On Dynamic Programming Decompositions of Static Risk Measures in Markov Decision Processes

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Summary

Primal methods

Augment state space to keep track of target value or time.[Wu, 1999], [Lin, 2003], [Bauerle, 2011], [Hau, 2023]

Robust methods

Define **MDP** for risk measure with augmented state space to keep track of risk levels. These popular dynamic program (DP) methods for solving risk averse MDP [Chow, 2015], [Jin, 2019], [Li, 2022], [Ni, 2022] are thought to be optimal with sufficiently discretized risk levels for policy optimization.

Contribution

Policy Evaluation

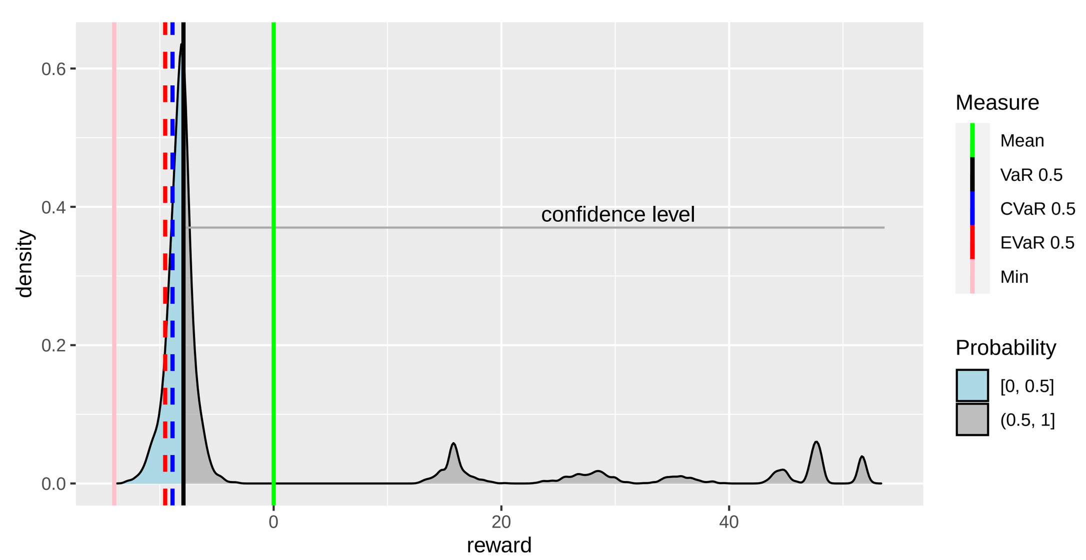
- Prove a new correct VaR and EVaR decomposition.

Policy Optimization

- Prove robust methods are suboptimal for CVaR [Chow 2015] and EVaR [Ni 2022] MDP.
- Propose optimal DP for robust methods Value-at-Risk (VaR) MDP.

Risk Averse MDPs

Risk Measures



$$\mathrm{VaR}_\alpha[\tilde{x}] = \sup\{z \in \mathbb{R} \mid \mathbb{P}[\tilde{x} < z] \le \alpha\} = \inf\{z \in \mathbb{R} \mid \mathbb{P}[\tilde{x} \le z] > \alpha\}$$

$$\mathrm{CVaR}_\alpha[\tilde{x}] = \sup_{z\in\mathbb{R}} \left(z - \alpha^{-1}\mathbb{E}[z-\tilde{x}]_+\right) = \inf_{\xi\in\Delta_m} \left\{\boldsymbol{\xi}^\top \boldsymbol{x} \mid \alpha\cdot\boldsymbol{\xi} \le \boldsymbol{q}\right\}$$

$$\mathrm{EVaR}_\alpha[\tilde{x}] = \sup_{\beta>0} -\frac{1}{\beta}\log\left(\alpha^{-1}\mathbb{E}\left[\exp(-\beta\cdot\tilde{x})\right]\right) = \inf_{\xi\in\Delta_m} \left\{\boldsymbol{\xi}^T \boldsymbol{x} \mid \mathrm{KL}(\boldsymbol{\xi}\|\boldsymbol{q}) \le -\log\alpha\right\}$$

Risk Averse Objectives

- Maximizes the risk measure $\psi[\cdot]$ of the total reward in a Markov decision process (MDP)

$$\max_{\pi\in\Pi} \psi\left[\sum_{t=0}^{T} r^\pi(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1})\right]$$

  Assume: Rewards $r(s,a,s') \in \mathbb{R}$, transition probabilities $P(s,a) \in \triangle^S$ and finite horizon.

