Non-Commutative Linear Logic in an Adjoint Model

Jiaming Jiang[1] and Harley Eades III[2]

1 **Computer Science, Augusta University, Augusta, Georgia, USA**
heades@augusta.edu

2 **Computer Science, North Carolina State University, Raleigh, North Carolina, USA**
jjiang13@ncsu.edu

Abstract

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1 Introduction

Linear logic is a well-known resource-sensitive logic. It has been used extensively to model attack trees. This paper concerns a non-commutative variant of linear logic and combines the non-commutative variant with Girard's linear logic []. We will only focus on the multiplicative (i.e. $\otimes$, $\multimap$) part of linear logic for simplicity. We construct the non-commutative variant by using a non-commutative tensor product $\triangleright$ instead of the commutative $\otimes$, and two implications $\leftharpoonup$ and $\rightharpoonup$ for the two directions of $\multimap$.

We model the non-commutative linear logic categorically using an ajunction between a symmetric monoidal closed category and a Lambek category. Our categorial adjoint model has a similar structure as Benton's adjoint model [], in which the multiplicative part of intuitionistic linear logic (ILL) is modeled using an adjunction between a cartesian closed category and a symmetric monoidal closed category. On the other hand, Moggi [] uses monad models to map intuitionistic logic into ILL. As discussed in [**?**], Benton's adjoint models only gives rise to commutative monad models and the non-commutative part remained as an open problem. Therefore, by combining our adjoint models with Benton's, we would be able to get non-commutative monad models and thus non-commutative ILL.

The rest of the paper is organized as follows. Section 2 discusses existing approaches on constructing non-commutative linear logic. Section 3 contains the basic definitions in category theory that we will be using in our adjoint model. Familiar readers may skip this section. Section 4 contains the definition and essential properties of our adjoint model. Section 5 discusses the sequent calculus and natural deduction rules for our non-commutative linear logic. We prove that our sequent calculus has the property of cut-elimination and the natural deduction is strongly normalizing. Section 6 talks about the preliminary result after combing our non-commutative model with Benton's commutative model. Section 7 briefly mentions how our model could be used in attack trees and other areas. Section 8 concludes this paper with future work.

2 Related Work

Polakow and Pfenning discuseed Ordered Linear Logic (OLL) [], which combines intuitionistic, commutative linear and non-commutative linear logic, OLL contains sequents of the form $\Gamma, \Delta, \Omega \vdash$

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A, where Γ is a multiset of intuitionistic assumptions, Δ is a multiset of commutative linear assumptions, and Ω is a list of non-commutative linear assumptions. OLL contains logical connectives from all three the logics. Therefore, our non-commutative adjoint model is a part of OLL and after combining with Benton's commutative adjoint model, we would get a simplification of OLL.

Greco and Palmigiano [] also presents a variant of the multiplicative fragment of non-commutative ILL. But they focus on proper display calculi while we use sequent calculi.

de Paiva and Eades [] also developed categorical models for the non-commutative ILL by adapting the Dialectica categorical models for linear logic.

3 Category Theory Basics

This section contains the basic definitions in category theory that we will be using in our adjoint model. Our model is based on special kinds of monoidal categories: Lambek categories and symmetric monoidal closed categories and Lambek categories, as defined in Definitions 2 and 3.

▶ **Definition 1.** A **monoidal category** $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ consists of

- a bifunctor $\triangleright : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- an object I, called the unit object;
- three natural isomorphisms α, λ, and ρ with components

$$\alpha_{A,B,C} : (A \triangleright B) \triangleright C \to A \triangleright (B \triangleright C)$$

$$\lambda_A : I \triangleright A \to A$$

$$\rho_A : A \triangleright I \to A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in $\mathcal{M}$:

$$\begin{tikzcd}
((A \triangleright B) \triangleright C) \triangleright D \arrow[r, "\alpha_{A,B,C} \triangleright id_D"] \arrow[d, "\alpha_{A \triangleright B,C,D}"'] & (A \triangleright (B \triangleright C)) \triangleright D \arrow[r, "\alpha_{A,B \triangleright C,D}"] & A \triangleright ((B \triangleright C) \triangleright D) \arrow[d, "id_A \triangleright \alpha_{B,C,D}"] \\
(A \triangleright B) \triangleright (C \triangleright D) \arrow[rr, "\alpha_{A,B,C \triangleright D}"'] & & A \triangleright (B \triangleright (C \triangleright D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \triangleright I) \triangleright B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \triangleright id_B"'] & & A \triangleright (I \triangleright B) \arrow[dl, "id_A \triangleright \lambda_B"] \\
& A \triangleright B &
\end{tikzcd}$$

▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\rightharpoonup : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and $\leftharpoonup : \mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \triangleright A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \triangleright X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

▶ **Definition 3.** A **symmetric monoidal category** (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\mathsf{ex}_{A,B} : A \otimes B \to B \otimes A$, called **exchange**, such that the following diagrams commute:

$$\begin{tikzcd} A \otimes I \arrow[r, "\mathsf{ex}_{A,I}"] \arrow[dr, "\rho_A"'] & I \otimes A \arrow[d, "\lambda_A"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[r, equal, "id_{A\otimes B}"] \arrow[d, "\mathsf{ex}_{A,B}"'] & A \otimes B \\ B \otimes A \arrow[ur, "\mathsf{ex}_{B,A}"'] \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathsf{ex}_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\mathsf{ex}_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,A,C}"] \\ (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \mathsf{ex}_{A,C}"'] & B \otimes (C \otimes A) \end{tikzcd}$$

We use $\triangleright$ for non-symmetric monoidal categories while $\otimes$ for symmetric ones.

▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap$: $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

The relation between SMMCs and Lambek categories are demonstrated in Lemma 5 and Corollary 6.

▶ **Lemma 5.** *Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightharpoonup B) \cong (B \leftharpoondown A)$.*

Proof. First, notice that for any object C we have

$$\begin{array}{rcll} Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\ & \cong & Hom[A \otimes C, B] & \text{By the exchange } \mathsf{ex}_{C,A} \\ & \cong & Hom[C, B \leftharpoondown A] & \mathcal{L} \text{ is a Lambek category} \end{array}$$

Thus, $A \rightharpoonup B \cong B \leftharpoondown A$ by the Yoneda lemma. ◀

▶ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

The essential component in our non-commutative adjoint model is a monoidal adjunction, defined in Definitions 7-11.

▶ **Definition 7.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \triangleright', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from $\mathcal{M}$ to $\mathcal{M}'$ is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \triangleright FB' \to F(A \triangleright B)$, such that the following diagrams commute for any objects A, B, and C in $\mathcal{M}$:

$$\begin{tikzcd} (FA \triangleright' FB) \triangleright' FC \arrow[r, "\alpha'_{FA,FB,FC}"] \arrow[d, "\mathsf{m}_{A,B} \triangleright' id_{FC}"'] & FA \triangleright' (FB \triangleright' FC) \arrow[r, "id_{FA} \triangleright' \mathsf{m}_{A,B}"] & FA \triangleright' F(B \triangleright C) \arrow[d, "\mathsf{m}_{A,B\triangleright C}"] \\ F(A \triangleright B) \triangleright' FC \arrow[r, "\mathsf{m}_{A\triangleright B,C}"'] & F((A \triangleright B) \triangleright C) \arrow[r, "F\alpha_{A,B,C}"'] & F(A \triangleright (B \triangleright C)) \end{tikzcd}$$

$$\begin{tikzcd} I' \triangleright' FA \arrow[r, "\lambda'_{FA}"] \arrow[d, "\mathsf{m}_I \triangleright id_{FA}"'] & FA \\ FI \triangleright' FA \arrow[r, "\mathsf{m}_{I,A}"'] & F(I \triangleright A) \arrow[u, "F\lambda_A"'] \end{tikzcd} \qquad \begin{tikzcd} FA \triangleright' I' \arrow[r, "\rho'_{FA}"] \arrow[d, "\mathsf{id}_{FA} \triangleright \mathsf{m}_I"'] & FA \\ FA \triangleright' FI \arrow[r, "\mathsf{m}_{A,I}"'] & F(A \triangleright I) \arrow[u, "F\rho_A"'] \end{tikzcd}$$

▶ **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "\mathsf{ex}_{FA,FB}"] \arrow[d, "\mathsf{m}_{A,B}"'] & FB \otimes' FA \arrow[d, "\mathsf{m}_{B,A}"] \\
F(A \otimes B) \arrow[r, "F\mathsf{ex}_{A,B}"'] & F(B \otimes A)
\end{tikzcd}$$

▶ **Definition 9.** An **adjunction** between categories C and $\mathcal{D}$ consists of two functors $F : \mathcal{D} \to C$, called the **left adjoint**, and $G : C \to \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta : id_{\mathcal{D}} \to GF$, called the **unit**, and $\varepsilon : FG \to id_C$, called the **counit**, such that the following diagrams commute for any object A in C and B in $\mathcal{D}$:

$$\begin{tikzcd}
FB \arrow[rr, "F\eta_B"] \arrow[dr, equal] & & FGFB \arrow[dl, "\varepsilon_{FB}"] \\
& FB &
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[rr, "\eta_{GA}"] \arrow[dr, equal] & & GFGA \arrow[dl, "G\varepsilon_A"] \\
& GA &
\end{tikzcd}$$

▶ **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ to a monoidal category $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$. A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, \mathsf{m}) \to (G, \mathsf{n})$ such that the following diagrams commute for any objects A and B in $\mathcal{M}$:

$$\begin{tikzcd}
FA \triangleright' FB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[d, "\theta_A \triangleright' \theta_B"'] & F(A \triangleright B) \arrow[d, "\theta_{A \triangleright B}"] \\
GA \triangleright' GB \arrow[r, "\mathsf{n}_{A,B}"'] & G(A \triangleright B)
\end{tikzcd}
\qquad
\begin{tikzcd}
FI \arrow[rr, "\theta_I"] & & GI \\
& I' \arrow[ul, "\mathsf{m}_I"] \arrow[ur, "\mathsf{n}_I"'] &
\end{tikzcd}$$

▶ **Definition 11.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \triangleright', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

In Moggi's monad model [], the monad is required to be strong, as defined in Definitions 12 and 13.

▶ **Definition 12.** Let C be a category. A **monad** on C consists of an endofunctor $T : C \to C$ together with two natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to T$, where id_C is the identity functor on C, such that the following diagrams commute:

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta_T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

▶ **Definition 13.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on $\mathcal{M}$. T is a **strong monad** if there is natural transformation τ, called the **tensorial strength**, with components

$\tau_{A,B} : A \triangleright TB \to T(A \triangleright B)$ such that the following diagrams commute:

$$\begin{tikzcd}
I \triangleright TA \arrow[r, "\tau_{I,A}"] \arrow[dr, "\lambda_{TA}"'] & T(I \triangleright A) \arrow[d, "T\lambda_A"] \\
& TA
\end{tikzcd}
\qquad
\begin{tikzcd}
A \triangleright B \arrow[r, "id_A \triangleright \eta_B"] \arrow[dr, "\eta_{A\triangleright B}"'] & A \triangleright TB \arrow[d, "\tau_{A,B}"] \\
& T(A \triangleright B)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \triangleright B) \triangleright TC \arrow[rr, "\tau_{A\triangleright B,C}"] \arrow[d, "\alpha_{A,B,TC}"'] & & T((A \triangleright B) \triangleright C) \arrow[d, "T\alpha_{A,B,C}"] \\
A \triangleright (B \triangleright TC) \arrow[r, "id_A \triangleright \tau_{B,C}"] & A \triangleright T(B \triangleright C) \arrow[r, "\tau_{A,B\triangleright C}"] & T(A \triangleright (B \triangleright C))
\end{tikzcd}$$

$$\begin{tikzcd}
A \triangleright T^2B \arrow[r, "\tau_{A,TB}"] \arrow[d, "id_A \triangleright \mu_B"'] & T(A \triangleright TB) \arrow[r, "T\tau_{A,B}"] & T^2(A \triangleright B) \arrow[d, "\mu_{A\triangleright B}"] \\
A \triangleright TB \arrow[rr, "\tau_{A,B}"'] & & T(A \triangleright B)
\end{tikzcd}$$

▶ **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex, and (T, η, μ) be a strong monad on $\mathcal{M}$. Then there is a **"twisted" tensorial strength** $\tau'_{A,B} : TA \otimes B \to T(A \otimes B)$ defined as $\tau'_{A,B} = T\mathsf{ex} \circ \tau_{B,A} \circ \mathsf{ex}$. We can construct a pair of natural transformations Φ, Φ' with components $\Phi_{A,B}, \Phi'_{A,B} : TA \otimes TB \to T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A\otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A\otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.

▶ **Definition 15.** Let $\mathcal{L}$ be a category. A **comonad** on $\mathcal{L}$ consists of an endofunctor $S : \mathcal{L} \to \mathcal{L}$ together with two natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S^2 \to S$ such that the following diagrams commute:

$$\begin{tikzcd}
S \arrow[r, "\delta"] \arrow[d, "\delta"'] & S^2 \arrow[d, "S\delta"] \\
S^2 \arrow[r, "\delta_S"'] & S^3
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2 \arrow[r, "S\varepsilon"] \arrow[d, "\varepsilon_S"'] & S \arrow[d, "\delta"] \\
S \arrow[r, "\delta"'] & S^2
\end{tikzcd}$$

4 Lambek Adjoint Model

Our adjoint model, Lambek Adjoint Model (LAM), has a similar structure as Benton's LNL model []. Benton's LNL model consists of a symmetric monoidal adjunction $F : C \dashv \mathcal{L} : G$ between a cartesian closed category C and a symmetric monoidal closed category $\mathcal{L}$. LAM consists of a monoidal adjunction between a symmetric monoidal closed category and a Lambek category.

▶ **Definition 16.** A **Lambek Adjoint Model (LAM)** $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ consists of

- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$;
- a monoidal adjunction $F : C \dashv \mathcal{L} : G$ with unit $\eta_X : X \to GFX$ and counit $\varepsilon : FG \to id_{\mathcal{L}}$, where $(F : C \to \mathcal{L}, m)$ and $(G : \mathcal{L} \to C, n)$ are monoidal functors.

