Algebraic Dynamics and Dynamical Iitaka Theory

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based on the joint work with Sheng Meng and Long Wang

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SCHOOL OF
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$$\delta_f = \alpha_f(x).$$

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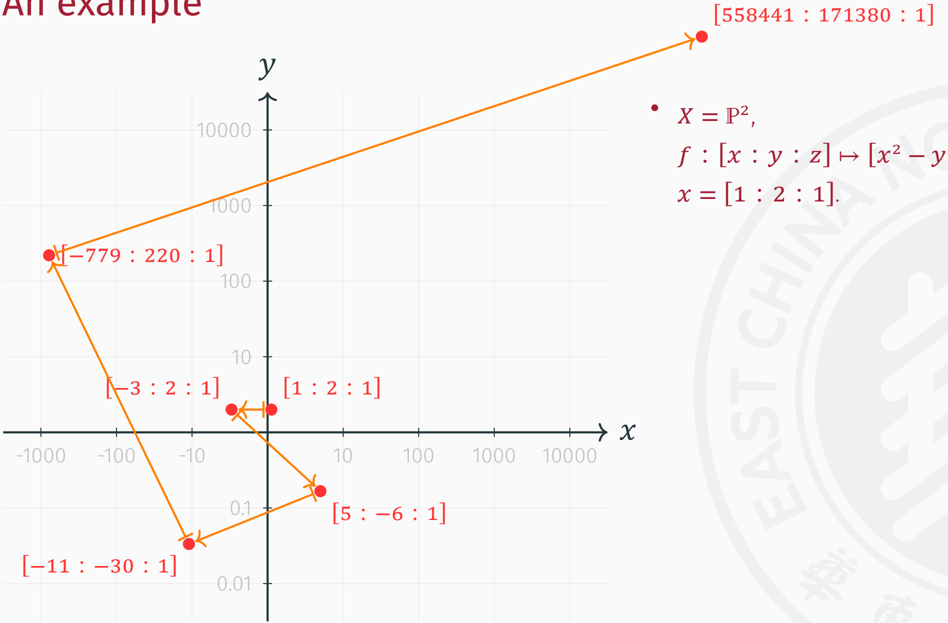
Slogan: GEOMETRY controls ARITHMETIC.

An example



- $X = \mathbb{P}^2$,
  $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
  $x = [1 : 2 : 1]$.

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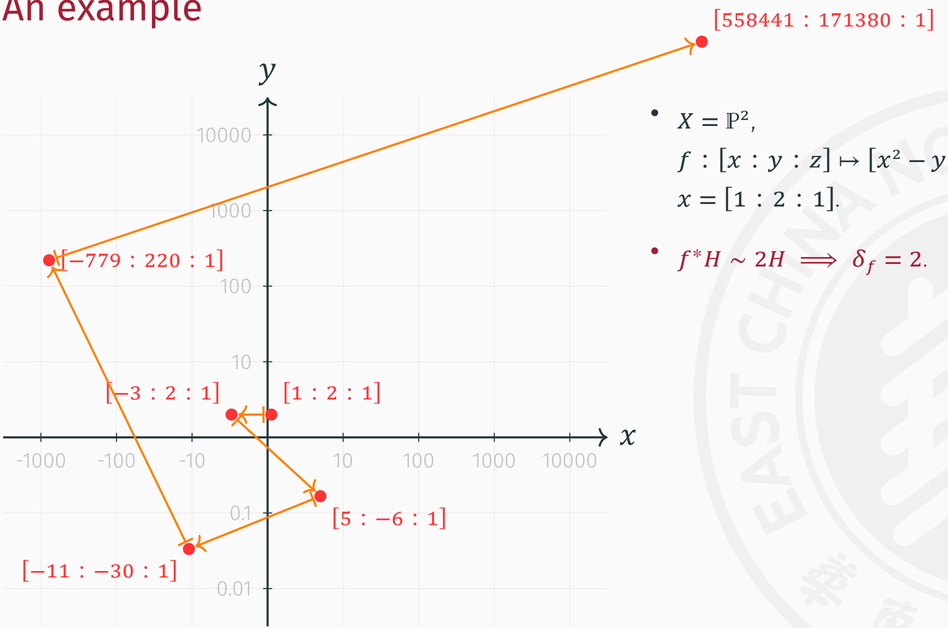

