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Chapter 1
SMAA in Robustness Analysis

Risto Lahdelma and Pekka Salminen

Abstract Stochastic multicriteria acceptability analysis (SMAA) is a simulation based method for discrete multicriteria decision aiding problems where information is uncertain, imprecise, or partially missing. In SMAA, different kind of uncertain information is represented by probability distributions. Because SMAA considers simultaneously the uncertainty in all parameters, it is particularly useful for robustness analysis. Depending on the problem setting, SMAA determines all possible rankings or classifications for the alternatives, and quantifies the possible results in terms of probabilities. This chapter describes SMAA in robustness analysis using a real-life decision problem as an example. Basic robustness analysis is demonstrated with respect to uncertainty in criteria and preference measurements. Then the analysis is extended to consider also the structure of the decision model.

1.1 Introduction

Robustness analysis of a computational model is a type of sensitivity analysis that considers simultaneous variations of all parameters in a given domain. More general robustness analysis would also consider the sensitivity of the analysis with respect to model structure derived from various assumptions. Robustness analysis is necessary in particular when some input parameters of the model are imprecise or uncertain.

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Stochastic multicriteria acceptability analysis (SMAA) is a simulation based method for discrete multicriteria decision aiding problems where information is uncertain, imprecise, or partially missing. In SMAA, different kind of uncertain information is represented by probability distributions. This approach is similar to metrology [22]. For example, if the cost of an alternative is not accurately known, it can be represented by a uniform distribution in a given range, or a normal distribution with specified expected value and standard deviation (Fig. 1.1). Uncertain preference information is similarly represented by distributions. Also subsequent computations in SMAA follow probability theory.

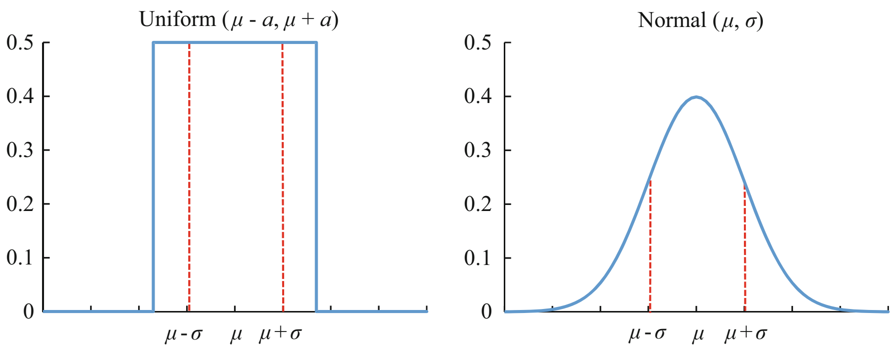

Fig. 1.1: Representing uncertain criteria measurements as distributions

Depending on the problem setting, SMAA computes statistically for each alternative the probability to be most preferred, dominate another alternative, be placed on a particular rank or fit in a specific category. The computation is implemented by Monte-Carlo simulation, where values for the uncertain variables are sampled from their distributions and alternatives are evaluated by applying the decision model.

SMAA can be applied with different decision models. These include linear and non-linear utility or value functions [8, 15, 16], ELECTRE methods [9, 26], reference point based methods [11, 17], efficiency score of Data Envelopment Analysis (DEA) [10], nominal classification method [29], and ordinal classification method [12]. For a surveys on different variants and applications of SMAA, see [13, 24]. Recent developments of SMAA include robustness analysis with respect to shape of the utility function by Lahdelma and Salminen [14], efficient Markov Chain Monte Carlo simulation technique to treat complex preference information by Tervonen et al. [27], the SMAA-PROMETHEE method by Corrente et al. [4], SMAA with Choquet Integral by Angilella et al. [1], and extensions for pairwise comparison methods such as the analytic hierarchy process (AHP) by Durbach et al. [6] and the Complementary Judgment Matrix (CJM) method by Wang et al. [28].

Because SMAA considers simultaneously the uncertainty in all parameters, it is particularly useful for robustness analysis of different multicriteria decision models. SMAA determines all possible rankings or classifications for the alternatives, and quantifies the possible results in terms of probabilities. The solution with highest probability is typically the recommended solution. However, the probabilities for other possible solutions are also provided for the decision makers (DMs). This means that SMAA describes how robust the model is subject to different uncertainties in the input data. SMAA can also be used to analyze the robustness of the decision problem with respect to the model structure. For example, robustness with respect to linearity assumptions in utility/value functions can be analyzed by choosing a more general parametrized utility function and exploring how the solutions change as a function of the degree of non-linearity.

In the following, we describe the SMAA method applied on a real-life decision problem of choosing a location for an air cargo hub in Morocco [21]. Section 1.2 describes problem representation in SMAA as a stochastic MCDA problem and how it is analysed using stochastic simulation. Section 1.3 presents the statistical measures of SMAA and shows how SMAA can be used to assess the robustness of an MCDA problem with respect to uncertainty in criteria and preference measurements. Section 1.4 extends the robustness analysis to consider the structure of the decision model.

1.2 Problem Representation in SMAA

1.2.1 Stochastic MCDA Problem

A discrete multi-criteria decision problem consists of a set of m alternatives that are measured in terms of n criteria. The alternatives are evaluated using a decision model $M(\mathbf{x}, \mathbf{w})$ that depends on the applied decision support method. The matrix $\mathbf{x} = [x_{ij}]$ contains the criteria measurements for each alternative i and criterion j. The preference information vector $\mathbf{w} = [w_j]$ represents the DM's preferences. Typically $\mathbf{w}$ contains importance weights for the criteria. Depending on the decision model, $\mathbf{w}$ can also contain other preference parameters, such as various shape parameters for non-linear models.

