

Linear Logic, Monads and Non-commutative Lambda Calculus

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Abstract

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1 Introduction

2 Main Ideas

3 Category Theory Basics

► **Definition 1.** A **monoidal category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a category \mathcal{M} consists of

- a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$, called the tensor product;
- an object I , called the unit object;
- three natural isomorphisms α , λ , and ρ with components

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$$

$$\lambda_A : I \otimes A \rightarrow A$$

$$\rho_A : A \otimes I \rightarrow A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in \mathcal{M} :

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes id_D} & (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D) \\ \downarrow \alpha_{A \otimes B, C, D} & & \downarrow id_A \otimes \alpha_{B,C,D} \\ (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} & A \otimes (B \otimes (C \otimes D)) \end{array}$$

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ \downarrow \rho_A \otimes id_B & & \downarrow id_A \otimes \lambda_B \\ A \otimes B & & A \otimes B \end{array}$$



► **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\multimap: \mathcal{M}^{op} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\multimap: \mathcal{M} \times \mathcal{M}^{op} \rightarrow \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\text{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \text{Hom}_{\mathcal{L}}(X, A \multimap B)$$

$$\text{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \text{Hom}_{\mathcal{L}}(X, B \multimap A)$$

► **Definition 3.** A **symmetric monoidal category** (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\text{ex}_{A,B}: A \otimes B \rightarrow B \otimes A$, called **exchange**, such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes I & \xrightarrow{\text{ex}_{A,I}} & I \otimes A \\ \rho_A \searrow & & \swarrow \lambda_A \\ & A & \end{array} \qquad \begin{array}{ccc} A \otimes B & \xrightarrow{id_{A \otimes B}} & A \otimes B \\ \text{ex}_{A,B} \searrow & & \swarrow \text{ex}_{B,A} \\ & B \otimes A & \end{array}$$

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\text{ex}_{A,B \otimes C}} & (B \otimes C) \otimes A \\ \downarrow \text{ex}_{A,B} \otimes id_C & & & & \downarrow \alpha_{B,A,C} \\ (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \text{ex}_{A,C}} & B \otimes (C \otimes A) \end{array}$$

► **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap: \mathcal{M}^{op} \times \mathcal{M} \rightarrow \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\text{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \text{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

► **Lemma 5.** Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \multimap B) \cong (B \multimap A)$.

Proof. First, notice that for any object C we have

$$\begin{aligned} \text{Hom}[C, A \multimap B] &\cong \text{Hom}[C \otimes A, B] && \mathcal{L} \text{ is a Lambek category} \\ &\cong \text{Hom}[A \otimes C, B] && \text{By the exchange } \text{ex}_{C,A} \\ &\cong \text{Hom}[C, B \multimap A] && \mathcal{L} \text{ is a Lambek category} \end{aligned}$$

Thus, $A \multimap B \cong B \multimap A$ by the Yoneda lemma. ◀

► **Corollary 6.** A Lambek category with exchange is symmetric monoidal closed.

► **Definition 7.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from \mathcal{M} to \mathcal{M}' is a functor $F: \mathcal{M} \rightarrow \mathcal{M}'$ together with a morphism $m_I: I' \rightarrow F(I)$ and a natural transformation $m_{A,B}: FA' \otimes FB' \rightarrow F(A \otimes B)$, such that the following diagrams commute for any objects A, B , and C in \mathcal{M} :

$$\begin{array}{ccccc} (FA' \otimes' FB') \otimes' FC & \xrightarrow{\alpha'_{FA',FB',FC}} & FA' \otimes' (FB' \otimes' FC) & \xrightarrow{id_{FA'} \otimes' m_{A,B}} & FA' \otimes' F(B \otimes C) \\ \downarrow m_{A,B} \otimes' id_{FC} & & & & \downarrow m_{A,B \otimes C} \\ F(A \otimes B) \otimes' FC & \xrightarrow{m_{A \otimes B, C}} & F((A \otimes B) \otimes C) & \xrightarrow{F\alpha_{A,B,C}} & F(A \otimes (B \otimes C)) \end{array}$$

$$\begin{array}{ccc} I' \otimes' FA & \xrightarrow{\lambda'_{FA}} & FA \\ \downarrow m_I \otimes id_{FA} & & \uparrow F\lambda_A \\ FI' \otimes' FA & \xrightarrow{m_{I,A}} & F(I \otimes A) \end{array} \qquad \begin{array}{ccc} FA' \otimes' I' & \xrightarrow{\rho'_{FA}} & FA \\ \downarrow id_{FA'} \otimes m_I & & \uparrow F\rho_A \\ FA' \otimes' FI & \xrightarrow{m_{A,I}} & F(A \otimes I) \end{array}$$

