Birational Geometry

“要知道你为什么出枪，你的心里有闷烧的火，那是大地上燃烧的煤矿，它的火焰终有一天烧破地面去点燃天空。你会吼叫，因为你若是不吐出那火焰，它会烧穿你的胸膛，它像是愤怒，又像是高亢的歌，龙虎的吼声让时间停止。”

Contents

1 Technical Preparation

1.1 Negativity Lemma

Theorem 1.1. Let $f : Y \to X$ be a proper birational morphism between normal varieties. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y such that $-D$ is f-nef. Then D is effective if and only if f_*D is.

Proof. Yang: To be completed. □

1.2 General adjunction formula

Theorem 1.2 (Adjuction formula)**.** Let X be a normal variety and S be a reduced divisor on X. Yang: Need to check the statement.

Proof. Yang: To be completed. □

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2 Kodaira Vanishing Theorem

2.1 Preliminary

Theorem 2.1 (Serre Duality)**.** Let X be a Cohen-Macaulay projective variety of dimension n over $\mathbf{k}$ and D a divisor on X. Then there is an isomorphism

$$H^i(X, D) \cong H^{n-i}(X, K_X - D)^\vee, \quad \forall i = 0, 1, \dots, n.$$

Theorem 2.2 (Log Resolution of Singularities)**.** Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighborhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there is a smooth variety (or analytic space) Y and a projective morphism $f : Y \to X$ such that

(a) f is an isomorphism over $X - (\operatorname{Sing}(X) \cup \operatorname{Supp} Z)$,

(b) $f^*I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and

(c) $\operatorname{Exc}(f) \cup D$ is an snc divisor.

Theorem 2.3 (Lefschetz Hyperplane Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is an isomorphism for $k < n - 1$ and an injection for $k = n - 1$.

Theorem 2.4 (Hodge Decomposition)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Then for any k, there is a functorial decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Combine Theorem 2.3 and Theorem 2.4, we have the following lemma.

Lemma 2.5. Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and Y a hyperplane section of X. Then the restriction map $r_k : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ decomposes as

$$r_k = \bigoplus_{p+q=k} r_{p,q}, \quad r_{p,q} : H^p(X, \Omega_X^q) \to H^p(Y, \Omega_Y^q).$$

And $r_{p,q}$ is an isomorphism for $p + q < n - 1$ and an injection for $p + q = n - 1$. In particular,

$$H^p(X, \mathcal{O}_X) \to H^p(Y, \mathcal{O}_Y)$$

is an isomorphism for $p < n - 1$ and an injection for $p = n - 1$.

Theorem 2.6 (Leray spectral sequence)**.** Let $f : Y \to X$ be a morphism of varieties and $\mathcal{F}$ a coherent sheaf on Y. Then there is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{F}) \implies H^{p+q}(Y, \mathcal{F}).$$

2.2 Kodaira Vanishing Theorem

Lemma 2.7. Let X be a smooth projective variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on X. Suppose there is an integer m and a smooth divisor $D \in H^0(X, \mathcal{L}^m)$. Then there exists a finite surjective morphism $f : Y \to X$ of smooth projective varieties such that $D' := f^{-1}(D)$ is smooth and satisfies that $bD' = af^*D$.

Proof. Let $s \in \mathcal{L}^m$ be the section defining D. It induces a homomorphism $\mathcal{L}^{-m} \to \mathcal{O}_X$. Consider the $\mathcal{O}_X$-algebra

$$\mathcal{A} := \left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} \right) \Big/ (\mathcal{L}^{-m} \to \mathcal{O}_X) \cong \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}.$$

Then $\mathcal{A}$ is a finite $\mathcal{O}_X$-algebra. Let $Y \coloneqq \operatorname{Spec}_X \mathcal{A}$. Then Y is a finite $\mathcal{O}_X$-scheme and the natural morphism $f : Y \to X$ is finite and surjective.

For every $x \in X$, let $\mathcal{L}$ locally generated by t near x. Then $\mathcal{O}_Y$ locally equal to $\mathcal{O}_X[t]/(t^m - s)$. Let D' be the divisor locally given by $t = 0$ on Y. Since X and D are smooth, then Y is a smooth variety and D' is smooth. Since f is finite, it is proper. Then Y is proper and hence Y is projective. □

Remark 2.8. Let D_i be reduced effective divisors on X such that $D + \sum_{i=1}^k D_i$ is snc. Set $D_i' = f^*(D_i)$. Then $D' + \sum_{i=1}^k D_i'$ is snc on Y by considering the local regular system of parameters.

Lemma 2.9. Let $f : Y \to X$ be a finite surjective morphism of projective varieties and $\mathcal{L}$ a line bundle on X. Suppose that X is normal. Then for any $i \geq 0$, $H^i(X, \mathcal{L})$ is a direct summand of $H^i(Y, f^*\mathcal{L})$.

Proof. Since f is finite, we have $H^i(Y, f^*\mathcal{L}) \cong H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L})$. Since X are normal, the inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits by the trace map $(1/n)\operatorname{Tr}_{Y/X}$. Thus we have $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}$ and hence

$$H^i(X, f_*\mathcal{O}_Y \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{L}).$$

Then the conclusion follows. □

Theorem 2.10 (Kodaira Vanishing Theorem)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and A an ample divisor on X. Then

$$H^i(X, \mathcal{O}_X(-A)) = 0, \quad \forall i < n.$$

Equivalently, we have

$$H^i(X, K_X + A) = 0, \quad \forall i > 0.$$

Proof. By Lemma 2.7 and 2.9, after taking a multiple of A, we can assume that A is effective. Then we have an exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0.$$

Taking the long exact sequence of cohomology, we have

$$H^{i-1}(X, \mathcal{O}_X) \to H^{i-1}(X, \mathcal{O}_A) \to H^i(X, \mathcal{O}_X(-A)) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_A).$$

Then the conclusion follows from Lemma 2.5 and Serre duality (Theorem 2.1). □

2.3 Kawamata-Viehweg Vanishing Theorem

We list three versions of Kawamata-Viehweg Vanishing Theorem here.