SMAA has been developed for real-life problems, where both criteria and preference information can be imprecise, uncertain or partially missing. To represent the incompleteness of the information explicitly, SMAA represents the problem as a *stochastic MCDA model*, where criteria and preference information is represented by suitable (joint) probability distributions:

- $f_X(\mathbf{x})$ the density function for stochastic criteria measurements.
- $f_W(\mathbf{w})$ the density function for stochastic importance weights or other preference parameters.

Because all information is represented uniformly as distributions, this allows using efficient simulation techniques for analyzing the problem and deriving results about prospective solutions and their robustness.

An example of a stochastic MCDA model is the problem of choosing a location for a centralized air cargo hub in Morocco [21]. In this problem, nine alternative locations were considered. Different socio-economic factors, the geographical location, and environmental impacts were formalized as six criteria: INVEST = investment cost, PROXIMITY = proximity to producers, POTENTIAL = potential of the site, TRANSPORT = transport cost, SERVICE = service level, ENVIRON = Environment. The alternatives, criteria and measurements are presented in Table 1.1.

The INVEST, POTENTIAL, TRANSPORT and SERVICE criteria were measured on cardinal scales. The values in Table 1.1 for these criteria are dimensionless quantities that have been obtained by scaling the actual measurements on linear scales where larger values are better. The uncertainty of these measurements appears on the last row as a plus/minus percentage. The measurements were then represented as independent, uniformly distributed random numbers in the plus/minus ranges around their expected values. In SMAA it is possible to use arbitrary distributions to represent uncertain criteria measurements. If the uncertainties of the criteria measurements are dependent, this can be represented by joint distributions, such as the multivariate Gaussian distribution [18, 19].

The PROXIMITY and ENVIRON criteria were evaluated ordinally, i.e. experts ranked the alternatives with respect to these criteria so that the best alternative obtained rank 1, second best rank 2 etc. Ordinal measurement can be necessary if cardinal measurement is too costly, or if it is difficult to form a measurable scale for the criterion.

Table 1.1: Alternatives and criteria measurements in air cargo hub case (alphabetical order)

Alt	INVEST (max)	PROXIMITY (min)	POTENTIAL (max)	TRANSPORT (max)	SERVICE (max)	ENVIRON (min)
Agadir	70	2	165	644	50	2
Benslimane	80	3	560	3718	40	1
Casablanca	65	1	585	3621	80	5
Dakhla	80	8	82	600	20	1
Fez	70	6	385	2872	30	4
Marrakesh	65	5	379	2589	45	1
Oujda	75	7	82	663	25	4
Rabat	65	4	542	3718	45	3
Tangier	70	3	357	1915	60	3
Uncertainty	$\pm 10\,\%$	Ordinal	$\pm 10\,\%$	$\pm 10\,\%$	$\pm 10\,\%$	Ordinal

1.2.2 Generic SMAA Simulation

Different variants of SMAA apply the generic simulation scheme of Algorithm 1 for analyzing stochastic MCDA problems. During each iteration, criteria measurements, weights, and possible other preference parameters are drawn from their distributions, and the decision model is used to evaluate the alternatives. Depending on the problem setting and decision model, different statistics are collected during the simulation and the SMAA measures are computed based on the statistics. For example, in the case of a ranking problem, statistics are collected about how frequently alternatives obtain a given rank.

Algorithm 1. Generic SMAA simulation

Assume a decision model M(**x, w**) for ranking or classifying the alternatives using precise information (criteria matrix **x** and preference information vector **w**)

Use Monte-Carlo simulation to treat stochastic criteria and preference parameters:
Repeat Ktimes {
    Draw <**x, w**> from their distributions
    Rank, sort or classify the alternatives using M(**x,w**)
    Update statistics about alternatives
}

Compute results based on the collected statistics

The efficient implementation and computational efficiency of SMAA methods have been described by Tervonen and Lahdelma [25]. The computational accuracy of the main results depends on the square root of the number of iterations, i.e. increasing the number of iterations by a factor of 100 will increase the accuracy by one decimal place. In practice about 10,000 iterations yield sufficient accuracy for the SMAA results.

1.2.3 Decision Model

SMAA can be used with arbitrarily shaped utility functions, and also with other kinds of decision models that are based on any kind of preference parameters. A common type of utility function is the *additive* utility function that defines the overall utility as a weighted sum of *partial utilities*:

$$u(x_i, \mathbf{w}) = w_1 u_{i1} + w_2 u_{i2} + \ldots + w_n u_{in} \tag{1.1}$$

The w_j are the importance weights for criteria and u_{ij} are the partial utilities obtained by mapping the original criteria measurements (expressed in various units) to unit-less scales so that the worst outcome is 0 and the best outcome is 1. The mappings can be linear or non-linear monotonic functions.

In the sample problem linear mappings were applied, leading to a linear overall utility function. In this study we consider also non-linear mappings in order to analyze the robustness of the problem with respect to the shape of the utility function.

The weights should be non-negative and normalized so that their sum is 1. By substituting 1 or 0 for each partial utility in (1.1) we see that the overall utility is 1 for an ideal alternative, and 0 for an anti-ideal alternative.

1.2.4 Preference Information

In SMAA, incomplete preference information is represented using probability distributions. In the following we consider incomplete weight information. However the same techniques can be used also for other preference parameters.

With an additive utility function, the weights express the relative importance of raising each criterion from its worst value to the best value. Ratios between weights correspond to trade-offs between criteria. In SMAA uncertain or imprecise weights are represented as a joint probability distribution in the *feasible weight space* defined as the set of non-negative and normalized weights

$$W = \{\mathbf{w} \,|\, w_j \geq 0 \text{ and } w_1 + w_2 + \ldots + w_n = 1\} \tag{1.2}$$

This means that the feasible weight space is an $(n-1)$-dimensional simplex. In the 3-criterion case, the feasible weight space is a triangle with corners (1,0,0), (0,1,0) and (0,0,1), as illustrated in Fig. 1.2a. In the absence of weight information, we assume that any feasible weights are equally possible, which is represented by a uniform distribution in W.

