Linear Logic, Monads and Non-commutative Lambda Calculus

Jiaming Jiang[1] **and Harley Eades III**[2]

1 **Computer Science, Augusta University, Augusta, Georgia, USA**
`heades@augusta.edu`
2 **Computer Science, North Carolina State University, Raleigh, North Carolina, USA**
`jjiang13@ncsu.edu`

Abstract
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1 Introduction

2 Main Ideas

3 Category Theory Basics

▶ **Definition 1.** A **monoidal category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ consists of

- a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- an object I, called the unit object;
- three natural isomorphisms α, λ, and ρ with components

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$\lambda_A : I \otimes A \to A$$

$$\rho_A : A \otimes I \to A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in $\mathcal{M}$:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes id_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B \otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "id_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\rightharpoonup : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and $\leftharpoonup : \mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

▶ **Definition 3.** A **symmetric monoidal category** (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\mathsf{ex}_{A,B} : A \otimes B \to B \otimes A$, called **exchange**, such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\mathsf{ex}_{A,I}"] \arrow[dr, "\rho_A"'] & & I \otimes A \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[rr, equal, "id_{A \otimes B}"] \arrow[dr, "\mathsf{ex}_{A,B}"'] & & A \otimes B \\
& B \otimes A \arrow[ur, "\mathsf{ex}_{B,A}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathsf{ex}_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\mathsf{ex}_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,A,C}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \mathsf{ex}_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

▶ **Lemma 5.** *Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$\begin{array}{rcll}
Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\
& \cong & Hom[A \otimes C, B] & \text{By the exchange } \mathsf{ex}_{C,A} \\
& \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category}
\end{array}$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◀

▶ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

▶ **Definition 7.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from $\mathcal{M}$ to $\mathcal{M}'$ is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \otimes FB' \to F(A \otimes B)$, such that the following diagrams commute for any objects A, B, and C in $\mathcal{M}$:

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "\alpha'_{FA,FB,FC}"] \arrow[d, "\mathsf{m}_{A,B} \otimes' id_{FC}"'] & FA \otimes' (FB \otimes' FC) \arrow[r, "id_{FA} \otimes' \mathsf{m}_{A,B}"] & FA \otimes' F(B \otimes C) \arrow[d, "\mathsf{m}_{A,B \otimes C}"] \\
F(A \otimes B) \otimes' FC \arrow[r, "\mathsf{m}_{A \otimes B,C}"'] & F((A \otimes B) \otimes C) \arrow[r, "F\alpha_{A,B,C}"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
I' \otimes' FA \arrow[r, "\lambda'_{FA}"] \arrow[d, "\mathsf{m}_I \otimes id_{FA}"'] & FA \\
FI \otimes' FA \arrow[r, "\mathsf{m}_{I,A}"'] & F(I \otimes A) \arrow[u, "F\lambda_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes' I' \arrow[r, "\rho'_{FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes \mathsf{m}_I"'] & FA \\
FA \otimes' FI \arrow[r, "\mathsf{m}_{A,I}"'] & F(A \otimes I) \arrow[u, "F\rho_A"']
\end{tikzcd}$$

▶ **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "\mathsf{ex}_{FA,FB}"] \arrow[d, "\mathsf{m}_{A,B}"'] & FB \otimes' FA \arrow[d, "\mathsf{m}_{B,A}"] \\
F(A \otimes B) \arrow[r, "F\mathsf{ex}_{A,B}"'] & F(B \otimes A)
\end{tikzcd}$$

▶ **Definition 9.** An **adjunction** between categories C and $\mathcal{D}$ consists of two functors $F : \mathcal{D} \to C$, called the **left adjoint**, and $G : C \to \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta : id_{\mathcal{D}} \to GF$, called the **unit**, and $\varepsilon : FG \to id_C$, called the **counit**, such that the following diagrams commute for any object A in C and B in $\mathcal{D}$:

$$\begin{tikzcd}
FB \arrow[rr, "F\eta_B"] \arrow[dr, equal] & & FGFB \arrow[dl, "\varepsilon_{FB}"] \\
& FB &
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[rr, "\eta_{GA}"] \arrow[dr, equal] & & GFGA \arrow[dl, "G\varepsilon_A"] \\
& GA &
\end{tikzcd}$$

