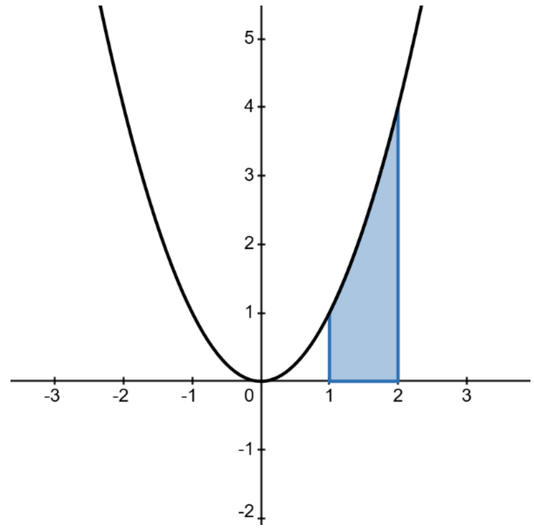
Estimating with Finite Sums

In *differential calculus***,** we learned a lot about rates of change, and how we can apply rates of change to a variety of different situations and contexts. Now, we will begin learning the "other half" of calculus, which is *integral calculus.* In this introductory lesson, we will introduce estimating with finite sums, an essential stepping stone toward understanding integrals. When we calculate the area under a curve, we gain valuable insights into the accumulation of quantities over time. In this lesson, we will learn how to *estimate* the area underneath a curve by dividing it into finite sub-intervals, summing the areas of rectangles or other shapes that fit under the curve.



Accumulation of Change

In differential calculus, we were given a function and asked to examine aspects of its rate of change. Now, we will be presented with a rate of change and will be asked to recover information about the original function.

Area Underneath a Curve

Finding the area underneath a graph of a *rate of change* (i.e. a graph of a first derivative) could tell us how much has been "accumulated" on the original function. For example, the area underneath a velocity graph can tell us how much *distance* was traveled.

Key Takeaway:

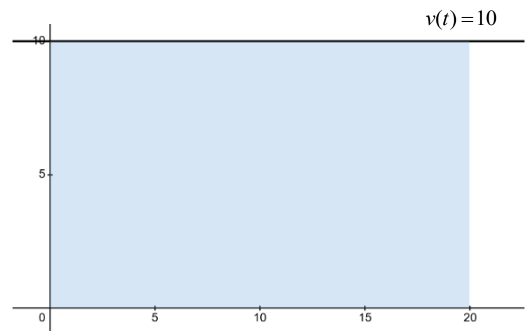
The area of a region bounded by the graph of the rate of change of a function and the horizontal axis tells us about the change in the original function

LEARNING GOALS

- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area
- Estimate the area under a curve using LRAM, MRAM, RRAM, and the Trapezoidal Rule
- Develop an understanding of the area under a curve as representing an accumulation of change
- Predict whether an estimation of area under a curve will be an overestimate or underestimate of the actual area

Re-examining Velocity

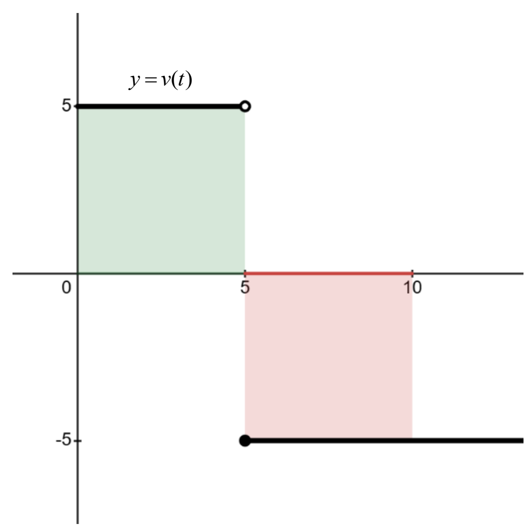
We know that if an object is moving at a steady rate of 10m/s for 20 seconds, then we can determine the total distance travelled in those 20 seconds by applying the formula distance = speed × time. Now we will look at making a connection between this result and the graph of the velocity function.



As we can see, calculating distance = speed × time is precisely the same as calculating the area of the rectangle that is found underneath the graph of $v(t)=10$ for $t\in[0,20]$. This, in fact, will be one of our main learning goals for this unit – determining the area *underneath a rate of change function* can help us to recover information about the *original function*. In other words, since velocity is the Rate of Change of position, we can use the area underneath the velocity graph to recover information about the object's position.

