Algebraic Dynamics and Dynamical Iitaka Theory

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based on the joint work with Sheng Meng and Long Wang

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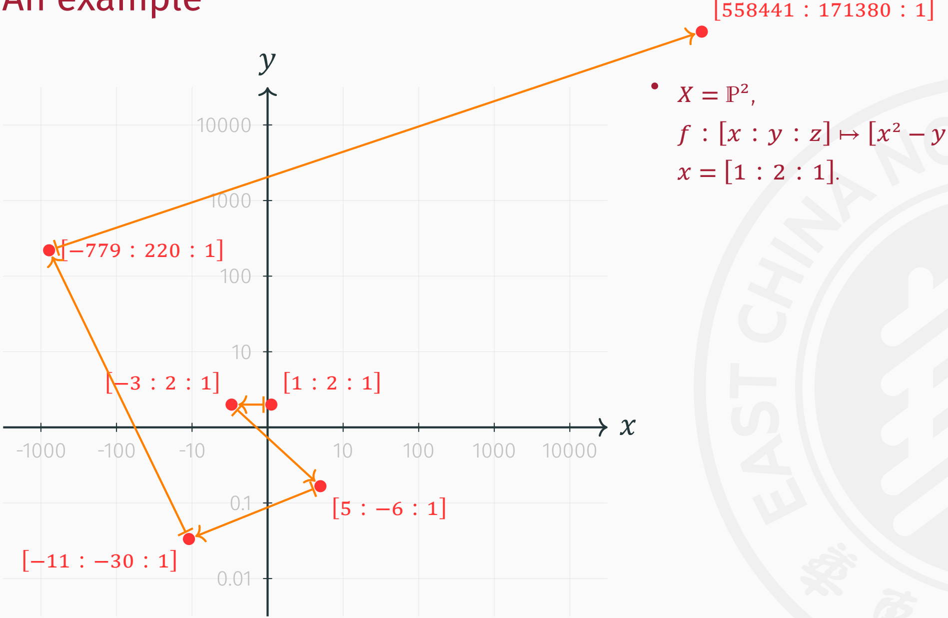
- $\delta_f := \lim_{n\to\infty} ((f^n)^*H \cdot H^{\dim X - 1})^{\frac{1}{n}}$ geometric, global invariant of f.

An example



- $X = \mathbb{P}^2$,
  $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
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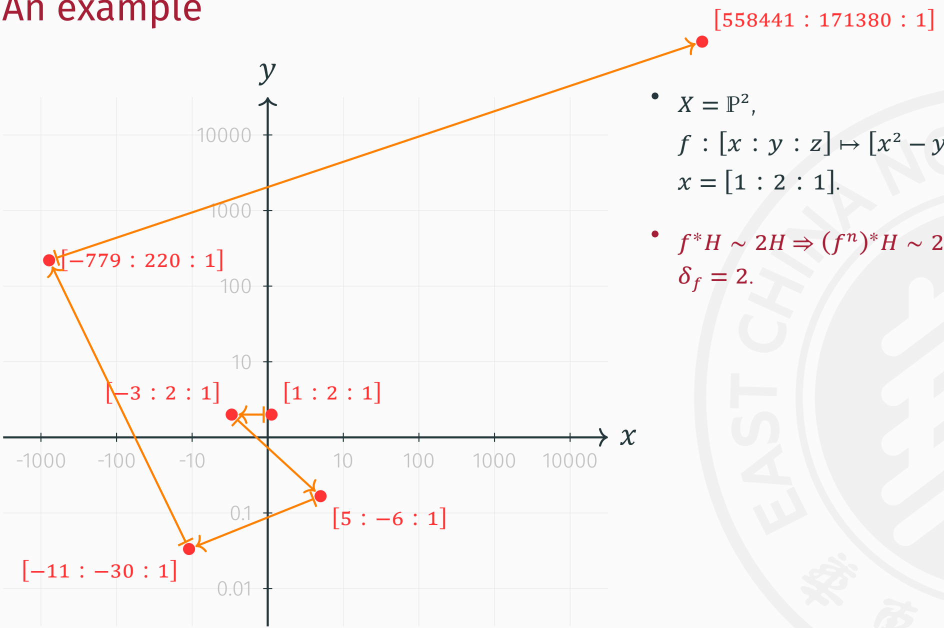