▶ **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category $\mathcal{M}$ to a monoidal category $\mathcal{M}'$. A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, \mathsf{m}) \to (G, \mathsf{n})$ such that the following diagrams commute for any objects A and B in $\mathcal{M}$:

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[d, "\theta_A \otimes' \theta_B"'] & F(A \otimes B) \arrow[d, "\theta_{A \otimes B}"] \\
GA \otimes' GB \arrow[r, "\mathsf{n}_{A,B}"'] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
FI \arrow[rr, "\theta_I"] & & GI \\
& I' \arrow[ul, "\mathsf{m}_I"] \arrow[ur, "\mathsf{n}_I"'] &
\end{tikzcd}$$

▶ **Definition 11.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

▶ **Definition 12.** Let C be a category. A **monad** on C consists of an endofunctor $T : C \to C$ together with two natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to T$, where id_C is the identity functor on C, such that the following diagrams commute:

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta_T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

▶ **Definition 13.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on $\mathcal{M}$. T is a **strong monad** if there is natural transformation τ, called the **tensorial strength**, with components

$\tau_{A,B} : A \otimes TB \to T(A \otimes B)$ such that the following diagrams commute:

$$
\begin{tikzcd}
I \otimes TA \arrow[rr, "\tau_{I,A}"] \arrow[dr, "\lambda_{TA}"'] & & T(I \otimes A) \arrow[dl, "T\lambda_A"] \\
& TA &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[rr, "id_A \otimes \eta_B"] \arrow[dr, "\eta_{A\otimes B}"'] & & A \otimes TB \arrow[dl, "\tau_{A,B}"] \\
& T(A \otimes B) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \otimes B) \otimes TC \arrow[rr, "\tau_{A\otimes B,C}"] \arrow[d, "\alpha_{A,B,TC}"'] & & T((A \otimes B) \otimes C) \arrow[d, "T\alpha_{A,B,C}"] \\
A \otimes (B \otimes TC) \arrow[r, "id_A \otimes \tau_{B,C}"] & A \otimes T(B \otimes C) \arrow[r, "\tau_{A,B\otimes C}"] & T(A \otimes (B \otimes C))
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes T^2B \arrow[r, "\tau_{A,TB}"] \arrow[d, "id_A \otimes \mu_B"'] & T(A \otimes TB) \arrow[r, "T\tau_{A,B}"] & T^2(A \otimes B) \arrow[d, "\mu_{A\otimes B}"] \\
A \otimes TB \arrow[rr, "\tau_{A,B}"'] & & T(A \otimes B)
\end{tikzcd}
$$

▶ **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex, and (T, η, μ) be a strong monad on $\mathcal{M}$. Then there is a **"twisted" tensorial strength** $\tau'_{A,B} : TA \otimes B \to T(A \otimes B)$ defined as $\tau'_{A,B} = T\mathsf{ex} \circ \tau_{B,A} \circ \mathsf{ex}$. We can construct a pair of natural transformations Φ, Φ' with components $\Phi_{A,B}, \Phi'_{A,B} : TA \otimes TB \to T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A\otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A\otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.

▶ **Definition 15.** Let $\mathcal{L}$ be a category. A **comonad** on $\mathcal{L}$ consists of an endofunctor $S : \mathcal{L} \to \mathcal{L}$ together with two natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S^2 \to S$ such that the following diagrams commute:

$$
\begin{tikzcd}
S \arrow[r, "\delta"] \arrow[d, "\delta"'] & S^2 \arrow[d, "S\delta"] \\
S^2 \arrow[r, "\delta_S"'] & S^3
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2 \arrow[r, "S\varepsilon"] \arrow[d, "\varepsilon_S"'] & S \arrow[d, "\delta"] \\
S \arrow[r, "\delta"'] & S^2
\end{tikzcd}
$$

4 Categorical Models

▶ **Definition 16.** A **SMCC-Lambek model** consists of

- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$;
- a monoidal adjunction $F : C \dashv \mathcal{L} : G$, where $F : C \to \mathcal{L}$ and $G : \mathcal{L} \to C$ are monoidal functors.

