Algebraic Dynamics and Dynamical Iitaka Theory

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based on the joint work with Sheng Meng and Long Wang

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EAST CHINA NORMAL UNIVERSITY

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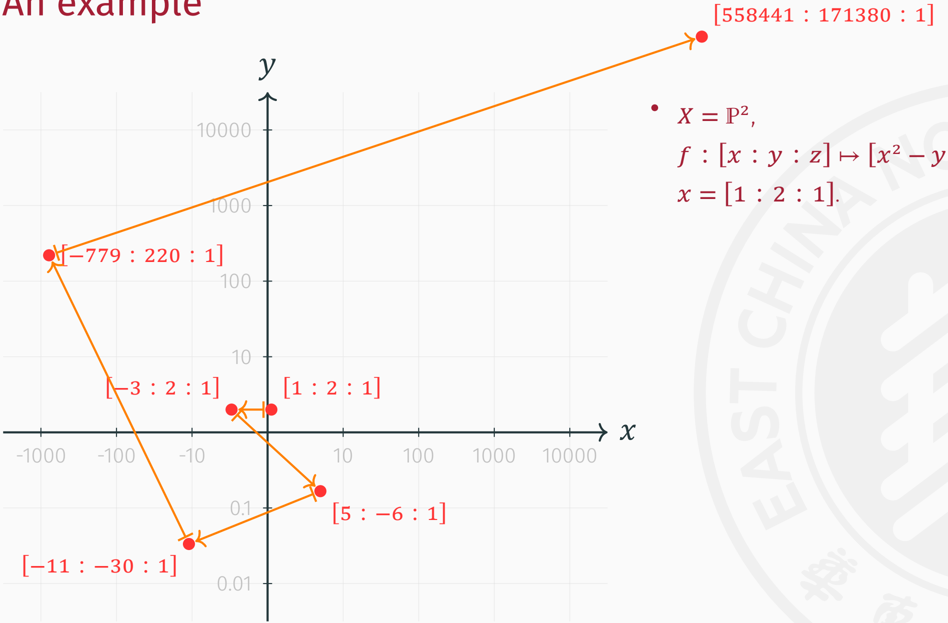
Slogan: GEOMETRY controls ARITHMETIC.

An example



- $X = \mathbb{P}^2$,
  $f : [x : y : z] \mapsto [x^2 - y^2 : xy : z^2]$,
  $x = [1 : 2 : 1]$.

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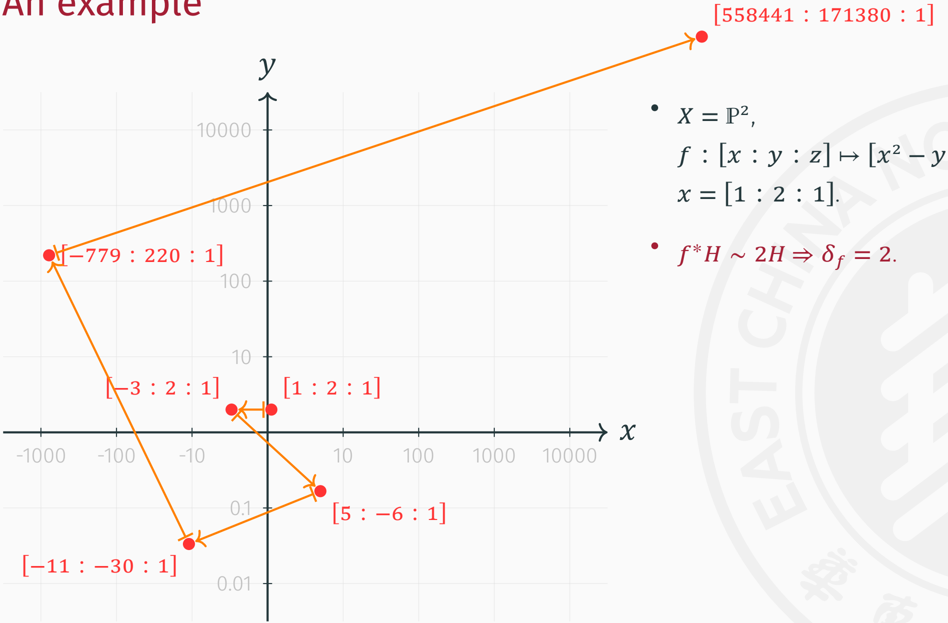

