# Linear Logic, Monads and Non-commutative Lambda Calculus

#### Jiaming Jiang<sup>1</sup> and Harley Eades III<sup>2</sup>

- 1 Computer Science, Augusta University, Augusta, Georgia, USA heades@augusta.edu
- 2 Computer Science, North Carolina State University, Raleigh, North Carolina, USA jjiang13@ncsu.edu

| _ | _  |     |            |
|---|----|-----|------------|
| Λ | hs | tra | <b>^</b> + |
| н | D5 | 117 | (:1        |

**TODO** 

1998 ACM Subject Classification TODO

Keywords and phrases TODO

Digital Object Identifier 10.4230/LIPIcs...

- 1 Introduction
- 2 Main Ideas
- 3 Category Theory Basics
- ▶ **Definition 1.** A monoidal category  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  is a category  $\mathcal{M}$  consists of
- a bifunctor  $\otimes$  :  $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ , called the tensor product;
- **a** an object *I*, called the unit object;
- three natural isomorphisms  $\alpha$ ,  $\lambda$ , and  $\rho$  with components

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$\lambda_A: I \otimes A \to A$$

$$\rho_A: A \otimes I \to A$$

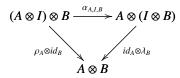
where  $\alpha$  is called associator,  $\lambda$  is left unitor, and  $\rho$  is right unitor,

such that the following diagrams commute for any objects A, B, C in  $\mathcal{M}$ :

$$((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha_{A,B,C} \otimes id_D} (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D)$$

$$\downarrow id_A \otimes \alpha_{B,C,D}$$

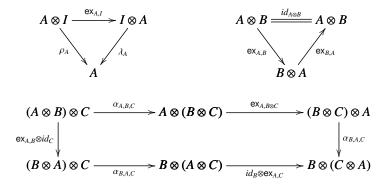
$$(A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C \otimes D}} A \otimes (B \otimes (C \otimes D))$$



▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  equipped with two bifunctors  $\rightarrow$ :  $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$  and  $\leftarrow$ :  $\mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$  that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\operatorname{\mathsf{Hom}}_f(X \otimes A, B) \cong \operatorname{\mathsf{Hom}}_f(X, A \rightharpoonup B)$$
  $\operatorname{\mathsf{Hom}}_f(A \otimes X, B) \cong \operatorname{\mathsf{Hom}}_f(X, B \leftharpoonup A)$ 

▶ **Definition 3.** A symmetric monoidal category (SMCC) is a monoidal category  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  together with a natural transformation with components  $ex_{A,B} : A \otimes B \to B \otimes A$ , called **exchange**, such that the following diagrams commute:



- ▶ **Definition 4.** A **symmetric monoidal closed category**  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  is a symmetric monoidal category equipped with a bifunctor  $\multimap$ :  $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$  that is right adjoint to the tensor product. That is, the following natural bijection  $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$  holds.
- ▶ **Lemma 5.** Let A and B be two objects in a Lambek category with the exchange natural transformation. Then  $(A \rightarrow B) \cong (B \leftarrow A)$ .

**Proof.** First, notice that for any object C we have

$$Hom[C, A \rightarrow B] \cong Hom[C \otimes A, B]$$
  $\mathcal{L}$  is a Lambek category  $\cong Hom[A \otimes C, B]$  By the exchange  $ex_{C,A}$   $\cong Hom[C, B \leftarrow A]$   $\mathcal{L}$  is a Lambek category

Thus,  $A \rightarrow B \cong B \leftarrow A$  by the Yoneda lemma.

