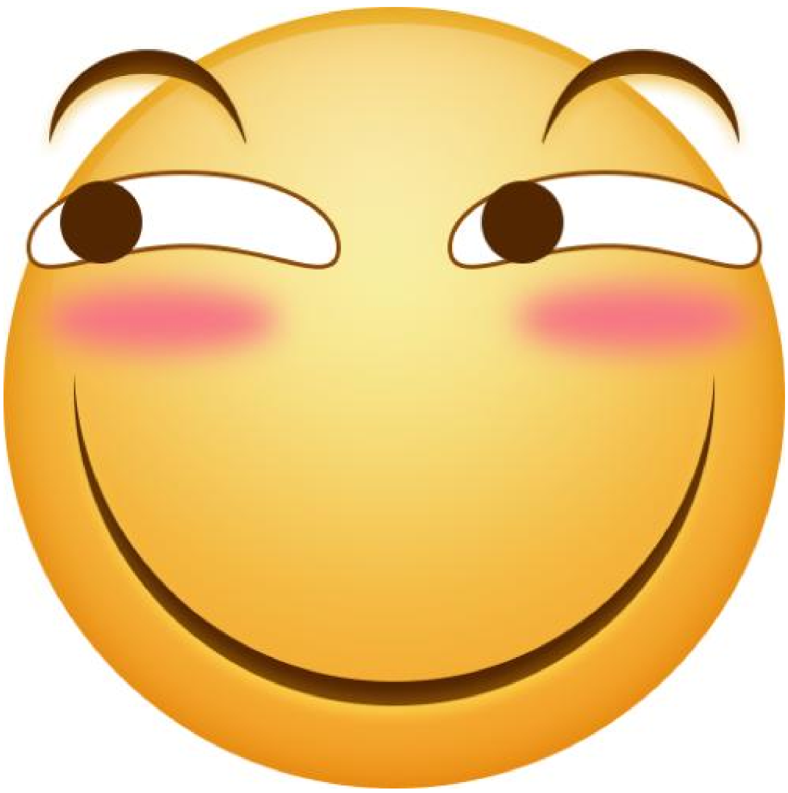
Mixed Characteristic MMP



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Contents

1 Basepoint Free Theorem on Positive Characteristic

This section refers to [Kee99]. For site and algebraic space, we refer to [Knu71], [Art70], [Stacks] and [FGA05]. Throughout this section, all schemes (or algebraic space) are of finite type over a base scheme S with S noetherian.

1.1 Preliminaries

Theorem 1.1 (Serre vanishing in relative setting, ref. [Laz04, Theorem 1.7.6])**.** Let $f : X \to S$ be a proper morphism of schemes, $\mathcal{L}$ a line bundle and $\mathcal{F}$ a coherent sheaf on X. Suppose that $\mathcal{L}$ is relatively ample. Then there exists $n_0 \in \mathbb{m}$ such that for all $n \geq n_0$, the higher direct image sheaves $R^i f_* \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ are zero for all $i > 0$.

Theorem 1.2 (ref. [Laz04, Proposition 1.4.37])**.** Let X be a projective scheme over a field $\mathbb{k}$. Then there exists a scheme T of finite type over $\mathbb{k}$ and a line bundle $\mathcal{L}$ on $X \times T$ such that every numerically trivial line bundle on X arises as the restriction $\mathcal{L}|_{X \times \{t\}}$ for some $t \in T$.

Theorem 1.3 (Theorem on Formal Functions, ref. [Har77, Chapter III, Theorem 11.1])**.** Let $f : X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{F}$ be a coherent sheaf on X, and let $y \in Y$. Then the natural map

$$(R^i f_* \mathcal{F})^{\wedge}_y \to \varprojlim H^i(X_n, \mathcal{F}_n)$$

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is an isomorphism for all $i \geq 0$, where $X_n = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}_y^n$ and $\mathcal{F}_n = \mathcal{F}|_{X_n}$.

Definition 1.4. Let X be a proper variety and $\mathcal{L}$ a nef line bundle on X. A closed subvariety $Z \subseteq X$ is called the *exceptional* for $\mathcal{L}$ if $\mathcal{L}^{\dim Z} \cdot Z = 0$. The *exceptional locus* of $\mathcal{L}$, denoted by $\operatorname{Exc} \mathcal{L}$, is defined as the closure of the union of all exceptional subvarieties of $\mathcal{L}$.

If $\mathcal{L}$ is semiample, then $\operatorname{Exc} \mathcal{L} = \operatorname{Exc} \varphi$ for the fibration $\varphi : X \to Y$ induced by $\mathcal{L}$.

Definition 1.5. Let X be a proper scheme and $\mathcal{L}$ a nef line bundle on X. We say that $\mathcal{L}$ is *endowed with a map (EWM)* if there is a proper morphism $\varphi : X \to Y$ to a proper algebraic space such that $\dim Z > \dim f(Z)$ if and only if Z is an exceptional subvariety of $\mathcal{L}$. If such a morphism is a fibration, then it is unique, called the *fibration associated to* $\mathcal{L}$.

