Here's your requested AP-style integrals practice packet, generated from the provided notes:

## AP-style Practice Packet – Integrals Unit

This practice packet is designed to reinforce the ideas surrounding *definite and indefinite integrals* using only the content of the provided notes on rational and reciprocal functions. The notes remind us that rational functions are a crucial "stepping stone in calculus, where rational functions frequently appear in the study of limits, derivatives, and integrals". When solving problems below, remember that rational functions have vertical asymptotes where the denominator is zero and that reciprocal functions preserve the original function's positive/negative intervals but reverse its intervals of increase and decrease. Inverse proportionality is another important theme—the notes state that if two quantities are inversely proportional, one decreases as the other increases. These observations will be used to reason about the sign of definite integrals and about whether a Riemann sum is an overestimate or underestimate.

## Section A – Conceptual & Interpretation

These questions emphasise the meaning of the definite integral as **signed area** and accumulation. For rational functions, note where the function is positive or negative and whether it increases or decreases using the reciprocal-function properties from the notes.

1. **Rational area and sign.** Let $f(x) = \frac{1}{x+1}$ on $0 \leq x \leq 2$.
   a. Explain whether $f(x)$ is positive or negative on this interval, and interpret $\int_0^2 f(x)\,dx$ as the area of a region with respect to the $x$–axis. Use the fact that rational functions appear in integral problems.
   b. Compute $\int_0^2 f(x)\,dx$ exactly and give a decimal approximation.

   **Solution:** Because $x + 1$ is always positive for $x \geq 0$, the rational function $f(x) = 1/(x+1)$ is positive. Therefore the definite integral represents the area under the curve above the $x$–axis. An antiderivative of $f(x)$ is $\ln|x+1|$, so

$$\int_0^2 \frac{1}{x+1}\,dx = \ln|x+1|\Big|_0^2 = \ln 3 - \ln 1 = \ln 3 \approx 1.0986.$$

2. **Symmetry and even functions.** Let $g(x) = \frac{1}{x^2+1}$.
   a. Explain why $g(x)$ is positive for all real $x$ and why $g(-x) = g(x)$.
   b. Use this symmetry to show that $\displaystyle\int_{-1}^{1} g(x)\,dx = 2\int_0^1 g(x)\,dx$.
   c. Evaluate $\int_0^1 g(x)\,dx$ exactly and numerically.

   **Solution:** The denominator $x^2+1$ is always positive, so the function is positive. Because squaring removes signs, $g(-x) = \frac{1}{(-x)^2+1} = \frac{1}{x^2+1} = g(x)$; such a function is even. On symmetric intervals $[-1,1]$, the areas on $[-1,0]$ and $[0,1]$ are equal, so the total integral is twice the integral over $[0,1]$. An antiderivative is $\arctan x$, so

$$\int_0^1 \frac{1}{x^2+1}\,dx = \arctan x\Big|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4} \approx 0.7854.$$

   Therefore $\int_{-1}^1 g(x)\,dx = 2 \times 0.7854 \approx 1.5708$.

3. **Odd integrands and cancellation.** Let $h(x) = \frac{x}{x^2+1}$.
   a. Show that $h(-x) = -h(x)$ and argue that $h$ is an odd function.
   b. Use symmetry to evaluate $\displaystyle\int_{-2}^{2} h(x)\,dx$ without doing any algebra.

   **Solution:** Substituting $-x$ yields $h(-x) = \frac{-x}{(-x)^2+1} = -\frac{x}{x^2+1} = -h(x)$, so $h$ is odd. For an odd function on a symmetric interval $[-a,a]$, the areas above and below the $x$–axis cancel; thus $\int_{-2}^2 h(x)\,dx = 0$. (No calculation is necessary.)

4. **Accumulation as displacement.** A particle moves with velocity $v(t) = \frac{1}{t+2}$ for $0 \le t \le 3$.
   a. Explain why the total distance travelled equals the net change in position on this interval.
   b. Evaluate $\int_0^3 v(t)\,dt$.

   **Solution:** For $t \ge 0$, $t+2 > 0$, so $v(t) > 0$. Because the velocity never changes sign, the definite integral gives both net displacement and total distance travelled. An antiderivative of $1/(t+2)$ is $\ln|t+2|$, so

   $\int_0^3 \frac{1}{t+2}\,dt = \ln|t+2|\Big|_0^3 = \ln 5 - \ln 2 \approx 0.9163.$

5. **Fundamental Theorem of Calculus (FTC).** Define $F(x) = \int_1^x \frac{1}{t+1}\,dt$.
   a. Use the FTC to find $F'(x)$.
   b. Compute $F(2)$.