Theorem 2.11 (Kawamata-Viehweg Vanishing Theorem I)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{R}$-divisor on X. Then

$$H^i(X, K_X + D) = 0, \quad \forall i > 0.$$

Theorem 2.12 (Kawamata-Viehweg Vanishing Theorem II)**.** Let X be a smooth projective variety of dimension n over $\mathbf{k}$ of characteristic 0 and D a nef and big $\mathbb{Q}$-divisor on X. Suppose that $\lceil D \rceil - D$ has snc support. Then

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall i > 0.$$

Theorem 2.13 (Kawamata-Viehweg Vanishing Theorem III)**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Let D be a nef $\mathbb{Q}$-divisor on X such that $D + K_{(X,B)}$ is a Cartier divisor. Then

$$H^i(X, K_{(X,B)} + D) = 0, \quad \forall i > 0.$$

If we replace the assumption "nef and big" of D by "ample" in II and III, we denote them as "II(ample)" and "III(ample)" respectively. Then the proof follows the following line:

$$\text{Kodaira Vanishing} \implies \text{II(ample)} \implies \text{III(ample)} \implies \text{I} \implies \text{II} \implies \text{III}.$$

The proofs leave here and the lemmas used in the proofs are collected in the end of this section.

Proof of II (Theorem 2.12). Set $M := \lceil D \rceil$. Let

$$B := \sum_{i=1}^{k} b_i B_i := \lceil D \rceil - D = M - A, \quad b_i \in (0,1) \cap \mathbb{Q}.$$

We do not require that B_i are irreducible but we require that B_i are smooth.

We induct on k. When $k = 0$, the conclusion follows from Theorem 2.11. (For the ample case, it follows Kodaira Vanishing Theorem (Theorem 2.10.)) Let $b_k = a/c$ with lowest terms. Then $a < c$. By Lemma 2.15 and 2.9, we can assume that $(1/c)B_k$ is a Cartier divisor (not necessarily effective). Applying Lemma 2.7 on B_k, we can find a finite surjective morphism $f : X' \to X$ such that $f^*B_k = cB'_k, B'_i = f^*B_i$ for $i < k$ and $\sum_{i=1}^{k} B'_i$ is an snc divisor on X'. Let $B' = \sum_{i=1}^{k-1} B'_i, A' = f^*A$ and $M' = f^*M$. Then $A' + B' = M' - aB'_k$ is Cartier. Hence by induction hypothesis, $H^i(X', -A' - B')$

vanishes for $i > 0$. On the other hand, we have

$$\mathcal{O}_{X'}(-M' + aB'_k) \cong \sum_{i=0}^{c-1} f^*\mathcal{O}_X(-M + (a-i)B_k).$$

Hence $H^i(X, \mathcal{O}_X(-M))$ is a direct summand of $H^i(X', \mathcal{O}_{X'}(-M' + aB'_k))$ by Lemma 2.9. ☐

Proof of III (Theorem 2.13). Let $f : \tilde{X} \to X$ be a resolution such that $\operatorname{Supp} f^*B \cup \operatorname{Exc} f$ is snc. We can write

$$f^*(K_{(X,B)} + D) + E = K_{(\tilde{X},\tilde{B})} + f^*D,$$

where $\tilde{B} \in (0,1)$ has snc support and E is an effective exceptional divisor.

By Lemma 2.14, we have

$$H^i(\tilde{X}, K_{(\tilde{X},\tilde{B})} + f^*D) = H^i(X, f_*\mathcal{O}_Y(f^*(K_{(X,B)} + D) + E)) = H^i(X, K_{(X,B)} + D),$$

and the left hand vanishes by Theorem 2.12 in either case relative to the assumption of D. ☐

Proof of I (Theorem 2.11). By Lemma 2.17, we can choose $k \gg 0$ such that $(X, 1/kB)$ is a klt pair with $D \sim_{\mathbb{Q}} A + \frac{1}{k}B$ for some ample divisor A. Then the theorem comes down to Theorem 2.13. ☐

Lemma 2.14. Let $f : Y \to X$ be a birational morphism of projective varieties with Y smooth and X has only rational singularities. Let E be an effective exceptional divisor on Y and D a divisor on X. Then we have

$$f_*(\mathcal{O}_Y(f^*D + E)) \cong \mathcal{O}_X(D), \quad R^if_*(\mathcal{O}_Y(f^*D + E)) = 0, \quad \forall i > 0.$$

Proof. Yang: I am unable to proof this lemma. ☐

Lemma 2.15. Let X be a projective variety, $\mathcal{L}$ a line bundle on X and $m \in \mathbb{Z}_{\geq 0}$. Then there exists a finite surjective morphism $f : Y \to X$ and a line bundle $\mathcal{L}'$ on Y such that $f^*\mathcal{L} \sim \mathcal{L}'^m$. If X is smooth, then we can take Y to be smooth. Moreover, if $D = \sum D_i$ is an snc divisor on X, then we can take f such that f^*D is an snc divisor on Y.