Proposition 1.6. Let X be a proper variety and $\mathcal{L}$ a nef line bundle on X endowed with a map. Let $\varphi : X \to Y$ be the associated fibration. Then TFAE:

(a) $\mathcal{L}$ is semiample;

(b) $\mathcal{L}^{\otimes m}$ is pulled back from an ample line bundle on Y for some $m \in \mathbb{Z}_{>0}$;

(c) $\mathcal{L}^{\otimes m}$ is pulled back from a line bundle on Y for some $m \in \mathbb{Z}_{>0}$;

Proof. (a) $\Leftrightarrow$ (b) $\implies$ (c) is clear. Replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ for some $m \in \mathbb{Z}_{>0}$, suppose that $\mathcal{L} = \varphi^* \mathcal{L}_Y$ for some line bundle $\mathcal{L}_Y$ on Y. We show that $\mathcal{L}_Y$ is ample. Indeed, for all closed subvarieties $Z \subset Y$, we can find $Z' \subset X$ such that $Z' \twoheadrightarrow Z$ and $\dim Z' = \dim Z$. Then

$$\mathcal{L}_Y^{\dim Z} \cdot Z = d \mathcal{L}^{\dim Z'} \cdot Z' > 0$$

where $d = \deg(Z' \to Z)$. Hence $\mathcal{L}_Y$ is ample. □

Definition 1.7. A morphism $f : X \to Y$ of schemes is called a *universal homeomorphism* if for every Y-scheme Y', the base change $X \times_Y Y' \to Y'$ is a homeomorphism between the underlying topological spaces.

Example 1.8. Let X be a scheme of finite type over $\mathbb{k}$. Then the natural morphism $X_{\mathrm{red}} \to X$ is a universal homeomorphism.

Let X be a scheme over S of characteristic p. Then the absolute and relative Frobenius morphisms are universal homeomorphisms. Yang: To be completed.

The morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ is not a universal homeomorphism.

Lemma 1.9. Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes with g finite. Let $\mathcal{F}$ be a coherent sheaf on X. Then the we have

$$R^i (g \circ f)_* \mathcal{F} = g_* (R^i f_* \mathcal{F}).$$

Proof. Yang: This is a simple application of the Grothendieck spectral sequence. However, I do not know anything about it. □

1.2 Algebraic space

Definition 1.10. Let $\mathbf{C}$ be a category. A *Grothendieck topology* on $\mathbf{C}$ is a collection of sets of arrows $\{U_i \to U\}_{i\in I}$, called *covering*, for each object U in $\mathbf{C}$ such that:

(a) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering;

(b) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a arrow, then the fiber product $U_i \times_U V \to V$ exists and $\{U_i \times_U V \to V\}$ is a covering of V;

(c) if $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ are coverings, then the collection of composition $\{U_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$ is a covering.

A *site* is a pair $(\mathbf{C}, \mathcal{J})$ where $\mathbf{C}$ is a category and $\mathcal{J}$ is a Grothendieck topology on $\mathbf{C}$.

Note that sheaf is indeed defined on a site.

Definition 1.11. Let $(\mathbf{C}, \mathcal{J})$ be a site. A *sheaf* on $(\mathbf{C}, \mathcal{J})$ is a functor $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{Set}$ satisfying the following condition: for every object U in $\mathbf{C}$ and every covering $\{U_i \to U\}_{i\in I}$ of U, if we have a collection of elements $s_i \in \mathcal{F}(U_i)$ such that for every i, j, the pullback $s_i|_{U_i\times_U U_j}$ and $s_j|_{U_i\times_U U_j}$ are equal, then there exists a unique element $s \in \mathcal{F}(U)$ such that for every i, the pullback $s|_{U_i} = s_i$.

Definition 1.12. Let X be a scheme. The *big étale site* of X, denoted by $(\mathbf{Sch}/X)_{\text{ét}}$, is the category of schemes over X with the Grothendieck topology generated by étale morphisms, that is, a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if and only if each U_i is étale over U and the union of their images is the whole U.

Let X be a scheme over S. By Yoneda's Lemma, it is equivalent to give a functor $h_X : \mathbf{Sch}_S^{op} \to \mathbf{Set}$ such that for any S-scheme T, $h_X(T) = \mathrm{Hom}_{\mathbf{Sch}_S}(T, X)$. Yang: Easy to check that h_X is a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

Definition 1.13. Let U be a scheme over a base scheme S. An *étale equivalence relation* on U is a morphism $R \to U \times_S U$ between schemes over S such that:

(a) the projections in two factors $R \to U$ are étale and surjective;

(b) for every S-scheme T, $h_R(T) \to h_U(T) \times h_U(T)$ gives an equivalence relation on $h_U(T)$ set-theoretically.

Definition 1.14. An *algebraic space* X over a base scheme S is an S-scheme U together with an étale equivalence relation $R \to U \times_S U$.

Let $X = (U, R)$ be an algebraic space over S. We explain X as a sheaf on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$.

