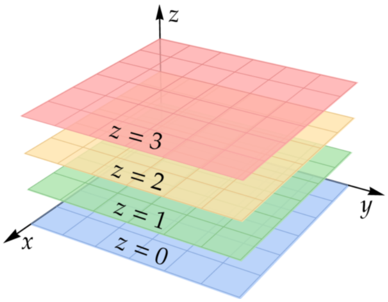
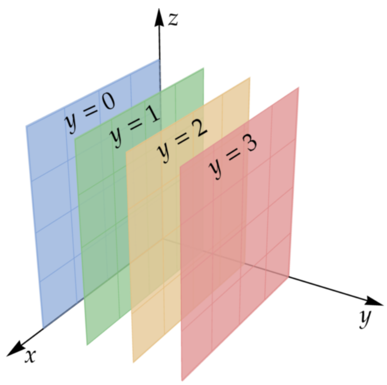
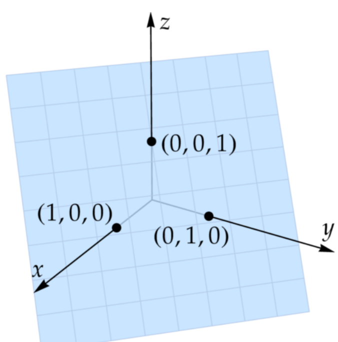
8.2 Planes and Hyperplanes



▲ **Figure 1:** The xy-plane and several other horizontal planes.



▲ **Figure 2:** The xz-plane and several parallel planes.



▲ **Figure 3:** The plane $x + y + z = 1$.

A **linear equation** in three variables x, y, and z is any equation of the form

$$ax + by + cz = d,$$

where a, b, c, d are constants and the coefficients a, b, c are not all zero. Any such equation defines a plane in $\mathbb{R}^3$.

Here are some examples of linear equations and the corresponding planes:

- The equation $z = 0$ defines the xy-plane in $\mathbb{R}^3$, since the points on the xy-plane are precisely those points whose z-coordinate is zero.
- If d is any constant, the equation $z = d$ defines a horizontal plane in $\mathbb{R}^3$, which is parallel to the xy-plane. Figure 1 shows several such planes.
- The equations $x = 0$ and $y = 0$ define the yz-plane and xz-plane, respectively, and equations of the form $x = d$ or $y = d$ define planes parallel to these. For example, Figure 2 shows several planes of the form $y = d$.
- The equation $x + y + z = 1$ defines a slanted plane in $\mathbb{R}^3$, which goes through the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. This plane is shown in Figure 3.

In general, two planes that do not intersect are said to be **parallel**. Such planes can be defined by equations having the same coefficients of x, y, and z, but different constant terms, i.e.

$$ax + by + cz = d \qquad \text{and} \qquad ax + by + cz = e$$

for $d \neq e$. No point (x, y, z) can simultaneously satisfy both of these equations, so two planes of this form do not intersect.

EXAMPLE 1

Find an equation for the plane that is parallel to the plane $4x + y + 2z = 8$ and goes through the point $(3, 1, 2)$.

SOLUTION The desired plane must have an equation of the form

$$4x + y + 2z = d$$

for some constant d. Plugging in the point $(3, 1, 2)$ gives

$$4(3) + (1) + 2(2) = d,$$

so $d = 17$. Thus the desired plane is defined by

$$4x + y + 2z = 17.$$

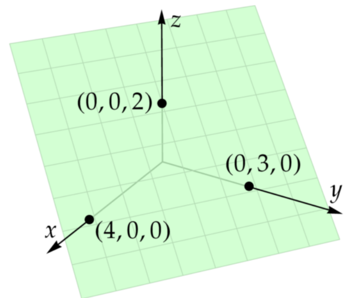
Note that the equation for a plane is not unique. For example, the planes defined by the equations

$$3x + 4y + 6z = 12 \qquad \text{and} \qquad 6x + 8y + 12z = 24.$$

are the same, since the second equation is just twice the first equation. In general, any nonzero scalar multiple of the equation for a plane gives another equation for the same plane.