   **Solution:** By the FTC, $F'(x) = \frac{1}{x+1}$. Evaluating $F$ at $x=2$ gives $F(2) = \int_1^2 \frac{1}{t+1}\,dt = \ln|t+1|\Big|_1^2 = \ln 3 - \ln 2 \approx 0.4055.$

6. **Domain restrictions and integrals.** Consider the rational function $r(x) = \frac{x-1}{x+2}$. From the notes we know the denominator controls restrictions for rational functions.

a. Determine the values of $x$ that are not allowed in the domain.
b. Evaluate the definite integral $\int_{-1}^{1} r(x)\,dx$.

**Solution:** The denominator $x+2$ vanishes at $x=-2$, so $x=-2$ is a vertical asymptote and must be excluded. The interval $[-1,1]$ does not cross this asymptote, so the integral is proper. Using long division, $\frac{x-1}{x+2} = 1 - \frac{3}{x+2}$. Thus
$\int_{-1}^{1} r(x)\,dx = \int_{-1}^{1}\left[1-\frac{3}{x+2}\right]dx = \left[x - 3\ln|x+2|\right]_{-1}^{1} = (1-3\ln 3) - \left(-1-3\ln 1\right) = 2 - 3\ln 3 \approx -1.2958.$

## Section B – Approximation Methods

Approximation methods such as left-, right-, and midpoint Riemann sums (LRAM, RRAM, MRAM) and the trapezoidal rule provide numerical estimates of definite integrals. Whether a particular Riemann sum over- or underestimates the true value depends on whether the integrand is increasing or decreasing. For functions that always decrease (e.g., reciprocals of linear functions), LRAM produces an overestimate and RRAM an underestimate, because the function values at the left endpoints exceed those at the right endpoints.

For the following problems, use **4 subintervals** unless stated otherwise. Clearly state whether each approximation is an overestimate or underestimate and give a numerical answer. (Approximate values were computed using uniform subintervals.)

7. **Approximating $\int_0^2 \frac{1}{x+1}\,dx$.**
   a. Compute the **Left Riemann Sum (LRAM)** and **Right Riemann Sum (RRAM)** with $n=4$.
   b. Decide which approximation overestimates and which underestimates the true integral, explaining your reasoning using the fact that $\frac{1}{x+1}$ is decreasing on $[0,2]$ (an instance of inverse proportionality).

   **Solution:** Partitioning $[0,2]$ into four subintervals of width $\Delta x = 0.5$ gives the sample points 0,0.5,1,1.5 and 0.5,1,1.5,2. Evaluating $f(x) = 1/(x+1)$ at these points and multiplying by $\Delta x$ gives:
   • LRAM $\approx 1.2833$.
   • RRAM $\approx 0.9500$.
   The true integral $\int_0^2 \frac{1}{x+1}\,dx = \ln 3 \approx 1.0986$. Because $f$ is decreasing, the left endpoints lie above the curve on each subinterval, so LRAM overestimates, while RRAM underestimates.

8. **Comparing MRAM and the trapezoidal rule.**
   Estimate $\int_0^1 \frac{1}{x^2+1}\,dx$ using the **midpoint** and **trapezoidal** rules with four subintervals. Which method is more accurate, and why? Recall that $\frac{1}{x^2+1}$ is decreasing on $[0,1]$.

   **Solution:** Using uniform subintervals of width 0.25, the midpoint rule yields $M_4 \approx 0.7867$ and the trapezoidal rule gives $T_4 \approx 0.7828$. The exact integral is $\pi/4 \approx 0.7854$. Both approximations are close, but the midpoint estimate is slightly closer. The trapezoidal rule averages the over- and underestimates from the left and right endpoints, while the midpoint rule samples where the function value better reflects the average height; for a moderately smooth function this tends to reduce error.

9. **Mixed monotonicity.** The function $h(x) = \frac{x}{x^2+1}$ increases on $[0,1]$ and decreases on $[1,2]$ (its derivative changes sign at $x=1$).

a. Compute LRAM, RRAM, MRAM, and the trapezoidal approximation for $\int_0^2 h(x)\,dx$ using four equal subintervals.
b. Compare each approximation with the true value $\int_0^2 h(x)\,dx = \frac{1}{2}\ln 5 \approx 0.8047$. Discuss why some approximations are above and others below the true value, relating your discussion to the changing monotonicity of $h$.

