})$. Yang: To be continued.

Definition 16. Let \mathcal{L} be a line bundle on a scheme X and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace. The *base locus* of V is the closed subset

$$\text{Bs}(V) = \{x \in X : s(x) = 0, \forall s \in V\}.$$

If $\text{Bs}(V) = \emptyset$, we say that V is *base-point free*. Yang: To be continued.

3 Asymptotic behavior

Definition 17. Let X be a scheme and \mathcal{L} a line bundle on X . The *section ring* of \mathcal{L} is the graded ring

$$R(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$

with multiplication induced by the tensor product of sections. Yang: To be continued.

Definition 18. A line bundle \mathcal{L} on a scheme X is *semiample* if for some $n > 0$, $\mathcal{L}^{\otimes n}$ is base-point free. **Yang:** To be continued.

Theorem 19. Let X be a scheme over a ring A and \mathcal{L} a semiample line bundle on X . Then there exists a morphism $f : X \rightarrow Y$ over A such that $\mathcal{L} \cong f^* \mathcal{O}_Y(1)$ for some very ample line bundle $\mathcal{O}_Y(1)$ on Y . Moreover, $Y = \text{Proj } R(X, \mathcal{L})$ and f is induced by the natural map $R(X, \mathcal{L}) \rightarrow \Gamma(X, \mathcal{L}^{\otimes n})$. **Yang:** To be continued.

Definition 20. A line bundle \mathcal{L} on a scheme X is *big* if the section ring $R(X, \mathcal{L})$ has maximal growth, i.e., there exists $C > 0$ such that

$$\dim \Gamma(X, \mathcal{L}^{\otimes n}) \geq Cn^{\dim X}$$

for all sufficiently large n . **Yang:** To be continued.

Example 21. Let $X = \mathbb{F}_2$ be the second Hirzebruch surface, i.e., the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over \mathbb{P}^1 . Let $\pi : X \rightarrow \mathbb{P}^1$ be the projection and E the unique section of π with self-intersection -2 . **Yang:** To be continued.