Thus, in LAM, the following four diagrams commute because η and ε are monoidal natural transformations:

$$\begin{tikzcd}
X \otimes Y \arrow[rr, equal, "id_{X\otimes Y}"] \arrow[d, "\eta_X \otimes \eta_Y"'] & & X \otimes Y \arrow[d, "\eta_{X\otimes Y}"] \\
GFX \otimes GFY \arrow[r, "\mathsf{n}_{FX,FY}"] & G(FX \otimes FY) \arrow[r, "G\mathsf{m}_{X,Y}"] & GF(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\eta_I"] \arrow[d, equal] & GFI \\
I \arrow[r, "\mathsf{n}_{I'}"'] & GI' \arrow[u, "G\mathsf{m}_I"']
\end{tikzcd}$$

$$\begin{tikzcd}
FGA \otimes FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes \varepsilon_B"'] & F(GA \otimes GB) \arrow[r, "F\mathsf{n}_{A,B}"] & FG(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\
A \otimes B \arrow[rr, equal] & & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
FGI' \arrow[r, "\varepsilon_{I'}"] & I' \arrow[d, equal] \\
FI \arrow[u, "F\mathsf{n}_{I'}"] & I' \arrow[l, "\mathsf{m}_I"]
\end{tikzcd}$$

And the following two triangles commute because of the adjunction:

$$\begin{tikzcd}
FX \arrow[rr, "F\eta_X"] \arrow[dr, equal] & & FGFX \arrow[dl, "\varepsilon_{FX}"] \\
& FX &
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[rr, "\eta_{GA}"] \arrow[dr, equal] & & GFGA \arrow[dl, "G\varepsilon_A"] \\
& GA &
\end{tikzcd}$$

Following the tradition, we use letters X, Y, Z for objects in C and A, B, C for objects in $\mathcal{L}$. The rest of this section proves essential properties of a LAM.

4.1 An Isomorphism

Let $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM, where (F, m) and (G, n) are monoidal functors. Similarly as in Benton's LNL model, $\mathsf{m}_{X,Y}$ are components of a natural isomorphism and m_I is an isomorphism. This is essential for deriving certain rules of our non-commutative linear logic, such as tensor elimination in natural deduction.

We define the inverses of $\mathsf{m}_{X,Y} : FX \triangleright FY \to F(X \otimes Y)$ and $\mathsf{m}_I : I' \to FI$ as:

$$\mathsf{p}_{X,Y} : F(X \otimes Y) \xrightarrow{F(\eta_X \otimes \eta_Y)} F(GFX \otimes GFY) \xrightarrow{F\mathsf{n}_{FX,FY}} FG(FX \triangleright FY) \xrightarrow{\varepsilon_{FX \triangleright FX}} FX \triangleright FY$$

$$\mathsf{p}_I : FI \xrightarrow{F\mathsf{n}_{I'}} FGI' \xrightarrow{\varepsilon_{I'}} I'$$

► **Theorem 17.** $\mathsf{m}_{X,Y}$ *are components of a natural isomorphism and their inverses are* $\mathsf{p}_{X,Y}$.

Proof. We need to show that $\mathsf{m}_{X,Y} \circ \mathsf{p}_{X,Y} = id_{F(X\otimes Y)}$ and $\mathsf{p}_{X,Y} \circ \mathsf{m}_{X,Y} = id_{FX \triangleright FX}$. The two equations hold because the following diagrams commute: (1)-adjunction; (2)-η is a monoidal natural transformation; (3)-naturality of ε; (4)-adjunction; (5)-naturality of m; (6)-ε is a monoidal natural transformation.

$$
\begin{tikzcd}
F(X\otimes Y) \arrow[rrrr, "F(\eta_X\otimes\eta_Y)"] \arrow[dd, equal] \arrow[drr, "F\eta_{X\otimes Y}"] & & & & F(GFX\otimes GFY) \arrow[dd, "F\mathsf{n}_{FX,FY}"] \\
& (1) & FGF(X\otimes Y) \arrow[dll, "\varepsilon_{F(X\otimes Y)}"'] \arrow[drr, "FG\mathsf{m}_{X,Y}"] \arrow[d, phantom, "(3)"] & (2) & \\
F(X\otimes Y) & & FX\triangleright FY \arrow[ll, "\mathsf{m}_{X,Y}"] & & FG(FX\triangleright FY) \arrow[ll, "\varepsilon_{FX\triangleright FY}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
FX\triangleright FY \arrow[rrrr, "\mathsf{m}_{X,Y}"] \arrow[dd, equal] \arrow[drr, "F\eta_X\triangleright F\eta_Y"] & & & & F(X\otimes Y) \arrow[dd, "F(\eta_X\otimes\eta_Y)"] \\
& (4) & FGFX\triangleright FGFY \arrow[dll, "\varepsilon_{FX}\triangleright\varepsilon_{FY}"'] \arrow[drr, "\mathsf{m}_{GFX,GFY}"] \arrow[d, phantom, "(6)"] & (5) & \\
FX\triangleright FY & & FG(FX\triangleright FY) \arrow[ll, "\varepsilon_{FX\triangleright FY}"] & & F(GFX\otimes GFY) \arrow[ll, "F\mathsf{n}_{FX,FY}"]
\end{tikzcd}
$$

◀

▶ **Theorem 18.** m_I *is an isomorphism and its inverse is* p_I.

Proof. This is equivalent to equations $\mathsf{m}_I \circ \mathsf{p}_I = id_{FI}$ and $\mathsf{p}_I \circ \mathsf{m}_I = id_{I'}$, equivalent to the following diagrams, which commute because ε is a monoidal natural transformation.

$$
\begin{tikzcd}
FI \arrow[r, "F\mathsf{n}_{I'}"] \arrow[d, equal] & FGI' \arrow[d, "\varepsilon_{I'}"] \\
FI & I' \arrow[l, "\mathsf{m}_I"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
I' \arrow[r, "\mathsf{m}_I"] \arrow[d, equal] & FI \arrow[d, "F\mathsf{n}_{I'}"] \\
I' & FGI' \arrow[l, "\varepsilon_{I'}"]
\end{tikzcd}
$$

◀

4.2 Monad on C

We first show that the monad on C in LAM is strong but non-commutative. In Benton's LNL model, the monad on the cartesian closed category is commutative.

▶ **Lemma 19.** *The monad on the symmetric monoidal closed category* C *in LAM is monoidal.*

Proof. Let $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM. We define the monad $(T, \eta : id_C \to T, \mu : T^2 \to T)$ on C as

$$T = GF \qquad \eta_X : X \to GFX \qquad \mu_X = G\varepsilon_{FX} : GFGFX \to GFX$$

Since (F, m) and (G, n) are monoidal functors, we have

$$\mathsf{t}_{X,Y} = G\mathsf{m}_{X,Y} \circ \mathsf{n}_{FX,FY} : TX \otimes TY \to T(X \otimes Y) \qquad \mathsf{t}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'} : I \to TI$$

The monad T being monoidal means:

1. T is a monoidal functor, i.e. the folllowing diagrams commute:

$$\begin{tikzcd}
(TX \otimes TY) \otimes TZ \arrow[r, "\alpha_{TX,TY,TZ}"] \arrow[d, "\mathsf{t}_{X,Y} \otimes id_{TZ}"'] & TX \otimes (TY \otimes TZ) \arrow[r, "id_{TX} \otimes \mathsf{t}_{Y,Z}"] \arrow[d, phantom, "(1)"] & TX \otimes T(Y \otimes Z) \arrow[d, "\mathsf{t}_{X,Y\otimes Z}"] \\
T(X \otimes Y) \otimes TZ \arrow[r, "\mathsf{t}_{X\otimes Y,Z}"'] & T((X \otimes Y) \otimes Z) \arrow[r, "T\alpha_{X,Y,Z}"'] & T(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes TX \arrow[r, "\lambda_{TX}"] \arrow[d, "\mathsf{t}_I \otimes id_{TX}"'] \arrow[dr, phantom, "(2)"] & TX \\
TI \otimes TX \arrow[r, "\mathsf{t}_{I,X}"'] & T(I \otimes X) \arrow[u, "T\lambda_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \otimes I \arrow[r, "\rho_{TX}"] \arrow[d, "id_{TX} \otimes \mathsf{t}_I"'] \arrow[dr, phantom, "(3)"] & TX \\
TX \otimes TI \arrow[r, "\mathsf{t}_{X,I}"'] & T(X \otimes I) \arrow[u, "T\rho_X"']
\end{tikzcd}$$

We write GF instead of T in the proof for clarity.
By replacing $\mathsf{t}_{X,Y}$ with its definition, diagram (1) above commutes by the following commutative diagram, in which the two hexagons commute because G and F are monoidal functors, and the two quadrilaterals commute by the naturality of n.

$$\begin{tikzcd}
(GFX \otimes GFY) \otimes GFZ \arrow[r, "\alpha_{GFX,GFY,GFZ}"] \arrow[d, "\mathsf{n}_{FX,FY} \otimes id_{GFZ}"'] & GFX \otimes (GFY \otimes GFZ) \arrow[r, "id_{GFX} \otimes \mathsf{n}_{FY,FZ}"] & GFX \otimes G(FY \triangleright FZ) \arrow[dl, "\mathsf{n}_{FX,FY\triangleright FZ}"'] \arrow[d, "id_{GFX} \otimes G\mathsf{m}_{Y,Z}"] \\
G(FX \triangleright FY) \otimes GFZ \arrow[d, "G\mathsf{m}_{X,Y} \otimes id_{GFZ}"'] \arrow[dr, "\mathsf{n}_{FX\triangleright FY,FZ}"] & G(FX \triangleright (FY \triangleright FZ)) \arrow[dr, "G(id_{FX} \triangleright \mathsf{m}_{Y,Z})"] & GFX \otimes GF(Y \otimes Z) \arrow[d, "\mathsf{n}_{FX,F(Y\otimes Z)}"] \\
GF(X \otimes Y) \otimes GFZ \arrow[d, "\mathsf{n}_{F(X\otimes Y),FZ}"'] & G((FX \triangleright FY) \triangleright FZ) \arrow[u, "G\alpha'_{FX,FY,FZ}"] \arrow[dl, "G(\mathsf{m}_{X,Y} \otimes id_{FZ})"'] & G(FX \triangleright F(Y \otimes Z)) \arrow[d, "G\mathsf{m}_{X,Y\otimes Z}"] \\
G(F(X \otimes Y) \triangleright FZ) \arrow[r, "G\mathsf{m}_{X\otimes Y,Z}"'] & GF((X \otimes Y) \otimes Z) \arrow[r, "GF\alpha_{X,Y,Z}"'] & GF(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Diagram (2) commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.

$$\begin{tikzcd}
I \otimes GFX \arrow[rr, "\lambda_{GFX}"] \arrow[d, "\mathsf{n}_{I'} \otimes id_{GFX}"'] & & GFX \\
GI' \otimes GFX \arrow[r, "\mathsf{n}_{I',FX}"'] \arrow[d, "G\mathsf{m}_I \otimes id_{GFX}"'] & G(I' \triangleright FX) \arrow[ur, "G\lambda'_{FX}"] \arrow[d, "G(\mathsf{m}_I \triangleright id_{FX})"] & \\
GFI \otimes GFX \arrow[r, "\mathsf{n}_{FI,FX}"'] & G(FI \triangleright FX) \arrow[r, "G\mathsf{m}_{I,X}"'] & GF(I \otimes X) \arrow[uu, "GF\lambda_X"']
\end{tikzcd}$$

Similarly, diagram (3) commutes as follows:

$$\begin{tikzcd}
GFX \otimes I \arrow[rr, "\rho_{GFX}"] \arrow[d, "id_{GFX}\otimes \mathsf{n}_{I'}"'] & & GFX \\
GFX \otimes GI' \arrow[r, "\mathsf{n}_{FX,I'}"'] \arrow[d, "id_{GFX}\otimes G\mathsf{m}_I"'] & G(FX \triangleright I') \arrow[ur, "G\rho'_{FX}"] \arrow[d, "G(id_{FX}\otimes \mathsf{m}_I)"] & \\
GFX \otimes GFI \arrow[r, "\mathsf{n}_{FX,FI}"'] & G(FX \triangleright FI) \arrow[r, "G\mathsf{m}_{X,I}"'] & GF(X \otimes I) \arrow[uu, "GF\rho_X"']
\end{tikzcd}$$

2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal by definition and thus the following two diagrams commute.

$$\begin{tikzcd}
X \otimes Y \arrow[r, equal] \arrow[d, "\eta_X\otimes\eta_Y"'] & X \otimes Y \arrow[d, "\eta_{X\otimes Y}"] \\
TX \otimes TY \arrow[r, "\mathsf{t}_{X,Y}"'] & T(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[rr, "\eta_I"] \arrow[dr, equal] & & TI \\
& I \arrow[ur, "\mathsf{t}_I"'] &
\end{tikzcd}$$

3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ. Thus the following diagrams commute.

$$\begin{tikzcd}
T^2X \otimes T^2Y \arrow[r, "\mathsf{t}_{TX,TY}"] \arrow[d, "\mu_X\otimes\mu_Y"'] & T(TX \otimes TY) \arrow[r, "T\mathsf{t}_{X,Y}"] & T^2(X \otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
TX \otimes TY \arrow[rr, "\mathsf{t}_{X,Y}"'] & & T(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2I \arrow[r, "\mu_I"] & TI \\
TI \arrow[u, "T\mathsf{t}_I"] & I \arrow[l, "\mathsf{t}_I"] \arrow[u, "\mathsf{t}_I"']
\end{tikzcd}$$

◀

However, the monad is not symmetric becauase the following diagram does not commute, for the Lambek category $\mathcal{L}$ is not symmetric.

$$\begin{tikzcd}
GFX \otimes GFY \arrow[r, "\mathsf{ex}_{GFX,GFY}"] \arrow[d, "\mathsf{n}_{FX,FY}"'] & GFY \otimes GFX \arrow[r, "\mathsf{n}_{FY,FX}"] & G(FY \triangleright FX) \arrow[d, "G\mathsf{m}_{Y,X}"] \\
G(FX \triangleright FY) \arrow[r, "G\mathsf{m}_{X,Y}"'] & GF(X \otimes Y) \arrow[r, "GF\mathsf{ex}_{X,Y}"'] & GF(Y \otimes X)
\end{tikzcd}$$

▶ **Lemma 20.** *The monad on the symmetric monoidal closed category in LAM is strong.*

Proof. Let $(\mathcal{C}, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM, where $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is symmetric monoidal closed, $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ is Lambek. In Lemma 19, we have proved that the monad $(T = GF, \eta, \mu)$ is monoidal with the natural transformation $\mathsf{t}_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ and the morphism $\mathsf{t}_I : I \to TI$. We define the tensorial strength $\tau_{X,Y} : X \otimes TY \to T(X \otimes Y)$ as $\tau_{X,Y} = \mathsf{t}_{X,Y} \circ (\eta_X \otimes id_{TY})$.
Since η is a monoidal natural transformation, we have $\eta_I = G\mathsf{m}_I \circ \mathsf{n}_{I'}$. Therefore $\eta_I = \mathsf{t}_I$. Thus the following diagram commutes because T is monoidal, where the composition $\mathsf{t}_{I,X} \circ (\mathsf{t}_I \otimes id_{TX})$ is the definition of $\tau_{I,X}$. So the first triangle in Defition 13 commutes.

$$\begin{tikzcd}
I \otimes TX \arrow[r, "\mathsf{t}_I\otimes id_{TX}"] \arrow[d, "\lambda_{TX}"'] & TI \otimes TX \arrow[d, "\mathsf{t}_{I,X}"] \\
TX & T(I \otimes X) \arrow[l, "T\lambda_X"]
\end{tikzcd}$$