For any scheme T over S, $h_R(T)$ is an equivalence relation on $h_U(T)$. The rule sending T to the set of equivalence classes of $h_R(T)$ gives a presheaf on the site $(\mathbf{Sch}/S)_{\text{ét}}$. The sheafification of this presheaf is the sheaf associated to the algebraic space X. Explicitly, we have

$$X(T) := \left\{ f = (f_i) \,\middle|\, \begin{array}{c} \{T_i \to T\} \text{ a covering, } f_i \in h_U(T_i) \text{ such} \\ \text{that } (f_i|_{T_i \times_T T_j}, f_j|_{T_i \times_T T_j}) \in h_R(T_i \times_T T_j) \end{array} \right\} / \sim,$$

where

$$\alpha \sim \beta \quad \text{if } \exists \{S_i \to T\} \text{ such that } (\alpha|_{S_i}, \beta|_{S_i}) \in h_R(S_i).$$

Definition 1.15. An *algebraic space* over a base scheme S is a sheaf F on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$ such that

(a) the diagonal morphism $F \to F \times_S F$ is representable;

(b) there exists a scheme U over S and a map $h_U \to F$ which is surjective and étale.

The *morphism between algebraic spaces* F_1, F_2 is defined as a natural transformation of functors F_1, F_2.

Remark 1.16. By Yoneda's Lemma, given a morphism $h_U \to F$ between sheaves is the same as giving an element of $F(U)$. We may abuse the notation.

Definition 1.17. Let $\mathcal{p}$ be a property of morphisms of schemes satisfying the following conditions:

(a) is preserved under any base change;

(b) is étale local on the base.Yang: In [Stacks], this requires that "fppf local".

Let $\alpha : F \to G$ be a representable morphism of sheaves on the big étale site $(\mathbf{Sch}/S)_{\text{ét}}$. We say that α has property $\mathcal{p}$ if for every $h_T \to G$, the base change $h_T \times_G F \to F$ has property $\mathcal{p}$.

Remark 1.18. The fiber product $F_1 \times_F F_2$ is just defined as $F_1 \times_F F_2(T) := F_1(T) \times_{F(T)} F_2(T)$ for any object $T \in \mathrm{Obj}(\mathbf{Sch}_S)$. We say that a morphism $f : F_1 \to F_2$ of sheaves is *representable* if for every $T \in \mathrm{Obj}(\mathbf{Sch}/S)$ and every $\xi \in F_2(T)$, the sheaf $F_1 \times_{F_2} h_T$ is representable as a functor. Here $h_T \to F_2$ is given by

$$h_T(U) \to F_2(U), \quad f \in \mathrm{Hom}(U, T) \mapsto F_2(f)(\xi) \in F_2(U).$$

In our case, given an arbitrary $h_U \to F \times F$ is equivalent to giving morphisms $h_{U_i} \to F$ for $i = 1, 2$. And the fiber product $F \times_{F \times F} (h_{U_1} \times h_{U_2})$ is just the fiber product $h_{U_1} \times_F h_{U_2}$. Hence the first condition in Definition 1.15 is equivalent to that $h_{U_1} \times_F h_{U_2}$ is representable for any U_1, U_2 over F. This implies that $h_U \to F$ is representable, whence the second condition in Definition 1.15 makes sense.

Definition 1.19. Let X be an algebraic space over a base scheme S. Two two morphisms form field $\operatorname{Spec} k_i \to X$ is called equivalent if there is a common extension $K \supset k_1, k_2$ such that we have $\operatorname{Spec} K \to \operatorname{Spec} k_i \to X$ are the same for $i = 1, 2$. The *underlying point set* of X, denote by $|X|$, is defined as the set of equivalence classes of morphisms $\operatorname{Spec} k \to X$ for all field k over the base field $\mathbb{k}$.

This definition coincides with the underlying set of a scheme. Let $\alpha : X \to Y$ be a morphism of algebraic spaces. It induces a map $|\alpha| : |X| \to |Y|$ by $x \mapsto \alpha \circ x$ (vertical composition).

Proposition 1.20 (ref. [Stacks, Lemma 66.4.6])**.** There is a unique topology on $|X|$ such that

(a) if X is a scheme, then the topology coincides with the usual topology.

(b) every morphism of algebraic spaces $f : X \to Y$ induces a continuous map $|f| : |X| \to |Y|$.

(c) if U is a scheme and $U \to X$ is étale, then the induced map $|U| \to |X|$ is open.

This topology is called the *Zariski topology* on $|X|$.

Definition 1.21. Let X be an algebraic space over a base scheme S. All étale morphisms $U \to X$ with U scheme form a small site $X_{\text{ét}}$. All étale morphisms $U \to X$ with U algebraic space form a small site $X_{\text{sp, ét}}$. The *structure sheaf* $\mathcal{O}_X$ of X is given by $U \mapsto \Gamma(U, \mathcal{O}_U)$ for every étale morphism $U \to X$ from a scheme. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

Example 1.22. Let $U = \mathbb{A}^1_{\mathbb{C}}$ and $R \subset U \times U$ given by $y = x + n, n \in \mathbb{Z}$. Then R is a disjoint union of lines in $U \times U$. Write $R = \coprod_{n\in\mathbb{Z}} R_n$ with $R_n = \{(x, x+n) : x \in \mathbb{C}\}$. Then the projection is given by

$$\pi_1|_{R_n} : R_n \to U, \quad (x, x+n) \mapsto x,$$
$$\pi_2|_{R_n} : R_n \to U, \quad (x, x+n) \mapsto x+n.$$

Easily see that the projection $\pi_i : R \to U$ is étale and surjective for $i = 1, 2$. Let $r_{ij} : R \times U \to U \times U \times U$ be the morphism which maps $((x, y), u)$ to (a_1, a_2, a_3) where $a_i = x$, $a_j = y$ and $a_k = u$ for $k \neq i, j$. Since $\Delta_U \to U \times U$ factors through R, $(\pi_1, \pi_2) = (\pi_2, \pi_1)$ and $r_{12} \times_{(U\times U\times U)} r_{23}$ factors through r_{13}, we have that $h_R(T)$ is an equivalence relation on $h_U(T)$ for all T over S. Then $X := (U, R)$ is an algebraic space.

