Kawaguchi-Silverman Conjecture
00000

Main Results
000

Dynamical Iitaka Theory
00000

Decomposition of cones
0000

Toric fibrations
00000000

Dynamical Iitaka Theory on Fano Contractions

Tianle Yang,
joint work with Sheng Meng and Long Wang

2025.08.20

Outline

Algebraic Dynamics

We work over $\overline{\mathbb{Q}}$. The fundamental objects in algebraic dynamics are (X, f), where X is a projective variety and $f : X \dashrightarrow X$ is a dominant rational self-map. Here we focus on the case f is a surjective endomorphism.

Dynamical degree

Definition (First dynamical degree)

Consider a projective variety X and a surjective endomorphism $f : X \to X$. The *first dynamical degree* δ_f of f is defined to be the following limit

$$\delta_f := \lim_{n \to \infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n} \in \mathbb{R}_{\geq 1},$$

where H is an ample Cartier divisor.

Dynamical degree

Definition (First dynamical degree)

Consider a projective variety X and a surjective endomorphism $f : X \to X$. The *first dynamical degree* δ_f of f is defined to be the following limit

$$\delta_f := \lim_{n \to \infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n} \in \mathbb{R}_{\geq 1},$$

where H is an ample Cartier divisor.

It is known that the limit always exists and is independent of the choice of H (see [DS04; DS05]; cf. [Dan20]).

Dynamical degree

Definition (First dynamical degree)

Consider a projective variety X and a surjective endomorphism $f : X \to X$. The *first dynamical degree* δ_f of f is defined to be the following limit

$$\delta_f := \lim_{n \to \infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n} \in \mathbb{R}_{\geq 1},$$

where H is an ample Cartier divisor.

It is known that the limit always exists and is independent of the choice of H (see [DS04; DS05]; cf. [Dan20]).
It is also known that the first dynamical degree δ_f coincides with the spectral radius of the induced linear operation $f^*|_{N^1(X)}$.

Height function

Recall that for every line bundle $\mathcal{L}$, there is a height function $h_{\mathcal{L}} : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ associated to $\mathcal{L}$, which measures the arithmetic complexity of $\overline{\mathbb{Q}}$-points and is unique up to a bounded function.

Height function

Recall that for every line bundle $\mathcal{L}$, there is a height function $h_{\mathcal{L}} \colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$ associated to $\mathcal{L}$, which measures the arithmetic complexity of $\overline{\mathbb{Q}}$-points and is unique up to a bounded function. For example, if $X = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ and $x = [x_0 : \cdots : x_n] \in X(\mathbb{Q})$ with $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$, then

$$h_{\mathcal{O}_{\mathbb{P}^n}(1)}(x) = \max\{|x_0|, \ldots, |x_n|\}$$

is the standard height function on $\mathbb{P}^n$.

Height function

Recall that for every line bundle $\mathcal{L}$, there is a height function $h_{\mathcal{L}} \colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$ associated to $\mathcal{L}$, which measures the arithmetic complexity of $\overline{\mathbb{Q}}$-points and is unique up to a bounded function.

For example, if $X = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ and $x = [x_0 : \cdots : x_n] \in X(\mathbb{Q})$ with $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$, then

$$h_{\mathcal{O}_{\mathbb{P}^n}(1)}(x) = \max\{|x_0|, \ldots, |x_n|\}$$

is the standard height function on $\mathbb{P}^n$.

In general, if H is a very ample Cartier divisor on X associated to a closed immersion $\varphi \colon X \hookrightarrow \mathbb{P}^N$, one can define the Weil height function h_H by

$$h_H(x) = h_{\mathcal{O}_{\mathbb{P}^N}(1)}(\varphi(x)).$$

Arithmetic degree

Definition (Arithmetic degree)

Let $h_H \geq 1$ be a Weil height function associated with an ample Cartier divisor H. Then for every $x \in X(\overline{\mathbb{Q}})$, we define the *arithmetic degree of f at x* by

$$\alpha_f(x) = \lim_{n \to \infty} h_H(f^n(x))^{1/n} \in \mathbb{R}_{\geq 1}.$$

Arithmetic degree

Definition (Arithmetic degree)

Let $h_H \geq 1$ be a Weil height function associated with an ample
Cartier divisor H. Then for every $x \in X(\overline{\mathbb{Q}})$, we define the
arithmetic degree of f at x by

$$\alpha_f(x) = \lim_{n \to \infty} h_H(f^n(x))^{1/n} \in \mathbb{R}_{\geq 1}.$$

Due to Kawaguchi-Silverman [KS16a], the limit always exists and
is also independent of the choice of the ample Cartier divisor and
the Weil height function.