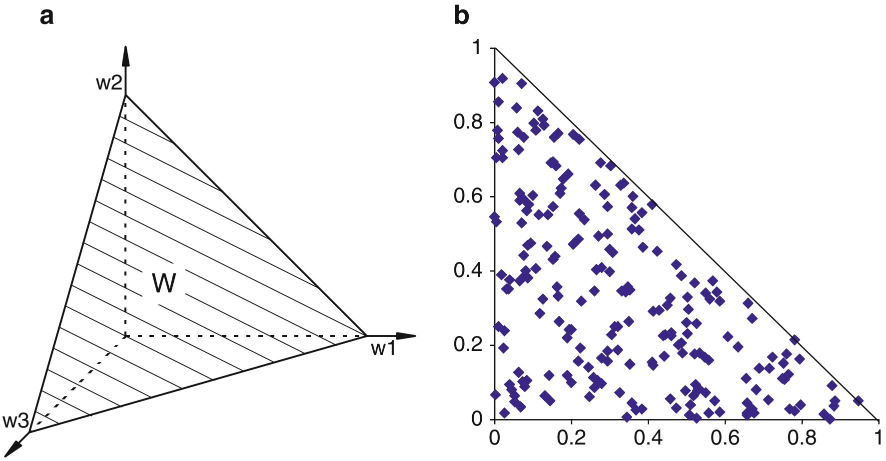

Fig. 1.2: (**a**) Feasible weight space in the 3-criterion case. (**b**) Sampling uniformly distributed weights in the 3-criterion case projected on the (w_1, w_2) plane

Uniformly distributed normalized weights need to be generated using a special technique [25]. First $n-1$ independent uniformly distributed random numbers in the interval [0,1] are generated and sorted together with 0 and 1 into ascending order to get $0 = r_0 \leq r_1 \leq \cdots \leq r_n = 1$. From these numbers the weights are computed as the intervals $w_1 = r_1 - r_0$, $w_2 = r_2 - r_1$, $\ldots$, $w_n = r_n - r_{n-1}$. It is obvious that the resulting weights will be non-negative and normalized. For the proof that the resulting joint distribution is uniform, see [5]. Figure 1.2b illustrates generation of uniformly distributed weights in the 3-dimensional case, projected on the (w_1, w_2) plane where $w_3 = 1 - w_1 - w_2$.

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Preference information can be treated in SMAA by restricting the uniform weight distribution with additional constraints. Another technique is to apply a non-uniform distribution for the weights. For example, if the DMs express precise weights with implicit imprecision, this can be represented by a distribution with decreasing density around the expressed weights. Suitable distributions are e.g. triangular distributions and (truncated) normal distributions.

Different ways to restrict the uniform or non-uniform weight distribution with additional constraints include the following:

- Weight intervals can be expressed as $w_j \in [w_j^{\min}, w_j^{\max}]$. Weight intervals may result from DMs' preference statements of type "the importance weight for criterion j is between $w_j^{\min}$ and $w_j^{\max}$". Weight intervals can also be computed to include precise weights or weight intervals of a group of DMs. Figure 1.3a illustrates weight intervals in the 3-criterion case.
- Intervals for trade-off ratios between criteria can be expressed as $w_j/w_k \in [w_{jk}^{\min}, w_{jk}^{\max}]$. Such intervals may result from preference statements like "criterion j is from $w_{jk}^{\min}$ to $w_{jk}^{\max}$ times more important than criterion k". These intervals can also be determined to include the preferences of a group of DMs. Figure 1.3b illustrates two constraints for trade-off ratios.
- Ordinal preference information can be expressed as linear constraints $w_1 \geq w_2 \geq \cdots \geq w_n$. Such constraints represent DMs preference statement that the criterion 1 is most important, 2 is second etc. It is also possible to allow unspecified importance ranking for some criteria or equal importance ($w_j = w_k$). Multiple DMs may either agree on a common partial ranking, or they can provide their own rankings, which can then be combined into a partial ranking that is consistent with each DM's preferences. Figure 1.3c illustrates ordinal preference information.
- DMs holistic preference statements "alternative x_i is more preferred than x_k" result in constraints $u(x_i, \mathbf{w}) \geq u(x_k, \mathbf{w})$ for the weights. In the case of an additive utility/value function, these constraints will be linear inequalities in the weight space. Figure 1.3d illustrates one such holistic preference statement. In the general case, with non-additive utility/value functions, outranking models etc., holistic constraints correspond to non-linear constraints in the weight space.

Weight constraints can be implemented by modifying the weight generation procedure to reject weights that do not satisfy the constraints. In most cases this technique is very efficient. In some cases the Markov Chain Monte Carlo simulation technique is more efficient [27].

1.2.5 Cardinal Criteria

In the case of a linear utility function, the partial utilities u_{ij} are computed from the actual cardinal criteria measurements x_{ij} through linear scaling. The best and worst values can be determined as some natural ideal and anti-ideal values, if such

