

Algebraic Dynamics and Dynamical Iitaka Theory

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based on the joint work with Sheng Meng and Long Wang

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SCHOOL OF
MATHEMATICAL SCIENCES
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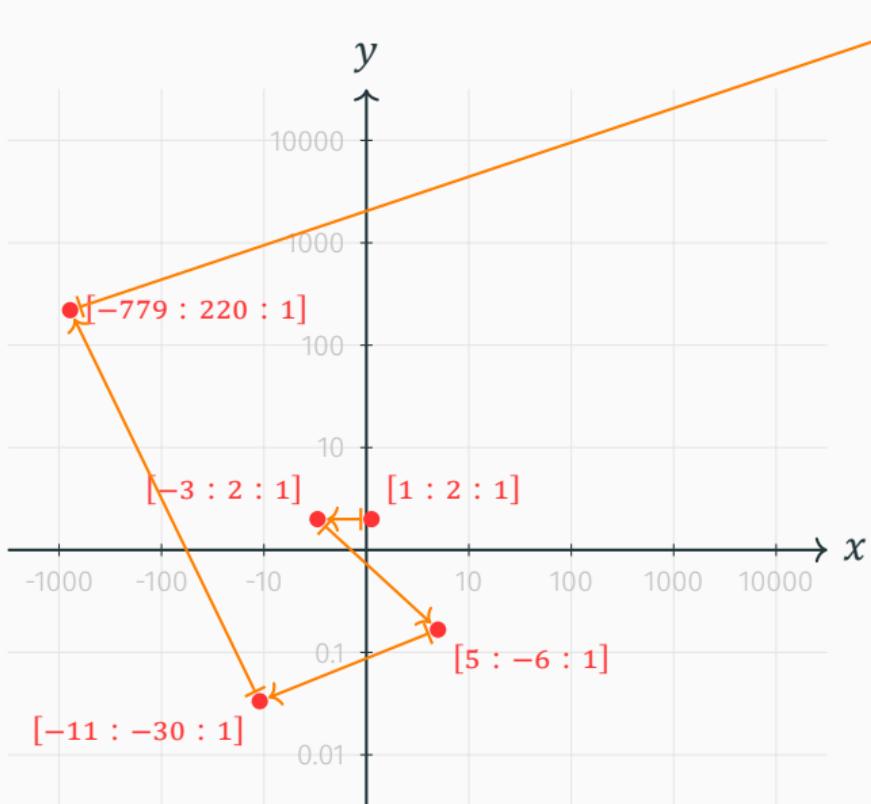
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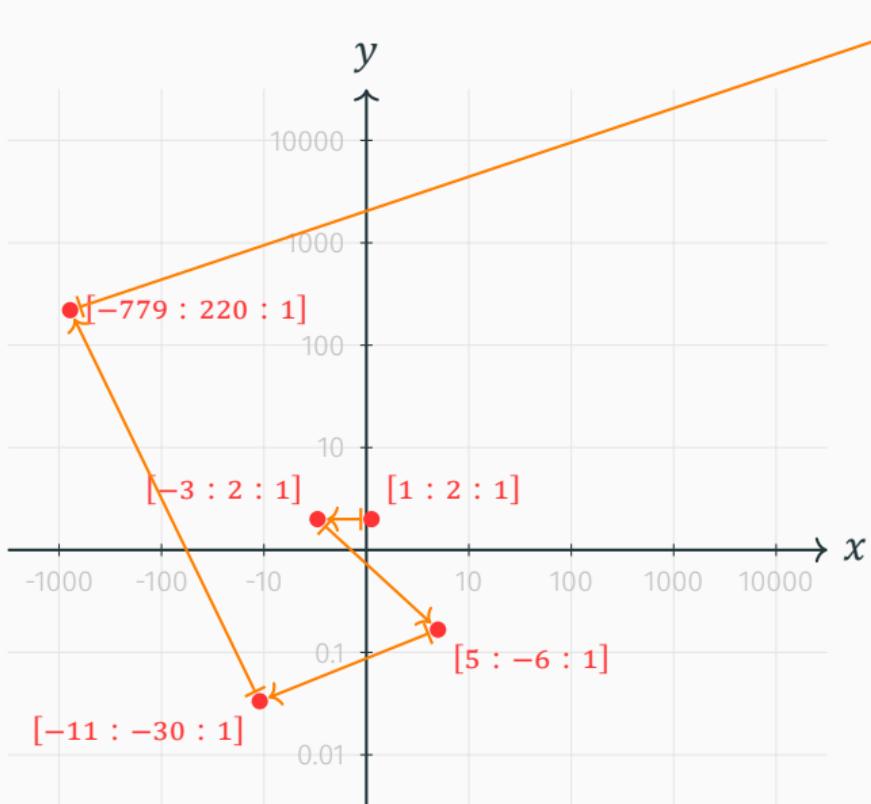
Slogan: GEOMETRY controls ARITHMETIC.

