On Linear Based Intuitionistic Substructural Logics

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──── **Abstract** ────

TODO

1   Introduction

2   Main Ideas

3   Category Theory Basics

▶ **Definition 1.** A **monoidal category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ consists of

- a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- an object I, called the unit object;
- three natural isomorphisms α, λ, and ρ with components

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

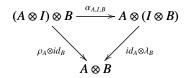$$\lambda_A : I \otimes A \to A$$

$$\rho_A : A \otimes I \to A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in $\mathcal{M}$:

$$
\begin{array}{ccc}
((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha_{A,B,C} \otimes id_D} (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D) \\
\downarrow{\alpha_{A \otimes B,C,D}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{id_A \otimes \alpha_{B,C,D}} \\
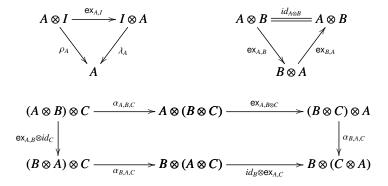(A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C \otimes D}} A \otimes (B \otimes (C \otimes D))
\end{array}
$$

$$
\begin{array}{ccc}
(A \otimes I) \otimes B \xrightarrow{\alpha_{A,I,B}} A \otimes (I \otimes B) \\
\searrow{\rho_A \otimes id_B} \qquad \swarrow{id_A \otimes \lambda_B} \\
A \otimes B
\end{array}
$$

▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\rightharpoonup: \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and $\leftharpoonup: \mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad\qquad \mathsf{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

▶ **Definition 3.** A **symmetric monoidal category** (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\mathsf{ex}_{A,B} : A \otimes B \to B \otimes A$, called **exchange**, such that the following diagrams commute:

$$
\begin{array}{ccc}
A \otimes I & \xrightarrow{\mathsf{ex}_{A,I}} & I \otimes A \\
& \rho_A \searrow \quad \swarrow \lambda_A & \\
& A &
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \otimes B & \xrightarrow{id_{A \otimes B}} & A \otimes B \\
\mathsf{ex}_{A,B} \searrow & & \nearrow \mathsf{ex}_{B,A} \\
& B \otimes A &
\end{array}
$$

$$
\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\mathsf{ex}_{A,B \otimes C}} & (B \otimes C) \otimes A \\
{\scriptstyle \mathsf{ex}_{A,B} \otimes id_C} \downarrow & & & & \downarrow {\scriptstyle \alpha_{B,A,C}} \\
(B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \mathsf{ex}_{A,C}} & B \otimes (C \otimes A)
\end{array}
$$

▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap: \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

▶ **Lemma 5.** *Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$
\begin{array}{llll}
Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\
& \cong & Hom[A \otimes C, B] & \text{By the exchange } \mathsf{ex}_{C,A} \\
& \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category}
\end{array}
$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◀

▶ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

▶ **Definition 7.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from $\mathcal{M}$ to $\mathcal{M}'$ is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \otimes FB' \to F(A \otimes B)$, such that the following diagrams commute for any objects A, B, and C in $\mathcal{M}$:

$$
\begin{array}{ccccc}
(FA \otimes' FB) \otimes' FC & \xrightarrow{\alpha'_{FA,FB,FC}} & FA \otimes' (FB \otimes' FC) & \xrightarrow{id_{FA} \otimes' \mathsf{m}_{A,B}} & FA \otimes' F(B \otimes C) \\
{\scriptstyle \mathsf{m}_{A,B} \otimes' id_{FC}} \downarrow & & & & \downarrow {\scriptstyle \mathsf{m}_{A,B \otimes C}} \\
F(A \otimes B) \otimes' FC & \xrightarrow{\mathsf{m}_{A \otimes B,C}} & F((A \otimes B) \otimes C) & \xrightarrow{F\alpha_{A,B,C}} & F(A \otimes (B \otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
I' \otimes' FA & \xrightarrow{\lambda'_{FA}} & FA \\
{\scriptstyle \mathsf{m}_I \otimes id_{FA}} \downarrow & & \uparrow {\scriptstyle F\lambda_A} \\
FI \otimes' FA & \xrightarrow{\mathsf{m}_{I,A}} & F(I \otimes A)
\end{array}
\qquad\qquad
\begin{array}{ccc}
FA \otimes' I' & \xrightarrow{\rho'_{FA}} & FA \\
{\scriptstyle id_{FA} \otimes \mathsf{m}_I} \downarrow & & \uparrow {\scriptstyle F\rho_A} \\
FA \otimes' FI & \xrightarrow{\mathsf{m}_{A,I}} & F(A \otimes I)
\end{array}
$$

