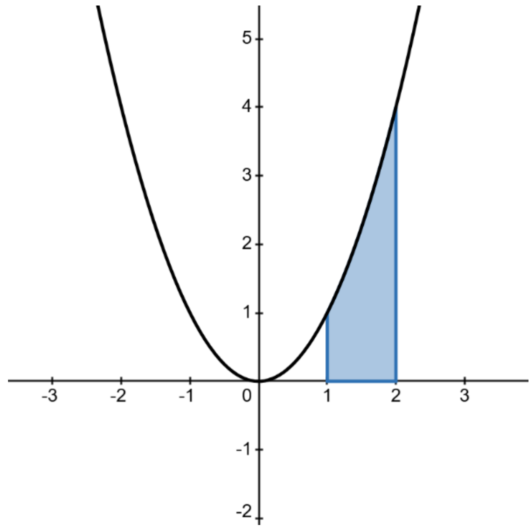
Estimating with Finite Sums

In *differential calculus***,** we learned a lot about rates of change, and how we can apply rates of change to a variety of different situations and contexts. Now, we will begin learning the "other half" of calculus, which is *integral calculus.* In this introductory lesson, we will introduce estimating with finite sums, an essential stepping stone toward understanding integrals. When we calculate the area under a curve, we gain valuable insights into the accumulation of quantities over time. In this lesson, we will learn how to *estimate* the area underneath a curve by dividing it into finite sub-intervals, summing the areas of rectangles or other shapes that fit under the curve.



Accumulation of Change

In differential calculus, we were given a function and asked to examine aspects of its rate of change. Now, we will be presented with a rate of change and will be asked to recover information about the original function.

Area Underneath a Curve

Finding the area underneath a graph of a *rate of change* (i.e. a graph of a first derivative) could tell us how much has been "accumulated" on the original function. For example, the area underneath a velocity graph can tell us how much *distance* was traveled.

Key Takeaway:

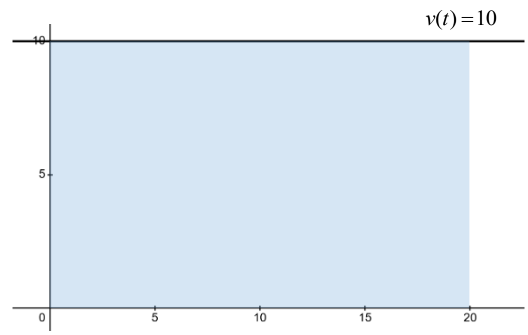
The area of a region bounded by the graph of the rate of change of a function and the horizontal axis tells us about the change in the original function

LEARNING GOALS

- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area
- Estimate the area under a curve using LRAM, MRAM, RRAM, and the Trapezoidal Rule
- Develop an understanding of the area under a curve as representing an accumulation of change
- Predict whether an estimation of area under a curve will be an overestimate or underestimate of the actual area

Re-examining Velocity

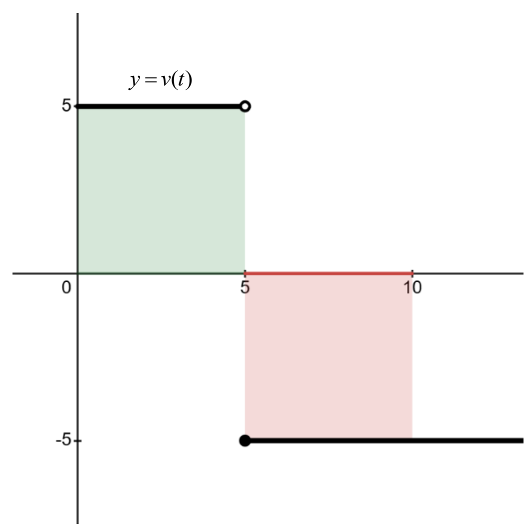
We know that if an object is moving at a steady rate of 10m/s for 20 seconds, then we can determine the total distance travelled in those 20 seconds by applying the formula distance = speed × time. Now we will look at making a connection between this result and the graph of the velocity function.



As we can see, calculating distance = speed × time is precisely the same as calculating the area of the rectangle that is found underneath the graph of $v(t)=10$ for $t \in [0,20]$. This, in fact, will be one of our main learning goals for this unit – determining the area *underneath a rate of change function* can help us to recover information about the *original function*. In other words, since velocity is the Rate of Change of position, we can use the area underneath the velocity graph to recover information about the object's position.

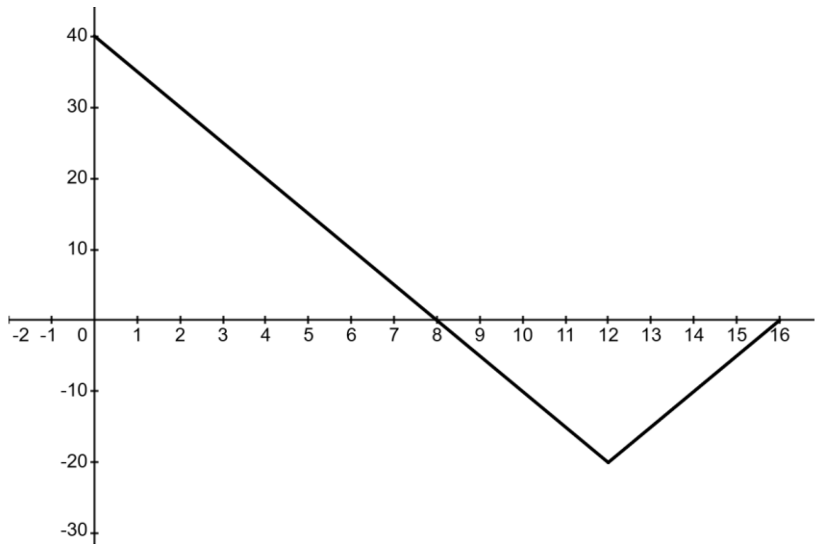
Positive and Negative Area

As we've learned previously, rates of change could be positive or negative. We also know that a positive rate of change indicates that a quantity is *increasing* and a negative rate of change indicates that a quantity is *decreasing.* We can extend these observations to our understanding of area underneath curves; areas that are found above the x-axis (i.e. underneath positive rates of change) are defined to be positive area, and areas that are found below the x-axis (i.e. underneath negative rates of change) are defined to be negative area.



In this mathematical context, "underneath" means "between the curve and the x-axis."

Estelle is in a hot air balloon. The function $R(t)$ models the rate of change in Estelle's altitude in feet per second, where t is measured in seconds. The graph of $R(t)$ is shown below.



a. On what interval(s) is Estelle's altitude *increasing*?

b. On what interval(s) is Estelle's altitude *decreasing*?

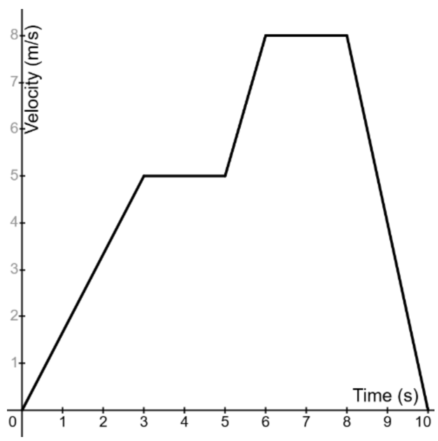
c. At what time(s) does Estelle reach her maximum altitude?

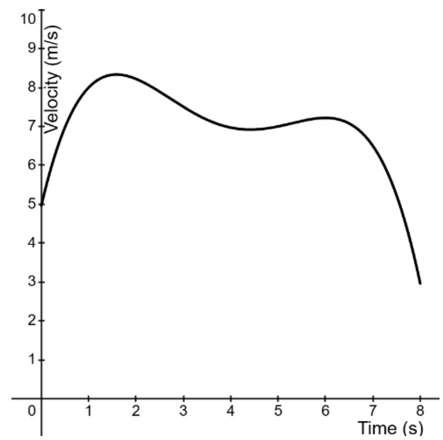
d. How much altitude did Estelle gain from $t = 0$ to $t = 8$?

e. What was the total change in Estelle's altitude over the first 16 seconds?

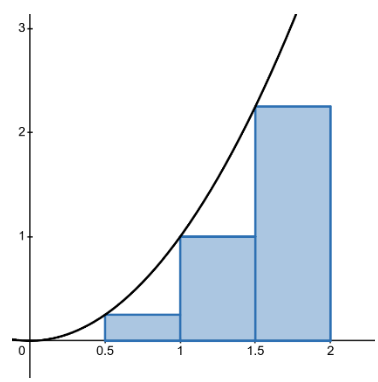
Area Under Contoured Curves

In the previous examples, we were able to easily find the area underneath the curve, since the curves were a simple rectangle and triangle, respectively. Consider the graphs below, which have a more complicated boundary. How might we estimate the area underneath these curves?