► **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \rightarrow \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$\begin{array}{ccc} FA \otimes' FB & \xrightarrow{\otimes_{FA, FB}} & FB \otimes' FA \\ \downarrow m_{A, B} & & \downarrow m_{B, A} \\ F(A \otimes B) & \xrightarrow{F\otimes_{A, B}} & F(B \otimes A) \end{array}$$

► **Definition 9.** An **adjunction** between categories \mathcal{C} and \mathcal{D} consists of two functors $F : \mathcal{D} \rightarrow \mathcal{C}$, called the **left adjoint**, and $G : \mathcal{C} \rightarrow \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta : id_{\mathcal{D}} \rightarrow GF$, called the **unit**, and $\varepsilon : FG \rightarrow id_{\mathcal{C}}$, called the **counit**, such that the following diagrams commute for any object A in \mathcal{C} and B in \mathcal{D} :

$$\begin{array}{ccc} FB & \xrightarrow{F\eta_B} & FGFB \\ & \searrow \varepsilon_{FB} & \swarrow \\ & FB & \end{array} \quad \begin{array}{ccc} GA & \xrightarrow{\eta_{GA}} & GFGA \\ & \searrow G\varepsilon_A & \swarrow \\ & GA & \end{array}$$

► **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category \mathcal{M} to a monoidal category \mathcal{M}' . A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, m) \rightarrow (G, n)$ such that the following diagrams commute for any objects A and B in \mathcal{M} :

$$\begin{array}{ccc} FA \otimes' FB & \xrightarrow{m_{A, B}} & F(A \otimes B) \\ \downarrow \theta_A \otimes' \theta_B & & \downarrow \theta_{A \otimes B} \\ GA \otimes' GB & \xrightarrow{n_{A, B}} & G(A \otimes B) \end{array} \quad \begin{array}{ccc} FI & \xrightarrow{\theta_I} & GI \\ \downarrow m_I & & \downarrow n_I \\ I' & & I' \end{array}$$

► **Definition 11.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \rightarrow \mathcal{M}'$ and $G : \mathcal{M}' \rightarrow \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

► **Definition 12.** Let \mathcal{C} be a category. A **monad** on \mathcal{C} consists of an endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ together with two natural transformations $\eta : id_{\mathcal{C}} \rightarrow T$ and $\mu : T^2 \rightarrow T$, where $id_{\mathcal{C}}$ is the identity functor on \mathcal{C} , such that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow \mu_T & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} \quad \begin{array}{ccc} T & \xrightarrow{\eta_T} & T^2 \\ \downarrow T\eta & \searrow & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

► **Definition 13.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on \mathcal{M} . T is a **strong monad** if there is natural transformation τ , called the **tensorial strength**, with components

$\tau_{A,B} : A \otimes TB \rightarrow T(A \otimes B)$ such that the following diagrams commute:

$$\begin{array}{ccc}
 I \otimes TA & \xrightarrow{\tau_{I,A}} & T(I \otimes A) \\
 \searrow \lambda_{TA} & & \swarrow T\lambda_A \\
 & TA &
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \otimes B & \xrightarrow{id_A \otimes \eta_B} & A \otimes TB \\
 \searrow \eta_{A \otimes B} & & \swarrow \tau_{A,B} \\
 & T(A \otimes B) &
 \end{array}$$

$$\begin{array}{ccc}
 (A \otimes B) \otimes TC & \xrightarrow{\tau_{A \otimes B, C}} & T((A \otimes B) \otimes C) \\
 \downarrow \alpha_{A,B,TC} & & \downarrow T\alpha_{A,B,C} \\
 A \otimes (B \otimes TC) & \xrightarrow{id_A \otimes \tau_{B,C}} A \otimes T(B \otimes C) \xrightarrow{\tau_{A, B \otimes C}} & T(A \otimes (B \otimes C))
 \end{array}$$