We do not check the representability here but give an example. Let $U \to X$ be the natural morphism given by $\mathrm{id}_U \in X(U)$. For any scheme T over $\mathbb{C}$, we have

$$(U \times_X U)(T) = \{(f, g) \in h_{U\times U}(T) : \exists \{T_i \to T\} \text{ s.t. } (f_i, g_i) \in h_R(T_i)\} = h_R(T).$$

Hence the fiber product $h_U \times_X h_U$ is represented by R.

We show that $X \ncong \mathbb{C}^\times$ by computing the the global sections. Consider the covering $U \to X$, a section $s \in \mathcal{O}_X(X)$ is given by a section $s \in \Gamma(U, \mathcal{O}_U) = \mathbb{C}[t]$ such that $\pi_1^* s = \pi_2^* s$ in $\Gamma(R, \mathcal{O}_R)$. This means that $s(x+n) = s(x)$ for all $n \in \mathbb{Z}$. Hence s is a constant function. In particular, $\mathcal{O}_X(X) = \mathbb{C} \neq \mathbb{C}[t, t^{-1}]$.

The underlying set $|X|$ is union of the quotient set $\mathbb{C}/\mathbb{Z}$ and a generic point. The Zariski topology on $|X|$ is the trivial topology.

In following, we will use the technique of *local construction* to construct many scheme-like objects on algebraic spaces. For local construction, see [Knu71]. Roughly speaking, for every étale morphism $U \to X$ with U a scheme, we construct a scheme-theoretic object on U which is compatible under base change. Then we glue these objects together to get a global object on X.

Definition 1.23. Let X be an algebraic space over a base scheme S. A *coherent sheaf* on X is a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ such that for every covering $\{U_i \to X\}$ with U_i schemes, the sheaf $\mathcal{F}|_{U_i}$ is coherent for every i. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.
An *ideal sheaf* on X is a coherent sheaf $\mathcal{I} \subset \mathcal{O}_X$. It defines a closed subspace $V(\mathcal{I}) \subset X$ by Yang: to be completed. And every closed subspace $Y \subset X$ is defined by an ideal sheaf $\mathcal{I}_Y$ such that $V(\mathcal{I}_Y) = Y$.

Definition 1.24. Let X be an algebraic space over a base scheme S. A *line bundle* on X is a coherent sheaf $\mathcal{L}$ on X such that for every covering $\{U_i \to X\}$ with U_i schemes, the sheaf $\mathcal{L}|_{U_i}$ is a line bundle on U_i. It extends to a sheaf on the site $X_{\text{sp, ét}}$ uniquely.

Theorem 1.25 (ref. [Stacks, Theorem 76.36.4])**.** Let $f : X \to Y$ be a proper morphism of algebraic spaces over a base scheme S. Then there exists a factorization

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y,$$

where f_1 has geometrically connected fibers and $(f_1)_*\mathcal{O}_X = \mathcal{O}_Z$ and f_2 is finite.

Definition 1.26. Let X be an algebraic space over a base scheme S and Y a closed subset of $|X|$. The *formal completion* of X along Y, denoted by $\mathfrak{X}$, is
Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined as $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$ where $\mathcal{I}$ is the ideal sheaf of Y in $\mathcal{O}_X$. Yang: to be completed.

Definition 1.27. Let X be an algebraic space and Y a closed subset of X. A *modification* of X along Y is a proper morphism $f : X' \to X$ and a closed subset $Y' \subset X'$ such that $X' \setminus Y' \to X \setminus Y$ is an isomorphism and $f^{-1}(Y) = Y'$.

Theorem 1.28 (ref. [Art70, Theorem 3.1])**.** Let Y' be a closed subset of an algebraic space X' of finite type over $\Bbbk$. Let $\mathfrak{X}'$ be the formal completion of X' along Y'. Suppose that there is a formal modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$. Then there is a unique modification

$$f : X' \to X, \quad Y \subset X$$

such that the formal completion of X along Y is isomorphic to $\mathfrak{X}$ and the induced morphism $\mathfrak{X}' \to \mathfrak{X}$ is isomorphic to $\mathfrak{f}$.

