Picard Groups of Abelian Varieties

Let $\mathbf{k}$ be a field and $\Bbbk$ its algebraic closure. Let A be an abelian variety over $\mathbf{k}$.

1 Pullback along group operations

Theorem 1 (Theorem of the cube)**.** Let X, Y, Z be proper varieties over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on $X \times Y \times Z$. Suppose that there exist $x \in X(\mathbf{k}), y \in Y(\mathbf{k}), z \in Z(\mathbf{k})$ such that the restriction $\mathcal{L}|_{\{x\}\times Y\times Z}$, $\mathcal{L}|_{X\times\{y\}\times Z}$ and $\mathcal{L}|_{X\times Y\times\{z\}}$ are trivial. Then $\mathcal{L}$ is trivial.

Proof. Yang: To be completed. □

Remark 2. If we assume the existence of the Picard scheme, then the Theorem 1 can be deduced from the Rigidity Lemma. Consider the morphism

$$\varphi : X \times Y \to \mathsf{Pic}(Z), \quad (x, y) \mapsto \mathcal{L}|_{\{x\}\times\{y\}\times Z}.$$

Since $\varphi(x, y) = \mathcal{O}_Z$, φ factors through $\mathsf{Pic}^0(Z)$. Then the assumption implies that φ contracts $\{x\}\times Y$, $X \times \{y\}$ and hence it maps $X \times Y$ to a point. Thus $\varphi(x', y') = \mathcal{O}_Z$ for every $(x', y') \in X \times Y$. Then by Grauert's theorem, we have $\mathcal{L} \cong p^*p_*\mathcal{L}$ where $p : X \times Y \times Z \to X \times Y$ is the projection. Note that $p_*\mathcal{L} \cong \mathcal{L}|_{X\times Y\times\{z\}} \cong \mathcal{O}_{X\times Y}$. Hence $\mathcal{L}$ is trivial.

Lemma 3. Let A be an abelian variety over $\mathbf{k}$, $f, g, h : X \to A$ morphisms from a variety X to A and $\mathcal{L}$ a line bundle on A. Then we have

$$(f + g + h)^*\mathcal{L} \cong (f + g)^*\mathcal{L} \otimes (f + h)^*\mathcal{L} \otimes (g + h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof. First consider $X = A\times A\times A$, $p : X \to A, (x_1, x_2, x_3) \mapsto x_1+x_2+x_3$, $p_{ij} : X \to A, (x_1, x_2, x_3) \mapsto x_i + x_j$ for $1 \le i < j \le 3$ and $p_i : X \to A, (x_1, x_2, x_3) \mapsto x_i$ for $1 \le i \le 3$. Then the conclusion follows from the theorem of the cube by taking $\mathcal{L}' = p^*\mathcal{L}^{-1}\otimes p_{12}^*\mathcal{L}\otimes p_{13}^*\mathcal{L}\otimes p_{23}^*\mathcal{L}\otimes p_1^*\mathcal{L}^{-1}\otimes p_2^*\mathcal{L}^{-1}\otimes p_3^*\mathcal{L}^{-1}$ and considering the restriction to $\{0\} \times A \times A$, $A \times \{0\} \times A$ and $A \times A \times \{0\}$.

In general, consider the morphism $\varphi = (f, g, h) : X \to A \times A \times A$ and pull back the above isomorphism along φ. □

Proposition 4. Let A be an abelian variety over $\mathbf{k}$, $n \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle on A. Then we have

$$[n]_A^*\mathcal{L} \cong \mathcal{L}^{\otimes\frac{1}{2}(n^2+n)} \otimes [-1]_A^*\mathcal{L}^{\otimes\frac{1}{2}(n^2-n)}.$$

Proof. For $n = 0, 1$, the conclusion is trivial. For $n \ge 2$, we can use the previous lemma on $[n-2]_A, [1]_A, [1]_A$ and induct on n. Hence we have

$$[n]_A^*\mathcal{L} \cong [n-1]_A^*\mathcal{L} \otimes [n-1]_A^*\mathcal{L} \otimes [2]_A^*\mathcal{L} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [1]_A^*\mathcal{L}^{-1} \otimes [n-2]_A^*\mathcal{L}^{-1}.$$

Then the conclusion follows from induction. Yang: To be completed. □

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Definition 5. Let A be an abelian variety over $\mathbf{k}$ and $\mathcal{L}$ a line bundle on A. We say that $\mathcal{L}$ is *symmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}$ and *antisymmetric* if $[-1]_A^*\mathcal{L} \cong \mathcal{L}^{-1}$.