Arithmetic degree

Definition (Arithmetic degree)

Let $h_H \geq 1$ be a Weil height function associated with an ample Cartier divisor H. Then for every $x \in X(\overline{\mathbb{Q}})$, we define the *arithmetic degree of f at x* by

$$\alpha_f(x) = \lim_{n \to \infty} h_H(f^n(x))^{1/n} \in \mathbb{R}_{\geq 1}.$$

Due to Kawaguchi-Silverman [KS16a], the limit always exists and is also independent of the choice of the ample Cartier divisor and the Weil height function.

Moreover, $\alpha_f(x)$ is equal to norm of an eigenvalue of $f^*|_{\mathrm{N}^1(X)_{\mathbb{C}}}$. In particular, $\alpha_f(x) \leq \delta_f$.

Kawaguchi-Silverman Conjecture

Conjecture (Kawaguchi-Silverman Conjecture = KSC)

Let $f : X \to X$ be a surjective endomorphism of a projective variety X defined over $\overline{\mathbb{Q}}$. Let $x \in X(\overline{\mathbb{Q}})$, and suppose that the (forward) orbit $O_f(x) = \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X. Then the arithmetic degree at x is equal to the dynamical degree of f, i.e., $\alpha_f(x) = \delta_f$.

Kawaguchi-Silverman Conjecture

Conjecture (Kawaguchi-Silverman Conjecture = KSC)

Let $f : X \to X$ be a surjective endomorphism of a projective variety X defined over $\overline{\mathbb{Q}}$. Let $x \in X(\overline{\mathbb{Q}})$, and suppose that the (forward) orbit $O_f(x) = \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X. Then the arithmetic degree at x is equal to the dynamical degree of f, i.e., $\alpha_f(x) = \delta_f$.

- KSC is known for abelian varieties ([KS16b],[Sil17]) or f is polarized ([KS16b]).

Kawaguchi-Silverman Conjecture

Conjecture (Kawaguchi-Silverman Conjecture = KSC)

Let $f : X \to X$ be a surjective endomorphism of a projective variety X defined over $\overline{\mathbb{Q}}$. Let $x \in X(\overline{\mathbb{Q}})$, and suppose that the (forward) orbit $O_f(x) = \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X. Then the arithmetic degree at x is equal to the dynamical degree of f, i.e., $\alpha_f(x) = \delta_f$.

- KSC is known for abelian varieties ([KS16b],[Sil17]) or f is polarized ([KS16b]).
- Suppose that there is an equivariant fibration $\pi : (X, f) \dashrightarrow (Y, g)$ such that $\delta_f = \delta_g$. If KSC holds for (Y, g), then KSC holds for (X, f).

Main Results

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an extremal Fano contraction $\pi : X \to Y$ to an abelian variety Y of positive dimension. Suppose f admits a Zariski dense orbit and $\delta_f > \delta_{f|_Y}$. Then the following hold.

1. *The ramification divisor satisfies $f^* R_f \equiv \delta_f R_f$.*

2. *There exists an f-equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f-Iitaka fibration of R_f, such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f-polarized.*

Main Results

Theorem

*Let f be a surjective endomorphism of a smooth projective variety X admitting an f-equivariant **smooth** extremal Fano contraction $\pi : X \to Y$ with $\rho(Y) = 1$. Suppose $\delta_f > \delta_{f|_Y} = 1$. Then the following hold.*

1. *The ramification divisor satisfies $f^* R_f \equiv \delta_f R_f$.*

2. *There exists an f-equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f-Iitaka fibration of R_f, such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f-polarized.*

Main Results

Corollary

KSC holds for any smooth projective variety X admitting an extremal Fano contraction to an abelian variety.