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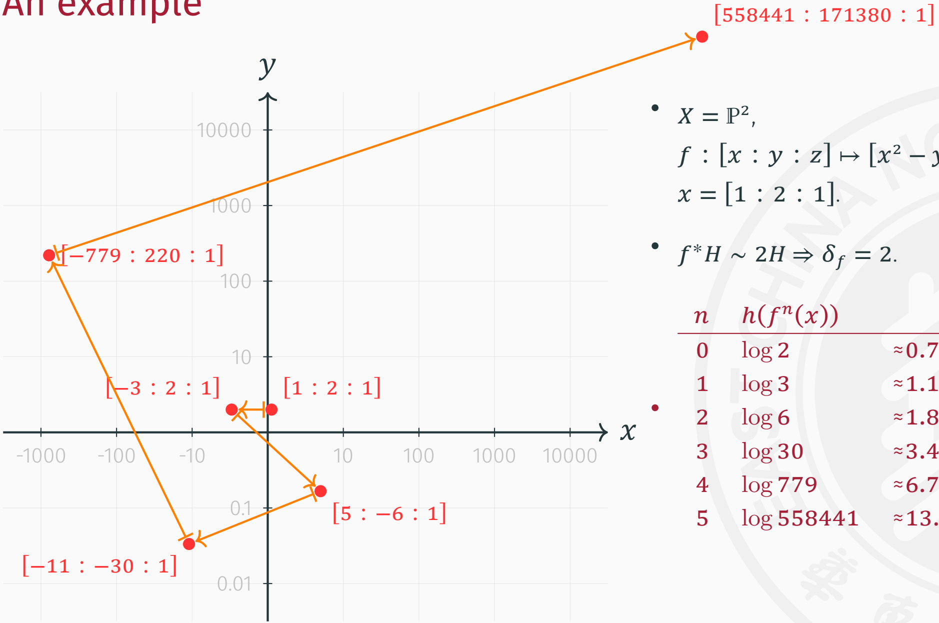

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0	$\log 2$	≈ 0.7
1	$\log 3$	≈ 1.1
2	$\log 6$	≈ 1.8
3	$\log 30$	≈ 3.4
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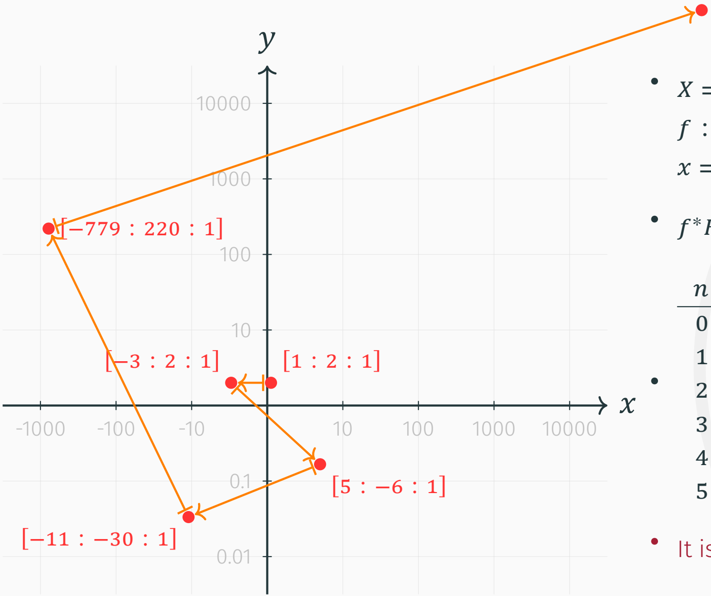

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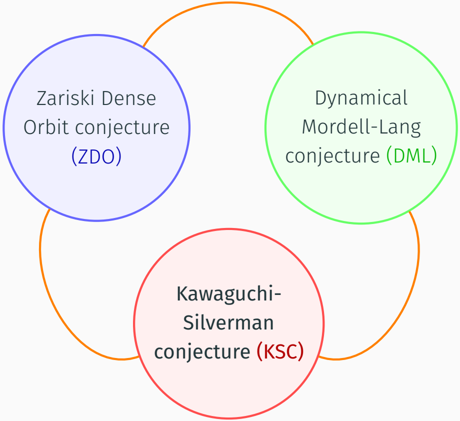
- It is expected that $\alpha_f(x) = 2$.

