Linear Logic, Monads and Non-commutative Lambda Calculus

Jiaming Jiang¹ and Harley Eades III²

- 1 Computer Science, Augusta University, Augusta, Georgia, USA heades@augusta.edu
- 2 Computer Science, North Carolina State University, Raleigh, North Carolina, USA jjiang13@ncsu.edu

_	_		
Λ	hs	tra	^ +
н	D5	117	(:1

TODO

1998 ACM Subject Classification TODO

Keywords and phrases TODO

Digital Object Identifier 10.4230/LIPIcs...

- 1 Introduction
- 2 Main Ideas
- 3 Category Theory Basics
- ▶ **Definition 1.** A monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a category \mathcal{M} consists of
- a bifunctor \otimes : $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- **a** an object *I*, called the unit object;
- three natural isomorphisms α , λ , and ρ with components

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$\lambda_A: I \otimes A \to A$$

$$\rho_A: A \otimes I \to A$$

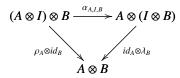
where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in \mathcal{M} :

$$((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha_{A,B,C} \otimes id_D} (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D)$$

$$\downarrow id_A \otimes \alpha_{B,C,D}$$

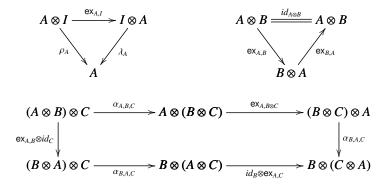
$$(A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C \otimes D}} A \otimes (B \otimes (C \otimes D))$$



▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ equipped with two bifunctors \rightarrow : $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and \leftarrow : $\mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\operatorname{\mathsf{Hom}}_f(X \otimes A, B) \cong \operatorname{\mathsf{Hom}}_f(X, A \rightharpoonup B)$$
 $\operatorname{\mathsf{Hom}}_f(A \otimes X, B) \cong \operatorname{\mathsf{Hom}}_f(X, B \leftharpoonup A)$

▶ **Definition 3.** A symmetric monoidal category (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $ex_{A,B} : A \otimes B \to B \otimes A$, called **exchange**, such that the following diagrams commute:



- ▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor \multimap : $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.
- ▶ **Lemma 5.** Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightarrow B) \cong (B \leftarrow A)$.

Proof. First, notice that for any object C we have

$$Hom[C, A \rightarrow B] \cong Hom[C \otimes A, B]$$
 \mathcal{L} is a Lambek category $\cong Hom[A \otimes C, B]$ By the exchange $ex_{C,A}$ $\cong Hom[C, B \leftarrow A]$ \mathcal{L} is a Lambek category

Thus, $A \rightarrow B \cong B \leftarrow A$ by the Yoneda lemma.

- ▶ Corollary 6. A Lambek category with exchange is symmetric monoidal closed.
- ▶ **Definition 7.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from \mathcal{M} to \mathcal{M}' is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \otimes FB' \to F(A \otimes B)$, such that the following diagrams commute for any objects A, B, and C in \mathcal{M} :

$$(FA \otimes' FB) \otimes' FC \xrightarrow{\alpha'_{FA,FB,FC}} FA \otimes' (FB \otimes' FC) \xrightarrow{id_{FA} \otimes' \mathsf{m}_{A,B}} FA \otimes' F(B \otimes C)$$

$$\downarrow^{\mathsf{m}_{A,B} \otimes' id_{FC}} \downarrow^{\mathsf{m}_{A,B} \otimes' id_{FC}} \downarrow$$

▶ **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$FA \otimes' FB \xrightarrow{\text{ex}_{FA,FB}} FB \otimes' FA$$

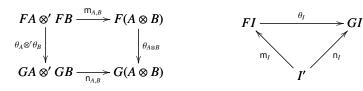
$$\downarrow m_{A,B} \downarrow \qquad \qquad \downarrow m_{B,A}$$

$$F(A \otimes B) \xrightarrow{F \in X_{A,B}} F(B \otimes A)$$

▶ **Definition 9.** An **adjunction** between categories C and \mathcal{D} consists of two functors $F: \mathcal{D} \to C$, called the **left adjoint**, and $G: C \to \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta: id_{\mathcal{D}} \to GF$, called the **unit**, and $\varepsilon: FG \to id_C$, called the **counit**, such that the following diagrams commute for any object A in C and B in D:



