1

Algebraic dynamics is "dynamics in the setting of algebraic geometry".

The base field is an algebraically closed field $\Bbbk$ with characteristic 0, usually $\mathbb{C}$ or $\overline{\mathbb{Q}}$.

An *algebraic dynamical system* is a pair (X, f) where X is a variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ is a dominant rational self-map.

Here we focus on the case that X is smooth and projective, and f is an endomorphism.

2

First we introduce some basic invariants of algebraic dynamical systems.

Definition 1 ((The first) dynamical degree)**.**

Definition 2 (Arithmetic degree)**.**

Note that dynamical degree is a global invariant of the map f, while arithmetic degree is a local invariant at the point x. Therefore, it is natural to ask about the relation between these two invariants. Kawaguchi-Silverman conjectured the following.

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Suppose that this point x has a Zariski dense (forward) orbit. Somehow this means that x can reflect the global dynamics of f on X. Then we expect that the arithmetic degree at x equals to the dynamical degree of f.

Conjecture 3 (Kawaguchi-Silverman Conjecture = KSC)**.**

So far KSC is still wide open in general. A useful method to study KSC is to use equivariant fibrations. That is, we have a dominant rational map $\pi : X \dashrightarrow Y$ and a dominant rational map $g : Y \dashrightarrow Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\pi"] & X \arrow[d, dashed, "\pi"] \\ Y \arrow[r, "g"] & Y. \end{tikzcd}$$

Suppose further that $\delta_f = \delta_g$. Then KSC for (X, f) can be reduced to KSC for (Y, g). Therefore, it is important to study when such fibrations exist.

4

A fibration $\pi : X \dashrightarrow Y$ is said to be f*-equivariant* if there is a dominant rational map $g : Y \dashrightarrow Y$ such that $\pi \circ f = g \circ \pi$.

Given a fibration $\pi : X \to Y$ induced by a semiample divisor D. Suppose that $f^*D \equiv qD$ for some $q > 0$, then by rigidity lemma and elementary intersection theory, π is f-equivariant.

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In algebraic geometry, Iitaka fibration is a very important fibration associated to an effective divisor. Therefore, it is natural to ask how to give a dynamical analogue of Iitaka fibration.

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Professor Meng and Zhang recently developed a dynamical analogue of Iitaka theory. Here I will briefly introduce some main results in their paper [MZ23].

First we need to extend the definition of Iitaka dimension to the setting of algebraic dynamics.

Definition 4 (Dynamical Iitaka dimension)**.**

Then we have the following result about the existence of dynamical Iitaka fibration.

Theorem 5 (ref. [MZ23, Theorem 4.6])**.**

6

Using dynamical Iitaka fibration, we can try to classify algebraic dynamical systems (X, f). In classical theory, varieties are classified according to the Kodaira dimension, i.e., Iitaka dimension of the canonical divisor K_X. In the dynamical setting, we will consider the dynamical Iitaka dimension of the ramification divisor $R_f = K_X - f^*K_X$. It is always effective. Hence we have the following three cases.

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- $\kappa_f(X, R_f) = 0$. (log étale case)
- $0 < \kappa_f(X, R_f) < \dim X$. (intermediate case)
- $\kappa_f(X, R_f) = \dim X$. (big case)

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We consider the case the (X, f) has admitted an f-equivariant extremal Fano contraction $\pi : X \to Y$ such that $\delta_f > \delta_{f|_Y}$.

Since π is of relative Picard number 1, the general fiber X_y is Fano and $\delta_f > \delta_{f|_Y} \geq 1$, the ramification divisor R_f is non-trivial and π-ample.

Hence we want to use the dynamical Iitaka fibration associated to R_f to get another f-equivariant fibration.

Further assume that the base Y is either an abelian variety or a smooth projective variety of Picard number one, we get our main results.

Theorem 6 (Main Theorem 1)**.**

Theorem 7 (Main Theorem 2)**.**

As a direct application of our main results, we can prove KSC for such varieties.