Non-Commutative Linear Logic in an Adjoint Model

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1 Introduction

Linear logic is a well-known resource-sensitive logic. It has been used extensively to model attack trees. This paper concerns a non-commutative variant of linear logic and combines the non-commutative variant with Girard's linear logic []. We will only focus on the multiplicative (i.e. $\otimes$, $\multimap$) part of linear logic for simplicity. We construct the non-commutative variant by using a non-commutative tensor product $\triangleright$ instead of the commutative $\otimes$, and two implications $\leftharpoonup$ and $\rightharpoonup$ for the two directions of $\multimap$.

We model the non-commutative linear logic categorically using an ajunction between a symmetric monoidal closed category and a Lambek category. Our categorial adjoint model has a similar structure as Benton's adjoint model [], in which the multiplicative part of intuitionistic linear logic (ILL) is modeled using an adjunction between a cartesian closed category and a symmetric monoidal closed category. On the other hand, Moggi [] uses monad models to map intuitionistic logic into ILL. As discussed in [**?**], Benton's adjoint models only gives rise to commutative monad models and the non-commutative part remained as an open problem. Therefore, by combining our adjoint models with Benton's, we would be able to get non-commutative monad models and thus non-commutative ILL.

The rest of the paper is organized as follows. Section 2 discusses existing approaches on constructing non-commutative linear logic. Section 3 contains the basic definitions in category theory that we will be using in our adjoint model. Familiar readers may skip this section. Section 4 contains the definition and essential properties of our adjoint model. Section 5 discusses the sequent calculus and natural deduction rules for our non-commutative linear logic. We prove that our sequent calculus has the property of cut-elimination and the natural deduction is strongly normalizing. Section 6 talks about the preliminary result after combing our non-commutative model with Benton's commutative model. Section 7 briefly mentions how our model could be used in attack trees and other areas. Section 8 concludes this paper with future work.

2 Related Work

Polakow and Pfenning discuseed Ordered Linear Logic (OLL) [], which combines intuitionistic, commutative linear and non-commutative linear logic, OLL contains sequents of the form $\Gamma, \Delta, \Omega \vdash$

A, where Γ is a multiset of intuitionistic assumptions, Δ is a multiset of commutative linear assumptions, and Ω is a list of non-commutative linear assumptions. OLL contains logical connectives from all three the logics. Therefore, our non-commutative adjoint model is a part of OLL and after combining with Benton's commutative adjoint model, we would get a simplification of OLL.

Greco and Palmigiano [] also presents a variant of the multiplicative fragment of non-commutative ILL. But they focus on proper display calculi while we use sequent calculi.

de Paiva and Eades [] also developed categorical models for the non-commutative ILL by adapting the Dialectica categorical models for linear logic.

3 Category Theory Basics

This section contains the basic definitions in category theory that we will be using in our adjoint model. Our model is based on special kinds of monoidal categories: Lambek categories and symmetric monoidal closed categories and Lambek categories, as defined in Definitions 2 and 3.

▶ **Definition 1.** A **monoidal category** $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ consists of

- a bifunctor $\triangleright : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- an object I, called the unit object;
- three natural isomorphisms α, λ, and ρ with components

$$\alpha_{A,B,C} : (A \triangleright B) \triangleright C \to A \triangleright (B \triangleright C)$$

$$\lambda_A : I \triangleright A \to A$$

$$\rho_A : A \triangleright I \to A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in $\mathcal{M}$:

$$\begin{tikzcd}
((A \triangleright B) \triangleright C) \triangleright D \arrow[r, "\alpha_{A,B,C} \triangleright id_D"] \arrow[d, "\alpha_{A \triangleright B,C,D}"'] & (A \triangleright (B \triangleright C)) \triangleright D \arrow[r, "\alpha_{A,B \triangleright C,D}"] & A \triangleright ((B \triangleright C) \triangleright D) \arrow[d, "id_A \triangleright \alpha_{B,C,D}"] \\
(A \triangleright B) \triangleright (C \triangleright D) \arrow[rr, "\alpha_{A,B,C \triangleright D}"'] & & A \triangleright (B \triangleright (C \triangleright D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \triangleright I) \triangleright B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \triangleright id_B"'] & & A \triangleright (I \triangleright B) \arrow[dl, "id_A \triangleright \lambda_B"] \\
& A \triangleright B &
\end{tikzcd}$$

▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\rightharpoonup : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and $\leftharpoonup : \mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \triangleright A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \triangleright X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

▶ **Definition 3.** A **symmetric monoidal category** (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\mathsf{ex}_{A,B} : A \otimes B \to B \otimes A$, called **exchange**, such that the following diagrams commute:

$$\begin{tikzcd} A \otimes I \arrow[rr, "\mathsf{ex}_{A,I}"] \arrow[dr, "\rho_A"'] & & I \otimes A \arrow[dl, "\lambda_A"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[rr, equal, "id_{A\otimes B}"] \arrow[dr, "\mathsf{ex}_{A,B}"'] & & A \otimes B \\ & B \otimes A \arrow[ur, "\mathsf{ex}_{B,A}"'] & \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathsf{ex}_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\mathsf{ex}_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,A,C}"] \\ (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \mathsf{ex}_{A,C}"'] & B \otimes (C \otimes A) \end{tikzcd}$$

We use $\triangleright$ for non-symmetric monoidal categories while $\otimes$ for symmetric ones.

▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap$: $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

The relation between SMMCs and Lambek categories are demonstrated in Lemma 5 and Corollary 6.

▶ **Lemma 5.** *Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$\begin{array}{rcll} Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\ & \cong & Hom[A \otimes C, B] & \text{By the exchange } \mathsf{ex}_{C,A} \\ & \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category} \end{array}$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◀

▶ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

The essential component in our non-commutative adjoint model is a monoidal adjunction, defined in Definitions 7-11.

▶ **Definition 7.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \triangleright', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from $\mathcal{M}$ to $\mathcal{M}'$ is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \triangleright FB' \to F(A \triangleright B)$, such that the following diagrams commute for any objects A, B, and C in $\mathcal{M}$:

$$\begin{tikzcd} (FA \triangleright' FB) \triangleright' FC \arrow[r, "\alpha'_{FA,FB,FC}"] \arrow[d, "\mathsf{m}_{A,B} \triangleright' id_{FC}"'] & FA \triangleright' (FB \triangleright' FC) \arrow[r, "id_{FA} \triangleright' \mathsf{m}_{A,B}"] & FA \triangleright' F(B \triangleright C) \arrow[d, "\mathsf{m}_{A,B\triangleright C}"] \\ F(A \triangleright B) \triangleright' FC \arrow[r, "\mathsf{m}_{A\triangleright B,C}"'] & F((A \triangleright B) \triangleright C) \arrow[r, "F\alpha_{A,B,C}"'] & F(A \triangleright (B \triangleright C)) \end{tikzcd}$$

$$\begin{tikzcd} I' \triangleright' FA \arrow[r, "\lambda'_{FA}"] \arrow[d, "\mathsf{m}_I \triangleright id_{FA}"'] & FA \\ FI \triangleright' FA \arrow[r, "\mathsf{m}_{I,A}"'] & F(I \triangleright A) \arrow[u, "F\lambda_A"'] \end{tikzcd} \qquad \begin{tikzcd} FA \triangleright' I' \arrow[r, "\rho'_{FA}"] \arrow[d, "\mathsf{id}_{FA} \triangleright \mathsf{m}_I"'] & FA \\ FA \triangleright' FI \arrow[r, "\mathsf{m}_{A,I}"'] & F(A \triangleright I) \arrow[u, "F\rho_A"'] \end{tikzcd}$$