$$\begin{array}{ccc}
 A \otimes T^2 B & \xrightarrow{\tau_{A, TB}} T(A \otimes TB) \xrightarrow{T\tau_{A,B}} & T^2(A \otimes B) \\
 \downarrow id_A \otimes \mu_B & & \downarrow \mu_{A \otimes B} \\
 A \otimes TB & \xrightarrow{\tau_{A,B}} & T(A \otimes B)
 \end{array}$$

► **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex , and (T, η, μ) be a strong monad on \mathcal{M} . Then there is a “**twisted**” **tensorial strength** $\tau'_{A,B} : TA \otimes B \rightarrow T(A \otimes B)$ defined as $\tau'_{A,B} = T\text{ex} \circ \tau_{B,A} \circ \text{ex}$. We can construct a pair of natural transformations Φ, Φ' with components $\Phi_{A,B}, \Phi'_{A,B} : TA \otimes TB \rightarrow T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A \otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A \otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.

► **Definition 15.** Let \mathcal{L} be a category. A **comonad** on \mathcal{L} consists of an endofunctor $S : \mathcal{L} \rightarrow \mathcal{L}$ together with two natural transformations $\varepsilon : S \rightarrow id_{\mathcal{L}}$ and $\delta : S^2 \rightarrow S$ such that the following diagrams commute:

$$\begin{array}{ccc}
 S & \xrightarrow{\delta} & S^2 \\
 \downarrow \delta & & \downarrow S\delta \\
 S^2 & \xrightarrow{\delta_S} & S^3
 \end{array}
 \qquad
 \begin{array}{ccc}
 S^2 & \xrightarrow{S\varepsilon} & S \\
 \downarrow \varepsilon_S & & \downarrow \delta \\
 S & \xrightarrow{\delta} & S^2
 \end{array}$$

4 Categorical Models

► **Definition 16.** A **SMCC-Lambek model** consists of

- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$;
- a monoidal adjunction $F : C \dashv \mathcal{L} : G$, where $F : C \rightarrow \mathcal{L}$ and $G : \mathcal{L} \rightarrow C$ are monoidal functors.

Thus, in a SMCC-Lambek model, the following four diagrams commute because η and ε are monoidal natural transformations:

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{id_{A \otimes B}} & A \otimes B \\
 \downarrow \eta_A \otimes \eta_B & & \downarrow \eta_{A \otimes B} \\
 GFA \otimes GFB & \xrightarrow{\eta_{FA, FB}} G(FA \otimes FB) \xrightarrow{Gm_{A,B}} & GF(A \otimes B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 I & \xrightarrow{\eta_I} & GFI \\
 \parallel & & \uparrow Gm_I \\
 I & \xrightarrow{\eta_{I'}} & GI'
 \end{array}$$

$$\begin{array}{ccc}
 FGA \otimes FGB & \xrightarrow{m_{GA,GB}} & F(GA \otimes GB) \xrightarrow{F\eta_{A,B}} FG(A \otimes B) \\
 \downarrow \varepsilon_A \otimes \varepsilon_B & & \downarrow \varepsilon_{A \otimes B} \\
 A \otimes B & \xlongequal{\quad} & A \otimes B
 \end{array}
 \quad
 \begin{array}{ccc}
 FGI' & \xrightarrow{\varepsilon_{I'}} & I' \\
 \uparrow F\eta_{I'} & & \uparrow m_I \\
 FI & \xleftarrow{m_I} & I'
 \end{array}$$

And the following two diagrams commute because of the adjunction:

$$\begin{array}{ccc}
 FA & \xrightarrow{F\eta_A} & FGFA \\
 \parallel & \searrow \varepsilon_{FA} & \\
 FA & &
 \end{array}
 \quad
 \begin{array}{ccc}
 GB & \xrightarrow{\eta_{GX}} & GFGB \\
 \parallel & \searrow G\varepsilon_B & \\
 GB & &
 \end{array}$$