Rectangular Approximation Method (RAM)



The Rectangular Approximation Method (RAM) allows us to *approximate* the area underneath a curve by constructing a series of rectangles (of uniform or non-uniform width) underneath the curve, then finding the sum of the areas of those rectangles.

The rectangles are said to have width Δx, represented by the change in the x-axis, and height $f(c_i)$, represented by the value of the function at some reference point c_i.

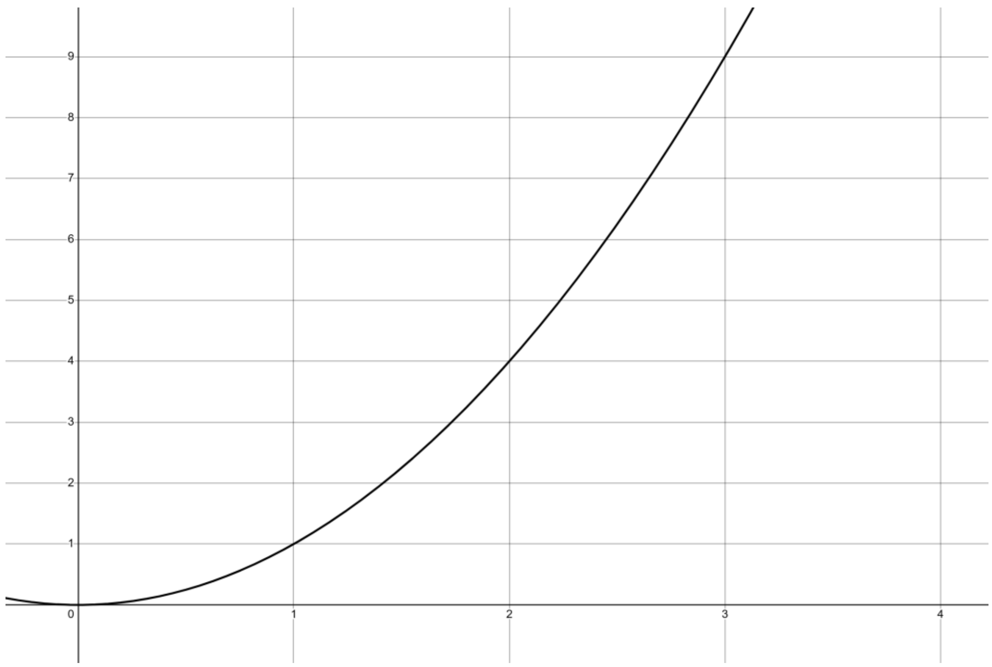
When we take a finite number of rectangles, this method is called a ***Riemann Sum,*** which we will study in much more detail in our next lesson. The reference point that we use in our RAM can be considered using:

- Left Endpoints (LRAM): The start of the sub-interval
- Right Endpoints (RRAM): The end of the sub-interval
- Midpoints (MRAM): The midpoint of the sub-interval

***Example 2** – Calculating MRAM*

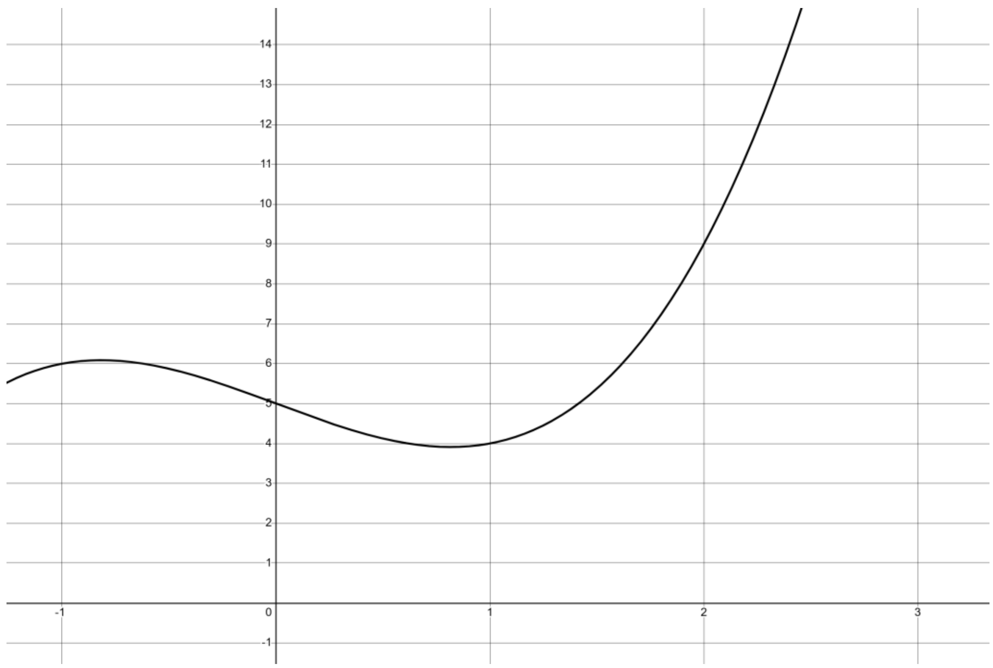
A particle starts at $t = 0$ and moves along the t-axis with velocity $v(t) = t^2$ for time $t \geq 0$.

Estimate the position of the particle at $t = 3$ by using the **Midpoint Rectangular Approximation Method** (MRAM) with 3 sub-intervals of uniform width.



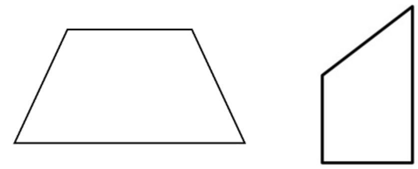
***Example 3** – Calculating with Riemann Sums LRAM*

A particle starts at $t=0$ and moves along the *t*-axis with velocity $v(t)=t^3-2t+5$ for time $t\geq 0.$ Estimate the position of the particle at $t=3$ by using the **Left Rectangular Approximation Method** (LRAM) with 3 sub-intervals of uniform width



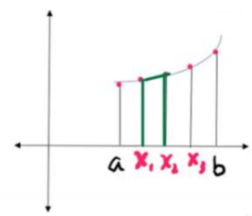
Trapezoidal Rule

A trapezoid is a quatrilateral with only one pair of parallel sides



The area of a trapezoid is given by $A = \left(\frac{b_1 + b_2}{2}\right)h$

Rather than construct rectangles underneath the curve, we can construct a series of trapezoids that have a flat bottom and sloped top. These trapezoids would more closely follow the shape of a curve

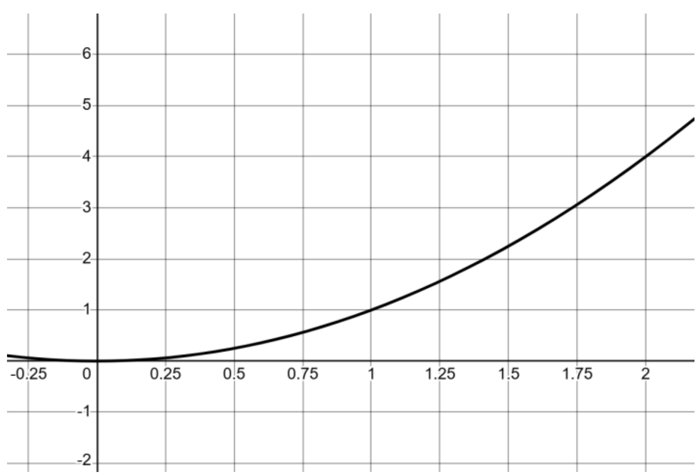


$$A = \frac{1}{2}\Delta x\left[\left(f(a) + f(x_1)\right) + \left(f(x_1) + f(x_2)\right) + \left(f(x_2) + f(x_3)\right) + \ldots\right]$$

$$A = \frac{\Delta x}{2}\left[f(a) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

Example 4 *– Trapezoidal Rule*

Use the Trapezoidal Rule with $n = 4$ to estimate the area underneath the curve $f(x) = x^2$ between $x = 1$ and $x = 2$.

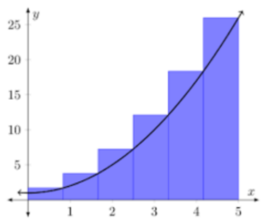


Overestimates vs Underestimates

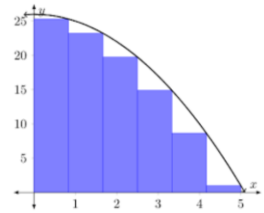
As we can see, using this method will always produce inaccuracies, since this method is merely an *estimate* of the area under a curve. How do we know if our number would be an overestimate or an underestimate?