Proof. We can assume that $\mathcal{L}$ is very ample by writing it as a difference of two very ample line bundles. Consider the fiber product $Y := \mathbb{P}^N \times_{\mathbb{P}^N} X$ as the following diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "g"] \\ X \arrow[r, "\varphi_{\mathcal{L}}"] & \mathbb{P}^N \end{tikzcd},$$

where $g : [x_0 : \ldots : x_N] \mapsto [x_0^m : \ldots : x_N^m]$. The morphism f is finite and surjective since so is g. Let $\mathcal{L}' \coloneqq \psi^*\mathcal{L}$.

For smoothness, we can compose g with a general automorphism of $\mathbb{P}^N$. Then the conclusion follows from [Har77, Chapter III, Theorem 10.8]. ☐

Lemma 2.16 (ref. [KM98, Theorem 5.10, 5.22])**.** Let (X, B) be a klt pair over $\mathbf{k}$ of characteristic 0. Then X has rational singularities and is Cohen-Macaulay.

Lemma 2.17. Let X be a projective variety of dimension n and D a nef and big divisor on X. Then there exists an effective divisor B such that for every k, there is an ample divisor A_k such that

$$D \sim_{\mathbb{Q}} A_k + \frac{1}{k}B.$$

Proof. By Yang: definition of big divisor, there exists an ample divisor A_1 and effective divisor B such that

$$D \sim_{\mathbb{Q}} A_1 + B.$$

Then we have

$$D \sim_{\mathbb{Q}} \frac{A + (k-1)D}{k} + \frac{1}{k}B.$$

Since A is ample and D is nef, we can take $A_k = (A + (k-1)D)/k$ which is ample. □

3 Cone Theorem

3.1 Preliminary

Theorem 3.1 (Iitaka fibration, semiample case, ref. [Laz04, Theorem 2.1.27])**.** Let X be a projective variety and $\mathcal{L}$ an semiample line bundle on X. Then there exists a fibration $\varphi : X \to Y$ of projective varieties such that for any $m \gg 0$ with $\mathcal{L}^m$ base point free, we have that the morphism $\varphi_{\mathcal{L}^m}$ induced by $\mathcal{L}^m$ is isomorphic to φ. Such a fibration is called the *Iitaka fibration* associated to $\mathcal{L}$.

Theorem 3.2 (Rigidity Lemma, ref. [Deb01, Lemma 1.15])**.** Let $\pi_i : X \to Y_i$ be proper morphisms of varieties over a field $\mathbf{k}$ for $i = 1, 2$. Suppose that π_1 is a fibration and π_2 contracts $\pi_1^{-1}(y_0)$. Then there exists a rational map $\varphi : Y_1 \dashrightarrow Y_2$ such that $\pi_2 \circ \varphi = \pi_1$ and φ is well-defined near $Y_1 \setminus \{y_0\}$.

Theorem 3.3. Let $A, B \subset \mathbb{R}^n$ be disjoint convex sets. Then there exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_A \leq c$ and $f|_B \geq c$ for some $c \in \mathbb{R}$.

Proposition 3.4. Let X be a normal projective variety of dimension n and H an ample divisor on X. Suppose that $K_X \cdot H^{n-1} < 0$. Then for a general point $x \in X$, there exists a rational curve Γ passing through x such that

$$0 < H \cdot \Gamma \leq -2n \cdot \frac{H^n}{K_X \cdot H^{n-1}}.$$

Schetch of proof. Take a resolution $f : Y \to X$, then f^*H is nef on Y and $K_Y \cdot f^*H^{n-1} < 0$ since $E \cdot f^*H^{n-1} = 0$. Choose an ample divisor H_Y on Y closed enough to f^*H such that $K_Y \cdot H_Y^{n-1} < 0$. By [MM86, Theorem 5] and take limit for H_Y. □

Lemma 3.5 (ref. [Kaw91, Lemma])**.** Let (X, B) be a projective klt pair and $f : X \to Y$ a birational projective morphism. Let E be an irreducible component of dimension d of the exceptional locus of f and $\nu : E^\nu \to X$ the normalization of E. Suppose that $f(E)$ is a point. Then for any ample divisor H on X, we have

$$K_{E^\nu} \cdot \nu^* H^{d-1} \leq K_{(X,B)}|_{E^\nu} \cdot \nu^* H^{d-1}.$$

3.2 Non-vanishing Theorem

Theorem 3.6 (Non-vanishing Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, we have

$$H^0(X, mD) \neq 0.$$

Proof. Yang: To be completed. □

3.3 Base Point Free Theorem

Theorem 3.7 (Base Point Free Theorem)**.** Let (X, B) be a projective klt pair and D a Cartier divisor on X. Suppose that D is nef and $aD - K_{(X,B)}$ is nef and big for some $a > 0$. Then for $m \gg 0$, mD is base point free.

Proof. Yang: To be completed. □

Remark 3.8. In general, we say that a Cartier divisor D is *semiample* if there exists a positive integer m such that mD is base point free. The statement in Base Point Free Theorem (Theorem 3.7) is strictly stronger than the semiample condition. For example, let $\mathcal{L}$ be a torsion line bundle, then $\mathcal{L}$ is semiample, but there exists no positive integer M such that $m\mathcal{L}$ is base point free for all $m > M$.

3.4 Rationality Theorem

Lemma 3.9 (ref. [KM98, Theorem 1.36])**.** Let X be a proper variety of dimension n and $D_1, \ldots, D_m$ Cartier divisors on X. Then the Euler characteristic $\chi(n_1 D_1, \ldots, n_m D_m)$ is a polynomial in $(n_1, \cdots, n_m)$ of degree at most n.

