Distortion Function and utility function

Monkie

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Utility function

Here we define utility function for a discrete random variable X given risk measure ρ and risk averse parameter α as $U^\alpha_\rho(X)$. Utility function has a property (P1) $\mathbb{E}[U^\alpha_\rho(X)] = \rho_\alpha[X]$. For certain risk measure (eg: $\rho \in \{\mathbb{E}, \mathrm{VaR}, \mathrm{CVaR}, \mathrm{EVaR}\}$), we can write $U^\alpha_\rho(x) = z(x) \cdot x$ where z is a function of x.

1. Expected value utility function

   $$U_{\mathbb{E}}(x) = x = 1 \cdot x$$

   Property (P1) follows trivially

   $$\mathbb{E}[U_{\mathbb{E}}(X)] = \mathbb{E}[X]$$

2. Value at risk utility function

   $$U^\alpha_{\mathrm{VaR}}(x) = \begin{cases} \frac{1}{\mathbb{P}[X = x_\alpha]} \cdot x & , x = x_\alpha \\ 0 & , \text{otherwise} \end{cases}$$

   where $x_\alpha = \inf\{x \in \mathbb{R} : P(X \leq x) \geq \alpha\} = \mathrm{VaR}_\alpha[X]$ is the α^{th} percentile of X. Property (P1) follows:

   $$\mathbb{E}[U^\alpha_{\mathrm{VaR}}(x)] = \mathbb{P}[X \neq x_\alpha] * 0 + \mathbb{P}[X = x_\alpha] * \frac{x_\alpha}{\mathbb{P}[X = x_\alpha]} = x_\alpha = \mathrm{VaR}_\alpha[X]$$

3. Conditional Value at risk utility function

   $$U^\alpha_{\mathrm{CVaR}}(x) = \begin{cases} \frac{1}{\alpha} \cdot x & , x < x_\alpha \\ \frac{\alpha \ - \ \mathbb{P}[X < x_\alpha]}{\alpha} \cdot x & , x = x_\alpha \\ 0 & , \text{otherwise} \end{cases}$$

   Property (P1) Follow:

   $$\mathbb{E}[U^\alpha_{\mathrm{CVaR}}(x)] = \frac{\mathbb{E}[X \ 1_{\{X < x_\alpha\}}]}{\alpha} + x_\alpha \cdot \frac{(\alpha - \mathbb{P}[X < x_\alpha])}{\alpha} = \mathrm{CVaR}_\alpha[X]$$

4. Entropic Value at risk utility function

   $$U^\alpha_{\mathrm{EVaR}}(x) = x \cdot \frac{e^{-\beta^\star x}}{\mathbb{E}[e^{-\beta^\star X}]}$$

   where $\beta^\star = \mathrm{argmax}_\beta\{-\beta^{-1} \cdot [\log(\mathbb{E}[e^{-\beta X}]) - \log(\alpha)]\}$. Furthermore, let $Z^\star = \frac{e^{-\beta^\star X}}{\mathbb{E}[e^{-\beta^\star X}]}$ we have $\mathbb{E}[Z^\star] = 1$ and $\mathbb{E}[Z^\star \log(Z^\star)] = -\log(\alpha)$. Property (P1) follows from the dual of EVaR:

   $$\mathbb{E}[U^\alpha_{\mathrm{EVaR}}(x)] = \mathbb{E}[X \cdot \frac{e^{-\beta^\star X}}{\mathbb{E}[e^{-\beta^\star X}]}] = \mathbb{E}[X \cdot Z^\star] = \inf_{Z > 0}\{\mathbb{E}[XZ] : \mathbb{E}[Z] = 1, \mathbb{E}[Z \log(Z)] \leq -\log(\alpha)\}$$

5. Entropic Risk measure utility function

   $$U^\beta_{\mathrm{erm}}(x) = \frac{1 - e^{-\beta(x - \mathrm{ERM}_\beta[X])}}{\beta} + \mathrm{ERM}_\beta[X]$$

Distortion function

Instead of changing the utility of the value in distribution X. Distortion function refer to the cumulative distribution of the (dual) robust distorted distribution $Q^\star$. Where $Q^\star$ is defined as

$$\rho_\alpha[X] = \mathbb{E}_{Q^\star}[X] = \inf_{Q\in\mathcal{Q}} (\ \mathbb{E}_Q[X]\ )$$

Instead of distorted (cumulative distribution) we first make a connection of utility function with distorted probability mass function (PMF). Here, we connect expected utility and the probability mass function (PMF) of the robust distribution $Q^\star$ with respect to certain risk measure (eg: $\rho \in \{\mathbb{E}, \mathrm{VaR}, \mathrm{CVaR}, \mathrm{EVaR}\}$) where $\rho_\alpha[X] = \mathbb{E}[U^\alpha_\rho(X)] = \mathbb{E}[z(X)\cdot X]$ over random variable X.

$$\rho_\alpha[X] = \mathbb{E}[U^\alpha_\rho(X)] = \sum_x p(x)\cdot U^\alpha_\rho(X) = \sum_x p(x)\cdot z(x)\cdot x = \sum_x q(x)\cdot x = \mathbb{E}_{Q^\star}[X]$$

Now we are ready to defined the Distortion (cumulative distribution) function $D^\alpha_\rho(\ F^P_X(x)\ ) = F^{Q^\star}_X(x)$. Given risk measure ρ and level α, the distortion function take in the CDF of the original distribution P of X and output the CDF of the robust distorted distribution $Q^\star$ of X .

1. Expected value distortion function

$$\mathrm{D}_{\mathbb{E}}(\ F^P_X(x))\ ) = F^{Q^\star}_X(x) = F^P_X(x)$$

2. Value at risk distortion function

$$\mathrm{D}^\alpha_{\mathrm{var}}(\ F^P_X(x)\ ) = F^{Q^\star}_X(x) = \begin{cases} 0 & , 0 \le F^P_X(x) < \alpha \\ 1 & , \alpha \le F^P_X(x) \le 1 \end{cases}$$

3. Conditional value at risk distortion function

$$\mathrm{D}^\alpha_{\mathrm{cvar}}(\ F^P_X(x)\ ) = F^{Q^\star}_X(x) = \begin{cases} \frac{F^P_X(x)}{\alpha} & , 0 \le F^P_X(x) < \alpha \\ 1 & , \alpha \le F^P_X(x) \le 1 \end{cases}$$

4. Entropic value at risk distortion function

$$\mathrm{D}^\alpha_{\mathrm{cvar}}(\ F^P_X(x)\ ) = F^{Q^\star}_X(x) = \sum_{\{\chi \le x\ :\ \chi\in X\}} \left(\ p(\chi)\cdot \frac{e^{-\beta^\star \chi}}{\mathbb{E}[e^{-\beta^\star X}]}\ \right)$$