Three orbit conjectures



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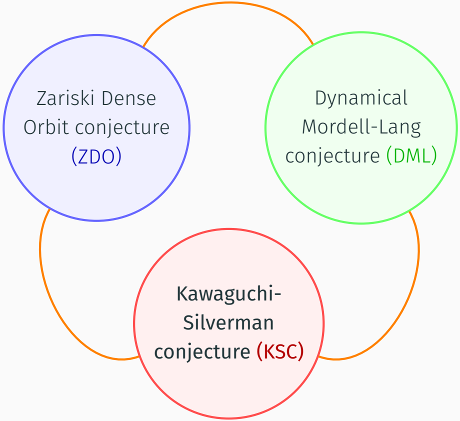

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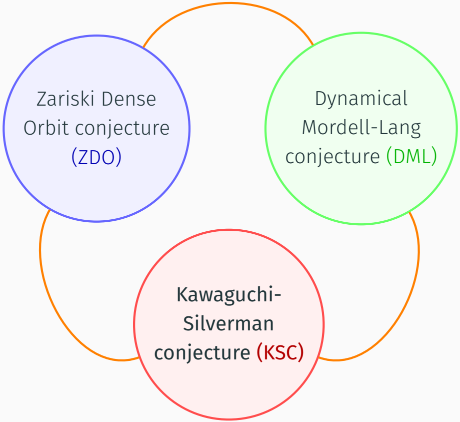

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The ZDO states that either

$$\exists x \text{ with } \overline{O_f(x)} = X,$$

or f descends to identity after iteration.

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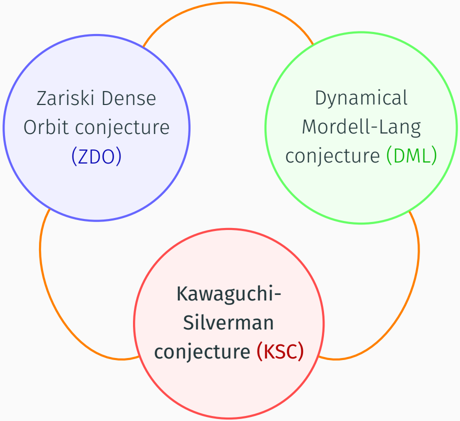

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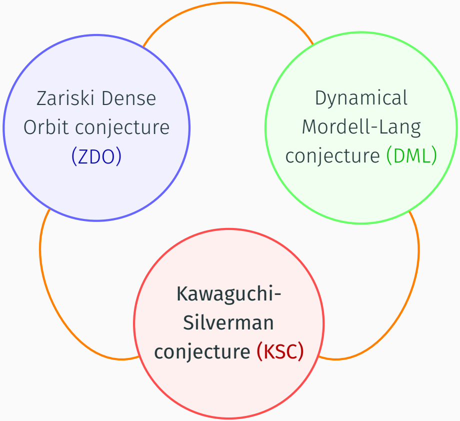

Main known cases:

Smooth projective surfaces [Matsuzawa-Sano-Shibata];

Quasi-projective surfaces (assume DML) [Wang];

Smooth projective 3folds with $\deg f > 1$ [Meng-Zhang];

Abelian varieties [Kawaguchi-Silverman];

Hyperkähler manifolds [Lesieutre-Santriano];

Mori dream spaces [Matsuzawa];

Polarized endomorphisms [Kawaguchi-Silverman];

Int-amplified endomorphisms [Meng-Zhong];

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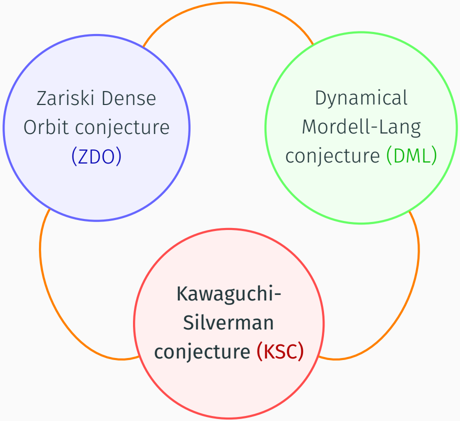

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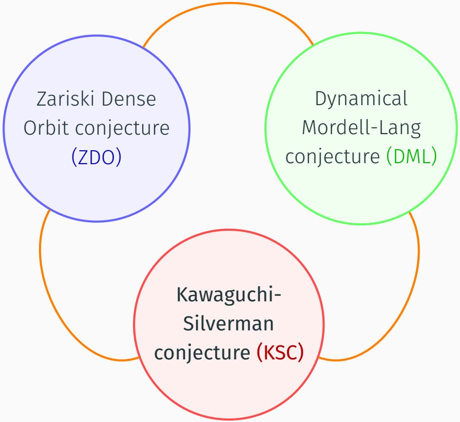

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Over $\mathbb{C}$ (uncountable field) [Amerik-Campana];

Existence of infinity orbits [Amerik];

Projective surfaces [Xie,Jia-Xie-Zhang];

Automorphisms of threefolds with positive entropy [Matsuzawa-Xie];

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Main results

Settings:

- $\pi : (X, f) \to (Y, g)$ smooth Fano fibration with X, Y smooth projective and $\rho(X) = \rho(Y) + 1$, for example, $\mathbb{P}^n$-bundles over Y;
- Y is an abelian variety or of picard number one;

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Essentially generalize the following known results:

- [Li-Matsuzawa 2021, Theorem 4.1], projective bundles on smooth Fano varieties of Picard number one;
- [Nasserden-Zotine 2021, Theorem 1.4], projective bundles on elliptic curves.