► **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$\begin{tikzcd} FA \otimes' FB \arrow[r, "\mathsf{ex}_{FA,FB}"] \arrow[d, "\mathsf{m}_{A,B}"'] & FB \otimes' FA \arrow[d, "\mathsf{m}_{B,A}"] \\ F(A \otimes B) \arrow[r, "F\mathsf{ex}_{A,B}"'] & F(B \otimes A) \end{tikzcd}$$

► **Definition 9.** An **adjunction** between categories C and $\mathcal{D}$ consists of two functors $F : \mathcal{D} \to C$, called the **left adjoint**, and $G : C \to \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta : id_{\mathcal{D}} \to GF$, called the **unit**, and $\varepsilon : FG \to id_C$, called the **counit**, such that the following diagrams commute for any object A in C and B in $\mathcal{D}$:

$$\begin{tikzcd} FB \arrow[rr, "F\eta_B"] \arrow[dr, equal] & & FGFB \arrow[dl, "\varepsilon_{FB}"] \\ & FB & \end{tikzcd} \qquad \begin{tikzcd} GA \arrow[rr, "\eta_{GA}"] \arrow[dr, equal] & & GFGA \arrow[dl, "G\varepsilon_A"] \\ & GA & \end{tikzcd}$$

► **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ to a monoidal category $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$. A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, \mathsf{m}) \to (G, \mathsf{n})$ such that the following diagrams commute for any objects A and B in $\mathcal{M}$:

$$\begin{tikzcd} FA \triangleright' FB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[d, "\theta_A \triangleright' \theta_B"'] & F(A \triangleright B) \arrow[d, "\theta_{A \triangleright B}"] \\ GA \triangleright' GB \arrow[r, "\mathsf{n}_{A,B}"'] & G(A \triangleright B) \end{tikzcd} \qquad \begin{tikzcd} FI \arrow[rr, "\theta_I"] & & GI \\ & I' \arrow[ul, "\mathsf{m}_I"] \arrow[ur, "\mathsf{n}_I"'] & \end{tikzcd}$$

► **Definition 11.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \triangleright', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

In Moggi's monad model [], the monad is required to be strong, as defined in Definitions 12 and 13.

► **Definition 12.** Let C be a category. A **monad** on C consists of an endofunctor $T : C \to C$ together with two natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to T$, where id_C is the identity functor on C, such that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

► **Definition 13.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on $\mathcal{M}$. T is a **strong monad** if there is natural transformation τ, called the **tensorial strength**, with components

$\tau_{A,B} : A \triangleright TB \to T(A \triangleright B)$ such that the following diagrams commute:

$$\begin{tikzcd}
I \triangleright TA \arrow[rr, "\tau_{I,A}"] \arrow[dr, "\lambda_{TA}"'] & & T(I \triangleright A) \arrow[dl, "T\lambda_A"] \\
& TA &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \triangleright B \arrow[rr, "id_A \triangleright \eta_B"] \arrow[dr, "\eta_{A \triangleright B}"'] & & A \triangleright TB \arrow[dl, "\tau_{A,B}"] \\
& T(A \triangleright B) &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \triangleright B) \triangleright TC \arrow[rr, "\tau_{A \triangleright B,C}"] \arrow[d, "\alpha_{A,B,TC}"'] & & T((A \triangleright B) \triangleright C) \arrow[d, "T\alpha_{A,B,C}"] \\
A \triangleright (B \triangleright TC) \arrow[r, "id_A \triangleright \tau_{B,C}"] & A \triangleright T(B \triangleright C) \arrow[r, "\tau_{A,B \triangleright C}"] & T(A \triangleright (B \triangleright C))
\end{tikzcd}$$

$$\begin{tikzcd}
A \triangleright T^2B \arrow[r, "\tau_{A,TB}"] \arrow[d, "id_A \triangleright \mu_B"'] & T(A \triangleright TB) \arrow[r, "T\tau_{A,B}"] & T^2(A \triangleright B) \arrow[d, "\mu_{A \triangleright B}"] \\
A \triangleright TB \arrow[rr, "\tau_{A,B}"'] & & T(A \triangleright B)
\end{tikzcd}$$

▶ **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange $\mathtt{ex}$, and (T, η, μ) be a strong monad on $\mathcal{M}$. Then there is a **"twisted" tensorial strength** $\tau'_{A,B} : TA \otimes B \to T(A \otimes B)$ defined as $\tau'_{A,B} = T\mathtt{ex} \circ \tau_{B,A} \circ \mathtt{ex}$. We can construct a pair of natural transformations Φ, Φ' with components $\Phi_{A,B}, \Phi'_{A,B} : TA \otimes TB \to T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A \otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A \otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.

▶ **Definition 15.** Let $\mathcal{L}$ be a category. A **comonad** on $\mathcal{L}$ consists of an endofunctor $S : \mathcal{L} \to \mathcal{L}$ together with two natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S^2 \to S$ such that the following diagrams commute:

$$\begin{tikzcd}
S \arrow[r, "\delta"] \arrow[d, "\delta"'] & S^2 \arrow[d, "S\delta"] \\
S^2 \arrow[r, "\delta_S"'] & S^3
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2 \arrow[r, "S\varepsilon"] \arrow[d, "\varepsilon_S"'] & S \arrow[d, "\delta"] \\
S \arrow[r, "\delta"'] & S^2
\end{tikzcd}$$

4 Lambek Adjoint Model

Our adjoint model, Lambek Adjoint Model (LAM), has a similar structure as Benton's LNL model []. Benton's LNL model consists of a symmetric monoidal adjunction $F : C \dashv \mathcal{L} : G$ between a cartesian closed category C and a symmetric monoidal closed category $\mathcal{L}$. LAM consists of a monoidal adjunction between a symmetric monoidal closed category and a Lambek category.

▶ **Definition 16.** A **Lambek Adjoint Model (LAM)** $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ consists of

- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$;
- a monoidal adjunction $F : C \dashv \mathcal{L} : G$ with unit $\eta_X : X \to GFX$ and counit $\varepsilon : FG \to id_{\mathcal{L}}$, where $(F : C \to \mathcal{L}, m)$ and $(G : \mathcal{L} \to C, n)$ are monoidal functors.

Thus, in LAM, the following four diagrams commute because η and ε are monoidal natural transformations:

$$\begin{tikzcd} X \otimes Y \arrow[rr, equal, "id_{X\otimes Y}"] \arrow[d, "\eta_X \otimes \eta_Y"'] & & X \otimes Y \arrow[d, "\eta_{X\otimes Y}"] \\ GFX \otimes GFY \arrow[r, "\mathsf{n}_{FX,FY}"] & G(FX \otimes FY) \arrow[r, "G\mathsf{m}_{X,Y}"] & GF(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\eta_I"] \arrow[d, equal] & GFI \\ I \arrow[r, "\mathsf{n}_{I'}"'] & GI' \arrow[u, "G\mathsf{m}_I"'] \end{tikzcd}$$

$$\begin{tikzcd} FGA \otimes FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes \varepsilon_B"'] & F(GA \otimes GB) \arrow[r, "F\mathsf{n}_{A,B}"] & FG(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\ A \otimes B \arrow[rr, equal] & & A \otimes B \end{tikzcd} \qquad \begin{tikzcd} FGI' \arrow[r, "\varepsilon_{I'}"] & I' \arrow[d, equal] \\ FI \arrow[u, "F\mathsf{n}_{I'}"] & I' \arrow[l, "\mathsf{m}_I"] \end{tikzcd}$$

And the following two triangles commute because of the adjunction:

$$\begin{tikzcd} FX \arrow[rr, "F\eta_X"] \arrow[dr, equal] & & FGFX \arrow[dl, "\varepsilon_{FX}"] \\ & FX & \end{tikzcd} \qquad \begin{tikzcd} GA \arrow[rr, "\eta_{GA}"] \arrow[dr, equal] & & GFGA \arrow[dl, "G\varepsilon_A"] \\ & GA & \end{tikzcd}$$

Following the tradition, we use letters X, Y, Z for objects in C and A, B, C for objects in $\mathcal{L}$. The rest of this section proves essential properties of a LAM.