- ▶ Corollary 6. A Lambek category with exchange is symmetric monoidal closed.
- ▶ **Definition 7.** Let  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  and  $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$  be monoidal categories. A **monoidal functor**  $(F, \mathsf{m})$  from  $\mathcal{M}$  to  $\mathcal{M}'$  is a functor  $F : \mathcal{M} \to \mathcal{M}'$  together with a morphism  $\mathsf{m}_I : I' \to F(I)$  and a natural transformation  $\mathsf{m}_{A,B} : FA' \otimes FB' \to F(A \otimes B)$ , such that the following diagrams commute for any objects A, B, and C in  $\mathcal{M}$ :

$$(FA \otimes' FB) \otimes' FC \xrightarrow{\alpha'_{FA,FB,FC}} FA \otimes' (FB \otimes' FC) \xrightarrow{id_{FA} \otimes' \mathsf{m}_{A,B}} FA \otimes' F(B \otimes C)$$

$$\downarrow^{\mathsf{m}_{A,B} \otimes' id_{FC}} \downarrow^{\mathsf{m}_{A,B} \otimes' id_{FC}} \downarrow$$

▶ **Definition 8.** Let  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  and  $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$  be symmetric monoidal categories. A **symmetric monoidal functor**  $F : \mathcal{M} \to \mathcal{M}'$  is a monoidal functor  $(F, \mathsf{m})$  that satisfies the following coherence diagram:

$$FA \otimes' FB \xrightarrow{\text{ex}_{FA,FB}} FB \otimes' FA$$

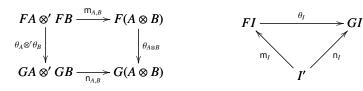
$$\downarrow m_{A,B} \downarrow \qquad \qquad \downarrow m_{B,A}$$

$$F(A \otimes B) \xrightarrow{F \in X_{A,B}} F(B \otimes A)$$

▶ **Definition 9.** An **adjunction** between categories C and  $\mathcal{D}$  consists of two functors  $F: \mathcal{D} \to C$ , called the **left adjoint**, and  $G: C \to \mathcal{D}$ , called the **right adjoint**, and two natural transformations  $\eta: id_{\mathcal{D}} \to GF$ , called the **unit**, and  $\varepsilon: FG \to id_C$ , called the **counit**, such that the following diagrams commute for any object A in C and B in D:



▶ **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category  $\mathcal{M}$  to a monoidal category  $\mathcal{M}'$ . A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation  $\theta : (F, m) \to (G, n)$  such that the following diagrams commute for any objects A and B in  $\mathcal{M}$ :



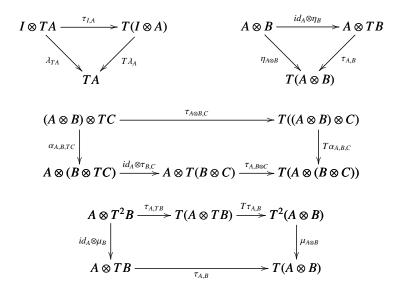
- ▶ **Definition 11.** Let  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  and  $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$  be monoidal categories,  $F : \mathcal{M} \to \mathcal{M}'$  and  $G : \mathcal{M}' \to \mathcal{M}$  be functors. The adjunction  $F : \mathcal{M} \to \mathcal{M}' : G$  is a **monoidal adjunction** if F and G are monoidal functors, and the unit  $\eta$  and the counit  $\varepsilon$  are monoidal natural transformations.
- ▶ **Definition 12.** Let C be a category. A **monad** on C consists of an endofunctor  $T: C \to C$  together with two natural transformations  $\eta: id_C \to T$  and  $\mu: T^2 \to T$ , where  $id_C$  is the identity functor on C, such that the following diagrams commute:



▶ **Definition 13.** Let  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  be a monoidal category and  $(T, \eta, \mu)$  be a monad on  $\mathcal{M}$ . T is a **strong monad** if there is natural transformation  $\tau$ , called the **tensorial strength**, with components

#### XX:4 Linear Logic, Monads and Non-commutative Lambda Calculus

 $\tau_{A,B}: A \otimes TB \to T(A \otimes B)$  such that the following diagrams commute:



- ▶ **Definition 14.** Let  $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$  be a symmetric monoidal category with exchange ex, and  $(T, \eta, \mu)$  be a strong monad on  $\mathcal{M}$ . Then there is a "**twisted**" **tensorial strength**  $\tau'_{A,B}: TA \otimes B \to T(A \otimes B)$  defined as  $\tau'_{A,B} = T$  ex  $\circ \tau_{B,A} \circ$  ex. We can construct a pair of natural transformations  $\Phi$ ,  $\Phi'$  with components  $\Phi_{A,B}, \Phi'_{A,B}: TA \otimes TB \to T(A \otimes B)$  defined as  $\Phi_{A,B} = \mu_{A \otimes B} \circ T \tau'_{A,B} \circ \tau_{TA,B}$  and  $\Phi'_{A,B} = \mu_{A \otimes B} \circ T \tau_{A,B} \circ \tau'_{A,TB}$ . If  $\Phi = \Phi'$ , then the monad T is **commutative**.
- ▶ **Definition 15.** Let  $\mathcal{L}$  be a category. A **comonad** on  $\mathcal{L}$  consists of an endofunctor  $S: \mathcal{L} \to \mathcal{L}$  together with two natural transformations  $\varepsilon: S \to id_{\mathcal{L}}$  and  $\delta: S^2 \to S$  such that the following diagrams commute:



#### 4 Categorical Models

- ▶ Definition 16. A SMCC-Lambek model consists of
- a symmetric monoidal closed category  $(C, \otimes, I, \alpha, \lambda, \rho)$ ;
- a Lambek category  $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$ ;
- **a** monoidal adjunction  $F: C \dashv \mathcal{L}: G$ , where  $F: C \rightarrow \mathcal{L}$  and  $G: \mathcal{L} \rightarrow C$  are monoidal functors.

Thus, in a SMCC-Lambek model, the following four diagrams commute because  $\eta$  and  $\varepsilon$  are monoidal natural transformations:

$$A \otimes B \xrightarrow{id_{A \otimes B}} A \otimes B \qquad I \xrightarrow{\eta_{I}} GFI$$

$$\downarrow^{\eta_{A \otimes \eta_{B}}} \downarrow^{\eta_{A \otimes B}} \qquad \downarrow^{GFI} \qquad \downarrow^{GFI}$$

$$GFA \otimes GFB \xrightarrow{n_{FA,FB}} G(FA \otimes FB) \xrightarrow{Gm_{A,B}} GF(A \otimes B) \qquad I \xrightarrow{n_{I'}} GI'$$

$$FGA \otimes FGB \xrightarrow{\mathsf{m}_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\mathsf{n}_{A,B}} FG(A \otimes B) \qquad FGI' \xrightarrow{\varepsilon_{I'}} I'$$

$$\downarrow_{\varepsilon_{A} \otimes \varepsilon_{B}} \downarrow \qquad \downarrow_{\varepsilon_{A \otimes B}} \qquad \downarrow_{F\mathsf{n}_{I'}} \downarrow \qquad \parallel$$

$$A \otimes B \xrightarrow{\mathsf{m}_{GA,GB}} A \otimes B \qquad FI \underset{\mathsf{m}_{I}}{\longleftarrow} I'$$

And the following two diagrams commute because of the adjunction:



▶ Lemma 17. The monad on the SMCC C in a SMCC-Lambek model is monoidal.

**Proof.** We define the monad T on the C in the adjunction of a SMCC-Lambek model as T = GF, and the two corresponding natural transformations  $\eta : id_C \to T$  and  $\mu : T^2 \to T$  are defined as

$$\eta_A:A\to GFA$$

$$\mu_A = G\varepsilon_{FA} : GFGFA \to GFA$$

where  $\eta$  is the unit and  $\varepsilon : FG \to id_{\mathcal{L}}$  is the counit in the adjunction  $F : C \dashv \mathcal{L} : G$ , and  $(F, \mathsf{m})$  and  $(G, \mathsf{n})$  are monoidal functors.