- If $\psi[r(\tilde{s},\tilde{a},\tilde{s}')]$ and $\max_{\pi\in\Pi}\psi[r^\pi(\tilde{s},\tilde{a},\tilde{s}')]$ can be written in term of $\psi[r(s,\tilde{a},\tilde{s}'|\tilde{s}=s)]$ and $\max_{\pi\in\Pi}\psi[r^\pi(s,\tilde{a},\tilde{s}'|\tilde{s}=s)]$ then exist DP for evaluation and policy optimization respectively.

Challenges: $\psi \in$ VaR, CVaR and EVaR, do not satisfy *tower property*: $\psi[X] = \psi[\,\psi[X \mid Y]\,]$.

$$\max_{\pi\in\Pi} \psi\left[\sum_{t=\tau}^{T} r^\pi(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1})\right] \ne \max_{a\in A} \psi_{s'}\left[r(s,a,s') + \max_{\pi\in\Pi}\psi\Big[\sum_{t=\tau+1}^{T} r^\pi(\tilde{s}_t,\tilde{a}_t,\tilde{s}_{t+1}) | \tilde{s}_{\tau+1} = s'\Big]\right]$$

Extended Conditional Coherent Risk Measure

Proposition 3.1 : Lemma 22 in [Pflug, 2016] Suppose that $\pi \in \Pi$ and $\tilde{s} \sim \hat{p}$, $\tilde{a} \sim \pi(\tilde{s})$, $\tilde{s}' \sim p_{s,a}$. Then,

$$\mathrm{CVaR}_\alpha[r(\tilde{s},\tilde{a},\tilde{s}')] = \min_{\zeta\in\mathcal{Z}_C} \sum_{s\in\mathcal{S}} \zeta_s \,\mathrm{CVaR}_{\alpha\zeta_s\hat{p}_s^{-1}}[r(s,\tilde{a},\tilde{s}') \mid \tilde{s} = s],$$

where the state s on the right-hand side is not random and $\mathcal{Z}_C = \{\zeta \in \Delta_S \mid \alpha\cdot\zeta \le \hat{p}\}$.

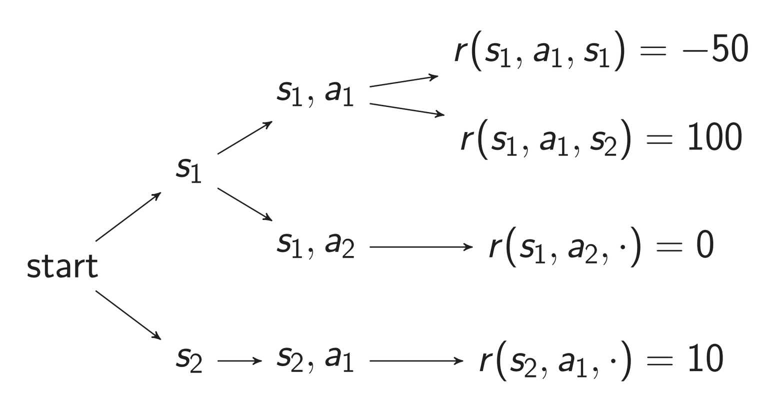
CVaR Decomposition Fails in Optimization