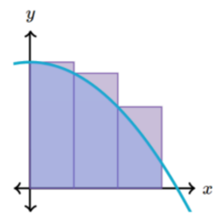
For LRAM and RRAM

Increasing Intervals:



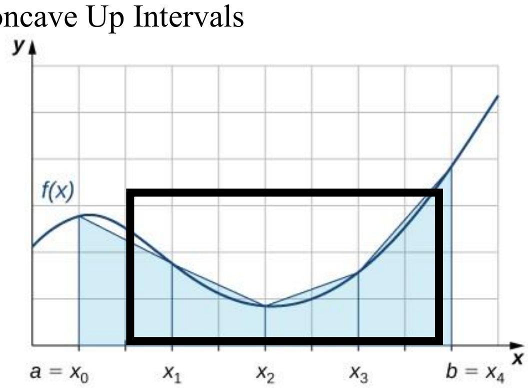
Decreasing Intervals:



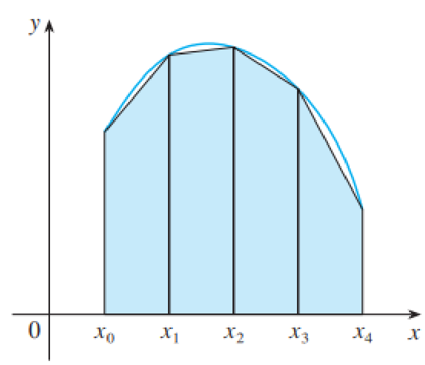


For Trapezoidal Rule

Concave Up Intervals



Concave Down Intervals



Chapter 2, Lesson 8

Strategies For Integration

Throughout this unit, we have been learning different ways that we may conceptualize integrals and how we might take the integral of a function. In our last lesson, we learned about the method of substitution to take the integral of a function. As we expand our understanding of integration, we encounter a wide variety of functions—some straightforward to integrate, while others require more advanced techniques. In this lesson, we will learn a couple of additional methods for integrating a function, and practice determining which strategy/strategies would be most appropriate to integrate a given function.

Integration Strategies

1. Apply integral rules
2. Remember common derivatives / antiderivatives
3. Geometry
4. Linear Composition
5. Short Division (i.e. splitting up a numerator)
6. *u*-substitution
7. Complete the Square
8. Long Division

Learning Goals

- Select and apply suitable strategies for integrating a variety of functions

Choosing Plans to Antidifferentiate

1. Look for a basic rule (or a combination of basic rules)
2. Look for a simple *u*-substitution
3. Try an algebraic technique like long division or completing the square

Example 1 *– Remembering Our Basic Rules*

Calculate $\int\left(2\cos x - x^4 + \frac{1}{x} + \frac{5}{1+x^2}\right)dx$

Example 2 *– Selecting Suitable Strategies*

Integrate the following two functions. What are the similarities / differences between them?

a. $\int \frac{x+5}{x^2+10x+26}\,dx$

b. $\int \frac{1}{x^2+10x+26}\,dx$

Example 3 *– Selecting Suitable Strategies*

Evaluate the following integrals. Note that while they all look similar, the each require a different approach.

a. $\int \frac{1}{x^2} dx$

b. $\int \frac{1}{x^2+1} dx$

c. $\int \frac{x}{x^2+1} dx$

d. $\int \frac{x^2}{x^2+1} dx$

Example 4 *– Geometry*

Calculate each integral by interpreting the integrand geometrically.

a. $\int_{-2}^{2} \sqrt{4-x^2}\, dx$

b. $\int_{0}^{6} 2x+\sqrt{9-(x-3)^2}\, dx$

Linear Compositions

If an integrand is a composition of functions where the inner function is *linear,* then we can technically use *u*-substitution to evaluate the integral. A simpler way to conceptualize this, though, is to simply integrate the function normally, then divide by the constant derivative of the linear function.

For instance:

$$\int \sin(3x+1)\,dx \qquad \text{let } u = 3x+1$$

$$du = 3dx$$

$$= \frac{1}{3}\int \sin(u)\,du \qquad \frac{1}{3}du = dx$$

$$= \frac{-\cos(u)}{3} + C$$

$$= \boxed{\frac{-\cos(3x+1)}{3} + C}$$

***Example 5** – Linear Compositions*

Integrate the following functions.

a. $\int \cos(5x+1)\,dx$

b. $\int e^{-2x}dx$

c. $\int \sec^2(3-7x)\,dx$

d. $\int \frac{1}{2\sqrt{5-3x}}dx$

A More Nuanced Look at Inverse Trigonometry…

We've been learning a lot about inverse trigonometric functions over the past year, especially in terms of their derivatives. Namely, we know the following to be true:

$$y = \arctan x \to y' = \frac{1}{x^2+1} \qquad y = \arcsin x \to y' = \frac{1}{\sqrt{1-x^2}} \qquad y = \arccos x \to y' = -\frac{1}{\sqrt{1-x^2}}$$

Consequently, we also know the following *integrals* to be true:

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \qquad \int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$

Look *very* closely at these forms though. You'll notice that the arguments in the denominator include 1 and some expression of x. In order to apply these integration formulas, your integrals *must* be in exactly this form.

So what happens if we get something of the form $\int \frac{1}{4+x^2}\,dx$? You have to factor out the 4!

We'll take a look at that in the next example…

***Example 6** – Integrating Inverse Trigonometric Functions*

Integrate the following functions.

a. $\int \frac{1}{4+x^2}\,dx$ b. $\int \frac{5}{\sqrt{36-x^2}}\,dx$ c. $\int -\frac{1}{\sqrt{7-x^2}}\,dx$

Example 7 *– Short Division vs Long Division*

Integrate the following two functions, once using short division and once using long division

a. $\int \frac{x^2}{x-1}\,dx$ (long division)

b. $\int \frac{x^2+3}{x^2+7}\,dx$ (short division)

Example 8 *– Selecting Suitable Strategies*

Evaluate each integral using the most suitable method

a. $\int \frac{1}{7x-5}\,dx$

b. $\int \frac{5}{x^2-6x+10}\,dx$

c. $\int \frac{2x+8}{x^2+8x+17}\,dx$

d. $\int \frac{w^3-3w^2+2}{w-3}\,dw$

e. $\int \frac{\left(2+\ln y\right)^3}{y}\,dy$

f. $\int \frac{3}{\sqrt{8-5x^2}}\,dx$

Chapter 2, Lesson 7

Anti-Differentiation by Substitution

In our study of integrals, we have thus far integrated very basic equations using simple antidifferentiation and/or areas of basic shapes. We will now learn a more sophisticaed method of integration called *integration by substitution*, which is a powerful technique that simplies complex integrals by reversing the Chain Rule of differentiation. Often, integrals contain nested functions or compositions that are difficult to evaluate directly. Substitution allows us to transform these expressions into more familiar forms, making them easier to integrate. In this lesson, we will learn to identify such cases, and use substitution to integrate the function.

Indefinite Integrals

Integrals which are *not* evaluated between given bounds. An indefinite integral gives a generalized anti-derivative of the integrand.

Substitution Rule

If an integrand is of the form $f'(g(x))\cdot g'(x)\,dx,$ then we can let $u=g(x)$ so that $du=g'(x)\,dx$ to solve the integral.

LEARNING GOALS

- Use substitution to evaluate indefinite integrals
- Use substitution to evaluate definite integrals

***Example 1** – Remembering Chain Rule*

Find the derivative of $y=(x^3+1)^7$

***Example 2** – Identifying "Inner Functions"*

Each of the functions here are composite functions. Identify the "inner function"

a. $y=(2x+1)^5$

b. $y=x\sin(x^2+7)$

c. $y=\dfrac{4x^3}{x^4+1}$

d. $y=x^2\sqrt{x^3-7}$

e. $y=\cos x\ln(\sin x)$

f. $y=6x^2e^{2x^3+8}$

***Example 3** – Understanding Substitution*

Consider the function $f(x) = \sin(x^2)$, and whose derivative is $f'(x) = 2x\cos(x^2)$. Use integration by substitution to show that $f(x) = \int f'(x)\,dx$.

Differentiation (Chain Rule)	**Integration (Substitution)**
$f(x) = \sin(x^2)$	
$f'(x) = \cos(x^2)\cdot\frac{d}{dx}(x^2)$	
$\boxed{f'(x) = 2x\cos(x^2)}$	

***Example 4** – Understanding Substitution (Arbitrary)*

Consider the function $y(x) = f\left(g(x)\right)$, and whose derivative is $y'(x) = f'\left(g(x)\right)\cdot g'(x)$. Use integration by substitution to show that $y(x) = \int y'(x)\,dx$.