Thus, we have

$$\mathsf{t}_{A,B} = G\mathsf{m}_{A,B} \circ \mathsf{n}_{FA,FB} : TA \otimes TB \to T(A \otimes B)$$
 
$$\mathsf{t}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'} : I \to TI$$

The monad T being monoidal means

1. T is a monoidal functor, i.e. the following diagrams commute:

$$(TA \otimes TB) \otimes TC \xrightarrow{\alpha_{TA,TB,TC}} TA \otimes (TB \otimes TC) \xrightarrow{id_{TA} \otimes t_{B,C}} TA \otimes T(B \otimes C)$$

$$\downarrow_{t_{A,B} \otimes id_{TC}} \downarrow \qquad \qquad \downarrow_{t_{A,B} \otimes C}$$

$$T(A \otimes B) \otimes TC \xrightarrow{t_{A \otimes B,C}} T((A \otimes B) \otimes C) \xrightarrow{T\alpha_{A,B,C}} T(A \otimes (B \otimes C))$$

$$I \otimes TA \xrightarrow{\lambda_{TA}} TA \qquad \qquad TA \otimes I \xrightarrow{\rho_{TA}} TA$$

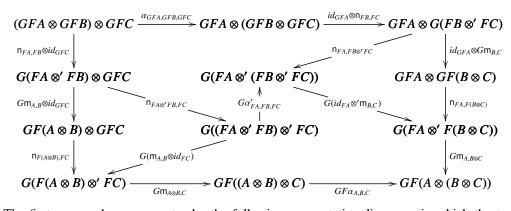
$$\downarrow_{t_{I} \otimes id_{TA}} \downarrow \qquad \qquad \uparrow_{t_{A,A}} TA \qquad \qquad \downarrow_{t_{A,A}} TA \otimes I \xrightarrow{t_{A,A}} T(A \otimes I)$$

$$TA \otimes TI \xrightarrow{t_{A,A}} T(A \otimes I)$$

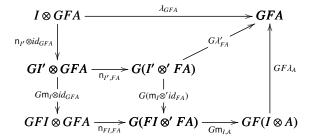
We write GF instead of T in the proof for clarity.

By replacing t with its definition, the first diagram above commutes by the following commutative diagram, where the two hexagons commute because G and F are monoidal functors, and the

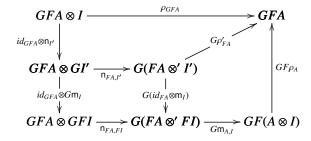
two quadrilaterals commute by the naturality of n.



The first square above commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.



Similarly, the second square above commutes by the following commutative diagram:



2.  $\eta$  is a monoidal natural transformation. In fact, since  $\eta$  is the unit of the monoidal adjunction,  $\eta$  is monoidal and thus the following two diagrams commute.



**3.**  $\mu$  is a monoidal natural transformation. It is obvious that since  $\mu = G\varepsilon_{FA}$  and  $\varepsilon$  is monoidal, so is  $\mu$ . Thus the following diagrams commute.

$$T^{2}A \otimes T^{2}B \xrightarrow{t_{TA,TB}} T(TA \otimes TB) \xrightarrow{Tt_{A,B}} T^{2}(A \otimes B) \qquad T^{2}I \xrightarrow{\mu_{I}} TI$$

$$\downarrow^{\mu_{A} \otimes \mu_{B}} \downarrow^{\mu_{A} \otimes B} \qquad Tt_{I} \uparrow^{t_{I}} \downarrow^{t_{I}} \downarrow^{t_{I}}$$

$$TA \otimes TB \xrightarrow{t_{A,B}} T(A \otimes B) \qquad TI \leftarrow \prod_{t_{I}} I$$

However, the monad is not symmetric because the following diagram does not commute, for the lambek category  $\mathcal{L}$  is not symmetric.

$$\begin{array}{c|c} GFA \otimes GFB \xrightarrow{\exp_{FA,GFB}} GFB \otimes GFA \xrightarrow{n_{FB,FA}} G(FB \otimes' FA) \\ \downarrow & & \downarrow \\ G(FA \otimes' FB) \xrightarrow{Gm_{A,B}} GF(A \otimes B) \xrightarrow{GFex_{A,B}} GF(B \otimes A) \end{array}$$