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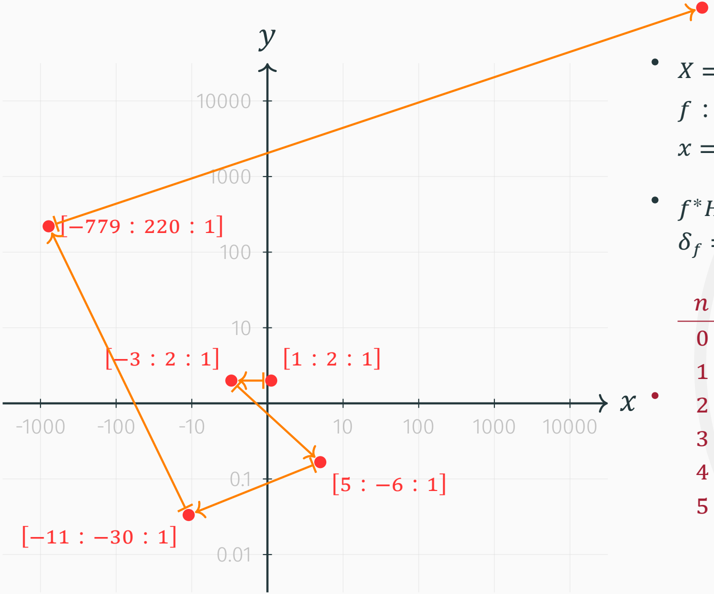

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0	$\log 2$	≈ 0.7
1	$\log 3$	≈ 1.1
2	$\log 6$	≈ 1.8
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5	$\log 558441$	≈ 13.2

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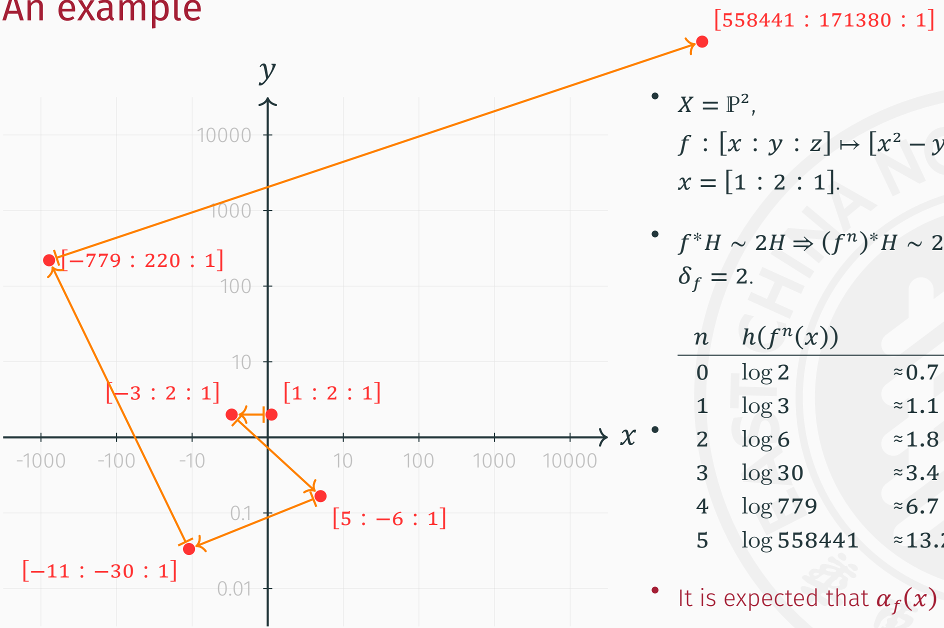