Similarly, by using the definition of τ, the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$
\begin{tikzcd}
X \otimes Y \arrow[r, "id_X \otimes \eta_Y"] \arrow[d, "\eta_{X\otimes Y}"'] \arrow[dr, "\eta_X \otimes \eta_Y"] & X \otimes TY \arrow[d, "\eta_X \otimes id_{TY}"] \\
T(X \otimes Y) & TX \otimes TY \arrow[l, "\mathrm{t}_{X,Y}"]
\end{tikzcd}
$$

The first pentagon in the definition commutes by the following commutative diagrams, because η and α are natural transformations and T is monoidal:

$$
\begin{tikzcd}
(X \otimes Y) \otimes TZ \arrow[r, "\eta_{X\otimes Y} \otimes id_{TZ}"] \arrow[d, "\alpha_{X,Y,TZ}"'] \arrow[dr, "(\eta_X \otimes \eta_Y) \otimes id_{TZ}"] & T(X \otimes Y) \otimes TZ \arrow[r, "\mathrm{t}_{X\otimes Y,Z}"] & T((X \otimes Y) \otimes Z) \arrow[d, "T\alpha_{X,Y,Z}"] \\
X \otimes (Y \otimes TZ) \arrow[dr, "\eta_X \otimes (\eta_Y \otimes id_{TZ})"] \arrow[dd, "id_X \otimes (\eta_Y \otimes id_{TZ})"'] & (TX \otimes TY) \otimes TZ \arrow[u, "\mathrm{t}_{X,Y} \otimes id_{TZ}"'] \arrow[d, "\alpha_{TX,TY,TZ}"] & T(X \otimes (Y \otimes Z)) \\
 & TX \otimes (TY \otimes TZ) \arrow[dr, "id_{TX} \otimes \mathrm{t}_{Y,Z}"] & \\
X \otimes (TY \otimes TZ) \arrow[ur, "\eta_X \otimes id_{TY\otimes TZ}"] \arrow[r, "id_X \otimes \mathrm{t}_{Y,Z}"'] & X \otimes T(Y \otimes Z) \arrow[r, "\eta_X \otimes id_{T(Y\otimes Z)}"'] & TX \otimes T(Y \otimes Z) \arrow[uu, "\mathrm{t}_{X,Y\otimes Z}"']
\end{tikzcd}
$$

The last diagram in the definition commtues by the following commutative diagram, because T is a monad, t is a natural transformation, and μ is a monoidal natural transformation:

$$
\begin{tikzcd}
X \otimes T^2Y \arrow[r, "\eta_X \otimes id_{T^2Y}"] \arrow[d, "id_X \otimes \mu_Y"'] & TX \otimes T^2Y \arrow[rr, "\mathrm{t}_{X,TY}"] \arrow[d, equal] \arrow[dr, "T\eta_X \otimes id_{T^2Y}"] \arrow[ddl, "id_{TX} \otimes \mu_Y"'] & & T(X \otimes TY) \arrow[d, "T(\eta_X \otimes id_{TY})"] \\
X \otimes TY \arrow[d, "\eta_X \otimes id_{TY}"'] & TX \otimes T^2Y \arrow[dl, "id_{TX} \otimes \mu_Y"] & T^2X \otimes T^2Y \arrow[l, "\mu_X \otimes id_{T^2Y}"'] \arrow[r, "\mathrm{t}_{TX,TY}"] \arrow[dll, "\mu_X \otimes \mu_Y"] & T(TX \otimes TY) \arrow[d, "T\mathrm{t}_{X,Y}"] \\
TX \otimes TY \arrow[r, "\mathrm{t}_{X,Y}"'] & T(X \otimes Y) & & T^2(X \otimes Y) \arrow[ll, "\mu_{X\otimes Y}"]
\end{tikzcd}
$$

◀

The following lemma is adopted from [].

► **Lemma 21.** *Let $\mathcal{M}$ be a symmetric monoidal category and T be a strong monad on $\mathcal{M}$. Then T is commutative iff it is symmetric.*

► **Theorem 22.** *The monad on the SMCC in LAM is strong but non-commutative.*

Proof. The proof is obvious. Based on Lemma 20 and Lemma 21, the monad is non-commutative. ◀

4.3 Comonad on $\mathcal{L}$

► **Lemma 23.** *The comonad on the Lambek category in a LAM is monoidal.*

Proof. We define the comonad $(S, \varepsilon : S \to id_{\mathcal{L}}, \delta : S \to S^2)$ on the Lambek category $\mathcal{L}$ as:

$$S = FG \qquad \varepsilon_A : SA \to A \qquad \delta_A = F\eta_{GA} : SA \to S^2A$$

Thus, we have natural transformation s and morphism s_I defined as:

$$\mathsf{s}_{A,B} = F\mathsf{n}_{A,B} \circ \mathsf{m}_{GA,GB} : SA \triangleright SB \to SA \triangleright SB \qquad \mathsf{s}_I = F\mathsf{n}_{I'} \circ \mathsf{m}_I : I' \to SI'$$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$\begin{tikzcd}
(SA \triangleright SB) \triangleright SC \arrow[r, "\alpha'_{SA,SB,SC}"] \arrow[d, "\mathsf{s}_{A,B} \triangleright id_{SC}"'] & SA \triangleright (SB \triangleright SC) \arrow[r, "id_{SA} \triangleright \mathsf{s}_{B,C}"] & SA \triangleright S(B \triangleright C) \arrow[d, "\mathsf{s}_{A,B \triangleright C}"] \\
S(A \triangleright B) \triangleright SC \arrow[r, "\mathsf{s}_{A \triangleright B,C}"'] & S((A \triangleright B) \triangleright C) \arrow[r, "S\alpha'_{A,B,C}"'] & S(A \triangleright (B \triangleright C))
\end{tikzcd}$$

$$\begin{tikzcd}
I' \triangleright SA \arrow[r, "\lambda'_{SA}"] \arrow[d, "\mathsf{s}_{I'} \triangleright id_{SA}"'] & SA \\
SI' \triangleright SA \arrow[r, "\mathsf{s}_{I',A}"'] & S(I' \triangleright A) \arrow[u, "S\lambda'_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
SA \triangleright I' \arrow[r, "\rho'_{SA}"] \arrow[d, "id'_{SA} \triangleright \mathsf{s}_{I'}"'] & SA \\
SA \triangleright SI' \arrow[r, "\mathsf{s}_{A,I'}"'] & S(A \triangleright I') \arrow[u, "S\rho'_A"']
\end{tikzcd}$$

2. ε is a monoidal natural transformation:

$$\begin{tikzcd}
SA \triangleright SB \arrow[r, "\mathsf{s}_{A,B}"] \arrow[d, "\varepsilon_A \triangleright \varepsilon_B"'] & S(A \triangleright B) \arrow[d, "\varepsilon_{A \triangleright B}"] \\
A \triangleright B \arrow[r, equal] & A \triangleright B
\end{tikzcd}
\qquad
\begin{tikzcd}
SI' \arrow[rr, "\varepsilon_{I'}"] & & I' \\
& I' \arrow[ul, "\mathsf{s}_{I'}"] \arrow[ur, equal] &
\end{tikzcd}$$

3. δ is a monoidal natural transformation:

$$\begin{tikzcd}
SA \triangleright SA \arrow[rr, "\mathsf{s}_{A,B}"] \arrow[d, "\delta_A \triangleright \delta_B"'] & & S(A \triangleright B) \arrow[d, "\delta_{A \triangleright B}"] \\
S^2A \triangleright S^2B \arrow[r, "\mathsf{s}_{SA,SB}"] & S(SA \triangleright SB) \arrow[r, "S\mathsf{s}_{A,B}"] & S^2(A \triangleright B)
\end{tikzcd}
\qquad
\begin{tikzcd}
SI' \arrow[r, "\delta_{I'}"] & S^2I' \\
I' \arrow[u, "\mathsf{s}_{I'}"] \arrow[r, "\mathsf{s}_{I'}"'] & SI' \arrow[u, "S\mathsf{s}_{I'}"']
\end{tikzcd}$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma 19. We do not include the proof here for simplicity. ◀

We then show that the co-Eilenberg-Moore category of the comonad S is symmetric monoidal closed.

▶ **Definition 24.** Let (S, ε, δ) be a comonad on a category $\mathcal{L}$. Then the **co-Eilenberg-Moore category $\mathcal{L}^S$** of the comonad has

- as objects the S-coalgebras $(A, h_A : A \to SA)$, where A is an object in $\mathcal{L}$, s.t. the following diagrams commute:

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] & SA \arrow[d, "\delta_A"] \\
SA \arrow[r, "Sh_A"'] & S^2A
\end{tikzcd}
\qquad
\begin{tikzcd}
& SA \arrow[dr, "\varepsilon_A"] & \\
A \arrow[ur, "h_A"] \arrow[rr, equal] & & A
\end{tikzcd}$$

- as morphisms the coalgebra morphisms, i.e. morphisms $f : (A, h_A) \to (B, h_B)$ between coalgebras s.t. the diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h_A"'] & B \arrow[d, "h_B"] \\ SA \arrow[r, "Sf"'] & SB \end{tikzcd}$$

▶ **Lemma 25.** *Given a LAM $(\mathcal{C}, \mathcal{L}, F, G, \eta, \varepsilon)$ and the comonad S on $\mathcal{L}$, the co-Eilenberg-Moore category $\mathcal{L}^S$ has an exchange natural transformation $\mathsf{ex}^S_{A,B} : A \triangleright B \to B \triangleright A$.*

Proof. We define the exchange $\mathsf{ex}^S_{A,B} : A \triangleright B \to B \triangleright A$ as

$$A \triangleright B \xrightarrow{h_A \triangleright h_B} FGA \triangleright FGB \xrightarrow{\mathsf{m}_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\mathsf{ex}_{GA,GB}} F(GB \otimes GA) \xrightarrow{F\mathsf{n}_{B,A}} FG(B \triangleright A) \xrightarrow{\varepsilon_{B \triangleright A}} B \triangleright A$$

in which (F, m) and (G, n) are monoidal functors, and ex is the exchange for $\mathcal{C}$. Then ex^S is a natural transformation because the following diagrams commute for morphisms $f : A \to A'$ and $g : B \to B'$:

$$\begin{tikzcd} A \triangleright B \arrow[r, "h_A \triangleright h_B"] \arrow[d, "f \triangleright g"'] & FGA \triangleright FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[d, "FGf \triangleright FGg"] & F(GA \otimes GB) \arrow[r, "F\mathsf{ex}_{A,B}"] \arrow[d, "F(Gf \otimes Gg)"] & F(GB \otimes GA) \arrow[r, "F\mathsf{n}_{B,A}"] \arrow[d, "F(Gg \otimes Gf)"] & FG(B \triangleright A) \arrow[r, "\varepsilon_{B \triangleright A}"] \arrow[d, "FG(g \triangleright f)"] & B \triangleright A \arrow[d, "g \triangleright f"] \\ A' \triangleright B' \arrow[r, "h_{A'} \triangleright h_{B'}"'] & FGA' \triangleright FGB' \arrow[r, "\mathsf{m}_{GA',GB'}"'] & F(GA' \otimes GB') \arrow[r, "F\mathsf{ex}_{A',B'}"'] & F(GB' \otimes GA') \arrow[r, "F\mathsf{n}_{B',A'}"'] & FG(B' \triangleright A') \arrow[r, "\varepsilon_{B' \triangleright A'}"'] & B' \triangleright A' \end{tikzcd}$$

◀

▶ **Lemma 26.** *The following diagrams commute in the co-Eilenberg-Moore category $\mathcal{L}^S$:*

$$\begin{tikzcd} F((GA \otimes GB) \otimes GC) \arrow[r, "F(\mathsf{n}_{A,B} \otimes id_{GC})"] \arrow[d, "F(\mathsf{ex}_{A,B} \otimes id_{GC})"'] & F(G(A \triangleright B) \otimes GC) \arrow[r, "F(\mathsf{ex}_{A,B} \otimes id_{GC})"] & FG((A \triangleright B) \triangleright C) \arrow[d, "\varepsilon_{(A \triangleright B) \triangleright C}"] \\ F(G(B \triangleright A) \otimes GC) \arrow[d, "F(\mathsf{n}_{B,A} \otimes id_{GC})"'] & & (A \triangleright B) \triangleright C \arrow[d, "\mathsf{ex}^S_{A,B} \triangleright id_C"] \\ F(G(B \triangleright A) \otimes GC) \arrow[r, "F\mathsf{n}_{B \triangleright A,C}"'] & FG((B \triangleright A) \triangleright C) \arrow[r, "\varepsilon_{(B \triangleright A) \triangleright C}"'] & (B \triangleright A) \triangleright C \end{tikzcd}$$

$$\begin{tikzcd} F(GB \otimes (GC \otimes GA)) \arrow[r, "F(id_{GB} \otimes \mathsf{n}_{C,A})"] \arrow[d, "F(id_{GB} \otimes \mathsf{ex}_{C,A})"'] & F(GB \otimes G(C \triangleright A)) \arrow[r, "F\mathsf{n}_{B,C \triangleright A}"] & FG(B \triangleright (C \triangleright A)) \arrow[d, "\varepsilon_{B \triangleright (C \triangleright A)}"] \\ F(GB \otimes (GA \otimes GC)) \arrow[d, "F(id_{GB} \otimes \mathsf{n}_{A,C})"'] & & B \triangleright (C \triangleright A) \arrow[d, "id_A \triangleright \mathsf{ex}^S_{C,A}"] \\ F(GB \otimes G(A \triangleright C)) \arrow[r, "F\mathsf{n}_{B,A \triangleright C}"'] & FG(B \triangleright (A \triangleright C)) \arrow[r, "\varepsilon_{B \triangleright (A \triangleright C)}"'] & B \triangleright (A \triangleright C) \end{tikzcd}$$

Proof. We only write the proof for the first diagram. The proof for the second one is similar. (1), (2), (3)–naturality of m; (4)–F is monoidal; (5), (12)–ε is monoidal; (6), (7), (8), (9), (10)–obvious;

(11)–coalgebra.

