

Algebraic Dynamics and Dynamical Iitaka Theory

Tianle Yang

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SCHOOL OF
MATHEMATICAL SCIENCES
EAST CHINA NORMAL UNIVERSITY

Algebraic Dynamics





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The fundamental objects in algebraic dynamics are (X, f) , where X is a variety and $f : X \dashrightarrow X$ is a dominant rational self-map.

Here we focus on the case X is projective and f is a surjective endomorphism.

For simplicity, we assume that X is smooth.



Definition: First dynamical degree

The *first dynamical degree* δ_f of f is defined to be the following limit

$$\delta_f := \lim_{n \rightarrow \infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n} \in \mathbb{R}_{\geq 1},$$

where H is an ample Cartier divisor on X .



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Definition: Arithmetic degree

Let $h : X(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ be the height function associated to an ample divisor on X . Then for every $x \in X(\mathbb{k})$, we define the *arithmetic degree of f at x* by

$$\alpha_f(x) = \lim_{n \rightarrow \infty} \max\{h(f^n(x)), 1\}^{1/n} \in \mathbb{R}_{\geq 1}.$$



Conjecture: Kawaguchi-Silverman Conjecture = KSC

Let $x \in X(\mathbb{K})$, and suppose that the (forward) orbit $O_f(x) = \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X . Then the arithmetic degree at x is equal to the dynamical degree of f , i.e., $\alpha_f(x) = \delta_f$.



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If we have an equivariant surjective $(X, f) \dashrightarrow (Y, g)$ such that $\delta_f = \delta_g$, then KSC for (Y, g) implies KSC for (X, f) .



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What about the case when L is only effective or when f^*L is not proportional to L ?

Dynamical Iitaka Theory





Definition: Dynamical Iitaka dimension

Let D be a Cartier divisor on X . The *dynamical f -Iitaka dimension* of D is defined as

$$\kappa_f(X, D) := \max \left\{ \kappa(X, D') \mid D' = \sum_{i=0}^m a_i (f^*)^i D, a_i \in \mathbb{Z} \right\}.$$



Theorem: ref. [MZ23, Theorem 4.6]

Suppose that $\kappa_f(X, D) \geq 0$. Then there is an f -equivariant dominant rational map

$$f \circ X \dashrightarrow Y \circ f|_Y$$

with Y normal projective of dimension $\kappa_f(X, D)$ and $f|_Y$ a surjective endomorphism.

The ramification divisor



The *ramification divisor* R_f is defined by

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It is effective and satisfies the ramification formula

$$K_X = f^*K_X + R_f.$$



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$$f|_{X \setminus \text{Supp } R_f} : X \setminus \text{Supp } R_f \rightarrow X \setminus \text{Supp } R_f$$

is (quasi-)étale.

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- $0 < \kappa_f(X, R_f) < \dim X$. We then have an f -equivariant dominant rational map

$$\varphi_{f, R_f} : X \dashrightarrow Y$$

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- $\kappa_f(X, R_f) = \dim X$. In this case, R_{f^s} is big when $s \gg 1$.

About our paper





We consider the case the (X, f) has admitted an f -equivariant extremal Fano contraction $\pi : X \rightarrow Y$ such that $\delta_f > \delta_{f|_Y}$.



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Hence we want to use the dynamical Iitaka fibration associated to R_f to get another f -equivariant fibration.

Further assume that the base Y is either an abelian variety or a smooth projective variety of Picard number one, we get our main results.



Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an extremal Fano contraction $\pi : X \rightarrow Y$ to an abelian variety Y of positive dimension. Suppose f admits a Zariski dense orbit and $\delta_f > \delta_{f|_Y}$. Then the following hold.

1. The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.
2. There exists an f -equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f -Iitaka fibration of R_f , such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f -polarized.



Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an f -equivariant **smooth** extremal Fano contraction $\pi : X \rightarrow Y$ with $\rho(Y) = 1$. Suppose $\delta_f > \delta_{f|_Y} = 1$. Then the following hold.

1. The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.
2. There exists an f -equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f -Iitaka fibration of R_f , such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f -polarized.



Corollary

KSC holds for any smooth projective variety X admitting an extremal Fano contraction to an abelian variety.

Corollary

KSC holds for any \mathbb{P}^n -bundle over either a Q -abelian variety or a smooth projective variety of Picard number one.

Thank You!

