

Algebraic Dynamics and Dynamical Itaka Theory

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based on the joint work with Sheng Meng and Long Wang

Undergraduate Forum, ICCM 2025, January 7, 2026



SCHOOL OF
MATHEMATICAL SCIENCES
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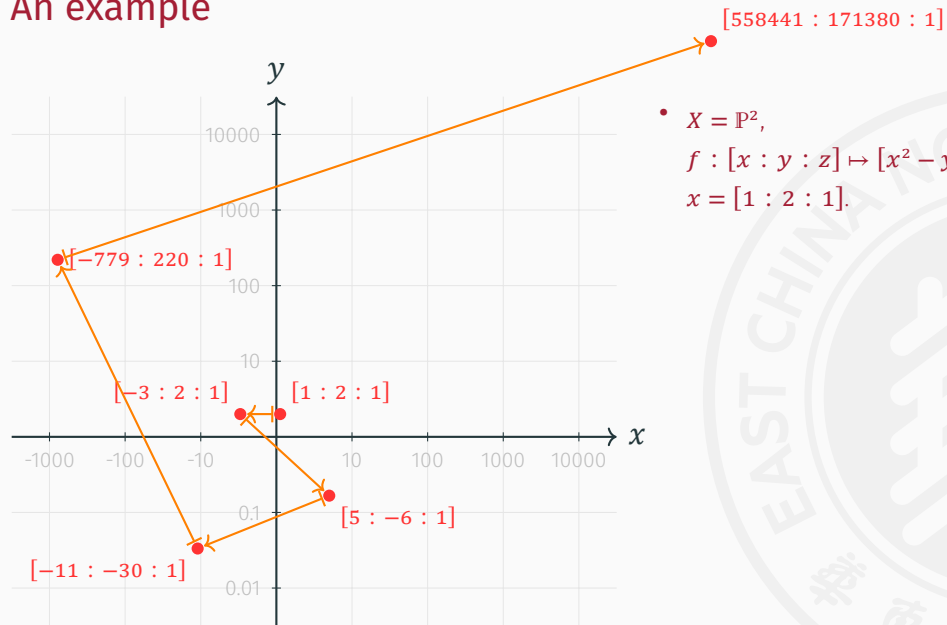
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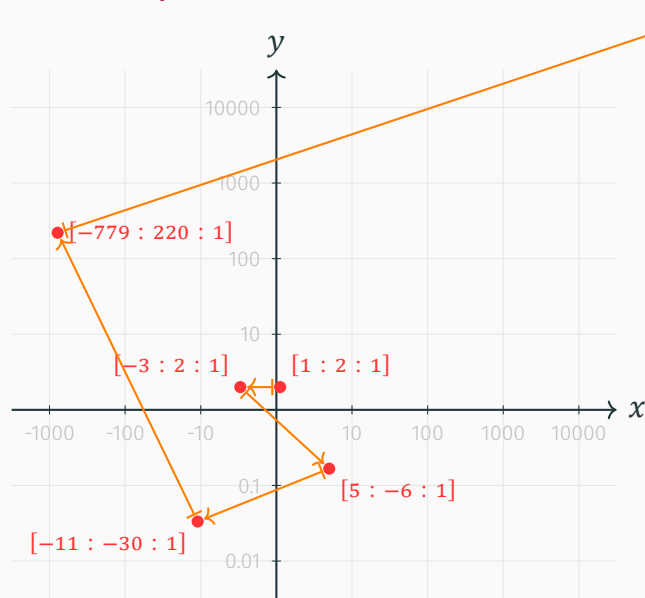
Slogan: GEOMETRY controls ARITHMETIC.

