

Deriving the cut rules in Elle-ND:

A Term Assignment for Natural Deduction Formulation of Elle

vars, n, a, x, y, z, w, m, o

ivar, i, k, j, l

const, b

A, B, C ::=
 | B
 | UnitS
 | $A \triangleright B$
 | $A \multimap B$
 | $A \multimapleft B$
 | FX

X, Y, Z ::=
 | B
 | UnitT
 | $X \otimes Y$
 | $X \multimapright Y$
 | GA

T ::=
 | A
 | X

p ::=
 | ★
 | x
 | trivT
 | trivS
 | $p \otimes p'$
 | $p \triangleright p'$
 | Fp
 | Gp

s ::=
 | x
 | b
 | trivS
 | let $s_1 : T$ be p in s_2
 | let $t : T$ be p in s

$$\begin{array}{l|l}
| & s_1 \triangleright s_2 \\
| & \lambda_l x : A.s \\
| & \lambda_r x : A.s \\
| & \text{app}_l s_1 s_2 \\
| & \text{app}_r s_1 s_2 \\
| & \text{derelict } t \\
| & \text{ex } s_1, s_2 \text{ with } x_1, x_2 \text{ in } s_3 \\
| & (s) \\
| & F_t
\end{array} \quad S$$

$$\begin{array}{l|l}
t & ::= \\
| & x \\
| & b \\
| & \text{trivT} \\
| & \text{let } t_1 : X \text{ be } p \text{ in } t_2 \\
| & t_1 \otimes t_2 \\
| & \lambda x : X.t \\
| & \text{app } t_1 t_2 \\
| & \text{ex } t_1, t_2 \text{ with } x_1, x_2 \text{ in } t_3 \\
| & (t) \\
| & Gs
\end{array} \quad S$$

$$\begin{array}{l|l}
\Phi, \Psi & ::= \\
| & \cdot \\
| & \Phi_1, \Phi_2 \\
| & x : X \\
| & (\Phi)
\end{array} \quad S$$

$$\begin{array}{l|l}
\Gamma, \Delta & ::= \\
| & \cdot \\
| & x : A \\
| & \Phi \\
| & \Gamma_1, \Gamma_2 \\
| & (\Gamma)
\end{array} \quad S$$

$\Phi \vdash_C t : X$

$$\begin{array}{c}
\frac{}{x : X \vdash_C x : X} \quad \text{T_ID} \\
\\
\frac{}{\cdot \vdash_C \text{trivT} : \text{UnitT}} \quad \text{T_UNITI} \\
\\
\frac{\Phi \vdash_C t_1 : \text{UnitT} \quad \Psi \vdash_C t_2 : Y}{\Phi, \Psi \vdash_C \text{let } t_1 : \text{UnitT} \text{ be } \text{trivT} \text{ in } t_2 : Y} \quad \text{T_UNIT E} \\
\\
\frac{\Phi \vdash_C t_1 : X \quad \Psi \vdash_C t_2 : Y}{\Phi, \Psi \vdash_C t_1 \otimes t_2 : X \otimes Y} \quad \text{T_TENI} \\
\\
\frac{\Phi \vdash_C t_1 : X \otimes Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_2 : Z}{\Psi_1, \Phi, \Psi_2 \vdash_C \text{let } t_1 : X \otimes Y \text{ be } x \otimes y \text{ in } t_2 : Z} \quad \text{T_TENE}
\end{array}$$

$$\begin{array}{c}
\frac{\Phi, x : X \vdash_C t : Y}{\Phi \vdash_C \lambda x : X. t : X \multimap Y} \quad \text{T_IMPI} \\
\frac{\Phi \vdash_C t_1 : X \multimap Y \quad \Psi \vdash_C t_2 : X}{\Phi, \Psi \vdash_C \text{app } t_1 t_2 : Y} \quad \text{T_IMPE} \\
\frac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_C \text{Gs} : \text{GA}} \quad \text{T_GI} \\
\frac{\Phi, x : X, y : Y, \Psi \vdash_C t : Z}{\Phi, z : Y, w : X, \Psi \vdash_C \text{ex } w, z \text{ with } x, y \text{ in } t : Z} \quad \text{T_BETA} \\
\frac{\Phi \vdash_C t_1 : X \quad \Psi_1, x : X, \Psi_2 \vdash_C t_2 : Y}{\Psi_1, \Phi, \Psi_2 \vdash_C [t_1/x]t_2 : Y} \quad \text{T_CUT}
\end{array}$$

