

Introduction to Moduli Problems

Let \mathcal{C} be a smooth projective curve of genus g over an algebraically closed field \mathbb{k} of characteristic 0. We are interested in the moduli space of vector bundles on \mathcal{C} .

1 Moduli functors

Let S be a noetherian scheme and T is a scheme of finite type over S . Recall the Yoneda lemma: there is a full and faithful functor

$$h : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \text{Fun}((\mathbf{Sch}_S)^{\text{op}}, \mathbf{Set}), \quad T \mapsto h_T(S) := \text{Hom}_{\mathbf{Sch}_S}(T, S).$$

A functor $F : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \mathbf{Set}$ is *representable* if there exists a scheme M over S such that $F \cong h_M$. We say that M is *the fine moduli space* of F .

Remark 1. If F is representable by M , then there is a universal object $\mathcal{U} \in F(M)$ given by $\text{id}_M \in h_M(M)$ satisfying the following universal property: for any $T \in \mathbf{Sch}_S$ and any $\xi \in F(T)$, there exists a unique morphism $f : T \rightarrow M$ such that $F(f)(\mathcal{U}) = \xi$.

The most famous example of representable functor is the Quot functor. Let S be a noetherian scheme, $\pi : X \rightarrow S$ a projective morphism, \mathcal{L} a relatively ample line bundle on X , \mathcal{F} a coherent sheaf on X , and $P \in \mathbb{Q}[t]$ a polynomial. We define a functor

$$\text{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}} : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \mathbf{Set}$$

$$T \mapsto \{\pi_T^* \mathcal{F} \twoheadrightarrow \mathcal{Q} \mid \mathcal{Q} \text{ is flat over } T, \forall t \in T, \mathcal{Q}_t \text{ has Hilbert polynomial } P\} / \sim,$$

where $\pi_T^* \mathcal{F} \twoheadrightarrow \mathcal{Q}$ and $\pi_T^* \mathcal{F} \twoheadrightarrow \mathcal{Q}'$ are equivalent if $\text{Ker}(\pi_T^* \mathcal{F} \twoheadrightarrow \mathcal{Q}) = \text{Ker}(\pi_T^* \mathcal{F} \twoheadrightarrow \mathcal{Q}')$.

By Grothendieck, $\text{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}}$ is representable by a projective S -scheme $\text{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}}$. Yang: Reference...

If we take $S = \text{Spec } \mathbb{k}$, X a projective variety and $\mathcal{F} = \mathcal{O}_X$. Then the Quot functor $\text{Quot}_{\mathcal{O}_X/X/\mathbb{k}}^{P, \mathcal{L}}$ becomes the Hilbert functor $\mathcal{Hilb}_{X/\mathbb{k}}^{P, \mathcal{L}}$, which is representable by a projective \mathbb{k} -scheme called the *Hilbert scheme* $\text{Hilb}_X^{P, \mathcal{L}}$.

2 Moduli functor of vector bundles

Consider the functor

$$\tilde{\mathcal{M}}_{r,d} : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \mathbf{Set}$$

$$T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a vector bundle on } X \times T \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\} / \sim,$$

where $\mathcal{E} \sim \mathcal{E}'$ if there exists a line bundle \mathcal{L} on T such that $\mathcal{E}' \cong \mathcal{E} \otimes \pi_T^* \mathcal{L}$, where $\pi_T : X \times T \rightarrow T$ is the projection.

Unfortunately, $\tilde{\mathcal{M}}_{r,d}$ is not representable. There are two main reasons:

- unboundedness and
- jumping phenomenon.

Definition 2. A family of vector bundles on a variety X is *bounded* if there exists a scheme S of finite type over \mathbb{k} and a vector bundle \mathcal{E} on $X \times S$ such that every vector bundle in the family is isomorphic to \mathcal{E}_s for some $s \in S$.

If $\tilde{\mathcal{M}}_{r,d}$ is representable by a scheme M of finite type over \mathbb{k} , then the family of vector bundles parametrized by M is bounded. This is impossible since if so, $\{h^0(X, \mathcal{E}) \mid \mathcal{E} \in \tilde{\mathcal{M}}_{r,d}(\mathbb{k})\}$ is bounded by semicontinuity theorem, which is not true. For example, consider the family $\mathcal{E}_n = \mathcal{O}_X(nP) \oplus \mathcal{O}_X(-nP) \in \tilde{\mathcal{M}}_{2,0}(\mathbb{k})$ for $n \geq 0$, where $P \in X(\mathbb{k})$ is a fixed point. By Riemann-Roch theorem, we have $h^0(X, \mathcal{E}_n) = n + 1 - g$ for n sufficiently large.