An example



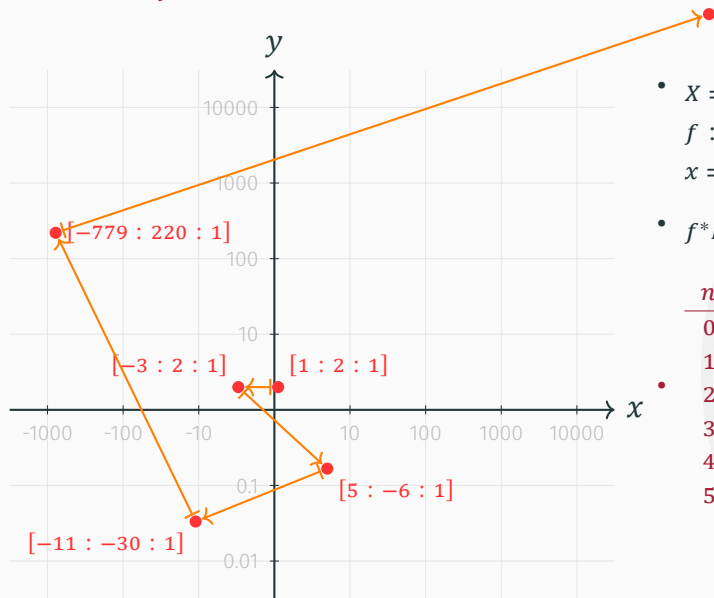
- $X = \mathbb{P}^2$,
 $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
 $x = [1 : 2 : 1]$.

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- $f^*H \sim 2H \Rightarrow \delta_f = 2$.

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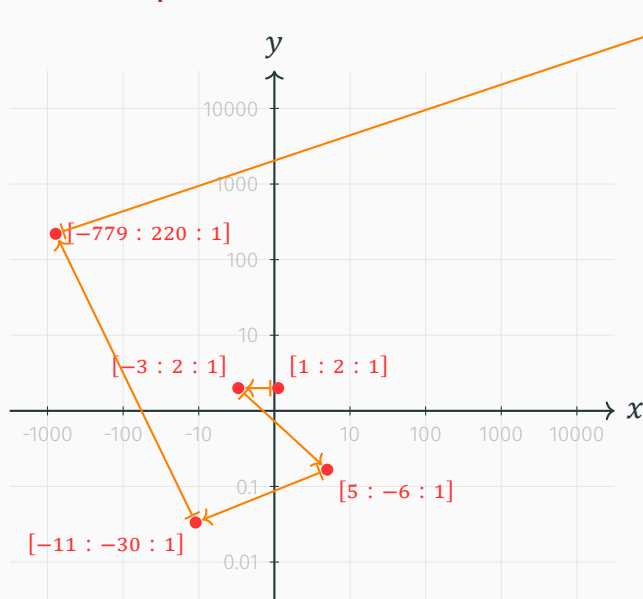


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0	$\log 2$	≈ 0.7
1	$\log 3$	≈ 1.1
2	$\log 6$	≈ 1.8
3	$\log 30$	≈ 3.4
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5	$\log 558441$	≈ 13.2

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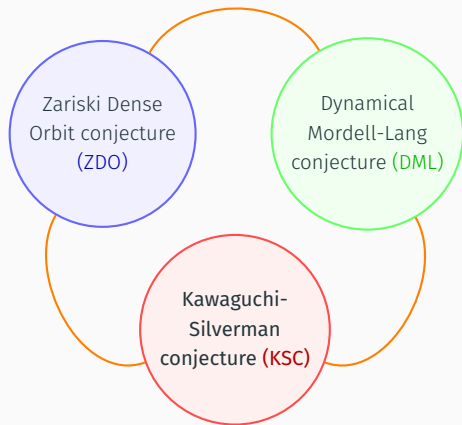
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- It is expected that $\alpha_f(x) = 2$.