$$\boxed{\Gamma \vdash_{\mathcal{L}} s : A}$$

$$\begin{array}{c}
\frac{}{x : A \vdash_{\mathcal{L}} x : A} \quad \text{S_ID} \\
\frac{}{\cdot \vdash_{\mathcal{L}} \text{trivS} : \text{UnitS}} \quad \text{S_UNITI} \\
\frac{\Phi \vdash_C t : \text{UnitT} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi, \Gamma \vdash_{\mathcal{L}} \text{let } t : \text{UnitT} \text{ be } \text{trivT} \text{ in } s : A} \quad \text{S_UNIT E1} \\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : \text{UnitS} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \text{let } s_1 : \text{UnitS} \text{ be } \text{trivS} \text{ in } s_2 : A} \quad \text{S_UNIT E2} \\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Delta \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B} \quad \text{S_TENI} \\
\frac{\Phi \vdash_C t : X \otimes Y \quad \Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} \text{let } t : X \otimes Y \text{ be } x \otimes y \text{ in } s : A} \quad \text{S_TENE1} \\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \text{let } s_1 : A \triangleright B \text{ be } x \triangleright y \text{ in } s_2 : C} \quad \text{S_TENE2} \\
\frac{\Gamma, x : A \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A. s : A \multimap B} \quad \text{S_IMPRI} \\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \multimap B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \text{app}_r s_1 s_2 : B} \quad \text{S_IMPRE} \\
\frac{x : A, \Gamma \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A. s : B \multimap A} \quad \text{S_IMPLI} \\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \multimap A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta, \Gamma \vdash_{\mathcal{L}} \text{app}_l s_1 s_2 : B} \quad \text{S_IMPLE} \\
\frac{\Phi \vdash_C t : X}{\Phi \vdash_{\mathcal{L}} \text{Ft} : \text{FX}} \quad \text{S_FI} \\
\frac{\Gamma \vdash_{\mathcal{L}} y : \text{FX} \quad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \text{let } \text{Ft} : \text{FX} \text{ be } y \text{ in } s : A} \quad \text{S_FE} \\
\frac{\Phi \vdash_C t : \text{GA}}{\Phi \vdash_{\mathcal{L}} \text{derelict } t : A} \quad \text{S_GE}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : Y, w : X, \Delta \vdash_{\mathcal{L}} \text{ex } w, z \text{ with } x, y \text{ in } s : A} \text{S_BETA} \\
\frac{\Phi \vdash_C t : X \quad \Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_1 \vdash_{\mathcal{L}} [t/x]s : A} \text{S_CUT1} \\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta_1, x : A, \Delta_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [s_1/x]s_2 : B} \text{S_CUT2}
\end{array}$$

$$t_1 \rightsquigarrow t_2$$

$$\begin{array}{c}
\frac{}{\text{let } \text{trivT} : \text{UnitT} \text{ be } \text{trivT} \text{ in } t \rightsquigarrow t} \text{TRED_LETU} \\
\frac{}{\text{let } t_1 \otimes t_2 : X \otimes Y \text{ be } x \otimes y \text{ in } t_3 \rightsquigarrow [t_1/x][t_2/y]t_3} \text{TRED_LET T} \\
\frac{}{\text{app } (\lambda x : X. t_1) t_2 \rightsquigarrow [t_2/x]t_1} \text{TRED_LAM} \\
\frac{t_1 \rightsquigarrow t'_1}{\text{app } t_1 t_2 \rightsquigarrow \text{app } t'_1 t_2} \text{TRED_APP1} \\
\frac{t_2 \rightsquigarrow t'_2}{\text{app } t_1 t_2 \rightsquigarrow \text{app } t_1 t'_2} \text{TRED_APP2}
\end{array}$$

$$s_1 \rightsquigarrow s_2$$

$$\begin{array}{c}
\frac{}{\text{let } \text{trivS} : \text{UnitS} \text{ be } \text{trivS} \text{ in } s \rightsquigarrow s} \text{SRED_LETU1} \\
\frac{}{\text{let } \text{trivT} : \text{UnitT} \text{ be } \text{trivT} \text{ in } s \rightsquigarrow s} \text{SRED_LETU2} \\
\frac{}{\text{let } t_1 \otimes t_2 : X \otimes Y \text{ be } x \otimes y \text{ in } s_3 \rightsquigarrow [t_1/x][t_2/y]s_3} \text{SRED_LET T1} \\
\frac{}{\text{let } s_1 \triangleright s_2 : A \triangleright B \text{ be } x \triangleright y \text{ in } s_3 \rightsquigarrow [s_1/x][s_2/y]s_3} \text{SRED_LET T2} \\
\frac{}{\text{let } Fx : FX \text{ be } F y \text{ in } s \rightsquigarrow [y/x]s} \text{SRED_LET F} \\
\frac{}{\text{app}_l (\lambda_l x : A. s_1) s_2 \rightsquigarrow [s_2/x]s_1} \text{SRED_LAM L} \\
\frac{}{\text{app}_r (\lambda_r x : A. s_1) s_2 \rightsquigarrow [s_2/x]s_1} \text{SRED_LAM R} \\
\frac{s_1 \rightsquigarrow s'_1}{\text{app}_l s_1 s_2 \rightsquigarrow \text{app}_l s'_1 s_2} \text{SRED_APPL1} \\
\frac{s_2 \rightsquigarrow s'_2}{\text{app}_l s_1 s_2 \rightsquigarrow \text{app}_l s_1 s'_2} \text{SRED_APPL2} \\
\frac{s_1 \rightsquigarrow s'_1}{\text{app}_r s_1 s_2 \rightsquigarrow \text{app}_r s'_1 s_2} \text{SRED_APPR1} \\
\frac{s_2 \rightsquigarrow s'_2}{\text{app}_r s_1 s_2 \rightsquigarrow \text{app}_r s_1 s'_2} \text{SRED_APPR2} \\
\frac{}{\text{derelict } Gs \rightsquigarrow s} \text{SRED_DERELICT}
\end{array}$$