Thus, in a SMCC-Lambek model, the following four diagrams commute because η and ε are monoidal natural transformations:

$$
\begin{tikzcd}
A \otimes B \arrow[rr, equal, "id_{A\otimes B}"] \arrow[d, "\eta_A \otimes \eta_B"'] & & A \otimes B \arrow[d, "\eta_{A\otimes B}"] \\
GFA \otimes GFB \arrow[r, "\mathsf{n}_{FA,FB}"] & G(FA \otimes FB) \arrow[r, "G\mathsf{m}_{A,B}"] & GF(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\eta_I"] \arrow[d, equal] & GFI \\
I \arrow[r, "\mathsf{n}_{I'}"'] & GI' \arrow[u, "G\mathsf{m}_I"']
\end{tikzcd}
$$

$$\begin{tikzcd} FGA \otimes FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes \varepsilon_B"'] & F(GA \otimes GB) \arrow[r, "F\mathsf{n}_{A,B}"] & FG(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\ A \otimes B \arrow[rr, equal] & & A \otimes B \end{tikzcd} \qquad \begin{tikzcd} FGI' \arrow[r, "\varepsilon_{I'}"] & I' \arrow[d, equal] \\ FI \arrow[u, "F\mathsf{n}_{I'}"] & I' \arrow[l, "\mathsf{m}_I"] \end{tikzcd}$$

And the following two diagrams commute because of the adjunction:

$$\begin{tikzcd} FA \arrow[rr, "F\eta_A"] \arrow[dr, equal] & & FGFA \arrow[dl, "\varepsilon_{FA}"] \\ & FA & \end{tikzcd} \qquad \begin{tikzcd} GB \arrow[rr, "\eta_{GX}"] \arrow[dr, equal] & & GFGB \arrow[dl, "G\varepsilon_B"] \\ & GB & \end{tikzcd}$$

▶ **Lemma 17.** *The monad on the SMCC $\mathcal{C}$ in a SMCC-Lambek model is monoidal.*

Proof. We define the monad T on the $\mathcal{C}$ in the adjunction of a SMCC-Lambek model as $T = GF$, and the two corresponding natural transformations $\eta : id_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$ are defined as

$$\eta_A : A \to GFA$$

$$\mu_A = G\varepsilon_{FA} : GFGFA \to GFA$$

where η is the unit and $\varepsilon : FG \to id_{\mathcal{L}}$ is the counit in the adjunction $F : \mathcal{C} \dashv \mathcal{L} : G$, and (F, m) and (G, n) are monoidal functors.
Thus, we have

$$\mathsf{t}_{A,B} = G\mathsf{m}_{A,B} \circ \mathsf{n}_{FA,FB} : TA \otimes TB \to T(A \otimes B)$$

$$\mathsf{t}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'} : I \to TI$$

The monad T being monoidal means

1. T is a monoidal functor, i.e. the folllowing diagrams commute:

$$\begin{tikzcd} (TA \otimes TB) \otimes TC \arrow[r, "\alpha_{TA,TB,TC}"] \arrow[d, "\mathsf{t}_{A,B} \otimes id_{TC}"'] & TA \otimes (TB \otimes TC) \arrow[r, "id_{TA} \otimes \mathsf{t}_{B,C}"] & TA \otimes T(B \otimes C) \arrow[d, "\mathsf{t}_{A,B\otimes C}"] \\ T(A \otimes B) \otimes TC \arrow[r, "\mathsf{t}_{A\otimes B,C}"'] & T((A \otimes B) \otimes C) \arrow[r, "T\alpha_{A,B,C}"'] & T(A \otimes (B \otimes C)) \end{tikzcd}$$

$$\begin{tikzcd} I \otimes TA \arrow[r, "\lambda_{TA}"] \arrow[d, "\mathsf{t}_I \otimes id_{TA}"'] & TA \\ TI \otimes TA \arrow[r, "\mathsf{t}_{I,A}"'] & T(I \otimes A) \arrow[u, "T\lambda_A"'] \end{tikzcd} \qquad \begin{tikzcd} TA \otimes I \arrow[r, "\rho_{TA}"] \arrow[d, "id_{TA} \otimes \mathsf{t}_I"'] & TA \\ TA \otimes TI \arrow[r, "\mathsf{t}_{A,I}"'] & T(A \otimes I) \arrow[u, "T\rho_A"'] \end{tikzcd}$$