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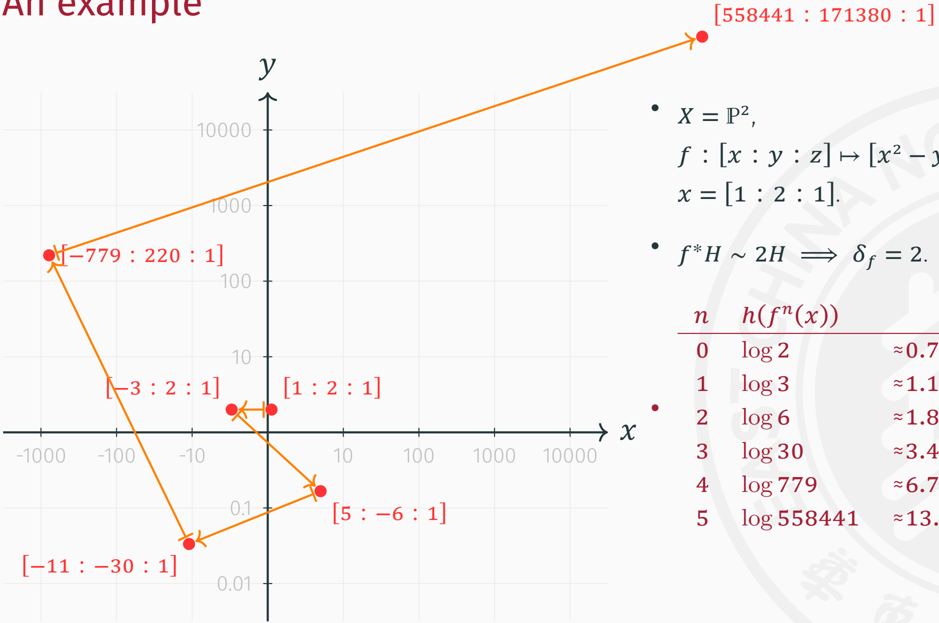

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n	$h(f^n(x))$	
0	$\log 2$	≈ 0.7
1	$\log 3$	≈ 1.1
2	$\log 6$	≈ 1.8
3	$\log 30$	≈ 3.4
4	$\log 779$	≈ 6.7
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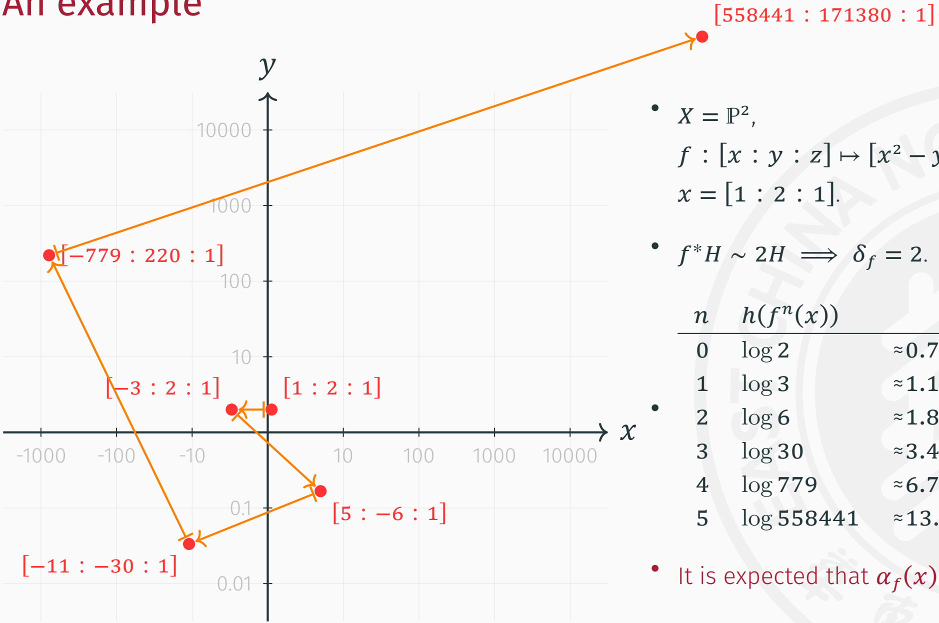

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- It is expected that $\alpha_f(x) = 2$.

Known cases for KSC

Projective surfaces [Matsuzawa-Sano-Shibata18,Meng-Zhang22];
Quasi-projective surfaces (assuming DML) [Wang23];
Birational map on surfaces [Xie24];
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Main results

Special Fano fibration case (*) (as the final output of MMP):

$X \xrightarrow{\pi} Y$ smooth Fano fibration of relative Picard number one (eg. $\mathbb{P}^n$-bundles) over

(*1) an abelian variety; or

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Generalize the following early results:

- [Li-Matsuzawa 21], projective bundles over smooth Fano varieties of Picard number one;
- [Nasserden-Zotine 21], projective bundles over elliptic curves.