Theorem 1.29 (ref. [Art70, Theorem 6.2])**.** Let $\mathfrak{X}'$ be a formal algebraic space and $Y' = V(\mathcal{J}')$ with $\mathcal{J}'$ the defining ideal sheaf of $\mathfrak{X}'$. Let $f : Y' \to Y$ be a proper morphism. Suppose that

(a) for every coherent sheaf $\mathcal{F}$ on $\mathfrak{X}'$, we have

$$R^1 f_* \mathcal{J}'^n \mathcal{F} / \mathcal{J}'^{n+1} \mathcal{F} = 0, \quad \forall n \gg 0;$$

(b) for every n, the homomorphism

$$f_*(\mathcal{O}_{\mathfrak{X}'}/\mathcal{J}'^n) \otimes_{f_* \mathcal{O}_{Y'}} \mathcal{O}_Y \to \mathcal{O}_Y$$

is surjective.

Then there exists a modification $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ and a defining ideal sheaf $\mathcal{J}$ of $\mathfrak{X}$ such that $V(\mathcal{J}) = Y$ and $\mathfrak{f}$ induces f on Y.

Theorem 1.30 (ref. [Art70, Theorem 6.1])**.** Let Y' be a closed algebraic subspace of an algebraic space X' and $f_0 : Y' \to Y$ a finite morphism. Then there exists a modification $f : X' \to X$ whose restriction to Y' is f_0. It is the amalgamated sum $X = X' \sqcup_{Y'} Y$ in the category of algebraic spaces **AlgSp**.

Example 1.31. Let $X = \mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y]$ and $Y = V(y)$ be the x-axis. Let $f_0 : Y' = \mathbb{A}^1 \to Y, x \mapsto x^2$. Then there exists a modification $f : X' \to X$ such that the restriction $f|_{Y'} : Y' \to Y$ is f_0. Yang: To be completed.

1.3 A sufficient and necessary condition for EWM

In this and next subsection, we assume that all schemes (algebraic spaces) are of finite type over a field $\Bbbk$ with characteristic $p > 0$.

Lemma 1.32. Let $f : X \to Y$ be a finite morphism of algebraic space which is of finite type over $\Bbbk$. Suppose that f is a universal homeomorphism. Then there exists $q = p^n$ such that the relative Frobinius morphism $\operatorname{Frob}^n_{X/\Bbbk}$ factors as

$$\operatorname{Frob}^n_{X/\Bbbk} : X \xrightarrow{f} Y \to X^{(q)}.$$

Proof. Yang: I can only prove this for schemes. Suppose that X, Y are affine. Factor it as $A \twoheadrightarrow B \hookrightarrow C$ with A, B, C $\Bbbk$-algebras.

For $A \twoheadrightarrow B$, let I be the kernel of the surjection. Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is finite universal homeomorphism, we have that I is a nilpotent ideal. Hence there exists q such that $I^q = 0$. Let $a, a' \in A$ with the same image b in B. Then we have $a^q - a'^q \in I^q = 0$. Hence $a^q = a'^q$ in A. This gives a map $B^q \to A, b^q \mapsto a^q$.

For $B \hookrightarrow C$, we induct on the dimension. If C is artinian, then $0 = C^q \subset B \subset C$. In general case, this shows that $B \cdot C^{q_1} \subset C$ is an isomorphism at generic points. Let $I := \mathrm{Ann}(B \cdot C^q/B) \subset B$. This is the conductor of extension $B \cdot C^{q_1} \subset C$, whence also an ideal of $B \cdot C^{q_1}$. To see this, for every $x \in B \cdot C^{q_1}$, $b \in I$, we have $xbB \cdot C^{q_1} = bB \cdot C^{q_1} \subset B$. By induction hypothesis, we have $(BC^{q_1}/I)^{q_2} \subset B/I$. For $x \in BC^{q_1}$, there exists $b \in B$ and $\delta \in I \subset B$ such that $x^{q_2} = b + \delta \in B$. Hence we have $(BC^{q_1})^{q_2} \subset B$. In particular, we have $C^{q_1 q_2} \subset (B \cdot C^{q_1})^{q_2} \subset B$.

In general case, we have

$$\begin{tikzcd}
C^{q_1 q_2} \arrow[r] & A' \arrow[r, two heads] \arrow[d] & C^{q_1} \arrow[d] & \\
& A \arrow[r, two heads] & B \arrow[r, hook] & C
\end{tikzcd},$$

where A' is the preimage of C^{q_1} in A. One we have $C^q \to A \to C$, note that $A \to C$ is over $\Bbbk$, then it gives

$$C^q \to C^{(q)} \to A \to C.$$

□

Corollary 1.33. Let $Z \to X$ be a finite universal homeomorphism of algebraic spaces and $Z \to Y$ any finite morphism of algebraic spaces. Suppose that X, Y, Z are all of finite type over $\Bbbk$. Then the amalgamated sum $X \amalg_Z Y$ exists in the category of algebraic spaces. Moreover, $Y \to X \amalg_Z Y$ is a finite universal homeomorphism.