An example



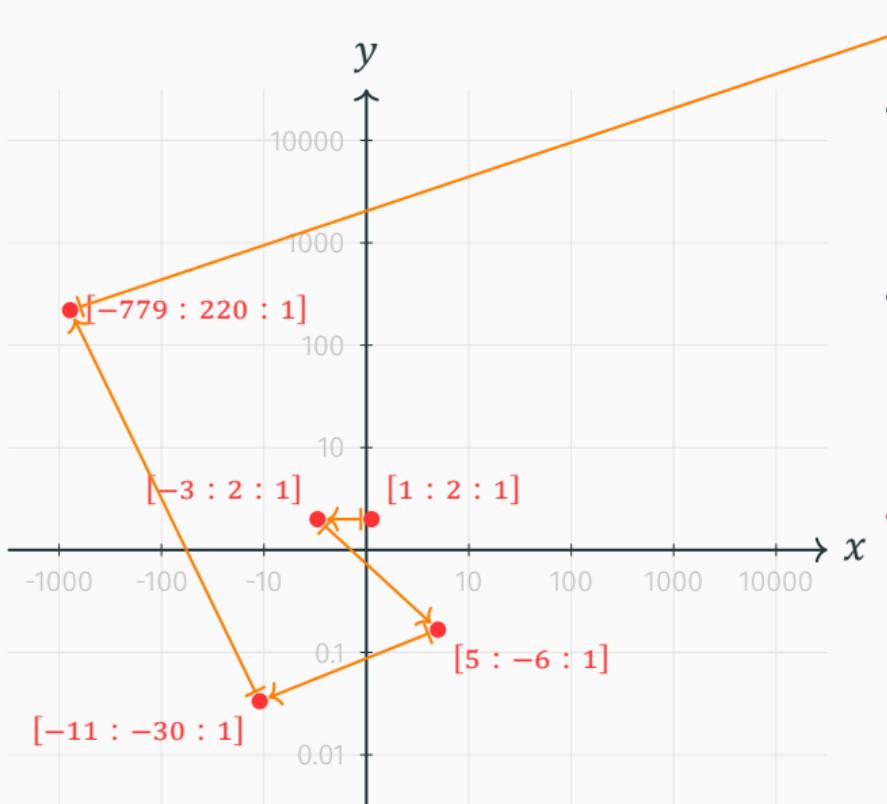
- $X = \mathbb{P}^2$,
- $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
- $x = [1 : 2 : 1]$.

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- $X = \mathbb{P}^2$,
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- $f^*H \sim 2H \Rightarrow \delta_f = 2$.