Example 3. Let us see a jumping phenomenon example due to Ress. Let \mathcal{E} be a vector bundle on X of rank r and degree d with a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}.$$

On $X \times \mathbb{A}^1$, we can construct a vector bundle \mathcal{F} by “deforming” \mathcal{E} to $\bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}$ as follows: let t be the coordinate of \mathbb{A}^1 , and define \mathcal{F} to be the subsheaf of $\pi_X^* \mathcal{E}$ generated by $t^{-i} \cdot \pi_X^* \mathcal{E}_i$ for $1 \leq i \leq r$, where $\pi_X : X \times \mathbb{A}^1 \rightarrow X$ is the projection. Then \mathcal{F} is a vector bundle on $X \times \mathbb{A}^1$ of rank r and degree d . We have

$$\mathcal{F}_t \cong \begin{cases} \mathcal{E}, & t \neq 0, \\ \bigoplus_{i=1}^r \mathcal{E}_i/\mathcal{E}_{i-1}, & t = 0. \end{cases}$$

This is called the *jumping phenomenon*. Yang: To be checked...

For a concrete example, let $X = \mathbb{P}^1$, we have an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0.$$

Let $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and consider the filtration $F : 0 \subset \mathcal{O}(-2) \subset \mathcal{E}$. Yang: To be continued...

If $\tilde{\mathcal{M}}_{r,d}$ is representable by a scheme M , then the family of vector bundles parametrized by M does not have jumping phenomenon. Indeed, if \mathcal{F} is a vector bundle on $X \times \mathbb{A}^1$ such that $\mathcal{F}_t \cong \mathcal{E}$ for $t \neq 0$, then by the universal property of M , there exists a unique morphism $f : \mathbb{A}^1 \rightarrow M$ such that $\mathcal{F} \cong (\text{id}_X \times f)^* \mathcal{U}$, where \mathcal{U} is the universal vector bundle on $X \times M$. Since f is constant on the open subset $\mathbb{A}^1 \setminus \{0\}$, it is constant on \mathbb{A}^1 . Thus, $\mathcal{F}_0 \cong \mathcal{E}$.

To fix the above problems, we need to

- restrict to a smaller family of vector bundles,
- kill jumping phenomenon, and
- weaken the notion of representability.

3 Coarse moduli space

Definition 4. Let $F : (\mathbf{Sch}_S)^{\text{op}} \rightarrow \mathbf{Set}$ be a functor, M a scheme over S , and $\eta : F \rightarrow h_M$ a natural transformation. We say that (M, η) *corepresents* F if for any scheme N over S and any natural transformation $\eta' : F \rightarrow h_N$, there exists a unique morphism $f : M \rightarrow N$ such that the following

diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\eta} & h_M \\ & \searrow \eta' & \downarrow h_f \\ & & h_N. \end{array}$$

Definition 5. A scheme M over S is called the *coarse moduli space* of F if

- (a) there exists a natural transformation $\eta : F \rightarrow h_M$ such that (M, η) corepresents F ;
- (b) $\eta_{\mathbb{k}} : F(\mathbb{k}) \rightarrow M(\mathbb{k})$ is a bijection.

Yang: To be continued...

4 Semistable vector bundles

Proposition 6. Let $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$ be an exact sequence of non-zero vector bundles on \mathcal{C} . Then $\mu(\mathcal{E}_2) \geq \mu(\mathcal{E}_1)$ (resp. $\mu(\mathcal{E}_2) > \mu(\mathcal{E}_1)$) if and only if $\mu(\mathcal{E}_2) \leq \mu(\mathcal{E}_3)$ (resp. $\mu(\mathcal{E}_2) < \mu(\mathcal{E}_3)$).

Proof. We have

$$\mu(\mathcal{E}_2) = \frac{\deg \mathcal{E}_2}{\text{rank } \mathcal{E}_2} = \frac{\deg \mathcal{E}_1 + \deg \mathcal{E}_3}{\text{rank } \mathcal{E}_1 + \text{rank } \mathcal{E}_3}.$$

Note that for any $a, b, c, d \in \mathbb{R}_{>0}$, we have

$$\frac{a+c}{b+d} \geq \frac{a}{b} \iff bc \geq ad \iff \frac{a+c}{b+d} \leq \frac{c}{d}.$$

The strict inequality case is similar. Then the proposition follows. \square

Definition 7. Let \mathcal{C} be a smooth projective curve over \mathbb{k} . For a vector bundle \mathcal{E} of rank r and degree d on \mathcal{C} , we define its slope to be $\mu(\mathcal{E}) := d/r$. We say that \mathcal{E} is *stable* (resp. *semistable*) if for any proper sub-bundle $\mathcal{F} \subset \mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$).

Proposition 8. Let \mathcal{E} and \mathcal{F} be vector bundles on \mathcal{C} . Suppose that they are semistable and $\mu(\mathcal{E}) > \mu(\mathcal{F})$. Then any homomorphism $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ is zero.

Suppose that they are stable and $\mu(\mathcal{E}) = \mu(\mathcal{F})$. Then any non-zero homomorphism $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ is an isomorphism.