Differentiation (Chain Rule)	**Integration (Substitution)**
$y(x) = f\left(g(x)\right)$	
$\boxed{y'(x) = f'\left(g(x)\right)\cdot g'(x)}$	

Example 5 – *Substitution*

Evaluate each integral using the Method of Substitution

a. $\int 2x\sqrt{1+x^2}\,dx$

b. $\int x^3 \cos(x^4+2)\,dx$

c. $\int \frac{\sin(\sqrt{x})}{2\sqrt{x}}\,dx$

d. $\int x^2 e^{x^3-7}\,dx$

Example 6 – *Definite Integrals with Substitution*

Integrate the following (familiar) integrals as *DEFINITE* integrals!

a. $\int_0^1 2x\sqrt{1+x^2}\,dx$

b. $\int_0^{\sqrt[4]{\pi-2}} x^3 \cos(x^4+2)\,dx$

__Example 7__ – Substitution (Slightly Trickier Ones)

Evaluate each integral using the Method of Substitution

a. $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$

b. $\displaystyle\int \tan(5x)\,dx$

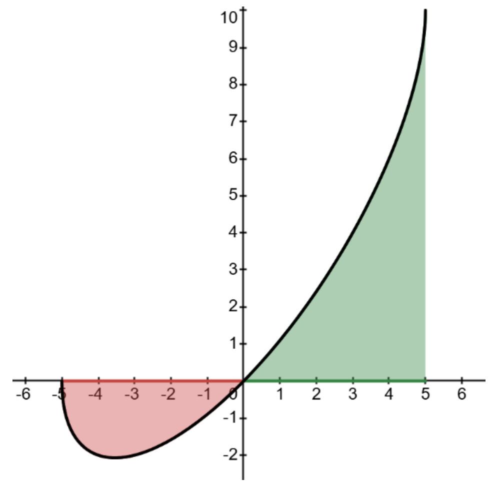
c. $\displaystyle\int \frac{1}{7+x^2}\,dx$

d. $\displaystyle\int \frac{\ln(\tan(\theta))}{\sin\theta\cos\theta}\,d\theta$

Chapter 2, Lesson 6

Applications of The Fundamental Theorem of Calculus

Now that we have learned about the Fundamental Theorem of Calculus, we will now focus on its applications in analyzing real-world scenarios where quantities change over time. As we established in our previous lesson, we can interpret a function as the rate of change of an integral. Knowing this allows us to analyze motion, accumulation, and area in a dynamic way. One key concept we will examine is the distinction between displacement and total distance traveled.. In today's lesson, we will continue to explore these applications of the Fundamental Theorem of Calculus, and of integrals in general.

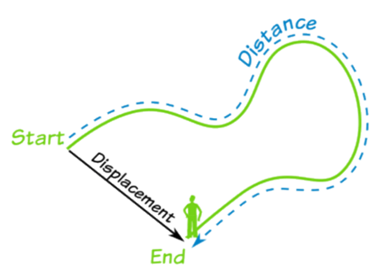


Displacement

The amount that an object has moved from its original position

Distance

The *total* amount an obect has travelled from its original position to its end destination.



LEARNING GOALS

- Apply the Fundamental Theorem of Calculus in a variety of contexts.
- Explain the difference between Distance and Displacement
- Recognize a function as the rate of change of an integral
- Recognize an integral as the accumulation of change of a function

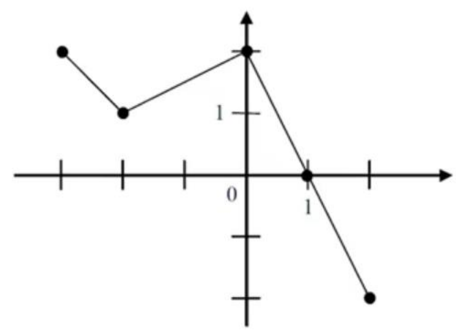
Example 1 *– Understanding FTC Part 1*

The graph of the piecewise linear function *f* is shown in the following figure. If $g(x)=\int_{-2}^{x} f(t)dt$, which of the following values is the greatest?

a. $g(-3)$ b. $g(-2)$

c. $g(0)$ d. $g(1)$

e. $g(2)$ f. None of these



Graph of f

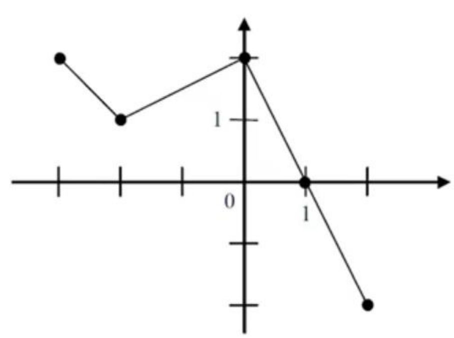
Example 2 *– Understanding the FTC Part 1*

For the same function as above, determine which of the following values is the greatest

a. $g'(-2)$ b. $g'(-1)$

c. $g'(0)$ d. $g'(1)$

e. $g'(2)$ f. None of these



Graph of f

Now consider the *second* derivatives! Which of the following values is the greatest?

a. $g''(-2)$ b. $g''(-1)$

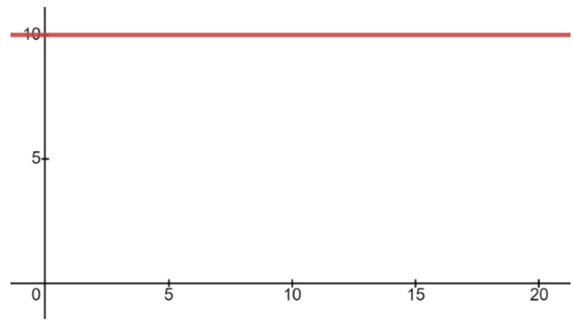
c. $g''(0)$ d. $g''(1)$

e. $|g''(1)|$ f. None of these

The Velocity Connection

Recall that *velocity* is the first derivative of position. Also recall that earlier in this unit, we discussed how finding the area under the graph of a velocity function can give us information about the function's position.

Recall this example, where we had constant velocity. In this example, an object was moving with a velocity of 10m/s for 20 seconds. To figure out the object's position, we simply multiplied together 10m/s × 20 seconds, which is precisely the same as finding the area under this line (which we have since come to understand is its *integral*).



So, if $s(t) = \int_a^t v(x)dx$ is the position of an object moving with velocity *v,* then the FTC says that $s'(t) = v(t)$ as we expect!

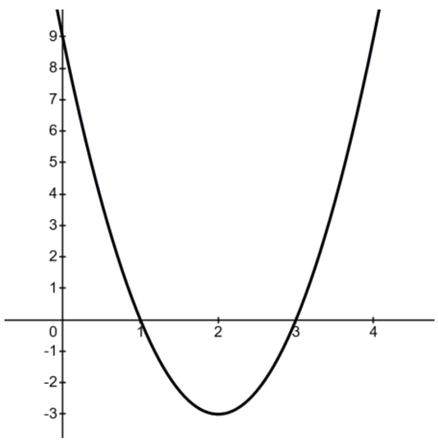
Example 3 *– FTC and Velocity*

King, the flying cat, has an altitude of $h(t) = \int_0^t \left(64 - 12u\right) du,$ where *t* is measured in seconds. What is the <u>rate of change</u> of King's altitude at $t = 4$ seconds?

***Example 4** – Finding Area Analytically*

The following function represents the velocity of some object moving through space. Find the total distance of the object analytically by partitioning the function at its roots. Then, discuss the difference between finding the *distance* and the *displacement* of the object.

