

# Algebraic Dynamics and Dynamical Itaka Theory

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# Algebraic Dynamics

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The fundamental objects in algebraic dynamics are  $(X, f)$ , where  $X$  is a variety and  $f : X \dashrightarrow X$  is a dominant rational self-map.

Here we focus on the case  $X$  is projective and  $f$  is a surjective endomorphism.

For simplicity, we assume that  $X$  is smooth.



## Definition: First dynamical degree

The *first dynamical degree*  $\delta_f$  of  $f$  is defined to be the following limit

$$\delta_f := \lim_{n \rightarrow \infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n} \in \mathbb{R}_{\geq 1},$$

where  $H$  is an ample Cartier divisor on  $X$ .



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## Definition: Arithmetic degree

Let  $h : X(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$  be the height function associated to an ample divisor on  $X$ . Then for every  $x \in X(\mathbb{k})$ , we define the *arithmetic degree of  $f$  at  $x$*  by

$$\alpha_f(x) = \lim_{n \rightarrow \infty} \max\{h(f^n(x)), 1\}^{1/n} \in \mathbb{R}_{\geq 1}.$$



## Conjecture: Kawaguchi-Silverman Conjecture = KSC

Let  $x \in X(\mathbb{k})$ , and suppose that the (forward) orbit  $O_f(x) = \{f^n(x) \mid n \geq 0\}$  is Zariski dense in  $X$ . Then the arithmetic degree at  $x$  is equal to the dynamical degree of  $f$ , i.e.,  $\alpha_f(x) = \delta_f$ .



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If we have an equivariant surjective  $(X, f) \dashrightarrow (Y, g)$  such that  $\delta_f = \delta_g$ , then KSC for  $(Y, g)$  implies KSC for  $(X, f)$ .



A fibration  $\pi : X \dashrightarrow Y$  is said to be *f-equivariant* if there is a dominant rational map  $g : Y \dashrightarrow Y$  such that  $\pi \circ f = g \circ \pi$ .





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Suppose that the fibration  $\pi : X \rightarrow Y$  is given by a semiample line bundle  $L$ . If  $f^*L \sim qL$  for some  $q \geq 1$ , then  $\pi$  is *f-equivariant* by the rigidity lemma and simple intersection theory.



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Suppose that the fibration  $\pi : X \rightarrow Y$  is given by a semiample line bundle  $L$ . If  $f^*L \sim qL$  for some  $q \geq 1$ , then  $\pi$  is *f-equivariant* by the rigidity lemma and simple intersection theory.

What about the case when  $L$  is only effective or when  $f^*L$  is not proportional to  $L$ ?

# Dynamical Itaka Theory

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## Definition: Dynamical Iitaka dimension

Let  $D$  be a Cartier divisor on  $X$ . The *dynamical  $f$ -Iitaka dimension* of  $D$  is defined as

$$\kappa_f(X, D) := \max \left\{ \kappa(X, D') \mid D' = \sum_{i=0}^m a_i (f^*)^i D, a_i \in \mathbb{Z} \right\}.$$



Theorem: ref. [MZ23, Theorem 4.6]

Suppose that  $\kappa_f(X, D) \geq 0$ . Then there is an  $f$ -equivariant dominant rational map

$$f \circ X \dashrightarrow Y \circ f|_Y$$

with  $Y$  normal projective of dimension  $\kappa_f(X, D)$  and  $f|_Y$  a surjective endomorphism.



The *ramification divisor*  $R_f$  is defined by

$$R_f := \sum_E (\text{mult}_E f^*(f(E)) - 1)E$$

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It is effective and satisfies the ramification formula

$$K_X = f^*K_X + R_f.$$



We consider three situations of  $\kappa_f(X, R_f)$  and briefly introduce how the dynamical Itaka theory works.





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- $\kappa_f(X, R_f) = 0$ . Then  $f^{-1}(\text{Supp } R_f) = \text{Supp } R_f$  and the restriction

$$f|_{X \setminus \text{Supp } R_f} : X \setminus \text{Supp } R_f \rightarrow X \setminus \text{Supp } R_f$$

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- $0 < \kappa_f(X, R_f) < \dim X$ . We then have an  $f$ -equivariant dominant rational map

$$\varphi_{f, R_f} : X \dashrightarrow Y$$

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- $\kappa_f(X, R_f) = \dim X$ . In this case,  $R_{f^s}$  is big when  $s \gg 1$ .

## About our paper

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We consider the case the  $(X, f)$  has admitted an  $f$ -equivariant extremal Fano contraction  $\pi : X \rightarrow Y$  such that  $\delta_f > \delta_{f|_Y}$ .





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Hence we want to use the dynamical Iitaka fibration associated to  $R_f$  to get another  $f$ -equivariant fibration.

Further assume that the base  $Y$  is either an abelian variety or a smooth projective variety of Picard number one, we get our main results.



## Theorem

Let  $f$  be a surjective endomorphism of a smooth projective variety  $X$  admitting an extremal Fano contraction  $\pi : X \rightarrow Y$  to an abelian variety  $Y$  of positive dimension. Suppose  $f$  admits a Zariski dense orbit and  $\delta_f > \delta_{f|_Y}$ . Then the following hold.

1. The ramification divisor satisfies  $f^*R_f \equiv \delta_f R_f$ .
2. There exists an  $f$ -equivariant dominant rational map  $\varphi : X \dashrightarrow Z$ , which is the  $f$ -Iitaka fibration of  $R_f$ , such that  $0 < \dim Z < \dim X$  and  $f|_Z$  is  $\delta_f$ -polarized.



## Theorem

Let  $f$  be a surjective endomorphism of a smooth projective variety  $X$  admitting an  $f$ -equivariant **smooth** extremal Fano contraction  $\pi : X \rightarrow Y$  with  $\rho(Y) = 1$ . Suppose  $\delta_f > \delta_{f|_Y} = 1$ . Then the following hold.

1. The ramification divisor satisfies  $f^*R_f \equiv \delta_f R_f$ .
2. There exists an  $f$ -equivariant dominant rational map  $\varphi : X \dashrightarrow Z$ , which is the  $f$ -Iitaka fibration of  $R_f$ , such that  $0 < \dim Z < \dim X$  and  $f|_Z$  is  $\delta_f$ -polarized.



## Corollary

KSC holds for any smooth projective variety  $X$  admitting an extremal Fano contraction to an abelian variety.

## Corollary

KSC holds for any  $\mathbb{P}^n$ -bundle over either a  $Q$ -abelian variety or a smooth projective variety of Picard number one.

# Thank You!