Theorem 3.10 (Rationality Theorem)**.** Let (X, B) be a projective klt pair, $a = a(X) \in \mathbb{Z}$ with $aK_{(X,B)}$ Cartier and H an ample divisor on X. Let

$$t \coloneqq \inf\{s \geq 0 : K_{(X,B)} + sH \text{ is nef}\}$$

be the nef threshold of (X, B) with respect to H. Then $t = v/u \in \mathbb{Q}$ and

$$0 \leq v \leq a(X) \cdot (\dim X + 1).$$

Proof. For every $r \in \mathbb{R}_{>0}$, let

$$v(r) \coloneqq \begin{cases} v, & \text{if } r = \frac{v}{u} \in \mathbb{Q} \text{ in lowest term;} \\ \infty, & \text{if } r \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We need to show that $v(t) \leq a(\dim X + 1)$. For every $(p, q) \in \mathbb{Z}^2_{>0}$, set $D(p, q) \coloneqq paK_{(X,B)} + qH$. If $(p, q) \in \mathbb{Z}^2_{>0}$ with $0 < atp - q < t$, then we have $D(p, q)$ is not nef and $D(p, q) - K_{(X,B)}$ is ample.

Step 1. We show that a polynomial $P(x,y)\neq 0\in\mathbb{Q}[x,y]$ of degree at most n is not identically zero on the set

$$\{(p,q)\in\mathbb{Z}^2 : p,q>M, 0<atp-q<t\varepsilon\},\quad \forall M>0,$$

if $v(t)\varepsilon>a(n+1)$.

If $v(t)=\infty$, for any n, we show that we can find infinitely many lines L such that $\#L\cap\Lambda\geq n+1$. If so, Λ is Zariski dense in $\mathbb{Q}^2$. Since $1/at\in\mathbb{R}\setminus\mathbb{Q}$, there exist $p_0,q_0>M$ such that

$$0<\frac{p_0}{q_0}-\frac{1}{at}<\frac{\varepsilon}{(n+1)a}\cdot\frac{1}{q_0},\ \text{i.e. } 0<atp_0-q_0<\frac{\varepsilon t}{n+1}.$$

Then $(ip_0,iq_0)\in\Lambda\cap\{p_0y=q_0x\}$ for $i=1,\cdots,n+1$. Since M is arbitrary, there are infinitely many such lines L.

Suppose $v(t)=v<\infty$ and $t=v/u$. Then the inequality is equivalent to $0<aup-vq<\varepsilon v$. Note that $\gcd(au,v)|a$, then $aup-vq=ai$ has integer solutions for $i=1,\cdots,n+1$. Since $v(t)\varepsilon>a(n+1)$, there are at least $n+1$ lines which intersect Λ in infinitely many points. This enforce any polynomial which vanishes on Λ has degree at least $n+1$.

Step 2. There exists an index set $\Lambda\subset\mathbb{Z}^2$ such that Λ contains all sufficiently large (p,q) with $0\leq atp-q\leq t$ and

$$Z:=\operatorname{Bs}|D(p,q)|=\operatorname{Bs}|D(p',q')|\neq\varnothing,\quad\forall(p,q),(p',q')\in\Lambda.$$

For every $(p,q)\in\mathbb{Z}^2_{>0}$ with $0<atp-q<t$, there exists $M>0$ such that

$$D(\alpha,\beta)=\alpha aK_{(X,B)}+\beta H$$

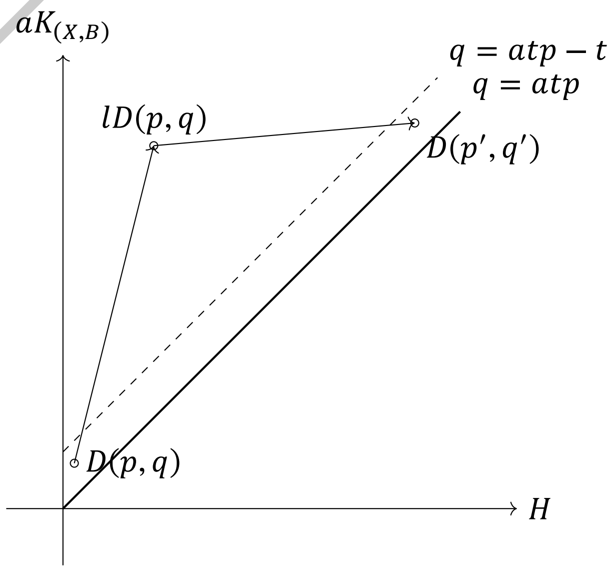
is base point free for all $\alpha=0,\cdots,p$ and $\beta>M$. Choose M' large enough such that for all $(p',q')\in\mathbb{Z}^2_{>0}$ with $p',q'>M'$ and $0<atp'-q'<t$, write

$$p'=lp+p_0,\quad q'=lq+q_0$$

for some $l\in\mathbb{Z}_{\geq 0}$ and $0\leq p_0<p$, we have $q_0>M$. The existence of such M' follows from the estimate

$$q_0=q'-lq=q'-\frac{p'-p_0}{p}q>q'-(p'-p_0)(at-\delta)>p'\delta,$$

where $\delta>0$ is a small enough number such that $at-\delta>q/p$.



