Algebraic Dynamics and Dynamical Iitaka Theory

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Algebraic Dynamics

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The fundamental objects in algebraic dynamics are (X, f), where X is a variety and $f : X \dashrightarrow X$ is a dominant rational self-map.

Here we focus on the case X is projective and f is a surjective endomorphism.

For simplicity, we assume that X is smooth.

Definition (First dynamical degree)

The *first dynamical degree* δ_f of f is defined to be the following limit

$$\delta_f := \lim_{n\to\infty}((f^n)^*H \cdot H^{\dim X-1})^{1/n} \in \mathbb{R}_{\geq 1},$$

where H is an ample Cartier divisor on X.

Definition (Arithmetic degree)

Let $h : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the height function associated to an ample divisor on X. Then for every $x \in X(\Bbbk)$, we define the *arithmetic degree of f at x* by

$$\alpha_f(x) = \lim_{n\to\infty} \max\{h(f^n(x)), 1\}^{1/n} \in \mathbb{R}_{\geq 1}.$$

Conjecture (Kawaguchi-Silverman Conjecture = KSC)

Let $x \in X(\Bbbk)$*, and suppose that the (forward) orbit* $O_f(x) = \{f^n(x) \mid n \geq 0\}$ *is Zariski dense in* X*. Then the arithmetic degree at* x *is equal to the dynamical degree of* f*, i.e.,* $\alpha_f(x) = \delta_f$*.*

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If we have an equivariant surjective $(X, f) \dashrightarrow (Y, g)$ such that $\delta_f = \delta_g$, then KSC for (Y, g) implies KSC for (X, f).

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What about the case when L is only effective or when f^*L is not proportional to L?

Dynamical Iitaka Theory

Definition (Dynamical Iitaka dimension)
Let D be a Cartier divisor on X. The *dynamical f-Iitaka dimension* of D is defined as

$$\kappa_f(X,D) := \max\{\kappa(X,D') \mid D' = \sum_{i=0}^{m} a_i (f^*)^i D, a_i \in \mathbb{Z}\}.$$

Theorem (ref. [MZ23b])
Suppose that $\kappa_f(X,D) \geq 0$. *Then there is an* f*-equivariant dominant rational map*

$$f \circlearrowright X \dashrightarrow Y \circlearrowleft f|_Y$$

with Y *normal projective of dimension* $\kappa_f(X,D)$ *and* $f|_Y$ *a surjective endomorphism.*

The ramification divisor

The *ramification divisor* R_f is defined by

$$R_f := \sum_E (\mathrm{mult}_E f^*(f(E)) - 1)E$$

where the sum runs over all prime divisors E of X.

It is effective and satisfies the ramification formula

$$K_X = f^*K_X + R_f.$$

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

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- $\kappa_f(X, R_f) = 0$. Then $f^{-1}(\operatorname{Supp} R_f) = \operatorname{Supp} R_f$ and the restriction

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- $0 < \kappa_f(X, R_f) < \dim X$. We then have an f-equivariant dominant rational map

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- $\kappa_f(X, R_f) = \dim X$. In this case, R_{f^s} is big when $s \gg 1$.

About our paper

Settings

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Hence we want to use the dynamical Iitaka fibration associated to R_f to get another f-equivariant fibration.

Further assume that the base Y is either an abelian variety or a smooth projective variety of Picard number one, we get our main results.

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an extremal Fano contraction $\pi : X \to Y$ to an abelian variety Y of positive dimension. Suppose f admits a Zariski dense orbit and $\delta_f > \delta_{f|_Y}$. Then the following hold.

1. *The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.*
2. *There exists an f-equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f-Iitaka fibration of R_f, such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f-polarized.*

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an f-equivariant **smooth** *extremal Fano contraction $\pi : X \to Y$ with $\rho(Y) = 1$. Suppose $\delta_f > \delta_{f|_Y} = 1$. Then the following hold.*

1. *The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.*
2. *There exists an f-equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f-Iitaka fibration of R_f, such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f-polarized.*

Corollary

KSC holds for any smooth projective variety X admitting an extremal Fano contraction to an abelian variety.