Theorem 3.2 : There exists an MDP and a risk level $\alpha \in [0,1]$ such that

$$\begin{aligned}\max_{\pi\in\Pi} \mathrm{CVaR}_\alpha[r^\pi(\tilde{s},\tilde{a},\tilde{s}')] &= \max_{\pi\in\Pi}\min_{\zeta\in\mathcal{Z}_C} \sum_{s\in\mathcal{S}} \zeta_s\,\mathrm{CVaR}_{\alpha\zeta_s\hat{p}_s^{-1}}[r^\pi(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s] \\ &< \min_{\zeta\in\mathcal{Z}_C}\max_{\pi\in\Pi} \sum_{s\in\mathcal{S}} \zeta_s\,\mathrm{CVaR}_{\alpha\zeta_s\hat{p}_s^{-1}}[r^\pi(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s] \\ &= \min_{\zeta\in\mathcal{Z}_C} \sum_{s\in\mathcal{S}} \zeta_s \max_{\pi\in\Pi}\mathrm{CVaR}_{\alpha\zeta_s\hat{p}_s^{-1}}[r^\pi(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s]\end{aligned}$$

Let $\theta_\pi(\zeta) = \sum_{s\in\mathcal{S}} \zeta_s\,\mathrm{CVaR}_{\alpha\zeta_s\hat{p}_s^{-1}}[r^\pi(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s]$ and $\alpha = 0.5$.

Figure right below plot the function $\min_{\zeta\in\mathcal{Z}_C}\max_{\pi\in\Pi}\theta_\pi(\zeta)$ and $\max_{\pi\in\Pi}\min_{\zeta\in\mathcal{Z}_C}\theta_\pi(\zeta)$ for the 2 states simple example in left below to illustrate the inaccurate approximation.



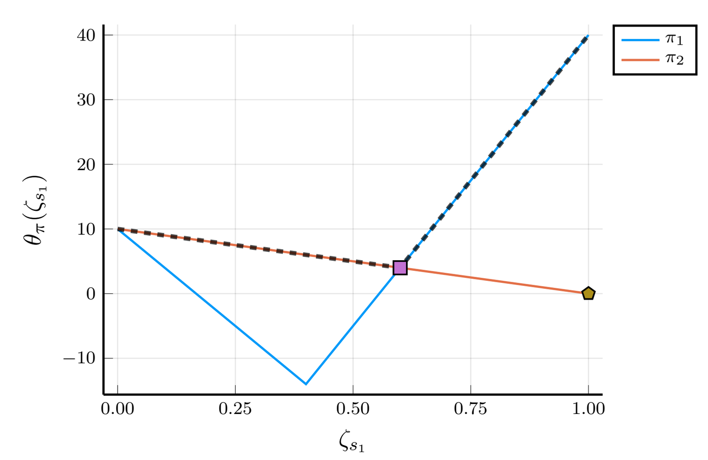
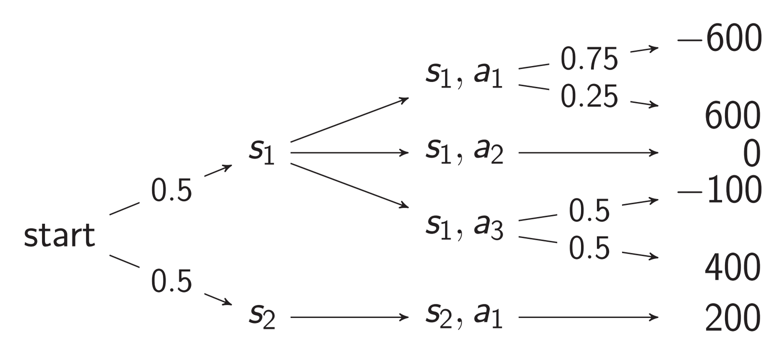
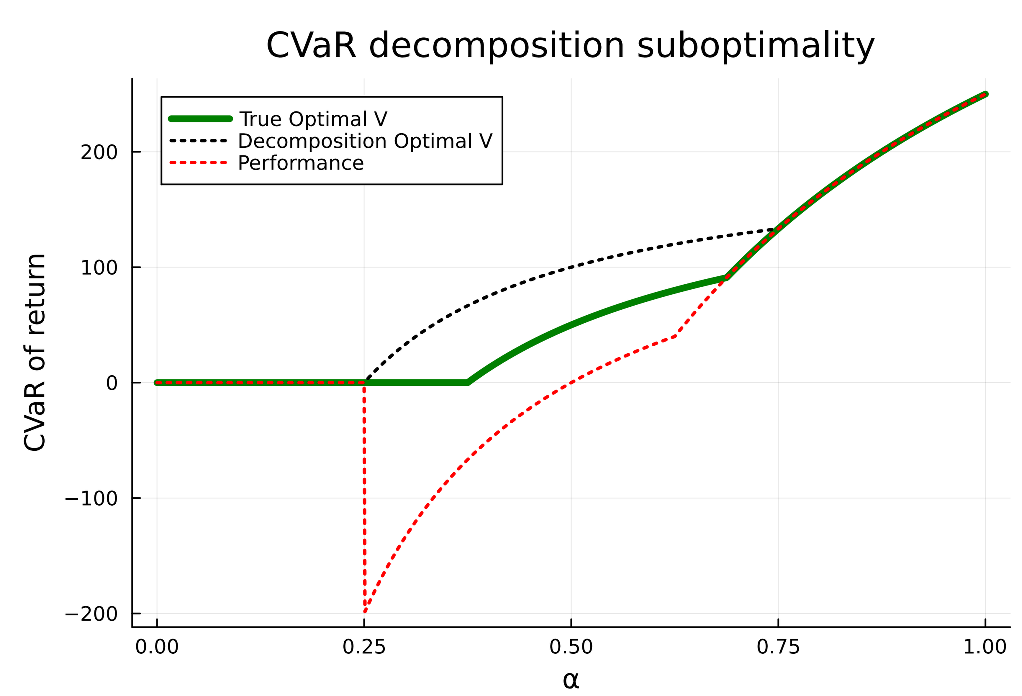