Then $D(p',q') - lD(p,q) = D(p_0,q_0)$ is base point free. It follows that $\operatorname{Bs}|D(p',q')| \subseteq \operatorname{Bs}|D(p,q)|$. By noetherian induction, there exists an index set Λ such that $\operatorname{Bs}|D(p,q)| = Z$ for all $(p,q) \in \Lambda$.

Step 3. Suppose the contradiction that $v(t) > a(\dim X+1)$. Then we show that $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$. This is an analogue of Non-vanishing Theorem in the proof of Base Point Free Theorem (Theorem 3.7).

Let $P(x,y) := \chi(D(x,y))$ be the Hilbert polynomial of $D(x,y)$. Note that $P(0,n) = \chi(nH) \neq 0$ since H is ample. Then $P(x,y) \neq 0$ and $\deg P \leq \dim X$. By Step 1, P is not identically zero on Λ. Note that $D(p,q) - K_{(X,B)}$ is ample for all $(p,q) \in \Lambda$, then $h^i(X, D(p,q)) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing theorem (Theorem 2.13). Then

$$P(p,q) = \chi(D(p,q)) = h^0(X, D(p,q)) \neq 0$$

for some $(p,q) \in \Lambda$. This is equivalent to that $Z \neq X$ and hence $H^0(X, D(p,q)) \neq 0$ for all $(p,q) \in \Lambda$.

Step 4. We follow the same line of the proof of Base Point Free Theorem (Theorem 3.7) to show that there is a section which does not vanish on Z.

Fix $(p,q) \in \Lambda$. If $v(t) < \infty$, we assume that $t = v/u$ and $atp - q = a(n+1)/u$. Let $f : Y \to X$ be a resolution such that

(a) $K_{Y,B_Y} = f^*K_{(X,B)} + E_Y$ for some effective exceptional divisor E_Y, and Y, B_Y is a klt pair;

(b) $f^*|D(p,q)| = |L| + F$ for some effective divisor F and a base point free divisor L, and $f(\operatorname{Supp} F) = Z$;

(c) $f^*D(p,q) - f^*K_{(X,B)} - E_0$ is ample for some effective $\mathbb{Q}$-divisor $E_0 \in (0,1)$, and coefficients of E_0 are sufficiently small;

(d) $B_Y + E_Y + F + E_0$ has snc support.

Yang: Such resolution exists by [KM98].

Let $c := \inf\{\lfloor B_Y + E_0 + tF \rfloor \neq 0\}$. Adjust the coefficients of E_0 slightly such that $\lfloor B_Y + E_0 + cF \rfloor = F_0$ for unique prime divisor F_0 with $F_0 \subset \operatorname{Supp} F$. Set $\Delta_Y := B_Y + cF + E_0 - F_0$. Then (Y, Δ_Y) is a klt pair.

Let

$$\begin{aligned} N(p',q') &:= f^*D(p',q') + E_Y - F_0 - K_{(Y,\Delta_Y)} \\ &= \Big(f^*D(p',q') - (1+c)f^*D(p,q)\Big) + \Big(f^*D(p,q) - f^*K_{(X,B)} - E_0\Big) + c\Big(f^*D(p,q) - F\Big). \end{aligned}$$

Note that on

$$\Lambda_0 := \{(p',q') \in \Lambda : 0 < atp' - q' < atp - q,\ p',q' > (1+c)\max\{p,q\}\},$$

the divisor $f^*D(p',q') - (1+c)f^*D(p,q) = f^*D(p' - (1+c)p, q' - (1+c)q)$ is ample, and hence $N(p',q')$ is ample.

By the exact sequence

$$0 \to o_Y(f^*D(p',q') + E_Y - F_0) \to o_Y(f^*D(p',q') + E_Y) \to o_{F_0}((f^*D(p',q') + E_Y)|_{F_0}) \to 0$$

and Kawamata-Viehweg Vanishing Theorem (Theorem 2.13), we get a surjective map

$$H^0(Y, f^*D(p', q') + E_Y) \twoheadrightarrow H^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}).$$

On F_0, consider the polynomial $\chi((f^*D(p', q') + E_Y)|_{F_0})$. Note that $\dim F_0 = n - 1$ and by the construction of $(p, q), \Lambda_0$, similar to Step 3, we can show that $\chi((f^*D(p', q') + E_Y)|_{F_0})$ is not identically zero on Λ_0. By adjunction, we have $(f^*D(p', q') + E_Y)|_{F_0} = N(p', q')|_{F_0} + K_{(F_0, \Delta_Y|_{F_0})}$ with $N(p', q')|_{F_0}$ ample and $(F_0, \Delta_Y|_{F_0})$ klt. Hence we can apply Kawamata-Viehweg Vanishing Theorem (Theorem 2.13) to get

$$h^0(F_0, (f^*D(p', q') + E_Y)|_{F_0}) = \chi(F_0, (D(p', q') + E_Y)|_{F_0}) \neq 0.$$

This combining with the surjective map contradict to the assumption that $f(F_0) \subset Z = \mathrm{Bs}\,|D(p', q')|$.

□

3.5 Cone Theorem and Contraction Theorem

Theorem 3.11 (Cone Theorem)**.** Let (X, B) be a projective klt pair. Then there exist countably many curves $C_i \subset X$ such that

(a) we have a decomposition of cones

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i];$$

(b) and for any $\varepsilon > 0$ and an ample divisor H on X, we have

$$\mathrm{Psef}_1(X) = \mathrm{Psef}_1(X)_{K_{(X,B)} + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Proof. Let $F_D := \mathrm{Psef}_1(X) \cap D^{\perp}$ for a nef divisor D on X. If $\dim F_D = 1$, we also write $R_D := F_D$. Let $H_1, \cdots, H_{\rho-1}$ be ample divisors on X such that they together with $K_{(X,B)}$ form a basis of $N^1(X)_{\mathbb{Q}}$. Fix a norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$ and let $S^{\rho-1} := S(N_1(X)_{\mathbb{R}})$ be the unit sphere in $N_1(X)_{\mathbb{R}}$.