We write GF instead of T in the proof for clarity.
By replacing t with its definition, the first diagram above commutes by the following commutative diagram, where the two hexagons commute because G and F are monoidal functors, and the

two quadrilaterals commute by the naturality of n.

$$\begin{tikzcd}
(GFA \otimes GFB) \otimes GFC \arrow[r, "\alpha_{GFA,GFB,GFC}"] \arrow[d, "\mathsf{n}_{FA,FB} \otimes id_{GFC}"'] & GFA \otimes (GFB \otimes GFC) \arrow[r, "id_{GFA} \otimes \mathsf{n}_{FB,FC}"] & GFA \otimes G(FB \otimes' FC) \arrow[dl, "\mathsf{n}_{FA,FB \otimes' FC}"'] \arrow[d, "id_{GFA} \otimes G\mathsf{m}_{B,C}"] \\
G(FA \otimes' FB) \otimes GFC \arrow[d, "G\mathsf{m}_{A,B} \otimes id_{GFC}"'] \arrow[dr, "\mathsf{n}_{FA \otimes' FB,FC}"] & G(FA \otimes' (FB \otimes' FC)) \arrow[dr, "G(id_{FA} \otimes' \mathsf{m}_{B,C})"] & GFA \otimes GF(B \otimes C) \arrow[d, "\mathsf{n}_{FA,F(B \otimes C)}"] \\
GF(A \otimes B) \otimes GFC \arrow[d, "\mathsf{n}_{F(A \otimes B),FC}"'] & G((FA \otimes' FB) \otimes' FC) \arrow[u, "G\alpha'_{FA,FB,FC}"] \arrow[dl, "G(\mathsf{m}_{A,B} \otimes id_{FC})"'] & G(FA \otimes' F(B \otimes C)) \arrow[d, "G\mathsf{m}_{A,B \otimes C}"] \\
G(F(A \otimes B) \otimes' FC) \arrow[r, "G\mathsf{m}_{A \otimes B,C}"'] & GF((A \otimes B) \otimes C) \arrow[r, "GF\alpha_{A,B,C}"'] & GF(A \otimes (B \otimes C))
\end{tikzcd}$$

The first square above commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.

$$\begin{tikzcd}
I \otimes GFA \arrow[rr, "\lambda_{GFA}"] \arrow[d, "\mathsf{n}_{I'} \otimes id_{GFA}"'] & & GFA \\
GI' \otimes GFA \arrow[r, "\mathsf{n}_{I',FA}"'] \arrow[d, "G\mathsf{m}_I \otimes id_{GFA}"'] & G(I' \otimes' FA) \arrow[ur, "G\lambda'_{FA}"] \arrow[d, "G(\mathsf{m}_I \otimes' id_{FA})"] & \\
GFI \otimes GFA \arrow[r, "\mathsf{n}_{FI,FA}"'] & G(FI \otimes' FA) \arrow[r, "G\mathsf{m}_{I,A}"'] & GF(I \otimes A) \arrow[uu, "GF\lambda_A"']
\end{tikzcd}$$

Similarly, the second square above commutes by the following commutative diagram:

$$\begin{tikzcd}
GFA \otimes I \arrow[rr, "\rho_{GFA}"] \arrow[d, "id_{GFA} \otimes \mathsf{n}_{I'}"'] & & GFA \\
GFA \otimes GI' \arrow[r, "\mathsf{n}_{FA,I'}"'] \arrow[d, "id_{GFA} \otimes G\mathsf{m}_I"'] & G(FA \otimes' I') \arrow[ur, "G\rho'_{FA}"] \arrow[d, "G(id_{FA} \otimes \mathsf{m}_I)"] & \\
GFA \otimes GFI \arrow[r, "\mathsf{n}_{FA,FI}"'] & G(FA \otimes' FI) \arrow[r, "G\mathsf{m}_{A,I}"'] & GF(A \otimes I) \arrow[uu, "GF\rho_A"']
\end{tikzcd}$$

2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal and thus the following two diagrams commute.