▶ **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category \mathcal{M} to a monoidal category \mathcal{M}' . A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, m) \to (G, n)$ such that the following diagrams commute for any objects A and B in \mathcal{M} :



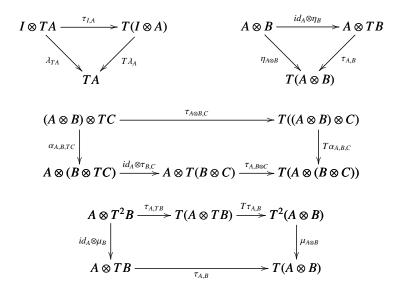
- ▶ **Definition 11.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \to \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.
- ▶ **Definition 12.** Let C be a category. A **monad** on C consists of an endofunctor $T: C \to C$ together with two natural transformations $\eta: id_C \to T$ and $\mu: T^2 \to T$, where id_C is the identity functor on C, such that the following diagrams commute:



▶ **Definition 13.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on \mathcal{M} . T is a **strong monad** if there is natural transformation τ , called the **tensorial strength**, with components

XX:4 Linear Logic, Monads and Non-commutative Lambda Calculus

 $\tau_{A,B}: A \otimes TB \to T(A \otimes B)$ such that the following diagrams commute:



- ▶ **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex, and (T, η, μ) be a strong monad on \mathcal{M} . Then there is a "**twisted**" **tensorial strength** $\tau'_{A,B}: TA \otimes B \to T(A \otimes B)$ defined as $\tau'_{A,B} = T$ ex $\circ \tau_{B,A} \circ$ ex. We can construct a pair of natural transformations Φ , Φ' with components $\Phi_{A,B}, \Phi'_{A,B}: TA \otimes TB \to T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A\otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A\otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.
- ▶ **Definition 15.** Let \mathcal{L} be a category. A **comonad** on \mathcal{L} consists of an endofunctor $S: \mathcal{L} \to \mathcal{L}$ together with two natural transformations $\varepsilon: S \to id_{\mathcal{L}}$ and $\delta: S^2 \to S$ such that the following diagrams commute:



4 Categorical Models

- ▶ Definition 16. A SMCC-Lambek model consists of
- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$;
- **a** monoidal adjunction $F: C \dashv \mathcal{L}: G$, where $F: C \rightarrow \mathcal{L}$ and $G: \mathcal{L} \rightarrow C$ are monoidal functors.

Thus, in a SMCC-Lambek model, the following four diagrams commute because η and ε are monoidal natural transformations:

$$A \otimes B \xrightarrow{id_{A \otimes B}} A \otimes B \qquad I \xrightarrow{\eta_{I}} GFI$$

$$\downarrow^{\eta_{A \otimes \eta_{B}}} \downarrow^{\eta_{A \otimes B}} \qquad \downarrow^{GFI} \qquad \downarrow^{GFI}$$

$$GFA \otimes GFB \xrightarrow{n_{FA,FB}} G(FA \otimes FB) \xrightarrow{Gm_{A,B}} GF(A \otimes B) \qquad I \xrightarrow{n_{I'}} GI'$$

$$FGA \otimes FGB \xrightarrow{\mathsf{m}_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\mathsf{n}_{A,B}} FG(A \otimes B) \qquad FGI' \xrightarrow{\varepsilon_{I'}} I'$$

$$\downarrow_{\varepsilon_{A} \otimes \varepsilon_{B}} \downarrow \qquad \downarrow_{\varepsilon_{A \otimes B}} \qquad \downarrow_{F\mathsf{n}_{I'}} \downarrow \qquad \parallel$$

$$A \otimes B \xrightarrow{\mathsf{m}_{GA,GB}} A \otimes B \qquad FI \underset{\mathsf{m}_{I}}{\longleftarrow} I'$$

And the following two diagrams commute because of the adjunction:



▶ Lemma 17. The monad on the SMCC C in a SMCC-Lambek model is monoidal.