Proof. By Lemma 1.32, we have a diagram

$$\begin{tikzcd}
Y^{(q)} & Y \arrow[l] \\
Z^{(q)} \arrow[u] & \\
X \arrow[u] & Z \arrow[l, "f"] \arrow[uu, "g"']
\end{tikzcd}.$$

Denote $X \to Y^{(q)}$ by f. Let

$$\mathcal{A} := \mathrm{Ker}(\mathcal{O}_X \times \mathcal{O}_Y \to \mathcal{O}_Z, \quad (s,t) \mapsto f^*s - g^*t).$$

Then $\mathcal{A}$ is an $\mathcal{O}_{Y(q)}$-algebra. Set $W := \mathrm{Spec}_{Y(q)} \mathcal{A}$. Then $W = X \amalg_Z Y$ is the amalgamated sum in the category of algebraic spaces. Yang: The most important point is that $Z \to W$ is finite. Yang: At least in the cat of schemes. □

Proposition 1.34. Let $g : X' \to X$ be a proper, finite universal homeomorphism between algebraic spaces. Then a line bundle $\mathcal{L}$ on X is endowed with a map if and only if $g^*\mathcal{L}$ is endowed with a map.

Proof. Let $f : X' \to Z$ be the map endowed on $g^*\mathcal{L}$. By Lemma 1.32, we have a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[dd, "f"] & X \arrow[d] \\
& X'^{(q)} \arrow[d] \\
Z \arrow[r] & Z^{(q)}
\end{tikzcd}.$$

Easy to check that $X \to Z^{(q)}$ is a map associated to $\mathcal{L}$. ☐

Proposition 1.35. Let X be a projective scheme and $\mathcal{L}$ a nef line bundle on X. Assume that $X = X_1 \cup X_2$ for closed subsets X_1 and X_2. Suppose that $\mathcal{L}|_{X_i}$ is endowed with a fibration $g_i : X_i \to Z_i$ for $i = 1, 2$. Then $\mathcal{L}$ is endowed with a map $g : X \to Z$.

Proof. Let $X_{12} := X_1 \cap X_2$. Let $X_{12} \to Z_{12}$ be the Stein factorization of the map $g_1|_{X_{12}}$. Then by Yang: Rigidity Lemma, it is also the Stein factorization of the map $g_2|_{X_{12}}$. Denote Y_i be the image of Z_{12} in Z_i for $i = 1, 2$. Then we have a commutative diagram

$$\begin{tikzcd}
& & Z_1 & \\
& X & X_1 \arrow[l] \arrow[u] & Y_1 \arrow[ul, "h'"] \\
Z_2 & X_2 \arrow[l] \arrow[u] & X_{12} \arrow[l] \arrow[u] \arrow[dr] & \\
& Y_2 \arrow[ul] & & Z_{12} \arrow[l] \arrow[ulll, "f"] \arrow[uu, "h"]
\end{tikzcd}.$$

Consider the sub-diagram

$$\begin{tikzcd}
& Z_1 \\
& Y_1 \arrow[u, "h'"] \\
Z_2 & Z_{12} \arrow[l, "f"] \arrow[u, "h"]
\end{tikzcd}.$$

Here f is finite, h is finite universal homeomorphism and h' is a closed immersion. By Corollary 1.33, we have the amalgamated sum $Z' := Y_1 \amalg_{Z_{12}} Z_2$ exists in the category of algebraic spaces. Since f is finite, so is the induced morphism $Y_1 \to Z'$. Then by Theorem 1.30, the amalgamated sum $Z := Z' \amalg_{Y_1} Z_1$ exists in the category of algebraic spaces.

Then we have a commutative diagram

$$\begin{tikzcd}
Z & & Z_1 \arrow[ll] \\
& X \arrow[ul, "g"] & X_1 \arrow[l] \arrow[u] \\
Z_2 \arrow[uu] & X_2 \arrow[l] \arrow[u] & X_{12} \arrow[l] \arrow[u]
\end{tikzcd}.$$

Directly check shows that g is a map associated to $\mathcal{L}$. □

Proposition 1.36. Let X be a projective scheme and D a nef and big divisor on X. Then we can write $D = A + E$ where A is an ample divisor and E is an effective divisor. Then D is endowed with a map iff $D|_{E_{red}}$ is endowed with a map.

Proof. By Proposition 1.34, we may assume that $D|_E$ is endowed with a map $f : E \to Z$. Let $\mathcal{L} = \mathcal{O}_X(-E)$ be the ideal sheaf of E. note that $-E = A - D$ and D is f-numerically trivial. Hence $\mathcal{L}|_E$ is f-ample. By Serre's vanishing, for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0 \in \mathfrak{m}$ such that for all $n \geq n_0$, we have

$$R^i f_* \mathcal{F}|_E \otimes \mathcal{L}|_E^{\otimes n} = 0$$

for all $i > 0$. In particular, let $n \in \mathbb{Z}$ such that $R^i f_* \mathcal{O}_X/\mathcal{L} \otimes \mathcal{L}^{\otimes m} = 0$ for all $i > 0, m \geq n$. Set $\mathcal{I} := \mathcal{L}^{\otimes n}$. Then by the exact sequence

$$0 \to \mathcal{L}^{n-1} \otimes \mathcal{O}_X/\mathcal{L} \to \mathcal{O}_X/\mathcal{L}^n \to \mathcal{O}_X/\mathcal{L} \to 0,$$

we have that $R^i f_*(\mathcal{O}_X/\mathcal{I} \otimes \mathcal{I}^t) = 0$ for all $i > 0, t \geq 1$. This implies that $f_*\mathcal{O}_X/\mathcal{I}^t \to f_*\mathcal{O}_X/\mathcal{I}$ is surjective for all $t \geq 1$.