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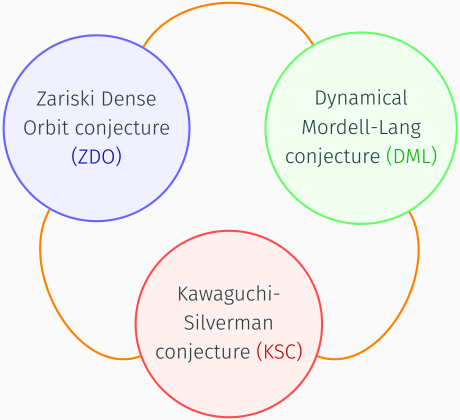
- It is expected that $\alpha_f(x) = 2$.

Three orbit conjectures



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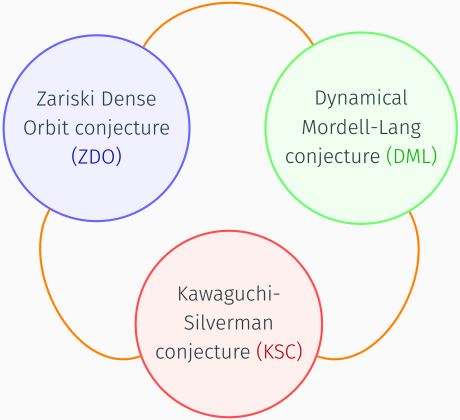

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$$\#O_f(x) \cap V = \infty \quad V \text{ a subvariety,}$$

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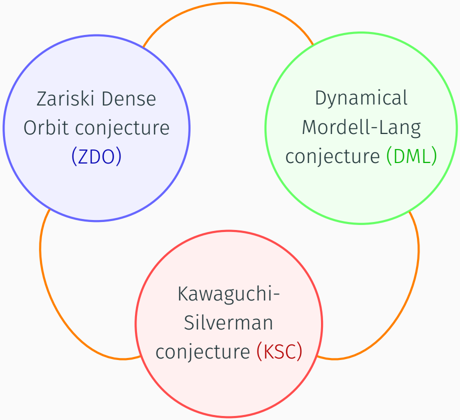

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The ZDO states that either

$$\exists x \text{ with } \overline{\mathcal{O}_f(x)} = X,$$

or

$$\exists\, f^n \circlearrowright X \dashrightarrow Y \circlearrowleft \mathrm{id}_Y.$$

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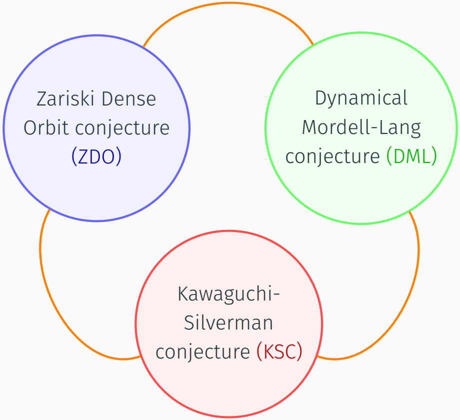

Main known cases:

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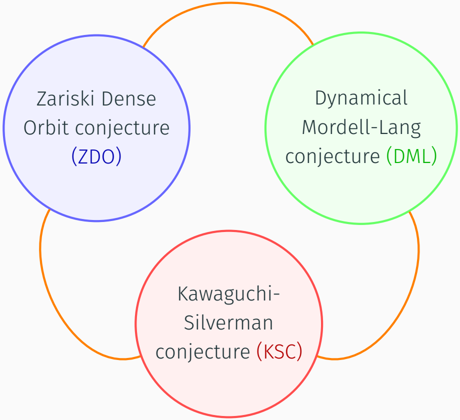

Main known cases:

Smooth projective surfaces [Matsuzawa-Sano-Shibata];

Quasi-projective surfaces (assume DML) [Wang];

Smooth projective 3folds with $\deg f > 1$ [Meng-Zhang];

Abelian varieties [Kawaguchi-Silverman];

Hyperkähler manifolds [Lesieutre-Santriano];

Mori dream spaces [Matsuzawa];

Polarized endomorphisms [Kawaguchi-Silverman];

Int-amplified endomorphisms [Meng-Zhong].

