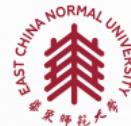


Algebraic Dynamics and Dynamical Iitaka Theory

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base on the joint work with Sheng Meng and Long Wang

ICCM 2025, January 7, 2026



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Kawaguchi-Silverman Conjecture

Conjecture: Kawaguchi-Silverman Conjecture = KSC

If the orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X , then

$$\alpha_f(x) = \delta_f.$$

Fibration on algebraic dynamics

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What about the case when L is only effective or when f^*L is not proportional to L ?

Dynamical Iitaka dimension

Definition: Dynamical Iitaka dimension

Let D be a Cartier divisor on X . The **dynamical f -Iitaka dimension** of D is defined as

$$\kappa_f(X, D) := \max \left\{ \kappa(X, D') \mid D' = \sum_{i=0}^m a_i (f^*)^i D, a_i \in \mathbb{Z} \right\}.$$

Dynamical Iitaka fibration

Theorem: ref. [MZ23b]

Suppose that $\kappa_f(X, D) \geq 0$. Then there is an f -equivariant dominant rational map

$$f : X \dashrightarrow Y \circ f|_Y$$

with Y normal projective of dimension $\kappa_f(X, D)$ and $f|_Y$ a surjective endomorphism.

The ramification divisor

The **ramification divisor** R_f is defined by

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It is effective and satisfies the ramification formula

$$R_f = K_X - f^*K_X.$$

Dynamical Iitaka Program for ramification divisor

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- $0 < \kappa_f(X, R_f) < \dim X$. We then have an f -equivariant dominant rational map

$$\varphi_{f, R_f} : X \dashrightarrow Y$$

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- $\kappa_f(X, R_f) = \dim X$. In this case, R_{f^s} is big when $s \gg 1$.

Settings

We consider the case the (X, f) has admitted an f -equivariant extremal Fano contraction $\pi : X \rightarrow Y$ such that $\delta_f > \delta_{f|_Y}$.

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Hence we want to use the dynamical Iitaka fibration associated to R_f to get another f -equivariant fibration.

Further assume that the base Y is either an abelian variety or a smooth projective variety of Picard number one, we get our main results.

Main Results

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an extremal Fano contraction $\pi : X \rightarrow Y$ to an abelian variety Y of positive dimension. Suppose f admits a Zariski dense orbit and $\delta_f > \delta_{f|_Y}$. Then the following hold.

1. The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.
2. There exists an f -equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f -Iitaka fibration of R_f , such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f -polarized.

Main Results

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an f -equivariant **smooth** extremal Fano contraction $\pi : X \rightarrow Y$ with $\rho(Y) = 1$. Suppose $\delta_f > \delta_{f|_Y} = 1$. Then the following hold.

1. The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.
2. There exists an f -equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f -Iitaka fibration of R_f , such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f -polarized.

Main Results

Corollary

KSC holds for any smooth projective variety X admitting an extremal Fano contraction to an abelian variety.

Corollary

KSC holds for any \mathbb{P}^n -bundle over either a Q -abelian variety or a smooth projective variety of Picard number one.

Thank You!