$f(x) = 3x^2 - 12x + 9, [0,4]$



A Note About Absolute Value…

For the above example, consider the difference between finding the following:

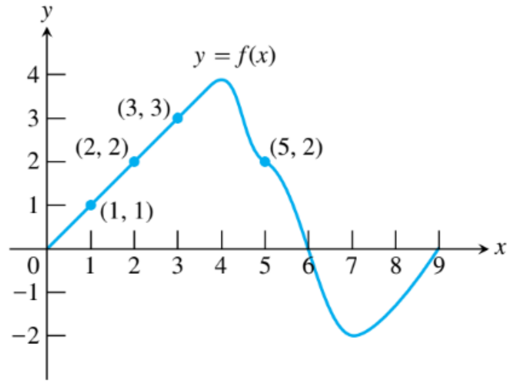
$$\int_0^4 3x^2 - 12x + 9\,dx$$

$$\int_0^4 \left|3x^2 - 12x + 9\right| dx$$

$$\left|\int_0^4 3x^2 - 12x + 9\,dx\right|$$

Let *f* be the differentiable function whose graph is shown to the right. The position at time *t* (sec) of a particle moving along a coordinate axis is

$$s(t) = \int_0^t f(x)dx$$

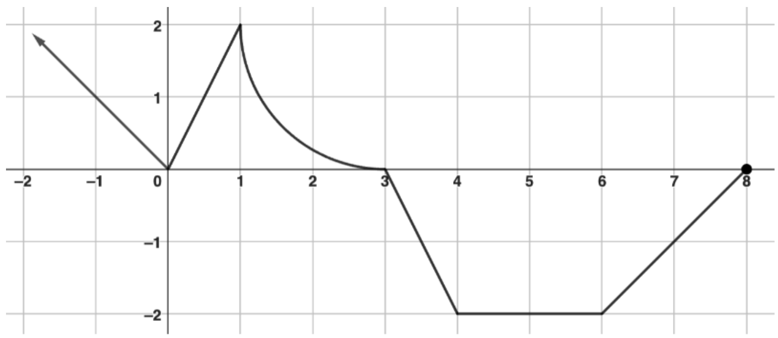


a. What is the particle's velocity at $t = 5$?

b. What is the particle's position at the time $t = 3$?

c. Give the *equation* which would describe the Displacement (D_1) and Distance (D_2) in the first 9 seconds

Let *f* be the function whose graph is given below. Note that the domain of *f* is $(-\infty, 8]$. The part of its graph between $x = 1$ and $x = 3$ is a quarter circle, and all other parts are straight lines.



Let *g* be the function defined by $g(x) = \int_0^x f(t)dt$, and let *h* be the function defined by $h(x) = \int_1^x f(t)dt$. Answer the following questions about *f, g,* and *h.*

a. Explain, conceptually, what *g* and *h* are. In other words, what quantity do they represent, as functions?

b. What are the domains of *g* and *h*?

c. At which point(s) on its domain is *f* non-differentiable?

d. At which points is *g* differentiable?

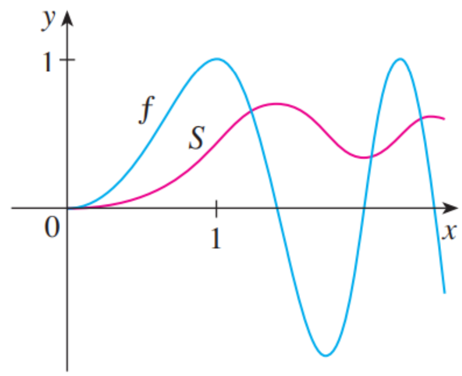
e. Let $G(x) = \int_x^{x^2+2} f(t)dt$. Find a value of *x* that is in the domain of *f* but *not* in the domain of *G*.

f. Compute $G'(1)$

CHAPTER 2, LESSON 5

THE FUNDAMENTAL THEOREM OF CALCULUS

Throughout our study of calculus, we have explored two seemingly separate ideas: differentiation, which measures rates of change, and integration, which calculates accumulation and areas under curves. The *Fundamental Theorem of Calculus (FTC)* serves as the critical bridge between these two concepts, revealing that differentiation and integration are, in fact, inverse operations. In this lesson, we will begin to learn about the Fundamental Theorem of Calculus and begin to explore the close relationship between an integral and a derivative.



Fundamental Theorem of Calculus (Part 1)

If f is a continuous function on $[a, b]$, then the function:

$F(x) = \int_a^x f(t)\,dt$ has a derivative at every point $x \in [a,b]$ which is:

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Fundamental Theorem of Calculus (Part 2)

(This is also called the **Integral Evaluation Theorem)**

If f is continuous on $[a, b]$, and if F is any antiderivative of f, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

LEARNING GOALS

- State the meaning of the Fundamental Theorem of Calculus, Part 1
- State the meaning of the Fundamental Theorem of Calculus, Part 2
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- Use the Integral Evaluation Theorem to evaluate definite integrals.
- Explain the relationship between differentiation and integration.

***Example 1** – FTC Part 2 (Integral Evaluation Theorem)*

Use the FTC Part 2 (the Integral Evaluation Theorem) to calculate each definite integral.

a. $\displaystyle\int_{-2}^{4} (x^2 + 2x)\,dx$

b. $\displaystyle\int_{1}^{4} \frac{2-\sqrt{x}}{x^2}\,dx$

c. $\displaystyle\int_{0}^{\frac{\pi}{2}} \left(\sin\theta - \cos\theta\right)d\theta$

Example 2 – *Conceptual Understanding of the FTC*

Suppose we have $F(x) = \int_1^x f(t)dt$. Find $F'(x)$.

Example 3 – *Conceptual Understanding of the FTC*

Suppose we have $F(x) = \int_{100,000}^{x} t^2\, dt$. Find $F'(x)$.

***Example 4** – Applying the FTC*

Use the FTC Part 1 to find $F'(x)$

a. $F(x)=\int_5^x (1-t)\,dt$

b. $F(x)=\int_{10}^x \sqrt{\cot t+4t^2}\,dt$

c. $F(x)=\int_{\pi}^x \dfrac{\ln\left(5-\sqrt{t+\sin t}\right)}{t-7}\,dt$

***Example 5** – FTC and the Chain Rule*

Use the FTC Part 1 to find $F'(x)$

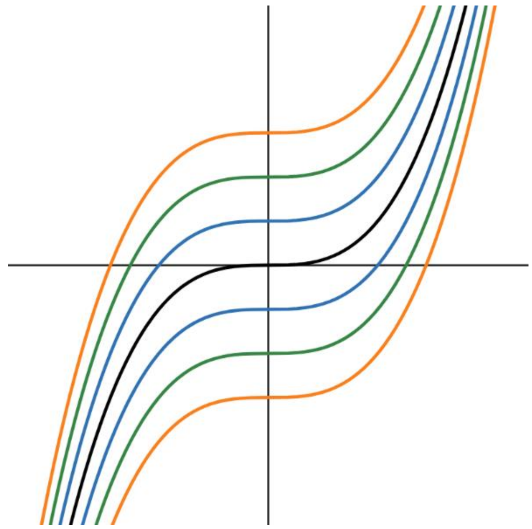
a. $F(x)=\int_6^{x^2} \csc(3t)\,dt$

b. $F(x)=\int_{3x^2}^{x^2+5} \ln(t+17)\,dt$

Chapter 2, Lesson 4

Antiderivatives

In our last lesson, we talked about some different integral properties. As we continue to learn more and more about integrals, we are inching closer and closer towards understanding the connection between integrals and derivatives. Today, we will learn the first step in that connection. In previous lessons, we explored derivatives as a way to measure rates of change and understand the behavior of functions. However, just as differentiation allows us to find the rate of change of a function, we now shift our focus to the reverse process; finding the original function given its derivative. In this lesson, we are going to solidify our understanding of the connection between an integral and a derivative by studying the *antiderivative.*



Antiderivative

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

LEARNING GOALS

- Determine a general antiderivative for a function
- Determine the antiderivative for a function that satisfies an initial value

General Antiderivatives

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x)+C$ where C is an arbitrary constant.

Function	Antiderivative
$cf(x)$	
$f(x)=x^n$ if $n \neq -1$	
$f(x)=\dfrac{1}{x}$	
$f(x)=\dfrac{1}{x\ln b}$	
$f(x)=e^x$	
$f(x)=b^x \ln b$	
$f(x)=\cos x$	
$f(x)=\sin x$	
$f(x)=\sec^2 x$	
$f(x)=\sec x \tan x$	
$f(x)=\dfrac{1}{\sqrt{1-x^2}}$	
$f(x)=-\dfrac{1}{\sqrt{1-x^2}}$	
$f(x)=\dfrac{1}{1+x^2}$	

***Example 1** – Anti-Derivatives of Power Functions*

Find the anti-derivative, $F(x)$, of each Power function, $f(x)$.

a. $f(x) = x$

b. $f(x) = x^3 + x^2$

c. $f(x) = \dfrac{1}{2\sqrt{x}}$

d. $f(x) = \pi$

e. $f(x) = \sqrt{x} + \sqrt[3]{x}$

f. $f(x) = x^{3.4} - x^{5.7}$

***Example 2** – Anti-Derivatives of Sinusoidal Functions*

Find the anti-derivative, $F(x)$, of each sinusoidal function, $f(x)$.

a. $f(x) = \sin x$

b. $f(x) = \cos x$

c. $f(x) = -\sin x$

d. $f(x) = \sec^2 x$

e. $f(x) = 2\sin x \cos x$

f. $f(x) = 1 + \cos x$

***Example 3** – Anti-Derivatives of Other Special Cases*

Find the anti-derivative, $F(x)$, of each function, $f(x)$.