► **Lemma 17.** *The monad on the SMCC \mathcal{C} in a SMCC-Lambek model is monoidal.*

Proof. We define the monad T on the \mathcal{C} in the adjunction of a SMCC-Lambek model as $T = GF$, and the two corresponding natural transformations $\eta : id_{\mathcal{C}} \rightarrow T$ and $\mu : T^2 \rightarrow T$ are defined as

$$\eta_A : A \rightarrow GFA$$

$$\mu_A = G\varepsilon_{FA} : GFGFA \rightarrow GFA$$

where η is the unit and $\varepsilon : FG \rightarrow id_{\mathcal{L}}$ is the counit in the adjunction $F : \mathcal{C} \dashv \mathcal{L} : G$, and (F, m) and (G, n) are monoidal functors.

Thus, we have

$$t_{A,B} = Gm_{A,B} \circ n_{FA,FB} : TA \otimes TB \rightarrow T(A \otimes B)$$

$$t_I = Gm_I \circ n_{I'} : I \rightarrow TI$$

The monad T being monoidal means

1. T is a monoidal functor, i.e. the following diagrams commute:

$$\begin{array}{ccccc}
 (TA \otimes TB) \otimes TC & \xrightarrow{\alpha_{TA,TB,TC}} & TA \otimes (TB \otimes TC) & \xrightarrow{id_{TA} \otimes t_{B,C}} & TA \otimes T(B \otimes C) \\
 \downarrow t_{A,B} \otimes id_{TC} & & & & \downarrow t_{A,B \otimes C} \\
 T(A \otimes B) \otimes TC & \xrightarrow{t_{A \otimes B, C}} & T((A \otimes B) \otimes C) & \xrightarrow{T\alpha_{A,B,C}} & T(A \otimes (B \otimes C))
 \end{array}$$

$$\begin{array}{ccc}
 I \otimes TA & \xrightarrow{\lambda_{TA}} & TA \\
 \downarrow t_I \otimes id_{TA} & & \uparrow T\lambda_A \\
 TI \otimes TA & \xrightarrow{t_{I,A}} & T(I \otimes A)
 \end{array}
 \quad
 \begin{array}{ccc}
 TA \otimes I & \xrightarrow{\rho_{TA}} & TA \\
 \downarrow id_{TA} \otimes t_I & & \uparrow T\rho_A \\
 TA \otimes TI & \xrightarrow{t_{A,I}} & T(A \otimes I)
 \end{array}$$

We write GF instead of T in the proof for clarity.

By replacing t with its definition, the first diagram above commutes by the following commutative diagram, where the two hexagons commute because G and F are monoidal functors, and the

two quadrilaterals commute by the naturality of n .

$$\begin{array}{ccccc}
 (GFA \otimes GFB) \otimes GFC & \xrightarrow{\alpha_{GFA,GFB,GFC}} & GFA \otimes (GFB \otimes GFC) & \xrightarrow{id_{GFA} \otimes n_{FB,FC}} & GFA \otimes G(FB \otimes FC) \\
 \downarrow n_{FA,FB} \otimes id_{GFC} & & & \swarrow n_{FA,FB} \otimes FC & \downarrow id_{GFA} \otimes Gm_{B,C} \\
 G(FA \otimes' FB) \otimes GFC & & G(FA \otimes' (FB \otimes' FC)) & & GFA \otimes GF(B \otimes C) \\
 \downarrow Gm_{A,B} \otimes id_{GFC} & \searrow n_{FA \otimes' FB, FC} & \uparrow G\alpha'_{FA,FB,FC} & \searrow G(id_{FA} \otimes' m_{B,C}) & \downarrow n_{FA,F(B \otimes C)} \\
 GF(A \otimes B) \otimes GFC & & G((FA \otimes' FB) \otimes' FC) & & G(FA \otimes' F(B \otimes C)) \\
 \downarrow n_{F(A \otimes B), FC} & \swarrow G(m_{A,B} \otimes id_{FC}) & & & \downarrow Gm_{A,B \otimes C} \\
 G(F(A \otimes B) \otimes' FC) & \xrightarrow{Gm_{A \otimes B, C}} & GF((A \otimes B) \otimes C) & \xrightarrow{GF\alpha_{A,B,C}} & GF(A \otimes (B \otimes C))
 \end{array}$$

