Algebraic Dynamics and Dynamical Iitaka Theory

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based on the joint work with Sheng Meng and Long Wang

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Kawaguchi-Silverman Conjecture

Work over $\overline{\mathbb{Q}}$. X: smooth projective variety, $f : X \to X$: surjective endomorphism. H: ample divisor on X. $h : X(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 1}$: a height function associated to H.

Conjecture: Kawaguchi-Silverman Conjecture = KSC

If the orbit $O_f(x) := \{f^n(x) \,|\, n \geq 0\}$ is Zariski dense in X, then

$$\alpha_f(x) = \delta_f.$$

here,

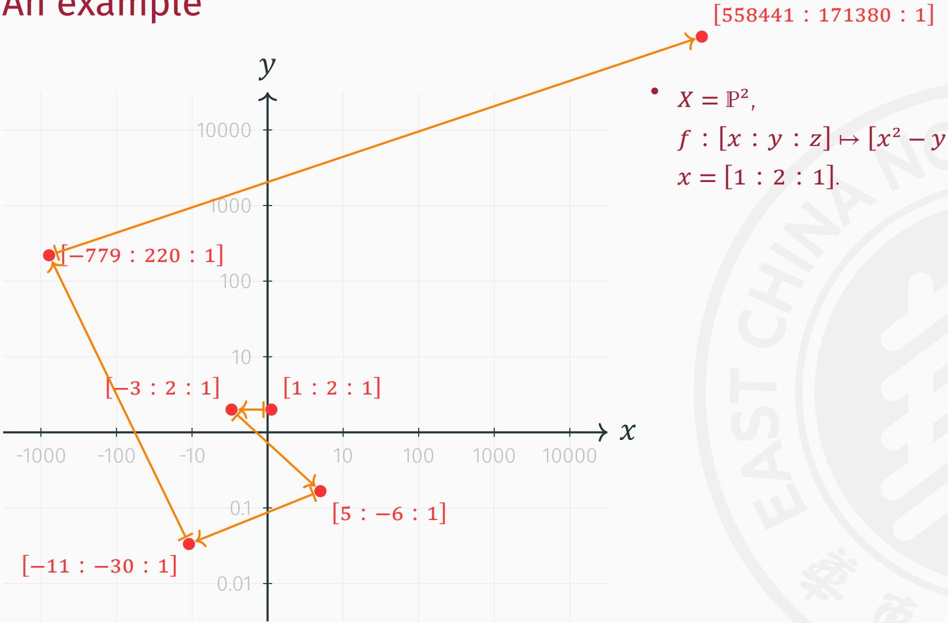
$$\alpha_f(x) := \lim_{n\to\infty} h(f^n(x))^{1/n},$$

arithmetic invariant at x,

$$\delta_f := \lim_{n\to\infty} ((f^n)^*H \cdot H^{\dim X - 1})^{1/n},$$

geometric invariant of f.

An example



- $X = \mathbb{P}^2$,
  $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
  $x = [1 : 2 : 1]$.

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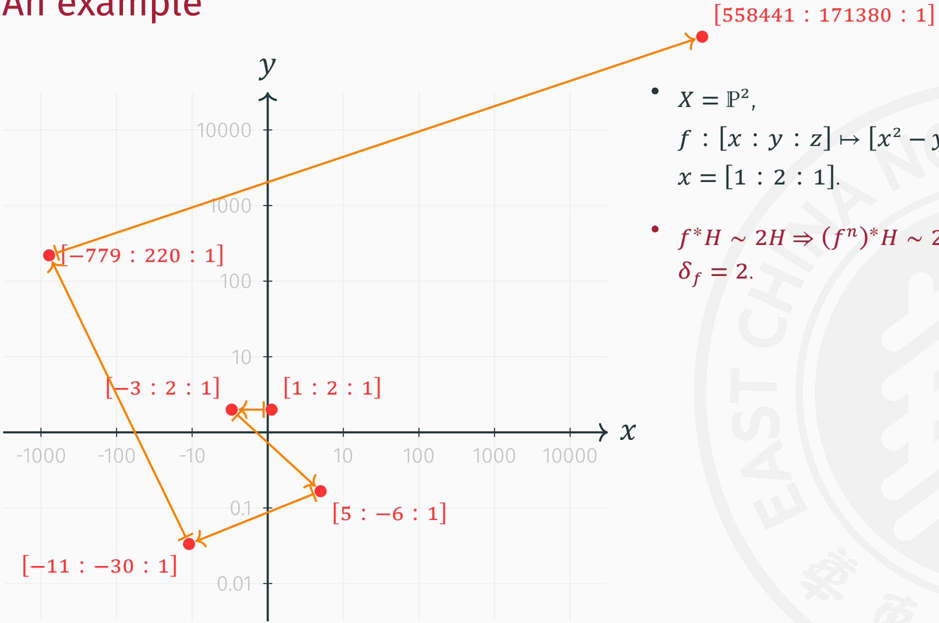