a. $f(x) = e^x$

b. $f(x) = \frac{1}{x}$

c. $f(x) = \frac{1}{1+x^2}$

d. $f(x) = \sec x \tan x$

e. $f(x) = \frac{1}{x \ln 3}$

f. $f(x) = 5^x \ln 5$

***Example 4** – Anti-Derivatives with Linear Arguments*

Find the anti-derivative, $F(x)$, of each function, $f(x)$.

a. $f(x) = \sin 2x$

b. $f(x) = e^{5x-3}$

c. $f(x) = \cos 3x$

d. $f(x) = \sec^2(5-3x)$

e. $f(x) = \frac{1}{7x+3}$

f. $f(x) = \frac{1}{7x^2+3}$

***Example 5** – Other Non-Obvious Cases*

Find the anti-derivative, $F(x)$, of each function, $f(x)$.

a. $f(x)=\dfrac{2x^4+4x^3-x}{x^3}$

b. $f(x)=\dfrac{1}{5}-\dfrac{2}{x}$

c. $f(x)=2\cos x-\dfrac{3}{\sqrt{1-x^2}}$

d. $f(x)=(4x-5)^3$

e. $f(x)=e^2$

f. $f(x)=\dfrac{\sqrt{x}}{1+x+x^2}$

***Example 6** – Anti-Derivatives with Initial Conditions*

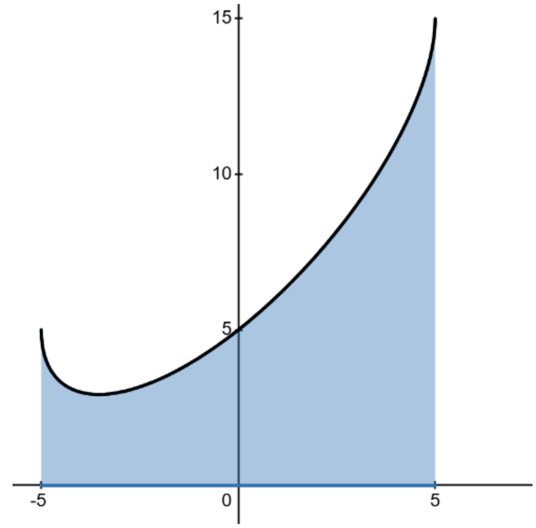
A particle is moving with the given data. Find the position of the particle given the initial condition.

a. $v(t)=\sin t-\cos t, \quad s(0)=0$

b. $v(t)=t^2-3\sqrt{t}, \quad s(4)=8$

Chapter 2, Lesson 3
Properties of Integrals

In differential calculus, we spent a lot of time taking derivatives. In integral calculus, the main focus of our discussions will revolve around Integrals. Just like with derivatives, there are many rules and properties about integrals that can help us to solve a variety of problems. In this lesson, we will begin to learn about those properties and learn how they can be used to solve a variety of problems involving integration, including interpreting an integral using geometric shapes. These properties allow us to break apart, combine, and scale integrals efficiently, much like we did with derivatives.



The Definite Integral

The signed area between a curve and the x-axis.

LEARNING GOALS

- Use geometry and the property of integrals to evaluate them

Properties of Integrals

The following properties of integrals can be used without verification or justification.

Name	Property	Interpretation
Order of Integration	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	We can reverse the bounds of integration by negating the integral.
Zero Property	$\int_a^a f(x)\,dx = 0$	There is *zero* area underneath a singular point.
Coefficient Property	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Coefficients can be factored out of definite integrals.
Sum / Difference Property	$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	Definite integrals are distributive across addition and subtraction.
Additivity Property	$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$	An interval of integration can be split into 2 segments by partitioning the interval at some value within its interior.
Odd Symmetry	If $f(x)$ is ODD, then $\int_{-a}^a f(x)dx = 0$	The integral of an odd function over a symmetrically spaced interval is 0
Even Symmertry	If $f(x)$ is EVEN, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$	When integrating an even function over a symmetrically spaced interval, you can integrate over half of that interval then double the result.

***Example 1** – Using the Integral Laws*

Suppose we have some continuous and integrable functions $f(x)$ and $g(x)$ and suppose we know the following:

$$\int_{-1}^{1} f(x)\,dx = 5 \qquad \int_{1}^{4} f(x)\,dx = -2 \qquad \int_{-1}^{1} g(x)\,dx = 7$$

It is also known that f is an *even* function. Evaluate each of the following integrals, if possible:

a. $\int_{4}^{1} f(x)\,dx$

b. $\int_{-1}^{4} f(x)\,dx$

c. $\int_{-1}^{1} [2f(x) + 3g(x)]\,dx$

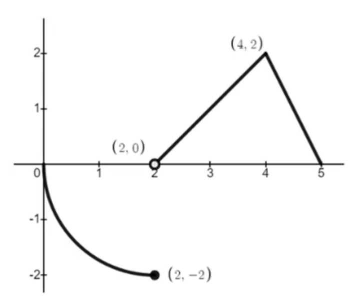
d. $\int_{0}^{1} f(x)\,dx$

e. $\int_{2}^{2} g(x)\,dx$

f. $\int_{-1}^{4} [f(x) + g(x)]\,dx$

***Example 2** – Graphical Representation*

Given the graph of $f(x)$, evaluate the following definite integrals



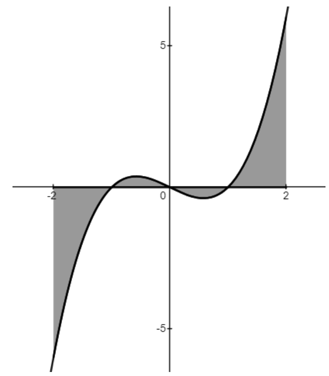
a. $\int_{0}^{2} f(x)dx$

b. $\int_{3}^{5} f(x)dx$

c. $\int_{0}^{5} f(x)dx$

Symmetry Properties

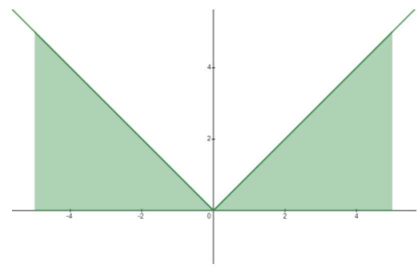
If $f(x)$ is an *ODD* function, and the interval of integration is evenly spaced around the origin, then the definite integral is 0 since there is an equal amount of area above and below the *x*-axis.



In other words:

If $f(x)$ is an ODD function, then $\int_{-a}^{a} f(x)dx = 0$

If $f(x)$ is an *EVEN* function, and the interval of integration is evenly spaced around the origin, then you can simply integrate *half* of the integral then double it, since there is an equal amount of area on either side of the *y*-axis.



In other words:

If $f(x)$ is an EVEN function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

Example 3 *– Symmetry Properties*

Evaluate the following definite integrals

a. $\int_{-3}^{3} x^3 - x\,dx$

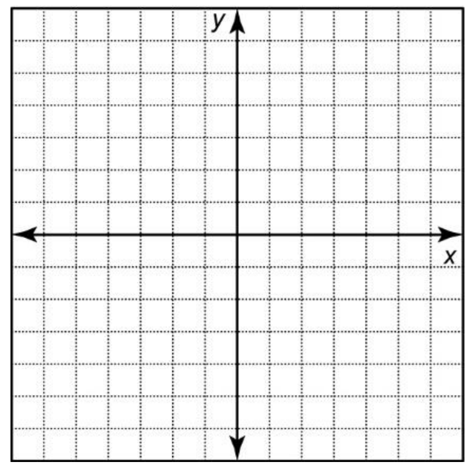
b. $\int_{-5}^{5} |\,x\,|\,dx$

c. $\int_{\pi}^{2\pi} \sin x\,dx$

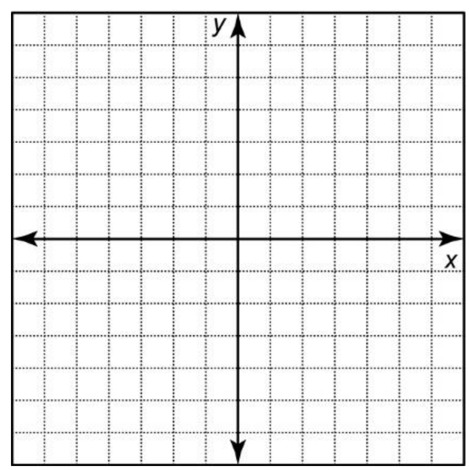
Example 4 – *Definite Integral*

Evaluate the integral by interpreting it as the area of a shape

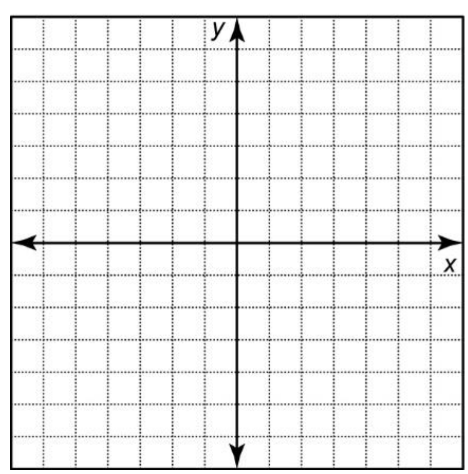
a. $\displaystyle\int_{-5}^{5} \sqrt{25-x^2}\,dx$



