Algebraic Dynamics and Dynamical Iitaka Theory

Tianle Yang
base on the joint work with Sheng Meng and Long Wang

ICCM 2025, January 7, 2026

SCHOOL OF
MATHEMATICAL SCIENCES
EAST CHINA NORMAL UNIVERSITY

Kawaguchi-Silverman Conjecture

Conjecture: Kawaguchi-Silverman Conjecture = KSC

If the orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is Zariski dense in X, then

$$\alpha_f(x) = \delta_f.$$

Fibration on algebraic dynamics

A fibration $\pi : X \dashrightarrow Y$ is said to be f-**equivariant** if there is a dominant rational map $g : Y \dashrightarrow Y$ such that $\pi \circ f = g \circ \pi$.

Fibration on algebraic dynamics

A fibration $\pi : X \dashrightarrow Y$ is said to be f-**equivariant** if there is a dominant rational map $g : Y \dashrightarrow Y$ such that $\pi \circ f = g \circ \pi$.

Suppose that the fibration $\pi : X \to Y$ is given by a semiample line bundle L. If $f^*L \equiv qL$ for some $q > 0$, then π is f-equivariant by the rigidity lemma and simple intersection theory.

Fibration on algebraic dynamics

A fibration $\pi : X \dashrightarrow Y$ is said to be f-**equivariant** if there is a dominant rational map $g : Y \dashrightarrow Y$ such that $\pi \circ f = g \circ \pi$.

Suppose that the fibration $\pi : X \to Y$ is given by a semiample line bundle L. If $f^*L \equiv qL$ for some $q > 0$, then π is f-equivariant by the rigidity lemma and simple intersection theory.

What about the case when L is only effective or when f^*L is not proportional to L?

Dynamical Iitaka dimension

Definition: Dynamical Iitaka dimension

Let D be a Cartier divisor on X. The **dynamical f-Iitaka dimension** of D is defined as

$$\kappa_f(X, D) := \max\left\{\kappa(X, D') \,\middle|\, D' = \sum_{i=0}^{m} a_i (f^*)^i D, a_i \in \mathbb{Z}\right\}.$$

Dynamical Iitaka fibration

Theorem: ref. [MZ23b]

Suppose that $\kappa_f(X,D) \geq 0$. Then there is an f-equivariant dominant rational map

$$f \circlearrowright X \dashrightarrow Y \circlearrowleft f|_Y$$

with Y normal projective of dimension $\kappa_f(X,D)$ and $f|_Y$ a surjective endomorphism.

The ramification divisor

The **ramification divisor** R_f is defined by

$$R_f := \sum_E (\mathrm{mult}_E f^*(f(E)) - 1)E$$

where the sum runs over all prime divisors E of X.

The ramification divisor

The **ramification divisor** R_f is defined by

$$R_f := \sum_E (\mathrm{mult}_E f^*(f(E)) - 1)E$$

where the sum runs over all prime divisors E of X.

It is effective and satisfies the ramification formula

$$R_f = K_X - f^*K_X.$$

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

- $\kappa_f(X, R_f) = 0$. Then $f^{-1}(\operatorname{Supp} R_f) = \operatorname{Supp} R_f$ and the restriction

$$f|_{X \setminus \operatorname{Supp} R_f} : X \setminus \operatorname{Supp} R_f \to X \setminus \operatorname{Supp} R_f$$

  is (quasi-)étale.

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

- $\kappa_f(X, R_f) = 0$. Then $f^{-1}(\operatorname{Supp} R_f) = \operatorname{Supp} R_f$ and the restriction

$$f|_{X \setminus \operatorname{Supp} R_f} : X \setminus \operatorname{Supp} R_f \to X \setminus \operatorname{Supp} R_f$$

  is (quasi-)étale.

- $0 < \kappa_f(X, R_f) < \dim X$. We then have an f-equivariant dominant rational map

$$\varphi_{f,R_f} : X \dashrightarrow Y$$

  with $0 < \dim Y = \kappa_f(X, R_f) < \dim X$.

Dynamical Iitaka Program for ramification divisor

We consider three situations of $\kappa_f(X, R_f)$ and briefly introduce how the dynamical Iitaka theory works.

- $\kappa_f(X, R_f) = 0$. Then $f^{-1}(\operatorname{Supp} R_f) = \operatorname{Supp} R_f$ and the restriction

  $$f|_{X \setminus \operatorname{Supp} R_f} : X \setminus \operatorname{Supp} R_f \to X \setminus \operatorname{Supp} R_f$$

  is (quasi-)étale.
- $0 < \kappa_f(X, R_f) < \dim X$. We then have an f-equivariant dominant rational map

  $$\varphi_{f,R_f} : X \dashrightarrow Y$$

  with $0 < \dim Y = \kappa_f(X, R_f) < \dim X$.
- $\kappa_f(X, R_f) = \dim X$. In this case, R_{f^s} is big when $s \gg 1$.

Settings

We consider the case the (X, f) has admitted an f-equivariant extremal Fano contraction $\pi : X \to Y$ such that $\delta_f > \delta_{f|_Y}$.

Settings

We consider the case the (X, f) has admitted an f-equivariant extremal Fano contraction $\pi : X \to Y$ such that $\delta_f > \delta_{f|_Y}$.

Since π is of relative Picard number 1, the general fiber X_y is Fano and $\delta_f > \delta_{f|_Y} \geq 1$, the ramification divisor R_f is non-trivial and π-ample.

Settings

We consider the case the (X, f) has admitted an f-equivariant extremal Fano contraction $\pi : X \to Y$ such that $\delta_f > \delta_{f|_Y}$.

Since π is of relative Picard number 1, the general fiber X_y is Fano and $\delta_f > \delta_{f|_Y} \geq 1$, the ramification divisor R_f is non-trivial and π-ample.

Hence we want to use the dynamical Iitaka fibration associated to R_f to get another f-equivariant fibration.

Settings

We consider the case the (X, f) has admitted an f-equivariant extremal Fano contraction $\pi : X \to Y$ such that $\delta_f > \delta_{f|_Y}$.

Since π is of relative Picard number 1, the general fiber X_y is Fano and $\delta_f > \delta_{f|_Y} \geq 1$, the ramification divisor R_f is non-trivial and π-ample.

Hence we want to use the dynamical Iitaka fibration associated to R_f to get another f-equivariant fibration.

Further assume that the base Y is either an abelian variety or a smooth projective variety of Picard number one, we get our main results.

Main Results

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an extremal Fano contraction $\pi : X \to Y$ to an abelian variety Y of positive dimension. Suppose f admits a Zariski dense orbit and $\delta_f > \delta_{f|_Y}$. Then the following hold.

1. The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.
2. There exists an f-equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f-Iitaka fibration of R_f, such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f-polarized.

Main Results

Theorem

Let f be a surjective endomorphism of a smooth projective variety X admitting an f-equivariant **smooth** extremal Fano contraction $\pi : X \to Y$ with $\rho(Y) = 1$. Suppose $\delta_f > \delta_{f|_Y} = 1$. Then the following hold.

1. The ramification divisor satisfies $f^*R_f \equiv \delta_f R_f$.
2. There exists an f-equivariant dominant rational map $\varphi : X \dashrightarrow Z$, which is the f-Iitaka fibration of R_f, such that $0 < \dim Z < \dim X$ and $f|_Z$ is δ_f-polarized.

Main Results

Corollary

KSC holds for any smooth projective variety X admitting an extremal Fano contraction to an abelian variety.

Corollary

KSC holds for any $\mathbb{P}^n$-bundle over either a Q-abelian variety or a smooth projective variety of Picard number one.

Thank You!