4.1 An Isomorphism

Let $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM, where (F, m) and (G, n) are monoidal functors. Similarly as in Benton's LNL model, $\mathsf{m}_{X,Y}$ are components of a natural isomorphism and m_I is an isomorphism. This is essential for deriving certain rules of our non-commutative linear logic, such as tensor elimination in natural deduction.

We define the inverses of $\mathsf{m}_{X,Y} : FX \triangleright FY \to F(X \otimes Y)$ and $\mathsf{m}_I : I' \to FI$ as:

$$\mathsf{p}_{X,Y} : F(X \otimes Y) \xrightarrow{F(\eta_X \otimes \eta_Y)} F(GFX \otimes GFY) \xrightarrow{F\mathsf{n}_{FX,FY}} FG(FX \triangleright FY) \xrightarrow{\varepsilon_{FX \triangleright FX}} FX \triangleright FY$$

$$\mathsf{p}_I : FI \xrightarrow{F\mathsf{n}_{I'}} FGI' \xrightarrow{\varepsilon_{I'}} I'$$

▶ **Theorem 17.** *$\mathsf{m}_{X,Y}$ are components of a natural isomorphism and their inverses are $\mathsf{p}_{X,Y}$.*

Proof. We need to show that $\mathsf{m}_{X,Y} \circ \mathsf{p}_{X,Y} = id_{F(X\otimes Y)}$ and $\mathsf{p}_{X,Y} \circ \mathsf{m}_{X,Y} = id_{FX \triangleright FX}$. The two equations hold because the following diagrams commute: (1)-adjunction; (2)-η is a monoidal natural transformation; (3)-naturality of ε; (4)-adjunction; (5)-naturality of m; (6)-ε is a monoidal natural transformation.

$$\begin{tikzcd}
F(X \otimes Y) \arrow[rrrr, "F(\eta_X \otimes \eta_Y)"] \arrow[dd, equal] \arrow[drr, "F\eta_{X\otimes Y}"] & & & & F(GFX \otimes GFY) \arrow[dd, "F\mathsf{n}_{FX,FY}"] \\
& (1) & FGF(X \otimes Y) \arrow[dll, "\varepsilon_{F(X\otimes Y)}"'] \arrow[drr, "FG\mathsf{m}_{X,Y}"] \arrow[d, phantom, "(3)"] & (2) & \\
F(X \otimes Y) & & FX \triangleright FY \arrow[ll, "\mathsf{m}_{X,Y}"] & & FG(FX \triangleright FY) \arrow[ll, "\varepsilon_{FX \triangleright FY}"]
\end{tikzcd}$$

$$\begin{tikzcd}
FX \triangleright FY \arrow[rrrr, "\mathsf{m}_{X,Y}"] \arrow[dd, equal] \arrow[drr, "F\eta_X \triangleright F\eta_Y"] & & & & F(X \otimes Y) \arrow[dd, "F(\eta_X \otimes \eta_Y)"] \\
& (4) & FGFX \triangleright FGFY \arrow[dll, "\varepsilon_{FX} \triangleright \varepsilon_{FY}"'] \arrow[drr, "\mathsf{m}_{GFX,GFY}"] \arrow[d, phantom, "(6)"] & (5) & \\
FX \triangleright FY & & FG(FX \triangleright FY) \arrow[ll, "\varepsilon_{FX \triangleright FY}"] & & F(GFX \otimes GFY) \arrow[ll, "F\mathsf{n}_{FX,FY}"]
\end{tikzcd}$$

◀

▶ **Theorem 18.** m_I *is an isomorphism and its inverse is* p_I.

Proof. This is equivalent to equations $\mathsf{m}_I \circ \mathsf{p}_I = id_{FI}$ and $\mathsf{p}_I \circ \mathsf{m}_I = id_{I'}$, equivalent to the following diagrams, which commute because ε is a monoidal natural transformation.

$$\begin{tikzcd}
FI \arrow[r, "F\mathsf{n}_{I'}"] \arrow[d, equal] & FGI' \arrow[d, "\varepsilon_{I'}"] \\
FI & I' \arrow[l, "\mathsf{m}_I"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
I' \arrow[r, "\mathsf{m}_I"] \arrow[d, equal] & FI \arrow[d, "F\mathsf{n}_{I'}"] \\
I' & FGI' \arrow[l, "\varepsilon_{I'}"]
\end{tikzcd}$$

◀

4.2 Monad on C

We first show that the monad on C in LAM is strong but non-commutative. In Benton's LNL model, the monad on the cartesian closed category is commutative.

▶ **Lemma 19.** *The monad on the symmetric monoidal closed category* C *in LAM is monoidal.*

Proof. Let $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM. We define the monad $(T, \eta : id_C \to T, \mu : T^2 \to T)$ on C as

$$T = GF \qquad \eta_X : X \to GFX \qquad \mu_X = G\varepsilon_{FX} : GFGFX \to GFX$$

Since (F, m) and (G, n) are monoidal functors, we have

$$\mathsf{t}_{X,Y} = G\mathsf{m}_{X,Y} \circ \mathsf{n}_{FX,FY} : TX \otimes TY \to T(X \otimes Y) \qquad \mathsf{t}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'} : I \to TI$$

The monad T being monoidal means:

1. T is a monoidal functor, i.e. the folllowing diagrams commute:

$$\begin{tikzcd}
(TX \otimes TY) \otimes TZ \arrow[r, "\alpha_{TX,TY,TZ}"] \arrow[d, "\mathsf{t}_{X,Y} \otimes id_{TZ}"'] & TX \otimes (TY \otimes TZ) \arrow[r, "id_{TX} \otimes \mathsf{t}_{Y,Z}"] \arrow[d, phantom, "(1)"] & TX \otimes T(Y \otimes Z) \arrow[d, "\mathsf{t}_{X,Y\otimes Z}"] \\
T(X \otimes Y) \otimes TZ \arrow[r, "\mathsf{t}_{X\otimes Y,Z}"'] & T((X \otimes Y) \otimes Z) \arrow[r, "T\alpha_{X,Y,Z}"'] & T(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes TX \arrow[r, "\lambda_{TX}"] \arrow[d, "\mathsf{t}_I \otimes id_{TX}"'] \arrow[dr, phantom, "(2)"] & TX \\
TI \otimes TX \arrow[r, "\mathsf{t}_{I,X}"'] & T(I \otimes X) \arrow[u, "T\lambda_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \otimes I \arrow[r, "\rho_{TX}"] \arrow[d, "id_{TX} \otimes \mathsf{t}_I"'] \arrow[dr, phantom, "(3)"] & TX \\
TX \otimes TI \arrow[r, "\mathsf{t}_{X,I}"'] & T(X \otimes I) \arrow[u, "T\rho_X"']
\end{tikzcd}$$

We write GF instead of T in the proof for clarity.
By replacing $\mathsf{t}_{X,Y}$ with its definition, diagram (1) above commutes by the following commutative diagram, in which the two hexagons commute because G and F are monoidal functors, and the two quadrilaterals commute by the naturality of n.