$$
\begin{tikzcd}
F(G(A \triangleright B) \otimes GC) \arrow[r, "F\mathsf{n}_{A \triangleright B,C}"] & FG((A \triangleright B) \triangleright C) \arrow[rr, "\varepsilon_{(A \triangleright B) \triangleright C}"] \arrow[d, phantom, "(5)"] & & (A \triangleright B) \triangleright C \arrow[r, equal] \arrow[dl, phantom, "(6)"] & (A \triangleright B) \triangleright C \arrow[d, "(h_A \triangleright h_B) \triangleright id_C"] \arrow[dl, phantom, "(11)"] \\
F((GA \otimes GB) \otimes GC) \arrow[u, "F(\mathsf{n}_{A,B} \otimes id_{GC})"] \arrow[d, "F(\mathsf{ex}_{A,B} \otimes id_{GC})"] \arrow[r, phantom, "(1)"] & FG(A \triangleright B) \triangleright FGC \arrow[ul, "\mathsf{m}_{G(A \triangleright B),GC}"] \arrow[urr, "\varepsilon_{A \triangleright B} \triangleright \varepsilon_C"] \arrow[rr, "id \triangleright \varepsilon_C"] \arrow[d, phantom, "(7)"] & & FG(A \triangleright B) \triangleright C \arrow[u, "\varepsilon_{A \triangleright B} \triangleright id_C"] \arrow[r, phantom, "(12)"] & (FGA \triangleright FGB) \triangleright C \arrow[ul, "(\varepsilon_A \triangleright \varepsilon_B) \triangleright id_C"] \arrow[d, "\mathsf{m}_{GA,GB} \triangleright id_C"] \\
F((GB \otimes GA) \otimes GC) \arrow[d, "F(\mathsf{n}_{B,A} \otimes id_{GC})"] \arrow[r, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & F(GA \otimes GB) \triangleright FGC \arrow[ul, "\mathsf{m}_{GA \otimes GB,GC}"] \arrow[u, "F\mathsf{n}_{A,B} \triangleright id_{FGC}"] \arrow[rrr, "id_{F(GA \otimes GB)} \triangleright \varepsilon_C"] \arrow[dr, "F\mathsf{ex}_{A,B} \triangleright id_{FGC}"] & & & F(GA \otimes GB) \triangleright C \arrow[ul, "F\mathsf{n}_{A,B} \otimes id_C"] \arrow[d, "F\mathsf{ex}_{A,B} \triangleright id_C"] \arrow[dl, phantom, "(8)"] \\
F(G(B \triangleright A) \otimes GC) \arrow[d, "F\mathsf{n}_{B \triangleright A,C}"] \arrow[dr, phantom, "(4)"] & FG(B \triangleright A) \triangleright FGC \arrow[l, "\mathsf{m}_{G(B \triangleright A),GC}"] \arrow[d, "\varepsilon_{B \triangleright A} \triangleright \varepsilon_C"] \arrow[drrr, "id_{FG(B \triangleright A)} \triangleright \varepsilon_C"] \arrow[dr, phantom, "(9)"] & F(GB \otimes GA) \triangleright FGC \arrow[ull, "\mathsf{m}_{GB \otimes GA,GC}"] \arrow[l, "F\mathsf{n}_{B,A} \triangleright id"] \arrow[rr, "id \triangleright \varepsilon_C"] \arrow[dr, phantom, "(10)"] & & F(GB \otimes GA) \triangleright C \arrow[d, "F\mathsf{n}_{B,A} \triangleright id_C"] \\
FG((B \triangleright A) \triangleright C) \arrow[r, "\varepsilon_{(B \triangleright A) \triangleright C}"] & (B \triangleright A) \triangleright C & & & FG(B \triangleright A) \triangleright C \arrow[lll, "\varepsilon_{B \triangleright A} \triangleright id_C"]
\end{tikzcd}
$$

◀

▶ **Theorem 27.** *The co-Eilenberg-Moore category $\mathcal{L}^S$ of S is symmetric monoidal closed.*

Proof. Let $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ be the Lambek cateogry in a SMCC-Lambek model and S be the comonad on $\mathcal{L}$. Since $\mathcal{L}$ is a Lambek category, it is obvious that $\mathcal{L}$ is also Lambek. By Corollary 6, we only need to prove the exchange defined in Lemma 25 satisfies the three commutative diagrams in Definition 3.

The first triangle in Definition 3 commutes as follows: (1)–coalgebra; (2)–ε is monoidal; (3)–naturality of ρ; (4)–naturality of ε; (5)–naturality of m; (6)–F is monoidal; (7)–C is symmetric; (8)–naturality of ex; (9)–G is monoidal.

$$
\begin{tikzcd}
A \triangleright I' \arrow[rr, "h_A \triangleright h_{I'}"] \arrow[ddd, "\rho'_A"] \arrow[ddr, equal] \arrow[dr, phantom, "(1)"] & & FGA \triangleright FGI' \arrow[rr, "\mathsf{m}_{GA,GI'}"] \arrow[dl, "id_{FGA} \triangleright \varepsilon_{I'}"] \arrow[dr, phantom, "(5)"] & & F(GA \otimes GI') \arrow[dddd, "F\mathsf{ex}_{GA,GI'}"] \\
& FGA \triangleright I' \arrow[d, "\varepsilon_{FGA} \triangleright id_{I'}"] \arrow[dr, equal] \arrow[r, phantom, "(2)"] & FGA \triangleright FI \arrow[u, "id_{FGA} \triangleright F\mathsf{n}_{I'}"] \arrow[r, "\mathsf{m}_{GA,I}"] \arrow[d, phantom, "(6)"] & F(GA \otimes I) \arrow[ur, "F(id_{GA} \otimes \mathsf{n}_{I'})"] \arrow[ddl, "F\rho_{GA}"] \arrow[dd, "F\mathsf{ex}_{GA,I}"] \arrow[ddr, phantom, "(8)"] & \\
& A \triangleright I' \arrow[d, phantom, "(3)"] & FGA \triangleright I' \arrow[u, "id_{FGA} \triangleright \mathsf{m}_I"] \arrow[d, "\rho'_{FGA}"] \arrow[r, phantom, "(7)"] & & \\
A \arrow[d, "\lambda'_A"] \arrow[dr, phantom, "(4)"] & & FGA \arrow[ll, "\varepsilon_A"] \arrow[dr, phantom, "(9)"] & F(I \otimes GA) \arrow[l, "F\lambda_{GA}"] \arrow[dr, "F(\mathsf{n}_{I'} \otimes id_{GA})"] & \\
I' \triangleright A & & FG(I' \triangleright A) \arrow[ll, "\varepsilon_{I' \triangleright A}"] \arrow[u, "FG\lambda'_A"] & & F(GI' \otimes GA) \arrow[ll, "F\mathsf{n}_{I',A}"]
\end{tikzcd}
$$

The second triangle in the proof commutes as follows: (1) and (5)–coalgebra; (2) and (4)–ε is monoidal; (3)–$\mathcal{C}$ is symmetric.

$$
\begin{tikzcd}
A \triangleright B \arrow[r, "h_A \triangleright h_B"] \arrow[dd, equal] \arrow[dr, phantom, "(1)"] & FGA \triangleright FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[ddl, "\varepsilon_A \triangleright \varepsilon_B"'] & F(GA \otimes GB) \arrow[r, "F\mathsf{ex}_{A,B}"] \arrow[dd, equal] & F(GB \otimes GA) \arrow[rr, "F\mathsf{n}_{B,A}"] \arrow[dd, equal] & & FG(B \triangleright A) \arrow[dd, "\varepsilon_{B \triangleright A}"] \\
& {} \arrow[r, phantom, "(2)"] & {} \arrow[r, phantom, "(3)"] & {} \arrow[r, phantom, "(4)"] & B \triangleright A \arrow[dr, equal] & \\
A \triangleright B & FG(A \triangleright B) \arrow[l, "\varepsilon_{A \triangleright B}"] & F(GA \otimes GB) \arrow[l, "F\mathsf{n}_{A,B}"] & F(GB \otimes GA) \arrow[l, "F\mathsf{ex}_{B,A}"] & FGB \triangleright FGA \arrow[l, "\mathsf{m}_{GB,GA}"] \arrow[u, "\varepsilon_B \triangleright \varepsilon_A"] \arrow[ur, phantom, "(5)"] & B \triangleright A \arrow[l, "h_A \triangleright h_A"]
\end{tikzcd}
$$

The third diagram commutes as follows: (1) and (7)–Lemma 26; (2)–naturality of α'; (3) and (8)–naturality of ε; (4), (6) and (12)–G is a monoidal functor; (5)–$\mathcal{C}$ is symmetrical monoidal closed; (9)–coalgebra; (10)–ε is a monoidal natural transformation; (11)–naturality of ex.

$$
\begin{tikzcd}
(A \triangleright B) \triangleright C \arrow[rr, "\alpha'_{A,B,C}"] \arrow[ddddd, "\mathsf{ex}^S_{A,B} \triangleright id_C"'] & & A \triangleright (B \triangleright C) \arrow[rr, "h_A \triangleright h_{B \triangleright C}"] \arrow[dr, equal] & {} \arrow[d, phantom, "(9)"] & FGA \triangleright FG(B \triangleright C) \arrow[dl, "\varepsilon_A \triangleright \varepsilon_{B \triangleright C}"] \arrow[dddd, "\mathsf{m}_{GA,G(B \triangleright C)}"] \\
& FG((A \triangleright B) \triangleright C) \arrow[ul, "\varepsilon_{(A \triangleright B) \triangleright C}"] \arrow[drr, "FG\alpha'_{A,B,C}"] \arrow[rr, phantom, "(3)"] & & A \triangleright (B \triangleright C) & \\
& F(G(A \triangleright B) \otimes GC) \arrow[u, "F\mathsf{n}_{A \triangleright B,C}"] & & FG(A \triangleright (B \triangleright C)) \arrow[u, "\varepsilon_{A \triangleright (B \triangleright C)}"] \arrow[r, phantom, "(10)"] & {} \\
{} \arrow[r, phantom, "(1)"] & F((GA \otimes GB) \otimes GC) \arrow[u, "F(\mathsf{n}_{A,B} \otimes id_{GC})"] \arrow[rr, "F\alpha_{GA,GB,GC}"] \arrow[d, "F(\mathsf{ex}_{A,B} \otimes id_{GC})"] & {} \arrow[u, phantom, "(4)"] & F(GA \otimes (GB \otimes GC)) \arrow[d, "F\mathsf{ex}_{GA,GB \otimes GC}"] \arrow[dr, "F(id_{GA} \otimes \mathsf{n}_{B,C})"] & \\
& F((GB \otimes GA) \otimes GC) \arrow[d, "F(\mathsf{n}_{B,A} \otimes id_{GC})"] \arrow[ddr, "F\alpha"] \arrow[rr, phantom, "(5)"] & & F((GB \otimes GC) \otimes GA) \arrow[d, "F\alpha_{GB,GC,GA}"] \arrow[dr, "F(\mathsf{n}_{B,C} \otimes id_{GA})"] \arrow[r, phantom, "(11)"] & F(GA \otimes G(B \triangleright C)) \arrow[uul, "F\mathsf{n}_{A,B \triangleright C}"'] \arrow[d, "F\mathsf{ex}_{A,B \triangleright C}"] \\
(B \triangleright A) \triangleright C \arrow[dddd, "\alpha'_{B,A,C}"'] & F(G(B \triangleright A) \otimes GC) \arrow[dd, "F\mathsf{n}_{B \triangleright A,C}"] & & F(GB \otimes (GC \otimes GA)) \arrow[dl, "F(id_{GC} \otimes \mathsf{ex}_{C,A})"'] \arrow[dd, "F(id_{GB} \otimes \mathsf{n}_{C,A})"] & F(G(B \triangleright C) \otimes GA) \arrow[ddddd, "F\mathsf{n}_{B \triangleright C,A}"] \\
& & F(GB \otimes (GA \otimes GC)) \arrow[dd, "F(id_{GB} \otimes \mathsf{n}_{A,C})"] & & \\
{} \arrow[r, phantom, "(2)"] & FG((B \triangleright A) \triangleright C) \arrow[uul, "\varepsilon_{(B \triangleright A) \triangleright C}"] \arrow[dd, "FG\alpha'_{B,A,C}"] & & F(GB \otimes G(C \triangleright A)) \arrow[dd, "F\mathsf{n}_{B,C \triangleright A}"] & \\
& {} \arrow[r, phantom, "(6)"] & F(GB \otimes G(A \triangleright C)) \arrow[dl, "F\mathsf{n}_{B,A \triangleright C}"] & {} \arrow[r, phantom, "(12)"] & {} \\
B \triangleright (A \triangleright C) \arrow[d, "\mathsf{id}_B \triangleright \mathsf{ex}^S_{A,C}"'] & FG(B \triangleright (A \triangleright C)) \arrow[l, "\varepsilon_{B \triangleright (A \triangleright C)}"] & {} \arrow[u, phantom, "(7)"] & FG(B \triangleright (C \triangleright A)) \arrow[dlll, "\varepsilon_{B \triangleright (C \triangleright A)}"] & \\
B \triangleright (C \triangleright A) & & (B \triangleright C) \triangleright A \arrow[ll, "\alpha'_{B,C,A}"] & {} \arrow[u, phantom, "(8)"] & FG((B \triangleright C) \triangleright A) \arrow[ul, "FG\alpha_{B,C,A}"'] \arrow[ll, "\varepsilon_{(B \triangleright C) \triangleright A}"]
\end{tikzcd}
$$

◀

5 Non-Commutative Linear Logic

In a LAM, the SMCC $\mathcal{C}$ models the commutative linear logic and the Lambeck category $\mathcal{L}$ models the non-commutative variant. In Section 5.1, we will present the term assignment for sequent calculus of both sides and prove the cut elimination theorem. In Section 5.2, we present the term assignment for natural deduction of both sides and prove the logic is strongly normalizing.

A sequent in the commutative side is of the form $\Psi, \Phi \vdash_{\mathcal{C}} t : X$. The types must be X, Y, Z, etc., which are objects in the SMCC $\mathcal{C}$. Tye typing contexts are multisets. Suppose Ψ is the set $x_1 : X_1, x_2 : X_2, ..., x_m : X_m$ and Φ is the set $y_1 : Y_1, y_2 : Y_2, ..., y_n : Y_n$, then the categorical interpretation of the sequent is the morphism $(X_1 \otimes X_2 \otimes ... \otimes X_m) \otimes (Y_1 \otimes Y_2 \otimes ... \otimes Y_n) \to X$.

A sequent in the non-commutative side is of the form $\Gamma, \Delta \vdash_{\mathcal{L}} s : A$. The types must be A, B, C, etc., which are objects in the Lambek category $\mathcal{L}$. Tye typing contexts are lists instead of multisets. The typing contexts are mixed in the sense that they could include contexts from the commutative side. When a commutative context $\Psi = \{x_1 : X_1, ..., x_m : X_m\}$ is included, it is interpreted as the object $F(X_1 \otimes ... \otimes X_m)$. Therefore, the interpretation of the sequent $\Psi, \Gamma \vdash_{\mathcal{L}} s : A$, where Ψ is defined as above and Γ is the list $y_1 : A_1, ..., y_n : A_n$, is the morphism $F(X_1 \otimes ... \otimes X_m) \triangleright (A_1 \triangleright ... \triangleright A_n) \to A$.