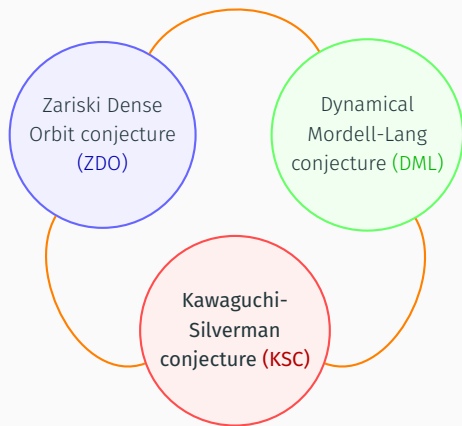
Three orbit conjectures



The KSC relates the growth rate of heights along orbits to dynamical degrees.



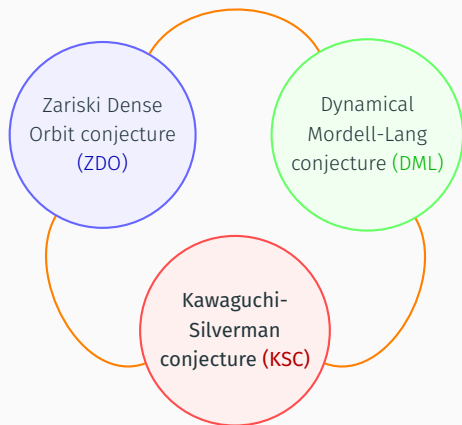
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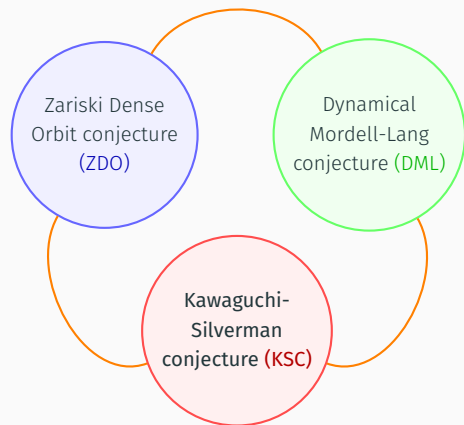
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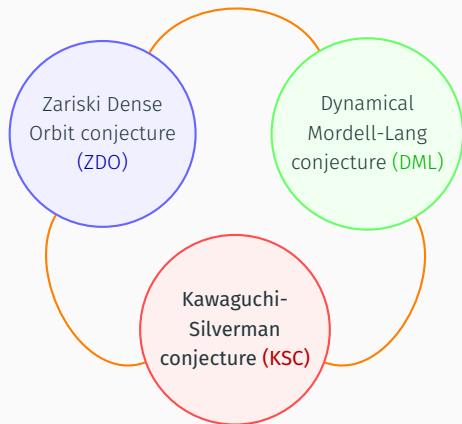
The ZDO states that either
 $\exists x$ with $\overline{O_f(x)} = X$,
or f descends to identity after iteration.

Three orbit conjectures

Main known cases:



Three orbit conjectures



Main known cases:

Smooth projective surfaces [Matsuzawa-Sano-Shibata];

Quasi-projective surfaces (assume DML) [Wang];

Smooth projective 3folds with $\deg f > 1$ [Meng-Zhang];

Abelian varieties [Kawaguchi-Silverman];

Hyperkähler manifolds [Lesieutre-Santriano];

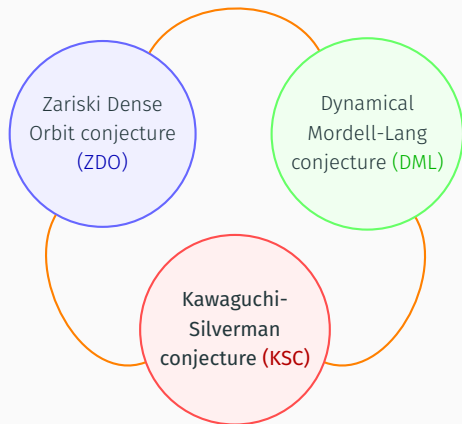
Mori dream spaces [Matsuzawa];

Polarized endomorphisms [Kawaguchi-Silverman];

Int-amplified endomorphisms [Meng-Zhong];

...