b. $\displaystyle\int_{\frac{1}{2}}^{\frac{3}{2}} -2t+4\,dt$

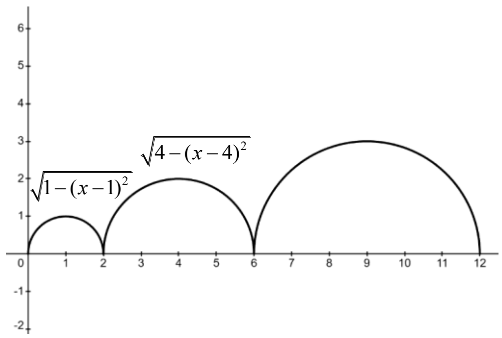


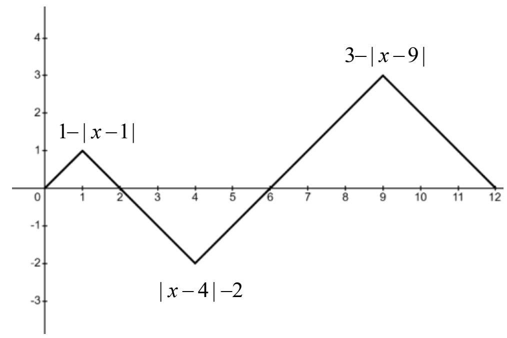
c. $\displaystyle\int_{-2}^{7} |\,x-1\,|\,dx$



Example 5 *– Definite Integral*

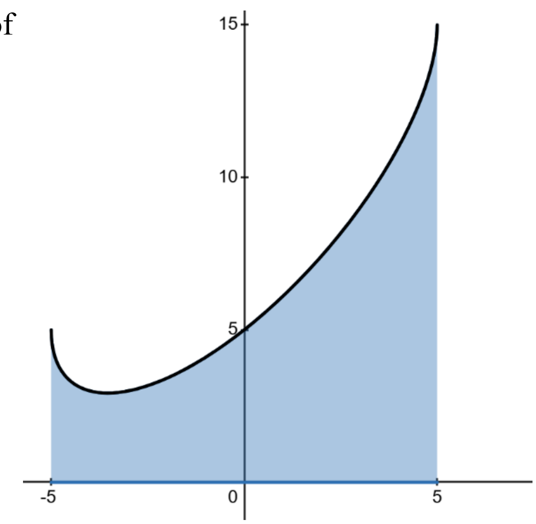
Shown below is the graph 2 functions. Set up an integral to represent the area underneath the curve, Then, evaluate the integral.





Example 6 *– Using Geometry*

Find the exact area between the x-axis and the graph of $f(x)=10+x-\sqrt{25-x^2}$ on the interval $[-5, 5]$.

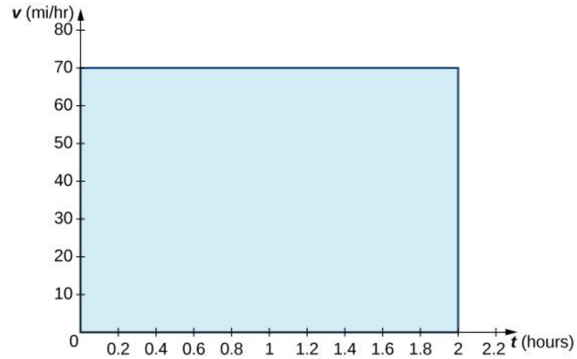


Integral of a Constant Function

The integral of a constant function is given by

$$\int_a^b c\,dx = c(b-a)$$

Integrate the function to the right



***Example** 7 – Revisiting Velocity*

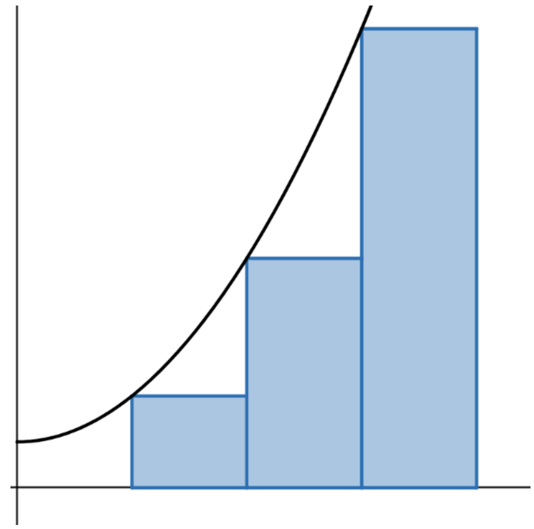
If an object is moving at a steady 10m/s from $t = 5$ to $t = 25$, we can express its total distance traveled as an integral, then evaluate. Remember, that an integral is an *accumulation,* so the integral of velocity is an accumulation of distance travelled!

Re-write the above situation as a function and integrate it to determine the total distance travelled.

Chapter 2, Lesson 2

Riemann Sums and the Definite Integral

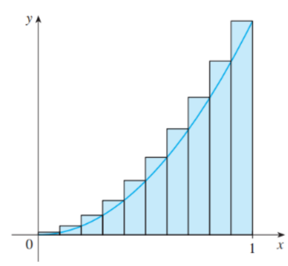
In our last lesson, we were introduced to the idea of the rectangular approximation method to estimate the area underneath a curve. As it turns out, this method is an example of a *Riemann Sum,* which is a type of finite sum that we use to approximate the area underneath a curve. We will now expand on those ideas by introducing the **Definite Integral.** In much the same way that First Principles develops the idea of Instantaneous Rate of Change, the Definite Integral develops the idea of a Riemann Sum. In this lesson, we will learn about the notation and fucntion of a Definite Integral and use it to interpret areas underneath curves.



Riemann Sum

A method of approximating the definite integral of a function by dividing the area under its curve into a series of finite rectangles and finding the sum of their areas.

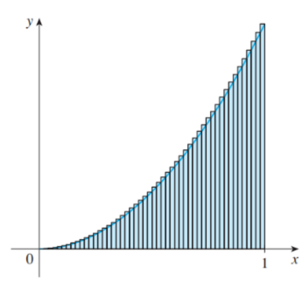
$$\sum_{i=1}^{n} \Delta x \cdot f(c_i)$$



The Definite Integral

The signed area between a curve and the x-axis. Theoretically, the definite integral involves finding a Riemann Sum with *infinitely many* rectangles.

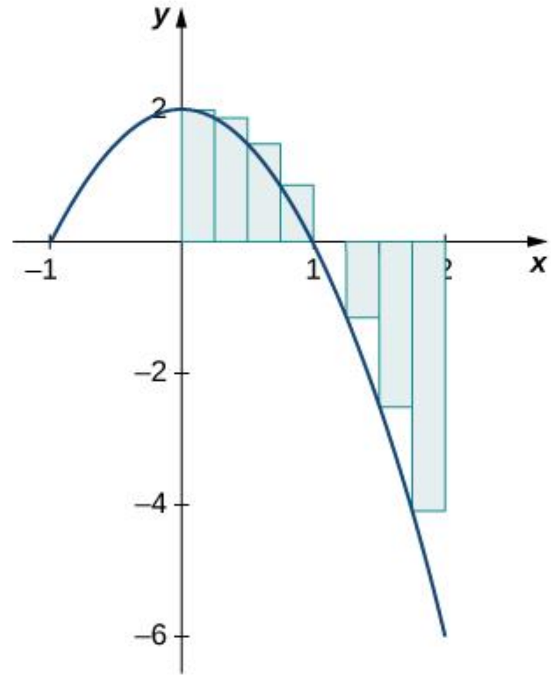
$$\lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \cdot f(c_i)$$



LEARNING GOALS

- State the definition of the definite integral
- Explain the terms integrand, limits of integration, and variable of integration
- Describe the relationship between the definite integral and net area
- Interpret the sigma expression for a definite integral using integral notation
- Use sigma notation to represent a left and right Riemann sum

Riemann Sums & Definite Integrals



In our last lesson, we constructed a finite number of rectangles underneath a curve in order to approximate the area under the curve, and concluded that the more rectangles we take, the more accurate our estimation will be. Let's now expand on that!