$$\begin{tikzcd}
(GFX \otimes GFY) \otimes GFZ \arrow[r, "\alpha_{GFX,GFY,GFZ}"] \arrow[d, "\mathsf{n}_{FX,FY} \otimes id_{GFZ}"'] & GFX \otimes (GFY \otimes GFZ) \arrow[r, "id_{GFX} \otimes \mathsf{n}_{FY,FZ}"] & GFX \otimes G(FY \triangleright FZ) \arrow[dl, "\mathsf{n}_{FX,FY\triangleright FZ}"'] \arrow[d, "id_{GFX} \otimes G\mathsf{m}_{Y,Z}"] \\
G(FX \triangleright FY) \otimes GFZ \arrow[d, "G\mathsf{m}_{X,Y} \otimes id_{GFZ}"'] \arrow[dr, "\mathsf{n}_{FX\triangleright FY,FZ}"] & G(FX \triangleright (FY \triangleright FZ)) \arrow[dr, "G(id_{FX} \triangleright \mathsf{m}_{Y,Z})"] & GFX \otimes GF(Y \otimes Z) \arrow[d, "\mathsf{n}_{FX,F(Y\otimes Z)}"] \\
GF(X \otimes Y) \otimes GFZ \arrow[d, "\mathsf{n}_{F(X\otimes Y),FZ}"'] & G((FX \triangleright FY) \triangleright FZ) \arrow[u, "G\alpha'_{FX,FY,FZ}"] \arrow[dl, "G(\mathsf{m}_{X,Y} \otimes id_{FZ})"'] & G(FX \triangleright F(Y \otimes Z)) \arrow[d, "G\mathsf{m}_{X,Y\otimes Z}"] \\
G(F(X \otimes Y) \triangleright FZ) \arrow[r, "G\mathsf{m}_{X\otimes Y,Z}"'] & GF((X \otimes Y) \otimes Z) \arrow[r, "GF\alpha_{X,Y,Z}"'] & GF(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Diagram (2) commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.

$$\begin{tikzcd}
I \otimes GFX \arrow[rr, "\lambda_{GFX}"] \arrow[d, "\mathsf{n}_{I'} \otimes id_{GFX}"'] & & GFX \\
GI' \otimes GFX \arrow[r, "\mathsf{n}_{I',FX}"'] \arrow[d, "G\mathsf{m}_I \otimes id_{GFX}"'] & G(I' \triangleright FX) \arrow[ur, "G\lambda'_{FX}"] \arrow[d, "G(\mathsf{m}_I \triangleright id_{FX})"'] & \\
GFI \otimes GFX \arrow[r, "\mathsf{n}_{FI,FX}"'] & G(FI \triangleright FX) \arrow[r, "G\mathsf{m}_{I,X}"'] & GF(I \otimes X) \arrow[uu, "GF\lambda_X"']
\end{tikzcd}$$

Similarly, diagram (3) commutes as follows:

$$
\begin{tikzcd}
GFX \otimes I \arrow[rr, "\rho_{GFX}"] \arrow[d, "id_{GFX} \otimes \mathsf{n}_{I'}"'] & & GFX \\
GFX \otimes GI' \arrow[r, "\mathsf{n}_{FX,I'}"'] \arrow[d, "id_{GFX} \otimes G\mathsf{m}_I"'] & G(FX \triangleright I') \arrow[ur, "G\rho'_{FX}"] \arrow[d, "G(id_{FX} \otimes \mathsf{m}_I)"] & \\
GFX \otimes GFI \arrow[r, "\mathsf{n}_{FX,FI}"'] & G(FX \triangleright FI) \arrow[r, "G\mathsf{m}_{X,I}"'] & GF(X \otimes I) \arrow[uu, "GF\rho_X"']
\end{tikzcd}
$$

2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal by definition and thus the following two diagrams commute.

$$
\begin{tikzcd}
X \otimes Y \arrow[r, equal] \arrow[d, "\eta_X \otimes \eta_Y"'] & X \otimes Y \arrow[d, "\eta_{X \otimes Y}"] \\
TX \otimes TY \arrow[r, "\mathsf{t}_{X,Y}"'] & T(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[rr, "\eta_I"] \arrow[dr, equal] & & TI \\
& I \arrow[ur, "\mathsf{t}_I"'] &
\end{tikzcd}
$$

3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ. Thus the following diagrams commute.

$$
\begin{tikzcd}
T^2X \otimes T^2Y \arrow[r, "\mathsf{t}_{TX,TY}"] \arrow[d, "\mu_X \otimes \mu_Y"'] & T(TX \otimes TY) \arrow[r, "T\mathsf{t}_{X,Y}"] & T^2(X \otimes Y) \arrow[d, "\mu_{X \otimes Y}"] \\
TX \otimes TY \arrow[rr, "\mathsf{t}_{X,Y}"'] & & T(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2 I \arrow[r, "\mu_I"] & TI \\
TI \arrow[u, "T\mathsf{t}_I"] & I \arrow[l, "\mathsf{t}_I"] \arrow[u, "\mathsf{t}_I"']
\end{tikzcd}
$$

◀

However, the monad is not symmetric becauase the following diagram does not commute, for the Lambek category $\mathcal{L}$ is not symmetric.

$$
\begin{tikzcd}
GFX \otimes GFY \arrow[r, "\mathsf{ex}_{GFX,GFY}"] \arrow[d, "\mathsf{n}_{FX,FY}"'] & GFY \otimes GFX \arrow[r, "\mathsf{n}_{FY,FX}"] & G(FY \triangleright FX) \arrow[d, "G\mathsf{m}_{Y,X}"] \\
G(FX \triangleright FY) \arrow[r, "G\mathsf{m}_{X,Y}"'] & GF(X \otimes Y) \arrow[r, "GF\mathsf{ex}_{X,Y}"'] & GF(Y \otimes X)
\end{tikzcd}
$$

▶ **Lemma 20.** *The monad on the symmetric monoidal closed category in LAM is strong.*

Proof. Let $(\mathcal{C}, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM, where $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is symmetric monoidal closed, $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ is Lambek. In Lemma 19, we have proved that the monad $(T = GF, \eta, \mu)$ is monoidal with the natural transformation $\mathsf{t}_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ and the morphism $\mathsf{t}_I : I \to TI$. We define the tensorial strength $\tau_{X,Y} : X \otimes TY \to T(X \otimes Y)$ as $\tau_{X,Y} = \mathsf{t}_{X,Y} \circ (\eta_X \otimes id_{TY})$.
Since η is a monoidal natural transformation, we have $\eta_I = G\mathsf{m}_I \circ \mathsf{n}_{I'}$. Therefore $\eta_I = \mathsf{t}_I$. Thus the following diagram commutes because T is monoidal, where the composition $\mathsf{t}_{I,X} \circ (\mathsf{t}_I \otimes id_{TX})$ is the definition of $\tau_{I,X}$. So the first triangle in Defition 13 commutes.

$$
\begin{tikzcd}
I \otimes TX \arrow[r, "\mathsf{t}_I \otimes id_{TX}"] \arrow[d, "\lambda_{TX}"'] & TI \otimes TX \arrow[d, "\mathsf{t}_{I,X}"] \\
TX & T(I \otimes X) \arrow[l, "T\lambda_X"]
\end{tikzcd}
$$

Similarly, by using the definition of τ, the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "id_X \otimes \eta_Y"] \arrow[d, "\eta_{X \otimes Y}"'] \arrow[dr, "\eta_X \otimes \eta_Y" description] & X \otimes TY \arrow[d, "\eta_X \otimes id_{TY}"] \\
T(X \otimes Y) & TX \otimes TY \arrow[l, "\mathsf{t}_{X,Y}"]
\end{tikzcd}$$