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Main known cases:

Étale case

Endomorphisms of \mathbb{A}^2

Projective surfaces in char $p > 0$

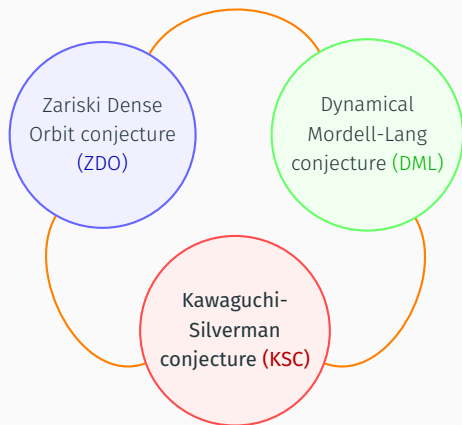
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Over \mathbb{C} (uncountable field)

Existence of infinity orbits

Projective surfaces

Automorphisms of threefolds with positive entropy

[Matsuzawa-Xie];

...

[Bell-Ghioca-Tucker];

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[Xie-Yang];

[Amerik-Campana];

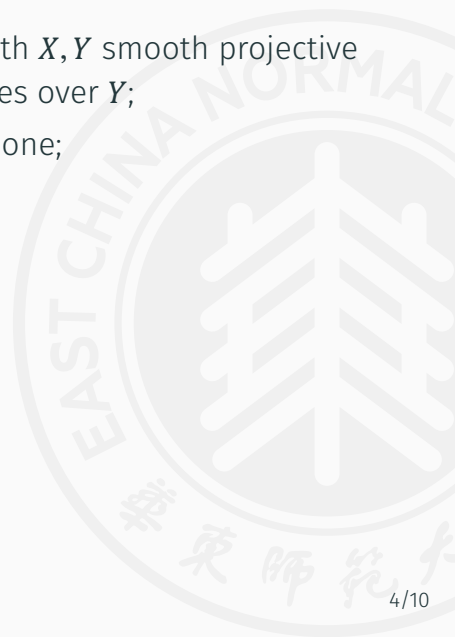
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Main results

Settings:

- $\pi : (X, f) \rightarrow (Y, g)$ smooth Fano fibration with X, Y smooth projective and $\rho(X) = \rho(Y) + 1$, for example, \mathbb{P}^n -bundles over Y ;
- Y is an abelian variety or of picard number one;



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Essentially generalize the following known results:

- [Li-Matsuzawa 2021, Theorem 4.1], projective bundles on smooth Fano varieties of Picard number one;
- [Nasserden-Zotine 2021, Theorem 1.4], projective bundles on elliptic curves.

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If $\delta_f > \delta_g$, under some allowable extra conditions, we have another fibration by dynamical litaka theory.

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- Y is an abelian variety or of picard number one;
- f has a Zariski dense orbit;
- $\delta_f > \delta_g$;
- if Y is of picard number one, then $\delta_g = 1$.

Theorem: [Meng-Wang-Y]

There is equivariant fibrations

$$f \circ X \overset{\varphi_f, R_f}{\dashrightarrow} Z \circ h$$

with $\dim Z > 0$ and h is δ_f -polarized and hence $\delta_h = \delta_f$.

Dynamical Iitaka Theory

Theorem: [Meng-Zhang]

Suppose that $f^*R_f \equiv qR_f$ for some $q > 1$. Then there exists an f -equivariant fibration (dynamical Iitaka fibration associated to R_f)

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Coarse classification of
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$\kappa(X, K_X)$	Typical geometry
$-\infty$	uniruled
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Dynamical analogue: classify (X, f) via $\kappa_f(X, R_f)$.