Corollary

KSC holds for any $\mathbb{P}^n$-bundle over either a Q-abelian variety or a smooth projective variety of Picard number one.

引言

引言

这个模版基于 Beamer 的METROPOLIS包，魔改自 overleaf 上已有的模版，部分参考了潘建瑜教授的LaTeX 模版（主要是最后的简笔画）。

笔者叠加了很多华东师大的元素在模版中，因此看起来比较花里胡哨。

好像没有什么需要特意在引言中交待的，笔者在奇怪的代码中都写了注释。

本模版目前有一些已知的 Bug，在本文档中一一记录。欢迎用户报告新错误。

如果对模版有疑问，或者对模版有 bug 反馈，可以用邮件联系笔者：dsi73@foxmail.com。

标题样式

Metropolis Titleformats

METROPOLIS 支持四种英文标题样式：

- Regular
- SMALLCAPS
- ALLSMALLCAPS
- ALLCAPS

它们既可以每页幻灯片设置，也可以单独设置。

这一页幻灯片用了 *smallcaps* 样式。

潜在的问题

请注意，并非所有字体都支持 *smallcaps* 样式。例如，若您使用 pdfLaTeX 排版演示文稿并选用 Computer Modern Sans Serif 字体，所有 smallcaps 文本都将被替换为 Computer Modern Serif 字体进行排版。

这一页幻灯片用了 `allsmallcaps` 样式。

潜在的问题

由于此样式同样使用了 smallcaps，因此会出现与 `smallcaps` 标题格式相同的问题。此外，该样式还可能引发其他问题，请参阅相关文档说明。

经验：只在纯文本标题中使用它。

这个样式会使右上角的华师大 Logo 异常上浮，这是一个 Bug，待修改。

这一页幻灯片用了 `allcaps` 样式。

潜在的问题

该样式虽不如 `allsmallcaps` 样式问题严重，但基本存在相同缺陷。若需使用，请查阅相关文档。

元素

排版

代码

```
本模版提供\emph{强调}、\alert{标注}与\textbf{粗体}。
```

编译为

本模版提供*强调*、标注与**粗体**。

字体

- Regular 常规
- *Italic 斜体*
- SMALLCAPS 小大写
- **Bold 粗体**
- ***Bold Italic 粗斜体***
- **BOLD SMALLCAPS 粗小大写**
- *`Monospace 等宽`*
- *`Monospace Italic 等宽斜体`*
- **`Monospace Bold 等宽粗体`**
- ***`Monospace Bold Italic 等宽粗斜体`***

中文字体中粗斜体的斜体去哪了？这是一个 Bug，待修改。

列表

Items

- Milk
- Eggs
- Potatos

Enumerations

1. First,
2. Second and
3. Last.

Descriptions

PowerPoint Meeh.

Beamer Yeeeha.

动画

- 这个很重要！

动画

- 这个很重要！
- 现在这个很重要。

动画

- 这个很重要!
- 现在这个很重要。
- 其实这个才重要。

动画

- 这个真的很重要！
- 现在这个很重要。
- 其实这个才重要。

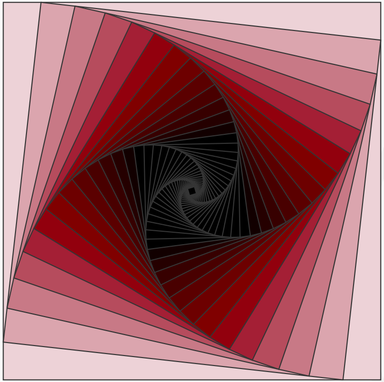

Figure 1: 旋转正方形，非常帅。来自 texample.net.

表格

Table 1: 世界上的大城市 (来源: Wikipedia)

City	Population
Mexico City	20,116,842
Shanghai	19,210,000
Peking	15,796,450
Istanbul	14,160,467

框

本模版预设了三种不同的框，可通过可选背景颜色进行样式设置。

注意：即使是左边的框，也会有白色条状物把背景的 logo 遮住。这可能不是一个 Bug，但仍然需要修改。

Default
Block content.