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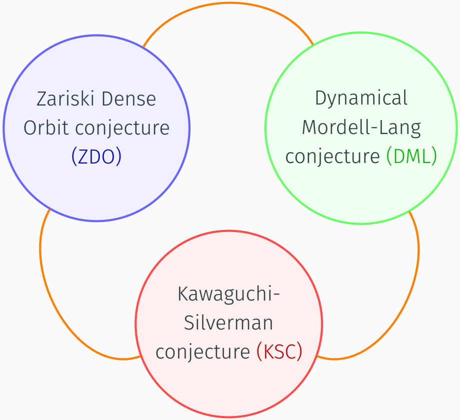

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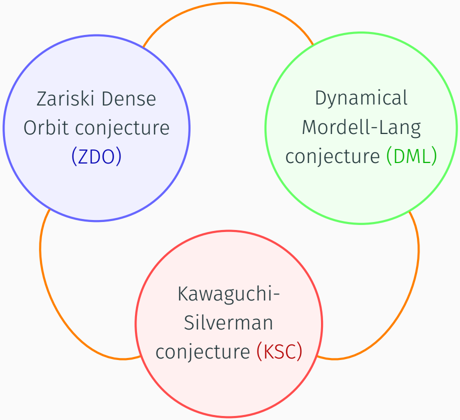

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Over $\mathbb{C}$ (uncountable field) [Amerik-Campana];

Existence of infinity orbits [Amerik];

Projective surfaces [Xie,Jia-Xie-Zhang];

Automorphisms of threefolds with positive entropy [Matsuzawa-Xie].

Dynamical Iitaka Theory

Coarse classification of varieties via Kodaira dimension $\kappa(X, K_X)$:

$\kappa(X, K_X)$	Typical geometry
$-\infty$	uniruled
0	Calabi-Yau
$0 < \kappa(X) < \dim X$	fibrations type
$\dim X$	general type

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Dynamical analogue: classify (X, f) via $\kappa_f(X, R_f)$.

$\kappa_f(X, R_f)$	Typical dynamics
0	f is log-étale
$0 < \kappa_f(X, R_f) < \dim X$	fibrations type
$\dim X$	R_{f^s} is big for $s \geq 0$

Main results

Settings:

- $\pi : (X, f) \to (Y, g)$ smooth Fano fibration with X, Y smooth projective and $\rho(X) = \rho(Y) + 1$, for example, $\mathbb{P}^n$-bundles over Y;
- Y is an abelian variety or of picard number one;

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KSC holds for (X, f) under the above settings.

Generalize the following known results:

- [Li-Matsuzawa 2021, Theorem 4.1], projective bundles on smooth Fano varieties of Picard number one;
- [Nasserden-Zotine 2021, Theorem 1.4], projective bundles on elliptic curves.

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Sketch of idea: If $\delta_f = \delta_g$, then

$$\begin{array}{c}\text{KSC for abelian varieties} \\ \text{or polarized endomorphisms}\end{array} \Rightarrow \text{KSC for } (Y, g) \Rightarrow \text{KSC for } (X, f).$$

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If $\delta_f > \delta_g$, under some extra conditions, we have another fibration by dynamical Iitaka theory.

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- Y is an abelian variety or of picard number one;
- f has a Zariski dense orbit;
- $\delta_f > \delta_g$;
- if Y is of picard number one, then $\delta_g = 1$.