Positive and Negative Area

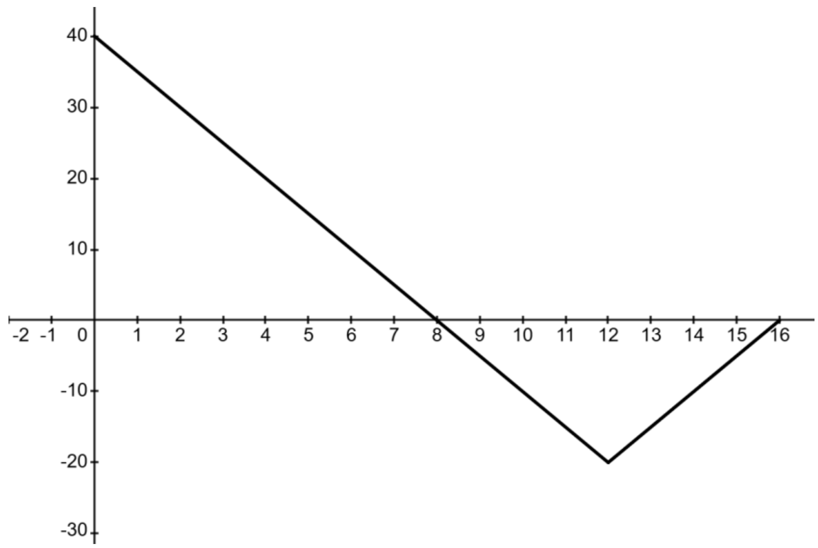
As we've learned previously, rates of change could be positive or negative. We also know that a positive rate of change indicates that a quantity is *increasing* and a negative rate of change indicates that a quantity is *decreasing.* We can extend these observations to our understanding of area underneath curves; areas that are found above the x-axis (i.e. underneath positive rates of change) are defined to be positive area, and areas that are found below the x-axis (i.e. underneath negative rates of change) are defined to be negative area.



In this mathematical context, "underneath" means "between the curve and the x-axis."

__Example 1__ – Net Change

Estelle is in a hot air balloon. The function $R(t)$ models the rate of change in Estelle's altitude in feet per second, where t is measured in seconds. The graph of $R(t)$ is shown below.



a. On what interval(s) is Estelle's altitude *increasing*?

b. On what interval(s) is Estelle's altitude *decreasing*?

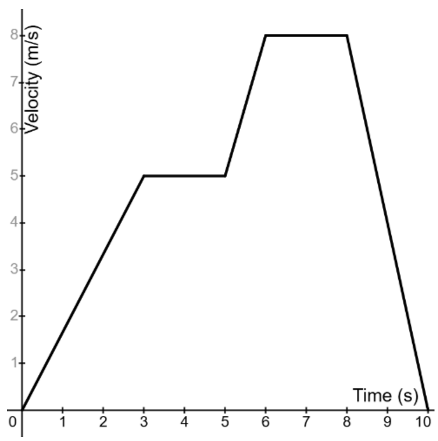
c. At what time(s) does Estelle reach her maximum altitude?

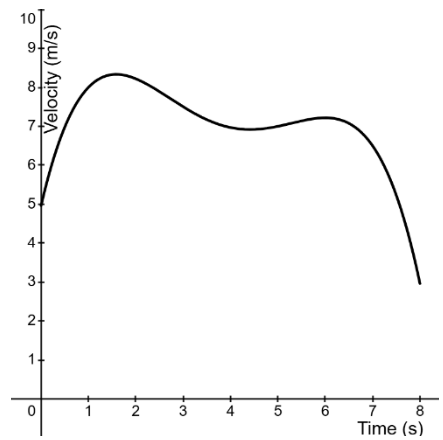
d. How much altitude did Estelle gain from $t=0$ to $t=8$?

e. What was the total change in Estelle's altitude over the first 16 seconds?

Area Under Contoured Curves

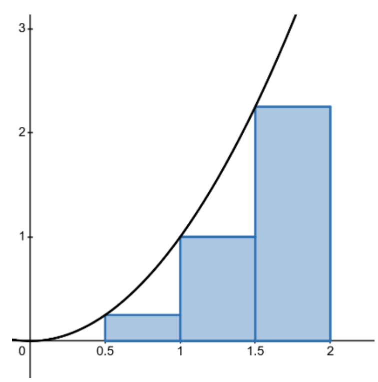
In the previous examples, we were able to easily find the area underneath the curve, since the curves were a simple rectangle and triangle, respectively. Consider the graphs below, which have a more complicated boundary. How might we estimate the area underneath these curves?





Rectangular Approximation Method (RAM)

The Rectangular Approximation Method (RAM) allows us to *approximate* the area underneath a curve by constructing a series of rectangles (of uniform or non-uniform width) underneath the curve, then finding the sum of the areas of those rectangles.



The rectangles are said to have width Δx, represented by the change in the x-axis, and height $f(c_i)$, represented by the value of the function at some reference point c_i.