$$\begin{tikzcd}
A \otimes B \arrow[r, equal] \arrow[d, "\eta_A \otimes \eta_B"'] & A \otimes B \arrow[d, "\eta_{A \otimes B}"] \\
TA \otimes TB \arrow[r, "\mathsf{t}_{A,B}"'] & T(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[rr, "\eta_I"] \arrow[dr, equal] & & TI \\
& I \arrow[ur, "\mathsf{t}_I"'] &
\end{tikzcd}$$

3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ. Thus the following diagrams commute.

$$\begin{tikzcd}
T^2A \otimes T^2B \arrow[r, "\mathsf{t}_{TA,TB}"] \arrow[d, "\mu_A \otimes \mu_B"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{t}_{A,B}"] & T^2(A \otimes B) \arrow[d, "\mu_{A \otimes B}"] \\
TA \otimes TB \arrow[rr, "\mathsf{t}_{A,B}"'] & & T(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2I \arrow[r, "\mu_I"] & TI \\
TI \arrow[u, "T\mathsf{t}_I"] & I \arrow[l, "\mathsf{t}_I"] \arrow[u, "\mathsf{t}_I"']
\end{tikzcd}$$

◀

However, the monad is not symmetric becauase the following diagram does not commute, for the lambek category $\mathcal{L}$ is not symmetric.

$$\begin{tikzcd}
GFA \otimes GFB \arrow[r, "\mathsf{ex}_{GFA,GFB}"] \arrow[d, "\mathsf{n}_{FA,FB}"'] & GFB \otimes GFA \arrow[r, "\mathsf{n}_{FB,FA}"] & G(FB \otimes' FA) \arrow[d, "G\mathsf{m}_{B,A}"] \\
G(FA \otimes' FB) \arrow[r, "G\mathsf{m}_{A,B}"'] & GF(A \otimes B) \arrow[r, "GF\mathsf{ex}_{A,B}"'] & GF(B \otimes A)
\end{tikzcd}$$

► **Lemma 18.** *The monad on the SMCC in a SMCC-Lambek model is strong.*

Proof. Let $F : C \vdash \mathcal{L} : G$ be a SMCC-Lambek model, where $(C, \otimes, I, \alpha, \lambda, \rho)$ is an SMCC, $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$ is a Lambek category, and (F, m) and (G, n) are monoidal functors. Let (T, η, μ) be the monad on C where $T = GF$. We have proved that T is monoidal with the natural transformation $\mathsf{t}_{A,B} : TA \otimes TB \to T(A \otimes B)$ and the morphism $\mathsf{t}_I : I \to TI$ defined as in Lemma **??**.
We define the tensorial strength $\tau_{A,B} : A \otimes TB \to T(A \otimes B)$ as $\tau_{A,B} = \mathsf{t}_{A,B} \circ \eta_A \otimes id_{TB}$.
Since η is a monoidal natural transformation, we have $\eta_I = G\mathsf{m}_I \circ \mathsf{n}_{I'}$. Therefore $\eta_I = \mathsf{t}_I$. Thus the following diagram commutes because T is monoidal, where the composition $\mathsf{t}_{I,A} \circ \mathsf{t}_I \otimes id_{TA}$ is the definition of $\tau_{I,A}$. So the first triangle in Defition **??** commutes.

$$\begin{tikzcd}
I \otimes TA \arrow[r, "\mathsf{t}_I \otimes id_{TA}"] \arrow[d, "\lambda_{TA}"'] & TI \otimes TA \arrow[d, "\mathsf{t}_{I,A}"] \\
TA & T(I \otimes A) \arrow[l, "T\lambda_A"]
\end{tikzcd}$$

Similarly, by using the definition of τ, the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$\begin{tikzcd}
A \otimes B \arrow[r, "id_A \otimes \eta_B"] \arrow[d, "\eta_{A \otimes B}"'] \arrow[dr, "\eta_A \otimes \eta_B"] & A \otimes TB \arrow[d, "\eta_A \otimes id_{TB}"] \\
T(A \otimes B) & TA \otimes TB \arrow[l, "\mathsf{t}_{A,B}"]
\end{tikzcd}$$