Corollary

KSC holds for any $\mathbb{P}^n$-bundle over either a Q-abelian variety or a smooth projective variety of Picard number one.

Dynamical Iitaka fibration

Theorem ([MZ23, Theorem 4.6])

Let f be a surjective endomorphism of a normal projective variety X. Let D be a $\mathbb{Q}$-Cartier divisor with $\kappa_f(X, D) \geq 0$. Then there is an f-equivariant dominant rational map $\phi_{f,D} : X \dashrightarrow Y$ to a normal projective variety Y (with $f|_Y$ a surjective endomorphism too) satisfying the following conditions.

1. $\dim Y = \kappa_f(X, D)$.
2. *Let Γ be the graph of $\phi_{f,D}$. Then the induced projection $\Gamma \to Y$ is equi-dimensional.*
3. $\phi_{f,D}$ *is birational to the Iitaka fibration of any $D' \in V_f(D)$ with $\kappa(X, D') = \kappa_f(X, D)$.*

Dynamical Iitaka dimension

Definition (Dynamical Iitaka dimension)

Let f be a surjective endomorphism of a normal projective variety X and D a $\mathbb{Q}$-Cartier divisor. Denote by $V_f(D)$ the subspace of $\mathrm{Pic}_{\mathbb{Q}}(X)$ spanned by $D_i := (f^*)^i(D)$ with $i \in \mathbb{Z}$. Note that $V_f(D)$ is finite dimensional (cf. [MZ22, Proposition 3.7]). We define the *dynamical f-Iitaka dimension* of D as

$$\kappa_f(X, D) := \max\{\kappa(X, D') \mid D' \in V_f(D)\}.$$

Dynamical Iitaka dimension

Definition (Dynamical Iitaka dimension)

Let f be a surjective endomorphism of a normal projective variety X and D a $\mathbb{Q}$-Cartier divisor. Denote by $V_f(D)$ the subspace of $\mathrm{Pic}_{\mathbb{Q}}(X)$ spanned by $D_i := (f^*)^i(D)$ with $i \in \mathbb{Z}$. Note that $V_f(D)$ is finite dimensional (cf. [MZ22, Proposition 3.7]). We define the *dynamical f-Iitaka dimension* of D as

$$\kappa_f(X, D) := \max\{\kappa(X, D') \mid D' \in V_f(D)\}.$$

Note that by ramification formula, we have $R_{f^s} = \sum_{i=0}^{s-1} (f^i)^* R_f$ and hence $\kappa_f(X, R_f) = \kappa(X, R_{f^s})$ for $s \gg 1$.

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

- $\kappa_f(X, R_f) = 0$. Then $f^{-1}(\operatorname{Supp} R_f) = \operatorname{Supp} R_f$ and the restriction

$$f|_{X \setminus \operatorname{Supp} R_f} \colon X \setminus \operatorname{Supp} R_f \to X \setminus \operatorname{Supp} R_f$$

is (quasi-)étale.

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

- $\kappa_f(X, R_f) = 0$. Then $f^{-1}(\mathrm{Supp}\, R_f) = \mathrm{Supp}\, R_f$ and the restriction

$$f|_{X \backslash \mathrm{Supp}\, R_f} \colon X \backslash \mathrm{Supp}\, R_f \to X \backslash \mathrm{Supp}\, R_f$$

  is (quasi-)étale.

- $0 < \kappa_f(X, R_f) < \dim X$. We then have an f-equivariant dominant rational map

$$\varphi_{f, R_f} \colon X \dashrightarrow Y$$

  with $0 < \dim Y = \kappa_f(X, R_f) < \dim X$.

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

- $\kappa_f(X, R_f) = 0$. Then $f^{-1}(\operatorname{Supp} R_f) = \operatorname{Supp} R_f$ and the restriction

$$f|_{X \setminus \operatorname{Supp} R_f} \colon X \setminus \operatorname{Supp} R_f \to X \setminus \operatorname{Supp} R_f$$

is (quasi-)étale.

