Algebraic Dynamics and Dynamical Iitaka Theory

Tianle Yang
based on the joint work with Sheng Meng and Long Wang

Undergraduate Forum, ICCM 2025, January 7, 2026

SCHOOL OF
MATHEMATICAL SCIENCES
EAST CHINA NORMAL UNIVERSITY

Kawaguchi-Silverman Conjecture

Setup: X sm. proj. var. $/\overline{\mathbb{Q}}$; $f : X \to X$ surj. endo.; H ample l.b.;
$h \geq 1$ height function ass. to H.

Kawaguchi-Silverman Conjecture

Setup: X sm. proj. var. $/\overline{\mathbb{Q}}$; $f : X \to X$ surj. endo.; H ample l.b.;
$h \geq 1$ height function ass. to H.

Conjecture: Kawaguchi-Silverman Conjecture = KSC

If the orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X, then

$$\delta_f = \alpha_f(x).$$

Kawaguchi-Silverman Conjecture

Setup: X sm. proj. var. $/\overline{\mathbb{Q}}$; $f : X \to X$ surj. endo.; H ample l.b.;
$h \geq 1$ height function ass. to H.

Conjecture: Kawaguchi-Silverman Conjecture = KSC

If the orbit $O_f(x) := \{f^n(x) \,|\, n \geq 0\}$ is Zariski dense in X, then

$$\delta_f = \alpha_f(x).$$

- dynamical degree:

  $\delta_f := \lim\limits_{n\to\infty}((f^n)^*H \cdot H^{\dim X-1})^{\frac{1}{n}}$ a geometric, global invariant of f.

Kawaguchi-Silverman Conjecture

Setup: X sm. proj. var. $/\overline{\mathbb{Q}}$; $f : X \to X$ surj. endo.; H ample l.b.; $h \geq 1$ height function ass. to H.

Conjecture: Kawaguchi-Silverman Conjecture = KSC

If the orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X, then

$$\delta_f = \alpha_f(x).$$

- dynamical degree:

  $\delta_f := \lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim X - 1})^{\frac{1}{n}}$ a geometric, global invariant of f.

- arithmetic degree:

  $\alpha_f(x) := \lim_{n\to\infty} h(f^n(x))^{\frac{1}{n}}$ an arithmetic, local invariant at x;

Kawaguchi-Silverman Conjecture

Setup: X sm. proj. var. $/\overline{\mathbb{Q}}$; $f : X \to X$ surj. endo.; H ample l.b.;
$h \geq 1$ height function ass. to H.

Conjecture: Kawaguchi-Silverman Conjecture = KSC

If the orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X, then

$$\delta_f = \alpha_f(x).$$

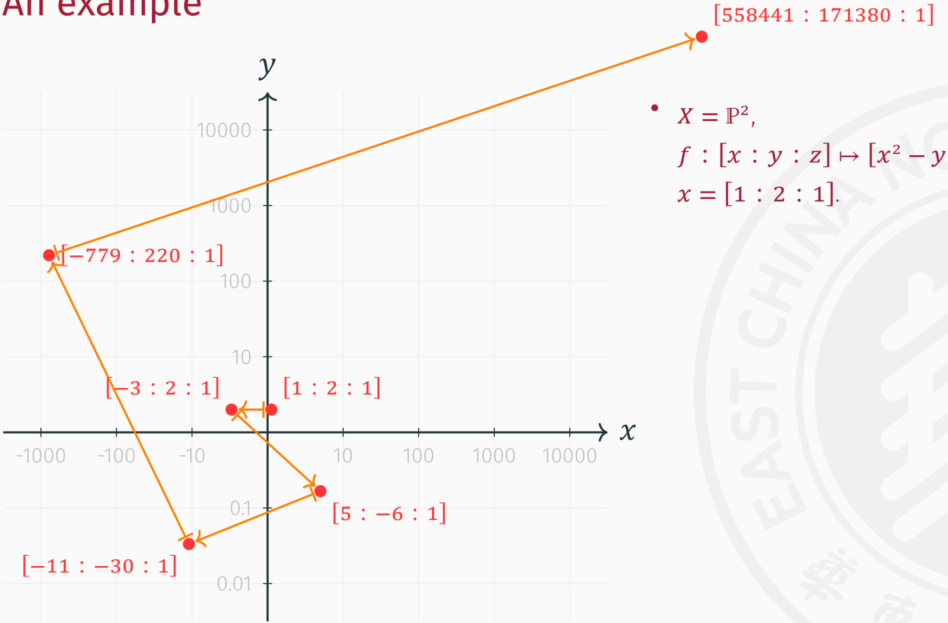
- dynamical degree:

  $\delta_f := \lim_{n\to\infty} ((f^n)^*H \cdot H^{\dim X - 1})^{\frac{1}{n}}$ a geometric, global invariant of f.
- arithmetic degree:

  $\alpha_f(x) := \lim_{n\to\infty} h(f^n(x))^{\frac{1}{n}}$ an arithmetic, local invariant at x;

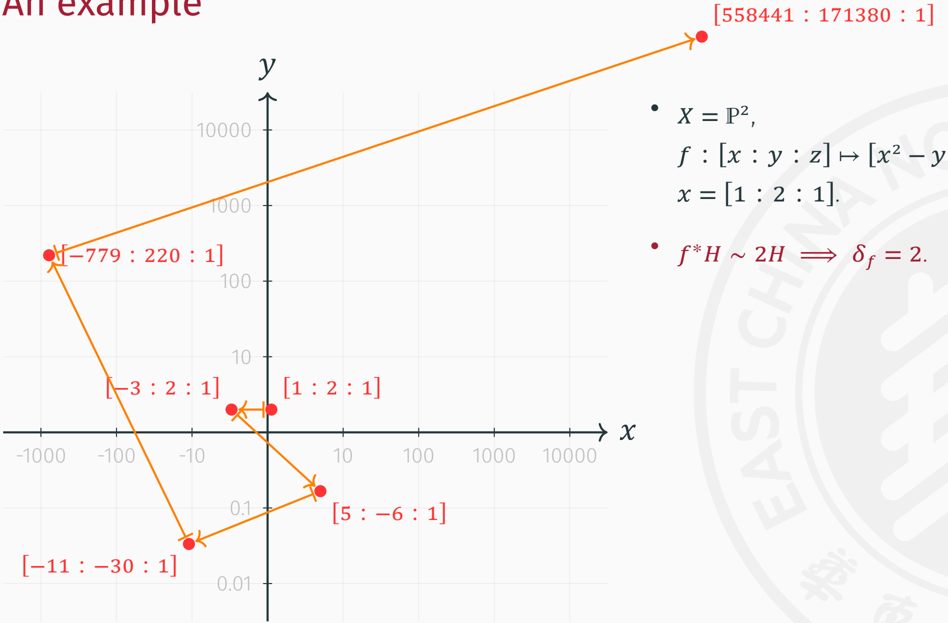
Slogan: GEOMETRY controls ARITHMETIC.

An example



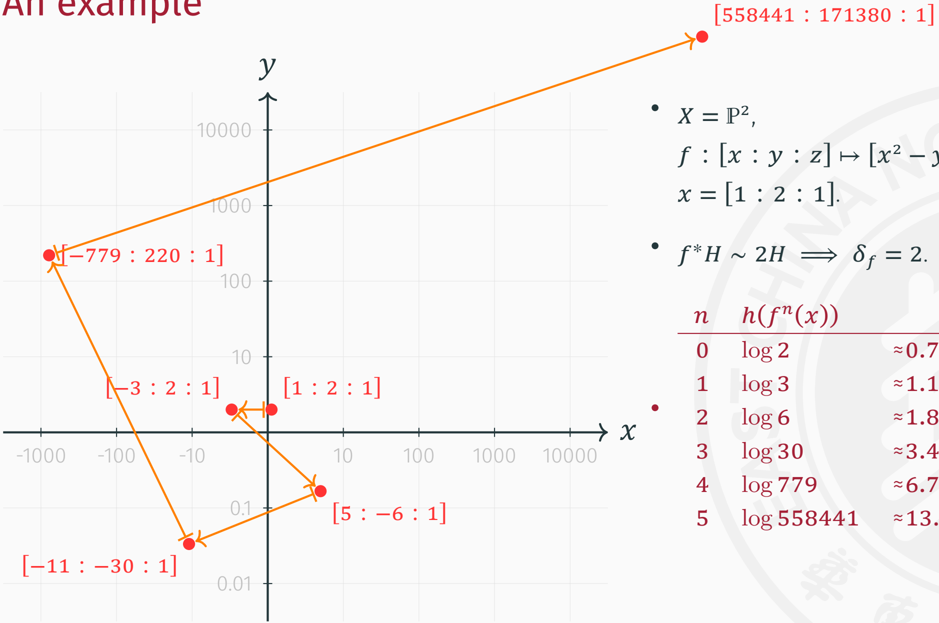
- $X = \mathbb{P}^2$,
  $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
  $x = [1 : 2 : 1]$.