▶ **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{\ \mathsf{ex}_{FA,FB}\ } & FB \otimes' FA \\
\mathsf{m}_{A,B} \downarrow & & \downarrow \mathsf{m}_{B,A} \\
F(A \otimes B) & \xrightarrow[\ F\mathsf{ex}_{A,B}\ ]{} & F(B \otimes A)
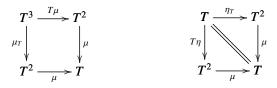\end{array}
$$

▶ **Definition 9.** An **adjunction** between categories C and $\mathcal{D}$ consists of two functors $F : \mathcal{D} \to C$, called the **left adjoint**, and $G : C \to \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta : id_{\mathcal{D}} \to GF$, called the **unit**, and $\varepsilon : FG \to id_C$, called the **counit**, such that the following diagrams commute for any object A in C and B in $\mathcal{D}$:

$$
\begin{array}{ccc}
FB & \xrightarrow{\ F\eta_B\ } & FGFB \\
 \| & & \swarrow {\varepsilon_{FB}} \\
 & FB &
\end{array}
\qquad\qquad
\begin{array}{ccc}
GA & \xrightarrow{\ \eta_{GA}\ } & GFGA \\
 \| & & \swarrow {G\varepsilon_A} \\
 & GA &
\end{array}
$$

▶ **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category $\mathcal{M}$ to a monoidal category $\mathcal{M}'$. A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, \mathsf{m}) \to (G, \mathsf{n})$ such that the following diagrams commute for any objects A and B in $\mathcal{M}$:

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{\ \mathsf{m}_{A,B}\ } & F(A \otimes B) \\
\theta_A \otimes' \theta_B \downarrow & & \downarrow \theta_{A \otimes B} \\
GA \otimes' GB & \xrightarrow[\ \mathsf{n}_{A,B}\ ]{} & G(A \otimes B)
\end{array}
\qquad\qquad
\begin{array}{ccc}
FI & \xrightarrow{\ \theta_I\ } & GI \\
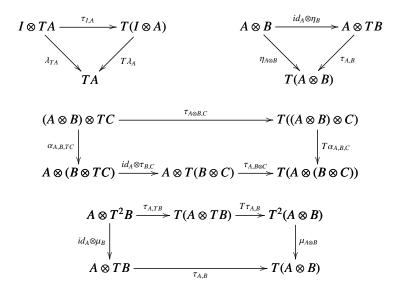 \mathsf{m}_I \nwarrow & & \nearrow \mathsf{n}_I \\
 & I' &
\end{array}
$$

▶ **Definition 11.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

▶ **Definition 12.** Let C be a category. A **monad** on C consists of an endofunctor $T : C \to C$ together with two natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to T$, where id_C is the identity functor on C, such that the following diagrams commute:

$$
\begin{array}{ccc}
T^3 & \xrightarrow{\ T\mu\ } & T^2 \\
\mu_T \downarrow & & \downarrow \mu \\
T^2 & \xrightarrow[\ \mu\ ]{} & T
\end{array}
\qquad\qquad
\begin{array}{ccc}
T & \xrightarrow{\ \eta_T\ } & T^2 \\
T\eta \downarrow & \ \| & \downarrow \mu \\
T^2 & \xrightarrow[\ \mu\ ]{} & T
\end{array}
$$

▶ **Definition 13.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on $\mathcal{M}$. T is a **strong monad** if there is natural transformation τ, called the **tensorial strength**, with components