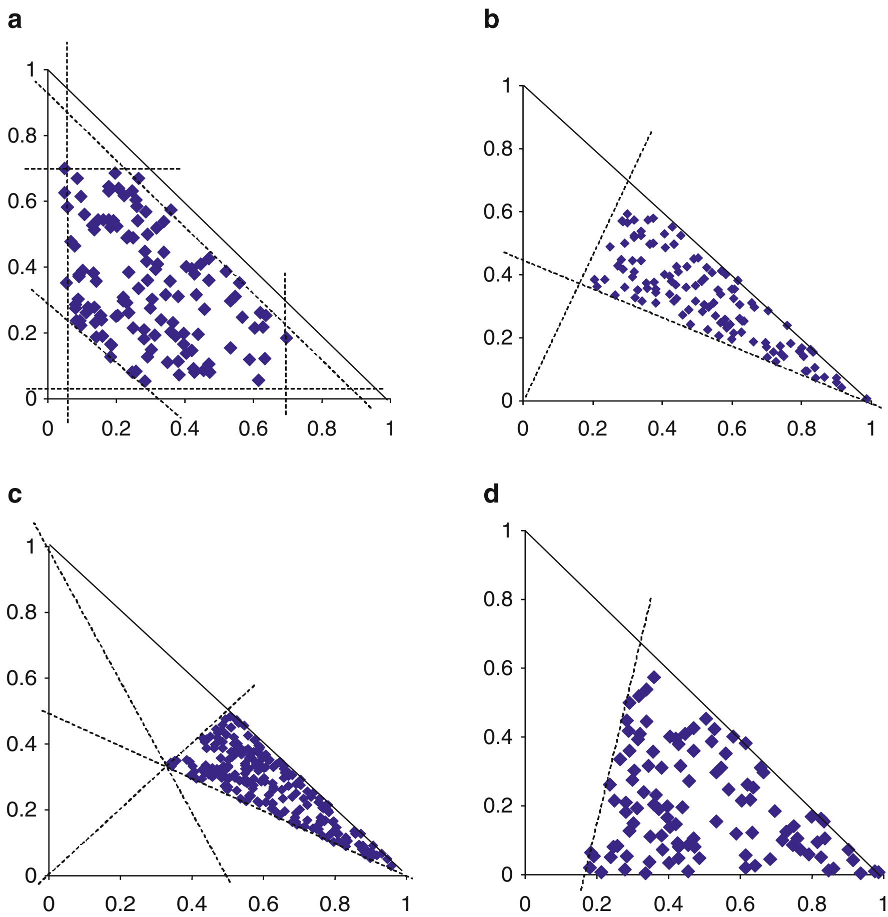

Fig. 1.3: Sampling uniformly distributed weights in the 3-criterion case projected on the (w_1, w_2) plane: (**a**) with interval constraints for weights; (**b**) with two constraints for trade-off ratios; (**c**) with ordinal preference information $w_1 \geq w_2 \geq w_3$; (**d**) with holistic preference information based on an additive utility/value function

exist. For example, the ideal value for costs could be 0 and the ideal value for an efficiency ratio could be 100 %. If such ideal and anti-ideal values cannot easily be defined, it is possible to do the scaling according to the best and worst measurements among the alternatives, as has been done for the sample problem in Table 1.2. Also the uncertainties have been scaled accordingly. A downside with scaling based on best and worst criteria measurements is that the scaling may change if the set of alternatives or their measurements change during the decision process.

As a result, the uncertainty intervals may contain values outside the [0, 1] range. This is not a problem, because the scaling interval is arbitrary; any other interval would order the alternatives identically according to their utilities.

Table 1.2: Scaled cardinal criteria measurements and their uncertainties in air cargo hub case

Alt	INVEST	POTENTIAL	TRANSPORT	SERVICE
Agadir	0.33±0.47	0.17±0.03	0.01±0.02	0.50±0.08
Benslimane	1.00±0.53	0.95±0.11	1.00±0.12	0.33±0.07
Casablanca	0.00±0.43	1.00±0.12	0.97±0.12	1.00±0.13
Dakhla	1.00±0.53	0.00±0.02	0.00±0.02	0.00±0.03
Fez	0.33±0.47	0.60±0.08	0.73±0.09	0.17±0.05
Marrakesh	0.00±0.43	0.59±0.08	0.64±0.08	0.42±0.08
Oujda	0.67±0.50	0.00±0.02	0.02±0.02	0.08±0.04
Rabat	0.00±0.43	0.91±0.11	1.00±0.12	0.42±0.08
Tangier	0.33±0.47	0.55±0.07	0.42±0.06	0.67±0.10

1.2.6 Ordinal Criteria

Ordinal criteria measurements are imprecise: we know the rank of each alternative with respect to the ordinal criterion, but we do not know how much better the first alternative is than the second or third one, etc. In SMAA, ordinal criteria are treated by simulating cardinal values that are consistent with the given ordinal ranks. The first rank corresponds to cardinal value $s_1 = 1$ and the last rank R corresponds to $s_R = 0$. The intermediate ranks 2, 3, …, $R-1$ should correspond to a descending sequence of unknown cardinal values between 1 and 0. To obtain the unknown intermediate values, $R-2$ independent uniformly distributed random numbers in the interval [0, 1] are generated. These values are then sorted together with 1 and 0 into descending order to obtain cardinal values that satisfy $1 = s_1 \geq s_2 \geq \cdots \geq s_{R-1} \geq s_R = 0$.

The process described converts ordinal criteria into stochastic cardinal criteria. Note that the intervals between subsequent values $s_r - s_{r+1}$ are non-negative and their sum is 1. Subject to these constraints, the intervals follow a uniform distribution [5].

In the air cargo hub case, the PROXIMITY and ENVIRON criteria were ordinal. Figure 1.4 shows some random cardinal mappings for these criteria. For the PROXIMITY criteria, alternatives Benslimane and Tangier were both ranked on level 3. Therefore rank levels 1–8 were assigned for the nine alternatives. Similarly, shared ranks for the ENVIRON criteria resulted in assigning five different rank levels for that criterion.