- $0 < \kappa_f(X, R_f) < \dim X$. We then have an f-equivariant dominant rational map

$$\varphi_{f, R_f} \colon X \dashrightarrow Y$$

with $0 < \dim Y = \kappa_f(X, R_f) < \dim X$.

- $\kappa_f(X, R_f) = \dim X$. In this case, R_{f^s} is big when $s \gg 1$.

Dynamical Iitaka Program for ramification divisor

Question

Let $f : X \to X$ be a surjective endomorphism of a smooth projective variety with $\varphi_{f,R_f} : X \dashrightarrow Y$ the induced dynamical Iitaka fibration. Suppose $\kappa_f(X, R_f) > 0$. When does the equality of first dynamical degrees $\delta_f = \delta_{f|_Y}$ hold and whether f splits into $f|_Y$ and a quasi-étale endomorphism?

Dynamical Iitaka Program for ramification divisor

Question

Let $f : X \to X$ be a surjective endomorphism of a smooth projective variety with $\varphi_{f,R_f} : X \dashrightarrow Y$ the induced dynamical Iitaka fibration. Suppose $\kappa_f(X, R_f) > 0$. When does the equality of first dynamical degrees $\delta_f = \delta_{f|_Y}$ hold and whether f splits into $f|_Y$ and a quasi-étale endomorphism?

Question ([MZ23, Question 6.9])

Let $f : X \to X$ be a surjective endomorphism of a smooth projective variety. Suppose R_f is big. Is f int-amplified?

Dynamical Iitaka fibration

A special case of the f-Iitaka fibration has been studied in [MZ22, Theorem 7.8].

Theorem ([MZ23, Theorem 4.8])

Let $f : X \to X$ be a surjective endomorphism of a projective variety X. Let D be a $\mathbb{Q}$-Cartier divisor such that $f^*D \equiv qD$ for some $q > 1$ and $\kappa(X, D) > 0$. Let $\phi_{f,D} : X \dashrightarrow Y$ be the f-Iitaka fibration of D. Then $f|_Y$ is q-polarized.

Dynamical Iitaka fibration

A special case of the f-Iitaka fibration has been studied in [MZ22, Theorem 7.8].

Theorem ([MZ23, Theorem 4.8])

Let $f : X \to X$ be a surjective endomorphism of a projective variety X. Let D be a $\mathbb{Q}$-Cartier divisor such that $f^*D \equiv qD$ for some $q > 1$ and $\kappa(X, D) > 0$. Let $\phi_{f,D} : X \dashrightarrow Y$ be the f-Iitaka fibration of D. Then $f|_Y$ is q-polarized.

By the dynamical Iitaka fibration, we hope to show that $f^*R_f \equiv \delta_f R_f$ and $0 < \kappa_f(X, R_f) < \dim X$ under the assumption of main theorems.

Decomposition of cones

First we show that $f^* R_f \equiv \delta_f R_f$. It follows from the following decomposition of the cone.

Decomposition of cones

Theorem (Cone Decomposition)

Let $\pi : X \to Y$ be an f-equivariant surjective morphism to a $\mathbb{Q}$-factorial normal projective variety Y with $\rho(X) = \rho(Y) + 1$ and $\delta_f > \delta_{f|_Y}$. Suppose further either one of the following two assumptions holds.

1. $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$.

2. X and Y are smooth and π is equi-dimensional.

Then we have the decompositions:

$$\mathrm{Nef}(X) = \pi^* \mathrm{Nef}(Y) \oplus \mathbb{R}_{\geq 0} D, \quad \mathrm{Psef}(X) = \pi^* \mathrm{Psef}(Y) \oplus \mathbb{R}_{\geq 0} D,$$

for some nef and π-ample Cartier divisor D with $f^* D \equiv \delta_f D$.

Sketch of proof of the decomposition of cones

- By the Perron-Frobenius Theorem, there is a nef and π-ample Cartier divisor D such that $f^*D \equiv \delta_f D$. We show that the numerical dimension $\nu(X, D) = \dim X - \dim Y$. Since it is π-ample, we have $\nu(X, D) \geq \dim X - \dim Y$.
  If $D^{\dim X - \dim Y + 1} \not\equiv_w 0$, then $\pi_* D^{\dim X - \dim Y + 1}$ will be a δ_f eigenvector of $f^*|_{N^1(Y)}$ by relative dynamical degree formula. This implies $\delta_f = \delta_{f|_Y}$, contradicting the assumption.