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- $f^*H \sim 2H \Rightarrow (f^n)^*H \sim 2^nH \Rightarrow$
  $\delta_f = 2$.

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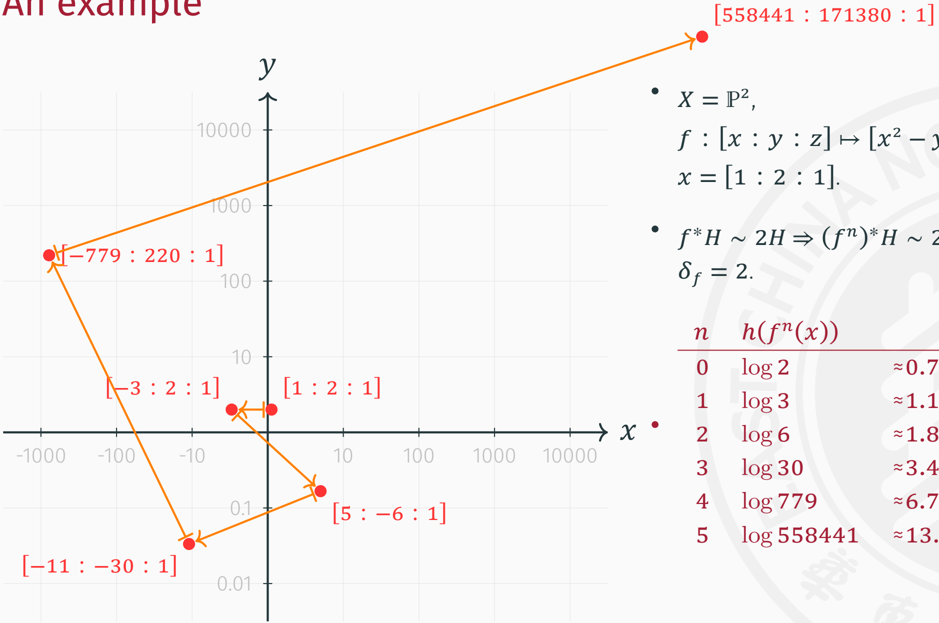

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n	$h(f^n(x))$	
0	$\log 2$	≈ 0.7
1	$\log 3$	≈ 1.1
2	$\log 6$	≈ 1.8
3	$\log 30$	≈ 3.4
4	$\log 779$	≈ 6.7
5	$\log 558441$	≈ 13.2

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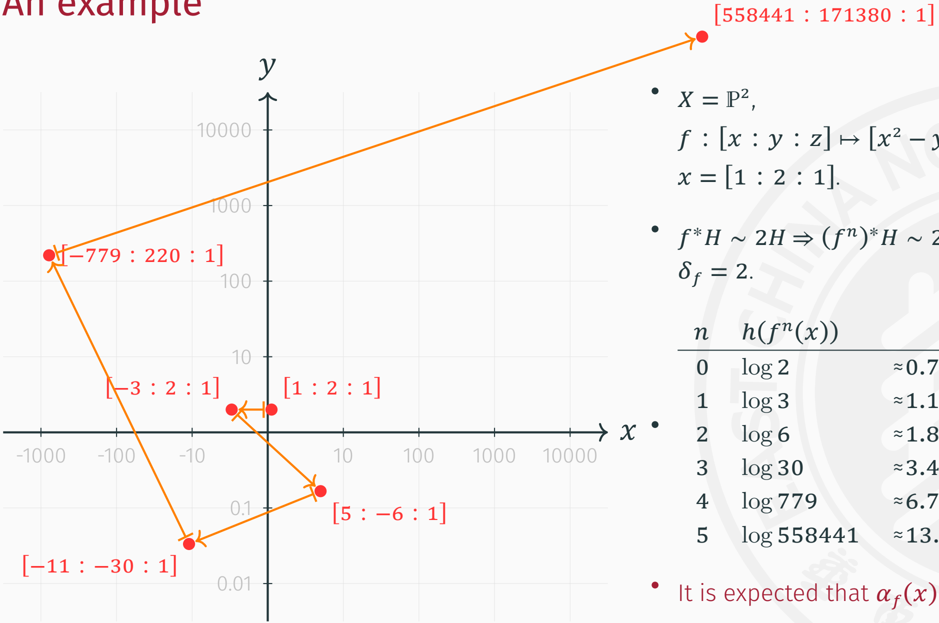