$\tau_{A,B} : A \otimes TB \to T(A \otimes B)$ such that the following diagrams commute:

$$
\begin{array}{ccc}
I \otimes TA & \xrightarrow{\ \tau_{I,A}\ } & T(I \otimes A) \\
 & \lambda_{TA} \searrow \quad \swarrow T\lambda_A & \\
 & TA &
\end{array}
\qquad
\begin{array}{ccc}
A \otimes B & \xrightarrow{\ id_A \otimes \eta_B\ } & A \otimes TB \\
 & \eta_{A \otimes B} \searrow \quad \swarrow \tau_{A,B} & \\
 & T(A \otimes B) &
\end{array}
$$

$$
\begin{array}{ccc}
(A \otimes B) \otimes TC & \xrightarrow{\qquad\qquad \tau_{A \otimes B, C} \qquad\qquad} & T((A \otimes B) \otimes C) \\
\alpha_{A,B,TC} \downarrow & & \downarrow T\alpha_{A,B,C} \\
A \otimes (B \otimes TC) \xrightarrow{id_A \otimes \tau_{B,C}} A \otimes T(B \otimes C) & \xrightarrow{\ \tau_{A,B \otimes C}\ } & T(A \otimes (B \otimes C))
\end{array}
$$

$$
\begin{array}{ccccc}
A \otimes T^2 B & \xrightarrow{\ \tau_{A,TB}\ } & T(A \otimes TB) & \xrightarrow{\ T\tau_{A,B}\ } & T^2(A \otimes B) \\
id_A \otimes \mu_B \downarrow & & & & \downarrow \mu_{A \otimes B} \\
A \otimes TB & & \xrightarrow{\qquad\qquad \tau_{A,B} \qquad\qquad} & & T(A \otimes B)
\end{array}
$$

▶ **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex, and (T, η, μ) be a strong monad on $\mathcal{M}$. Then there is a **"twisted" tensorial strength** $\tau'_{A,B} : TA \otimes B \to T(A \otimes B)$ defined as $\tau'_{A,B} = T\mathsf{ex} \circ \tau_{B,A} \circ \mathsf{ex}$. We can construct a pair of natural transformations Φ, Φ' with components $\Phi_{A,B}, \Phi'_{A,B} : TA \otimes TB \to T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A \otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A \otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.

▶ **Definition 15.** Let $\mathcal{L}$ be a category. A **comonad** on $\mathcal{L}$ consists of an endofunctor $S : \mathcal{L} \to \mathcal{L}$ together with two natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S^2 \to S$ such that the following diagrams commute:

$$
\begin{array}{ccc}
S & \xrightarrow{\ \delta\ } & S^2 \\
\delta \downarrow & & \downarrow S\delta \\
S^2 & \xrightarrow{\ \delta_S\ } & S^3
\end{array}
\qquad\qquad
\begin{array}{ccc}
S^2 & \xrightarrow{\ S\varepsilon\ } & S \\
\varepsilon_S \downarrow & & \downarrow \delta \\
S & \xrightarrow{\ \delta\ } & S^2
\end{array}
$$

4 Categorical Models

▶ **Definition 16.** A **SMCC-Lambek model** consists of

- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$;
- a monoidal adjunction $F : C \dashv \mathcal{L} : G$, where $F : C \to \mathcal{L}$ and $G : \mathcal{L} \to C$ are monoidal functors.

Thus, in a SMCC-Lambek model, the following four diagrams commute because η and ε are monoidal natural transformations:

$$
\begin{array}{ccc}
A \otimes B & =\!\!=\!\!\xrightarrow{\ id_{A \otimes B}\ }\!\!=\!\!= & A \otimes B \\
\eta_A \otimes \eta_B \downarrow & & \downarrow \eta_{A \otimes B} \\
GFA \otimes GFB \xrightarrow{n_{FA,FB}} G(FA \otimes FB) & \xrightarrow{\ Gm_{A,B}\ } & GF(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\ \eta_I\ } & GFI \\
\| & & \uparrow Gm_I \\
I & \xrightarrow{\ n_{I'}\ } & GI'
\end{array}
$$

$$FGA \otimes FGB \xrightarrow{\mathsf{m}_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\mathsf{n}_{A,B}} FG(A \otimes B)$$

with vertical maps $\varepsilon_A \otimes \varepsilon_B$ on the left and $\varepsilon_{A \otimes B}$ on the right, and bottom $A \otimes B = A \otimes B$.

$$FGI' \xrightarrow{\varepsilon_{I'}} I'$$

with $F\mathsf{n}_{I'}$ on the left, identity on the right, and $FI \xleftarrow{\mathsf{m}_I} I'$.