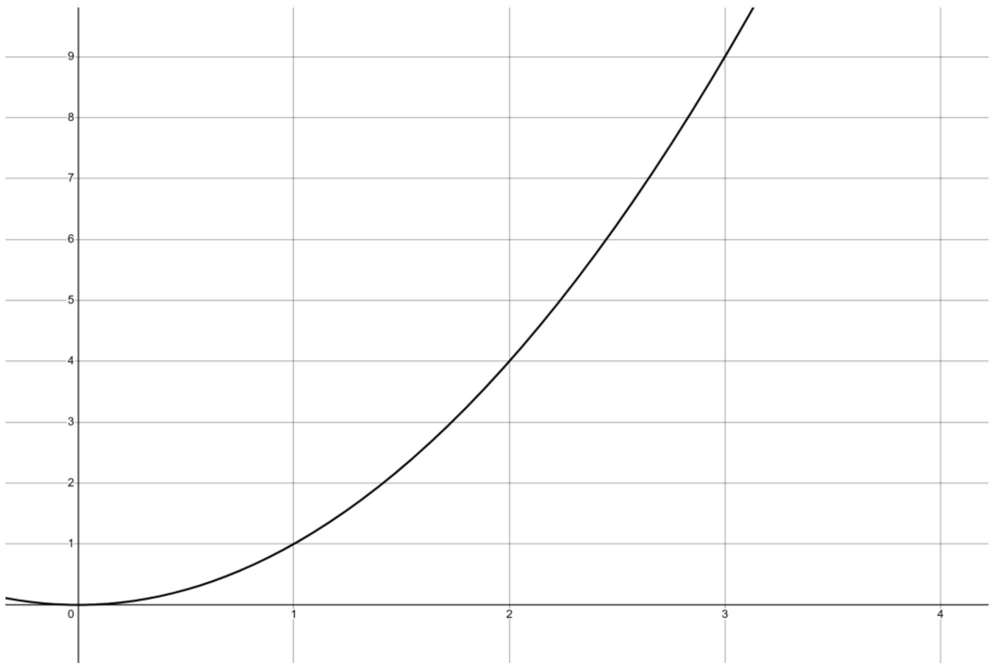
When we take a finite number of rectangles, this method is called a ***Riemann Sum,*** which we will study in much more detail in our next lesson. The reference point that we use in our RAM can be considered using:

- Left Endpoints (LRAM): The start of the sub-interval
- Right Endpoints (RRAM): The end of the sub-interval
- Midpoints (MRAM): The midpoint of the sub-interval

Example 2 – *Calculating MRAM*

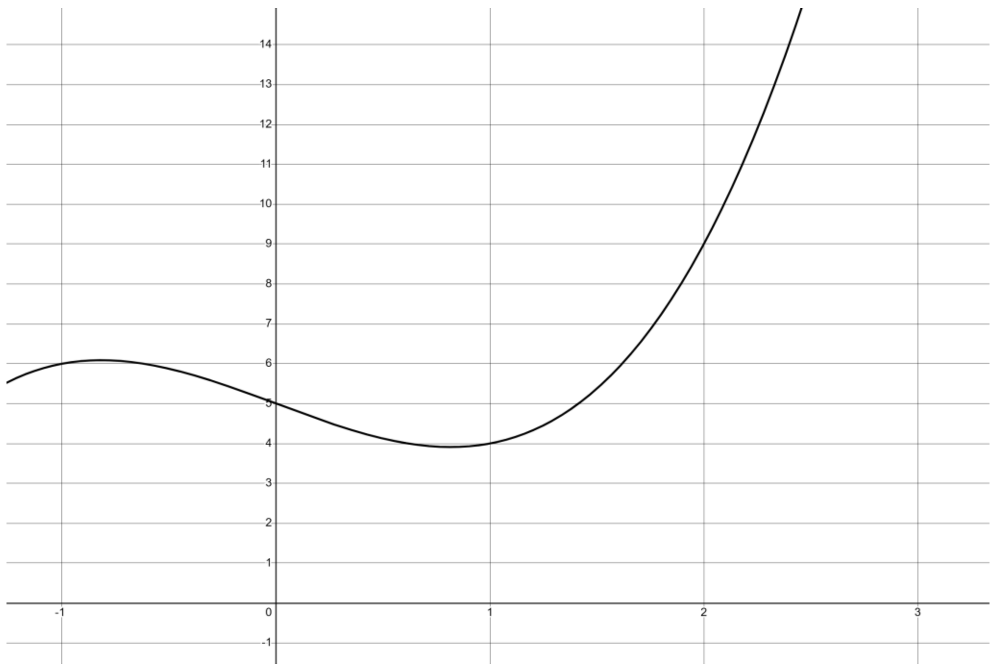
A particle starts at $t = 0$ and moves along the t-axis with velocity $v(t) = t^2$ for time $t \geq 0$.