The first pentagon in the definition commutes by the following commutative diagrams, because η and α are natural transformations and T is monoidal:

$$\begin{tikzcd}
(X \otimes Y) \otimes TZ \arrow[r, "\eta_{X \otimes Y} \otimes id_{TZ}"] \arrow[d, "\alpha_{X,Y,TZ}"'] \arrow[dr, "(\eta_X \otimes \eta_Y) \otimes id_{TZ}" description] & T(X \otimes Y) \otimes TZ \arrow[r, "\mathsf{t}_{X \otimes Y,Z}"] & T((X \otimes Y) \otimes Z) \arrow[d, "T\alpha_{X,Y,Z}"] \\
X \otimes (Y \otimes TZ) \arrow[dr, "\eta_X \otimes (\eta_Y \otimes id_{TZ})" description] \arrow[dd, "id_X \otimes (\eta_Y \otimes id_{TZ})"'] & (TX \otimes TY) \otimes TZ \arrow[u, "\mathsf{t}_{X,Y} \otimes id_{TZ}" description] \arrow[d, "\alpha_{TX,TY,TZ}" description] & T(X \otimes (Y \otimes Z)) \\
 & TX \otimes (TY \otimes TZ) \arrow[dr, "id_{TX} \otimes \mathsf{t}_{Y,Z}" description] & \\
X \otimes (TY \otimes TZ) \arrow[r, "id_X \otimes \mathsf{t}_{Y,Z}"'] \arrow[ur, "\eta_X \otimes id_{TY \otimes TZ}" description] & X \otimes T(Y \otimes Z) \arrow[r, "\eta_X \otimes id_{T(Y \otimes Z)}"'] & TX \otimes T(Y \otimes Z) \arrow[uu, "\mathsf{t}_{X,Y \otimes Z}"']
\end{tikzcd}$$

The last diagram in the definition commtues by the following commutative diagram, because T is a monad, t is a natural transformation, and μ is a monoidal natural transformation:

$$\begin{tikzcd}
X \otimes T^2Y \arrow[r, "\eta_X \otimes id_{T^2Y}"] \arrow[d, "id_X \otimes \mu_Y"'] & TX \otimes T^2Y \arrow[rr, "\mathsf{t}_{X,TY}"] \arrow[d, equal] \arrow[dr, "T\eta_X \otimes id_{T^2Y}" description] \arrow[ddl, "id_{TX} \otimes \mu_Y" description] & & T(X \otimes TY) \arrow[d, "T(\eta_X \otimes id_{TY})"] \\
X \otimes TY \arrow[d, "\eta_X \otimes id_{TY}"'] & TX \otimes T^2Y \arrow[dl, "id_{TX} \otimes \mu_Y" description] & T^2X \otimes T^2Y \arrow[l, "\mu_X \otimes id_{T^2Y}" description] \arrow[r, "\mathsf{t}_{TX,TY}" description] \arrow[dll, "\mu_X \otimes \mu_Y" description] & T(TX \otimes TY) \arrow[d, "T\mathsf{t}_{X,Y}"] \\
TX \otimes TY \arrow[r, "\mathsf{t}_{X,Y}"'] & T(X \otimes Y) & & T^2(X \otimes Y) \arrow[ll, "\mu_{X \otimes Y}"]
\end{tikzcd}$$

◀

The following lemma is adopted from [].

► **Lemma 21.** *Let $\mathcal{M}$ be a symmetric monoidal category and T be a strong monad on $\mathcal{M}$. Then T is commutative iff it is symmetric.*

► **Theorem 22.** *The monad on the SMCC in LAM is strong but non-commutative.*

Proof. The proof is obvious. Based on Lemma 20 and Lemma 21, the monad is non-commutative. ◀

4.3 Comonad on $\mathcal{L}$

► **Lemma 23.** *The comonad on the Lambek category in a LAM is monoidal.*

Proof. We define the comonad $(S, \varepsilon : S \to id_{\mathcal{L}}, \delta : S \to S^2)$ on the Lambek category $\mathcal{L}$ as:

$$S = FG \qquad \varepsilon_A : SA \to A \qquad \delta_A = F\eta_{GA} : SA \to S^2A$$

Thus, we have natural transformation s and morphism s_I defined as:

$$\mathsf{s}_{A,B} = F\mathsf{n}_{A,B} \circ \mathsf{m}_{GA,GB} : SA \triangleright SB \to SA \triangleright SB \qquad \mathsf{s}_I = F\mathsf{n}_{I'} \circ \mathsf{m}_I : I' \to SI'$$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$\begin{tikzcd}
(SA \triangleright SB) \triangleright SC \arrow[r, "\alpha'_{SA,SB,SC}"] \arrow[d, "\mathsf{s}_{A,B} \triangleright id_{SC}"'] & SA \triangleright (SB \triangleright SC) \arrow[r, "id_{SA} \triangleright \mathsf{s}_{B,C}"] & SA \triangleright S(B \triangleright C) \arrow[d, "\mathsf{s}_{A,B \triangleright C}"] \\
S(A \triangleright B) \triangleright SC \arrow[r, "\mathsf{s}_{A \triangleright B,C}"'] & S((A \triangleright B) \triangleright C) \arrow[r, "S\alpha'_{A,B,C}"'] & S(A \triangleright (B \triangleright C))
\end{tikzcd}$$

$$\begin{tikzcd}
I' \triangleright SA \arrow[r, "\lambda'_{SA}"] \arrow[d, "\mathsf{s}_{I'} \triangleright id_{SA}"'] & SA \\
SI' \triangleright SA \arrow[r, "\mathsf{s}_{I',A}"'] & S(I' \triangleright A) \arrow[u, "S\lambda'_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
SA \triangleright I' \arrow[r, "\rho'_{SA}"] \arrow[d, "id'_{SA} \triangleright \mathsf{s}_{I'}"'] & SA \\
SA \triangleright SI' \arrow[r, "\mathsf{s}_{A,I'}"'] & S(A \triangleright I') \arrow[u, "S\rho'_A"']
\end{tikzcd}$$

2. ε is a monoidal natural transformation:

$$\begin{tikzcd}
SA \triangleright SB \arrow[r, "\mathsf{s}_{A,B}"] \arrow[d, "\varepsilon_A \triangleright \varepsilon_B"'] & S(A \triangleright B) \arrow[d, "\varepsilon_{A \triangleright B}"] \\
A \triangleright B \arrow[r, equal] & A \triangleright B
\end{tikzcd}
\qquad
\begin{tikzcd}
SI' \arrow[rr, "\varepsilon_{I'}"] & & I' \\
& I' \arrow[ul, "\mathsf{s}_{I'}"] \arrow[ur, equal] &
\end{tikzcd}$$

3. δ is a monoidal natural transformation:

$$\begin{tikzcd}
SA \triangleright SA \arrow[rr, "\mathsf{s}_{A,B}"] \arrow[d, "\delta_A \triangleright \delta_B"'] & & S(A \triangleright B) \arrow[d, "\delta_{A \triangleright B}"] \\
S^2A \triangleright S^2B \arrow[r, "\mathsf{s}_{SA,SB}"] & S(SA \triangleright SB) \arrow[r, "S\mathsf{s}_{A,B}"] & S^2(A \triangleright B)
\end{tikzcd}
\qquad
\begin{tikzcd}
SI' \arrow[r, "\delta_{I'}"] & S^2I' \\
I' \arrow[u, "\mathsf{s}_{I'}"] \arrow[r, "\mathsf{s}_{I'}"'] & SI' \arrow[u, "S\mathsf{s}_{I'}"']
\end{tikzcd}$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma 19. We do not include the proof here for simplicity. ◀

We then show that the co-Eilenberg-Moore category of the comonad S is symmetric monoidal closed.