The first pentagon in the definition commutes by the following commutative diagrams, because η are α natural transformations and T is monoidal:

$$\begin{tikzcd}
(A \otimes B) \otimes TC \arrow[r, "\eta_{A \otimes B} \otimes id_{TC}"] \arrow[d, "\alpha_{A,B,TC}"'] \arrow[dr, "(\eta_A \otimes \eta_B) \otimes id_{TC}"] & T(A \otimes B) \otimes TC \arrow[r, "\mathsf{t}_{A \otimes B,C}"] & T((A \otimes B) \otimes C) \arrow[d, "T\alpha_{A,B,C}"] \\
A \otimes (B \otimes TC) \arrow[dd, "id_A \otimes (\eta_B \otimes id_{TC})"'] \arrow[dr, "\eta_A \otimes (\eta_B \otimes id_{TC})"] & (TA \otimes TB) \otimes TC \arrow[u, "\mathsf{t}_{A,B} \otimes id_{TC}"'] \arrow[d, "\alpha_{TA,TB,TC}"] & T(A \otimes (B \otimes C)) \\
 & TA \otimes (TB \otimes TC) \arrow[dr, "id_{TA} \otimes \mathsf{t}_{B,C}"] & \\
A \otimes (TB \otimes TC) \arrow[ur, "\eta_A \otimes id_{TB \otimes TC}"] \arrow[r, "id_A \otimes \mathsf{t}_{B,C}"'] & A \otimes T(B \otimes C) \arrow[r, "\eta_A \otimes id_{T(B \otimes C)}"'] & TA \otimes T(B \otimes C) \arrow[uu, "\mathsf{t}_{A,B \otimes C}"']
\end{tikzcd}$$

The last diagram in the definition commtues by the following commutative diagram, because T is a monad, t is a natural transformation, and μ is a monoidal natural transformation:

$$\begin{tikzcd}
A \otimes T^2B \arrow[r, "\eta_A \otimes id_{T^2B}"] \arrow[d, "id_A \otimes \mu_B"'] & TA \otimes T^2B \arrow[rr, "\mathsf{t}_{A,TB}"] \arrow[d, equal] \arrow[dr, "T\eta_A \otimes id_{T^2B}"] \arrow[ddl, "id_{TA} \otimes \mu_B"'] & & T(A \otimes TB) \arrow[d, "T(\eta_A \otimes id_{TB})"] \\
A \otimes TB \arrow[d, "\eta_A \otimes id_{TB}"'] & TA \otimes T^2B \arrow[dl, "id_{TA} \otimes \mu_B"] & T^2A \otimes T^2B \arrow[l, "\mu_A \otimes id_{T^2B}"'] \arrow[r, "\mathsf{t}_{TA,TB}"] \arrow[dll, "\mu_A \otimes \mu_B"] & T(TA \otimes TB) \arrow[d, "T\mathsf{t}_{A,B}"] \\
TA \otimes TB \arrow[r, "\mathsf{t}_{A,B}"'] & T(A \otimes B) & & T^2(A \otimes B) \arrow[ll, "\mu_{A \otimes B}"]
\end{tikzcd}$$

◀

▶ **Lemma 19** ([?]). *Let $\mathcal{M}$ be a symmetric monoidal category and T be a strong monad on $\mathcal{M}$. Then T is a symmetric monoidal functor iff it is commutative.*

▶ **Theorem 20.** *The monad on the SMCC in a SMCC-Lambek model is monoidal and non-commutative.*

▶ **Lemma 21.** *The comonad on the Lambek category in a SMCC-Lambek model is monoidal.*

Proof. We define the comonad S on the Lambek category $\mathcal{L}$ in the adjunction $F : C \vdash \mathcal{L} : G$ of a SMCC-Lambek model as $S = FG$, and the two corresponding natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S \to S^2$ are defined as

$$\varepsilon_A : SA \to A$$

$$\delta_A = F\eta_{GA} : SA \to S^2A$$

where ε is the counit and $\eta : id_{\mathcal{L}} \to GF$ is the unit in the adjunction, and (F, m) and (G, n) are monoidal functors. Thus, we have

$$\mathsf{s}_{A,B} = F\mathsf{n}_{A,B} \circ \mathsf{m}_{GA,GB} : SA \otimes' SB \to SA \otimes' SB$$