Figure below demonstrate an example and illustrate the sub-optimality of a policy for CVaR decomposition.





EVaR Decomposition Fails in Optimization

Theorem 4.2: EVaR evaluation decomposition Given $\alpha \in (0,1]$, we have that

$$\mathrm{EVaR}_\alpha[r(\tilde{s},\tilde{a},\tilde{s}')] = \inf_{\zeta\in(0,1]^S,\,\xi\in\mathcal{Z}'_E(\zeta)} \sum_{s\in\mathcal{S}} \xi_s\,\mathrm{EVaR}_{\zeta_s}[r(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s],$$

where $\mathcal{Z}'_E(\zeta) = \left\{\boldsymbol{\xi} \in \Delta_S \mid \boldsymbol{\xi} \ll \hat{\boldsymbol{p}},\ \sum_{s\in\mathcal{S}}\xi_s\big(\log(\xi_s/\hat{p}_s) - \log(\zeta_s)\big) \le -\log\alpha\right\}$.

Note : EVaR Decomposition also fails in optimization with the same reason in Theorem 3.2.

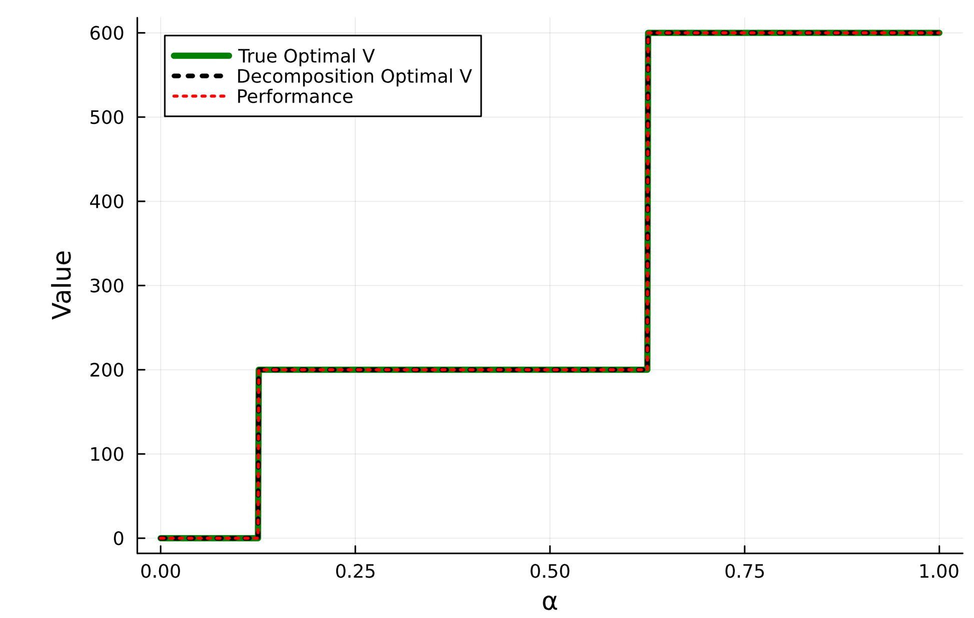
Value at Risk (VaR) MDP

Theorem 5.1: VaR policy evaluation Given any finite MDP for risk level $\alpha \in [0,1]$ we have

$$\mathrm{VaR}_\alpha[r(\tilde{s},\tilde{a},\tilde{s}')] = \sup_{\zeta\in\Delta_N}\left\{\min_{s} \mathrm{VaR}_{\alpha\zeta_s\hat{p}_s^{-1}}[r(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s],\ \alpha\cdot\boldsymbol{\zeta} \le \hat{\boldsymbol{p}}\right\},$$

Theorem 5.2: VaR policy optimization Given any finite MDP with $\alpha \in [0,1]$, we have

$$\begin{aligned}\max_{\pi\in\Pi}\mathrm{VaR}_\alpha^{\tilde{a}\sim\pi(\tilde{s})}[r(\tilde{s},\tilde{a},\tilde{s}')] &= \max_{\pi\in\Pi}\sup_{\zeta\in\Delta_S}\left\{\min_{s\in\mathcal{S}}\mathrm{VaR}^{\tilde{a}\sim\pi}_{\alpha\zeta_s\hat{p}_s^{-1}}\left[r(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s\right] \mid \alpha\cdot\boldsymbol{\zeta}\le\hat{\boldsymbol{p}}\right\} \\ &= \sup_{\zeta\in\Delta_S}\left\{\max_{\pi\in\Pi}\min_{s\in\mathcal{S}}\mathrm{VaR}^{\tilde{a}\sim\pi}_{\alpha\zeta_s\hat{p}_s^{-1}}\left[r(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s\right] \mid \alpha\cdot\boldsymbol{\zeta}\le\hat{\boldsymbol{p}}\right\} \\ &= \sup_{\zeta\in\Delta_S}\left\{\min_{s\in\mathcal{S}}\max_{\pi\in\Pi}\mathrm{VaR}^{\tilde{a}\sim\pi}_{\alpha\zeta_s\hat{p}_s^{-1}}\left[r(s,\tilde{a},\tilde{s}') \mid \tilde{s}=s\right] \mid \alpha\cdot\boldsymbol{\zeta}\le\hat{\boldsymbol{p}}\right\}\end{aligned}$$



Value at Risk (VaR) Value Iteration

$$q^\star_\tau(s,\alpha,a) = \max_{\pi\in\Pi}\mathrm{VaR}^\pi_\alpha\left[\sum_{t=\tau}^{T} r(\tilde{s}_t,\tilde{a}_t,\tilde{s}_{t+1})\right] = \sup_{\zeta\in\Delta_S}\left\{\min_{s'\in\mathcal{S}}\left[r(s,a,s') + \max_{a'\in A} q^\star_{\tau+1}\Big(s', \frac{\alpha\zeta_{s'}}{p_{sas'}}, a'\Big)\right]\right\}$$

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