Sketch of proof of the decomposition of cones

- By the Perron-Frobenius Theorem, there is a nef and π-ample Cartier divisor D such that $f^*D \equiv \delta_f D$. We show that the numerical dimension $\nu(X, D) = \dim X - \dim Y$. Since it is π-ample, we have $\nu(X, D) \geq \dim X - \dim Y$.
  If $D^{\dim X - \dim Y + 1} \not\equiv_w 0$, then $\pi_* D^{\dim X - \dim Y + 1}$ will be a δ_f eigenvector of $f^*|_{N^1(Y)}$ by relative dynamical degree formula. This implies $\delta_f = \delta_{f|_Y}$, contradicting the assumption.

- Under the assumption that $\mathrm{Nef}(Y) = \mathrm{Psef}(Y)$. Note that $\pi^* \mathrm{Nef}(Y) + \mathbb{R}_{\geq 0} D$ is a closed subcone of $\mathrm{Nef}(X)$ and it also spans $N^1(X)$. Then it suffices to show $D + \pi^* B$ is not big for any nef and non-ample $\mathbb{R}$-Cartier divisor B on Y. Note that $(D + \pi^* B)^{\dim X} = 0$ because $D^i \equiv_w 0$ for any $i > \dim X - \dim Y$ and $(\pi^* B)^j \equiv_w 0$ for any $j \geq \dim Y$.

Proof of the decomposition of cones

- Under the assumption that X and Y are smooth and π is equi-dimensional.

  Let $B \in \mathrm{Psef}(X)$ and write $B = \pi^* B_Y + aD$. Let C be a nef 1-cycle on Y. Note that $\pi^* C$ is a nef $(d+1)$-cycle on X by the projection formula. Then $B \cdot D^d \cdot \pi^* C \geq 0$. So we have

  $$\pi_*(B \cdot D^d \cdot \pi^* C) = \pi_*(\pi^*(B_Y \cdot C) \cdot D^d) + \pi_*(\pi^* C \cdot D^{d+1})$$
  $$= B_Y \cdot C \cdot \pi_*(D^d) \geq 0$$

  and hence B_Y is pseudo-effective. This gives

  $$\mathrm{Psef}(X) = \pi^* \mathrm{Psef}(Y) + \mathbb{R}_{\geq 0} D$$

  and the nef case is similar.

Polarization of R_f

Then we can show that $f^* R_f \equiv \delta_f R_f$ by the cone decomposition.

Polarization of R_f

Write

$$K_X \equiv \pi^* B + aD$$

for some $\mathbb{Q}$-Cartier divisor B on Y and $a < 0$. Since Y is an abelian variety, by the Cone theorem, we can show that B is nef.

Polarization of R_f

Write

$$K_X \equiv \pi^* B + aD$$

for some $\mathbb{Q}$-Cartier divisor B on Y and $a < 0$. Since Y is an abelian variety, by the Cone theorem, we can show that B is nef.
By the ramification formula, we have

$$R_f = K_X - f^* K_X \equiv \pi^*(B - g^* B) + a(\delta_f - 1)D$$

where $g = f|_Y$. Then $\Delta = B - g^* B$ is pseudo-effective by the decomposition of $\operatorname{Psef}(X)$.

Polarization of R_f

Write

$$K_X \equiv \pi^* B + aD$$

for some $\mathbb{Q}$-Cartier divisor B on Y and $a < 0$. Since Y is an abelian variety, by the Cone theorem, we can show that B is nef.

By the ramification formula, we have

$$R_f = K_X - f^* K_X \equiv \pi^* (B - g^* B) + a(\delta_f - 1)D$$

where $g = f|_Y$. Then $\Delta = B - g^* B$ is pseudo-effective by the decomposition of $\mathrm{Psef}(X)$.