Intercepts

The **intercepts** of a plane are the locations at which the plane intersects the x, y, and z axes. Most planes intersect each axis at exactly one point, and finding these intercepts can help to give a sense of how a plane sits in space.



▲ **Figure 4:** The plane $3x + 4y + 6z = 12$.

EXAMPLE 2

Find the points at which the plane $3x + 4y + 6z = 12$ intersects the three axes.

SOLUTION A point lies on the x-axis if and only if its y and z coordinates are both zero. Thus, we can figure out where the plane intersects the x-axis by setting y and z equal to 0 and then solving for x:

$$3x + 4(0) + 6(0) = 12.$$

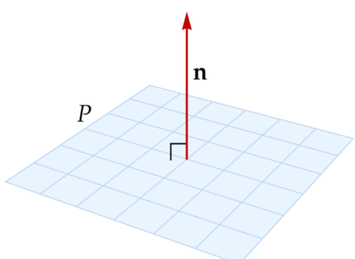
Solving for x gives $x = 4$, so the plane intersects the x-axis at the point $(4, 0, 0)$.

A similar procedure can be used to determine the intersection with the y and z axes. In particular, this plane intersects the y axis at the point $(0, 3, 0)$, and it intersects the z-axis at the point $(0, 0, 2)$, as shown in Figure 4.

Normal Vectors

A **normal vector** to a plane is any vector whose direction is perpendicular to that of the plane, as shown in Figure 5. For example, the vector $(0, 0, 1)$ is normal to any horizontal plane.

There is a close relationship between the linear equation for a plane and the normal vector.



▲ **Figure 5:** The vector $\mathbf{n}$ is normal to the plane P.

Normal Vector to a Plane

If P is the plane in $\mathbb{R}^3$ defined by the equation

$$ax + by + cz = d,$$

then $\mathbf{n} = (a, b, c)$ is a normal vector for P.

For example, the vector $(3, 4, 2)$ is normal to the plane $3x + 4y + 2z = 15$, and the vector $(1, 0, 1)$ is normal to the plane $x + z = 3$.

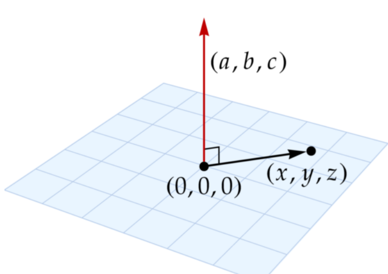
We can justify this formula using the dot product. First, consider a plane P that goes through the origin $(0, 0, 0)$. Such a plane has an equation of the form

$$ax + by + cz = 0.$$

Using the dot product, we can rewrite this equation as

$$(a, b, c) \cdot (x, y, z) = 0.$$

Geometrically, this equation says that the plane P consists of all points (x, y, z) whose radial vector is orthogonal to (a, b, c), as shown in Figure 6. It follows that (a, b, c) is a normal vector for P. Since parallel planes have the same normal vectors, this also holds for any plane of the form $ax + by + cz = d$.



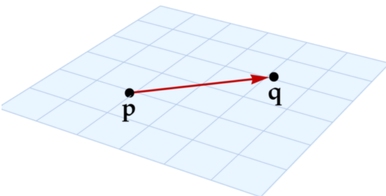
▲ **Figure 6:** The plane $ax + by + cz = 0$ consists of all points (x, y, z) whose radial vector is orthogonal to (a, b, c).

Normal vectors are useful because a normal vector to a plane completely determines the direction of the plane. Indeed, specifying a normal vector is probably the most common way to describe how a plane is oriented in space. Note, however, that the normal vector is not uniquely determined, since any nonzero scalar multiple of a normal vector is again a normal vector.

Parallel Vectors

A vector $\mathbf{v}$ is said to be **parallel** to a given plane if $\mathbf{v}$ can be moved so that its arrow lies entirely on the plane, as shown in Figure 7. Equivalently, $\mathbf{v}$ is parallel to a given plane if there exist points $\mathbf{p}$ and $\mathbf{q}$ on the plane so that $\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Note that any vector parallel to a plane must be orthogonal to the normal vector. Conversely, any vector that is orthogonal to the normal vector must be a parallel vector.



▲ **Figure 7:** A parallel vector to a plane.

