F-singularities

Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$. Let X be a projective variety over $\mathbf{k}$. Let F denote the relative Frobenius morphism on X.

Definition 1. We say that X is *F-finite* if $F : X \to X^{(p)}$ is finite.

Definition 2. We say that X is *globally F-split* if $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ splits as $\mathcal{O}_X$-modules for some $e \geq 0$. This is equivalent to for every $e \in \mathbb{Z}_{>0}$, $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ splits as $\mathcal{O}_X$-modules.

Definition 3. Fix $\phi : F^e_* L \to \mathcal{O}_X$ a splitting of $\mathcal{O}_X \to F^e_* \mathcal{O}_X$. Define $\phi^n : F^{ne}_* L^{1+p^e+\cdots+p^{(n-1)e}} \to \mathcal{O}_X$ by induction:

$$\phi^n := \phi \circ F^e_*(\phi^{n-1}).$$

Theorem 4. Above ϕ^n will be stable. That is, $\operatorname{Im} \phi^n = \operatorname{Im} \phi^{n+1}$ for all $n \gg 0$.

Definition 5. Let $\sigma(X, \phi) := \operatorname{Im} \phi^n$. We say that (X, ϕ) is *F-pure* if $\sigma(X, \phi) = \mathcal{O}_X$.

Proposition 6. There is a bijection between

$$\{\text{effective } \mathbb{Q}\text{-divisor } \Delta \text{ such that } (p^e - 1)(K_X + \Delta) \text{ is Cartier}\}/\sim$$

and

$$\{\text{line bundles } \mathcal{L} \text{ and } \phi : F^e_* \mathcal{L} \to \mathcal{O}_X\}.$$

Proof. We have

$$F^e_X \mathcal{O}_X((1-p^e)K_X) \to \mathcal{O}_X$$

given by $F^e \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X)$ and reflexivity of $\mathcal{O}_X(K_X)$. Since Δ is effective, we have

$$F^e(\mathcal{O}_X((1-p^e)(K_X+\Delta))) \to F^e \mathcal{O}_X((1-p^e)(K_X)) \to \mathcal{O}_X.$$

The another direction is by Grothendieck's duality

$$\mathcal{H}om_{\mathcal{O}_X}(F^e \mathcal{L}, \mathcal{O}_X) \cong F^e_*(\mathcal{L}^{-1} \otimes \mathcal{O}_X((1-p^e)K_X)).$$

□

Definition 7. Let $\phi_{e,\Delta} : F^e_*(\mathcal{O}_X((1-p^e)(K_X+\Delta))) \to \mathcal{O}_X$ be the morphism corresponding to the effective $\mathbb{Q}$-divisor Δ.
We say that (X, Δ) is *F-pure* if $(X, \phi_{e,\Delta})$ is F-pure.
We say that (X, Δ) is *globally F-split* if for every Weil divisor $D \geq 0$, $\mathcal{O}_X \to F^e_*(\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + D))$ admits a splitting for some $e \geq 0$.

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We say that (X,Δ) is *strongly F-split* if for every Weil divisor $D \geq 0$, $\mathcal{O}_X \to F^e_*(\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + D))$ admits a local splitting for some $e \geq 0$.

Definition 8.

Definition 9. $S^0(X, \sigma(X,\Delta) \otimes \mathcal{M})$

Proposition 10. Let X be a globally F-split projective variety. Then we have

(a) suppose that $H^i(X,\mathcal{L}^n) = 0$ for all $i > 0$ and all $n \gg 0$, then $H^i(X,\mathcal{L}) = 0$ for all $i > 0$;

(b) for every ample divisor A on X, we have $H^i(X,\mathcal{O}_X(A)) = 0$ for all $i > 0$;

(c) suppose that X is Cohen-Macaulay and A-ample, then $H^i(X,\mathcal{O}_X(-A)) = 0$ for all $i < \dim X$;

(d) suppose that X is normal and A-ample, then $H^i(X,\omega_X(A)) = 0$ for all $i > 0$.