Sites, algebraic spaces and stacks

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1 Sites

1.1 Grothendieck topology

Definition 1.1. Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of sets of arrows $\{U_i \to U\}_{i\in I}$, called *covering*, for each object U in $\mathbf{C}$ such that:

(a) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering;

(b) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a arrow, then the fiber product $U_i \times_U V \to V$ exists and $\{U_i \times_U V \to V\}$ is a covering of V;

(c) if $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ are coverings, then the collection of composition $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{J})$ where $\mathbf{C}$ is a category and $\mathcal{J}$ is a Grothendieck topology on $\mathbf{C}$.

Note that sheaf is indeed defined on a site.

Definition 1.2. Let $(\mathbf{C}, \mathcal{J})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{J})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ satisfying the following condition: for every object U in $\mathbf{C}$ and every covering $\{U_i \to U\}_{i\in I}$ of U, if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every i, j, the pullback $s_i|_{U_i \times_U U_j}$ and $s_j|_{U_i \times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every i, the pullback $s|_{U_i} = s_i$.

Definition 1.3. Let X be a scheme. The *big étale site* of X, denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over X with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if and only if each U_i is étale over U and the union of their images is the whole U.

Let X be a scheme over S. By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \to \mathbf{Set}$ such that for any S-scheme T, $h_X(T) = \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$. Yang: Easy to check that h_X is a sheaf on the

Date: January 14, 2026, **Author**: Tianle Yang, My Homepage big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

2 Algebraic spaces

Definition 2.1. Let U be a scheme over a base scheme S. An *étale equivalence relation* on U is a morphism $R \to U \times_S U$ between schemes over S such that:

(a) the projections in two factors $R \to U$ are étale and surjective;

(b) for every S-scheme T, $h_R(T) \to h_U(T) \times h_U(T)$ gives an equivalence relation on $h_U(T)$ set-theoretically.

Definition 2.2. An *algebraic space* X over a base scheme S is an S-scheme U together with an étale equivalence relation $R \to U \times_S U$.

Let $X = (U, R)$ be an algebraic space over S. We explain X as a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. For any scheme T over S, $h_R(T)$ is an equivalence relation on $h_U(T)$. The rule sending T to the set of equivalence classes of $h_R(T)$ gives a presheaf on the site $(\mathbf{Sch}/S)_{\text{ét}}$. The sheafification of this presheaf is the sheaf associated to the algebraic space X. Explicitly, we have

$$X(T) := \left\{ f = (f_i) \,\middle|\, \begin{array}{c} \{T_i \to T\} \text{ a covering, } f_i \in h_U(T_i) \text{ such} \\ \text{that } (f_i|_{T_i \times_T T_j}, f_j|_{T_i \times_T T_j}) \in h_R(T_i \times_T T_j) \end{array} \right\} \Big/ \sim,$$

where

$$\alpha \sim \beta \quad \text{ if } \exists \{S_i \to T\} \text{ such that } (\alpha|_{S_i}, \beta|_{S_i}) \in h_R(S_i).$$

Definition 2.3. An *algebraic space* over a base scheme S is a sheaf F on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$ such that

(a) the diagonal morphism $F \to F \times_S F$ is representable;

(b) there exists a scheme U over S and a map $h_U \to F$ which is surjective and étale.

The *morphism between algebraic spaces* F_1, F_2 is defined as a natural transformation of functors F_1, F_2.

Remark 2.4. By Yoneda's Lemma, given a morphism $h_U \to F$ between sheaves is the same as giving an element of $F(U)$. We may abuse the notation.

Definition 2.5. Let $\mathcal{P}$ be a property of morphisms of schemes satisfying the following conditions:

(a) is preserved under any base change;

(b) is étale local on the base. Yang: In [Stacks], this requires that "fppf local".

Let $\alpha : F \to G$ be a representable morphism of sheaves on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. We say that α has property $\mathcal{P}$ if for every $h_T \to G$, the base change $h_T \times_G F \to F$ has property $\mathcal{P}$.

Remark 2.6. The fiber product $F_1 \times_F F_2$ is just defined as $F_1 \times_F F_2(T) := F_1(T) \times_{F(T)} F_2(T)$ for any object $T \in \mathsf{Obj}(\mathbf{Sch}_S)$. We say that a morphism $f : F_1 \to F_2$ of sheaves is *representable* if for every $T \in \mathsf{Obj}(\mathbf{Sch}/S)$ and every $\xi \in F_2(T)$, the sheaf $F_1 \times_{F_2} h_T$ is representable as a functor. Here $h_T \to F_2$ is given by

$$h_T(U) \to F_2(U), \quad f \in \mathrm{Hom}(U,T) \mapsto F_2(f)(\xi) \in F_2(U).$$

In our case, given an arbitrary $h_U \to F \times F$ is equivalent to giving morphisms $h_{U_i} \to F$ for $i = 1, 2$. And the fiber product $F \times_{F \times F} (h_{U_1} \times h_{U_2})$ is just the fiber product $h_{U_1} \times_F h_{U_2}$. Hence the first condition in Definition 2.3 is equivalent to that $h_{U_1} \times_F h_{U_2}$ is representable for any U_1, U_2 over F. This implies that $h_U \to F$ is representable, whence the second condition in Definition 2.3 makes sense.