EXAMPLE 3

Find the value of t for which the vector $(3, 1, t)$ is parallel to the plane $2x + 4y + 5z = 12$.

SOLUTION The normal vector to this plane is $(2, 4, 5)$. We take the dot product of this with the given vector:

$$(2,4,5) \cdot (3,1,t) \;=\; (2)(3) + (4)(1) + (5)(t) \;=\; 10 + 5t.$$

The given vector will be parallel to the plane when this dot product is zero, which occurs for $\boxed{t = -2}$.

Because the normal vector is orthogonal to all of the parallel vectors, the cross product of any two parallel vectors that point in different directions will yield a normal vector.

Finding an Equation for a Plane

A plane in $\mathbb{R}^3$ can be specified using its normal vector as well as a point on the plane. This is similar to specifying a line in $\mathbb{R}^2$ using the slope of the line as well as a point in the line.

The following example illustrates how to find the equation of a plane from a point and a normal vector.

EXAMPLE 4

Find and equation for the plane through the point $(2, 1, 3)$ that has normal vector $(4, 2, 3)$.

SOLUTION From the normal vector, we know that the plane has an equation of the form

$$4x + 2y + 3z \;=\; d$$

for some constant d. Plugging in the point $(2, 1, 3)$ gives the equation

$$4(2) + 2(1) + 3(3) \;=\; d$$

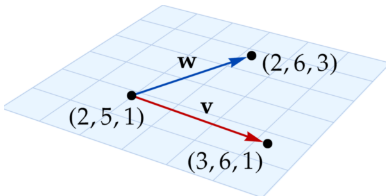
and thus $d = 19$. Thus one equation for the plane is

$$4x + 2y + 3z \;=\; 19.$$

In the same way that any two points determine a line, any three points determine a plane. More precisely, if $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are three points in $\mathbb{R}^3$ that do not lie on a single line, then there exists a unique plane in $\mathbb{R}^3$ going through all three point.

We say that three points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are **collinear** if there is a line that goes through all three of them. Thus any three non-collinear points determine a plane.

To find the equation of such a plane, observe that the vectors $\mathbf{v} = \mathbf{q} - \mathbf{p}$ and $\mathbf{w} = \mathbf{r} - \mathbf{p}$ are parallel to the plane, and therefore the cross product $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is a normal vector.



▲ **Figure 8:** The plane from Example 5.

Note that $(2, -2, 1)$ is orthogonal to both $(1, 1, 0)$ and $(0, 1, 2)$.

It is easy to check that all three of the given points satisfy this equation.

EXAMPLE 5

Find the equation of the plane that goes through the points $(2, 5, 1)$, $(3, 6, 1)$, and $(2, 6, 3)$.

SOLUTION Let $\mathbf{v}$ and $\mathbf{w}$ be the vectors shown in Figure 8. Then

$$\mathbf{v} = (3, 6, 1) - (2, 5, 1) = (1, 1, 0) \qquad \text{and} \qquad \mathbf{w} = (2, 6, 3) - (2, 5, 1) = (0, 1, 2).$$

Both of these vectors are parallel to the plane, so their cross product is a normal vector:

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (2, -2, 1).$$

Then the plane has an equation of the form

$$2x - 2y + z = d$$

for some constant d. Plugging in any one of the three points gives $d = -5$, so one equation for the plane is $\boxed{2x - 2y + z = -5}$.

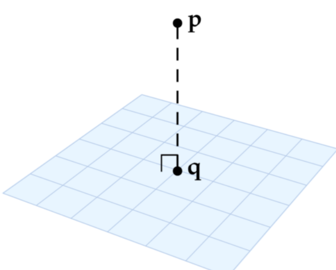
Distance from a Point to a Plane

Given a plane in $\mathbb{R}^3$ and a point $\mathbf{p}$ not on the plane, there is always exactly one point $\mathbf{q}$ on the plane that is closest to $\mathbf{p}$, as shown in Figure 9. The point $\mathbf{q}$ is known as the **projection of p onto the plane**, and the distance from $\mathbf{p}$ to $\mathbf{q}$ is the **distance from the point p to the plane**.