**Solution:** With $\Delta x = 0.5$ the approximations are:
– **LRAM:** $0.6808$.
– **RRAM:** $0.8808$.
– **MRAM:** $0.8169$.
– **Trapezoidal:** $0.7808$.
Since $h$ increases on $[0,1]$ and decreases on $[1,2]$, the left rectangles underestimate the first half and overestimate the second half, while the right rectangles do the opposite. These opposing errors lead to LRAM being slightly below the exact value and RRAM above it. The midpoint and trapezoidal approximations balance the over- and underestimates, yielding values closer to $0.8047$.

10. **Longer interval trapezoidal estimate.** Approximate $\int_0^4 \frac{1}{x^2+1}\,dx$ using the trapezoidal rule with four subintervals. Compare with the exact value $\arctan 4 \approx 1.3258$. Is the trapezoidal estimate an overestimate or underestimate?

**Solution:** Partitioning $[0,4]$ into four subintervals of width 1 gives trapezoidal estimate $T_4 \approx 1.3294$. Because $\frac{1}{x^2+1}$ is decreasing and concave upward on this interval, the trapezoids lie slightly above the curve at the left end and below at the right end; however the concavity makes the trapezoidal rule slightly overestimate the area. Hence $T_4$ is a small overestimate of $\arctan 4 \approx 1.3258$.

11. **Riemann sums for an inversely proportional function.** For $k(x) = \frac{1}{x+3}$ on $[1,5]$, take 5 subintervals.
a. Compute the LRAM and RRAM approximations for $\int_1^5 k(x)\,dx$.
b. Given that $k(x)$ decreases as $x$ increases, which approximation is an overestimate?
c. Compare both approximations with the exact value $\ln \frac{8}{4} = \ln 2 \approx 0.6931$.

**Solution:** With $\Delta x = 0.8$, the approximations are:
– **LRAM:** $0.7456$.
– **RRAM:** $0.6456$.
Since $k(x)$ is decreasing (as an inversely proportional function), the left rectangles overshoot the function on each subinterval and the right rectangles undershoot it. Thus LRAM overestimates and RRAM underestimates the true value $\ln 2 \approx 0.6931$. The midpoint of these two approximations $(0.7456 + 0.6456)/2 \approx 0.6956$ is very close to the actual integral.

## Section C – Analytical Integration

These problems ask you to compute antiderivatives and evaluate definite integrals of rational functions. Remember that rational functions often appear in calculus problems and that domain restrictions occur at values where the denominator is zero.

12. **Antiderivative of a simple reciprocal.** Find $\displaystyle\int \frac{1}{x+1}\,dx$. Then evaluate $\int_0^2 \frac{1}{x+1}\,dx$.

**Solution:** An antiderivative of $1/(x+1)$ is $\ln|x+1|+C$. Evaluating from 0 to 2 gives $\ln 3 \approx 1.0986$, as in Section A.

13. **Decomposing a rational integrand.** Compute $\displaystyle\int \frac{x-1}{x+2}\,dx$, and then use it to evaluate $\int_{-1}^1 \frac{x-1}{x+2}\,dx$ (confirming the result from Section A).

**Solution:** Perform long division: $\frac{x-1}{x+2} = 1 - \frac{3}{x+2}$. Therefore
$\int \frac{x-1}{x+2}\,dx = \int \left[1 - \frac{3}{x+2}\right] dx = x - 3\ln|x+2| + C.$
Evaluating from $-1$ to $1$ yields $2 - 3\ln 3 \approx -1.2958$ (as before).

14. **Integrating an even rational function.** Evaluate $\displaystyle\int \frac{1}{x^2+1}\,dx$ and compute $\int_0^1 \frac{1}{x^2+1}\,dx$ and $\int_{-1}^1 \frac{1}{x^2+1}\,dx$.

**Solution:** The antiderivative is $\arctan x + C$. Thus

and $\int_{-1}^1 \frac{1}{x^2+1}\,dx = 2 \times \frac{\pi}{4} = \frac{\pi}{2} \approx 1.5708.$

15. **Integrating an odd rational function.** Evaluate $\displaystyle\int \frac{x}{x^2+1}\,dx$ and $\int_0^2 \frac{x}{x^2+1}\,dx$.