And the following two diagrams commute because of the adjunction:

$$FA \xrightarrow{F\eta_A} FGFA \qquad \varepsilon_{FA} \qquad FA$$

$$GB \xrightarrow{\eta_{GX}} GFGB \qquad G\varepsilon_B \qquad GB$$

▶ **Lemma 17.** *The monad on the SMCC C in a SMCC-Lambek model is monoidal.*

Proof. We define the monad T on the C in the adjunction of a SMCC-Lambek model as $T = GF$, and the two corresponding natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to T$ are defined as

$$\eta_A : A \to GFA$$

$$\mu_A = G\varepsilon_{FA} : GFGFA \to GFA$$

where η is the unit and $\varepsilon : FG \to id_{\mathcal{L}}$ is the counit in the adjunction $F : C \dashv \mathcal{L} : G$, and (F, m) and (G, n) are monoidal functors.
Thus, we have

$$\mathsf{t}_{A,B} = G\mathsf{m}_{A,B} \circ \mathsf{n}_{FA,FB} : TA \otimes TB \to T(A \otimes B)$$

$$\mathsf{t}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'} : I \to TI$$

The monad T being monoidal means

1. T is a monoidal functor, i.e. the folllowing diagrams commute:

$$(TA \otimes TB) \otimes TC \xrightarrow{\alpha_{TA,TB,TC}} TA \otimes (TB \otimes TC) \xrightarrow{id_{TA} \otimes \mathsf{t}_{B,C}} TA \otimes T(B \otimes C)$$

with left vertical $\mathsf{t}_{A,B} \otimes id_{TC}$, right vertical $\mathsf{t}_{A,B \otimes C}$,

$$T(A \otimes B) \otimes TC \xrightarrow{\mathsf{t}_{A \otimes B,C}} T((A \otimes B) \otimes C) \xrightarrow{T\alpha_{A,B,C}} T(A \otimes (B \otimes C))$$

$$I \otimes TA \xrightarrow{\lambda_{TA}} TA \qquad TA \otimes I \xrightarrow{\rho_{TA}} TA$$

with $\mathsf{t}_I \otimes id_{TA}$ and $T\lambda_A$, and $TI \otimes TA \xrightarrow{\mathsf{t}_{I,A}} T(I \otimes A)$; and with $id_{TA} \otimes \mathsf{t}_I$ and $T\rho_A$, and $TA \otimes TI \xrightarrow{\mathsf{t}_{A,I}} T(A \otimes I)$.

We write GF instead of T in the proof for clarity.
By replacing t with its definition, the first diagram above commutes by the following commutative diagram, where the two hexagons commute because G and F are monoidal functors, and the

two quadrilaterals commute by the naturality of n.

$$
\begin{array}{ccccc}
(GFA \otimes GFB) \otimes GFC & \xrightarrow{\alpha_{GFA,GFB,GFC}} & GFA \otimes (GFB \otimes GFC) & \xrightarrow{id_{GFA} \otimes n_{FB,FC}} & GFA \otimes G(FB \otimes' FC) \\
\downarrow{\scriptstyle n_{FA,FB} \otimes id_{GFC}} & & \searrow{\scriptstyle n_{FA,FB \otimes' FC}} & & \downarrow{\scriptstyle id_{GFA} \otimes Gm_{B,C}} \\
G(FA \otimes' FB) \otimes GFC & & G(FA \otimes' (FB \otimes' FC)) & & GFA \otimes GF(B \otimes C) \\
\downarrow{\scriptstyle Gm_{A,B} \otimes id_{GFC}} & \searrow{\scriptstyle n_{FA \otimes' FB,FC}} & \uparrow{\scriptstyle G\alpha'_{FA,FB,FC}} & \searrow{\scriptstyle G(id_{FA} \otimes' m_{B,C})} & \downarrow{\scriptstyle n_{FA,F(B \otimes C)}} \\
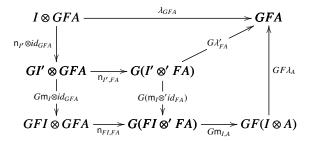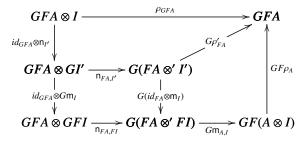GF(A \otimes B) \otimes GFC & & G((FA \otimes' FB) \otimes' FC) & & G(FA \otimes' F(B \otimes C)) \\
\downarrow{\scriptstyle n_{F(A \otimes B),FC}} & \searrow{\scriptstyle G(m_{A,B} \otimes id_{FC})} & & & \downarrow{\scriptstyle Gm_{A,B \otimes C}} \\
G(F(A \otimes B) \otimes' FC) & \xrightarrow{Gm_{A \otimes B,C}} & GF((A \otimes B) \otimes C) & \xrightarrow{GF\alpha_{A,B,C}} & GF(A \otimes (B \otimes C))
\end{array}
$$