▶ **Definition 24.** Let (S, ε, δ) be a comonad on a category $\mathcal{L}$. Then the **co-Eilenberg-Moore category $\mathcal{L}^S$** of the comonad has

- as objects the S-coalgebras $(A, h_A : A \to SA)$, where A is an object in $\mathcal{L}$, s.t. the following diagrams commute:

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] & SA \arrow[d, "\delta_A"] \\
SA \arrow[r, "Sh_A"'] & S^2A
\end{tikzcd}
\qquad
\begin{tikzcd}
& SA \arrow[dr, "\varepsilon_A"] & \\
A \arrow[ur, "h_A"] \arrow[rr, equal] & & A
\end{tikzcd}$$

- as morphisms the coalgebra morphisms, i.e. morphisms $f : (A, h_A) \to (B, h_B)$ between coalgebras s.t. the diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h_A"'] & B \arrow[d, "h_B"] \\
SA \arrow[r, "Sf"'] & SB
\end{tikzcd}$$

▶ **Lemma 25.** *Given a LAM $(\mathcal{C}, \mathcal{L}, F, G, \eta, \varepsilon)$ and the comonad S on $\mathcal{L}$, the co-Eilenberg-Moore category $\mathcal{L}^S$ has an exchange natural transformation $\mathsf{ex}^S_{A,B} : A \triangleright B \to B \triangleright A$.*

Proof. We define the exchange $\mathsf{ex}^S_{A,B} : A \triangleright B \to B \triangleright A$ as

$$A \triangleright B \xrightarrow{h_A \triangleright h_B} FGA \triangleright FGB \xrightarrow{\mathsf{m}_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\mathsf{ex}_{GA,GB}} F(GB \otimes GA) \xrightarrow{F\mathsf{n}_{B,A}} FG(B \triangleright A) \xrightarrow{\varepsilon_{B \triangleright A}} B \triangleright A$$

in which (F, m) and (G, n) are monoidal functors, and ex is the exchange for $\mathcal{C}$. Then ex^S is a natural transformation because the following diagrams commute for morphisms $f : A \to A'$ and $g : B \to B'$:

$$\begin{tikzcd}
A \triangleright B \arrow[r, "h_A \triangleright h_B"] \arrow[d, "f \triangleright g"'] & FGA \triangleright FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[d, "FGf \triangleright FGg"] & F(GA \otimes GB) \arrow[r, "F\mathsf{ex}_{A,B}"] \arrow[d, "F(Gf \otimes Gg)"] & F(GB \otimes GA) \arrow[r, "F\mathsf{n}_{B,A}"] \arrow[d, "F(Gg \otimes Gf)"] & FG(B \triangleright A) \arrow[r, "\varepsilon_{B \triangleright A}"] \arrow[d, "FG(g \triangleright f)"] & B \triangleright A \arrow[d, "g \triangleright f"] \\
A' \triangleright B' \arrow[r, "h_{A'} \triangleright h_{B'}"'] & FGA' \triangleright FGB' \arrow[r, "\mathsf{m}_{GA',GB'}"'] & F(GA' \otimes GB') \arrow[r, "F\mathsf{ex}_{A',B'}"'] & F(GB' \otimes GA') \arrow[r, "F\mathsf{n}_{B',A'}"'] & FG(B' \triangleright A') \arrow[r, "\varepsilon_{B' \triangleright A'}"'] & B' \triangleright A'
\end{tikzcd}$$

◀

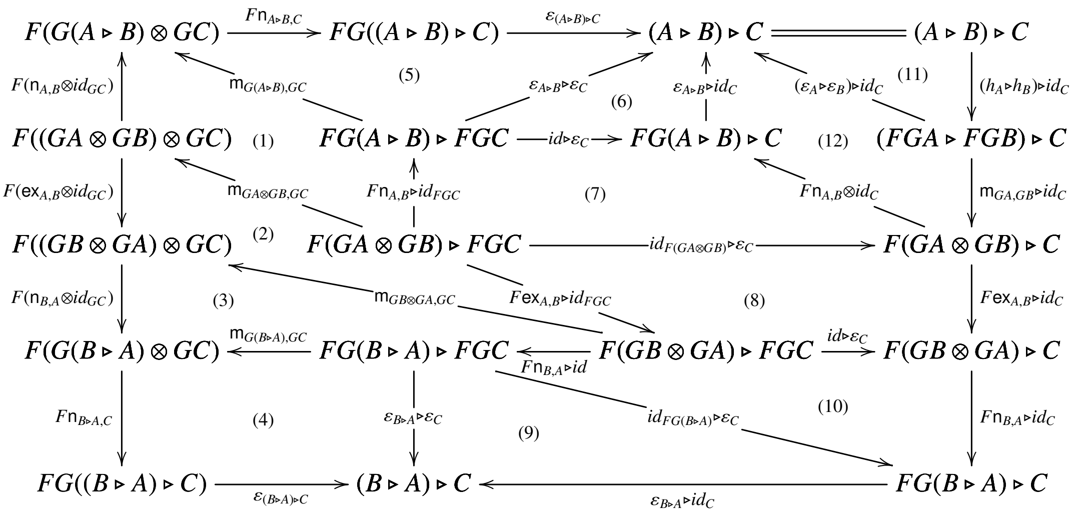
▶ **Lemma 26.** *The following diagrams commute in the co-Eilenberg-Moore category $\mathcal{L}^S$:*

$$\begin{tikzcd}
F((GA \otimes GB) \otimes GC) \arrow[r, "F(\mathsf{n}_{A,B} \otimes id_{GC})"] \arrow[d, "F(\mathsf{ex}_{A,B} \otimes id_{GC})"'] & F(G(A \triangleright B) \otimes GC) \arrow[r, "F(\mathsf{ex}_{A,B} \otimes id_{GC})"] & FG((A \triangleright B) \triangleright C) \arrow[d, "\varepsilon_{(A \triangleright B) \triangleright C}"] \\
F(G(B \triangleright A) \otimes GC) \arrow[d, "F(\mathsf{n}_{B,A} \otimes id_{GC})"'] & & (A \triangleright B) \triangleright C \arrow[d, "\mathsf{ex}^S_{A,B} \triangleright id_C"] \\
F(G(B \triangleright A) \otimes GC) \arrow[r, "F\mathsf{n}_{B \triangleright A,C}"'] & FG((B \triangleright A) \triangleright C) \arrow[r, "\varepsilon_{(B \triangleright A) \triangleright C}"'] & (B \triangleright A) \triangleright C
\end{tikzcd}$$

$$\begin{tikzcd}
F(GB \otimes (GC \otimes GA)) \arrow[r, "F(id_{GB} \otimes \mathsf{n}_{C,A})"] \arrow[d, "F(id_{GB} \otimes \mathsf{ex}_{C,A})"'] & F(GB \otimes G(C \triangleright A)) \arrow[r, "F\mathsf{n}_{B,C \triangleright A}"] & FG(B \triangleright (C \triangleright A)) \arrow[d, "\varepsilon_{B \triangleright (C \triangleright A)}"] \\
F(GB \otimes (GA \otimes GC)) \arrow[d, "F(id_{GB} \otimes \mathsf{n}_{A,C})"'] & & B \triangleright (C \triangleright A) \arrow[d, "id_A \triangleright \mathsf{ex}^S_{C,A}"] \\
F(GB \otimes G(A \triangleright C)) \arrow[r, "F\mathsf{n}_{B,A \triangleright C}"'] & FG(B \triangleright (A \triangleright C)) \arrow[r, "\varepsilon_{B \triangleright (A \triangleright C)}"'] & B \triangleright (A \triangleright C)
\end{tikzcd}$$

Proof. We only write the proof for the first diagram. The proof for the second one is similar. (1), (2), (3)–naturality of m; (4)–F is monoidal; (5), (12)–ε is monoidal; (6), (7), (8), (9), (10)–obvious;

(11)–coalgebra.