Step 1. There exists an integer N such that for every $K_{(X,B)}$-negative extremal face F_D and for every ample divisor H, there exists $n_0, r \in \mathbb{Z}_{>0}$ such that for all $n > n_0$, $\{0\} \neq F_{nD + rK_{(X,B)} + NH} \subset F_D$.

Let $N := (a(X)(\dim X + 1))!$, where $a(X)$ is the number in Theorem 3.10. For every n, $nD + H$ is an ample divisor and by Theorem 3.10, the nef threshold of $K_{(X,B)}$ with respect to $nD + H$ is of form

$$\inf\{s \geq 0 : K_{(X,B)} + s(nD + H) \text{ is nef}\} = \frac{N}{r_n}, \quad r_n \in \mathbb{Z}_{\geq 0}.$$

Since $K_{(X,B)} + (N/r_n)((n+1)D + H)$ is nef, we have $r_n \leq r_{n+1}$. On the other hand, let $\xi \in F_D \setminus \{0\}$. Then $\xi \cdot (K_{(X,B)} + (N/r_n)(nD + H)) \geq 0$ implies that

$$r_n \leq -N \cdot \frac{K_{(X,B)} \cdot \xi}{H \cdot \xi}.$$

Hence $r_n \to r \in \mathbb{Z}_{\geq 0}$. It follows that $rK_{(X,B)} + nND + NH$ is a nef but not ample divisor for all $n \gg 0$. Note that for every nef divisors N_1, N_2, we have $F_{N_1 + N_2} = F_{N_1} \cap F_{N_2}$. Then for all $n \gg 0$, there exists m large enough such that

$$\{0\} \neq F_{rK_{(X,B)}+mND+NH} \subset F_{rK_{(X,B)}+nD+NH} \subset F_D.$$

Step 2. Let $\Phi : N_1(X)_{K_{(X,B)}<0} \to \mathbb{R}^{\rho-1}$ be the map defined by

$$\alpha \mapsto \left(\frac{H_1 \cdot \alpha}{K_{(X,B)} \cdot \alpha}, \dots, \frac{H_{\rho-1} \cdot \alpha}{K_{(X,B)} \cdot \alpha}\right).$$

We show that the image of R_D under Φ lies in a $\mathbb{Z}$-lattice in $\mathbb{R}^{\rho-1}$.

Suppose $R = \mathbb{R}_{\geq 0}\xi$ for a class ξ. By Step 1, we have $R_{nD+rK_{(X,B)}+NH_i} = R_D$ for some integers n, r. Then $\xi \cdot (nD + rK_{(X,B)} + NH_i) = 0$ implies that

$$\frac{H_i \cdot \xi}{K_{(X,B)} \cdot \xi} = \frac{-r}{N} \in \frac{1}{N}\mathbb{Z}.$$

It follows that the image of R_D under Φ lies in $\frac{1}{N}\mathbb{Z}^{\rho-1}$.

Step 3. We show that every $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ is of the form R_D for some nef divisor D on X.

Let $R = \mathbb{R}_{\geq 0}\xi$ be a $K_{(X,B)}$-negative exposed ray. Then R is of form $D^{\perp} \cap \mathrm{Psef}_1(X)$ for some nef $\mathbb{R}$-divisor D on X. We need to show that D can be choose as a nef $\mathbb{Q}$-divisor. There is a sequence of nef but not ample $\mathbb{Q}$-divisors D_m such that $D_m \to D$ as $m \to \infty$. We adjust D_m such that $\dim F_{D_m} = 1$ for all n.

By re-choosing H_i, we can assume that $D = a_1 H_1 + \cdots + a_{\rho-1}H_{\rho-1} + a_\rho K_{(X,B)}$ for $a_i > 0$ since $aD - K$ is ample for $a \gg 0$. After truncation, we can assume that so is D_m. Then F_{D_m} is $K_{(X,B)}$-negative. Note that $F_{nD_m + r_i K_{(X,B)} + NH_i} \subset F_{D_m}$ for some $r_i > 0$ and $n \gg 0$ by Step 1. If $\dim F_{D_m} > 1$, then not all $H_i|_{F_{D_m}}$ are proportional to $K_{(X,B)}|_{D_m}$. We can assume that$r_1 K_{(X,B)} + NH_1$ is not identically zero on F_{D_m}. Then we can choose n large enough such that $\|r_1 K_{(X,B)} + NH_1\|/n < 1/m$. Replace D_m by $D_m + (r_1 K_{(X,B)} + NH_1)/n$. Inductively we construct D_m nef $\mathbb{Q}$-divisor with $D_m \to D$ and $\dim F_{D_m} = 1$.

Let $R_{D_m} = \mathbb{R}_{\geq 0}\xi_m$. Suppose that $\|\xi_m\| = \|\xi\| = 1$. By passing to a subsequence, we can assume that ξ_m converges. Then $\xi_m \to \xi$ since $\lim D_m \cdot \xi_m = D \cdot \lim \xi_m = 0$. However, Φ is well-defined at ξ and the image of ξ_m under Φ is discrete. Hence $\xi = \xi_m$ for all m large enough. It follows that $R = R_{D_m}$ for a nef $\mathbb{Q}$-divisor D_m.