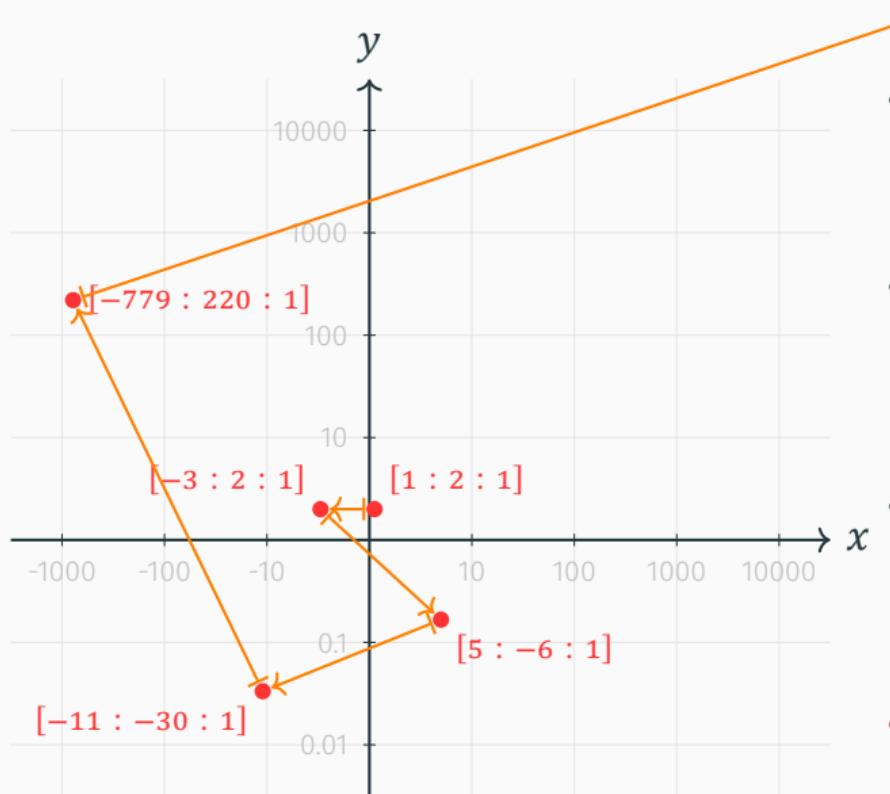
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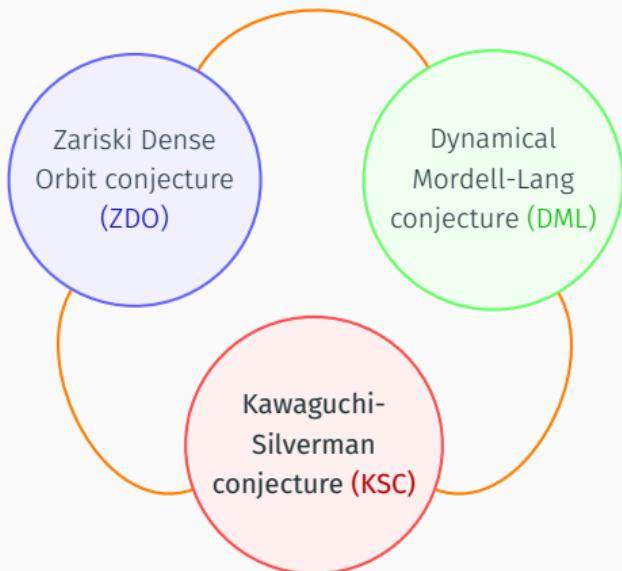
n	$h(f^n(x))$	
0	$\log 2$	≈ 0.7
1	$\log 3$	≈ 1.1
2	$\log 6$	≈ 1.8
3	$\log 30$	≈ 3.4
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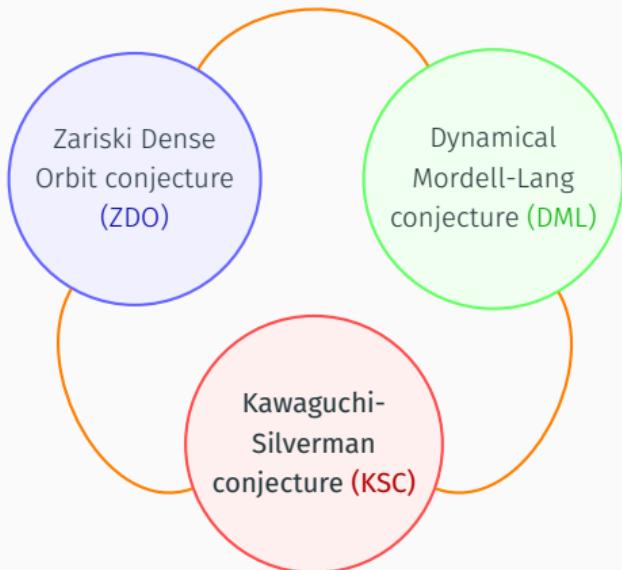
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- It is expected that $\alpha_f(x) = 2$.