Proof. We define the monad T on the C in the adjunction of a SMCC-Lambek model as T = GF, and the two corresponding natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to T$ are defined as

$$\eta_A:A\to GFA$$

$$\mu_A = G\varepsilon_{FA} : GFGFA \to GFA$$

where η is the unit and $\varepsilon : FG \to id_{\mathcal{L}}$ is the counit in the adjunction $F : C \dashv \mathcal{L} : G$, and (F, m) and (G, n) are monoidal functors.

Thus, we have

$$\mathsf{t}_{A,B} = G\mathsf{m}_{A,B} \circ \mathsf{n}_{FA,FB} : TA \otimes TB \to T(A \otimes B)$$

$$\mathsf{t}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'} : I \to TI$$

The monad T being monoidal means

1. T is a monoidal functor, i.e. the following diagrams commute:

$$(TA \otimes TB) \otimes TC \xrightarrow{\alpha_{TA,TB,TC}} TA \otimes (TB \otimes TC) \xrightarrow{id_{TA} \otimes t_{B,C}} TA \otimes T(B \otimes C)$$

$$\downarrow_{t_{A,B} \otimes id_{TC}} \downarrow \qquad \qquad \downarrow_{t_{A,B} \otimes C}$$

$$T(A \otimes B) \otimes TC \xrightarrow{t_{A \otimes B,C}} T((A \otimes B) \otimes C) \xrightarrow{T\alpha_{A,B,C}} T(A \otimes (B \otimes C))$$

$$I \otimes TA \xrightarrow{\lambda_{TA}} TA \qquad \qquad TA \otimes I \xrightarrow{\rho_{TA}} TA$$

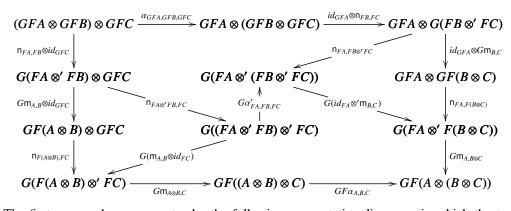
$$\downarrow_{t_{I} \otimes id_{TA}} \downarrow \qquad \qquad \uparrow_{t_{A,A}} TA \qquad \qquad \downarrow_{t_{A,A}} TA \otimes I \xrightarrow{t_{A,A}} T(A \otimes I)$$

$$TA \otimes TI \xrightarrow{t_{A,A}} T(A \otimes I)$$

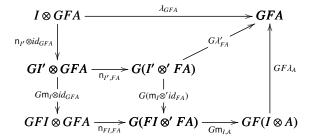
We write GF instead of T in the proof for clarity.

By replacing t with its definition, the first diagram above commutes by the following commutative diagram, where the two hexagons commute because G and F are monoidal functors, and the

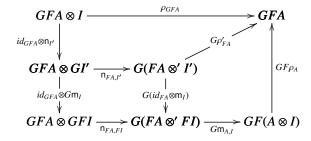
two quadrilaterals commute by the naturality of n.



The first square above commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.



Similarly, the second square above commutes by the following commutative diagram:



2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal and thus the following two diagrams commute.



3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ . Thus the following diagrams commute.

$$T^{2}A \otimes T^{2}B \xrightarrow{t_{TA,TB}} T(TA \otimes TB) \xrightarrow{Tt_{A,B}} T^{2}(A \otimes B) \qquad T^{2}I \xrightarrow{\mu_{I}} TI$$

$$\downarrow^{\mu_{A} \otimes \mu_{B}} \downarrow^{\mu_{A} \otimes B} \qquad Tt_{I} \uparrow^{t_{I}} \downarrow^{t_{I}} \downarrow^{t_{I}}$$

$$TA \otimes TB \xrightarrow{t_{A,B}} T(A \otimes B) \qquad TI \leftarrow \prod_{t_{I}} I$$