Let

$$\mathcal{A} := \mathcal{O}_X \oplus \mathcal{I}T \oplus \mathcal{I}^2T^2 \oplus \cdots,$$
$$\mathcal{M} := \mathcal{F} \oplus \mathcal{I}\mathcal{F}T \oplus \mathcal{I}^2\mathcal{F}T^2 \oplus \cdots,$$

where T is a formal variable to denote the grading. Then $\mathcal{A}$ is a graded $\mathcal{O}_X$-algebra of finite type and $\mathcal{M}$ is a finite graded $\mathcal{A}$-module. We have an exact sequence of graded $\mathcal{A}$-modules

$$0 \to \mathcal{K} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{I}T \to \mathcal{M} \to 0,$$

where $\mathcal{K} = \bigoplus \mathcal{K}_r T^r$ is a finite graded $\mathcal{A}$-module. Hence for $r \gg 1$, we have that $\mathcal{I}T \cdot \mathcal{K}_r T^r = \mathcal{K}_{r+1}T^{r+1}$. It implies that the image of $\mathcal{K}_{r+1}T^{r+1} \to \mathcal{M}_r T^r \otimes_{\mathcal{A}} \mathcal{I}T$ is contained in $\mathcal{I}\mathcal{M}_r$ for all $r \gg 1$. Tensor with $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}$, we have that

$$\mathcal{K}_{r+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I} \to 0 \to \mathcal{M}_r \otimes_{\mathcal{O}_X} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I} \to \mathcal{M}_{r+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I} \to 0.$$

That is, $\mathcal{I}^r\mathcal{F}/\mathcal{I}^{r+1}\mathcal{F} \otimes_{\mathcal{O}_X/\mathcal{I}} \mathcal{I}/\mathcal{I}^2 \cong \mathcal{I}^{r+1}\mathcal{F}/\mathcal{I}^{r+2}\mathcal{F}$ for all $r \gg 1$. Hence we have that

$$R^i f_*(\mathcal{I}^{r-1}\mathcal{F}/\mathcal{I}^r\mathcal{F}) = 0$$

for all $i > 0, r \gg 1$.

Let $E' := V(\mathcal{I})$, we have that $D|_{E'}$ is endowed with a map $f' : E' \to Z'$ by Proposition 1.34. Moreover, we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & Z \\ \downarrow & & \downarrow g \\ E' & \xrightarrow{f'} & Z' \end{array}$$

with g finite. Then by Grothendieck Spectral Sequence, we have that

$$R^i f'_*(\mathcal{J}^{r-1}\mathcal{F}/\mathcal{J}^r\mathcal{F}) = 0$$

for all $i > 0, r \gg 1$.

Then we can apply Theorems 1.28 and 1.29 to get a modification $X \to Y$. Note that $\operatorname{Exc} D \subset \operatorname{Supp} E$. It follows that $X \to Y$ is a map associated to D. ☐

Theorem 1.37. Let X be a proper variety and $\mathcal{L}$ a nef line bundle on X. Then $\mathcal{L}$ is endowed with a map if and only if $\mathcal{L}|_{\operatorname{Exc}\mathcal{L}}$ is endowed with a map.

Proof. By Proposition 1.35, we can assume that $\mathcal{L}$ is big. Then the result follows from Proposition 1.36 and induction on dimension. ☐

1.4 For semiample

Lemma 1.38. Let X be a projective scheme over $\Bbbk = \overline{\mathbb{F}_p}$. Then $\mathcal{L}$ is numerically trivial if and only if $\mathcal{L}$ is torsion in $\operatorname{Pic}(X)$.

Proof. Let T be the scheme in Theorem 1.2. Then $\mathcal{L}$ corresponds to a $\mathbb{F}_q$-point of T. Note that there are only finitely many $\mathbb{F}_q$-points in T. Hence $\mathcal{L}$ is torsion in $\operatorname{Pic}(X)$. ☐

Proposition 1.39. Let $f : X \to Y$ be a finite universal homeomorphism between algebraic spaces of finite type over $\Bbbk$ and $\mathcal{L}$ a line bundle on Y. Then there exists $q = p^n$ such that

(a) for every section $s \in H^0(X, f^*\mathcal{L})$, we have $s^q \in \operatorname{Im}(H^0(Y, \mathcal{L}^{\otimes q}) \to H^0(X, f^*\mathcal{L}^{\otimes q}))$;

(b) $\mathcal{L}$ is semiample if and only if $f^*\mathcal{L}$ is semiample;

(c) the map

$$f^* : \operatorname{Pic}(Y) \otimes \mathbb{Z}[1/q] \to \operatorname{Pic}(X) \otimes \mathbb{Z}[1/q]$$

is an isomorphism;

(d) if $f^*s_1 = f^*s_2$ for two sections $s_1, s_2 \in H^0(Y, \mathcal{L})$, then $s_1^q = s_2^q$ in $H^0(X, \mathcal{L}^{\otimes q})$.

