Algebraic Dynamics and Dynamical Iitaka Theory

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ICCM 2025, January 7, 2026

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Algebraic Dynamics

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The fundamental objects in algebraic dynamics are (X, f), where X is a variety and $f : X \dashrightarrow X$ is a dominant rational self-map.

Here we focus on the case X is projective and f is a surjective endomorphism.

For simplicity, we assume that X is smooth.

Dynamical degree and arithmetic degree

Definition: (The first) dynamical degree

The **first dynamical degree** δ_f of f is defined to be the following limit

$$\delta_f := \lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n} \in \mathbb{R}_{\geq 1},$$

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Definition: Arithmetic degree

Let $h : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the height function associated to an ample divisor on X. Then for every $x \in X(\Bbbk)$, we define the **arithmetic degree of f at x** by

$$\alpha_f(x) = \lim_{n\to\infty} \max\{h(f^n(x)), 1\}^{1/n} \in \mathbb{R}_{\geq 1}.$$

Kawaguchi-Silverman Conjecture

Conjecture: Kawaguchi-Silverman Conjecture = KSC

Let $x \in X(\Bbbk)$, and suppose that the (forward) orbit $O_f(x) = \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X. Then the arithmetic degree at x is equal to the dynamical degree of f, i.e., $\alpha_f(x) = \delta_f$.

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If we have an equivariant surjective $(X, f) \dashrightarrow (Y, g)$ such that $\delta_f = \delta_g$, then KSC for (Y, g) implies KSC for (X, f).

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Fibration on algebraic dynamics

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What about the case when L is only effective or when f^*L is not proportional to L?

Dynamical Iitaka Theory

Dynamical Iitaka dimension

Definition: Dynamical Iitaka dimension

Let D be a Cartier divisor on X. The **dynamical f-Iitaka dimension** of D is defined as

$$\kappa_f(X,D) := \max\left\{\kappa(X,D') \,\middle|\, D' = \sum_{i=0}^{m} a_i (f^*)^i D, a_i \in \mathbb{Z}\right\}.$$

Theorem: ref. [MZ23, Theorem 4.6]

Suppose that $\kappa_f(X,D) \geq 0$. Then there is an f-equivariant dominant rational map

$$f \circlearrowright X \dashrightarrow Y \circlearrowleft f|_Y$$

with Y normal projective of dimension $\kappa_f(X,D)$ and $f|_Y$ a surjective endomorphism.

The **ramification divisor** R_f is defined by

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It is effective and satisfies the ramification formula

$$K_X = f^*K_X + R_f.$$

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- $0 < \kappa_f(X, R_f) < \dim X$. We then have an f-equivariant dominant rational map

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- $\kappa_f(X, R_f) = \dim X$. In this case, R_{f^s} is big when $s \gg 1$.

About our paper

Settings

We consider the case the (X, f) has admitted an f-equivariant extremal Fano contraction $\pi : X \to Y$ such that $\delta_f > \delta_{f|_Y}$.

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Hence we want to use the dynamical Iitaka fibration associated to R_f to get another f-equivariant fibration.

Further assume that the base Y is either an abelian variety or a smooth projective variety of Picard number one, we get our main results.

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an extremal Fano contraction $\pi : X \to Y$ to an abelian variety Y of positive dimension. Suppose f admits a Zariski dense orbit and $\delta_f > \delta_{f|_Y}$. Then the following hold.

1. The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.
2. There exists an f-equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f-Iitaka fibration of R_f, such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f-polarized.

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an f-equivariant **smooth** extremal Fano contraction $\pi : X \to Y$ with $\rho(Y) = 1$. Suppose $\delta_f > \delta_{f|_Y} = 1$. Then the following hold.

1. The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.
2. There exists an f-equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f-Iitaka fibration of R_f, such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f-polarized.

Corollary

KSC holds for any smooth projective variety X admitting an extremal Fano contraction to an abelian variety.

Corollary

KSC holds for any $\mathbb{P}^n$-bundle over either a Q-abelian variety or a smooth projective variety of Picard number one.

Thank You!