If $\Delta \not\equiv 0$, we can assume that Δ is an integral Cartier divisor. Fix an ample Cartier divisor H on Y. We have

$$B \cdot H^{\dim X - 1} \geq (B - (g^{n+1})^* B) \cdot H^{\dim X - 1} = \sum_{i=0}^{n} (g^i)^* \Delta \cdot H^{\dim X - 1} \geq n$$

Toric fibrations

The equation $f^*R_f \equiv \delta_f R_f$ and $\delta_f > \delta_{f|_Y}$ imply that R_{f^s} is not big for any $s \geq 1$. Then we have that $\kappa_f(X, R_f) < \dim X$. It is sufficient to show that $\kappa_f(X, R_f) > 0$ to proof the main theorems.

Toric fibrations

The equation $f^*R_f \equiv \delta_f R_f$ and $\delta_f > \delta_{f|_Y}$ imply that R_{f^s} is not big for any $s \geq 1$. Then we have that $\kappa_f(X, R_f) < \dim X$. It is sufficient to show that $\kappa_f(X, R_f) > 0$ to proof the main theorems. If $\kappa_f(X, R_f) = 0$, let $D := \operatorname{Supp} R_f$. Then D is prime and $f^{-1}(D) = D$. Since $\rho(X) = \rho(Y) + 1$ and $\delta_f > \delta_g$, we can apply following theorem to show that $(X_y, D|_{X_y})$ is a toric pair for general $y \in Y$.

Toric fibrations

The equation $f^*R_f \equiv \delta_f R_f$ and $\delta_f > \delta_{f|_Y}$ imply that R_{f^s} is not big for any $s \geq 1$. Then we have that $\kappa_f(X, R_f) < \dim X$. It is sufficient to show that $\kappa_f(X, R_f) > 0$ to proof the main theorems. If $\kappa_f(X, R_f) = 0$, let $D := \operatorname{Supp} R_f$. Then D is prime and $f^{-1}(D) = D$. Since $\rho(X) = \rho(Y) + 1$ and $\delta_f > \delta_g$, we can apply following theorem to show that $(X_y, D|_{X_y})$ is a toric pair for general $y \in Y$.

Theorem

Let f be a surjective endomorphism of smooth projective variety X. Let $\pi : X \to Y$ be an f-equivariant Fano fibration. Suppose there is a reduced divisor D such that $K_X + D$ is π-trivial and $f^*D = qD$ for some $q > 1$. Then $(X_y, D|_{X_y})$ is a toric pair for general $y \in Y$.

Sketch of proof of toric fibrations

We use the strategy of [MZ19] to prove the theorem. In fact, the theorem is a relative version of [MZ19, Theorem 1.2].

Sketch of proof of toric fibrations

We use the strategy of [MZ19] to prove the theorem. In fact, the theorem is a relative version of [MZ19, Theorem 1.2].

- [BMSZ18] defines an invariant $c(X_y, D|_{X_y})$, called *complexity*, and show that $(X_y, D|_{X_y})$ is a toric pair if $c(X_y, D|_{X_y}) \leq 0$.

Sketch of proof of toric fibrations

We use the strategy of [MZ19] to prove the theorem. In fact, the theorem is a relative version of [MZ19, Theorem 1.2].

- [BMSZ18] defines an invariant $c(X_y, D|_{X_y})$, called *complexity*, and show that $(X_y, D|_{X_y})$ is a toric pair if $c(X_y, D|_{X_y}) \leq 0$.
- [MZ19] gives a upper bound of $c(X_y, D|_{X_y})$ using the sheaf of logarithmic differential:

$$c(X_y, D|_{X_y}) \leq \dim X_y + h^1(X_y, \mathcal{O}_{X_y}) - h^0(X_y, \hat{\Omega}^1_{X_y}(\log D|_{X_y})).$$

Sketch of proof of toric fibrations

We use the strategy of [MZ19] to prove the theorem. In fact, the theorem is a relative version of [MZ19, Theorem 1.2].

- [BMSZ18] defines an invariant $c(X_y, D|_{X_y})$, called *complexity*, and show that $(X_y, D|_{X_y})$ is a toric pair if $c(X_y, D|_{X_y}) \leq 0$.