1.3 Robustness with Imprecise Criteria and Weights

In the following we demonstrate the SMAA method using the air cargo hub case presented in Sect. 1.2. A linear utility/value function was used as the decision model in this application. The simulation scheme presented in Algorithm 1 is applied and the utility function is used to rank the alternatives. Observe that this approach

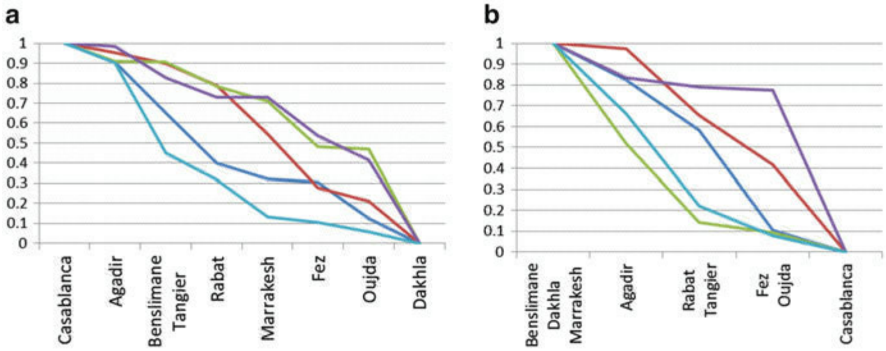

Fig. 1.4: Sample of simulated cardinal values for the air cargo hub case. (**a**) PROXIMITY criterion. (**b**) ENVIRON criterion

differs from traditional utility function methods that compute the expected utility. This means that SMAA does not require a cardinal utility function—an ordinal utility/value function is sufficient. Based on the ranking, the following statistics are collected during the simulation:

- B_{ir}: The number of times alternative x_i obtained rank r.
- C_{ik}: The number of times alternative x_i was more preferred than x_k.
- $\mathbf{W}_i$: Sum of the weight vectors that made alternative x_i most preferred.

Based on the collected statistics the basic SMAA measures are computed. These include *rank acceptability indices*, *pairwise winning indices*, *central weight vectors*, and *confidence factors*, as presented in the following sections.

1.3.1 Rank Acceptability Indices

The primary SMAA measure is the *rank acceptability index* b_i^r. It measures the variety of different preferences that place alternative x_i on rank r. It is the share of all feasible weights that make the alternative acceptable for a particular rank. In other words, it is the probability that the alternative obtains a certain rank. Particularly interesting is the first rank acceptability index b_i^1, which is the probability that the alternative is the most preferred one. For inefficient alternatives the first rank acceptability index is zero. The rank acceptability indices are estimated from the simulation statistics (with K iterations) as

$$b_i^k \approx B_{ir}/K \tag{1.3}$$

The rank acceptability indices can be used for ***robust choice of one or a few best alternatives*** from a large set. Alternatives with high acceptability for the best ranks are candidates for the most acceptable solution. Alternatives with large acceptability

Table 1.3: Rank acceptability indices for air cargo hub case (sorted by b_1)

Alt	b^1	b^2	b^3	b^4	b^5	b^6	b^7	b^8	b^9
Benslimane	72.00	23.00	4.00	1.00	0.00	0.00	0.00	0.00	0.00
Casablanca	25.00	33.00	14.00	7.00	5.00	5.00	5.00	3.00	3.00
Dakhla	1.00	7.00	6.00	5.00	5.00	6.00	15.00	41.00	15.00
Agadir	0.40	5.00	9.00	11.00	13.00	20.00	30.00	9.00	3.00
Tangier	0.38	7.00	17.00	28.00	25.00	16.00	4.00	2.00	0.00
Rabat	0.28	14.00	37.00	17.00	13.00	9.00	5.00	3.00	2.00
Marrakesh	0.03	11.00	11.00	22.00	25.00	19.00	6.00	3.00	3.00
Oujda	0.02	0.00	1.00	2.00	2.00	2.00	4.00	20.00	70.00
Fez	0.00	1.00	2.00	7.00	12.00	23.00	31.00	19.00	4.00

for the worst ranks should be avoided when searching for a robust most preferred alternative even if they would have fairly high acceptability for the best ranks. If none of the alternatives receive high acceptability indices for the best ranks, it indicates a need to measure the criteria, preferences or both more accurately.

Table 1.3 presents the rank acceptability indices for the air cargo hub case and Fig. 1.5 shows the corresponding *acceptability profile*. To make the acceptability profile easy to read, the alternatives are sorted by their first rank acceptability index. In case of equal first rank indices, order is determined based on the second index etc. This is called *lexicographic order*. The most acceptable (best) alternatives are Benslimane and Casablanca with clearly highest acceptability for the highest ranks. Benslimane receives 72 % acceptability for the first rank, 23 % for the second rank, 4 % for the third rank, 1 % for the fourth rank, and 0 for the ranks 5–9. This means that Benslimane is a robust choice subject to many different possible preferences. Also Casablanca with 25 % acceptability for the first rank and 33 % for the second rank is a possible choice subject to suitable preferences. However, Casablanca is not as robust subject to different preferences, because it can obtain also all other ranks with some probability.

The rank acceptability indices can also be used for ***eliminating some of the worst alternatives***. Among the less acceptable alternatives, in particular Oujda receives either the last or next to last rank with 90 % probability. Eliminating Oujda from the set of best alternatives would a robust choice.

The acceptability profile will provide only a ***rough ranking*** of the alternatives because there is no objective way to combine acceptability indices for different ranks to reach a complete ranking. For forming a complete ranking, Lahdelma and Salminen [8] suggested the *holistic acceptability index*, which is a weighted sum of the rank acceptability indices for different ranks. However, the holistic acceptability index depends on meta-weights in the weighted sum, and meta-weights are subjective. Another problem with using the acceptability indices to form a complete ranking is that if alternatives are removed from or added to the problem, acceptability indices may change, and the mutual order of alternatives may change. This is known as the *rank reversal problem*, present in several MCDA methods. In SMAA the above ranking problems can be resolved by the pairwise winning index, which is presented next.