By Step 2, the $K_{(X,B)}$-negative extremal rays form a discrete set in $\{\alpha \in \mathrm{Psef}_1(X) : K_{(X,B)} \cdot \alpha < 0\}$. Hence every $K_{(X,B)}$-negative extremal ray is an exposed ray by Straszewicz's Theorem.

Step 4. Proof of the theorem.

Given an ample divisor H on X, note that εH has positive minimum δ on $\mathrm{Psef}_1(X) \cap S^{\rho-1}$. Note that the set

$$\{\alpha \in \mathrm{Psef}_1(X) \cap S^{\rho-1} : K_{(X,B)} \cdot \alpha \leq -\varepsilon H \cdot \alpha\} \subset \{\alpha : K_{(X,B)} \cdot \alpha \leq -\delta\}$$

is compact, and Φ is well-defined on it. By Steps 2 and 3, there are only finitely many extremal rays on $\mathrm{Psef}_1(X)_{K_{(X,B)}+\varepsilon H \leq 0}$. Hence we get (b).

For (a), note that any closed cone is equal to the closure of the cone generated by its extremal ray. We only need to show that the cone

$$c \coloneqq \mathrm{Psef}_1(X)_{K_{(X,B)} \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

is closed. Choose a Cauchy sequence $\{\alpha_n\} \subset c$ such that $\alpha_n \to \alpha \in N_1(X)_\mathbb{R}$. Note that $\mathrm{Psef}_1(X)$ is closed, hence $\alpha \in \mathrm{Psef}_1(X)$. We only need to consider the case $\alpha \cdot K_{(X,B)} < 0$. We can choose an ample divisor and $\varepsilon > 0$ such that $\alpha \cdot (K_{(X,B)} + \varepsilon H) < 0$. Then $\alpha_n \cdot (K_{(X,B)} + \varepsilon H) < 0$ for all n large enough. Note that $c \cap \{K_{(X,B)} + \varepsilon H \leq 0\}$ is a polyhedral cone by Step 2 and hence is closed. Then $\alpha \in c$ and the conclusion follows. ☐

Remark 3.12. Thanks for my friend Qin for pointing out that the extremal ray may not be exposed.

Theorem 3.13 (Contraction Theorem)**.** Let (X, B) be a projective klt pair and $F \subset \mathrm{Psef}_1(X)$ a $K_{(X,B)}$-negative extremal face of $\mathrm{Psef}_1(X)$. Then there exists a fibration $\varphi_F : X \to Y$ of projective varieties such that

(a) an irreducible curve $C \subset X$ is contracted by φ_F if and only if $[C] \in F$;

(b) up to linearly equivalence, any Cartier divisor G with $F \subset G^\perp = \{\alpha \in N_1(X) : \alpha \cdot G = 0\}$ comes from a Cartier divisor on Y, i.e., there exists a Cartier divisor G_Y on Y such that $G \sim \varphi_F^* G_Y$.

Proof. We follow the following steps to prove the theorem.

Step 1. We show that there exists a nef divisor D on X such that $F = D^\perp \cap \mathrm{Psef}_1(X)$. In other words, F is defined on $N_1(X)_\mathbb{Q}$.

We can choose an ample divisor H and $n > 0$ such that $K_{(X,B)} + (1/n)H$ is negative on F since $F \cap S^{\rho-1}$ is compact and $K_{(X,B)}$ is strictly negative on it, where $S^{\rho-1}$ is the unit sphere in $N_1(X)_\mathbb{R}$. Then by Cone Theorem (Theorem 3.11), F is an extremal face of a rational polyhedral cone, namely $\mathrm{Psef}_1(X)_{K_{(X,B)}+(1/n)H\leq 0}$. It follows that $F^\perp \subset N^1(X)_\mathbb{R}$ is defined on $\mathbb{Q}$. Since F is extremal and $K_{(X,B)} + (1/n)H$-negative, the set $\{L \in F^\perp : L|_{\mathrm{Psef}_1(X)\setminus F} > 0\}$ has non-empty interior in $F^\perp$ by Theorems 3.3 and 3.11. Then there exists a Cartier divisor D such that $D \in F^\perp$ and $D|_{\mathrm{Psef}_1(X)\setminus F} > 0$. It follows that D is nef and $F = D^\perp \cap \mathrm{Psef}_1(X)$.

Step 2. Let $\varphi : X \to Y$ be the Iitaka fibration associated to D by Theorem 3.1. We show that φ is the desired fibration.

Note that $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$ is compact and D is strictly positive on it. Then there exist $a \geq 0$ such that $aD - K_{(X,B)}$ is strictly positive on $\mathrm{Psef}_1(X)_{K_{(X,B)}\geq 0} \cap S^{\rho-1}$. And $K_{(X,B)}$ is strictly negative on $F \setminus \{0\}$ since F is $K_{(X,B)}$-negative. Then by Base Point Free Theorem (Theorem 3.7), we know that mD is base point free for all $m \gg 0$. Hence we can apply Theorem 3.1 to get a fibration $\varphi_D : X \to Y$.

First we show that D comes from Y. Note that mD and $(m+1)D$ induces the same fibration φ_D for $m \gg 0$. Then there exists $D_{Y,m}$ and $D_{Y,m+1}$ such that $\varphi_D^* D_{Y,m} \sim mD$ and $\varphi_D^* D_{Y,m+1} \sim (m+1)D$. Then set $D_Y = D_{Y,m+1} - D_{Y,m}$, we have $\varphi_D^* D_Y \sim D$.