Proof. Let $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ be a non-zero homomorphism of vector bundles on \mathcal{C} . We have an exact sequence

$$0 \rightarrow \text{Ker } \varphi \rightarrow \mathcal{E} \rightarrow \text{Im } \varphi \rightarrow 0.$$

Since \mathcal{F} is vector bundle, hence torsion-free, $\text{Im } \varphi$ is also torsion-free, thus a vector bundle.

If \mathcal{E} and \mathcal{F} are semistable with $\mu(\mathcal{E}) > \mu(\mathcal{F})$, clearly $\text{Ker } \varphi \neq 0$, then by [Proposition 6](#), we have

$$\mu(\mathcal{E}) \leq \mu(\text{Im } \varphi) \leq \mu(\mathcal{F}).$$

This is a contradiction, thus $\varphi = 0$.

Suppose that \mathcal{E} and \mathcal{F} are stable with $\mu(\mathcal{E}) = \mu(\mathcal{F})$. If $\text{Ker } \varphi \neq 0$, then by [Proposition 6](#), we

have

$$\mu(\mathcal{E}) < \mu(\operatorname{Im} \varphi) \leq \mu(\mathcal{F}).$$

This is a contradiction, thus φ is injective. Since \mathcal{F} is stable and $\operatorname{Im} \varphi \subset \mathcal{F}$ has the same slope as \mathcal{F} , we have $\operatorname{Im} \varphi = \mathcal{F}$. \square

Corollary 9. A stable vector bundle is simple as a coherent sheaf, i.e., $\operatorname{End}(\mathcal{E}) \cong \mathbb{k}$.

Proof. Let $\varphi \in \operatorname{End}(\mathcal{E})$ be a non-zero endomorphism. Then there exists $P \in \mathcal{C}(\mathbb{k})$ such that $\varphi_P : \mathcal{E}_P \rightarrow \mathcal{E}_P$ is non-zero. Let $a \in \mathbb{k}$ be an eigenvalue of φ_P and consider the endomorphism $\varphi - a \cdot \operatorname{id}_{\mathcal{E}}$. Then $(\varphi - a \cdot \operatorname{id}_{\mathcal{E}})_P : \mathcal{E}_P \rightarrow \mathcal{E}_P$ is not an isomorphism, so is $\varphi - a \cdot \operatorname{id}_{\mathcal{E}}$. By Proposition 8, $\varphi - a \cdot \operatorname{id}_{\mathcal{E}} = 0$, thus $\varphi = a \cdot \operatorname{id}_{\mathcal{E}}$. \square

Lemma 10. Let \mathcal{E} be a semistable vector bundle on X .

- (a) if $\mu(\mathcal{E}) > 2g - 2$, then $H^1(X, \mathcal{E}) = 0$;
- (b) if $\mu(\mathcal{E}) > 2g - 1$, then \mathcal{E} is globally generated.

Proof. Yang: To be continued... \square

Let $S_{r,d}$ be set of isomorphism classes of semistable vector bundles on X of rank r and degree d .

Proposition 11. The family $S_{r,d}$ is bounded.

Proof. Yang: To be continued... \square

Definition 12 (Jordan-Hölder filtration). Let \mathcal{E} be a semistable vector bundle on \mathcal{C} . A *Jordan-Hölder filtration* of \mathcal{E} is a filtration

$$F : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ are stable with $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E})$ for all $1 \leq i \leq n$.

Proposition 13. Any semistable vector bundle on \mathcal{C} admits a Jordan-Hölder filtration. Moreover, the associated graded object

$$\operatorname{gr}(\mathcal{E}) := \bigoplus_{i=1}^n \mathcal{E}_i/\mathcal{E}_{i-1}$$

is independent of the choice of Jordan-Hölder filtration up to isomorphism.

Proof. Yang: To be continued... \square

Definition 14 (S-equivalence). Two semistable vector bundles \mathcal{E} and \mathcal{F} of the same rank and degree on \mathcal{C} are called *S-equivalent* if their associated graded objects $\operatorname{gr}(\mathcal{E})$ and $\operatorname{gr}(\mathcal{F})$ (from their Jordan-Hölder filtrations) are isomorphic.

Definition 15. We define a functor

$$\mathcal{M}_{r,d}^{ss} : (\mathbf{Sch}_{\mathbb{k}})^{\operatorname{op}} \rightarrow \mathbf{Set}$$

$$T \mapsto \{\mathcal{E} \mid \mathcal{E} \text{ is a family of semistable vector bundles on } X \text{ of rank } r, \forall t \in T, \deg(\mathcal{E}_t) = d\} / \sim,$$

where $\mathcal{E} \sim \mathcal{E}'$ if for any $t \in T$, the vector bundles \mathcal{E}_t and \mathcal{E}'_t are S-equivalent. Yang: To be continued...

DRAFT