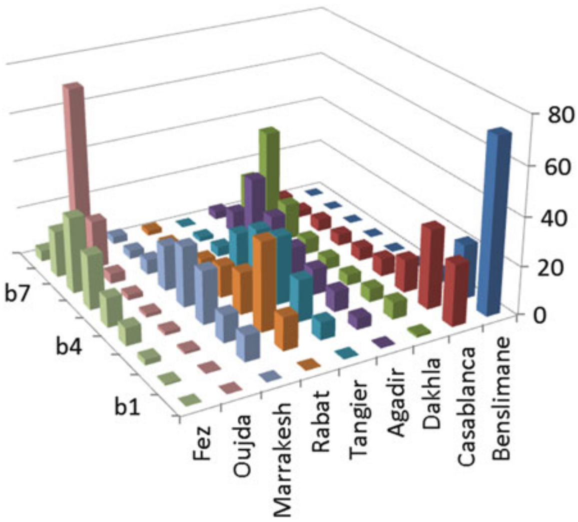

Fig. 1.5: Acceptability profile for alternatives in air cargo hub case

1.3.2 Pairwise Winning Indices

The *pairwise winning index* c_{ik} is the probability for alternative x_i being more preferred than x_k, considering the uncertainty in criteria and preferences [20]. The pairwise winning index is estimated from the simulation statistics as

$$c_{ik} \approx C_{ik}/K \tag{1.4}$$

The pairwise winning indices are useful when comparing the mutual performance of two alternatives. This information can be used e.g. when it is necessary to eliminate inferior alternatives that are dominated by other alternatives.

Unlike the rank acceptability index, the pairwise winning index between one pair of alternatives is independent on the other alternatives. This means that the pairwise winning index can be used to form a ranking among the alternatives. The ranking is obtained by ordering the alternatives so that each alternative x_i precedes all alternatives x_k for which $c_{ik} > 50\,\%$ or some bigger threshold value.

Table 1.4 shows the pairwise winning indices for the air cargo hub case. In this table the alternatives have been ordered to form a complete ranking, which means that all pairwise winning indices in the upper triangle are $>50\,\%$ and $<50\,\%$ in the lower triangle. Observe that there are problems where a complete ranking cannot be obtained. For example, three or more alternatives may win each other in a cyclic manner. In that case such subsets of alternatives obtain the same rank.

Table 1.4: Pairwise winning indices for air cargo hub case (complete ranking)

Alt	Benslimane	Casablanca	Rabat	Tangier	Marrakesh	Agadir	Fez	Dakhla	Oujda
Benslimane	-	74	98	97	100	98	100	99	100
Casablanca	26	-	72	78	77	82	91	81	94
Rabat	2	28	-	62	69	72	90	80	94
Tangier	3	22	38	-	54	78	82	81	96
Marrakesh	0.2	23	31	46	-	69	80	82	93
Agadir	2	18	28	22	31	-	58	72	93
Fez	0.01	9	10	18	20	42	-	64	91
Dakhla	1	19	20	19	18	28	36	-	83
Oujda	0.1	6	6	4	7	7	9	17	-

1.3.3 Central Weight Vectors

The *central weight vector* $\mathbf{w}_i^c$ is the expected center of gravity of the weights that make an alternative most preferred. The central weight vector represents the preferences of a ‘typical’ DM supporting an alternative. The central weight vectors can be presented to the DMs in order to help them understand how different weights correspond to different alternative choices. To justify their decision, the DMs can, instead of expressing their own trade-off weights for the different criteria, judge if they are willing to accept the central weights of some alternative. The central weight vector for an alternative is estimated from the simulation statistics as

$$\mathbf{w}_i^c \approx \mathbf{W}_i / B_{i1} \tag{1.5}$$

Figure 1.6 (and Table 1.5) shows the central weight vectors for the air cargo hub case. The central weight vector for Fez is not defined, because Fez is an inefficient alternative (first rank acceptability index is zero). For the remaining alternatives the central weight vectors reveal what kind of preferences favor each alternative. For example, Benslimane, which is the most widely acceptable alternative, is most preferred with relatively uniform weights for each criterion. In contrast, Oujda, which is a nearly inefficient alternative, would require very much weight (68 %) on the INVEST criterion alone, and very little weight (2 %) on the POTENTIAL and ENVIRON criteria.

1.3.4 Confidence Factors

The *confidence factor* p_i^c is the probability for an alternative to obtain the first rank when its central weight vector is chosen. The confidence factors measure how robust choice for the first rank an alternative can be if the DMs accept the central weight vector to represent their preferences. A second simulation, presented in Algorithm 2 below, is needed to compute the confidence factors from collected statistics: P_i. The number of times alternative xi was most preferred using weights $\mathbf{w}_i^c$.

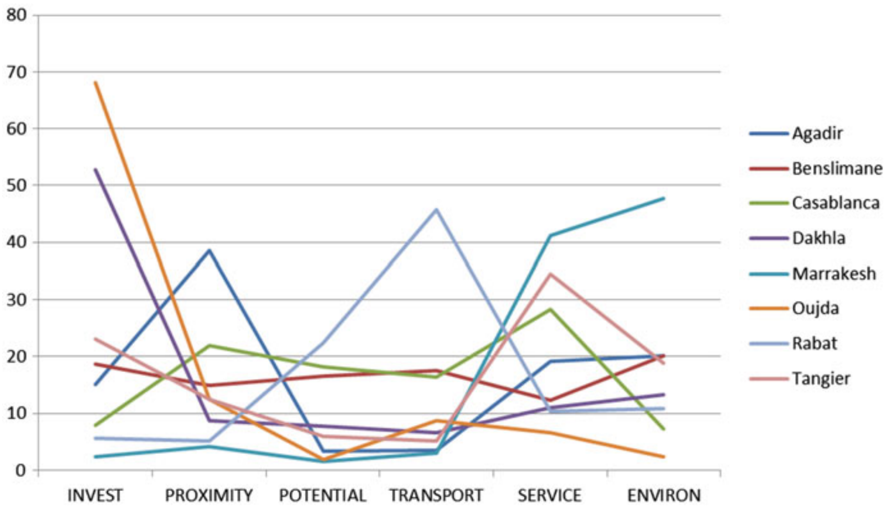

Fig. 1.6: Central weight vectors for air cargo hub case

Algorithm 2. Computation of confidence factors in SMAA

```
Repeat K times {
        Draw x from its distribution
        For the central weight vector w_i^c of each alternative {
                Rank the alternatives using u(x_i, w_i^c)
                Update statistics (P_i) about alternatives
        }
}
```

The confidence factor is estimated from the simulation results as

$$p_i^c \approx P_i/K \tag{1.6}$$

If the confidence factors for all alternatives are low, it means that the criteria measurements are not accurate enough for discriminating the alternatives robustly. In such a situation, collecting more accurate preference information is not sufficient; instead the criteria should be measured more accurately. In the opposite case, when some alternatives have high confidence factors, but low acceptability indices for the best ranks, collecting more accurate preference information may be sufficient.