Estimate the position of the particle at $t = 3$ by using the **Midpoint Rectangular Approximation Method** (MRAM) with 3 sub-intervals of uniform width.



__Example 3__ – Calculating with Riemann Sums LRAM

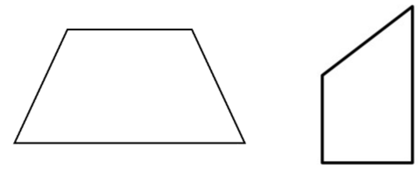
A particle starts at $t=0$ and moves along the t-axis with velocity $v(t)=t^3-2t+5$ for time $t \geq 0$. Estimate the position of the particle at $t=3$ by using the **Left Rectangular Approximation Method** (LRAM) with 3 sub-intervals of uniform width



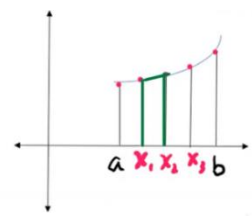
Trapezoidal Rule

A trapezoid is a quatrilateral with only one pair of parallel sides

The area of a trapezoid is given by $A = \left(\frac{b_1 + b_2}{2}\right)h$



Rather than construct rectangles underneath the curve, we can construct a series of trapezoids that have a flat bottom and sloped top. These trapezoids would more closely follow the shape of a curve

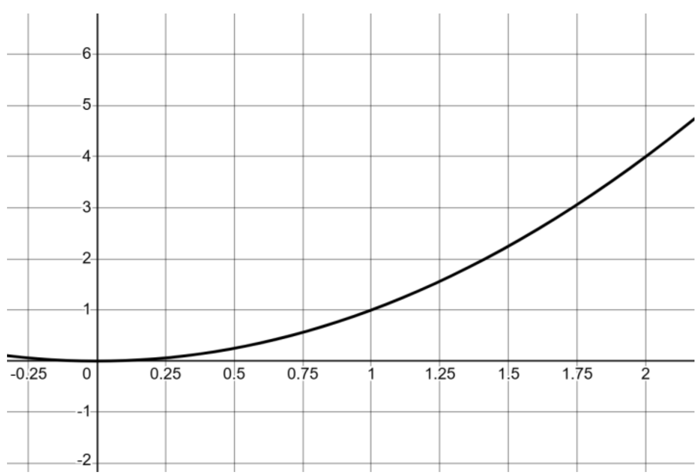


$$A = \frac{1}{2}\Delta x\left[\left(f(a) + f(x_1)\right) + \left(f(x_1) + f(x_2)\right) + \left(f(x_2) + f(x_3)\right) + \ldots\right]$$

$$A = \frac{\Delta x}{2}\left[f(a) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

Example 4 **–** *Trapezoidal Rule*

Use the Trapezoidal Rule with $n = 4$ to estimate the area underneath the curve $f(x) = x^2$ between $x = 1$ and $x = 2$.

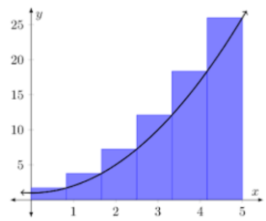


Overestimates vs Underestimates

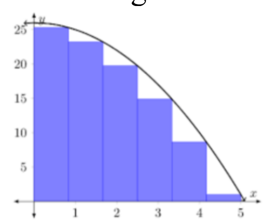
As we can see, using this method will always produce inaccuracies, since this method is merely an *estimate* of the area under a curve. How do we know if our number would be an overestimate or an underestimate?

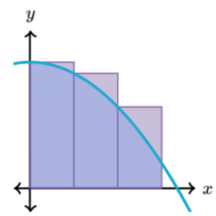
For LRAM and RRAM

Increasing Intervals:



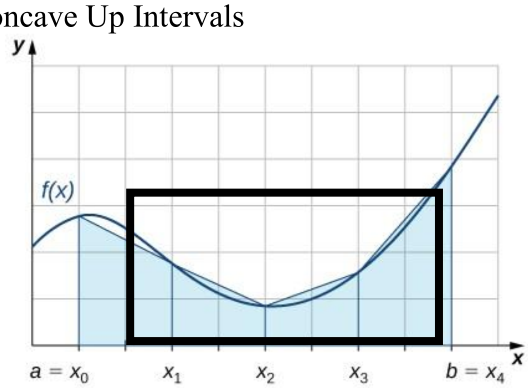
Decreasing Intervals:





For Trapezoidal Rule

Concave Up Intervals



Concave Down Intervals