#### ▶ **Lemma 18.** The monad on the SMCC in a SMCC-Lambek model is strong.

**Proof.** Let  $F: C \vdash \mathcal{L}: G$  be a SMCC-Lambek model, where  $(C, \otimes, I, \alpha, \lambda, \rho)$  is an SMCC,  $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$  is a Lambek category, and (F, m) and (G, n) are monoidal functors. Let  $(T, \eta, \mu)$  be the monad on C where T = GF. We have proved that T is monoidal with the natural transformation  $t_{A,B}: TA \otimes TB \to T(A \otimes B)$  and the morphism  $t_I: I \to TI$  defined as in Lemma 17.

We define the tensorial strength  $\tau_{A,B}: A \otimes TB \to T(A \otimes B)$  as  $\tau_{A,B} = \mathsf{t}_{A,B} \circ \eta_A \otimes id_{TB}$ .

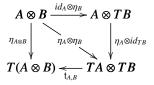
Since  $\eta$  is a monoidal natural transformation, we have  $\eta_I = Gm_I \circ n_{I'}$ . Therefore  $\eta_I = t_I$ . Thus the following diagram commutes because T is monoidal, where the composition  $t_{I,A} \circ t_I \otimes id_{TA}$  is the definition of  $\tau_{I,A}$ . So the first triangle in Defition 13 commutes.

$$I \otimes TA \xrightarrow{t_I \otimes id_{TA}} TI \otimes TA$$

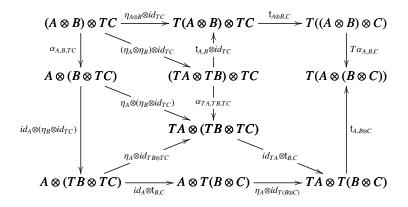
$$\downarrow t_{I,A} \qquad \qquad \downarrow t_{I,A}$$

$$TA \leftarrow TA \leftarrow T(I \otimes A)$$

Similarly, by using the definition of  $\tau$ , the second triangle in the definition is equivalent to the following diagram, which commutes because  $\eta$  is a monoidal natural transformation:

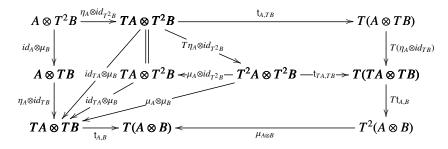


The first pentagon in the definition commutes by the following commutative diagrams, because  $\eta$  are  $\alpha$  natural transformations and T is monoidal:



◀

The last diagram in the definition commtues by the following commutative diagram, because T is a monad, t is a natural transformation, and  $\mu$  is a monoidal natural transformation:



- ▶ **Lemma 19** ([?]). *Let*  $\mathcal{M}$  *be a symmetric monoidal category and* T *be a strong monad on*  $\mathcal{M}$ . *Then* T *is a symmetric monoidal functor iff it is commutative.*
- ▶ Theorem 20. The monad on the SMCC in a SMCC-Lambek model is monoidal and non-commutative.
- ▶ **Lemma 21.** The comonad on the Lambek category in a SMCC-Lambek model is monoidal.

**Proof.** We define the comonad S on the Lambek category  $\mathcal{L}$  in the adjunction  $F: C \vdash \mathcal{L}: G$  of a SMCC-Lambek model as S = FG, and the two corresponding natural transformations  $\varepsilon: S \to id_{\mathcal{L}}$  and  $\delta: S \to S^2$  are defined as

$$\varepsilon_A : SA \to A$$

$$\delta_A = F\eta_{GA} : SA \to S^2A$$

where  $\varepsilon$  is the counit and  $\eta: id_{\mathcal{L}} \to GF$  is the unit in the adjunction, and (F, m) and (G, n) are monoidal functors. Thus, we have

$$s_{A,B} = Fn_{A,B} \circ m_{GA,GB} : SA \otimes' SB \rightarrow SA \otimes' SB$$
  
