Discretize

EVaR

2022-11-16

```
setwd("C:/GITHUB/Rasr_submit")
source("Code/RASR_code.R")
```

```
## Loading required package: proto
## Loading required package: iterators
## Loading required package: parallel
```

Entropic Risk Measure

We define ERM for $beta > 0$ as:

$$\mathrm{ERM}_\beta[X] = -\beta^{-1}\log(\mathbb{E}[e^{-\beta X}])$$

furthermore, it can be written as

$$\mathrm{ERM}_\beta[X] = \inf_Z\{\mathbb{E}[ZX] - \frac{\mathbb{E}[Z\log(Z)]}{\beta} : \mathbb{E}[Z\cdot e^{-\beta X}] \leq 1\}$$

$$\mathrm{ERM}_\beta[X] = \inf_\alpha\{\mathrm{EVaR}_\alpha[X] - \frac{\log(\alpha)}{\beta}\}$$

ERM has utility function

$$U^\beta_{\mathrm{ERM}}(x) = \frac{1-e^{-\beta x}}{\beta}$$

Entropic Value at Risk

Here we define EVaR as

$$\mathrm{EVaR}_\alpha[X] = \max_\beta\{\mathrm{ERM}_\beta[X] + \log(\alpha)\}$$

EVaR has utility function

$$U^\alpha_{\mathrm{EVaR}}(x) = x\cdot\frac{e^{-\beta^\star x}}{\mathbb{E}[e^{-\beta^\star X}]}$$

where $\beta^\star = \mathrm{argmax}_\beta\{-\beta^{-1}\cdot[\log(\mathbb{E}[e^{-\beta X}]) - \log(\alpha)]\}$. Furthermore, let $Z^\star = \frac{e^{-\beta^\star X}}{\mathbb{E}[e^{-\beta^\star X}]}$ we have $\mathbb{E}[Z^\star] = 1$ and $\mathbb{E}[Z^\star\log(Z^\star)] = -\log(\alpha)$. Property (P1) follows from the dual of EVaR:

$$\mathbb{E}[U^\alpha_{\mathrm{EVaR}}(x)] = \mathbb{E}[X\cdot\frac{e^{-\beta^\star X}}{\mathbb{E}[e^{-\beta^\star X}]}] = \mathbb{E}[X\cdot Z^\star] = \inf_{Z>0}\{\mathbb{E}[XZ] : \mathbb{E}[Z] = 1, \mathbb{E}[Z\log(Z)] \leq -\log(\alpha)\}$$

The conditional EVaR can be formulate as

$$\mathrm{EVaR}_\alpha[X] = \inf_Z\{\mathbb{E}[Z\cdot\mathrm{EVaR}_{\alpha\cdot\exp(-\mathbb{E}[Z\log(Z)])}[X|S']]\}$$

Proof:

$$\mathrm{EVaR}_\alpha[X] = \sup_\beta\{\mathrm{ERM}_\beta[X] + \frac{\log(\alpha)}{\beta}\} = \mathrm{ERM}_{\beta^\star}[X] + \frac{\log(\alpha)}{\beta^\star}$$

$$= \mathrm{ERM}_{\beta^\star}[\mathrm{ERM}_{\beta^\star}[X|S']] + \frac{\log(\alpha)}{\beta^\star}$$

$$= \mathrm{ERM}_{\beta^\star}[\inf_{\alpha'}\{\mathrm{EVaR}_{\alpha'}[X|S'] - \frac{\log(\alpha')}{\beta}\}] + \frac{\log(\alpha)}{\beta^\star}$$

$$= \mathrm{ERM}_{\beta^\star}[\inf_{Z_{S'}}\{\mathrm{EVaR}_{\alpha Z_{S'}}[X|S'] - \frac{\log(\alpha Z_{S'})}{\beta^\star}\}] + \frac{\log(\alpha)}{\beta^\star}$$

$$= \mathrm{ERM}_{\beta^\star}[\inf_{Z_{S'}}\{\mathrm{EVaR}_{\alpha Z_{S'}}[X|S'] - \frac{\log(\alpha Z_{S'})}{\beta^\star}\}] + \frac{\log(\alpha)}{\beta^\star}$$

$$\inf_Z\{\mathbb{E}[Z \cdot \inf_{Z_{S'}}\{\mathrm{EVaR}_{\alpha Z_{S'}}[X|S'] - \frac{\log(\alpha Z_{S'})}{\beta^\star}\}] - \frac{\mathbb{E}[Z\log(Z)]}{\beta^\star} : \mathbb{E}[Z \cdot e^{-\beta^\star X}] \leq 1\} + \frac{\log(\alpha)}{\beta^\star}$$

$$\inf_Z\{\inf_{Z_{S'}}\{\mathbb{E}[Z \cdot \mathrm{EVaR}_{\alpha Z_{S'}}[X|S']] + \frac{-\log(\alpha Z_{S'}) + \log(\alpha) - \mathbb{E}[Z\log(Z)]}{\beta^\star} : \mathbb{E}[Z \cdot e^{-\beta^\star X}] \leq 1\}\}$$

If $-\log(\alpha Z_{S'}) + \log(\alpha) - \mathbb{E}[Z\log(Z)] = 0$ implies $Z_{S'} = e^{-\mathbb{E}[Z\log(Z)]}$

$$= \inf_Z\{\mathbb{E}[Z \cdot \mathrm{EVaR}_{\alpha\cdot\exp(-\mathbb{E}[Z\log(Z)])}[X|S']]\}$$

$$\neq \inf_Z\{\mathbb{E}[Z \cdot \mathrm{EVaR}_{\alpha\cdot Z}[X|S']]\}$$

Example

```
X1 = data.frame(x = c(1,10),p = c(0.2,0.8))
X2 = data.frame(x = c(2,10),p = c(0.9,0.1))
condp = c(0.3,0.7)

Xjoint = data.frame(x = c(1,2,10),p =c(0.06,0.63,0.31))

beta = 0.5
logZ = -beta*(Xjoint$x - ERM(Xjoint$x,alpha=beta,prob=Xjoint$p))
Z = exp(logZ)
alpha = exp(-E(X=exp(logZ)*logZ,prob=Xjoint$p))
soln = sum(Z*Xjoint$x*Xjoint$p)
soln
```

```
## [1] 1.927171
```

```
logZ1 = -beta*(X1$x - ERM(X1$x,alpha=beta,prob=X1$p))
logZ2 = -beta*(X2$x - ERM(X2$x,alpha=beta,prob=X2$p))
Z1 = exp(logZ1)
Z2 = exp(logZ2)
alpha1 = exp(-E(X=exp(logZ1)*logZ1,prob=X1$p))
alpha2 = exp(-E(X=exp(logZ2)*logZ2,prob=X2$p))
zt1 = alpha1/alpha
zt2 = alpha2/alpha
# The wrong solution can be computed as
cat("wrong method",E(X = c(zt1 * E(Z1*X1$x,prob=X1$p),zt2 * E(Z2*X2$x,prob=X2$p)),prob = condp))
```

```
## wrong method 1.959746
```

```
# The right solution can be computed as
library(lamW)
```

```
## Warning: package 'lamW' was built under R version 4.2.2
```

```
mixz1 = exp(lambertWm1(log(zt1)))
mixz2 = exp(lambertW0(log(zt2)))
cat("maybe right method",E(X = c(mixz1 * E(Z1*X1$x,prob=X1$p),mixz2 * E(Z2*X2$x,prob=X2$p)),prob = cond
```

```
## maybe right method NaN
```