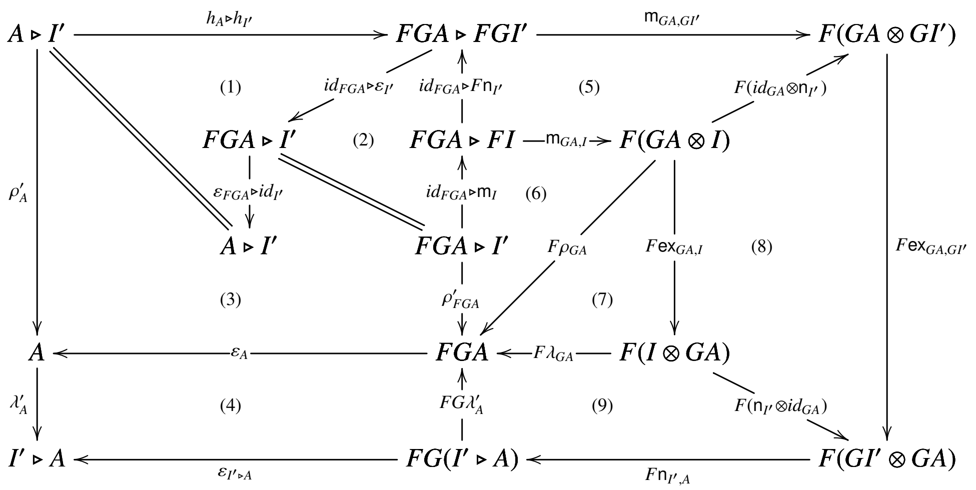


◀

▶ **Theorem 27.** *The co-Eilenberg-Moore category $\mathcal{L}^S$ of S is symmetric monoidal closed.*

Proof. Let $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ be the Lambek cateogry in a SMCC-Lambek model and S be the comonad on $\mathcal{L}$. Since $\mathcal{L}$ is a Lambek category, it is obvious that $\mathcal{L}$ is also Lambek. By Corollary 6, we only need to prove the exchange defined in Lemma 25 satisfies the three commutative diagrams in Definition 3.

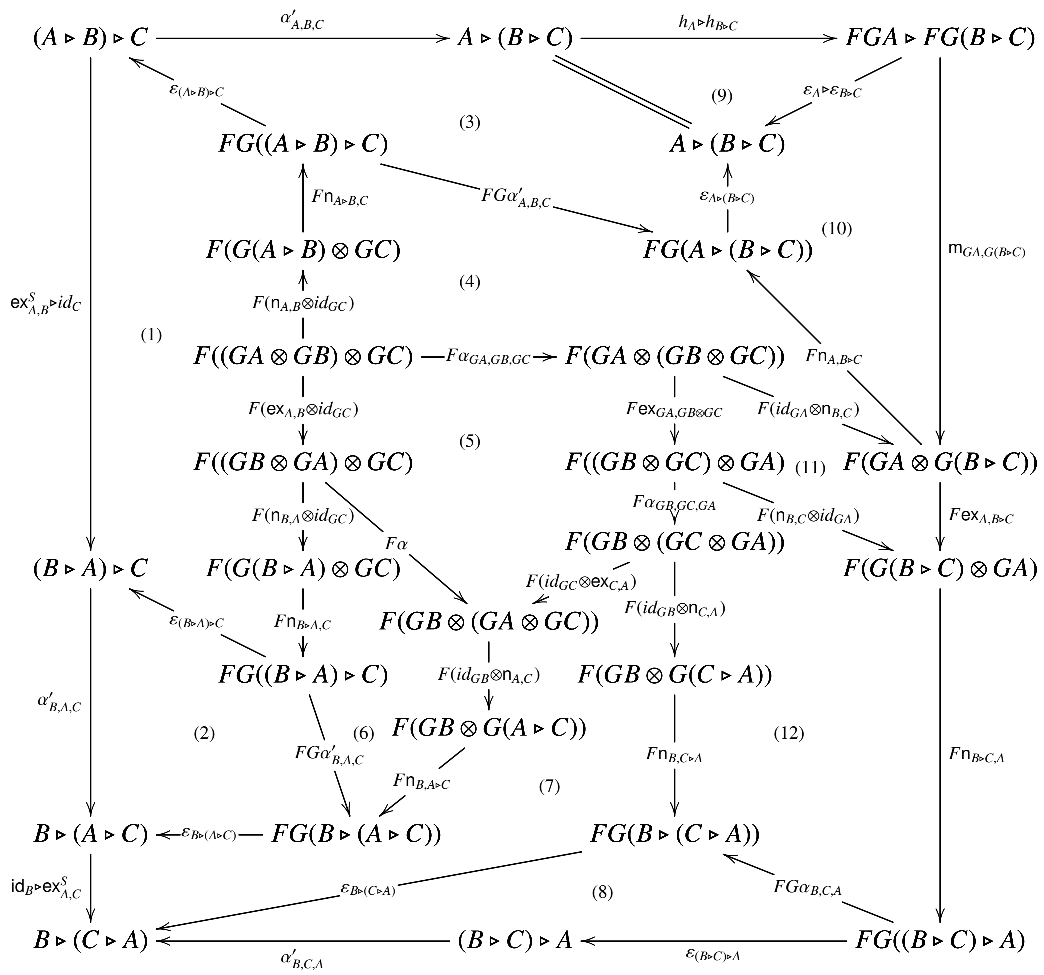
The first triangle in Definition 3 commutes as follows: (1)–coalgebra; (2)–ε is monoidal; (3)–naturality of ρ; (4)–naturality of ε; (5)–naturality of m; (6)–F is monoidal; (7)–C is symmetric; (8)–naturality of ex; (9)–G is monoidal.



The second triangle in the proof commutes as follows: (1) and (5)–coalgebra; (2) and (4)–ε is monoidal; (3)–C is symmetric.

$$
\begin{tikzcd}
A \triangleright B \arrow[r, "h_A \triangleright h_B"] \arrow[dd, equal] \arrow[ddr, phantom, "(1)"] & FGA \triangleright FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[ddl, "\varepsilon_A \triangleright \varepsilon_B"'] \arrow[ddr, phantom, "(2)"] & F(GA \otimes GB) \arrow[r, "F\mathsf{ex}_{A,B}"] \arrow[dd, equal] \arrow[ddr, phantom, "(3)"] & F(GB \otimes GA) \arrow[rr, "F\mathsf{n}_{B,A}"] \arrow[dd, equal] \arrow[ddr, phantom, "(4)"] & & FG(B \triangleright A) \arrow[dd, "\varepsilon_{B \triangleright A}"] \\
& & & & B \triangleright A \arrow[dr, equal] \arrow[dr, phantom, "(5)"'] & \\
A \triangleright B & FG(A \triangleright B) \arrow[l, "\varepsilon_{A \triangleright B}"] & F(GA \otimes GB) \arrow[l, "F\mathsf{n}_{A,B}"] & F(GB \otimes GA) \arrow[l, "F\mathsf{ex}_{B,A}"] & FGB \triangleright FGA \arrow[l, "\mathsf{m}_{GB,GA}"] \arrow[u, "\varepsilon_B \triangleright \varepsilon_A"] & B \triangleright A \arrow[l, "h_A \triangleright h_A"]
\end{tikzcd}
$$

The third diagram commutes as follows: (1) and (7)–Lemma 26; (2)–naturality of α'; (3) and (8)–naturality of ε; (4), (6) and (12)–G is a monoidal functor; (5)–C is symmetrical monoidal closed; (9)–coalgebra; (10)–ε is a monoidal natural transformation; (11)–naturality of ex.



◀

5 Non-Commutative Linear Logic

In a LAM, the SMCC C models the commutative linear logic and the Lambeck category $\mathcal{L}$ models the non-commutative variant. In Section 5.1, we will present the term assignment for sequent calculus of both sides and prove the cut elimination theorem. In Section 5.2, we present the term assignment for natural deduction of both sides and prove the logic is strongly normalizing.

A sequent in the commutative side is of the form

The typing contexts in the commutative side are of form [[P, I]], each of which is a multiset of variables with types X, Y, Z, ...