Table 1.5 presents the confidence factors and corresponding central weight vectors for the alternatives in the air cargo hub case. We can see that only Benslimane and Casablanca are robust choices with suitable preferences falling at or near their central weight vectors. The remaining alternatives are very unlikely to be most preferred even with their central weight vectors. Choosing any of the remaining alternatives would require, besides favorable weights, also more accurate criteria measurement and a new analysis to reassess their robustness.

Table 1.5: Confidence factors and central weights for alternatives in air cargo hub case

Alt	pc	INVEST	PROXIMITY	POTENTIAL	TRANSPORT	SERVICE	ENVIRON
Agadir	6.80	15	39	3	4	19	20
Benslimane	99.98	19	15	17	17	12	20
Casablanca	96.87	8	22	18	16	28	7
Dakhla	17.10	53	9	8	7	11	13
Fez	–	–	–	–	–	–	–
Marrakesh	21.68	2	4	2	3	41	48
Oujda	3.81	68	13	2	9	7	2
Rabat	1.68	6	5	22	46	10	11
Tangier	3.38	23	13	6	5	35	19

1.4 Robustness with Respect to Model Structure

SMAA can be used to analyze the robustness of the decision problem with respect the structure of the decision model. For example, robustness with respect to linearity assumptions in utility/value functions can be analyzed by choosing a more general parametrized utility function and exploring how the solutions change as a function of the degree of non-linearity [14]. As an example, we consider additive utility functions (1.1) where the partial utility functions $u_j(\cdot)$ are non-linear, exponential functions (similar to the Constant Absolute Risk Aversion (CARA) model):

$$u_j(x_j) = \frac{1 - e^{-cx_j}}{1 - e^{-c}} \tag{1.7}$$

The parameter c measures the curvature of the function. Positive values of c result in concave shapes and negative values yield convex shapes. When $c \to 0$, the function approaches a linear function.

Partial utility functions with positive curvature compose into an overall utility function favoring alternatives that are uniformly good on each criterion. Negative curvature favors alternatives that are superior on any single criterion. In any case, a dominated alternative can never be the most preferred.

To analyze the robustness of the air cargo hub case, we study how the first rank acceptability indices (b_i^1) and lexicographic ranks of alternatives depend on the curvature of the partial utility functions. For the cardinally measured criteria (INVEST, POTENTIAL, TRANSPORT, SERVICE) we consider 11 curvature levels: $c \in \{-8, -4, -2, -1, -0.5, 0, 0.5, 1, 2, 4, 8\}$. Figure 1.7 illustrates the corresponding partial utility functions. The curvature for $c = 8$ is very high; the marginal value at $x_j = 0$ is 2980 times higher than at 1. The different partial utility functions may have different shapes. In this example we consider only the situation where each cardinal criterion has the same curvature.

In the following we analyze how much the acceptability indices and the lexicographic rankings of alternatives change when moving from the linear model to each of the non-linear models. Table 1.6 shows that the acceptability indices are very robust subject to small non-linearities. Significant (>5 %) changes in acceptability indices occur only for Benslimane and Casablanca at $c > 2$, for Benslimane at $c < -1$, for Casablanca at $c < -2$, and for Dakhla at $c < -4$.

Table 1.7 shows that the lexicographic ranking of the top alternatives is very robust subject to non-linearity. Benslimane and Casablanca preserve their first and second rank regardless the curvature. Dakhla preserves its third rank for negative curvature but for positive curvature it loses its position.

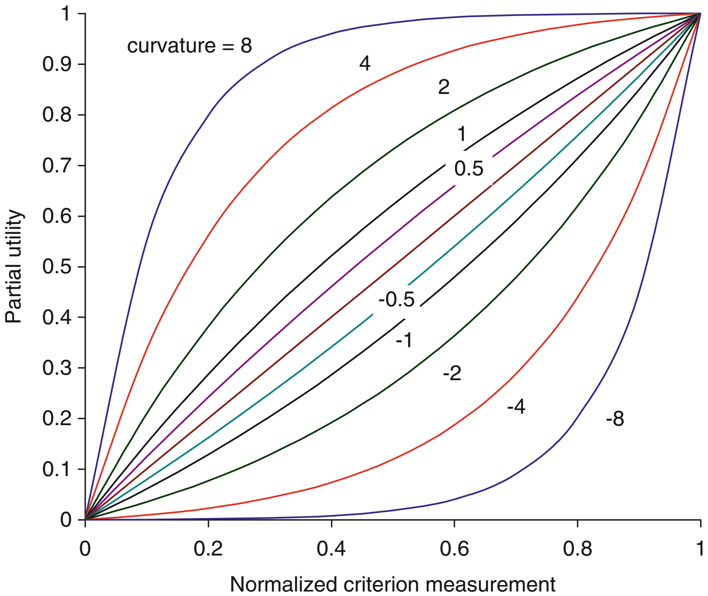

Fig. 1.7: Partial utility functions with different amounts of non-linearity

1.5 Recent Developments of SMAA

Recent developments of SMAA include more efficient computational methods and extensions to different decision models.