Definition 2.7. Let X be an algebraic space over a base scheme S. Two two morphisms form field $\mathsf{Spec}\, k_i \to X$ is called equivalent if there is a common extension $K \supset k_1, k_2$ such that we have $\mathsf{Spec}\, K \to \mathsf{Spec}\, k_i \to X$ are the same for $i = 1, 2$. The *underlying point set* of X, denote by $|X|$, is defined as the set of equivalence classes of morphisms $\mathsf{Spec}\, k \to X$ for all field k over the base field $\Bbbk$.

This definition coincides with the underlying set of a scheme. Let $\alpha : X \to Y$ be a morphism of algebraic spaces. It induces a map $|\alpha| : |X| \to |Y|$ by $x \mapsto \alpha \circ x$ (vertical composition).

Proposition 2.8 (ref. [Stacks, Lemma 66.4.6])**.** There is a unique topology on $|X|$ such that

(a) if X is a scheme, then the topology coincides with the usual topology.

(b) every morphism of algebraic spaces $f : X \to Y$ induces a continuous map $|f| : |X| \to |Y|$.

(c) if U is a scheme and $U \to X$ is étale, then the induced map $|U| \to |X|$ is open.

This topology is called the *Zariski topology* on $|X|$.

Definition 2.9. Let X be an algebraic space over a base scheme S. All étale morphisms $U \to X$ with U scheme form a small site $X_{\text{ét}}$. All étale morphisms $U \to X$ with U algebraic space form a small site $X_{\text{sp, ét}}$. The *structure sheaf* $\mathcal{O}_X$ of X is given by $U \mapsto \Gamma(U, \mathcal{O}_U)$ for every étale morphism $U \to X$ from a scheme. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

Example 2.10. Let $U = \mathbb{a}^1_{\mathbb{c}}$ and $R \subset U \times U$ given by $y = x + n, n \in \mathbb{Z}$. Then R is a disjoint union of lines in $U \times U$. Write $R = \coprod_{n \in \mathbb{Z}} R_n$ with $R_n = \{(x, x+n) : x \in \mathbb{c}\}$. Then the projection is given by

$$\pi_1|_{R_n} : R_n \to U, \quad (x, x+n) \mapsto x,$$
$$\pi_2|_{R_n} : R_n \to U, \quad (x, x+n) \mapsto x+n.$$

Easily see that the projection $\pi_i : R \to U$ is étale and surjective for $i = 1, 2$. Let $r_{ij} : R \times U \to U \times U \times U$ be the morphism which maps $((x, y), u)$ to (a_1, a_2, a_3) where $a_i = x$, $a_j = y$ and $a_k = u$ for $k \neq i, j$. Since $\Delta_U \to U \times U$ factors through R, $(\pi_1, \pi_2) = (\pi_2, \pi_1)$ and $r_{12} \times_{(U \times U \times U)} r_{23}$ factors through r_{13}, we have that $h_R(T)$ is an equivalence relation on $h_U(T)$ for all T over S. Then $X := (U, R)$ is an algebraic space.

We do not check the representability here but give an example. Let $U \to X$ be the natural morphism given by $\mathbf{id}_U \in X(U)$. For any scheme T over $\mathbb{c}$, we have

$$(U \times_X U)(T) = \{(f,g) \in h_{U\times U}(T) : \exists\{T_i \to T\} \text{ s.t. } (f_i, g_i) \in h_R(T_i)\} = h_R(T).$$

Hence the fiber product $h_U \times_X h_U$ is represented by R.

We show that $X \ncong \mathbb{c}^\times$ by computing the the global sections. Consider the covering $U \to X$, a section $s \in \mathcal{O}_X(X)$ is given by a section $s \in \Gamma(U, \mathcal{O}_U) = \mathbb{c}[t]$ such that $\pi_1^* s = \pi_2^* s$ in $\Gamma(R, \mathcal{O}_R)$. This means that $s(x+n) = s(x)$ for all $n \in \mathbb{Z}$. Hence s is a constant function. In particular, $\mathcal{O}_X(X) = \mathbb{c} \neq \mathbb{c}[t, t^{-1}]$.

The underlying set $|X|$ is union of the quotient set $\mathbb{c}/\mathbb{Z}$ and a generic point. The Zariski topology on $|X|$ is the trivial topology.

In following, we will use the technique of *local construction* to construct many scheme-like objects on algebraic spaces. For local construction, see [Knu71]. Roughly speaking, for every étale morphism $U \to X$ with U a scheme, we construct a scheme-theoretic object on U which is compatible under base change. Then we glue these objects together to get a global object on X.