$\kappa_f(X, R_f)$	Typical dynamics
0	f is log-étale
$0 < \kappa_f(X, R_f) < \dim X$	fibrations type
$\dim X$	R_{f^s} is big for $s \gg 0$

Technique: decomposition of cones

Theorem: [Meng-Wang-Y]

$\pi : (X, f) \rightarrow (Y, g)$ flat;
 X, Y smooth projective;
 $\delta_f > \delta_g; \rho(X) = \rho(Y) + 1.$

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$\exists D$ nef with $f^*D \equiv \delta_f D$ s.t.
 $\text{Psef}^1(X) = \pi^* \text{Psef}^1(Y) \oplus \mathbb{R}_{\geq 0} D,$
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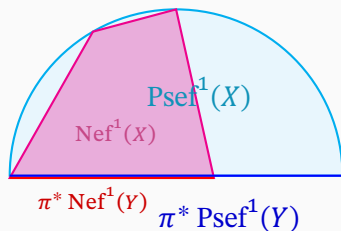
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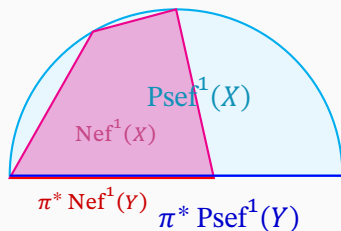
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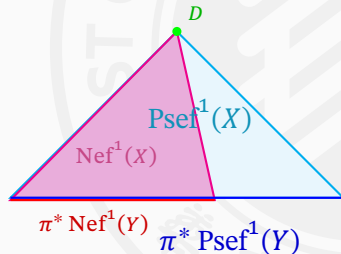
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Cones of fibration with dynamical restrictions:



Technique: criterion for toric bundles

Theorem: [Meng-Wang-Y]

Let $\pi : (X, f) \rightarrow (Y, g)$ be a Fano fibration with X smooth. Suppose that \exists reduced divisor D on X with $f^{-1}(D) = D, f^*D \sim qD$ for some $q > 1$ and $K_X + D \equiv_{\pi} 0$. Then (X_y, D_y) is a toric pair for general $y \in Y$.

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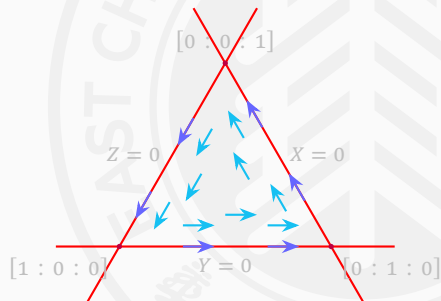
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The right sides is an example in the absolute case.



$$f : \mathbb{P}^2 \rightarrow \mathbb{P}^2,$$

$$[X : Y : Z] \mapsto [X^q : Y^q : Z^q];$$

$$D = \{XYZ = 0\}.$$

Further questions

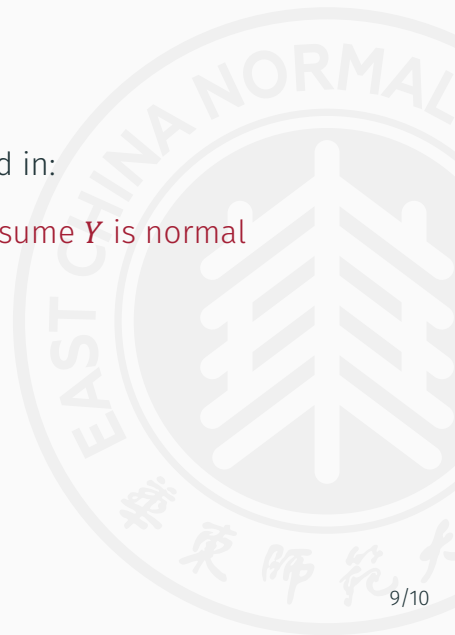
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- weaken the conditions on Y : can we just assume Y is normal projective?
- deal the case R_f is big in the dynamical Iitaka theory.

Thank You!