- [MZ19] gives a upper bound of $c(X_y, D|_{X_y})$ using the sheaf of logarithmic differential:

$$c(X_y, D|_{X_y}) \leq \dim X_y + h^1(X_y, \mathcal{O}_{X_y}) - h^0(X_y, \hat{\Omega}^1_{X_y}(\log D|_{X_y})).$$

- [GKP16] shows that under some mild condition (X_y is smooth and Fano), a reflexive and μ-slope semi-stable coherent sheaf $\mathcal{F}$ with vanishing intersection numbers

$$c_1(\mathcal{F}) \cdot H^{n-1} = 0, \quad c_1(\mathcal{F})^2 \cdot H^{n-2} = 0, \quad c_2(\mathcal{F}) \cdot H^{n-2} = 0$$

is trivial.

Sketch of proof of toric fibrations

- Notice that for general $y \in Y$, we have

$$(f^* \hat{\Omega}_X^1(\log D))|_{X_y} \cong \hat{\Omega}_X^1(\log D)|_{X_y} \tag{1}$$

since f is étale on $X \setminus D$. On the other hand, we have $f^* H \equiv_Y qH$ for π-ample Cartier divisor H and $q > 1$.

Sketch of proof of toric fibrations

- Notice that for general $y \in Y$, we have

$$(f^* \hat{\Omega}_X^1(\log D))|_{X_y} \cong \hat{\Omega}_X^1(\log D)|_{X_y} \tag{1}$$

  since f is étale on $X \setminus D$. On the other hand, we have
  $f^* H \equiv_Y qH$ for π-ample Cartier divisor H and $q > 1$.

- Then we have

$$\begin{aligned}
q^d c_1(\hat{\Omega}_X^1(\log D)|_{X_y}) \cdot H|_{X_y}^{d-1} &= c_1(\hat{\Omega}_X^1(\log D)|_{X_y}) \cdot (f|_{X_y})^* H^{n-1} \\
&= c_1((f|_{X_y})^* \hat{\Omega}_X^1(\log D)|_{X_y}) \cdot f^* H|_{X_y}^{d-1} \\
&= c_1(\hat{\Omega}_X^1(\log D)|_{X_y}) \cdot (qH)|_{X_y}^{d-1} \\
&= q^{d-1} c_1(\hat{\Omega}_X^1(\log D)|_{X_y}) \cdot H|_{X_y}^{d-1}.
\end{aligned}$$

  This implies the vanishing intersection numbers for
  $\hat{\Omega}_X^1(\log D)|_{X_y}$.

Sketch of proof of toric fibrations

- Let $\mathcal{G} \subset \hat{\Omega}^1_X(\log D)$ be the relative maximal destabilizing subsheaf. By eq. (1), we have

$$(f^*\mathcal{G})|_{X_y} = (f|_{X_y})^*(\mathcal{G}|_{X_{g(y)}})$$
$$\subset (f|_{X_y})^*(\hat{\Omega}^1_X(\log D)|_{X_{g(y)}}) \cong \hat{\Omega}^1_X(\log D)|_{X_y}.$$

By the similar argument, we have

$$\mu_H(\mathcal{G}|_{X_y}) = \frac{c_1(\mathcal{G}|_{X_y}) \cdot H|_{X_y}^{d-1}}{\operatorname{rank} \mathcal{G}|_{X_y}} = 0.$$

Sketch of proof of toric fibrations

- By the fundamental exact sequence

$$0 \to \mathcal{O}_{X_y}^{\oplus m} \to \hat{\Omega}_X^1(\log D)|_{X_y} \to \hat{\Omega}_{X_y}^1(\log D|_{X_y}) \to 0,$$

we have the vanishing intersection numbers for $\hat{\Omega}_{X_y}^1(\log D|_{X_y})$. For every subsheaf $\mathcal{E} \subset \hat{\Omega}_{X_y}^1(\log D|_{X_y})$, we can consider its preimage in $\hat{\Omega}_X^1(\log D)|_{X_y}$. This shows that $\hat{\Omega}_{X_y}^1(\log D|_{X_y})$ is μ-slope semi-stable.