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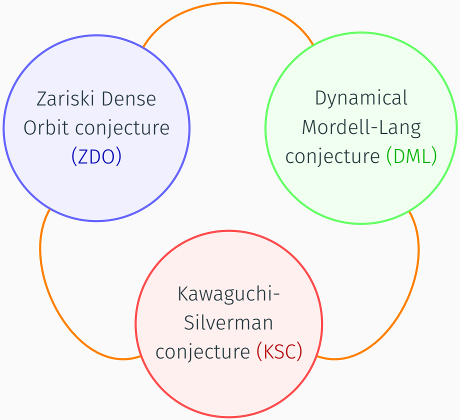
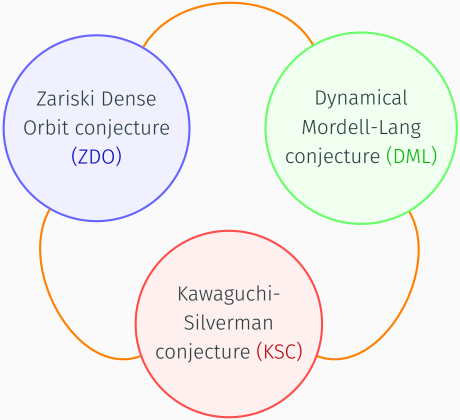
- It is expected that $\alpha_f(x) = 2$.

Three orbit conjectures



The KSC relates the growth rate of heights along orbits to dynamical degrees.

Three orbit conjectures



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for some $y = f^r(x), g = f^s$.

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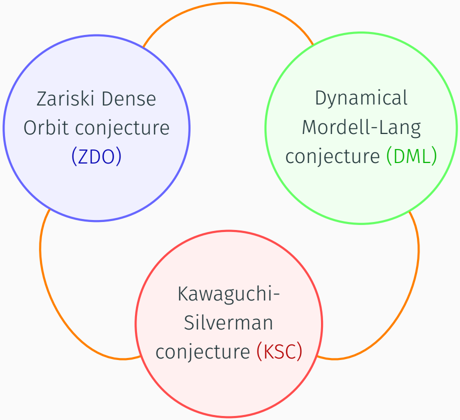

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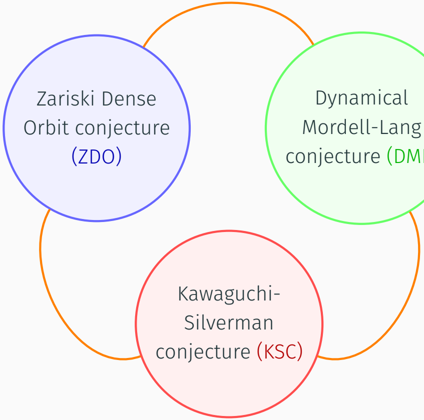
The ZDO states that either

$$\exists x \text{ with } \overline{\mathcal{O}_f(x)} = X,$$

or

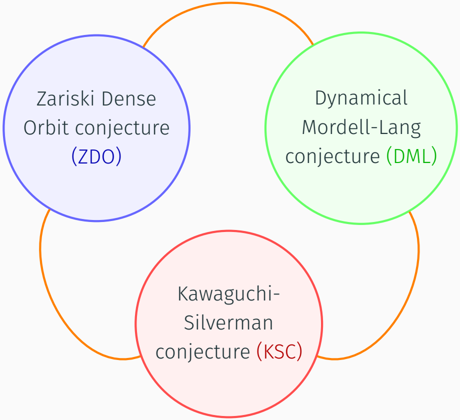
$$\exists\, f^n \circlearrowright X \dashrightarrow Y \circlearrowleft \mathrm{id}_Y.$$