The first square above commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.

$$
\begin{array}{ccc}
I \otimes GFA & \xrightarrow{\lambda_{GFA}} & GFA \\
\downarrow{\scriptstyle n_{I'} \otimes id_{GFA}} & \nearrow{\scriptstyle G\lambda'_{FA}} & \uparrow \\
GI' \otimes GFA \xrightarrow{n_{I',FA}} G(I' \otimes' FA) & & {\scriptstyle GF\lambda_A} \\
\downarrow{\scriptstyle Gm_I \otimes id_{GFA}} \quad \downarrow{\scriptstyle G(m_I \otimes' id_{FA})} & & \uparrow \\
GFI \otimes GFA \xrightarrow{n_{FI,FA}} G(FI \otimes' FA) & \xrightarrow{Gm_{I,A}} & GF(I \otimes A)
\end{array}
$$

Similarly, the second square above commutes by the following commutative diagram:

$$
\begin{array}{ccc}
GFA \otimes I & \xrightarrow{\rho_{GFA}} & GFA \\
\downarrow{\scriptstyle id_{GFA} \otimes n_{I'}} & \nearrow{\scriptstyle G\rho'_{FA}} & \uparrow \\
GFA \otimes GI' \xrightarrow{n_{FA,I'}} G(FA \otimes' I') & & {\scriptstyle GF\rho_A} \\
\downarrow{\scriptstyle id_{GFA} \otimes Gm_I} \quad \downarrow{\scriptstyle G(id_{FA} \otimes m_I)} & & \uparrow \\
GFA \otimes GFI \xrightarrow{n_{FA,FI}} G(FA \otimes' FI) & \xrightarrow{Gm_{A,I}} & GF(A \otimes I)
\end{array}
$$

2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal and thus the following two diagrams commute.

$$
\begin{array}{ccc}
A \otimes B & = & A \otimes B \\
\downarrow{\scriptstyle \eta_A \otimes \eta_B} & & \downarrow{\scriptstyle \eta_{A \otimes B}} \\
TA \otimes TB & \xrightarrow{t_{A,B}} & T(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\eta_I} & TI \\
\| & & \nearrow{\scriptstyle t_I} \\
I & &
\end{array}
$$

3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ. Thus the following diagrams commute.

$$
\begin{array}{ccccc}
T^2A \otimes T^2B & \xrightarrow{t_{TA,TB}} & T(TA \otimes TB) & \xrightarrow{Tt_{A,B}} & T^2(A \otimes B) \\
\downarrow{\scriptstyle \mu_A \otimes \mu_B} & & & & \downarrow{\scriptstyle \mu_{A \otimes B}} \\
TA \otimes TB & & \xrightarrow{t_{A,B}} & & T(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
T^2I & \xrightarrow{\mu_I} & TI \\
\uparrow{\scriptstyle Tt_I} & & \uparrow{\scriptstyle t_I} \\
TI & \xleftarrow{t_I} & I
\end{array}
$$

◀

However, the monad is not symmetric becauase the following diagram does not commute, for the lambek category $\mathcal{L}$ is not symmetric.

$$
\begin{array}{ccc}
GFA \otimes GFB & \xrightarrow{\text{ex}_{GFA,GFB}} GFB \otimes GFA \xrightarrow{\text{n}_{FB,FA}} & G(FB \otimes' FA) \\
\text{n}_{FA,FB} \downarrow & & \downarrow G\text{m}_{B,A} \\
G(FA \otimes' FB) \xrightarrow{\quad G\text{m}_{A,B} \quad} GF(A \otimes B) & \xrightarrow{\quad GF\text{ex}_{A,B} \quad} & GF(B \otimes A)
\end{array}
$$

▶ **Lemma 18.** *The monad on the SMCC in a SMCC-Lambek model is strong.*

Proof. Let $F : C \vdash \mathcal{L} : G$ be a SMCC-Lambek model, where $(C, \otimes, I, \alpha, \lambda, \rho)$ is an SMCC, $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$ is a Lambek category, and (F, m) and (G, n) are monoidal functors. Let (T, η, μ) be the monad on C where $T = GF$. We have proved that T is monoidal with the natural transformation $\text{t}_{A,B} : TA \otimes TB \to T(A \otimes B)$ and the morphism $\text{t}_I : I \to TI$ defined as in Lemma **??**.