Three orbit conjectures



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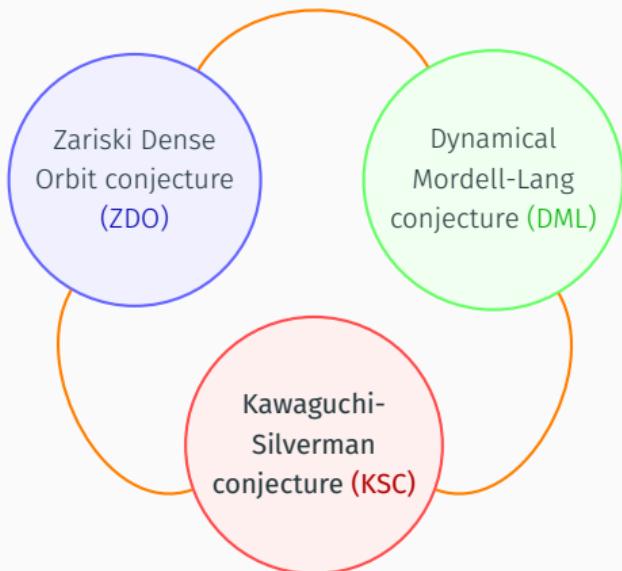
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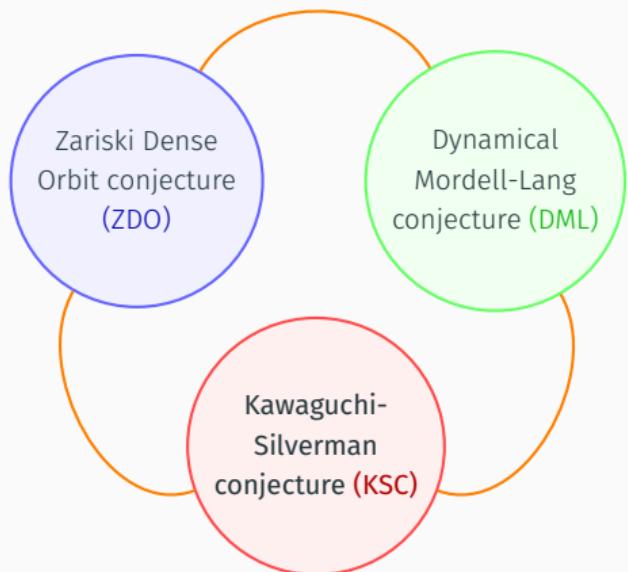
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The ZDO states that either

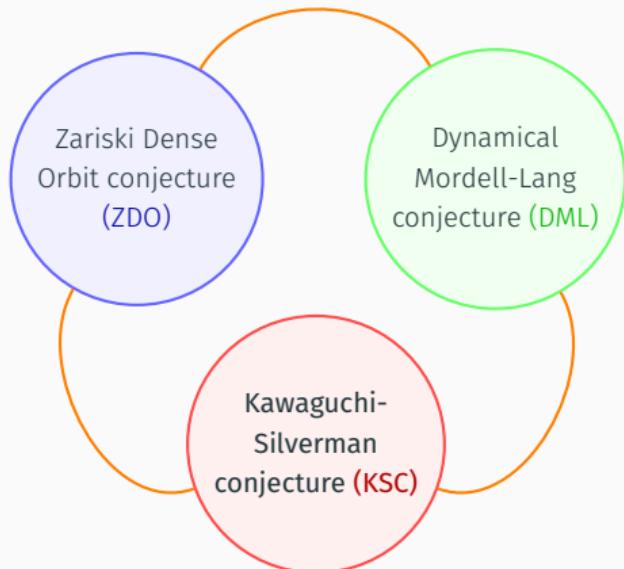
$\exists x$ with $O_f(x) = X$,
or f descends to identity after iteration.

Three orbit conjectures

Main known cases:



Three orbit conjectures



Main known cases:

Smooth projective surfaces [Matsuzawa-Sano-Shibata];

Quasi-projective surfaces (assume DML) [Wang];

Smooth projective 3folds with $\deg f > 1$ [Meng-Zhang];

Abelian varieties

[Kawaguchi-Silverman];

Hyperkähler manifolds

[Lesieutre-Santriano];

Mori dream spaces

[Matsuzawa];