An example



- $X = \mathbb{P}^2$,
  $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
  $x = [1 : 2 : 1]$.
- $f^*H \sim 2H \implies \delta_f = 2$.

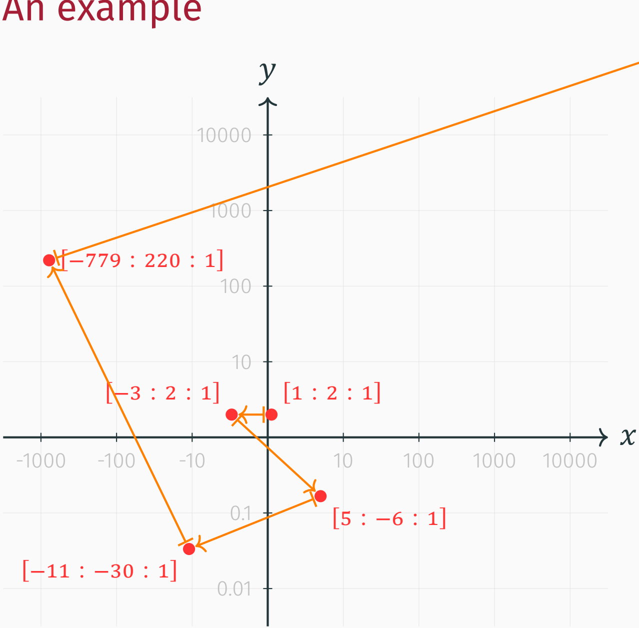
An example



- $X = \mathbb{P}^2$,
  $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
  $x = [1 : 2 : 1]$.
- $f^*H \sim 2H \implies \delta_f = 2$.
-

n	$h(f^n(x))$	
0	$\log 2$	≈ 0.7
1	$\log 3$	≈ 1.1
2	$\log 6$	≈ 1.8
3	$\log 30$	≈ 3.4
4	$\log 779$	≈ 6.7
5	$\log 558441$	≈ 13.2

An example



- $X = \mathbb{P}^2$,
  $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
  $x = [1 : 2 : 1]$.
- $f^*H \sim 2H \implies \delta_f = 2$.

n	$h(f^n(x))$	
0	$\log 2$	≈ 0.7
1	$\log 3$	≈ 1.1
2	$\log 6$	≈ 1.8
3	$\log 30$	≈ 3.4
4	$\log 779$	≈ 6.7
5	$\log 558441$	≈ 13.2

- It is expected that $\alpha_f(x) = 2$.

Known cases for KSC

Projective surfaces ([MSS18,MZ22]);

Quasi-projective surfaces (assuming DML) ([Wang23]);

Birational map on surfaces ([Xie24]);

Smooth projective threefolds with $\deg f > 1$ ([MZ23]);

Known cases for KSC

Projective surfaces ([MSS18,MZ22]);
Quasi-projective surfaces (assuming DML) ([Wang23]);
Birational map on surfaces ([Xie24]);
Smooth projective threefolds with $\deg f > 1$ ([MZ23]);

Abelian varieties ([KS16]);
Hyperkähler manifolds ([LS21]);
Mori dream spaces ([Matsuzawa20]);

Known cases for KSC

Projective surfaces ([MSS18,MZ22]);
Quasi-projective surfaces (assuming DML) ([Wang23]);
Birational map on surfaces ([Xie24]);
Smooth projective threefolds with $\deg f > 1$ ([MZ23]);

Abelian varieties ([KS16]);
Hyperkähler manifolds ([LS21]);
Mori dream spaces ([Matsuzawa20]);

Polarized endomorphisms ([KS14]);
Int-amplified endomorphisms ([MZ24]);

...