Theorem: [Meng-Wang-Y]

There is equivariant fibrations

$$f \circlearrowright X \overset{\varphi_{f,R_f}}{\dashrightarrow\!\!\!\!\to} Y \circlearrowleft g$$

with $\dim Y > 0$ and g is q-polarized.

Main results

Theorem: [Meng-Zhang]

Suppose that $f^*R_f \equiv qR_f$ for some $q > 1$. Then there exists an f-equivariant fibration (dynamical Iitaka fibration associated to R_f)

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- $\kappa_f(X, R_f) \geq 1$; $\Longleftarrow$ criterion for toric bundles
- $f^*R_f \equiv \delta_f R_f$. $\Longleftarrow$ decomposition of cones + no rational curve on Y

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KSC for polarized endomorphisms $\Longrightarrow$ KSC for (Y, g) $\Longrightarrow$ KSC for (X, f).

Technique: decomposition of cones

Theorem: [Meng-Wang-Y]

$\pi : (X, f) \to (Y, g)$ flat; X, Y smooth projective; $\delta_f > \delta_g; \rho(X) = \rho(Y) + 1.$	$\Longrightarrow$	$\exists D$ nef with $f^* D \equiv \delta_f D$ s.t. $\mathrm{Psef}^1(X) = \pi^* \mathrm{Psef}^1(Y) \oplus \mathbb{R}_{\geq 0} D,$ $\mathrm{Nef}^1(X) = \pi^* \mathrm{Nef}^1(Y) \oplus \mathbb{R}_{\geq 0} D.$

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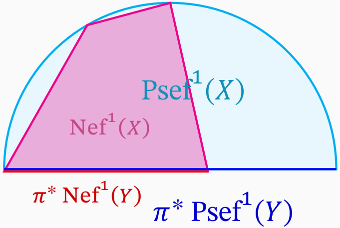

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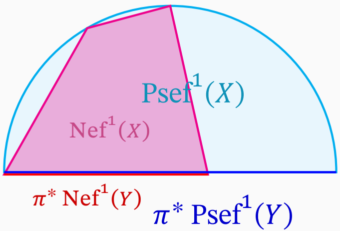
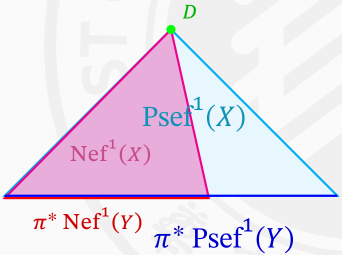


Cones of fibration with dynamical restrictions:



Technique: criterion for toric bundles

Theorem: [Meng-Wang-Y]

Let $\pi : (X, f) \to (Y, g)$ be a Fano fibration with X smooth. Suppose that $\exists$ reduced divisor D on X with $f^{-1}(D) = D, f^*D \sim qD$ for some $q > 1$ and $K_X + D \equiv_\pi 0$. Then $\pi : X \to Y$ is a toric bundle.

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This theorem generalizes the following absolute case ($Y = \{\mathrm{pt}\}$) criterion:

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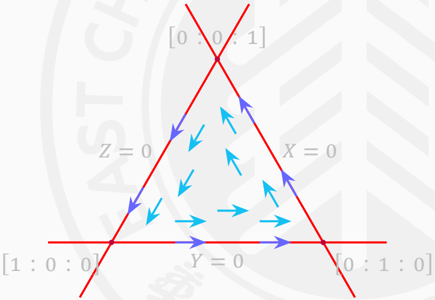
The right sides is an example in the absolute case.



$$
\begin{aligned}
f :& \mathbb{P}^2 \to \mathbb{P}^2, \\
& [X : Y : Z] \mapsto [X^q : Y^q : Z^q]; \\
D =& \{XYZ = 0\}.
\end{aligned}
$$

Further questions

Here are two further questions we are interested in:

- weaken the conditions on Y: can we just assume Y is normal projective?
-

Thank You!