**Solution:** Let $u = x^2+1$; then $du = 2x\,dx$, so $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{u}\,dx = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$. Evaluating from 0 to 2 gives

$$\int_0^2 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)\Big|_0^2 = \frac{1}{2}(\ln 5 - \ln 1) = \frac{1}{2}\ln 5 \approx 0.8047.$$

16. **Substitution with a shifted denominator.** Compute $\int_0^3 \frac{2x}{x^2+4}\,dx$.

**Solution:** Using $u = x^2+4$ and $du = 2x\,dx$, the integral becomes $\int \frac{du}{u} = \ln|u| + C$. Thus

$$\int_0^3 \frac{2x}{x^2+4}\,dx = \ln|x^2+4|\Big|_0^3 = \ln(13) - \ln 4 \approx 1.1787.$$

17. **Arctangent substitution.** Evaluate $\int_0^2 \frac{1}{x^2+4}\,dx$.

**Solution:** Rewrite the integrand as $\frac{1}{4}\cdot\frac{1}{(x/2)^2+1}$. Let $u = x/2$; then $dx = 2\,du$. The integral becomes
$\int_0^2 \frac{1}{x^2+4}\,dx = \int_0^1 \frac{1}{4}\cdot\frac{2}{u^2+1}\,du = \frac{1}{2}\int_0^1 \frac{1}{u^2+1}\,du = \frac{1}{2}\left[\arctan u\right]_0^1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8} \approx 0.3927.$

18. **Partial fractions (optional challenge).** For $x \neq -1, -2$, decompose and integrate $\dfrac{1}{(x+1)(x+2)}$. Then evaluate $\int_0^1 \frac{1}{(x+1)(x+2)}\,dx$.

**Solution:** The partial-fraction decomposition is $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$. Therefore
$\int \frac{1}{(x+1)(x+2)}\,dx = \int \left[\frac{1}{x+1} - \frac{1}{x+2}\right] dx = \ln|x+1| - \ln|x+2| + C.$
Evaluating from 0 to 1 yields $\ln 1 - \ln 2 - \big(\ln 2 - \ln 3\big) = \ln 3 - 2\ln 2 \approx -0.2877.$

## Section D – Theorem Applications & Symmetry

This section uses properties of definite integrals—such as additivity and symmetry—to simplify calculations. Although the notes discuss rational and reciprocal functions, the principles apply to any integrable function.

19. **Additivity of integrals.** Show that $\int_0^2 \frac{1}{x+1}\,dx$ can be split at $x = 1$ into the sum of two simpler integrals. Compute both parts and verify that they add up to $\ln 3$.

**Solution:** Because rational functions appear frequently in integral problems, we can use the property $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$. Splitting at $x = 1$ gives

$$\int_0^2 \frac{1}{x+1}\,dx = \int_0^1 \frac{1}{x+1}\,dx + \int_1^2 \frac{1}{x+1}\,dx.$$

Each part evaluates to $\ln|x+1|\Big|_0^1 = \ln 2 - \ln 1 = \ln 2$ and $\ln|x+1|\Big|_1^2 = \ln 3 - \ln 2$. Summing them yields $\ln 3$.

20. **Using odd/even properties.**
    a. Explain why $g(x) = \frac{1}{x^2+1}$ is an even function and use this to evaluate $\int_{-2}^2 g(x)\,dx$.
    b. Explain why $h(x) = \frac{x}{x^2+1}$ is odd and use symmetry to find $\int_{-2}^2 h(x)\,dx$.

**Solution:** As shown earlier, $g(-x) = g(x)$, so it is even. Thus $\int_{-2}^2 g(x)\,dx = 2\int_0^2 g(x)\,dx = 2\,\arctan 2 \approx 2 \times 1.1071 \approx 2.2142$. The function $h(x)$ satisfies $h(-x) = -h(x)$, so it is odd and integrates to zero over symmetric limits: $\int_{-2}^2 h(x)\,dx = 0$.

21. **Piecewise accumulation.** Suppose a function $p(x) = \frac{1}{x+1}$ is used to compute the total accumulation over two consecutive intervals: $[0,1]$ and $[1,3]$.
    a. Compute $\int_0^1 p(x)\,dx$.
    b. Compute $\int_1^3 p(x)\,dx$.
    c. Interpret the sum in terms of total accumulation.

