Relative objects

1 Relative schemes

Definition 1. Let X be a scheme. An $\mathcal{O}_X$-algebra is a sheaf . Yang: To be continued...

Definition 2. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. The relative Spec of $\mathcal{A}$, denoted by $\operatorname{Spec}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Spec} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. Yang: To be continued...

Proposition 3. Let X be a scheme and $\mathcal{E}$ be a locally free sheaf of finite rank on X. Then the relative Spec of the symmetric algebra of $\mathcal{E}$, denoted by $\mathbb{V}(\mathcal{E}) = \operatorname{Spec}_{X\ \mathcal{O}_X} \mathcal{E}$, is called the geometric vector bundle associated to $\mathcal{E}$. The projection morphism $\pi : \mathbb{V}(\mathcal{E}) \to X$ is affine and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Spec}_{\mathcal{O}_X(U)} \mathcal{E}(U)$. Yang: To be continued...

Definition 4. Let X be a scheme and $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_0 = \mathcal{O}_X$ and $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{O}_X$-algebra. The relative Proj of $\mathcal{A}$, denoted by $\operatorname{Proj}_X \mathcal{A}$, is the scheme obtained by gluing the affine schemes $\operatorname{Proj} \mathcal{A}(U)$ for all affine open subsets $U \subset X$. The projection morphism $\pi : \operatorname{Proj}_X \mathcal{A} \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{A}(U)$. Yang: To be continued...

2 Blowing up

Definition 5. Let X be a scheme and $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The blowing up of X along $\mathcal{J}$, denoted by $\operatorname{Bl}_{\mathcal{J}} X$, is defined to be the relative Proj of the Rees algebra of $\mathcal{J}$:

$$\operatorname{Bl}_{\mathcal{J}} X = \operatorname{Proj}_X \bigoplus_{n=0}^{\infty} \mathcal{J}^n.$$

The projection morphism $\pi : \operatorname{Bl}_{\mathcal{J}} X \to X$ is projective and for any open subset $U \subset X$, we have $\pi^{-1}(U) \cong \operatorname{Bl}_{\mathcal{J}(U)} U$. The exceptional divisor of the blowing up is defined to be the closed subscheme $E = \pi^{-1}(V(\mathcal{J}))$ of $\operatorname{Bl}_{\mathcal{J}} X$. Yang: To be continued...

3 Relative ampleness and relative morphisms

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