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KSC holds for X in the special Fano fibration case (*).

Sketch of idea:

After iteration, the dynamics $f \circlearrowright X$ descends to $g \circlearrowright Y$. If $\delta_f = \delta_g$, then

$$\begin{array}{l}\text{KSC for abelian varieties} \\ \text{or polarized endomorphisms}\end{array} \Longrightarrow \text{KSC for } (Y, g) \Longrightarrow \text{KSC for } (X, f).$$

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Theorem: [Meng-Wang-Y]

KSC holds for X in the special Fano fibration case (*).

Sketch of idea:

In the case (*2), if $\delta_g > 1$, then f is int-amplified and KSC holds by [Meng-Zhong24].

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Dynamical conditions (†):

The fibration $(X, f) \xrightarrow{\pi} (Y, g)$ does not preserve the dynamical degree, i.e., $\delta_f > \delta_g$;
If in the case (*2), then we require $\delta_g = 1$.

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Theorem: New Fibration, [Meng-Wang-Y]

In the special Fano fibration case (*) with dynamical conditions (†), there exists an equivariant fibration

$$f \circlearrowright X \overset{\varphi_{f,R_f}}{\dashrightarrow\!\!\!\!\rightarrow} Z \circlearrowleft h$$

with $\dim Z > 0$ and h is δ_f-polarized (in particular, $\delta_h = \delta_f$).

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Using the dynamical Iitaka theory [Meng-Zhang23], to prove the theorem **New Fibration**, we only need to show that after iterating f:

① $f^*R_f \equiv \delta_f R_f$;

② $\kappa(X, R_f) \geq 1$.

Proof of ①: $f^*R_f \equiv \delta_f R_f$

Case (*2):

In the special Fano fibration case (*2) with dynamical conditions (†),

Proof of ①: $f^*R_f \equiv \delta_f R_f$

Case (*2):

$\boxed{\text{In the special Fano fibration case (*2) with dynamical conditions } (\dagger),} \Longrightarrow \boxed{f^* \text{ has exactly two eigenvalues } \delta_f \text{ and } \delta_g = 1 \text{ on } N^1(X).}$

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Case (*2):

In the special Fano fibration case (*2) with dynamical conditions (†), $\Longrightarrow$ f^* has exactly two eigenvalues δ_f and $\delta_g = 1$ on $N^1(X)$.

By the ramification formula, we have $R_f = K_X - f^*K_X \in \mathrm{Im}(\mathrm{id} - f^*)$.

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Case (*1):

In the special Fano fibration case (*1) with dynamical conditions (†), $\Longrightarrow$ $\mathrm{Psef}^1(X) = \pi^* \mathrm{Psef}^1(Y) \oplus \mathbb{R}_{\geq 0} D$ with D nef and $f^*D \equiv \delta_f D$

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Positivity of abelian varieties: there is no rational curve on it.

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Theorem: Decompositions of cones, [Meng-Wang-Y]

In the special Fano fibration case (*) with dynamical conditions (†), there exist decompositions of cones:

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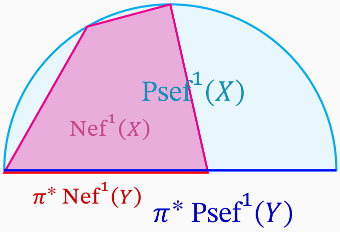
Without dynamical restrictions:

complicated cones



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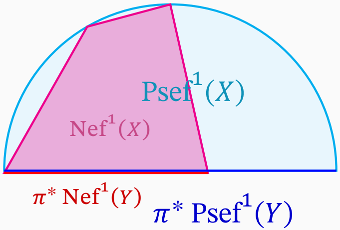
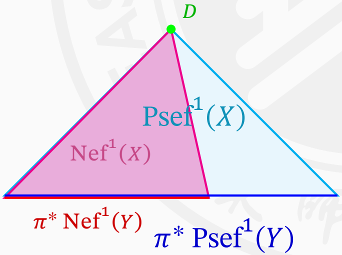
Without dynamical restrictions: complicated cones

With dynamical restrictions: simple cones





Proof of ②: positivity of $\kappa(X, R_f)$

Set $D := \mathrm{Supp}(R_f)$

Special Fano fibration case (*)
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Assume
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D is prime even after base change

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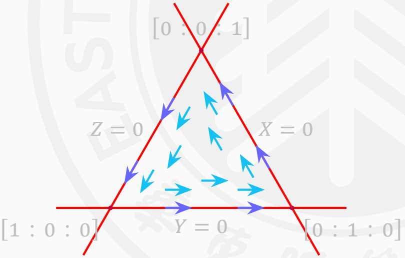
and $D = \{XYZ = 0\}$.