 $s_I = Fn_{I'} \circ m_I : I' \rightarrow SI'$ 

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$(SA \otimes' SB) \otimes' SC \xrightarrow{\alpha'_{SA,SB,SC}} \rightarrow SA \otimes' (SB \otimes' SC) \xrightarrow{id_{SA} \otimes' S_{B,C}} \rightarrow SA \otimes' S(B \otimes' C)$$

$$\downarrow_{S_{A,B} \otimes' id_{SC}} \downarrow_{S_{A,B} \otimes' C} \rightarrow S((A \otimes' B) \otimes' C) \xrightarrow{S\alpha'_{A,B,C}} \rightarrow S(A \otimes' (B \otimes' C))$$

$$I' \otimes' SA \xrightarrow{\lambda'_{SA}} SA \qquad SA \otimes' I' \xrightarrow{\rho'_{SA}} SA$$

$$\downarrow SI' \otimes' SA \xrightarrow{SI'} S(I' \otimes' A) \qquad id'_{SA} \otimes' SI' \xrightarrow{\rho'_{SA}} S(A \otimes' I')$$

$$\downarrow SA \otimes' SI' \xrightarrow{\rho'_{SA}} S(A \otimes' I')$$

$$\downarrow SA \otimes' SI' \xrightarrow{SA,I'} S(A \otimes' I')$$

**2.**  $\varepsilon$  is a monoidal natural transformation:



**3.**  $\delta$  is a monoidal natural transformation:

$$SA \otimes' SA \xrightarrow{S_{A,B}} S(A \otimes' B) \qquad SI' \xrightarrow{\delta_{I'}} S^2I'$$

$$\downarrow \delta_{A\otimes'\delta_B} \downarrow \qquad \downarrow \delta_{A\otimes'B} \qquad \downarrow \delta_{A\otimes'B} \qquad \downarrow S_{I'} \xrightarrow{S_{I'}} S_{S_{I'}}$$

$$S^2A \otimes' S^2B \xrightarrow{S_{SA,SB}} S(SA \otimes' SB) \xrightarrow{S_{SA,B}} S^2(A \otimes' B) \qquad I' \xrightarrow{S_{I'}} SI'$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma 17. We do not include the proof here for simplicity.

### 5 Logic

#### 5.1 Categorical Interpretation of Natural Deductions

[[x : G | -t : X]]

T rules: in the symmetric monoidal closed category of the adjunction model

T\_identity:  $id_X : X \to X$ 

T\_unitI:

T\_unitE: given  $t_1: \Delta \to Unit$  and  $t_2: \Gamma \to Y$ , returns  $\lambda_Y \circ (t_1 \otimes t_2): \Gamma \otimes \Delta \to Unit \otimes Y \to Y$ 

T\_tenI: given  $t_1: \Gamma \to X$  and  $t_2: \Delta \to Y$ , returns  $t_1 \otimes t_2: \Gamma \otimes \Delta \to X \otimes Y$ 

T\_tenE: given  $t_1: \Gamma \to X \otimes Y$  and  $t_2: \Delta \otimes X \otimes Y \to Z$ , returns

 $t_2 \circ \mathsf{ex}_{X \otimes Y, \Delta} \circ t_1 \otimes id_\Delta : \Gamma \otimes \Delta \to (X \otimes Y) \otimes \Delta \to \Delta \otimes (X \otimes Y) \to Z$ 

T\_implI:

T\_implE:

T imprI:

T\_imprE:

T\_GI: given  $s: FX_1 \otimes' ... \otimes' FX_n \to A$ , returns

 $Gs \circ Gm^{-1} \circ \eta : X_1 \otimes ... \otimes X_n \to GF(X_1 \otimes ... \otimes X_n) \to G(FX_1 \otimes' ... \otimes' FX_n) \to GA$ 

S rules: in the Lambek category of the adjunction model

S\_identity:  $id_A: A \rightarrow A$ 

S\_unitI:

S\_unitE:

#### 5.2 Normaalization and Reduction

## 6 Applications

### 7 Related Work

TODO

## 8 Conclusion

**TODO** 

## A Appendix