For the commutative side, since the contexts are multisets, the following exchange rule is implicit in both sequent calculus and natural deduction:

$$\frac{\Phi, x : X, y : Y, \Psi \vdash_{\mathcal{C}} t : Z}{\Phi, z : Y, w : X, \Psi \vdash_{\mathcal{C}} \mathsf{ex}\, w, z\, \mathsf{with}\, x, y\, \mathsf{in}\, t : Z}\ \text{T\_BETA}$$

5.1 Sequent Calculus

The term assignment for sequent calculus of the commutative part of the model, i.e. the SMCC of the adjunction, is defined in Figure 1. And the term assignme for the non-commutative part, i.e. the Lambek category of the adjunction, is defined in Figure 2. We do not have the structural rules except for exchange because the calculus is for linear logic.

$$\frac{}{x : X \vdash_{\mathcal{C}} x : X}\ \text{T\_AX} \qquad \frac{\Phi, \Psi \vdash_{\mathcal{C}} t : X}{\Phi, x : \mathsf{UnitT}, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, t : X}\ \text{T\_UNITL} \qquad \frac{}{\cdot \vdash_{\mathcal{C}} \mathsf{trivT} : \mathsf{UnitT}}\ \text{T\_UNITR}$$

$$\frac{\Phi, x : X, y : Y, \Psi \vdash_{\mathcal{C}} t : Z}{\Phi, z : X \otimes Y, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t : Z}\ \text{T\_TENL} \qquad \frac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi \vdash_{\mathcal{C}} t_2 : Y}{\Phi, \Psi \vdash_{\mathcal{C}} t_1 \otimes t_2 : X \otimes Y}\ \text{T\_TENR}$$

$$\frac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi_1, x : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, y : X \multimap Y, \Phi, \Psi_2 \vdash_{\mathcal{C}} [\mathsf{app}\, y\, t_1/x]t_2 : Z}\ \text{T\_IMPL} \qquad \frac{\Phi, x : X \vdash_{\mathcal{C}} t : Y}{\Phi \vdash_{\mathcal{C}} \lambda x : X.t : X \multimap Y}\ \text{T\_IMPR}$$

$$\frac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_{\mathcal{C}} \mathsf{G}s : \mathsf{G}A}\ \text{T\_GR} \qquad \frac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi_1, x : X, \Psi_2 \vdash_{\mathcal{C}} t_2 : Y}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} [t_1/x]t_2 : Y}\ \text{T\_CUT}$$

Figure 1 Sequent Calculus: Commutative Part

Next, we prove cut elimination for the sequent calculus. We define the **degree** $|X|$ **(or** $|A|$**) of a commutative (or non-commutative) formula** to be the number of logical connectives $|X|$ plus 1. For instance, $|X \otimes Y| = |X| + |Y| + 1$. And the **degree of a cut rule** is the degree of the cut formula. The following key cases demonstrate how we can replace a cut with at most two cuts with lower degree. The **degree** $|\Pi|$ **of a proof** Π is the maximum of the degrees of all cut fules in the proof and $|\Pi| = 0$ if Π is cut-free. Finally, the **height** $h(\Pi)$ **of a proof** Π is the length of the longest path in the proof tree and the height of an axiom is 0.

We consider the following 11 key cases in proving cut elimination, each of which is a (R, L) pair for the same connective.

- (T_UNITR, T_UNITL):

$$\frac{\dfrac{}{\cdot \vdash_{\mathcal{C}} \mathsf{triv} : \mathsf{Unit}}\ \text{UNITR} \qquad \dfrac{\Phi, \Psi \vdash_{\mathcal{C}} t : X}{\Phi, x : \mathsf{Unit}, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t : X}\ \text{UNITL}}{\Phi, \Psi \vdash_{\mathcal{C}} [\mathsf{triv}/x](\mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t) : X}\ \text{CUT}$$

is transformed to

$$\frac{}{x : A \vdash_{\mathcal{L}} x : A}\ \text{S\_AX} \qquad \frac{}{\cdot \vdash_{\mathcal{L}} \mathsf{trivS} : \mathsf{UnitS}}\ \text{S\_UNITR} \qquad \frac{\Gamma, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, x : \mathsf{UnitT}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitT}\ \mathsf{be}\ \mathsf{trivT}\ \mathsf{in}\ s : A}\ \text{S\_UNITL1}$$

$$\frac{\Gamma, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, x : \mathsf{UnitS}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitS}\ \mathsf{be}\ \mathsf{trivS}\ \mathsf{in}\ s : A}\ \text{S\_UNITL2} \qquad \frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : Y, w : X, \Delta \vdash_{\mathcal{L}} \mathsf{ex}\, w, z\, \mathsf{with}\, x, y\ \mathsf{in}\ s : A}\ \text{S\_BETA}$$

$$\frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \otimes Y, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ s : A}\ \text{S\_TENL1} \qquad \frac{\Gamma, x : A, y : B, \Delta \vdash_{\mathcal{L}} s : C}{\Gamma, z : A \triangleright B, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\ \mathsf{be}\ x \triangleright y\ \mathsf{in}\ s : C}\ \text{S\_TENL2}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \qquad \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Delta \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B}\ \text{S\_TENR} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X \qquad \Gamma, x : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : X \multimap Y, \Phi, \Delta \vdash_{\mathcal{L}} [\mathsf{app}\, y\, t/x]s : A}\ \text{S\_IMPL}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \qquad \Delta, x : B \vdash_{\mathcal{L}} s_2 : C}{\Delta, y : A \rightharpoonup B, \Gamma \vdash_{\mathcal{L}} [\mathsf{app}_r\, y\, s_1/x]s_2 : C}\ \text{S\_IMPRL} \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \qquad x : B, \Delta \vdash_{\mathcal{L}} s_2 : C}{\Gamma, y : B \leftharpoonup A, \Delta \vdash_{\mathcal{L}} [\mathsf{app}_l\, y\, s_1/x]s_2 : C}\ \text{S\_IMPLL}$$

$$\frac{\Gamma, x : A \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A.s : A \rightharpoonup B}\ \text{S\_IMPRR} \qquad \frac{x : A, \Gamma \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A.s : B \leftharpoonup A}\ \text{S\_IMPLR} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X}{\Phi \vdash_{\mathcal{L}} \mathsf{F}t : \mathsf{F}X}\ \text{S\_FR}$$

$$\frac{\Gamma, x : X, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : \mathsf{F}X, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{F}X\ \mathsf{be}\ \mathsf{F}\, x\ \mathsf{in}\ s : A}\ \text{S\_FL} \qquad \frac{\Gamma, x : A, \Delta \vdash_{\mathcal{L}} s : B}{\Gamma, y : \mathsf{G}A, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{G}A\ \mathsf{be}\ \mathsf{G}\, x\ \mathsf{in}\ s : B}\ \text{S\_GL}$$

$$\frac{\Phi \vdash_{\mathcal{C}} t : X \qquad \Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_1 \vdash_{\mathcal{L}} [t/x]s : A}\ \text{S\_CUT1} \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \qquad \Delta_1, x : A, \Delta_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [s_1/x]s_2 : B}\ \text{S\_CUT2}$$

Figure 2 Sequent Calculus: Non-Commutative Part

$$\Phi, \Psi \vdash_{\mathcal{C}} t : X$$

- (T_UNITR, S_UNITL1):

$$\frac{\dfrac{}{\cdot \vdash_{\mathcal{C}} \mathsf{triv} : \mathsf{Unit}}\ \text{UNITR} \qquad \dfrac{\Gamma, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, x : \mathsf{Unit}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\ \mathsf{be}\ \mathsf{triv}\ \mathsf{in}\ s : A}\ \text{UNITL}}{\Gamma, \Delta \vdash_{\mathcal{L}} [\mathsf{triv}/x](\mathsf{let}\, x : \mathsf{Unit}\ \mathsf{be}\ \mathsf{triv}\ \mathsf{in}\ s) : A}\ \text{CUT}$$

is transformed to

$$\Gamma, \Delta \vdash_{\mathcal{L}} s : A$$

- (T_TENR, T_TENL):

$$\frac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : X \qquad \Phi_2 \vdash_{\mathcal{C}} t_2 : Y}{\Phi_1, \Phi_2 \vdash_{\mathcal{C}} t_1 \otimes t_2 : X \otimes Y}\ \text{TENR} \qquad \dfrac{\Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Psi_1, z : X \otimes Y, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, z : X \otimes Y\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t_3 : Z}\ \text{TENL}}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} [t_1 \otimes t_2/z](\mathsf{let}\, z : X \otimes Y\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t_3) : Z}\ \text{CUT}$$

is transformed to

$$\frac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : X \qquad \Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Psi_1, \Phi_1, y : Y, \Psi_2 \vdash_{\mathcal{C}} [t_1/x]t_3 : Z}\ \text{CUT} \qquad \Phi_2 \vdash_{\mathcal{C}} t_2 : Y}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} [t_2/y][t_1/x]t_3 : Z}\ \text{CUT}$$

- (T_TENR, S_TENL1):

$$\frac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : X \qquad \Phi_2 \vdash_{\mathcal{C}} t_2 : Y}{\Phi_1, \Phi_2 \vdash_{\mathcal{C}} t_1 \otimes t_2 : X \otimes Y}\ \text{TENR} \qquad \dfrac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \otimes Y, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ s : A}\ \text{TENL1}}{\Gamma, \Phi_1, \Phi_2, \Delta \vdash_{\mathcal{L}} [t_1 \otimes t_2/z](\mathsf{let}\, z : X \otimes Y\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ s) : A}\ \text{CUT1}$$

is transformed to

$$\frac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : X \qquad \Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, \Phi_1, y : Y, \Delta \vdash_{\mathcal{L}} [t_1/x]s : A}\ \text{CUT1} \qquad \Phi_2 \vdash_{\mathcal{C}} t_2 : Y}{\Gamma, \Phi_1, \Phi_2, \Delta \vdash_{\mathcal{L}} [t_2/y][t_1/x]s : A}\ \text{CUT1}$$

- (T_IMPR, T_IMPL)

$$\frac{\dfrac{\Phi_1, x : X \vdash_{\mathcal{C}} t_1 : Y}{\Phi_1 \vdash_{\mathcal{C}} \lambda x : X.t_1 : X \multimap Y}\ \text{IMPR} \qquad \dfrac{\Phi_2 \vdash_{\mathcal{C}} t_2 : X \qquad \Psi_1, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Psi_1, z : X \multimap Y, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} [\mathsf{app}\, z\, t_2/y]t_3 : Z}\ \text{IMPL}}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} [(\lambda x : X.t_1)/z][\mathsf{app}\, z\, t_2/y]t_3 : Z}\ \text{CUT}$$

is transformed to

$$\dfrac{\dfrac{\Phi_1, x : X \vdash_{\mathcal{C}} t_1 : Y \qquad \Phi_2 \vdash_{\mathcal{C}} t_2 : X}{\Phi_1, \Phi_2 \vdash_{\mathcal{C}} [t_2/x]t_1 : Y}\,\textsc{cut} \qquad \Psi_1, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} [([t_2/x]t_1)/y]t_3 : Z}\,\textsc{cut}$$

- (S_$\textsc{unit}$R, S_$\textsc{unit}$L2):

$$\dfrac{\dfrac{}{\cdot \vdash_{\mathcal{L}} \mathsf{trivS} : \mathsf{Unit}}\,\textsc{unit}\mathrm{R} \qquad \dfrac{\Gamma, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, x : \mathsf{Unit}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\,\textsc{unit}\mathrm{L2}}{\Gamma, \Delta \vdash_{\mathcal{L}} [\mathsf{trivS}/x](\mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s) : A}\,\textsc{cut}2$$

is transformed to

$$\Gamma, \Delta \vdash_{\mathcal{L}} s : A$$

- (S_$\textsc{ten}$R, S_$\textsc{ten}$L2):

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : B}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B}\,\textsc{ten}\mathrm{R} \qquad \dfrac{\Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, z : A \triangleright B, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s : C}\,\textsc{ten}\mathrm{L1}}{\Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} [s_1 \triangleright s_2/z](\mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s) : C}\,\textsc{cut}2$$

is transformed to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, y : B, \Delta_2 \vdash_{\mathcal{L}} [s_1/x]s_3 : C}\,\textsc{cut}2 \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} [s_2/y][s_1/x]s_3 : C}\,\textsc{cut}2$$

- (S_$\textsc{impr}$R, S_$\textsc{impr}$L):

$$\dfrac{\dfrac{\Gamma, x : A \vdash_{\mathcal{L}} s_1 : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A.s_1 : A \rightharpoonup B}\,\textsc{impr}\mathrm{R} \qquad \dfrac{\Delta_1 \vdash_{\mathcal{L}} s_2 : A \qquad \Delta_2, y : B \vdash_{\mathcal{L}} s_3 : C}{\Delta_2, z : A \rightharpoonup B, \Delta_1 \vdash_{\mathcal{L}} [\mathsf{app}_r\, z\, s_2/y]s_3 : C}\,\textsc{impr}\mathrm{L}}{\Delta_2, \Gamma, \Delta_1 \vdash_{\mathcal{L}} [(\lambda_r x : A.s_1)/z][\mathsf{app}_r\, z\, s_2/y]s_3 : C}\,\textsc{cut}2$$

is transformed to

$$\dfrac{\dfrac{\Gamma, x : A \vdash_{\mathcal{L}} s_1 : B \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta_1 \vdash_{\mathcal{L}} [s_2/x]s_1 : B}\,\textsc{cut}2 \qquad \Delta_2, y : B \vdash_{\mathcal{L}} s_3 : C}{\Delta_2, \Gamma, \Delta_1 \vdash_{\mathcal{L}} [([s_2/x]s_1)/y]s_3 : C}\,\textsc{cut}2$$

- (S_$\textsc{impl}$R, S_$\textsc{impl}$L):

$$\dfrac{\dfrac{x : A, \Gamma \vdash_{\mathcal{L}} s_1 : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A.s_1 : B \leftharpoonup A}\,\textsc{impl}\mathrm{R} \qquad \dfrac{\Delta_1 \vdash_{\mathcal{L}} s_2 : A \qquad y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, z : B \leftharpoonup A, \Delta_2 \vdash_{\mathcal{L}} [\mathsf{app}_l\, z\, s_2/y]s_3 : C}\,\textsc{impl}\mathrm{L}}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [(\lambda_l x : A.s_1)/z][\mathsf{app}_l\, z\, s_2/y]s_3 : C}\,\textsc{cut}2$$

is transformed to

$$\dfrac{\dfrac{x : A, \Gamma \vdash_{\mathcal{L}} s_1 : B \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A}{\Delta_1, \Gamma \vdash_{\mathcal{L}} [s_2/x]s_1 : B}\,\textsc{cut}2 \qquad y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [([s_2/x]s_1)/y]s_3 : C}\,\textsc{cut}2$$

- (S_$\textsc{Fr}$, S_$\textsc{Fl}$):

$$\dfrac{\dfrac{\Phi \vdash_{\mathcal{C}} t : X}{\Phi \vdash_{\mathcal{L}} \mathsf{F}t : \mathsf{F}X}\,\mathrm{FR} \qquad \dfrac{\Gamma, x : X, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : \mathsf{F}X, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}\, x\, \mathsf{in}\, s : A}\,\mathrm{FL}}{\Gamma, \Phi, \Delta \vdash_{\mathcal{L}} [\mathsf{F}t/y](\mathsf{let}\, y : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}\, x\, \mathsf{in}\, s) : A}\,\textsc{cut}2$$

is transformed to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : X \qquad \Gamma, x : A, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, \Phi, \Delta \vdash_{\mathcal{L}} [t/x]s : A}\,\textsc{cut}1$$

- (T_$\textsc{Gr}$, S_$\textsc{Gl}$):

$$\dfrac{\dfrac{\Phi \vdash_{\mathcal{L}} s_1 : A}{\Phi \vdash_{\mathcal{C}} \mathsf{G}s_1 : \mathsf{G}A}\,\mathrm{GR} \qquad \dfrac{\Gamma, x : A, \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, y : \mathsf{G}A, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{G}A\, \mathsf{be}\, \mathsf{G}\, x\, \mathsf{in}\, s_2 : B}\,\mathrm{GL}}{\Gamma, \Phi, \Delta \vdash_{\mathcal{L}} [\mathsf{G}s_1/y](\mathsf{let}\, y : \mathsf{G}A\, \mathsf{be}\, \mathsf{G}\, x\, \mathsf{in}\, s_2) : B}\,\textsc{cut}1$$

is transformed to

$$\dfrac{\Phi \vdash_{\mathcal{L}} s_1 : A \qquad \Gamma, x : A, \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Phi, \Delta \vdash_{\mathcal{L}} [s_1/x]s_2 : B}\,\textsc{cut}$$

Based on the key cases, given a formula (either commutative or non-commutative) L and proofs Π, Π', of sequents $M \vdash N$ and $M' \vdash N'$ respectively with degrees less than $|L|$, there is a proof of $M, N \vdash M', N'$ with degree less than $|L|$, s.t. all currences of formula L is removed. This can be proved by induction on $h(\Pi) + h(\Pi')$. Therefore, we have the result that given a proof of a sequent with degree $d > 0$, there is a proof of the same sequent. As a result, we have the cut elimination theorem.