Note that $D_Y \equiv (1/m)D_{Y,m}$ and $D_{Y,m}$ is ample. Hence D_Y is ample. Then for any curve $C \subset X$, we have

$$D \cdot C = \varphi^* D_Y \cdot C = D_Y \cdot (\varphi_D)_* C.$$

It follows that C is contracted by φ_D if and only if $D \cdot C = 0$, which is equivalent to $[C] \in F$.

Let G be arbitrary Cartier divisor on X such that $F \subset G^\perp$. Since D is strictly positive on $\mathrm{Psef}_1(X) \setminus F$, for $m \gg 0$, let $D' := mD + G$, we have $D'^\perp \cap \mathrm{Psef}_1(X) = F$. Then by the same argument as above, we get an other fibration $\varphi_{D'} : X \to Y'$ such that a curve C is contracted by $\varphi_{D'}$ if and only if $[C] \in F$. Then by Rigidity Lemma (Theorem 3.2), we see that $\varphi_D = \varphi_{D'}$ up to an isomorphism on Y. In particular, $D' \sim \varphi_D^* D'_Y$ for some Cartier divisor D'_Y on Y. Then $G = D' - mD$ also comes from Y. □

Remark 3.14. The Step 1 is amazing. If F is not $K_{(X,B)}$-negative, then it may not be rational. For example, let $X = E \times E$ for a general elliptic curve E. By [Laz04, Lemma 1.5.4], we know that $\mathrm{Psef}_1(X)$ is a circular cone. The we see there indeed exist some irrational extremal faces of $\mathrm{Psef}_1(X)$.

Theorem 3.15 (Length of extremal rays)**.** Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Then there exists a rational curve $C \subset X$ such that $[C] \in R$ and

$$0 < -K_{(X,B)} \cdot C \leq 2 \dim X.$$

Proof. By Theorem 3.13, let $\varphi_D : X \to Y$ be the contraction associated to R_D (note that we do not need the step to proof Theorem 3.13). If $\dim Y < \dim X$, let F be a general fiber of φ_D. Yang: By adjunction, $(F, B|_F)$ is a klt pair and $K_{(F,B|_F)} = K_{(X,B)}|_F$. Take $H = aD - K_{(X,B)}$ for some $a > 0$ such that H is ample on F. By Proposition 3.4. Yang: In birational case, by adjunction, suppose $\varphi_D(E)$ is a point. By Lemma 3.5, we can use Proposition 3.4 to get the result. Yang: To be completed. □

Definition 3.16. Let (X, B) be a projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$ with contraction $\varphi_R : X \to Y$. There are three types of contractions:

(a) *Divisorial contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension one;

(b) *Small contraction*: if $\dim X = \dim Y$ and the exceptional locus of φ_R is of codimension at least two;

(c) *Mori fiber space*: if $\dim X > \dim Y$.

Proposition 3.17. Let (X, B) be a $\mathbb{Q}$-factorial projective klt pair and R a $K_{(X,B)}$-negative extremal ray of $\mathrm{Psef}_1(X)$. Suppose that the contraction $\varphi : X \to Y$ associated to R is either divisorial or a Mori fiber space. Then Y is $\mathbb{Q}$-factorial.

Proof. Let D be a prime Weil divisor on Y and $U \subset Y$ a big open smooth subset. Let $R = \mathbb{R}_{\geq 0}[C]$ for an irreducible curve C contracted by φ. Set $D_X := \overline{\varphi|_{\varphi^{-1}(U)}^{-1} D}$. Then D_X is a prime Weil divisor on X and hence is $\mathbb{Q}$-Cartier.

If φ is a Mori fiber space, then $D_X|_F \equiv 0$ for general fiber F of φ. Then by Contraction Theorem (Theorem 3.13), we see that $mD_X \sim \varphi^* D'$ for some Cartier divisor D' on Y. We have $mD|_U \sim D'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is a fibration. Then $mD \sim D'$ and hence D is $\mathbb{Q}$-Cartier.

If φ is a divisorial contraction, let E be the exceptional divisor of φ and assume that $\varphi^{-1}|_U$ is an isomorphism. Then $E \cdot C \neq 0$ (otherwise $E \sim_{\mathbb{Q}} f^* E_Y$ for some Cartier $\mathbb{Q}$-divisor E_Y on Y). Then we can choose $a \in \mathbb{Q}$ such that $(D_X + aE) \cdot C = 0$. By Contraction Theorem (Theorem 3.13), we have $mD_X + maE \sim \varphi^* D'$ for some Cartier divisor D' on Y. Then we also have $D|_U \sim mD'|_U$ since $\varphi|_{\varphi^{-1}(U)}$ is an isomorphism. Hence D is $\mathbb{Q}$-Cartier. □

Remark 3.18. If φ is a small contraction, then Y is never $\mathbb{Q}$-factorial. Otherwise, let B_Y be the strict transform of B on Y. Note that $K_{(Y,B_Y)}|_U \sim K_{(X,B)}|_U$ on a big open subset U. Suppose $K_{(Y,B_Y)}$

is $\mathbb{Q}$-Cartier. Then $\varphi^* K_{(Y,B_Y)} \sim_{\mathbb{Q}} K_{(X,B)}$. Then we have

$$\varphi^* K_{(Y,B_Y)} \cdot C = 0 = K_{(X,B)} \cdot C < 0.$$

This is a contradiction.

Example 3.19. Let $X = E \times E \times \mathbb{P}^1$. Yang: To be completed.

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