The first square above commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n .

$$\begin{array}{ccc}
 I \otimes GFA & \xrightarrow{\lambda_{GFA}} & GFA \\
 \downarrow n_{I'} \otimes id_{GFA} & & \uparrow G\lambda'_{FA} \\
 G I' \otimes GFA & \xrightarrow{n_{I',FA}} & G(I' \otimes' FA) \\
 \downarrow Gm_I \otimes id_{GFA} & & \downarrow G(m_I \otimes' id_{FA}) \\
 GF I \otimes GFA & \xrightarrow{n_{FI,FA}} & G(FI \otimes' FA) \xrightarrow{Gm_{I,A}} GF(I \otimes A)
 \end{array}$$

Similarly, the second square above commutes by the following commutative diagram:

$$\begin{array}{ccc}
 GFA \otimes I & \xrightarrow{\rho_{GFA}} & GFA \\
 \downarrow id_{GFA} \otimes n_{I'} & & \uparrow G\rho'_{FA} \\
 GFA \otimes G I' & \xrightarrow{n_{FA,I'}} & G(FA \otimes' I') \\
 \downarrow id_{GFA} \otimes Gm_I & & \downarrow G(id_{FA} \otimes m_I) \\
 GFA \otimes GF I & \xrightarrow{n_{FA,FI}} & G(FA \otimes' FI) \xrightarrow{Gm_{A,I}} GF(A \otimes I)
 \end{array}$$

2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal and thus the following two diagrams commute.

$$\begin{array}{ccc}
 A \otimes B & \xlongequal{\quad} & A \otimes B \\
 \downarrow \eta_A \otimes \eta_B & & \downarrow \eta_{A \otimes B} \\
 TA \otimes TB & \xrightarrow{t_{A,B}} & T(A \otimes B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 I & \xrightarrow{\eta_I} & TI \\
 \parallel & \searrow t_I & \uparrow \\
 I & & I
 \end{array}$$

3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ . Thus the following diagrams commute.

$$\begin{array}{ccc}
 T^2 A \otimes T^2 B & \xrightarrow{t_{TA,TB}} T(TA \otimes TB) \xrightarrow{Tt_{A,B}} & T^2(A \otimes B) \\
 \downarrow \mu_A \otimes \mu_B & & \downarrow \mu_{A \otimes B} \\
 TA \otimes TB & \xrightarrow{t_{A,B}} & T(A \otimes B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 T^2 I & \xrightarrow{\mu_I} & TI \\
 \uparrow Tt_I & & \uparrow t_I \\
 TI & \xleftarrow{t_I} & I
 \end{array}$$

However, the monad is not symmetric because the following diagram does not commute, for the lambek category \mathcal{L} is not symmetric.

$$\begin{array}{ccccc}
 GFA \otimes GFB & \xrightarrow{\text{ex}_{GFA,GFB}} & GFB \otimes GFA & \xrightarrow{\eta_{FB,FA}} & G(FB \otimes' FA) \\
 \downarrow \eta_{FA,FB} & & & & \downarrow Gm_{B,A} \\
 G(FA \otimes' FB) & \xrightarrow{Gm_{A,B}} & GF(A \otimes B) & \xrightarrow{GF\text{ex}_{A,B}} & GF(B \otimes A)
 \end{array}$$

► **Lemma 18.** *The monad on the SMCC in a SMCC-Lambek model is strong.*

Proof. Let $F : C \vdash \mathcal{L} : G$ be a SMCC-Lambek model, where $(C, \otimes, I, \alpha, \lambda, \rho)$ is an SMCC, $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$ is a Lambek category, and (F, m) and (G, n) are monoidal functors. Let (T, η, μ) be the monad on C where $T = GF$. We have proved that T is monoidal with the natural transformation $t_{A,B} : TA \otimes TB \rightarrow T(A \otimes B)$ and the morphism $t_I : I \rightarrow TI$ defined as in Lemma ??.