▶ **Theorem 28** (Cut Elimination)**.** *Let Π be a proof of a sequent $\Phi \vdash_{\mathcal{C}} t : X$ or a sequent $\Gamma \vdash_{\mathcal{L}} s : A$ s.t. $|\Pi| > 0$. Then there is a cut-free proof of the same sequent.*

5.2 Natural Deduction

The term assignment for natural deduction of the commutative part of the model, i.e. the SMCC of the adjunction, is defined in Figure 3. And the term assignme for the non-commutative part, i.e. the Lambek category of the adjunction, is defined in Figure 4.

$$\frac{}{x : X \vdash_{\mathcal{C}} x : X}\ \text{T\_ID} \qquad \frac{}{\cdot \vdash_{\mathcal{C}} \mathsf{triv} : \mathsf{Unit}}\ \text{T\_UNITI} \qquad \frac{\Phi \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \quad \Psi \vdash_{\mathcal{C}} t_2 : Y}{\Phi, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t_2 : Y}\ \text{T\_UNITE}$$

$$\frac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi \vdash_{\mathcal{C}} t_2 : Y}{\Phi, \Psi \vdash_{\mathcal{C}} t_1 \otimes t_2 : X \otimes Y}\ \text{T\_TENI} \qquad \frac{\Phi \vdash_{\mathcal{C}} t_1 : X \otimes Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2 : Z}\ \text{T\_TENE}$$

$$\frac{\Phi, x : X \vdash_{\mathcal{C}} t : Y}{\Phi \vdash_{\mathcal{C}} \lambda x : X.t : X \multimap Y}\ \text{T\_IMPI} \qquad \frac{\Phi \vdash_{\mathcal{C}} t_1 : X \multimap Y \quad \Psi \vdash_{\mathcal{C}} t_2 : X}{\Phi, \Psi \vdash_{\mathcal{C}} \mathsf{app}\, t_1\, t_2 : Y}\ \text{T\_IMPE} \qquad \frac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_{\mathcal{C}} \mathsf{G}s : \mathsf{G}A}\ \text{T\_GI}$$

$$\frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : Y, w : X, \Delta \vdash_{\mathcal{L}} \mathsf{ex}\, w, z\, \mathsf{with}\, x, y\, \mathsf{in}\, s : A}\ \text{S\_BETA}$$

Figure 3 Natural Deduction: Commutative Part

$$\frac{}{x : A \vdash_{\mathcal{L}} x : A}\ \text{S\_ID} \qquad \frac{}{\cdot \vdash_{\mathcal{L}} \mathsf{trivS} : \mathsf{Unit}}\ \text{S\_UNITI} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : \mathsf{Unit} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \text{S\_UNITE1}$$

$$\frac{\Phi \vdash_{\mathcal{C}} t : \mathsf{Unit} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \text{S\_UNITE1} \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_2 : A}\ \text{S\_UNITE2}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Delta \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B}\ \text{S\_TENI} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X \otimes Y \quad \Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, t : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\ \text{S\_TENE1}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C}\ \text{S\_TENE2} \qquad \frac{\Gamma, x : A \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A.s : A \rightharpoonup B}\ \text{S\_IMPRI}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_2 : B}\ \text{S\_IMPRE} \qquad \frac{x : A, \Gamma \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A.s : B \leftharpoonup A}\ \text{S\_IMPLI}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftharpoonup A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta, \Gamma \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_2 : B}\ \text{S\_IMPLE} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : \mathsf{G}A}{\Phi \vdash_{\mathcal{L}} \mathsf{derelict}\, t : A}\ \text{S\_GE} \qquad \frac{\Phi \vdash_{\mathcal{C}} t : X}{\Phi \vdash_{\mathcal{L}} \mathsf{F}t : \mathsf{F}X}\ \text{S\_FI}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} y : \mathsf{F}X \quad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\, \mathsf{be}\, y\, \mathsf{in}\, s : A}\ \text{S\_FE}$$

Figure 4 Natural Deduction: Non-Commutative Part

We could derive exchange comonadically as follows:

$$\dfrac{\dfrac{}{z : \mathsf{FG}A \triangleright \mathsf{FG}B \vdash_{\mathcal{L}} z : \mathsf{FG}A \triangleright \mathsf{FG}B}\,\textsc{id} \quad \dfrac{\dfrac{}{x_2 : \mathsf{FG}A \vdash_{\mathcal{L}} x_2 : \mathsf{FG}A}\,\textsc{id} \quad \dfrac{\dfrac{}{y_2 : \mathsf{FG}B \vdash_{\mathcal{L}} y_2 : \mathsf{FG}B}\,\textsc{id} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{y_0 : \mathsf{G}B \vdash_{\mathcal{C}} y_0 : \mathsf{G}B}\,\textsc{id}}{y_0 : \mathsf{G}B \vdash_{\mathcal{L}} \mathsf{F}y_0 : \mathsf{FG}B}\,\mathrm{FI} \quad \dfrac{\dfrac{}{x_0 : \mathsf{G}A \vdash_{\mathcal{C}} x_0 : \mathsf{G}A}\,\textsc{id}}{x_0 : \mathsf{G}A \vdash_{\mathcal{L}} \mathsf{F}x_0 : \mathsf{FG}A}\,\mathrm{FI}}{y_0 : \mathsf{G}B, x_0 : \mathsf{G}A \vdash_{\mathcal{L}} \mathsf{F}y_0 \triangleright \mathsf{F}x_0 : \mathsf{FG}B \triangleright \mathsf{FG}A}\,\mathrm{FE}}{x_1 : \mathsf{G}A, y_1 : \mathsf{G}B \vdash_{\mathcal{L}} \mathsf{ex}\, y_1, x_1 \,\mathsf{with}\, y_0, x_0 \,\mathsf{in}\, (\mathsf{F}y_0 \triangleright \mathsf{F}x_0) : \mathsf{FG}B \triangleright \mathsf{FG}A}\,\textsc{beta}}{x_1 : \mathsf{G}A, y_2 : \mathsf{FG}B \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}y_1 : \mathsf{FG}B \,\mathsf{be}\, y_2 \,\mathsf{in}\, (\mathsf{ex}\, y_1, x_1 \,\mathsf{with}\, y_0, x_0 \,\mathsf{in}\, (\mathsf{F}y_0 \triangleright \mathsf{F}x_0)) : \mathsf{FG}B \triangleright \mathsf{FG}A}\,\mathrm{FE}}{x_2 : \mathsf{FG}A, y_2 : \mathsf{FG}B \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x_1 : \mathsf{FG}A \,\mathsf{be}\, x_2 \,\mathsf{in}\, (\mathsf{let}\, \mathsf{F}y_1 : \mathsf{FG}B \,\mathsf{be}\, y_2 \,\mathsf{in}\, (\mathsf{ex}\, y_1, x_1 \,\mathsf{with}\, y_0, x_0 \,\mathsf{in}\, (\mathsf{F}y_0 \triangleright \mathsf{F}x_0))) : \mathsf{FG}B \triangleright \mathsf{FG}A}\,\mathrm{FE}}{\dfrac{z : \mathsf{FG}A \triangleright \mathsf{FG}B \vdash_{\mathcal{L}} \mathsf{let}\, z : \mathsf{FG}A \triangleright \mathsf{FG}B \,\mathsf{be}\, x_2 \triangleright y_2 \,\mathsf{in}\, (\mathsf{let}\, \mathsf{F}x_1 : \mathsf{FG}A \,\mathsf{be}\, x_2 \,\mathsf{in}\, (\mathsf{let}\, \mathsf{F}y_1 : \mathsf{FG}B \,\mathsf{be}\, y_2 \,\mathsf{in}\, (\mathsf{ex}\, y_1, x_1 \,\mathsf{with}\, y_0, x_0 \,\mathsf{in}\, (\mathsf{F}y_0 \triangleright \mathsf{F}x_0)))) : \mathsf{FG}B \triangleright \mathsf{FG}A}{\cdot \vdash_{\mathcal{L}} \lambda_r z : \mathsf{FG}A \triangleright \mathsf{FG}B. \mathsf{let}\, z : \mathsf{FG}A \triangleright \mathsf{FG}B \,\mathsf{be}\, x_2 \triangleright y_2 \,\mathsf{in}\, (\mathsf{let}\, \mathsf{F}x_1 : \mathsf{FG}A \,\mathsf{be}\, x_2 \,\mathsf{in}\, (\mathsf{let}\, \mathsf{F}y_1 : \mathsf{FG}B \,\mathsf{be}\, y_2 \,\mathsf{in}\, (\mathsf{ex}\, y_1, x_1 \,\mathsf{with}\, y_0, x_0 \,\mathsf{in}\, (\mathsf{F}y_0 \triangleright \mathsf{F}x_0)))) : (\mathsf{FG}A \triangleright \mathsf{FG}B) \rightharpoonup (\mathsf{FG}B \triangleright \mathsf{FG}A)}\,\textsc{impr}\mathrm{I}}\,\textsc{ten}\mathrm{E2}$$

We also have the three cut rules derivable in the natural deduction: (NOTE: Don't know how to prove the third one S_cut2.)

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi_1, x : X, \Psi_2 \vdash_{\mathcal{C}} t_2 : Y}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} [t_1/x]t_2 : Y}\ \mathrm{T\_}\textsc{cut} \qquad \dfrac{\Phi \vdash_{\mathcal{C}} t : X \quad \Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_1 \vdash_{\mathcal{L}} [t/x]s : A}\ \mathrm{S\_}\textsc{cut}1 \qquad \dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta_1, x : A, \Delta_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [s_1/x]s_2 : B}\ \mathrm{S\_}\textsc{cut}2$$

5.2.1 Commuting Conversions

- Commutation of UnitT_E:
  - (T_unitE, T_unitE):

$$\dfrac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \quad \Phi_2 \vdash_{\mathcal{C}} t_2 : \mathsf{Unit}}{\Phi_2, \Phi_1 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_1 : \mathsf{Unit}}\,\textsc{unit}\mathrm{E} \quad \Phi_3 \vdash_{\mathcal{C}} t_3 : X}{\Phi_2, \Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{let}\, (\mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_1) : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_3 : X}\,\textsc{unit}\mathrm{E}$$

    commutes to

$$\dfrac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \quad \Phi_3 \vdash_{\mathcal{C}} t_3 : X}{\Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_3 : X}\,\textsc{unit}\mathrm{E} \quad \Phi_2 \vdash_{\mathcal{C}} t_2 : \mathsf{Unit}}{\Phi_2, \Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, (\mathsf{let}\, t_1 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_3) : X}\,\textsc{unit}\mathrm{E}$$

  - (T_unitE, T_tenE) need multiple exchanges at the end:

$$\dfrac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : X \otimes Y \quad \Phi_2 \vdash_{\mathcal{C}} t_2 : \mathsf{Unit}}{\Phi_2, \Phi_1 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_1 : X \otimes Y}\,\textsc{ten}\mathrm{E} \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Psi_1, \Phi_2, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, (\mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_1) : X \otimes Y \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t_3 : Z}\,\textsc{unit}\mathrm{E}$$

    commutes to

$$\dfrac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : X \otimes Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Psi_1, \Phi_1, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : X \otimes Y \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t_3 : Z}\,\textsc{unit}\mathrm{E} \quad \Phi_2 \vdash_{\mathcal{C}} t_2 : \mathsf{Unit}}{\Phi_2, \Psi_1, \Phi_1, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, (\mathsf{let}\, t_1 : X \otimes Y \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t_3) : Z}\,\textsc{ten}\mathrm{E}$$

  - (T_unitE, T_impE):

$$\dfrac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : X \multimap Y \quad \Phi_2 \vdash_{\mathcal{C}} t_2 : \mathsf{Unit}}{\Phi_2, \Phi_1 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_1 : X \multimap Y}\,\textsc{ten}\mathrm{E} \quad \Phi_3 \vdash_{\mathcal{C}} t_3 : X}{\Phi_2, \Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{app}\, (\mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_1)\, t_3 : Y}\,\textsc{unit}\mathrm{E}$$

    commutes to

$$\dfrac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : X \multimap Y \quad \Phi_3 \vdash_{\mathcal{C}} t_3 : X}{\Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{app}\, t_1\, t_3 : Y}\,\textsc{unit}\mathrm{E} \quad \Phi_2 \vdash_{\mathcal{C}} t_2 : \mathsf{Unit}}{\Phi_2, \Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, (\mathsf{app}\, t_1\, t_3) : Y}\,\textsc{ten}\mathrm{E}$$

- Commutation of $\otimes_E$:
  - (T_tenE, T_unitE):

$$\dfrac{\dfrac{\Phi_1, x : X, y : Y, \Phi_2 \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \quad \Psi_1 \vdash_{\mathcal{C}} t_2 : X \otimes Y}{\Phi_1, \Psi_1, \Phi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : X \otimes Y \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t_1 : \mathsf{Unit}}\,\textsc{ten}\mathrm{E} \quad \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, (\mathsf{let}\, t_2 : X \otimes Y \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t_1) : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_3 : Z}\,\textsc{unit}\mathrm{E}$$

    commutes to

$$\dfrac{\dfrac{\Phi_1, x : X, y : Y, \Phi_2 \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \quad \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Phi_1, x : X, y : Y, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_3 : Z}\,\textsc{unit}\mathrm{E} \quad \Psi_1 \vdash_{\mathcal{C}} t_2 : X \otimes Y}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : X \otimes Y \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, (\mathsf{let}\, t_1 : \mathsf{Unit} \,\mathsf{be}\, \mathsf{triv} \,\mathsf{in}\, t_3) : Z}\,\textsc{ten}\mathrm{E}$$