Main results

Special Fano fibration case (*) (as a final output of MMP):

smooth Fano fibration $X \xrightarrow{\pi} Y$ of relative Picard number one (eg. $\mathbb{P}^n$-bundles) over

(*1) an abelian variety; or

(*2) a smooth projective variety of Picard number one.

Main results

Special Fano fibration case (*) (as a final output of MMP):

smooth Fano fibration $X \xrightarrow{\pi} Y$ of relative Picard number one (eg. $\mathbb{P}^n$-bundles) over

(*1) an abelian variety; or

(*2) a smooth projective variety of Picard number one.

Theorem: [Meng-Wang-Y]

KSC holds for X in the special Fano fibration case (*).

Main results

Special Fano fibration case (*) (as a final output of MMP):

smooth Fano fibration $X \xrightarrow{\pi} Y$ of relative Picard number one (eg. $\mathbb{P}^n$-bundles) over

(*1) an abelian variety; or

(*2) a smooth projective variety of Picard number one.

Dynamical conditions (†):

the fibration $(X, f) \xrightarrow{\pi} (Y, g)$ does not preserve the dynamical degree, i.e., $\delta_f > \delta_g$;
if in the case (*2), we further require $\delta_g = 1$.

Main results

Special Fano fibration case (*) (as a final output of MMP):

smooth Fano fibration $X \xrightarrow{\pi} Y$ of relative Picard number one (eg. $\mathbb{P}^n$-bundles) over

(*1) an abelian variety; or

(*2) a smooth projective variety of Picard number one.

Dynamical conditions (†):

the fibration $(X, f) \xrightarrow{\pi} (Y, g)$ does not preserve the dynamical degree, i.e., $\delta_f > \delta_g$;
if in the case (*2), we further require $\delta_g = 1$.

Theorem: [Meng-Wang-Y]

In the special Fano fibration case (*) with dynamical conditions (†), there exists an equivariant fibration

$$f \circlearrowright X \overset{\varphi_{f,R_f}}{\dashrightarrow\!\!\!\rightarrow} Z \circlearrowleft h$$

with $\dim Z > 0$ and h is δ_f-polarized (in particular, $\delta_h = \delta_f$).

Sketch of proof

Special Fano fibration case (*)
with dynamical conditions (†)

Sketch of proof

Special Fano fibration case (*) with dynamical conditions (†) $\Longrightarrow$ Decomposition of the pseudo-effective cone $\mathrm{Psef}^1(X)$

Sketch of proof

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Special Fano fibration case } (*) \\ \text{with dynamical conditions } (\dagger)\end{array}} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \boxed{\begin{array}{c}\text{Decomposition of the} \\ \text{pseudo-effective cone } \mathrm{Psef}^1(X)\end{array}} \arrow[d, Rightarrow] \\
 & \boxed{\begin{array}{c} f^*R_f \equiv \delta_f R_f \\ \text{(this make the} \\ \text{base polarized)}\end{array}}
\end{tikzcd}
$$

Sketch of proof

Special Fano fibration case (*) with dynamical conditions (†)