We define the tensorial strength $\tau_{A,B} : A \otimes TB \rightarrow T(A \otimes B)$ as $\tau_{A,B} = t_{A,B} \circ \eta_A \otimes id_{TB}$.

Since η is a monoidal natural transformation, we have $\eta_I = Gm_I \circ \eta_{I'}$. Therefore $\eta_I = t_I$. Thus the following diagram commutes because T is monoidal, where the composition $t_{I,A} \circ t_I \otimes id_{TA}$ is the definition of $\tau_{I,A}$. So the first triangle in Definition ?? commutes.

$$\begin{array}{ccc}
 I \otimes TA & \xrightarrow{t_I \otimes id_{TA}} & TI \otimes TA \\
 \downarrow \lambda_{TA} & & \downarrow t_{I,A} \\
 TA & \xleftarrow{T\lambda_A} & T(I \otimes A)
 \end{array}$$

Similarly, by using the definition of τ , the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{id_A \otimes \eta_B} & A \otimes TB \\
 \downarrow \eta_{A \otimes B} & \searrow \eta_A \otimes \eta_B & \downarrow \eta_A \otimes id_{TB} \\
 T(A \otimes B) & \xleftarrow{t_{A,B}} & TA \otimes TB
 \end{array}$$

The first pentagon in the definition commutes by the following commutative diagrams, because η are α natural transformations and T is monoidal:

$$\begin{array}{ccccc}
 (A \otimes B) \otimes TC & \xrightarrow{\eta_{A \otimes B} \otimes id_{TC}} & T(A \otimes B) \otimes TC & \xrightarrow{t_{A \otimes B, C}} & T((A \otimes B) \otimes C) \\
 \downarrow \alpha_{A,B,TC} & \searrow (\eta_A \otimes \eta_B) \otimes id_{TC} & \uparrow t_{A,B} \otimes id_{TC} & & \downarrow T\alpha_{A,B,C} \\
 A \otimes (B \otimes TC) & & (TA \otimes TB) \otimes TC & & T(A \otimes (B \otimes C)) \\
 \downarrow id_A \otimes (\eta_B \otimes id_{TC}) & \searrow \eta_A \otimes (\eta_B \otimes id_{TC}) & \downarrow \alpha_{TA,TB,TC} & & \uparrow t_{A,B \otimes C} \\
 & & TA \otimes (TB \otimes TC) & & \\
 \downarrow id_A \otimes \eta_{B \otimes TC} & \searrow \eta_A \otimes id_{TB \otimes TC} & \downarrow id_{TA} \otimes t_{B,C} & & \\
 A \otimes (TB \otimes TC) & \xrightarrow{id_A \otimes t_{B,C}} & A \otimes T(B \otimes C) & \xrightarrow{\eta_A \otimes id_{T(B \otimes C)}} & TA \otimes T(B \otimes C)
 \end{array}$$

The last diagram in the definition commutes by the following commutative diagram, because T is a monad, η is a natural transformation, and μ is a monoidal natural transformation:

$$\begin{array}{ccccc}
 A \otimes T^2 B & \xrightarrow{\eta_A \otimes id_{T^2 B}} & TA \otimes T^2 B & \xrightarrow{t_{A, TB}} & T(A \otimes TB) \\
 \downarrow id_A \otimes \mu_B & & \parallel & \searrow T\eta_A \otimes id_{T^2 B} & \downarrow T(\eta_A \otimes id_{TB}) \\
 A \otimes TB & \xrightarrow{id_{TA} \otimes \mu_B} & TA \otimes T^2 B & \xleftarrow{\mu_A \otimes id_{T^2 B}} & T^2 A \otimes T^2 B \xrightarrow{t_{TA, TB}} T(TA \otimes TB) \\
 \downarrow \eta_A \otimes id_{TB} & \swarrow id_{TA} \otimes \mu_B & \swarrow \mu_A \otimes \mu_B & & \downarrow T\eta_{A, B} \\
 TA \otimes TB & \xrightarrow{t_{A, B}} & T(A \otimes B) & \xleftarrow{\mu_{A \otimes B}} & T^2(A \otimes B)
 \end{array}$$

► **Lemma 19** ([?]). *Let \mathcal{M} be a symmetric monoidal category and T be a strong monad on \mathcal{M} . Then T is a symmetric monoidal functor iff it is commutative.*