  - (T_tenE, T_tenE):

$$\dfrac{\dfrac{\Phi_1, x : X_2, y : Y_2, \Phi_2 \vdash_{\mathcal{C}} t_1 : X_1 \otimes Y_1 \quad \Psi \vdash_{\mathcal{C}} t_2 : X_2 \otimes Y_2}{\Phi_1, \Psi, \Phi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_2 : X_2 \otimes Y_2 \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t_1 : X_1 \otimes Y_1}\,\textsc{ten}\mathrm{E} \quad \Psi_1, w : X_1, z : Y_1, \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Psi_1, \Phi_1, \Psi, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, (\mathsf{let}\, t_2 : X_2 \otimes Y_2 \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t_1) : X_1 \otimes Y_1 \,\mathsf{be}\, w \otimes z \,\mathsf{in}\, t_3 : Z}\,\textsc{ten}\mathrm{E}$$

$\mathcal{C}$-reductions:

$$\frac{}{\mathsf{let}\,\mathsf{triv} : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t \rightsquigarrow t}\ \textsc{Tred\_letU}$$

$$\frac{}{\mathsf{let}\,t_1 \otimes t_2 : X \otimes Y\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_3 \rightsquigarrow [t_1/x][t_2/y]t_3}\ \textsc{Tred\_letT}$$

$$\frac{}{\mathsf{app}\,(\lambda x : X.t_1)\,t_2 \rightsquigarrow [t_2/x]t_1}\ \textsc{Tred\_lam}$$

$\mathcal{L}$-reductions:

$$\frac{}{\mathsf{let}\,\mathsf{F}t : \mathsf{F}X\,\mathsf{be}\,\mathsf{F}\,y\,\mathsf{in}\,s \rightsquigarrow [t/y]s}\ \textsc{Sred\_letF} \qquad \frac{}{\mathsf{app}_l\,(\lambda_l x : A.s_1)\,s_2 \rightsquigarrow [s_2/x]s_1}\ \textsc{Sred\_lamL}$$

$$\frac{}{\mathsf{app}_r\,(\lambda_r x : A.s_1)\,s_2 \rightsquigarrow [s_2/x]s_1}\ \textsc{Sred\_lamR} \qquad \frac{}{\mathsf{derelict}\,\mathsf{G}s \rightsquigarrow s}\ \textsc{Sred\_derelict}$$

Figure 5 β-Reduction for LAM Logic

Figure 6 Commuting Conversions for LAM Logic

commutes to

$$\dfrac{\dfrac{\Phi_1, x : X_2, y : Y_2, \Phi_2 \vdash_{\mathcal{C}} t_1 : X_1 \otimes Y_1 \qquad \Psi_1, w : X_1, z : Y_1, \Psi_2 \vdash_{\mathcal{C}} t_3 : Z}{\Psi_1, \Phi_1, x : X_2, y : Y_2, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\,t_1 : X_1 \otimes Y_1\,\mathsf{be}\,w \otimes z\,\mathsf{in}\,t_3 : Z}\ \textsc{tenE} \qquad \Psi \vdash_{\mathcal{C}} t_2 : X_2 \otimes Y_2}{\Psi_1, \Phi_1, \Psi, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\,t_2 : X_2 \otimes Y_2\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,(\mathsf{let}\,t_1 : X_1 \otimes Y_1\,\mathsf{be}\,w \otimes z\,\mathsf{in}\,t_3) : Z}\ \textsc{tenE}$$

- (T_tenE, T_impE):

$$\dfrac{\dfrac{\Phi_1, x : X_2, y : Y_2, \Phi_2 \vdash_{\mathcal{C}} t_1 : X_1 \multimap Y_1 \qquad \Psi_1 \vdash_{\mathcal{C}} t_2 : X_2 \otimes Y_2}{\Phi_1, \Psi_1, \Phi_2 \vdash_{\mathcal{C}} \mathsf{let}\,t_2 : X_2 \otimes Y_2\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_1 : X_1 \multimap Y_1}\ \textsc{impE} \qquad \Psi_2 \vdash_{\mathcal{C}} t_3 : X_1}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{app}\,(\mathsf{let}\,t_2 : X_2 \otimes Y_2\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,t_1)\,t_3 : Y_1}\ \textsc{tenE}$$

commutes to

$$\dfrac{\dfrac{\Phi_1, x : X_2, y : Y_2, \Phi_2 \vdash_{\mathcal{C}} t_1 : X_1 \multimap Y_1 \qquad \Psi_2 \vdash_{\mathcal{C}} t_3 : X_1}{\Phi_1, x : X_2, y : Y_2, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{app}\,t_1\,t_3 : Y_1}\ \textsc{tenE} \qquad \Psi_1 \vdash_{\mathcal{C}} t_2 : X_2 \otimes Y_2}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\,t_2 : X_2 \otimes Y_2\,\mathsf{be}\,x \otimes y\,\mathsf{in}\,(\mathsf{app}\,t_1\,t_3) : Y_1}\ \textsc{impE}$$

- Commutation of $\multimap_E$:
  - (T_impE, T_unitE):

$$\dfrac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \qquad \Phi_2 \vdash_{\mathcal{C}} t_2 : \mathsf{Unit} \multimap \mathsf{Unit}}{\Phi_2, \Phi_1 \vdash_{\mathcal{C}} \mathsf{app}\,t_2\,t_1 : \mathsf{Unit}}\ \textsc{impE} \qquad \Phi_3 \vdash_{\mathcal{C}} t_3 : \mathsf{Unit}}{\Phi_2, \Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{let}\,(\mathsf{app}\,t_2\,t_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3 : \mathsf{Unit}}\ \textsc{tenE}$$

commutes to

$$\dfrac{\dfrac{\Phi_1 \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \qquad \Phi_3 \vdash_{\mathcal{C}} t_3 : \mathsf{Unit}}{\Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{let}\,t_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3 : \mathsf{Unit}}\ \textsc{tenE} \qquad \Phi_2 \vdash_{\mathcal{C}} t_2 : \mathsf{Unit} \multimap \mathsf{Unit}}{\Phi_2, \Phi_1, \Phi_3 \vdash_{\mathcal{C}} \mathsf{app}\,t_2\,(\mathsf{let}\,t_1 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,t_3) : \mathsf{Unit}}\ \textsc{impE}$$

  - (T_impE, T_tenE): ?
  - (T_impE, T_impE): ?
- Commutation of $\triangleright_E$:
  - (S_unitE2, S_unitE2):

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{Unit}}{\Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\,s_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_1 : \mathsf{Unit}}\ \textsc{unitE} \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{let}\,(\mathsf{let}\,s_2 : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_1) : \mathsf{Unit}\,\mathsf{be}\,\mathsf{triv}\,\mathsf{in}\,s_3 : A}\ \textsc{unitE}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_3 : A}\,\textsc{unit}\mathrm{E} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{Unit}}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, (\mathsf{let}\, s_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_3) : A}\,\textsc{unit}\mathrm{E}$$

- (S_$\textsc{unit}$E2, S_$\textsc{ten}$E2): Does NOT commute

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{Unit}}{\Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_1 : A \triangleright B}\,\textsc{unit}\mathrm{E} \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_2, \Gamma_1, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_1) : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_3 : C}\,\textsc{ten}\mathrm{E2}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_3 : C}\,\textsc{ten}\mathrm{E2} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{Unit}}{\Gamma_2, \Delta_1, \Gamma_1, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, (\mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_3) : C}\,\textsc{unit}\mathrm{E}$$

- (S_$\textsc{unit}$E2, S_$\textsc{impr}$E):

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{Unit}}{\Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_1 : A \rightharpoonup B}\,\textsc{unit}\mathrm{E} \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{app}_r\, (\mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_1)\, s_3 : B}\,\textsc{impr}\mathrm{E}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_3 : B}\,\textsc{impr}\mathrm{E} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{Unit}}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, (\mathsf{app}_r\, s_1\, s_3) : B}\,\textsc{unit}\mathrm{E}$$

- (S_$\textsc{unit}$E2, S_$\textsc{impl}$E): Does NOT commute.

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : B \leftharpoondown A \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{Unit}}{\Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_1 : B \leftharpoondown A}\,\textsc{unit}\mathrm{E} \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_3, \Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{app}_l\, (\mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_1)\, s_3 : B}\,\textsc{impr}\mathrm{E}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : B \leftharpoondown A \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_3, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_3 : B}\,\textsc{impr}\mathrm{E} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{Unit}}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, (\mathsf{app}_l\, s_1\, s_3) : B}\,\textsc{unit}\mathrm{E}$$

- (S_$\textsc{ten}$E2, S_$\textsc{unit}$E2):

$$\dfrac{\dfrac{\Gamma_1, x : A, y : B, \Gamma_2 \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A \triangleright B}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_1 : \mathsf{Unit}}\,\textsc{ten}\mathrm{E2} \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, s_2 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_1) : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_3 : C}\,\textsc{unit}\mathrm{E2}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1, x : A, y : B, \Gamma_2 \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Gamma_1, x : A, y : B, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_3 : C}\,\textsc{unit}\mathrm{E2} \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A \triangleright B}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, (\mathsf{let}\, s_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_3) : C}\,\textsc{ten}\mathrm{E2}$$

- (S_$\textsc{ten}$E2, S_$\textsc{ten}$E2):

$$\dfrac{\dfrac{\Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A_1 \triangleright B_1 \qquad \Gamma \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Gamma_1, \Gamma, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_1 : A_1 \triangleright B_1}\,\textsc{ten}\mathrm{E2} \qquad \Delta_1, w : A_1, z : B_1, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, \Gamma, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, s_2 : A_2 \triangleright B_2\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_1) : A_1 \triangleright B_1\, \mathsf{be}\, w \triangleright z\, \mathsf{in}\, s_3 : C}\,\textsc{ten}\mathrm{E2}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A_1 \triangleright B_1 \qquad \Delta_1, w : A_1, z : B_1, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, x : A_2, y : B_2, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A_1 \triangleright B_1\, \mathsf{be}\, w \triangleright z\, \mathsf{in}\, s_3 : C}\,\textsc{ten}\mathrm{E2} \qquad \Gamma \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Delta_1, \Gamma_1, \Gamma, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, (\mathsf{let}\, s_1 : A_1 \triangleright B_1\, \mathsf{be}\, x \triangleright z\, \mathsf{in}\, s_3) : C}\,\textsc{ten}\mathrm{E2}$$

- (S_$\textsc{ten}$E2, S_$\textsc{impr}$E):

$$\dfrac{\dfrac{\Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A_1 \rightharpoonup B_1 \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_1 : A_1 \rightharpoonup B_1}\,\textsc{ten}\mathrm{E2} \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : A_1}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{app}_r\, (\mathsf{let}\, s_2 : A_2 \triangleright B_2\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_1)\, s_3 : B_1}\,\textsc{impr}\mathrm{E}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A_1 \rightharpoonup B_1 \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : A_1}{\Gamma_1, x : A_2, y : B_2, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_3 : B_1}\,\textsc{impr}\mathrm{E} \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, (\mathsf{app}_r\, s_1\, s_3) : B_1}\,\textsc{ten}\mathrm{E2}$$

- (S_$\textsc{ten}$E2, S_$\textsc{impl}$E):

$$\dfrac{\dfrac{\Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : B_1 \leftharpoondown A_1 \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_1 : B_1 \leftharpoondown A_1}\,\textsc{ten}\mathrm{E2} \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : A_1}{\Delta_2, \Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{app}_l\, (\mathsf{let}\, s_2 : A_2 \triangleright B_2\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_1)\, s_3 : B_1}\,\textsc{impr}\mathrm{E}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : B_1 \leftharpoonup A_1 \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : A_1}{\Delta_2, \Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_3 : B_1}\ \textsc{impr}\mathrm{E} \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Delta_2, \Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2\ \mathsf{be}\ x \triangleright y\ \mathsf{in}\ (\mathsf{app}_l\, s_1\, s_3) : B_1}\ \textsc{ten}\mathrm{E2}$$

- Commutation of F_E:
  - (S_FE, S_UNITE2):

$$\dfrac{\dfrac{\Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \Delta_1 \vdash_{\mathcal{L}} y : \mathsf{F}X}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ s_1 : \mathsf{Unit}}\ \mathrm{FE} \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ s_1) : \mathsf{Unit}\ \mathsf{be}\ \mathsf{triv}\ \mathsf{in}\ s_2 : A}\ \textsc{unit}\mathrm{E2}$$

    commutes to

$$\dfrac{\dfrac{\Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Gamma_1, x : X, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{Unit}\ \mathsf{be}\ \mathsf{triv}\ \mathsf{in}\ s_2 : A}\ \textsc{unit}\mathrm{E2} \qquad \Delta_1 \vdash_{\mathcal{L}} y : \mathsf{F}X}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ (\mathsf{let}\, s_1 : \mathsf{Unit}\ \mathsf{be}\ \mathsf{triv}\ \mathsf{in}\ s_2) : A}\ \mathrm{FE}$$

  - (S_FE, S_TENE2):

$$\dfrac{\dfrac{\Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \Delta \vdash_{\mathcal{L}} y : \mathsf{F}X}{\Gamma_1, \Delta, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ s_1 : A \triangleright B}\ \mathrm{FE} \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma_1, \Delta, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ s_1) : A \triangleright B\ \mathsf{be}\ x \triangleright y\ \mathsf{in}\ s_2 : C}\ \textsc{ten}\mathrm{E2}$$

    commutes to

$$\dfrac{\dfrac{\Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma_1, x : X, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\ \mathsf{be}\ x \triangleright y\ \mathsf{in}\ s_2 : C}\ \textsc{ten}\mathrm{E2} \qquad \Delta \vdash_{\mathcal{L}} y : \mathsf{F}X}{\Delta_1, \Gamma_1, \Delta, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ (\mathsf{let}\, s_1 : A \triangleright B\ \mathsf{be}\ x \triangleright y\ \mathsf{in}\ s_2) : C}\ \mathrm{FE}$$

  - (S_FE, S_IMPRE):

$$\dfrac{\dfrac{\Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \qquad \Delta_1 \vdash_{\mathcal{L}} y : \mathsf{F}X}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ s_1 : A \rightharpoonup B}\ \mathrm{FE} \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{app}_r\, (\mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ s_1)\, s_2 : B}\ \textsc{impr}\mathrm{E}$$

    commutes to

$$\dfrac{\dfrac{\Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Gamma_1, x : X, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_2 : B}\ \textsc{impr}\mathrm{E} \qquad \Delta_1 \vdash_{\mathcal{L}} y : \mathsf{F}X}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ (\mathsf{app}_r\, s_1\, s_2) : B}\ \mathrm{FE}$$

  - (S_FE, S_IMPLE):

$$\dfrac{\dfrac{\Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \leftharpoonup B \qquad \Delta_1 \vdash_{\mathcal{L}} y : \mathsf{F}X}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ s_1 : A \leftharpoonup B}\ \mathrm{FE} \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Delta_2, \Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{app}_l\, (\mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ s_1)\, s_2 : B}\ \textsc{impl}\mathrm{E}$$

    commutes to

$$\dfrac{\dfrac{\Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \leftharpoonup B \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Delta_2, \Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_2 : B}\ \textsc{impr}\mathrm{E} \qquad \Delta_1 \vdash_{\mathcal{L}} y : \mathsf{F}X}{\Delta_2, \Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\ \mathsf{be}\ y\ \mathsf{in}\ (\mathsf{app}_l\, s_1\, s_2) : B}\ \mathrm{FE}$$