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General $y \in Y$: D_y has ≥ 2 irreducible components

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General $y \in Y$: D_y has ≥ 2 irreducible components

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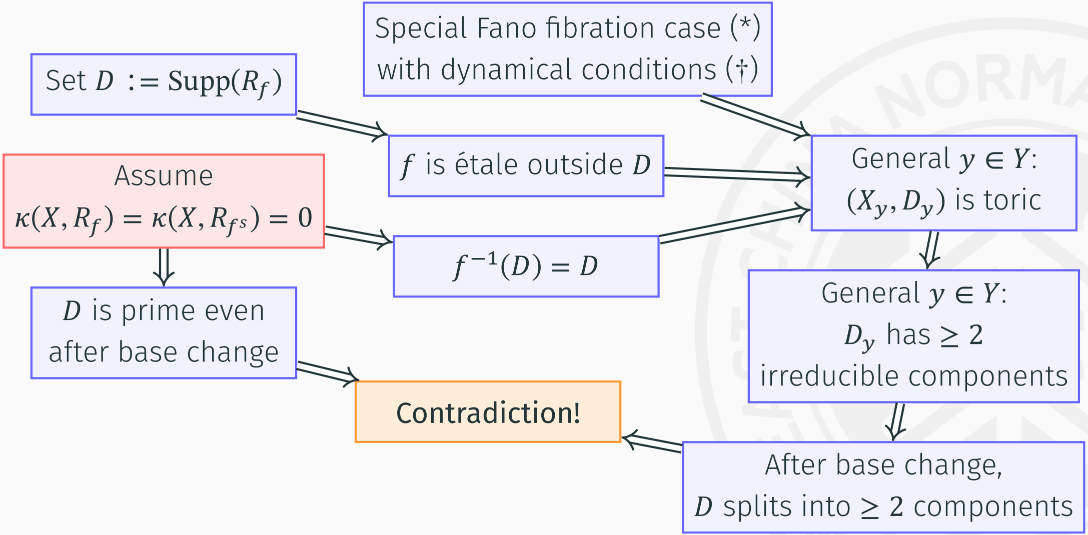
After base change, D splits into ≥ 2 components

Proof of ②: positivity of $\kappa(X, R_f)$



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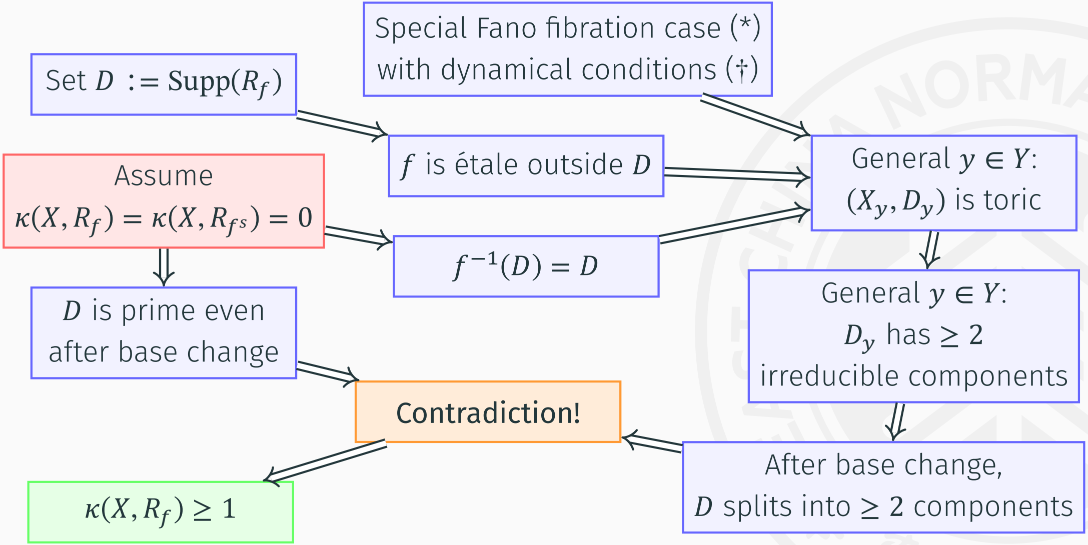

Thank You!