$\kappa_f(X, R_f) > 0$ and base change trick

Now we return to the proof of $\kappa_f(X, R_f) > 0$. We can assume that $\kappa_f(X, R_f) = 0$ and hence $D = \operatorname{Supp} R_f$ is prime and $f^{-1}(D) = D$. By above theorem, $D|_{X_y}$ has $\rho(X_y) + \dim X_y$ components for general $y \in Y$.

$\kappa_f(X, R_f) > 0$ and base change trick

Now we return to the proof of $\kappa_f(X, R_f) > 0$. We can assume that $\kappa_f(X, R_f) = 0$ and hence $D = \operatorname{Supp} R_f$ is prime and $f^{-1}(D) = D$. By above theorem, $D|_{X_y}$ has $\rho(X_y) + \dim X_y$ components for general $y \in Y$.

Consider the equivariant dynamics $(D^\nu, f|_{D^\nu}) \to (Y, g)$, where D^ν is the normalization of D. Let $(\hat{Y}, \hat{g}) \to (Y, g)$ be the Stein factorization of it. Let $\hat{X} := X \times_Y \hat{Y}$. Then we have an equivariant diagram

$$\begin{array}{ccc} \widehat{f} \circlearrowright \widehat{X} & \xrightarrow{\widehat{\pi}} & \widehat{Y} \circlearrowright \widehat{g}. \\ {\scriptstyle p_X} \downarrow & & \downarrow {\scriptstyle p_Y} \\ f \circlearrowright X & \xrightarrow{\pi} & Y \circlearrowright g \end{array}$$

Repeating this process, we can assume that $p_X^* D$ has $\rho(X) + \dim X$ components.

$\kappa_f(X, R_f) > 0$ and base change trick

Suppose that all varieties in the diagram are smooth.

$\kappa_f(X, R_f) > 0$ and base change trick

Suppose that all varieties in the diagram are smooth.
After iteration, we can assume that every component of $p_X^* D$ is
$\hat{f}^{-1}$-invariant.

$\kappa_f(X, R_f) > 0$ and base change trick

Suppose that all varieties in the diagram are smooth.

After iteration, we can assume that every component of $p_X^* D$ is $\hat{f}^{-1}$-invariant.

Since their number are more than $\rho(X_y)$ for general $y \in Y$, we can find

$$\sum a_i D_i - \sum b_j D_j \in \pi^* \operatorname{Pic} Y, \quad a_i, b_j > 0,$$

where D_i, D_j are the components of $p_X^* D$.

$\kappa_f(X, R_f) > 0$ and base change trick

Suppose that all varieties in the diagram are smooth.

After iteration, we can assume that every component of $p_X^* D$ is $\hat{f}^{-1}$-invariant.

Since their number are more than $\rho(X_y)$ for general $y \in Y$, we can find

$$\sum a_i D_i - \sum b_j D_j \in \pi^* \operatorname{Pic} Y, \quad a_i, b_j > 0,$$

where D_i, D_j are the components of $p_X^* D$.

Since $\delta_f > \delta_g$, we have $\kappa(\hat{X}, p_X^* D) > 0$. Then $\kappa(X, D) = \kappa_f(\hat{X}, p_X^* D) > 0$.

Singularities of the base change and abelian varieties

However, in general, we can say nothing about the singularities of $\hat{Y}$ now.

Singularities of the base change and abelian varieties

However, in general, we can say nothing about the singularities of $\hat{Y}$ now.

But in the case that Y is an abelian variety, we can show that $\hat{Y}$ is also an abelian variety and hence smooth. It is sufficient to show that p_Y is étale.

Singularities of the base change and abelian varieties

However, in general, we can say nothing about the singularities of $\hat{Y}$ now.

But in the case that Y is an abelian variety, we can show that $\hat{Y}$ is also an abelian variety and hence smooth. It is sufficient to show that p_Y is étale.

Note that our g is étale and hence the branch divisor B_{p_Y} is g^{-1}-invariant.

However, if an endomorphism of an abelian variety has Zariski dense orbit, then it has no totally invariant prime divisor (by the fact that all periodic subvariety of an abelian variety is a translation of an abelian subvariety).