In most cases the SMAA computations can be performed very efficiently using straight forward Monte Carlo simulation. However, the computation may slow down in case of complex preference information. In such cases, the Markov Chain Monte Carlo (MCMC) simulation technique can be used to speed up the computation [27]. The JSMAA open source implementation of SMAA includes the MCMC technique and performs the simulation as a background process while the user views or edits the model (see www.smaa.fi, [23]).

Table 1.6: Acceptability indices (%) of alternatives with different amount of curvature. Over 5 % changes highlighted for illustrative purposes

	Curvature c										
Alternative	-8	-4	-2	-1	-0.5	0	0.5	1	2	4	8
Agadir	0.05	0.09	0.16	0.26	0.31	0.40	0.48	0.53	0.75	1.40	2.00
Benslimane	**47.00**	**54.00**	**62.00**	**67.00**	70.00	72.00	75.00	76.00	**79.00**	**82.00**	**85.00**
Casablanca	25.00	30.00	**31.00**	29.00	28.00	25.00	24.00	22.00	**18.00**	**12.00**	**9.00**
Dakhla	**23.00**	**13.00**	5.80	2.90	1.70	1.00	0.53	0.17	0.01	0.00	0.00
Fez	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Marrakesh	0.00	0.00	0.01	0.01	0.01	0.03	0.04	0.07	0.14	0.72	1.00
Oujda	0.84	0.41	0.19	0.09	0.05	0.02	0.01	0.01	0.00	0.00	0.00
Rabat	3.60	2.00	0.99	0.52	0.43	0.28	0.15	0.14	0.16	0.19	0.08
Tangier	0.00	0.00	0.05	0.12	0.22	0.38	0.54	0.73	1.50	2.80	2.80

Table 1.7: Lexicographic ranks of alternatives with different amount of curvature

	Curvature c										
Alternative	-8	-4	-2	-1	-0.5	0	0.5	1	2	4	8
Agadir	6	6	6	5	5	4	5	4	4	4	4
Benslimane	1	1	1	1	1	1	1	1	1	1	1
Casablanca	2	2	2	2	2	2	2	2	2	2	2
Dakhla	3	3	3	3	3	3	4	5	7	8	8
Fez	9	9	9	9	9	9	9	9	8	7	7
Marrakesh	7	7	8	8	8	7	7	7	6	5	5
Oujda	5	5	5	7	7	8	8	8	9	9	9
Rabat	4	4	4	4	4	6	6	6	5	6	6
Tangier	8	8	7	6	6	5	3	3	3	3	3

Extensions to different decision models include different shaped utility or value functions and also decision models not based on utility functions. Cohen et al. [3] applied SMAA with an additive value function where the partial value functions (marginal value functions) were piecewise linear monotonic mappings. They varied the mappings during simulation using a random process resembling treatment of ordinal criteria measurements in SMAA. Babalos et al. [2] applied the SMAA-2 framework and considered three different aggregate evaluation measures: the holistic acceptability index, Borda count method, and average score. Kontu et al. [7] extended the SMAA method to handle a hierarchy of criteria and sub-criteria. A criteria hierarchy is useful when the number of criteria is large.

Additive utility function models assume independence between criteria. SMAA with Choquet integral by Angilella et al. [1] considers interaction between criteria. The Choquet integral can be seen as a value function where positive or negative interaction between criteria is also contributing to the evaluation of alternatives. The Choquet integral is thus a more general decision model than the additive value function. Lahdelma and Salminen [14] studied the robustness of decision problems with respect to the shape of the utility function, as demonstrated in the previous section.

The SMAA-PROMETHEE method by Corrente et al. [4] is a recent extension of SMAA to non-utility function based methods. PROMETHEE is based on an outranking procedure where fuzzy preference relations between alternatives are aggregated together to yield a partial order (PROMETHEE I) or complete order (PROMETHEE II). Durbach et al. [6] extended the analytic hierarchy process (AHP) to consider imprecise or uncertain pairwise comparisons by probability distributions. The resulting SMAA-AHP method is suitable for group decision making problems, where it is difficult to agree on precise pairwise comparisons. Wang et al. [28] extended the Complementary Judgement Matrix (CJM) method in a similar manner. CJM differs from AHP in the way how the pairwise comparisons are expressed, and in how the weights are solved from inconsistent comparisons. In particular, the weights in CJM are determined by minimizing the square sum of inconsistency errors.

1.6 Discussion

In SMAA uniform distributions are used to represent absence of information both in criteria and preferences. Ordinal criteria are transformed into cardinal measurements by simulating consistent ordinal to cardinal mappings. The simulation process is equivalent to treating the absence of interval information of ordinal scales as uniform joint distributions. Similarly, absence of weight information is treated as a uniform joint distribution in the feasible weight space.

Although SMAA can be used with arbitrarily shaped utility functions, in real-life applications simple forms, such as linear or some concave shapes are most commonly applied. Assessing the precise preference structure of DMs can be difficult and time-consuming in practice. SMAA can be used to test the robustness of the problem also with respect to the decision model, as illustrated in the previous section. If the problem can be identified as robust with respect to model structure, it may be possible to assume a simpler model in the interaction between the DMs.

The strength of SMAA in robustness analysis of multicriteria decision aiding problems is that it is able to handle the whole range of uncertain, imprecise or partially missing information flexibly using suitable probability distributions. Typically, a real-life decision process may start with very vague and uncertain criteria and preference information. The information will become gradually more accurate during the process. SMAA can be used in such processes repeatedly after any refinement of information, until a robust decision can be identified and agreed on. SMAA reveals if the information is accurate enough for making the decision, and also pinpoints which parts of the information need to be refined. This can (1) protect the DMs from making wrong decisions based on insufficient information and also (2) cause significant savings in information collection if less accurate information is sufficient for making a robust decision.

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