**Solution:** From previous calculations, $\int_0^1 p(x)\,dx = \ln 2 \approx 0.6931$. Evaluating $\int_1^3 p(x)\,dx = \ln|x+1|\Big|_1^3 = \ln 4 - \ln 2 = \ln 2 \approx 0.6931$. The total accumulation on $[0,3]$ is their sum $\approx 1.3862$, which equals $\ln 4$. This confirms that splitting the interval and adding the results yields the same total change.

22. **Zero net area but non-zero total area.** Let $m(x) = \frac{x}{x^2+4}$.
    a. Show that $m(x)$ is odd about the origin and thus $\int_{-a}^a m(x)\,dx = 0$.
    b. Explain why the **total area** $\int_{-a}^a |m(x)|\,dx$ is positive even though the net area is zero.

**Solution:** Substituting $-x$ gives $m(-x) = \frac{-x}{(-x)^2+4} = -m(x)$; hence $m$ is odd. By symmetry the positive and negative regions cancel, so $\int_{-a}^a m(x)\,dx = 0$. However, taking the absolute value makes all areas positive; $\int_{-a}^a |m(x)|\,dx$ counts the area in both halves and is therefore strictly positive.

23. **Application of domain restrictions.** Suppose $s(x) = \frac{1}{x(x+3)}$. The notes stress that the denominator of a rational function indicates its restrictions.
   a. Determine the values excluded from the domain.
   b. Evaluate $\int_1^2 s(x)\,dx$.
   c. Why can't we integrate $s(x)$ across $x = 0$ or $x = -3$?

**Solution:** The denominator is zero at $x = 0$ and $x = -3$; therefore those points are vertical asymptotes (restrictions). The integral on $[1, 2]$ is proper because it avoids the singularities. Using partial fractions, $\frac{1}{x(x+3)} = \frac{1}{3x} - \frac{1}{3(x+3)}$. Thus
$\int_1^2 \frac{1}{x(x+3)}\,dx = \frac{1}{3}\int_1^2 \left[\frac{1}{x} - \frac{1}{x+3}\right]dx = \frac{1}{3}\Big[\ln|x| - \ln|x+3|\Big]_1^2 = \frac{1}{3}\big(\ln 2 - \ln 5 - (\ln 1 - \ln 4)\big) = \frac{1}{3}\big(\ln 8 - \ln 5\big) \approx -0.0680.$
Attempting to integrate across $x = 0$ or $x = -3$ would require an improper integral because the integrand blows up at those points; the definite integral is not defined across a vertical asymptote.

24. **Changing limits and reversing signs.** Show that $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ using $f(x) = \frac{1}{x+1}$ as an example on $[0, 2]$. Then verify numerically.

**Solution:** The reversal property of definite integrals states that swapping the limits introduces a negative sign. For $f(x) = 1/(x+1)$, we have
$\int_2^0 \frac{1}{x+1}\,dx = \left[-\int_0^2 \frac{1}{x+1}\,dx\right] = -(\ln 3) \approx -1.0986.$
Evaluating directly confirms this relationship. Geometrically, integrating from 2 down to 0 accumulates negative "area" because the direction is reversed.

25. **Combining properties.** Use both additivity and odd/even symmetry to evaluate $\displaystyle\int_{-3}^{3} \frac{x}{x^2+1}\,dx + \int_0^3 \frac{1}{x^2+1}\,dx$. Explain the reasoning without computing each integral from scratch.

**Solution:** The first integral has an odd integrand $\frac{x}{x^2+1}$; over $[-3, 3]$ it evaluates to zero (by symmetry). The second integral is half of the integral over $[-3, 3]$ of the even function $\frac{1}{x^2+1}$. Using the even property, $\int_{-3}^3 \frac{1}{x^2+1}\,dx = 2\int_0^3 \frac{1}{x^2+1}\,dx = 2\,\arctan 3$. Therefore

$$\int_{-3}^{3} \frac{x}{x^2+1}\,dx + \int_0^3 \frac{1}{x^2+1}\,dx = 0 + \int_0^3 \frac{1}{x^2+1}\,dx = \arctan 3 \approx 1.2490.$$

---

## Notes on Using This Packet

- All problems are crafted using concepts from the provided notes. The notes emphasise that rational and reciprocal functions often appear in calculus and that their behaviour (positive/negative intervals, increasing/decreasing intervals, and inverse proportionality) is important.
- Where overestimates or underestimates are discussed, the sign of the derivative or the changing monotonicity of the rational function is used for justification.
- Domain restrictions come from the zeros of the denominator of rational functions and must be respected when setting up integrals.