We can use the dot product to find the distance from a point $\mathbf{p}$ to a plane. The trick is to first choose *any* point $\mathbf{r}$ that lies on the plane. Then the point $\mathbf{p}$, its projection $\mathbf{q}$, and the point $\mathbf{r}$ make a right triangle, as shown in Figure 10. In this case, the distance from $\mathbf{p}$ to $\mathbf{q}$ is given by the formula

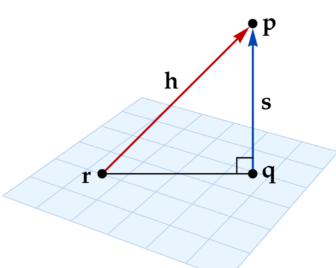
$$\boxed{|\mathbf{s}| = |\mathbf{h} \cdot \mathbf{u}|}$$

where $\mathbf{u}$ is a unit vector normal to the plane.



▲ **Figure 9:** The point $\mathbf{q}$ is the projection of the point $\mathbf{p}$ onto this plane.

Here the absolute value is necessary in the case where $\mathbf{u}$ and $\mathbf{s}$ point in opposite directions.



▲ **Figure 10:** The right triangle made by a point $\mathbf{p}$, its projection $\mathbf{q}$, and another point $\mathbf{r}$ on the plane.

EXAMPLE 6

Find the distance from the point $\mathbf{p} = (8, 0, 9)$ to the plane $3x - 2y + 4z = 2$.

SOLUTION We start by choosing any point $\mathbf{r}$ on the plane, i.e. any values of x, y, and z that satisfy the given equation. There are many possible choices, but let's use $\mathbf{r} = (0, 1, 1)$. Then

$$\mathbf{h} = \mathbf{p} - \mathbf{r} = (8, 0, 9) - (0, 1, 1) = (8, -1, 8).$$

The vector $\mathbf{n} = (3, -2, 4)$ is normal to the plane, so a unit normal vector is

$$\mathbf{u} = \frac{1}{|\mathbf{n}|}\mathbf{n} = \frac{1}{\sqrt{29}}(3, -2, 4).$$

Then

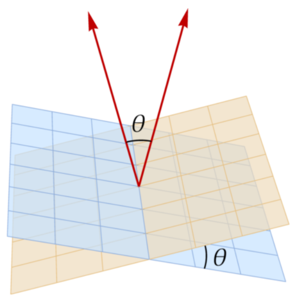
$$\mathbf{h} \cdot \mathbf{u} = (8, -1, 8) \cdot \frac{1}{\sqrt{29}}(3, -2, 4) = \frac{58}{\sqrt{29}} = 2\sqrt{29}.$$

so the distance is $\boxed{2\sqrt{29}}$.

Angle Between Planes

Two planes that intersect form an angle, sometimes called a **dihedral angle**. As Figure 11 illustrates, the angle between two planes is the same as the angle between their corresponding normal vectors.

Of course, there are two possible directions for each normal vector, which are opposite from one another. There are also two different angles between the planes, namely the acute angle θ shown in Figure 11, and the obtuse angle $180° - \theta$. Depending on which pair of normal vectors we choose, the angle between them might be either θ or $180° - \theta$.



▲ **Figure 11:** The angle between two planes is the same as the angle between the two normal vectors.

EXAMPLE 7

Find the angle between the planes $x + z = 1$ and $x + y + z = 2$.

SOLUTION The corresponding normal vectors are $\mathbf{v} = (1, 0, 1)$ and $\mathbf{w} = (1, 1, 1)$. The formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$ gives

$$2 = \sqrt{2}\sqrt{3}\cos\theta.$$

Then

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 35.26°.$$

If we had switched the direction of one the normal vectors, such as using

$$\mathbf{w} = (-1, -1, -1),$$

then the angle between $\mathbf{v}$ and $\mathbf{w}$ would have been

$$180° - 35.26° = 144.74°.$$

Note that each plane has normal vectors in two possible directions, which are opposite from one another.

Hyperplanes

A linear equation in four variables has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b,$$

where a_1, a_2, a_3, a_4, and b are constants. Such an equation defines a **hyperplane** in $\mathbb{R}^4$. A hyperplane is similar to a plane, except that it is three-dimensional. That is, in the same way that a plane is like a copy of $\mathbb{R}^2$ sitting inside of $\mathbb{R}^3$, a hyperplane is like a copy of $\mathbb{R}^3$ sitting inside of $\mathbb{R}^4$.