We define the tensorial strength $\tau_{A,B} : A \otimes TB \to T(A \otimes B)$ as $\tau_{A,B} = \text{t}_{A,B} \circ \eta_A \otimes id_{TB}$.

Since η is a monoidal natural transformation, we have $\eta_I = G\text{m}_I \circ \text{n}_{I'}$. Therefore $\eta_I = \text{t}_I$. Thus the following diagram commutes because T is monoidal, where the composition $\text{t}_{I,A} \circ \text{t}_I \otimes id_{TA}$ is the definition of $\tau_{I,A}$. So the first triangle in Defition **??** commutes.
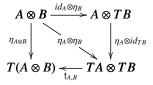
$$
\begin{array}{ccc}
I \otimes TA & \xrightarrow{\text{t}_I \otimes id_{TA}} & TI \otimes TA \\
\lambda_{TA} \downarrow & & \downarrow \text{t}_{I,A} \\
TA & \xleftarrow{\quad T\lambda_A \quad} & T(I \otimes A)
\end{array}
$$

Similarly, by using the definition of τ, the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{id_A \otimes \eta_B} & A \otimes TB \\
\eta_{A \otimes B} \downarrow & \searrow{\scriptstyle \eta_A \otimes \eta_B} & \downarrow \eta_A \otimes id_{TB} \\
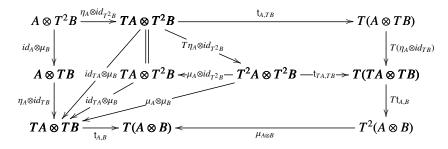T(A \otimes B) & \xleftarrow{\quad \text{t}_{A,B} \quad} & TA \otimes TB
\end{array}
$$

The first pentagon in the definition commutes by the following commutative diagrams, because η are α natural transformations and T is monoidal:

The last diagram in the definition commtues by the following commutative diagram, because T is a monad, t is a natural transformation, and μ is a monoidal natural transformation:

$$
\begin{array}{ccccc}
A \otimes T^2B & \xrightarrow{\eta_A \otimes id_{T^2B}} & TA \otimes T^2B & \xrightarrow{\quad \mathsf{t}_{A,TB} \quad} & T(A \otimes TB) \\
{\scriptstyle id_A \otimes \mu_B} \downarrow & & \| \quad {\scriptstyle T\eta_A \otimes id_{T^2B}} & & \downarrow {\scriptstyle T(\eta_A \otimes id_{TB})} \\
A \otimes TB \quad {\scriptstyle id_{TA} \otimes \mu_B} & TA \otimes T^2B & \xleftarrow{\mu_A \otimes id_{T^2B}} T^2A \otimes T^2B \xrightarrow{\mathsf{t}_{TA,TB}} & & T(TA \otimes TB) \\
{\scriptstyle \eta_A \otimes id_{TB}} \downarrow \quad {\scriptstyle id_{TA} \otimes \mu_B} & {\scriptstyle \mu_A \otimes \mu_B} & & & \downarrow {\scriptstyle T\mathsf{t}_{A,B}} \\
TA \otimes TB & \xrightarrow{\mathsf{t}_{A,B}} & T(A \otimes B) & \xleftarrow{\quad \mu_{A \otimes B} \quad} & T^2(A \otimes B)
\end{array}
$$

◄

▶ **Lemma 19** ([?]). *Let $\mathcal{M}$ be a symmetric monoidal category and T be a strong monad on $\mathcal{M}$. Then T is a symmetric monoidal functor iff it is commutative.*

▶ **Theorem 20.** *The monad on the SMCC in a SMCC-Lambek model is monoidal and non-commutative.*

▶ **Lemma 21.** *The comonad on the Lambek category in a SMCC-Lambek model is monoidal.*