However, the monad is not symmetric because the following diagram does not commute, for the lambek category \mathcal{L} is not symmetric.

$$\begin{array}{c|c} GFA \otimes GFB \xrightarrow{\exp_{FA,GFB}} GFB \otimes GFA \xrightarrow{n_{FB,FA}} G(FB \otimes' FA) \\ \downarrow & & \downarrow \\ G(FA \otimes' FB) \xrightarrow{Gm_{A,B}} GF(A \otimes B) \xrightarrow{GFex_{A,B}} GF(B \otimes A) \end{array}$$

▶ **Lemma 18.** The monad on the SMCC in a SMCC-Lambek model is strong.

Proof. Let $F: C \vdash \mathcal{L}: G$ be a SMCC-Lambek model, where $(C, \otimes, I, \alpha, \lambda, \rho)$ is an SMCC, $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$ is a Lambek category, and (F, m) and (G, n) are monoidal functors. Let (T, η, μ) be the monad on C where T = GF. We have proved that T is monoidal with the natural transformation $\mathsf{t}_{A,B}: TA \otimes TB \to T(A \otimes B)$ and the morphism $\mathsf{t}_I: I \to TI$ defined as in Lemma \ref{loop} ?

We define the tensorial strength $\tau_{A,B}: A \otimes TB \to T(A \otimes B)$ as $\tau_{A,B} = \mathsf{t}_{A,B} \circ \eta_A \otimes id_{TB}$.

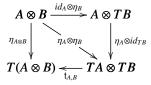
Since η is a monoidal natural transformation, we have $\eta_I = Gm_I \circ n_{I'}$. Therefore $\eta_I = t_I$. Thus the following diagram commutes because T is monoidal, where the composition $t_{I,A} \circ t_I \otimes id_{TA}$ is the definition of $\tau_{I,A}$. So the first triangle in Defition ?? commutes.

$$I \otimes TA \xrightarrow{t_{I} \otimes id_{TA}} TI \otimes TA$$

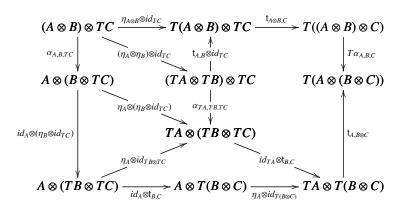
$$\downarrow t_{I,A} \qquad \qquad \downarrow t_{I,A}$$

$$TA \underset{T\lambda_{A}}{\longleftarrow} T(I \otimes A)$$

Similarly, by using the definition of τ , the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

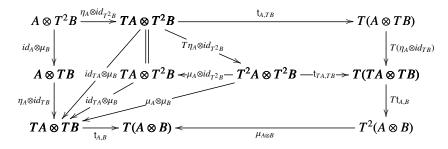


The first pentagon in the definition commutes by the following commutative diagrams, because η are α natural transformations and T is monoidal:



•

The last diagram in the definition commtues by the following commutative diagram, because T is a monad, t is a natural transformation, and μ is a monoidal natural transformation:



- ▶ **Lemma 19** ([?]). *Let* \mathcal{M} *be a symmetric monoidal category and* T *be a strong monad on* \mathcal{M} . *Then* T *is a symmetric monoidal functor iff it is commutative.*
- ▶ Theorem 20. The monad on the SMCC in a SMCC-Lambek model is monoidal and non-commutative.
- ▶ **Lemma 21.** The comonad on the Lambek category in a SMCC-Lambek model is monoidal.