Theorem 6 (Theorem of the square)**.** Let A be an abelian variety over $\mathbf{k}$, $x, y \in A(\mathbf{k})$ two points and $\mathcal{L}$ a line bundle on A. Then
$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Proof. Yang: To be completed. □

Example 7. Let E be an elliptic curve over $\Bbbk$ with origin 0. For $x \in E(\Bbbk)$, let P_x be the corresponding prime divisor on E. Denote $\mathrm{Pic}^0(E)$ the subgroup of $\mathrm{Pic}(E)$ consisting of line bundles of degree zero. Given $x, y \in E(\Bbbk)$, by the theorem of the square, we have
$$t_{-x-y}^*\mathcal{O}_E(P_0) \otimes \mathcal{O}_E(P_0) \cong t_{-x}^*\mathcal{O}_E(P_0) \otimes t_{-y}^*\mathcal{O}_E(P_0).$$
Note that $t_{-x}^*\mathcal{O}_E(P_0) \cong \mathcal{O}_E(P_x)$. Hence
$$\mathcal{O}_E(P_{x+y} - P_0) \cong \mathcal{O}_E(P_x - P_0 + P_y - P_0).$$
This shows that the map $E(\Bbbk) \to \mathrm{Pic}^0(E), x \mapsto \mathcal{O}_E(P_x - P_0)$ is a group homomorphism. It is injective since if $\mathcal{O}_E(P_x - P_0) \cong \mathcal{O}_E$, then $P_x \sim P_0$ and hence $x = 0$. It is also surjective since for any $\mathcal{L} \in \mathrm{Pic}^0(E)$, we can write $\mathcal{L} \cong \mathcal{O}_E(\sum a_i P_{x_i})$ with $\sum a_i = 0$ and then
$$\mathcal{L} \cong \mathcal{O}_E\left(\sum a_i(P_{x_i} - P_0)\right) \cong \mathcal{O}_E\left(P_{\sum a_i x_i} - P_0\right).$$
Hence we have an isomorphism of groups $E(\Bbbk) \cong \mathrm{Pic}^0(E)$.

Remark 8. We can define a map
$$\Phi_{\mathcal{L}} : A(\mathbf{k}) \to \mathrm{Pic}(A), \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$
Then theorem of the square implies that $\Phi_{\mathcal{L}}$ is a homomorphism of groups. When we vary $\mathcal{L}$, the map
$$\Phi_{\square} : \mathrm{Pic}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A(\mathbf{k}), \mathrm{Pic}(A)), \quad \mathcal{L} \mapsto \Phi_{\mathcal{L}}$$
is also a group homomorphism. For any $x \in A(\mathbf{k})$, we have
$$\Phi_{t_x^*\mathcal{L}} = \Phi_{\mathcal{L}}$$
by Theorem 6. In the other words,
$$\Phi_{\mathcal{L}}(x) \in \operatorname{Ker}\Phi_{\square}, \quad \forall \mathcal{L} \in \mathrm{Pic}(A), x \in A(\mathbf{k}).$$

If we assume the scheme structure on $\mathrm{Pic}(A)$, then $\Phi_{\mathcal{L}}$ is a morphism of scheme and factors through $\mathrm{Pic}^0(A)$. Let $K(\mathcal{L}) := \operatorname{Ker}\Phi_{\mathcal{L}}$, then $K(\mathcal{L})$ is a subgroup scheme of A. We give another description of $K(\mathcal{L})$. From this point, when $K(\mathcal{L})$ is finite, we can recover the dual abelian variety $A^\vee = \mathrm{Pic}^0_{A/\mathbf{k}}$ as the quotient $A/K(\mathcal{L})$.

Example 9. Let E be an elliptic curve over $\Bbbk$ with origin 0. We have $\mathrm{Pic}^0(E) \cong E(\Bbbk)$ by sending $x \in E(\Bbbk)$ to $\mathcal{O}_E(P_x - P_0)$ where P_x is the point on E corresponding to x.

Then

$$\Phi_{n\mathcal{O}(P_x)}(y) = t_y^*\mathcal{O}_E(nP_x) \otimes \mathcal{O}_E(-nP_x) \cong \mathcal{O}_E(nP_{x-y} - nP_0 + nP_0 - nP_x) \leftrightarrow -ny$$

Hence

$$\Phi_{n\mathcal{O}(P_x)} : E(\Bbbk) \to E(\Bbbk), \quad y \mapsto -ny.$$

Hence $K(\mathcal{L}) = \{x \in E(\Bbbk) : n \cdot x \sim n \cdot 0\} = E[n](\Bbbk)$ is the subgroup of n-torsion points of E.

2 Projectivity

In this subsection, we work over the algebraically closed field $\Bbbk$.

Proposition 10. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. Then $|2D|$ is base point free.

Proof. Yang: To be completed. □

Theorem 11. Let A be an abelian variety over $\Bbbk$ and D an effective divisor on A. TFAE:

(a) the stabilizer $\mathsf{Stab}(D)$ of D is finite;

(b) the morphism $\phi_{|2D|}$ induced by the complete linear system $|2D|$ is finite;

(c) D is ample;

(d) $K(\mathcal{O}_A(D))$ is finite.

Proof. Yang: To be completed. □

Theorem 12. Let A be an abelian variety over $\Bbbk$. Then A is projective.

Proof. Yang: To be completed. □

Corollary 13. Let A be an abelian variety over $\Bbbk$ and D a divisor on A. Then D is pseudo-effective if and only if it is nef, i.e. $\mathsf{Psef}^1(A) = \mathsf{Nef}^1(A)$.