Alert
Block content.

Example
Block content.

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Block content.

Theorem

微积分基本定理：

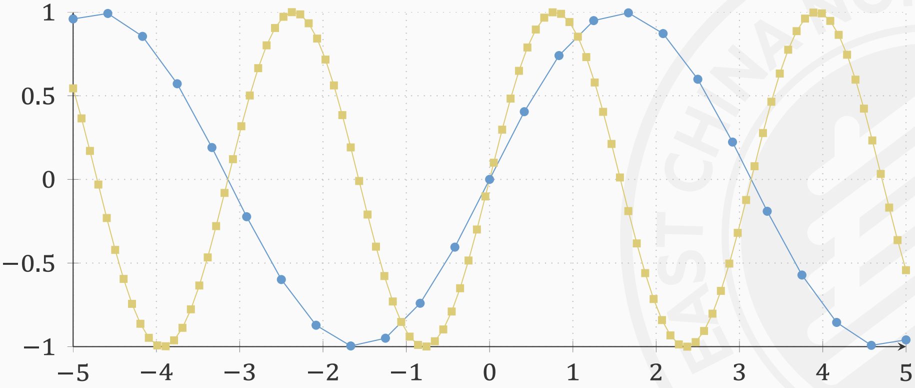
$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Proof.

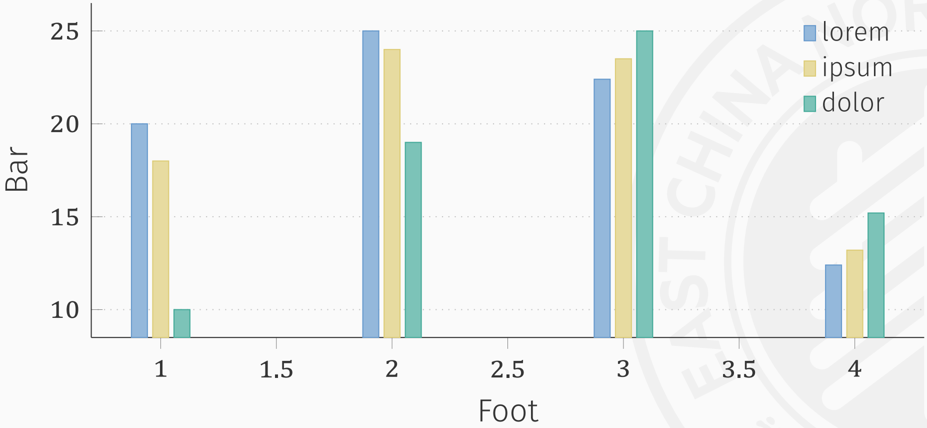
令 $g(x) = e^x - x - 1$。则当 $x > 1$ 时，有 $g'(x) = e^x - 1 > 0$，因此 $g(x) > g(1) = 0$。即有 $x > 1$ 时 $e^x > 1 + x$。 □

$$\mathrm{e} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

折线图



柱状图



引用

> *I can eat glass, it doesn’t hurt me.*
> 我能吞下玻璃而不伤身体。

注意：这里中文字体理论上应该不一样。这可能不是一个 Bug，但仍然需要修改。

页脚

METROPOLIS 定义了一个自定义的 beamer 模板，用于在页脚添加文本。可通过以下方式设置：

```
\setbeamertemplate{frame footer}{My custom footer}
```

文献引用

部分参考文献展示：[knuth92; ConcreteMath; Simpson; Er01; greenwade93]

总结

获取主题

访问 *github.com/matze/mtheme*以获取 METROPOLIS 主题。

要获取本魔改包，可以在微信公众号：**73Dsi 的小站**下留言。

本主题遵守 知识共享署名-相同方式共享 4.0 国际许可协议。

Questions?

页码删除

有时，在文档末尾添加幻灯片以便在观众提问时参考会很有用。

引用*appendixnumberbeamer*包，并在备用幻灯片前调用*appendix*。

METROPOLIS 将自动关闭附录中幻灯片的编号和进度条。

Thank You!