Definition 2.11. Let X be an algebraic space over a base scheme S. A *coherent sheaf* on X is a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ such that for every covering $\{U_i \to X\}$ with U_i schemes, the sheaf $\mathcal{F}|_{U_i}$ is coherent for every i. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.
An *ideal sheaf* on X is a coherent sheaf $\mathcal{I} \subset \mathcal{O}_X$. It defines a closed subspace $V(\mathcal{I}) \subset X$ by Yang: to be completed. And every closed subspace $Y \subset X$ is defined by an ideal sheaf $\mathcal{I}_Y$ such that $V(\mathcal{I}_Y) = Y$.

Definition 2.12. Let X be an algebraic space over a base scheme S. A *line bundle* on X is a coherent sheaf $\mathcal{L}$ on X such that for every covering $\{U_i \to X\}$ with U_i schemes, the sheaf $\mathcal{L}|_{U_i}$ is a line bundle on U_i. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

Theorem 2.13 (ref. [Stacks, Theorem 76.36.4])**.** Let $f : X \to Y$ be a proper morphism of algebraic spaces over a base scheme S. Then there exists a factorization

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y,$$

where f_1 has geometrically connected fibers and $(f_1)_* \mathcal{O}_X = \mathcal{O}_Z$ and f_2 is finite.

Definition 2.14. Let X be an algebraic space over a base scheme S and Y a closed subset of $|X|$. The *formal completion* of X along Y, denoted by $\mathfrak{X}$, is
Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined as $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$ where $\mathcal{I}$ is the ideal sheaf of Y in $\mathcal{O}_X$. Yang: to be completed.

Definition 2.15. Let X be an algebraic space and Y a closed subset of X. A *modification* of X along Y is a proper morphism $f : X' \to X$ and a closed subset $Y' \subset X'$ such that $X' \setminus Y' \to X \setminus Y$ is an isomorphism and $f^{-1}(Y) = Y'$.

Theorem 2.16 (ref. [Art70, Theorem 3.1])**.** Let Y' be a closed subset of an algebraic space X' of finite type over $\mathbb{k}$. Let $\mathfrak{X}'$ be the formal completion of X' along Y'. Suppose that there is a formal modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$. Then there is a unique modification

$$f : X' \to X, \quad Y \subset X$$

such that the formal completion of X along Y is isomorphic to $\mathfrak{X}$ and the induced morphism $\mathfrak{X}' \to \mathfrak{X}$ is isomorphic to $\mathfrak{f}$.

Theorem 2.17 (ref. [Art70, Theorem 6.2])**.** Let $\mathfrak{X}'$ be a formal algebraic space and $Y' = V(\mathcal{J}')$ with $\mathcal{J}'$ the defining ideal sheaf of $\mathfrak{X}'$. Let $f : Y' \to Y$ be a proper morphism. Suppose that

(a) for every coherent sheaf $\mathcal{F}$ on $\mathfrak{X}'$, we have

$$R^1 f_* \mathcal{J}'^n \mathcal{F} / \mathcal{J}'^{n+1} \mathcal{F} = 0, \quad \forall n \gg 0;$$

(b) for every n, the homomorphism

$$f_*(\mathcal{O}_{\mathfrak{X}'}/\mathcal{J}'^n) \otimes_{f_* \mathcal{O}_{Y'}} \mathcal{O}_Y \to \mathcal{O}_Y$$

is surjective.

Then there exists a modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ and a defining ideal sheaf $\mathcal{J}$ of $\mathfrak{X}$ such that $V(\mathcal{J}) = Y$ and $\mathfrak{f}$ induces f on Y.

Theorem 2.18 (ref. [Art70, Theorem 6.1])**.** Let Y' be a closed algebraic subspace of an algebraic space X' and $f_0 : Y' \to Y$ a finite morphism. Then there exists a modification $f : X' \to X$ whose restriction to Y' is f_0. It is the amalgamated sum $X = X' \amalg_{Y'} Y$ in the category of algebraic spaces **AlgSp**.

Example 2.19. Let $X = \mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y]$ and $Y = V(y)$ be the x-axis. Let $f_0 : Y' = \mathbb{A}^1 \to Y, x \mapsto x^2$. Then there exists a modification $f : X' \to X$ such that the restriction $f|_{Y'} : Y' \to Y$ is f_0. Yang: To be completed.

3 Stacks in category theory

4 Algebraic stacks

References

[Art70] Michael Artin. "Algebraization of formal moduli: II. Existence of modifications". In: *Annals of Mathematics* 91.1 (1970), pp. 88–135 (cit. on pp. 4, 5).

[Knu71] Donald Knutson. *Algebraic Spaces*. Vol. 203. Lecture Notes in Mathematics. Berlin, Heidelberg, New York: Springer-Verlag, 1971. ISBN: 978-3-540-05496-2 (cit. on p. 4).

[Stacks] The Stacks Project Authors. *Stacks Project*. URL: https://stacks.math.columbia.edu/ (cit. on pp. 2–4).