Polarized endomorphisms

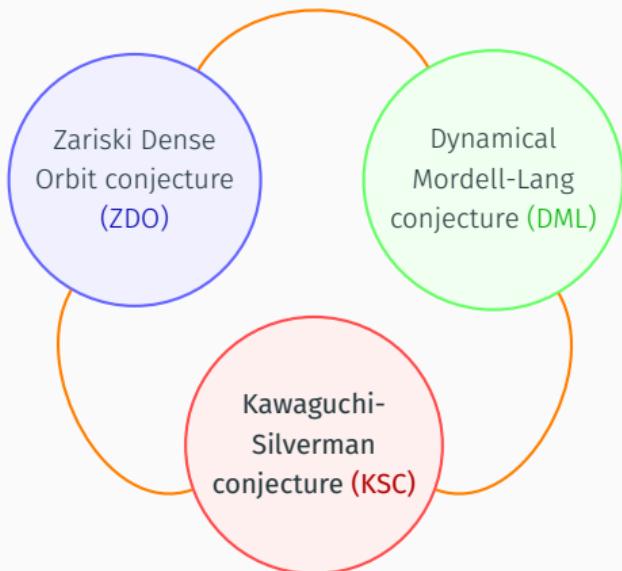
[Kawaguchi-Silverman];

Int-amplified endomorphisms

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Étale case

Endomorphisms of \mathbb{A}^2

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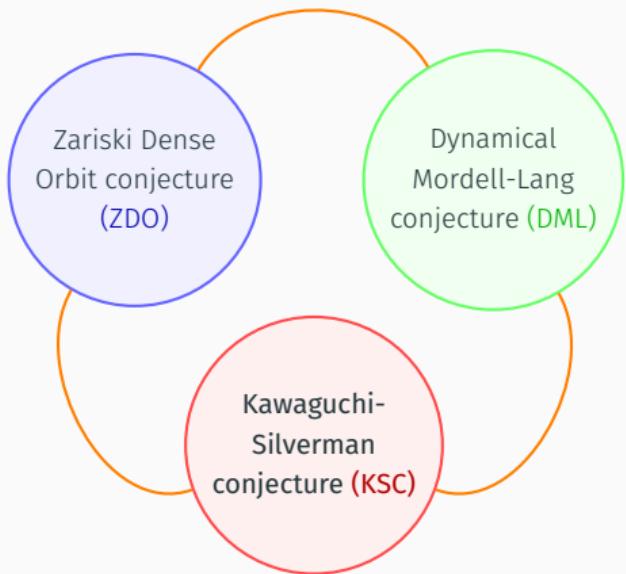
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[Xie-Yang];

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Over \mathbb{C} (uncountable field)

[Amerik-Campana];

Existence of infinity orbits

[Amerik];

Projective surfaces

[Xie,Jia-Xie-Zhang];

Automorphisms of threefolds with positive entropy

[Matsuzawa-Xie];

...

Main results

Settings:

- $\pi : (X, f) \rightarrow (Y, g)$ smooth Fano fibration with X, Y smooth projective and $\rho(X) = \rho(Y) + 1$, for example, \mathbb{P}^n -bundles over Y ;
- Y is an abelian variety or of picard number one;

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Essentially generalize the following known results:

- [Li-Matsuzawa 2021, Theorem 4.1], projective bundles on smooth Fano varieties of Picard number one;
- [Nasserden-Zotine 2021, Theorem 1.4], projective bundles on elliptic curves.

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If $\delta_f > \delta_g$, under some allowable extra conditions, we have another
fibration by dynamical Iitaka theory.

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- Y is an abelian variety or of picard number one;
- f has a Zariski dense orbit;
- $\delta_f > \delta_g$;
- if Y is of picard number one, then $\delta_g = 1$.

Theorem: [Meng-Wang-Y]

There is equivariant fibrations

$$f \circ X \dashrightarrow Z \circ h$$

with $\dim Z > 0$ and h is δ_f -polarized and hence $\delta_h = \delta_f$.

Dynamical Iitaka Theory

Theorem: [Meng-Zhang]

Suppose that $f^*R_f \equiv qR_f$ for some $q > 1$. Then there exists an f -equivariant fibration (dynamical Iitaka fibration associated to R_f)

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Coarse classification of varieties via Kodaira dimension $\kappa(X, K_X)$:

$\kappa(X, K_X)$	Typical geometry
$-\infty$	uniruled
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Dynamical analogue: classify (X, f) via $\kappa_f(X, R_f)$.