Proof. We define the comonad S on the Lambek category $\mathcal{L}$ in the adjunction $F : C \vdash \mathcal{L} : G$ of a SMCC-Lambek model as $S = FG$, and the two corresponding natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S \to S^2$ are defined as

$$\varepsilon_A : SA \to A$$

$$\delta_A = F\eta_{GA} : SA \to S^2A$$

where ε is the counit and $\eta : id_{\mathcal{L}} \to GF$ is the unit in the adjunction, and (F, m) and (G, n) are monoidal functors. Thus, we have

$$\mathsf{s}_{A,B} = F\mathsf{n}_{A,B} \circ \mathsf{m}_{GA,GB} : SA \otimes' SB \to SA \otimes' SB$$

$$\mathsf{s}_I = F\mathsf{n}_{I'} \circ \mathsf{m}_I : I' \to SI'$$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$
\begin{array}{ccccc}
(SA \otimes' SB) \otimes' SC & \xrightarrow{\alpha'_{SA,SB,SC}} & SA \otimes' (SB \otimes' SC) & \xrightarrow{id_{SA} \otimes' \mathsf{s}_{B,C}} & SA \otimes' S(B \otimes' C) \\
{\scriptstyle \mathsf{s}_{A,B} \otimes' id_{SC}} \downarrow & & & & \downarrow {\scriptstyle \mathsf{s}_{A,B \otimes' C}} \\
S(A \otimes' B) \otimes' SC & \xrightarrow{\mathsf{s}_{A \otimes' B,C}} & S((A \otimes' B) \otimes' C) & \xrightarrow{S\alpha'_{A,B,C}} & S(A \otimes' (B \otimes' C))
\end{array}
$$

$$
\begin{array}{ccc}
I' \otimes' SA & \xrightarrow{\lambda'_{SA}} & SA \\
{\scriptstyle \mathsf{s}_{I'} \otimes' id_{SA}} \downarrow & & \uparrow {\scriptstyle S\lambda'_A} \\
SI' \otimes' SA & \xrightarrow{\mathsf{s}_{I',A}} & S(I' \otimes' A)
\end{array}
\qquad
\begin{array}{ccc}
SA \otimes' I' & \xrightarrow{\rho'_{SA}} & SA \\
{\scriptstyle id'_{SA} \otimes' \mathsf{s}_{I'}} \downarrow & & \uparrow {\scriptstyle S\rho'_A} \\
SA \otimes' SI' & \xrightarrow{\mathsf{s}_{A,I'}} & S(A \otimes' I')
\end{array}
$$

2. ε is a monoidal natural transformation:

$$
\begin{array}{ccc}
SA \otimes' SB & \xrightarrow{\mathsf{s}_{A,B}} & S(A \otimes' B) \\
{\scriptstyle \varepsilon_A \otimes' \varepsilon_B} \downarrow & & \downarrow {\scriptstyle \varepsilon_{A \otimes' B}} \\
A \otimes' B & = \!\!=\!\!= & A \otimes' B
\end{array}
\qquad
\begin{array}{ccc}
SI' & \xrightarrow{\varepsilon_{I'}} & I' \\
& {\scriptstyle \mathsf{s}_{I'}} \nwarrow & \| \\
& I' &
\end{array}
$$

3. δ is a monoidal natural transformation:

$$
\begin{array}{ccc}
SA \otimes' SA & \xrightarrow{\ \mathsf{s}_{A,B}\ } & S(A \otimes' B) \\
{\scriptstyle \delta_A \otimes' \delta_B}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta_{A \otimes' B}} \\
S^2 A \otimes' S^2 B \xrightarrow{\ \mathsf{s}_{SA,SB}\ } S(SA \otimes' SB) & \xrightarrow{\ S\mathsf{s}_{A,B}\ } & S^2(A \otimes' B)
\end{array}
\qquad
\begin{array}{ccc}
SI' & \xrightarrow{\ \delta_{I'}\ } & S^2 I' \\
{\scriptstyle \mathsf{s}_{I'}}\Big\uparrow & & \Big\uparrow{\scriptstyle S\mathsf{s}_{I'}} \\
I' & \xrightarrow[\ \mathsf{s}_{I'}\ ]{} & SI'
\end{array}
$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma **??**. We do not include the proof here for simplicity. ◄

5   Logic

6   Applications

7   Related Work

TODO

8   Conclusion

TODO

A   Appendix