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If $\delta_f > \delta_g$, under some allowable extra conditions, we have another fibration by dynamical Iitaka theory.

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- $\pi : (X, f) \to (Y, g)$ smooth Fano fibration with X, Y smooth projective and $\rho(X) = \rho(Y) + 1$, for example, $\mathbb{P}^n$-bundles over Y;
- Y is an abelian variety or of picard number one;
- f has a Zariski dense orbit;
- $\delta_f > \delta_g$;
- if Y is of picard number one, then $\delta_g = 1$.

Theorem: [Meng-Wang-Y]

There is equivariant fibrations

$$f \circlearrowright X \overset{\varphi_{f,R_f}}{\dashrightarrow\!\!\!\rightarrow} Y \circlearrowleft g$$

with $\dim Y > 0$ and g is δ_f-polarized and hence $\delta_g = \delta_f$.

Dynamical Iitaka Theory

Theorem: [Meng-Zhang]

Suppose that $f^*R_f \equiv qR_f$ for some $q > 1$. Then there exists an f-equivariant fibration (dynamical Iitaka fibration associated to R_f)

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Coarse classification of varieties via Kodaira dimension $\kappa(X, K_X)$:

$\kappa(X, K_X)$	Typical geometry
$-\infty$	uniruled
0	Calabi-Yau
$0 < \kappa(X) < \dim X$	fibrations type
$\dim X$	general type

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Dynamical analogue: classify (X, f) via $\kappa_f(X, R_f)$.

$\kappa_f(X, R_f)$	Typical dynamics
0	f is log-étale
$0 < \kappa_f(X, R_f) < \dim X$	fibrations type
$\dim X$	R_{f^s} is big for $s \gg 0$

Technique: decomposition of cones

Theorem: [Meng-Wang-Y]

$\pi : (X, f) \to (Y, g)$ flat;
X, Y smooth projective;
$\delta_f > \delta_g; \rho(X) = \rho(Y) + 1.$

$\Longrightarrow$

$\exists D$ nef with $f^*D \equiv \delta_f D$ s.t.
$$\mathrm{Psef}^1(X) = \pi^* \mathrm{Psef}^1(Y) \oplus \mathbb{R}_{\geq 0} D,$$
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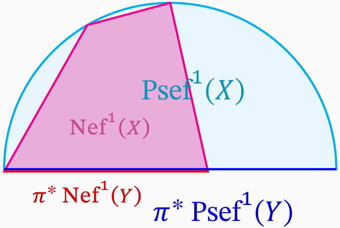

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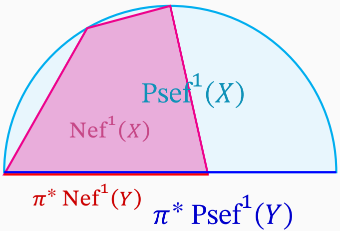
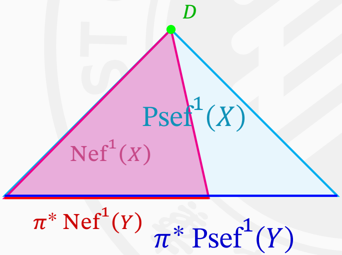


Cones of fibration with dynamical restrictions:



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Theorem: [Meng-Wang-Y]

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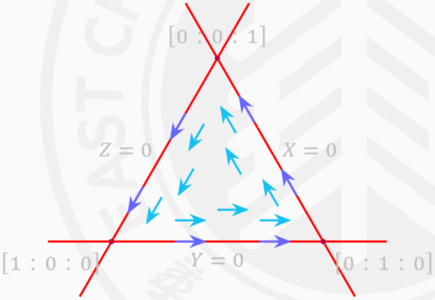
The right sides is an example in the absolute case.



$$f : \mathbb{P}^2 \to \mathbb{P}^2,$$
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$$D = \{XYZ = 0\}.$$

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- weaken the conditions on Y: can we just assume Y is normal projective?
- deal the case R_f is big in the dynamical Iitaka theory.

Thank You!