Proof. Note that $\operatorname{Frob}^* \mathcal{L} \cong \mathcal{L}^{\otimes p}$. Then all the properties follows from Lemma 1.32. ☐

Proposition 1.40. Let X be a projective scheme and $\mathcal{L}$ a nef line bundle on X. Assume that $X = X_1 \cup X_2$ for closed subsets X_1 and X_2. Suppose that $\mathcal{L}|_{X_i}$ is semiample for $i = 1, 2$. Then $\mathcal{L}$ is semiample.

Proof. Yang: To be learned. ☐

Lemma 1.41. Let $f : X \to Y$ be a proper map between algebraic spaces with $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $\mathcal{L}$ a line bundle on X. Let $D = V(\mathcal{J}) \subset X$ be a closed subspace defined by an ideal sheaf $\mathcal{J}$, $Z = f(D)$ and $D_k := V(\mathcal{J}^k)$. Suppose that f is a modification with respect to D, Z and $R^1 f_*\mathcal{J}^k/\mathcal{J}^{k+1} = 0$ for all

$k \gg 0$. Suppose for every k, there exists $r > 0$ such that $\mathcal{L}^{\otimes r}|_{D_k}$ is pulled back from $f(D_k)$. Then $\mathcal{L}^{\otimes r}$ is pulled back from Y for some $r > 0$.

Proof. Replace D by D_k and $\mathcal{L}$ by $\mathcal{L}^{\otimes r}$ for some $k, r > 0$, we can assume that $R^1 f_* \mathcal{I}^k/\mathcal{I}^{k+1} = 0$ for all k and $\mathcal{L}|_D$ is pulled back from $f(D)$. Then we show that $f_*\mathcal{L}$ is a line bundle and $f^*f_*\mathcal{L} \cong \mathcal{L}$. Both of them are local, so we can assume that $X = \operatorname{Spec} B, Z = \operatorname{Spec} A$ are spectrum of local rings. Hence $\mathcal{L}|_{D_k}$ is trivial for all k. By vanishing of $R^1 f_* \mathcal{I}^k/\mathcal{I}^{k+1}$, we have a surjection $H^0(D_{k+1}, \mathcal{L}|_{D_{k+1}}) \twoheadrightarrow H^0(D_k, \mathcal{L}|_{D_k})$ for all k. This allow us to choose a section $s_k \in H^0(D_k, \mathcal{L}|_{D_k})$ such that $s_k = s_{k+1}|_{D_k}$ for all k. Then we have a section $s \in H^0(D, \mathcal{L}|_D)$ such that $s|_{D_k} = s_k$ for all k. By Nakayama's Lemma, we can assume that s_k is nowhere vanishing. Yang: To be completed. □

Proposition 1.42. Let X be a projective scheme and D a nef and big divisor on X. Then we can write $D = A + E$ where A is an ample divisor and E is an effective divisor. Then D is semiample iff $D|_{E_{red}}$ is semiample.

Proof. Yang: To be completed. □

Theorem 1.43. Let X be a proper variety and $\mathcal{L}$ a nef line bundle on X. Then $\mathcal{L}$ is semiample if and only if $\mathcal{L}|_{\operatorname{Exc}\mathcal{L}}$ is semiample.

Proof. Yang: To be completed. □

1.5 Basepoint free theorem on positive characteristic

Proposition 1.44 (ref. Yang:). Let $T \subset X$ be a reduced Weil divisor on a normal variety X. Let $T^\nu \to T$ be the normalization, $C \subset T^\nu$ the effective Weil divisor defined by the conductor and $p : T^\nu \to T \hookrightarrow X$ the composition. Suppose that $K_X + T$ is $\mathbb{Q}$-Cartier. Then there exists an effective $\mathbb{Q}$-Weil divisor D on T^ν such that

$$K_{T^\nu} + C + D = p^*(K_X + T).$$

Theorem 1.45. Let X be a normal projective $\mathbb{Q}$-factorial threefold and $B \in (0, 1)$ a $\mathbb{Q}$-divisor. Let $\mathcal{L}$ be a nef and big line bundle on X such that $\mathcal{L} - K_{(X,B)}$ is nef and big. Then $\mathcal{L}$ is endowed with a map. Moreover, if $\Bbbk = \overline{\mathbb{F}_p}$, $\mathcal{L}$ is semiample.

Proof. Let $\mathcal{L} = \mathcal{O}_X(A + E)$ with A an ample divisor and E an effective divisor. Write $E = E_0 + E_1 + E_2$ such that the restriction of $\mathcal{L}$ to every irreducible component of E_i is of numerical dimension i. Let $S := \operatorname{Supp} E_1$ and $S = \sum S_i$ with S_i irreducible components. Let $S^\nu \to S$ and $S_i^\nu \to S_i$ be the normalizations.

Step 1. Reduce to show that $\mathcal{L}|_S$ is endowed with a map (semiample).

Yang: To be completed.

Step 2. Reduce to show that $\mathcal{L}|_{S_i^\nu}$ is endowed with a map (semiample).

Yang: To be completed.

Step 3. Show that $\mathcal{L}|_{S_i^\nu}$ is endowed with a map (semiample).

Yang: To be completed. □

2 lifting + stable sections

2.1 + stable sections

Definition 2.1. $\mathfrak{I}, \mathcal{L}$,Frac Im are defined as

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