Proof. Yang: To be completed. □

3 Dual abelian varieties

In this subsection, we work over the algebraically closed field $\Bbbk$.

Definition 14. Let A be an abelian variety over $\Bbbk$. We define the *dual abelian variety* of A to be $A/K(\mathcal{L})$ for some ample line bundle $\mathcal{L}$ on A. We denote it by $A^\vee$.

Yang: We have a natural map $A^\vee(\Bbbk) \to \mathsf{Pic}^0(A)$ by sending $x + K(\mathcal{L}) \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$. We will show that this map is an isomorphism.

Lemma 15. There exists a unique line bundle $\mathcal{P}$ on $A \times A^\vee$ such that for every $y = \mathcal{L} \in A^\vee = \mathsf{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{y\}} \cong \mathcal{L}$.

Proof. Yang: To be completed. ☐

Lemma 16. Let A be an abelian variety over $\Bbbk$ and B a group variety over $\Bbbk$. Then there is a natural bijection between the morphisms $f : B \to A^\vee$ and the line bundles $\mathcal{L}$ on $A \times B$ such that for every $b \in B(\Bbbk)$, we have $\mathcal{L}|_{A\times\{b\}} \in \mathrm{Pic}^0(A)$. The bijection is given by $f \mapsto (1_A \times f)^*\mathcal{P}$ where $\mathcal{P}$ is the Poincaré line bundle on $A \times A^\vee$. Yang: To be completed.

Proof. Yang: To be completed. ☐

Theorem 17. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ and the Poincaré line bundle $\mathcal{P}$ on $A \times A^\vee$ do not depend on the choice of the ample line bundle $\mathcal{L}$. Moreover, there is a natural bijection $A^\vee(\mathbf{k}) \to \mathrm{Pic}^0(A)$ of groups. Under this bijection, for every $x = \mathcal{L} \in A^\vee(\mathbf{k}) = \mathrm{Pic}^0(A)$, we have $\mathcal{P}|_{A\times\{x\}} \cong \mathcal{L}$.

Proof. Yang: To be completed. ☐

Proposition 18. Let A be an abelian variety over $\mathbf{k}$. Then the dual abelian variety $A^\vee$ is also an abelian variety and the natural morphism $A \to A^{\vee\vee}$ is an isomorphism.

Proof. Yang: To be completed. ☐

4 The Néron-Severi group

Theorem 19. Let A be an abelian variety over $\Bbbk$. The we have an inclusion $\mathrm{NS}(A) \hookrightarrow \mathrm{Hom}_{\mathbf{Av}}(A, A^\vee)$ of groups given by

$$\mathcal{L} \mapsto \left(\Phi_{\mathcal{L}} : A \to A^\vee, \quad x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}\right).$$

Yang: To be completed.

Example 20. Let E be an elliptic curve over $\Bbbk$ without complex multiplication and $A = E^n$ for some $n \geq 1$. Set $D_i = E^{i-1} \times \{0\} \times E^{n-i}$ for $1 \leq i \leq n$ and $D_{ij} = \Delta_{ij} - D_i - D_j$ for $1 \leq i < j \leq n$ where Δ_{ij} is the pullback of the diagonal divisor $\Delta_E \subseteq E \times E$ along the projection $A \to E \times E$ to the i-th and j-th factors. Then $\mathrm{NS}(A)$ is generated by the classes of D_i's and D_{ij}'s.Yang: why?

The homomorphism $\Phi : \mathrm{NS}(A) \to \mathrm{Hom}_{\mathbf{Av}}(A, A^\vee)$ can be described as follows. Note that $A^\vee \cong (E^\vee)^n \cong E^n = A$. For D_i, $\Phi_{D_i} : A \to A^\vee$ is given by

$$\Phi_{D_i}(x_1, \dots, x_n) = t^*_{(x_1,\dots,x_n)}\mathcal{O}_A(D_i) \otimes \mathcal{O}_A(D_i)^{-1} \cong (0, \dots, 0, x_i, 0, \dots, 0).$$

For D_{ij}, $\Phi_{D_{ij}} : A \to A^\vee$ is given by

$$\Phi_{D_{ij}}(x_1, \dots, x_n) = t^*_{(x_1,\dots,x_n)}\mathcal{O}_A(D_{ij}) \otimes \mathcal{O}_A(D_{ij})^{-1} \cong (0, \dots, 0, x_j, 0, \dots, 0, x_i, 0, \dots, 0).$$

Hence under the identification $\mathrm{Hom}_{\mathbf{Av}}(A, A^\vee) \cong M_n(\mathbb{Z})$, the map $\Phi : \mathrm{NS}(A) \to \mathrm{Hom}_{\mathbf{Grp}}(A, A^\vee)$ is given by

$$D_i \mapsto E_{ii}, \quad D_{ij} \mapsto E_{ij} + E_{ji}$$

where E_{ij} is the matrix with 1 at the (i,j)-th entry and 0 elsewhere.