5.1 Sequent Calculus

The term assignment for sequent calculus of the commutative part of the model, i.e. the SMCC of the adjunction, is defined in Figure **??**. And the term assignme for the non-commutative part, i.e. the Lambek category of the adjunction, is defined in Figure **??**. [[P]] and [[I]] are contexts for the non-commutative part and they are lists. [[G]] and [[D]] are contexts for the commutative part and they are multisets, therefore the following exchange rules are implicit.

Figure 1 Commutative Part

5.2 Natural Deduction

The term assignment for natural deduction of the commutative part of the model, i.e. the SMCC of the adjunction, is defined in Figure 2. And the term assignme for the non-commutative part, i.e. the Lambek category of the adjunction, is defined in Figure 3. [[P]] and [[I]] are contexts for the commutative part and they are multisets. [[G]] and [[D]] are contexts for the mix of the commutative part and the non-commutative part, and they are lists. Therefore the following exchange rule is implicit.

$$\frac{\Phi, x : X, y : Y, \Psi \vdash_{C} t : Z}{\Phi, z : Y, w : X, \Psi \vdash_{C} \mathsf{ex}\, w, z\, \mathsf{with}\, x, y\, \mathsf{in}\, t : Z}\ \text{T\_BETA}$$

$$\frac{}{x : X \vdash_{C} x : X}\ \text{T\_ID} \qquad \frac{}{\cdot \vdash_{C} \mathsf{trivT} : \mathsf{UnitT}}\ \text{T\_UNITI} \qquad \frac{\Phi \vdash_{C} t_1 : \mathsf{UnitT} \quad \Psi \vdash_{C} t_2 : Y}{\Phi, \Psi \vdash_{C} \mathsf{let}\, t_1 : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, t_2 : Y}\ \text{T\_UNITE}$$

$$\frac{\Phi \vdash_{C} t_1 : X \quad \Psi \vdash_{C} t_2 : Y}{\Phi, \Psi \vdash_{C} t_1 \otimes t_2 : X \otimes Y}\ \text{T\_TENI} \qquad \frac{\Phi \vdash_{C} t_1 : X \otimes Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_{C} t_2 : Z}{\Psi_1, \Phi, \Psi_2 \vdash_{C} \mathsf{let}\, t_1 : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2 : Z}\ \text{T\_TENE}$$

$$\frac{\Phi, x : X \vdash_{C} t : Y}{\Phi \vdash_{C} \lambda x : X.t : X \multimap Y}\ \text{T\_IMPI} \qquad \frac{\Phi \vdash_{C} t_1 : X \multimap Y \quad \Psi \vdash_{C} t_2 : X}{\Phi, \Psi \vdash_{C} \mathsf{app}\, t_1\, t_2 : Y}\ \text{T\_IMPE} \qquad \frac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_{C} \mathsf{G}s : \mathsf{G}A}\ \text{T\_GI}$$

$$\frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : Y, w : X, \Delta \vdash_{\mathcal{L}} \mathsf{ex}\, w, z\, \mathsf{with}\, x, y\, \mathsf{in}\, s : A}\ \text{S\_BETA}$$

Figure 2 Commutative Part

We could derive exchange comonadically as follows:

$$\frac{}{x : A \vdash_{\mathcal{L}} x : A}\ \text{S\_ID} \qquad \frac{}{\cdot \vdash_{\mathcal{L}} \mathsf{trivS} : \mathsf{UnitS}}\ \text{S\_UNITI} \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_2 : A}\ \text{S\_UNITE1}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_2 : A}\ \text{S\_UNITE1} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : \mathsf{UnitT} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, s : A}\ \text{S\_UNITE2}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Delta \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B}\ \text{S\_TENI} \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C}\ \text{S\_TENE1}$$

$$\frac{\Phi \vdash_{\mathcal{C}} t : X \otimes Y \quad \Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, t : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\ \text{S\_TENE2} \qquad \frac{\Gamma, x : A \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A.s : A \rightharpoonup B}\ \text{S\_IMPRI}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_2 : B}\ \text{S\_IMPRE} \qquad \frac{x : A, \Gamma \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A.s : B \leftharpoonup A}\ \text{S\_IMPLI}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftharpoonup A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_2 : B}\ \text{S\_IMPLE} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : \mathsf{G}A}{\Phi \vdash_{\mathcal{L}} \mathsf{derelict}\, t : A}\ \text{S\_GE} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X}{\Phi \vdash_{\mathcal{L}} \mathsf{F}t : \mathsf{F}X}\ \text{S\_FI}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} y : \mathsf{F}X \quad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\, \mathsf{be}\, y\, \mathsf{in}\, s : A}\ \text{S\_FE}$$

Figure 3 Non-Commutative Part

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$[[z : h(FGfA)(>)FGfB| - lz : h(FGfA)(>)FGfB]]$}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$[[x2 : FGfA| - lx2 : FGfA]]$}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$[[y2 : FGfB| - ly2 : FGfB]]$}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$[[y0 : GfB| - cy0 : GfB]]$}
\RightLabel{FI}
\UnaryInfC{$[[y0 : GfB| - lFy0 : FGfB]]$}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$[[x0 : GfA| - cx0 : GfA]]$}
\RightLabel{FI}
\UnaryInfC{$[[x0 : GfA| - lFx0 : FGfA]]$}
\RightLabel{FE}
\BinaryInfC{$[[y0 : GfB, x0 : GfA| - lh(Fy0)(>)Fx0 : h(FGfB)(>)FGfA]]$}
\RightLabel{BETA}
\UnaryInfC{$[[x1 : GfA, y1 : GfB| - lexy1, x1withy0, x0in(h(Fy0)(>)Fx0) : h(FGfB)(>)FGfA]]$}
\RightLabel{FE}
\BinaryInfC{$[[x1 : GfA, y2 : FGfB| - lletFy1 : FGfBbey2in(exy1, x1withy0, x0in(h(Fy0)(>)Fx0)) : h(FGfB)(>)FGfA]]$}
\RightLabel{FE}
\BinaryInfC{$[[x2 : FGfA, y2 : FGfB| - lletFx1 : FGfAbex2in(letFy1 : FGfBbey2in(exy1, x1withy0, x0in(h(Fy0)(>)Fx0))) : h(FGfB)(>)FGfA]]$}
\RightLabel{TENE2}
\BinaryInfC{$[[z : h(FGfA)(>)FGfB| - lletz : h(FGfA)(>)FGfBbex2(>)y2in(letFx1 : FGfAbex2in(letFy1 : FGfBbey2in(exy1, x1withy0, x0in(h(Fy0)(>)Fx0)))) : h(FGfB)(>)FGfA]]$}
\RightLabel{IMPRI}
\UnaryInfC{$[[.| - lr_z : h(FGfA)(>)FGfB.letz : h(FGfA)(>)FGfBbex2(>)y2in(letFx1 : FGfAbex2in(letFy1 : FGfBbey2in(exy1, x1withy0, x0in(h(Fy0)(>)Fx0)))) : (h(FGfA)(>)FGfB)- > (h(FGfB)(>)FGfA)]]$}
\end{prooftree}
$$

We also have the three cut rules derivable in the natural deduction:

$$\frac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi_1, x : X, \Psi_2 \vdash_{\mathcal{C}} t_2 : Y}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} [t_1/x]t_2 : Y}\ \text{T\_CUT} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X \quad \Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_1 \vdash_{\mathcal{L}} [t/x]s : A}\ \text{S\_CUT1} \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta_1, x : A, \Delta_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [s_1/x]s_2 : B}\ \text{S\_CUT2}$$

$$\frac{[[I| - ct : X]]}{[[I| - lFt : FX]]}\ \text{FI}$$

6 Combining with Benton's Adjoint Model

7 Applications

8 Conclusion

TODO

A Appendix