$\kappa_f(X, R_f)$	Typical dynamics
0	f is log-étale
$0 < \kappa_f(X, R_f) < \dim X$	fibrations type
$\dim X$	R_{fs} is big for $s \gg 0$

Technique: decomposition of cones

Theorem: [Meng-Wang-Y]

$$\begin{array}{l} \pi : (X, f) \rightarrow (Y, g) \text{ flat;} \\ X, Y \text{ smooth projective;} \\ \delta_f > \delta_g; \rho(X) = \rho(Y) + 1. \end{array} \implies \begin{array}{l} \exists D \text{ nef with } f^*D \equiv \delta_f D \text{ s.t.} \\ \text{Psef}^1(X) = \pi^* \text{Psef}^1(Y) \oplus \mathbb{R}_{\geq 0} D, \\ \text{Nef}^1(X) = \pi^* \text{Nef}^1(Y) \oplus \mathbb{R}_{\geq 0} D. \end{array}$$

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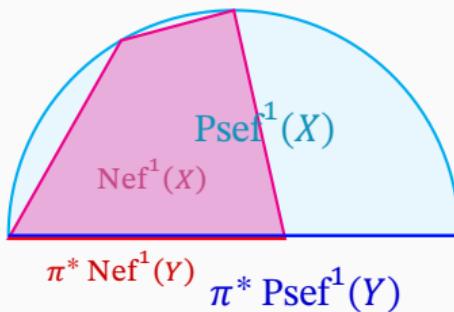
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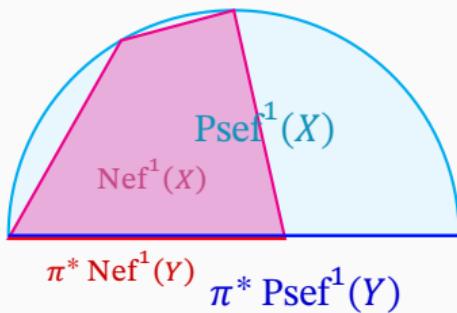
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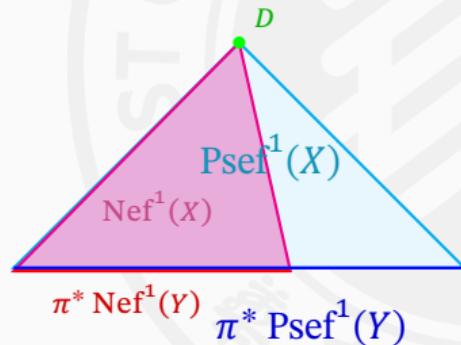
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Cones of fibration with dynamical restrictions:



Technique: criterion for toric bundles

Theorem: [Meng-Wang-Y]

Let $\pi : (X, f) \rightarrow (Y, g)$ be a Fano fibration with X smooth. Suppose that \exists reduced divisor D on X with $f^{-1}(D) = D, f^*D \sim qD$ for some $q > 1$ and $K_X + D \equiv_{\pi} 0$. Then (X_y, D_y) is a **toric pair** for general $y \in Y$.

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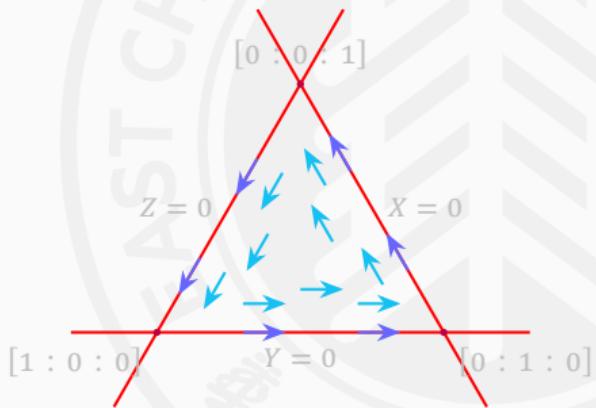
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The right side is an example in the absolute case.



$$f : \mathbb{P}^2 \rightarrow \mathbb{P}^2,$$

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$$D = \{XYZ = 0\}.$$

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- weaken the conditions on Y : can we just assume Y is normal projective?
- deal the case R_f is big in the dynamical Iitaka theory.

Thank You!