Proof. We define the comonad S on the Lambek category \mathcal{L} in the adjunction $F: C \vdash \mathcal{L}: G$ of a SMCC-Lambek model as S = FG, and the two corresponding natural transformations $\varepsilon: S \to id_{\mathcal{L}}$ and $\delta: S \to S^2$ are defined as

$$\varepsilon_A : SA \to A$$

$$\delta_A = F\eta_{GA} : SA \to S^2A$$

where ε is the counit and $\eta: id_{\mathcal{L}} \to GF$ is the unit in the adjunction, and (F, m) and (G, n) are monoidal functors. Thus, we have

$$s_{A,B} = Fn_{A,B} \circ m_{GA,GB} : SA \otimes' SB \rightarrow SA \otimes' SB$$

 $s_I = Fn_{I'} \circ m_I : I' \rightarrow SI'$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$(SA \otimes' SB) \otimes' SC \xrightarrow{\alpha'_{SA,SB,SC}} \rightarrow SA \otimes' (SB \otimes' SC) \xrightarrow{id_{SA} \otimes' S_{B,C}} \rightarrow SA \otimes' S(B \otimes' C)$$

$$\downarrow_{S_{A,B} \otimes' id_{SC}} \downarrow_{S_{A,B} \otimes' C} \rightarrow S((A \otimes' B) \otimes' C) \xrightarrow{S\alpha'_{A,B,C}} \rightarrow S(A \otimes' (B \otimes' C))$$

$$I' \otimes' SA \xrightarrow{\lambda'_{SA}} SA \qquad SA \otimes' I' \xrightarrow{\rho'_{SA}} SA$$

$$\downarrow SI' \otimes' SA \xrightarrow{SI'} S(I' \otimes' A) \qquad id'_{SA} \otimes' SI' \xrightarrow{\rho'_{SA}} S(A \otimes' I')$$

$$\downarrow SA \otimes' SI' \xrightarrow{\rho'_{SA}} S(A \otimes' I')$$

$$\downarrow SA \otimes' SI' \xrightarrow{SA,I'} S(A \otimes' I')$$

2. ε is a monoidal natural transformation:



3. δ is a monoidal natural transformation:

$$SA \otimes' SA \xrightarrow{S_{A,B}} S(A \otimes' B) \qquad SI' \xrightarrow{\delta_{I'}} S^2I'$$

$$\downarrow \delta_{A\otimes'\delta_B} \downarrow \qquad \downarrow \delta_{A\otimes'B} \qquad \downarrow \delta_{A\otimes'B} \qquad \downarrow S_{I'} \xrightarrow{S_{I'}} S_{S_{I'}}$$

$$S^2A \otimes' S^2B \xrightarrow{S_{SA,SB}} S(SA \otimes' SB) \xrightarrow{S_{SA,B}} S^2(A \otimes' B) \qquad I' \xrightarrow{S_{I'}} SI'$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma ??. We do not include the proof here for simplicity.

5 Logic

5.1 Categorical Interpretation of Natural Deductions

T rules: in the symmetric monoidal closed category of the adjunction model

T_identity: $id_X : X \to X$

T_unitI:

T_unitE: given $t_1: \Delta \to Unit$ and $t_2: \Gamma \to Y$, returns $\lambda_Y \circ (t_1 \otimes t_2): \Gamma \otimes \Delta \to Unit \otimes Y \to Y$

T_tenI: given $t_1: \Gamma \to X$ and $t_2: \Delta \to Y$, returns $t_1 \otimes t_2: \Gamma \otimes \Delta \to X \otimes Y$

T_tenE: given $t_1: \Gamma \to X \otimes Y$ and $t_2: \Delta \otimes X \otimes Y \to Z$, returns

 $t_2 \circ \mathsf{ex}_{X \otimes Y, \Delta} \circ t_1 \otimes id_\Delta : \Gamma \otimes \Delta \to (X \otimes Y) \otimes \Delta \to \Delta \otimes (X \otimes Y) \to Z$

T_implI:

T_implE:

T_imprI:

T_imprE:

T_GI: given $s: FX_1 \otimes' ... \otimes' FX_n \to A$, returns

 $Gs \circ Gm^{-1} \circ \eta : X_1 \otimes ... \otimes X_n \to GF(X_1 \otimes ... \otimes X_n) \to G(FX_1 \otimes' ... \otimes' FX_n) \to GA$

S rules: in the Lambek category of the adjunction model

S_identity: $id_A: A \rightarrow A$

S_unitI:

S_unitE:

5.2 Normaalization and Reduction

- 6 Applications
- 7 Related Work

TODO

8 Conclusion

TODO

A Appendix