$\Longrightarrow$

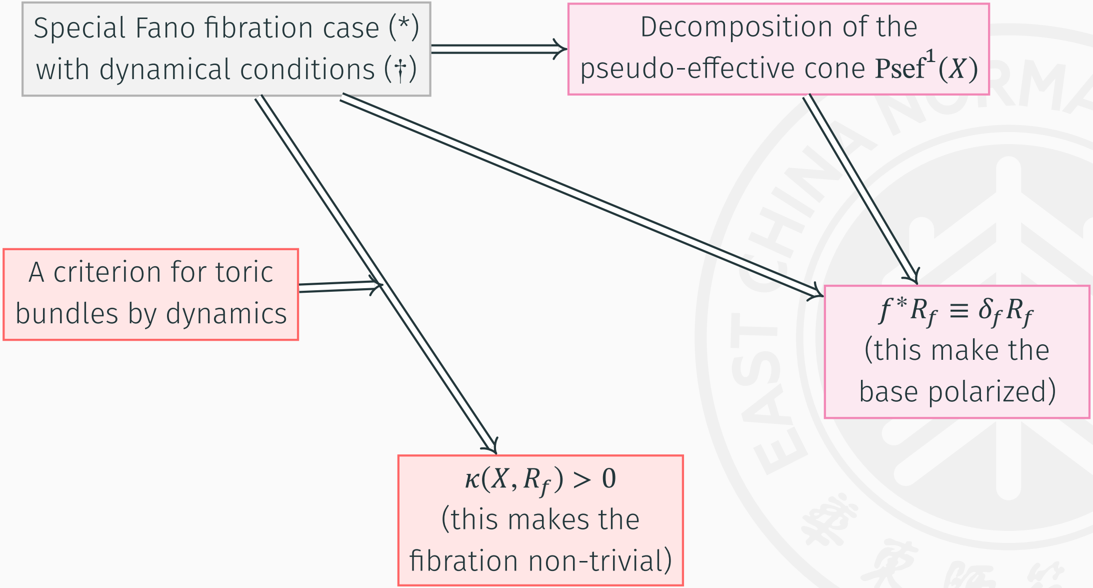
Decomposition of the pseudo-effective cone $\mathrm{Psef}^1(X)$

$\Longrightarrow$

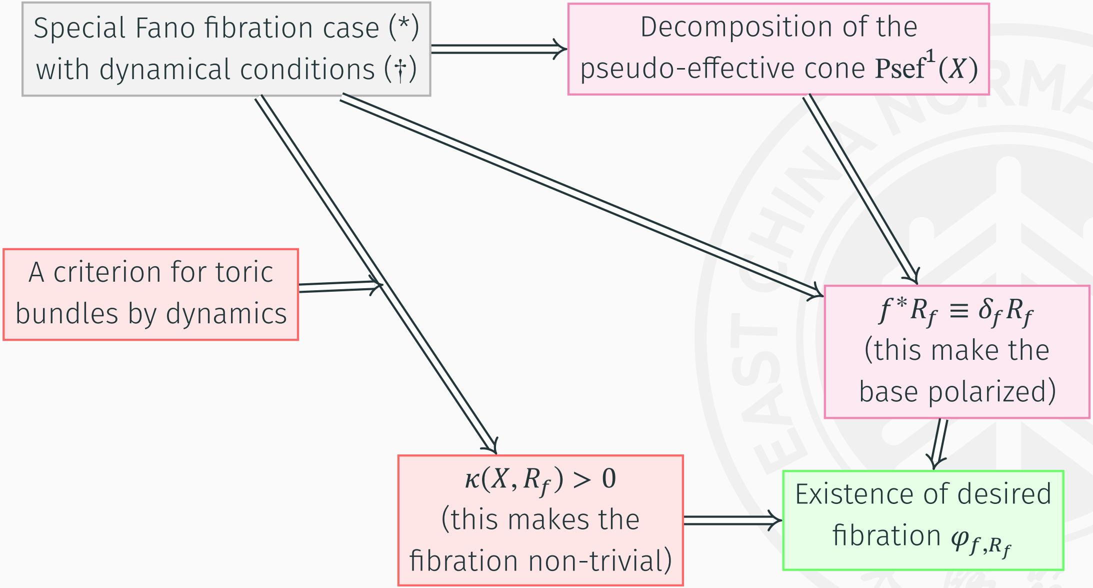
$f^*R_f \equiv \delta_f R_f$ (this make the base polarized)

A criterion for toric bundles by dynamics

Sketch of proof



Sketch of proof



Thank You!