Here are some examples of linear equations in $\mathbb{R}^4$ and the corresponding hyperplanes:

- The equation $x_4 = 0$ defines the $x_1x_2x_3$-hyperplane in $\mathbb{R}^4$, i.e. the hyperplane that contains the x_1, x_2, and x_3 axes.
- If b is any constant, the equation $x_4 = b$ defines a hyperplane in $\mathbb{R}^4$ that is parallel to the $x_1x_2x_3$-hyperplane.
- The equations $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ define the $x_2x_3x_4$-hyperplane, the $x_1x_3x_4$-hyperplane, and the $x_1x_2x_4$-hyperplane, respectively.
- The equation $x_1 + x_2 + x_3 + x_4 = 1$ defines a slanted hyperplane in $\mathbb{R}^4$, which goes through the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$.

Any vector between two points in a hyperplane is said to be **parallel** to the hyperplane. A vector is **normal** to a hyperplane if it is orthogonal to every parallel vector. As in $\mathbb{R}^3$, the hyperplane in $\mathbb{R}^4$ defined by the equation $a_1x_1 + a_2x_2 + a_3x_3 + a - 4x_4 = b$ has (a_1, a_2, a_3, a_4) as a normal vector.

Here the word **flat** refers to any infinite, boundless shape that does not bend or curve. Lines and planes are examples of flats, but in $\mathbb{R}^n$ a flat may have any number of dimensions from 1 to $n-1$.

In general, the word "hyperplane" refers to an $(n-1)$-dimensional flat in $\mathbb{R}^n$. For example, hyperplanes in $\mathbb{R}^5$ are 4-dimensional flats and hyperplanes in $\mathbb{R}^{17}$ are 16-dimensional flats. Any hyperplane in $\mathbb{R}^n$ can be defined by a linear equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are constants. Such a hyperplane always has $(a_1, a_2, \ldots, a_n)$ as a normal vector.

EXERCISES

1. Find the equation of the plane that is parallel to $x - 3y + 2z = 4$ and goes through the point $(2, 1, 5)$.

2. Find the points at which the plane $4x - 2y + 5z = 20$ intersects the x, y, and z axes.

3–4 ■ Find an equation for the plane through the point $\mathbf{p}$ that has normal vector $\mathbf{n}$.

3. $\mathbf{p} = (-2, 3, 1)$, $\mathbf{n} = (1, -2, 4)$

4. $\mathbf{p} = (3, 4, 2)$, $\mathbf{n} = (1, 0, 2)$

5. Find an equation for the plane that goes through the points $(1, 1, 0)$, $(2, 0, 1)$, and $(3, 1, 3)$.

6. Let L be the line in $\mathbb{R}^3$ that goes through the points $(2, -4, 5)$ and $(3, 0, 7)$. Find an equation for the plane through $(2, -4, 5)$ that is perpendicular to L.

7. Find a vector that is parallel to the plane $-x + 5y - 2z = 3$ and orthogonal to $(1, 1, 2)$.

8. (a) Find the distance from the point $(5, 6, 3)$ to the plane $x + y + z = 2$.

(b) Find the projection of the point $(5, 6, 3)$ onto this plane.

(c) Find the reflection of the point $(5, 6, 3)$ across this plane.

9. Find the distance between the planes $x + 2y - 2z = 2$ and $x + 2y - 2z = 17$.

10–11 ■ Find the angle between the given planes.

10. $x + y + 2z = 3$, $2x - y + z = 1$

11. $x + z = 5$, $3x + 5y - 3z = 7$

12. Find an equation for the hyperplane in $\mathbb{R}^4$ that goes through the point $(2, 1, 5, 2)$ and has normal vector $(1, -1, 1, -1)$.

13. Let H be the hyperplane in $\mathbb{R}^4$ defined by the equation $x_1 + 2x_2 + 2x_3 + 4x_4 = 4$. Find the distance from the point $(4, 5, 6, 7)$ to H.