► **Theorem 20.** *The monad on the SMCC in a SMCC-Lambek model is monoidal and non-commutative.*

► **Lemma 21.** *The comonad on the Lambek category in a SMCC-Lambek model is monoidal.*

Proof. We define the comonad S on the Lambek category \mathcal{L} in the adjunction $F : \mathcal{C} \vdash \mathcal{L} : G$ of a SMCC-Lambek model as $S = FG$, and the two corresponding natural transformations $\varepsilon : S \rightarrow id_{\mathcal{L}}$ and $\delta : S \rightarrow S^2$ are defined as

$$\varepsilon_A : SA \rightarrow A$$

$$\delta_A = F\eta_{GA} : SA \rightarrow S^2 A$$

where ε is the counit and $\eta : id_{\mathcal{L}} \rightarrow GF$ is the unit in the adjunction, and (F, m) and (G, n) are monoidal functors. Thus, we have

$$s_{A, B} = F n_{A, B} \circ m_{GA, GB} : SA \otimes' SB \rightarrow SA \otimes' SB$$

$$s_I = F n_{I'} \circ m_I : I' \rightarrow S I'$$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$\begin{array}{ccccc}
 (SA \otimes' SB) \otimes' SC & \xrightarrow{\alpha'_{SA, SB, SC}} & SA \otimes' (SB \otimes' SC) & \xrightarrow{id_{SA} \otimes' s_{B, C}} & SA \otimes' S(B \otimes' C) \\
 \downarrow s_{A, B} \otimes' id_{SC} & & & & \downarrow s_{A, B \otimes' C} \\
 S(A \otimes' B) \otimes' SC & \xrightarrow{s_{A \otimes' B, C}} & S((A \otimes' B) \otimes' C) & \xrightarrow{s_{A \otimes' B, C}} & S(A \otimes' (B \otimes' C))
 \end{array}$$

$$\begin{array}{ccc}
 I' \otimes' SA & \xrightarrow{\lambda'_{SA}} & SA \\
 \downarrow s_{I'} \otimes' id_{SA} & & \uparrow s_{I', A} \\
 S I' \otimes' SA & \xrightarrow{s_{I', A}} & S(I' \otimes' A)
 \end{array}$$

$$\begin{array}{ccc}
 SA \otimes' I' & \xrightarrow{\rho'_{SA}} & SA \\
 \downarrow id'_{SA} \otimes' s_{I'} & & \uparrow s_{I', A} \\
 SA \otimes' S I' & \xrightarrow{s_{A, I'}} & S(A \otimes' I')
 \end{array}$$

2. ε is a monoidal natural transformation:

$$\begin{array}{ccc}
 SA \otimes' SB & \xrightarrow{s_{A, B}} & S(A \otimes' B) \\
 \downarrow \varepsilon_A \otimes' \varepsilon_B & & \downarrow \varepsilon_{A \otimes' B} \\
 A \otimes' B & \xrightarrow{=} & A \otimes' B
 \end{array}$$

$$\begin{array}{ccc}
 S I' & \xrightarrow{\varepsilon_{I'}} & I' \\
 \downarrow s_{I'} & & \uparrow \\
 I' & & I'
 \end{array}$$

3. δ is a monoidal natural transformation:

$$\begin{array}{ccc} SA \otimes' SA & \xrightarrow{S_{A,B}} & S(A \otimes' B) \\ \delta_A \otimes' \delta_B \downarrow & & \downarrow \delta_{A \otimes' B} \\ S^2A \otimes' S^2B & \xrightarrow{S S_{A,B}} S(SA \otimes' SB) \xrightarrow{S S_{A,B}} & S^2(A \otimes' B) \end{array}$$

$$\begin{array}{ccc} SI' & \xrightarrow{\delta_{I'}} & S^2I' \\ \uparrow s_{I'} & & \uparrow S s_{I'} \\ I' & \xrightarrow{s_{I'}} & SI' \end{array}$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma ???. We do not include the proof here for simplicity. ◀

5

Logic

6

Applications

7

Related Work

TODO

8

Conclusion

TODO

A

Appendix