5.3 Mappings Between Sequent Calculus and Natural Deduction

Function $S : ND \to SE$ maps a proof in the natural deduction to a proof of the same sequent in the sequent calculus. The function is defined as follows:

- The axioms map to axioms.
- Introduction rules map to right rules.
- Elimination rules map to combinations of left rules with cuts:
  - T_UNITE:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \quad \Psi \vdash_{\mathcal{C}} t_2 : Y}{\Phi, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : \mathsf{Unit}\ \mathsf{be}\ \mathsf{triv}\ \mathsf{in}\ t_2 : Y}\ \mathrm{T\_}\textsc{unit}\mathrm{E}$$

    maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : \mathsf{Unit} \qquad \dfrac{\Psi \vdash_{\mathcal{C}} t_2 : Y}{x : \mathsf{Unit}, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, x : \mathsf{Unit}\ \mathsf{be}\ \mathsf{triv}\ \mathsf{in}\ t_2 : Y}\ \textsc{unit}\mathrm{L}}{\Phi, \Psi \vdash_{\mathcal{C}} [t_1/x](\mathsf{let}\, x : \mathsf{Unit}\ \mathsf{be}\ \mathsf{triv}\ \mathsf{in}\ t_2) : Y}\ \textsc{cut}$$

  - T_TENE:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \otimes Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : X \otimes Y\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t_2 : Z}\ \mathrm{T\_}\textsc{ten}\mathrm{E}$$

    maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \otimes Y \qquad \dfrac{\Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, z : X \otimes Y, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2 : Z}\,\textsc{unitL}}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} [t_1/z](\mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2) : Z}\,\textsc{cut}$$

- T_IMPE:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \multimap Y \quad \Psi \vdash_{\mathcal{C}} t_2 : X}{\Phi, \Psi \vdash_{\mathcal{C}} \mathsf{app}\, t_1\, t_2 : Y}\ \text{T\_IMPE}$$

  maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \multimap Y \qquad \dfrac{\Psi \vdash_{\mathcal{C}} t_2 : X \qquad x : Y \vdash_{\mathcal{C}} x : Y}{y : X \multimap Y, \Psi \vdash_{\mathcal{C}} [\mathsf{app}\, y\, t_2/x]x : Y}\,\textsc{unitL}}{\Phi, \Psi \vdash_{\mathcal{C}} [t_1/y][\mathsf{app}\, y\, t_2/x]x : Y}\,\textsc{cut}$$

- S_UNITE1:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : \mathsf{Unit} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \text{S\_UNITE1}$$

  maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : \mathsf{Unit} \qquad \dfrac{\Gamma \vdash_{\mathcal{L}} s : A}{x : \mathsf{Unit}, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\,\textsc{unitL1}}{\Phi, \Psi \vdash_{\mathcal{L}} [t/x](\mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s) : A}\,\textsc{cut1}$$

- S_UNITE2:

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_2 : A}\ \text{S\_UNITE2}$$

  maps to

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{Unit} \qquad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A}{x : \mathsf{Unit}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_2 : A}\,\textsc{unitL2}}{\Gamma, \Psi \vdash_{\mathcal{L}} [s_1/x](\mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s_2) : A}\,\textsc{cut2}$$

- S_TENE1:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : X \otimes Y \quad \Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, t : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\ \text{S\_TENE1}$$

  maps to

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : X \otimes Y \qquad \dfrac{\Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, z : X \otimes Y, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\,\textsc{tenL1}}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} [t/z](\mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s) : A}\,\textsc{cut1}$$

- S_TENE2:

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C}\ \text{S\_TENE2}$$

  maps to

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \dfrac{\Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, z : A \triangleright B, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C}\,\textsc{tenL2}}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [s_1/z](\mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2) : C}\,\textsc{cut2}$$

- S_IMPRE: (NOT SURE)

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_2 : B}\ \text{S\_IMPRE}$$

  maps to

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \qquad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A \qquad x : B \vdash_{\mathcal{L}} x : B}{y : A \rightharpoonup B, \Delta \vdash_{\mathcal{L}} [\mathsf{app}_r\, y\, s_2/x]x : B}\,\textsc{imprL}}{\Gamma, \Delta \vdash_{\mathcal{L}} [s_1/y][\mathsf{app}_r\, y\, s_2/x]x : B}\,\textsc{cut2}$$

- S_IMPLE: (NOT SURE)

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftharpoondown A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta, \Gamma \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_2 : B}\ \text{S\_IMPLE}$$

  maps to

$$\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftharpoondown A \qquad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A \qquad x : B \vdash_{\mathcal{L}} x : B}{\Delta, y : B \leftharpoondown A \vdash_{\mathcal{L}} [\mathsf{app}_l\, y\, s_2/x]x : B}\,\textsc{implL}}{\Delta, \Gamma \vdash_{\mathcal{L}} [s_1/y][\mathsf{app}_l\, y\, s_2/x]x : B}\,\textsc{cut2}$$

- S_FE:

$$\dfrac{\Gamma \vdash_{\mathcal{L}} y : \mathsf{F}X \quad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X\, \mathsf{be}\, y\, \mathsf{in}\, s : A}\ \text{S\_FE}$$

  maps to

$$\dfrac{\Gamma \vdash_{\mathcal{L}} y : \mathsf{F}X \qquad \dfrac{\Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, z : \mathsf{F}X, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}\, x\, \mathsf{in}\, s : A}\,\text{FL}}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [y/z](\mathsf{let}\, y : \mathsf{F}X\, \mathsf{be}\, \mathsf{F}\, x\, \mathsf{in}\, s) : A}\,\textsc{cut2}$$

- S_GE:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t : \mathsf{G}A}{\Phi \vdash_{\mathcal{L}} \mathsf{derelict}\, t : A}\ \text{S\_GE}$$

  maps to

$$\dfrac{\dfrac{x : A \vdash_{\mathcal{L}} x : A}{y : \mathsf{G}A \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{G}A\, \mathsf{be}\, \mathsf{G}\, x\, \mathsf{in}\, x : A}\ \text{GL} \qquad \Phi \vdash_{\mathcal{C}} t : \mathsf{G}A}{\Phi \vdash_{\mathcal{L}} [t/y](\mathsf{let}\, y : \mathsf{G}A\, \mathsf{be}\, \mathsf{G}\, x\, \mathsf{in}\, x) : A}\ \textsc{cut}1$$

Function $N : SE \to ND$ maps a proof in the sequent calculus to a proof of the same sequent in the natural deduction. The function is defined as follows:

- Axioms map to axioms.
- Instances of cut rules map to the admissible substitution rules.
- Right rules map to introductions.
- Left rules map to eliminations modulo some structural fiddling.
  - T_unitL: Need multiple exchanges

$$\dfrac{\Phi, \Psi \vdash_{\mathcal{C}} t : X}{\Phi, x : \mathsf{UnitT}, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, t : X}\ \text{T\_}\textsc{unit}\text{L}$$

    maps to

$$\dfrac{x : \mathsf{Unit} \vdash_{\mathcal{C}} x : \mathsf{Unit} \qquad \Phi, \Psi \vdash_{\mathcal{C}} t : X}{x : \mathsf{Unit}, \Phi, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, t : X}\ \textsc{unit}\text{E}$$

  - T_tenL:

$$\dfrac{\Phi, x : X, y : Y, \Psi \vdash_{\mathcal{C}} t : Z}{\Phi, z : X \otimes Y, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t : Z}\ \text{T\_}\textsc{ten}\text{L}$$

    maps to

$$\dfrac{z : X \otimes Y \vdash_{\mathcal{C}} z : X \otimes Y \qquad \Phi, x : X, y : Y, \Psi \vdash_{\mathcal{C}} t : Z}{\Phi, z : X \otimes Y, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t : Z}\ \textsc{ten}\text{E}$$

  - T_impL:

$$\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi_1, x : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, y : X \multimap Y, \Phi, \Psi_2 \vdash_{\mathcal{C}} [\mathsf{app}\, y\, t_1 / x] t_2 : Z}\ \text{T\_}\textsc{imp}\text{L}$$

    maps to

$$\dfrac{\dfrac{z : X \multimap Y \vdash_{\mathcal{C}} z : X \multimap Y \qquad \Phi \vdash_{\mathcal{C}} t_1 : X}{z : X \multimap Y, \Phi \vdash_{\mathcal{C}} \mathsf{app}\, z\, t_1 : Y}\ \textsc{imp}\text{E} \qquad \Psi_1, x : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, z : X \multimap Y, \Phi, \Psi_2 \vdash_{\mathcal{C}} [\mathsf{app}\, z\, t_2 / x] t_2 : Z}\ \textsc{cut}1$$

  - S_unitL1: Does NOT work

$$\dfrac{\Gamma, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, x : \mathsf{UnitT}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, s : A}\ \text{S\_}\textsc{unit}\text{L1}$$

    maps to

$$\dfrac{x : \mathsf{Unit} \vdash_{\mathcal{C}} x : \mathsf{Unit} \qquad \Gamma, \Delta \vdash_{\mathcal{L}} s : A}{x : \mathsf{Unit}, \Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \textsc{unit}\text{E1}$$

  - S_unitL2: Does NOT work

$$\dfrac{\Gamma, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, x : \mathsf{UnitS}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s : A}\ \text{S\_}\textsc{unit}\text{L2}$$

    maps to

$$\dfrac{x : \mathsf{Unit} \vdash_{\mathcal{L}} x : \mathsf{Unit} \qquad \Gamma, \Delta \vdash_{\mathcal{L}} s : A}{x : \mathsf{Unit}, \Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{Unit}\, \mathsf{be}\, \mathsf{triv}\, \mathsf{in}\, s : A}\ \textsc{unit}\text{E2}$$

  - S_tenL1:

$$\dfrac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \otimes Y, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\ \text{S\_}\textsc{ten}\text{L1}$$

    maps to

$$\dfrac{z : X \otimes Y \vdash_{\mathcal{C}} z : X \otimes Y \qquad \Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \otimes Y, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\ \textsc{ten}\text{E1}$$

  - S_tenL2:

$$\dfrac{\Gamma, x : A, y : B, \Delta \vdash_{\mathcal{L}} s : C}{\Gamma, z : A \triangleright B, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s : C}\ \text{S\_}\textsc{ten}\text{L2}$$

    maps to

$$\frac{z : A \triangleright B \vdash_{\mathcal{L}} z : A \triangleright B \qquad \Gamma, x : A, y : B, \Delta \vdash_{\mathcal{L}} s : C}{\Gamma, z : A \triangleright B, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, s : C}\ \textsc{tenE2}$$

- S_impL:

$$\frac{\Phi \vdash_{\mathcal{C}} t : X \quad \Gamma, x : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : X \multimap Y, \Phi, \Delta \vdash_{\mathcal{L}} [\mathsf{app}\, y\, t/x]s : A}\ \text{S\_}\textsc{impL}$$

maps to

$$\frac{\dfrac{z : X \multimap Y \vdash_{\mathcal{C}} z : X \multimap Y \qquad \Phi \vdash_{\mathcal{C}} t : X}{z : X \multimap Y, \Phi \vdash_{\mathcal{C}} \mathsf{app}\, z\, t : Y}\ \textsc{impE} \qquad \Gamma, x : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \multimap Y, \Phi, \Delta \vdash_{\mathcal{L}} [\mathsf{app}\, z\, t/x]s : A}\ \textsc{cut1}$$

- S_imprL:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta, x : B \vdash_{\mathcal{L}} s_2 : C}{\Delta, y : A \rightharpoonup B, \Gamma \vdash_{\mathcal{L}} [\mathsf{app}_r\, y\, s_1 /x]s_2 : C}\ \text{S\_}\textsc{imprL}$$

maps to

$$\frac{\dfrac{z : A \rightharpoonup B \vdash_{\mathcal{L}} z : A \rightharpoonup B \qquad \Gamma \vdash_{\mathcal{L}} s_1 : A}{z : A \rightharpoonup B, \Gamma \vdash_{\mathcal{L}} \mathsf{app}_r\, z\, s_1 : B}\ \textsc{imprE} \qquad \Delta, x : B \vdash_{\mathcal{L}} s_2 : C}{\Delta, z : A \rightharpoonup B, \Gamma \vdash_{\mathcal{L}} [\mathsf{app}_r\, z\, s_1 /x]s_2 : C}\ \textsc{cut2}$$

- S_implL:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad x : B, \Delta \vdash_{\mathcal{L}} s_2 : C}{\Gamma, y : B \leftharpoonup A, \Delta \vdash_{\mathcal{L}} [\mathsf{app}_l\, y\, s_1 /x]s_2 : C}\ \text{S\_}\textsc{implL}$$

maps to

$$\frac{\dfrac{z : B \leftharpoonup A \vdash_{\mathcal{L}} z : B \leftharpoonup A \qquad \Gamma \vdash_{\mathcal{L}} s_1 : A}{\Gamma, z : B \leftharpoonup A \vdash_{\mathcal{L}} \mathsf{app}_l\, z\, s_1 : B}\ \textsc{implE} \qquad x : B, \Delta \vdash_{\mathcal{L}} s_2 : C}{\Gamma, z : B \leftharpoonup A, \Delta \vdash_{\mathcal{L}} [\mathsf{app}_l\, z\, s_1 /x]s_2 : C}\ \textsc{cut2}$$

- S_FL:

$$\frac{\Gamma, x : X, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : \mathsf{F}X, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{F}X \,\mathsf{be}\, \mathsf{F}\, x \,\mathsf{in}\, s : A}\ \text{S\_}\textsc{Fl}$$

maps to

$$\frac{z : \mathsf{F}X \vdash_{\mathcal{L}} z : \mathsf{F}X \qquad \Gamma, x : X, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : \mathsf{F}X, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, \mathsf{F}x : \mathsf{F}X \,\mathsf{be}\, z \,\mathsf{in}\, s : A}\ \mathrm{FE}$$

- S_GL:

$$\frac{\Gamma, x : A, \Delta \vdash_{\mathcal{L}} s : B}{\Gamma, y : \mathsf{G}A, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : \mathsf{G}A \,\mathsf{be}\, \mathsf{G}\, x \,\mathsf{in}\, s : B}\ \text{S\_}\textsc{Gl}$$

maps to

$$\frac{\dfrac{y : \mathsf{G}A \vdash_{\mathcal{C}} y : \mathsf{G}A}{y : \mathsf{G}A \vdash_{\mathcal{L}} \mathsf{derelict}\, y : A}\ \mathrm{GE} \qquad \Gamma, x : A, \Delta \vdash_{\mathcal{L}} s : B}{\Gamma, y : \mathsf{G}A, \Delta \vdash_{\mathcal{L}} [\mathsf{derelict}\, y/x]s : B}\ \textsc{cut2}$$

6 Combining with Benton's Adjoint Model

7 Applications

8 Conclusion

TODO

A Appendix