Three orbit conjectures



Main known cases:

Three orbit conjectures



Main known cases:

Endomorphisms of projective surfaces [Matsuzawa-Sano-Shibata];
Birational map of projective surfaces [Wang, to be checked];
Int-amplified endomorphisms [Meng-Zhong];
Non-isomorphic surjective endomorphism on threefolds [Meng-Zhang];

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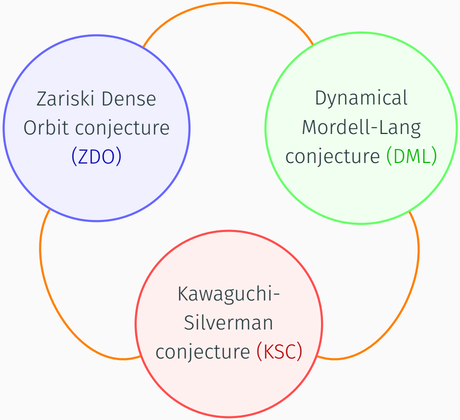

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DML in the étale case [Bell-Ghioca-Tucker] ;
DML for endomorphisms of $\mathbb{A}^2$ [Xie];

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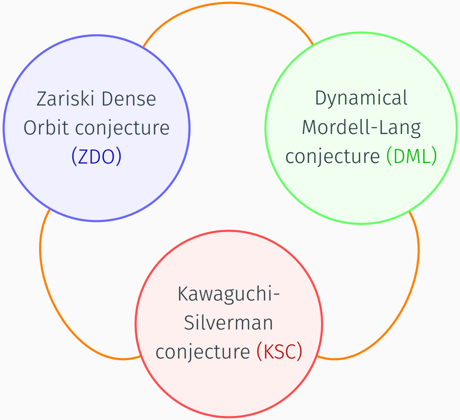

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DML in the étale case [Bell-Ghioca-Tucker] ;
DML for endomorphisms of $\mathbb{A}^2$ [Xie];

The ZDO for endomorphisms of surfaces [Xie,Jia-Xie-Zhang];
The ZDO for automorphisms of threefolds with positive entropy [Matsuzawa-Xie]

Dynamical Iitaka Theory

Coarse classification of varieties via Kodaira dimension $\kappa(X, K_X)$:

$\kappa(X, K_X)$	Typical geometry
$-\infty$	uniruled
0	Calabi-Yau
$0 < \kappa(X) < \dim X$	fibrations type
$\dim X$	general type

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Dynamical analogue: classify (X, f) via $\kappa_f(X, R_f)$.

$\kappa_f(X, R_f)$	Typical dynamics
0	f is log-étale
$0 < \kappa_f(X, R_f) < \dim X$	fibrations type
$\dim X$	R_{f^s} is big for $s \geq 0$

Main results

Settings:

- $\pi : (X, f) \to (Y, g)$ smooth Fano fibration with X, Y smooth projective and $\rho(X) = \rho(Y) + 1$;
- Y is an abelian variety or of picard number one;

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Generalize the following known results:

-

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Sketch of idea: If $\delta_f = \delta_g$, then

KSC for abelian varieties
or polarized endomorphisms $\Rightarrow$ KSC for $(Y, g) \Rightarrow$ KSC for (X, f).

If $\delta_f > \delta_g$, under some extra conditions, we have another fibration by dynamical Iitaka theory.

Main results

Settings:

- $\pi : (X, f) \to (Y, g)$ smooth Fano fibration with X, Y smooth projective and $\rho(X) = \rho(Y) + 1$;
- Y is an abelian variety or of picard number one;
- f has a Zariski dense orbit;
- $\delta_f > \delta_g$;
- if Y is of picard number one, then $\delta_g = 1$.

Theorem: [Meng-Wang-Y]

There is equivariant fibrations

$$f \circlearrowright X \overset{\varphi_{f,R_f}}{\dashrightarrow\!\!\!\!\!\rightarrow} Y \circlearrowleft g$$

with $\dim Y > 0$ and g is q-polarized.