1. Take a function $f(x)$ over a closed interval $[a, b]$

2. Partition the function into n sub-intervals. In the last lesson, we had $n = 3$.

3. The width of one of these sub-intervals is $\Delta x = \dfrac{b-a}{n}$

4. Within each sub-interval, choose a point (either left endpoint, right endpoint, or midpoint) as a reference point to build the height of your rectangle. Call this point c_i, then the height of the rectangle is called $f(c_i)$.

5. The area of ONE rectangle would be $\Delta x \cdot f(c_i)$

6. We repeat this process for as many rectangles as we have. This is notated by $\sum_{i=1}^{n} \Delta x \cdot f(c_i)$, where n is the number of sub-intervals/rectangles that we are taking. This sum is called the **Riemann Sum.**

7. Our end goal is to take *INFINITELY* many rectangles, and so we obtain $\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(c_i)$.

   This gives us a true indication of the area underneath the curve, and is called a **Definite Integral.**

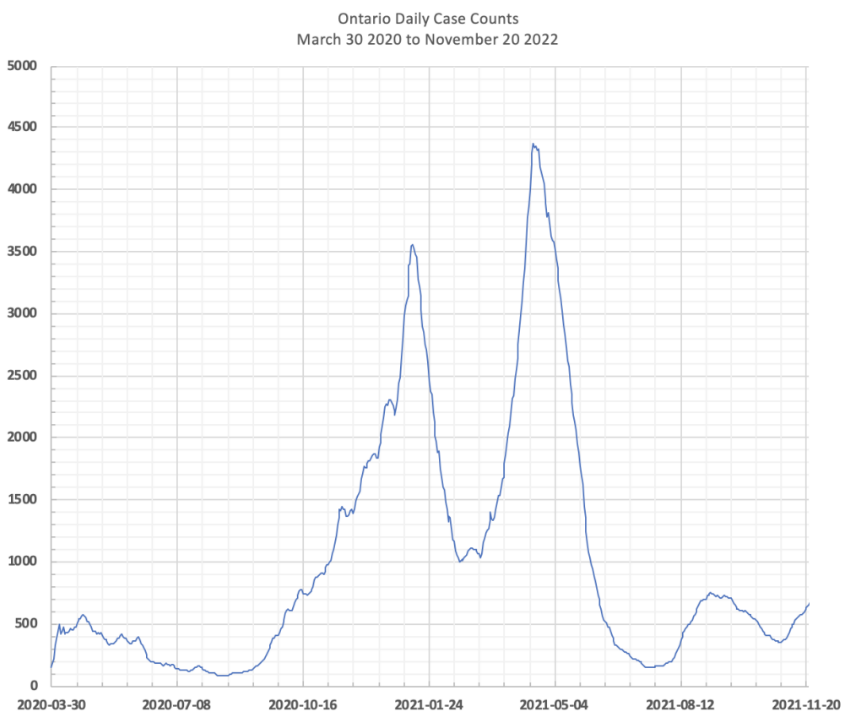
Summation Formulas & Sigma Notation

Sigma Notation is a mathematical notation that is used to denote a finite/infinite sum of numbers. The notation for sigma notation is shown below:

$$\sum_{i=a}^{n} f(i)$$

Example 1 *– Understanding the Riemann Sum*

Below shows a graph of daily case counts of COVID-19 cases in Ontario from March 30, 2020 to November 20, 2021 (600 days). The y-axis scale is 100 cases/tick and the x-axis scale is 20 days/tick. Find the value of the Riemann Sum of the curve on the entire interval with $n = 6$ and using Midpoints as sample points. Round your answer in some appropriate way.



The Definite Integral

The Definite Integral is the signed area between a curve and the *x*-axis. Theoretically, the definite integral involves finding a Riemann Sum with *infinitely many* rectangles. When this is the case, we have a new notation to represent this area, called *integral notation.*

$$\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\cdot f(c_i)=\int_a^b f(x)dx$$

Example 2 *– Integral Notation*

In each of these examples, c_i is chosen from the i^{th} subinterval of a partition of the interval [1, 10]. Express the limit as a definite integral

a. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(c_i^{\,2}-3c_i\right)\Delta x$

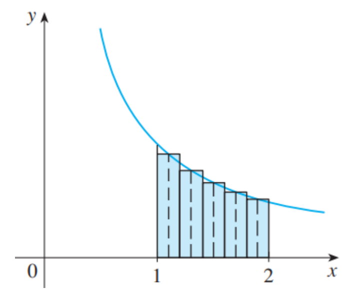
b. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\sin(k_i)-3k_i^{\,3}\right)\Delta x$

c. $\lim\limits_{n\to\infty}\sum\limits_{m=1}^{n}\left(\dfrac{5}{m_i+2\sqrt{m_i}}\right)\Delta x$

A Closer Look at the Definite Integral

Definite integrals can be further deconstructed by giving mathematical equations to represent the value of Δx and the location of *all* the reference points, c_i.

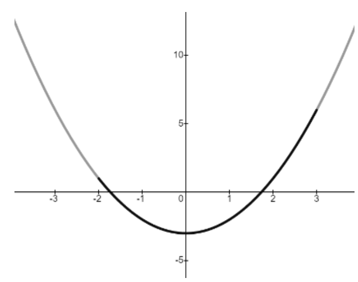
$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x \cdot f(c_i)$$



Where $\Delta x = \dfrac{b-a}{n}$ and $c_i = a + \Delta x i$

For Instance:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{5}{n}\left[\left(-2+\frac{5i}{n}\right)^2 - 3\right] = \int_{-2}^{3} x^2 - 3\,dx$$

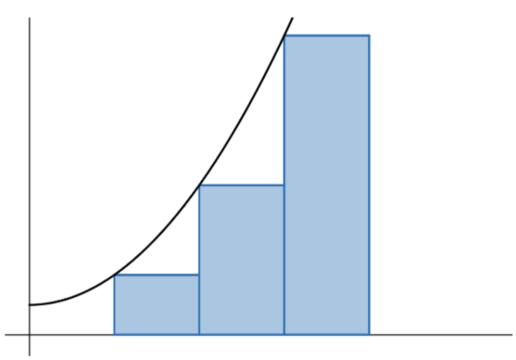


Left & Right Endpoints

Immediately adding a Δx puts us at the *right* endpoint of a sub-interval. To start at the *left* endpoint, we either have to start sigma notation at 0 *or* subtract 1 from i in our expression.

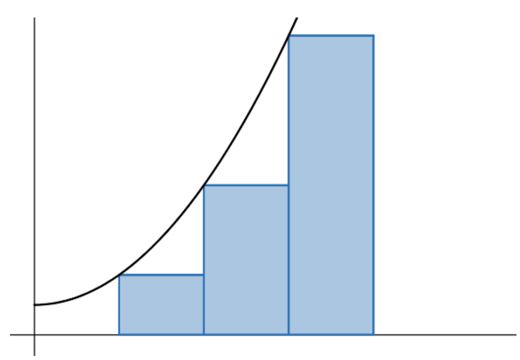
Right Endpoints

$$\sum_{i=1}^{n}\Delta x \cdot f(a+\Delta x i)$$



Left Endpoints

$$\sum_{i=0}^{n-1}\Delta x \cdot f(a+\Delta x i) \;\; or \;\; \sum_{i=1}^{n}\Delta x \cdot f\big(a+\Delta x(i-1)\big)$$



__Example 3__ – Interpreting Definite Integrals as Riemann Sums

Write each of the following definite integrals as right Riemann sum approximation with n subintervals of equal length. Then, write the same expression as a left Riemann sum approximation with n subintervals of equal length.

a. $\int_1^6 3x-4\,dx$ b. $\int_{-2}^4 x^3\,dx$ c. $\int_{-2}^0 \sqrt{x^2+1}\,dx$

__Example 4__ – Interpreting Definite Integrals as Riemann Sums

Determine whether each of the following expressions is a right or left Riemann sum. Then, interpret the expression as a definite integral and write its equivalent sigma expression

Expression	Integral Expression	Equivalent Sigma Notation
a. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left[5\left(2+\frac{4i}{n}\right)+7\right]$		
b. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left[\ln\left(2+\frac{4(i-1)}{n}\right)\right]$		
c. $\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{2}{n}\left[4\left(1+\frac{4i}{n}\right)^3-1\right]$		