Main results

Theorem: [Meng-Zhang]

Suppose that $f^*R_f \equiv qR_f$ for some $q > 1$. Then there exists an f-equivariant fibration (dynamical Iitaka fibration associated to R_f)

$$f \circlearrowright X \overset{\varphi_{f,R_f}}{\dashrightarrow} Y \circlearrowleft g$$

with $\dim Y = \kappa_f(X, R_f)$ and g is q-polarized.

With help of this result, we only need to show that under our settings,

- $\kappa_f(X, R_f) \geq 1$;
- $f^*R_f \equiv \delta_f R_f$.

Then

KSC for polarized endomorphisms $\Longrightarrow$ KSC for (Y, g) $\Longrightarrow$ KSC for (X, f).

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With help of this result, we only need to show that under our settings,

- $\kappa_f(X, R_f) \geq 1$; $\Longleftarrow$ criterion for toric bundles
- $f^*R_f \equiv \delta_f R_f$. $\Longleftarrow$ decomposition of cones + no rational curve on Y

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KSC for polarized endomorphisms $\implies$ KSC for (Y, g) $\implies$ KSC for (X, f).

Technique: decomposition of cones

Theorem: [Meng-Wang-Y]

$\pi : (X, f) \to (Y, g)$ flat;
X, Y smooth projective;
$\delta_f > \delta_g$;$\rho(X) = \rho(Y) + 1$.

$\Longrightarrow$

$\exists D$ nef with $f^*D \equiv \delta_f D$ s.t.
$\mathrm{Psef}^1(X) = \pi^* \mathrm{Psef}^1(Y) \oplus \mathbb{R}_{\geq 0} D$,
$\mathrm{Nef}^1(X) = \pi^* \mathrm{Nef}^1(Y) \oplus \mathbb{R}_{\geq 0} D$.

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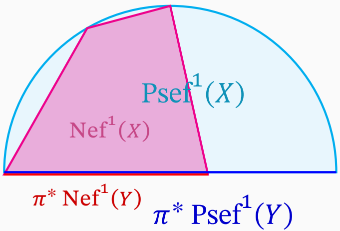
Cones of a general fibration $\pi : X \to Y$:



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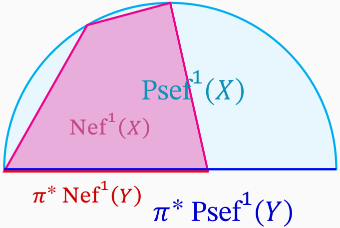
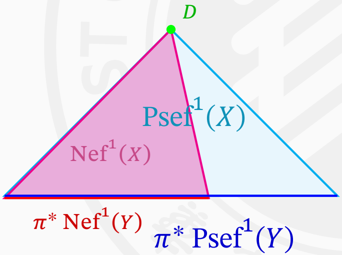


Cones of fibration with dynamical restrictions:

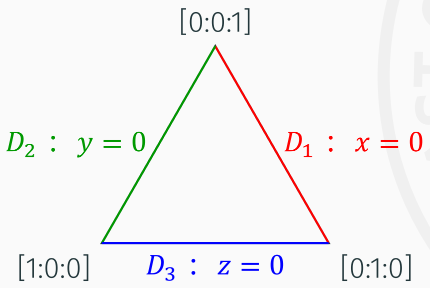


Technique: criterion for toric bundles

Theorem: [Meng-Wang-Y]

Let $\pi : (X, f) \to (Y, g)$ be a Fano fibration with X smooth. Suppose that $\exists$ reduced divisor D on X with $f^{-1}(D) = D, f^*D \sim qD$ for some $q > 1$ and $K_X + D \equiv_\pi 0$. Then $\pi : X \to Y$ is a toric bundle.

$\mathbb{P}^2$ as a toric variety with toric boundary $D = D_1 + D_2 + D_3$:



Further questions

Here are two further questions we are interested in:

- generalize the case $\rho(Y) = 1$ (hence $\rho(X) = 2$) to any X with picard number 2;
- study the decomposition of cones with the decomposition of K_X and R_f and use them to study the case R_f is big.

Thank You!