$$\mathsf{s}_I = F\mathsf{n}_{I'} \circ \mathsf{m}_I : I' \to SI'$$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$\begin{tikzcd}
(SA \otimes' SB) \otimes' SC \arrow[r, "\alpha'_{SA,SB,SC}"] \arrow[d, "\mathsf{s}_{A,B} \otimes' id_{SC}"'] & SA \otimes' (SB \otimes' SC) \arrow[r, "id_{SA} \otimes' \mathsf{s}_{B,C}"] & SA \otimes' S(B \otimes' C) \arrow[d, "\mathsf{s}_{A,B \otimes' C}"] \\
S(A \otimes' B) \otimes' SC \arrow[r, "\mathsf{s}_{A \otimes' B,C}"'] & S((A \otimes' B) \otimes' C) \arrow[r, "S\alpha'_{A,B,C}"'] & S(A \otimes' (B \otimes' C))
\end{tikzcd}$$

$$\begin{tikzcd}
I' \otimes' SA \arrow[r, "\lambda'_{SA}"] \arrow[d, "\mathsf{s}_{I'} \otimes' id_{SA}"'] & SA \\
SI' \otimes' SA \arrow[r, "\mathsf{s}_{I',A}"'] & S(I' \otimes' A) \arrow[u, "S\lambda'_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
SA \otimes' I' \arrow[r, "\rho'_{SA}"] \arrow[d, "id'_{SA} \otimes' \mathsf{s}_{I'}"'] & SA \\
SA \otimes' SI' \arrow[r, "\mathsf{s}_{A,I'}"'] & S(A \otimes' I') \arrow[u, "S\rho'_A"']
\end{tikzcd}$$

2. ε is a monoidal natural transformation:

$$\begin{tikzcd}
SA \otimes' SB \arrow[r, "\mathsf{s}_{A,B}"] \arrow[d, "\varepsilon_A \otimes' \varepsilon_B"'] & S(A \otimes' B) \arrow[d, "\varepsilon_{A \otimes' B}"] \\
A \otimes' B \arrow[r, equal] & A \otimes' B
\end{tikzcd}
\qquad
\begin{tikzcd}
SI' \arrow[rr, "\varepsilon_{I'}"] & & I' \\
& I' \arrow[ul, "\mathsf{s}_{I'}"] \arrow[ur, equal] &
\end{tikzcd}$$

3. δ is a monoidal natural transformation:

$$
\begin{tikzcd}
SA \otimes' SA \arrow[rr, "\mathsf{s}_{A,B}"] \arrow[d, "\delta_A \otimes' \delta_B"'] & & S(A \otimes' B) \arrow[d, "\delta_{A \otimes' B}"] \\
S^2A \otimes' S^2B \arrow[r, "\mathsf{s}_{SA,SB}"] & S(SA \otimes' SB) \arrow[r, "S\mathsf{s}_{A,B}"] & S^2(A \otimes' B)
\end{tikzcd}
\qquad
\begin{tikzcd}
SI' \arrow[r, "\delta_{I'}"] & S^2I' \\
I' \arrow[u, "\mathsf{s}_{I'}"] \arrow[r, "\mathsf{s}_{I'}"'] & SI' \arrow[u, "S\mathsf{s}_{I'}"']
\end{tikzcd}
$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma **??**. We do not include the proof here for simplicity. ◀

5 Logic

5.1 Categorical Interpretation of Natural Deductions

T rules: in the symmetric monoidal closed category of the adjunction model

T_identity: $id_X : X \to X$

T_unitI:

T_unitE: given $t_1 : \Delta \to Unit$ and $t_2 : \Gamma \to Y$, returns $\lambda_Y \circ (t_1 \otimes t_2) : \Gamma \otimes \Delta \to Unit \otimes Y \to Y$

T_tenI: given $t_1 : \Gamma \to X$ and $t_2 : \Delta \to Y$, returns $t_1 \otimes t_2 : \Gamma \otimes \Delta \to X \otimes Y$

T_tenE: given $t_1 : \Gamma \to X \otimes Y$ and $t_2 : \Delta \otimes X \otimes Y \to Z$, returns
$t_2 \circ \mathsf{ex}_{X \otimes Y, \Delta} \circ t_1 \otimes id_\Delta : \Gamma \otimes \Delta \to (X \otimes Y) \otimes \Delta \to \Delta \otimes (X \otimes Y) \to Z$

T_implI:

T_implE:

T_imprI:

T_imprE:

T_GI: given $s : FX_1 \otimes' ... \otimes' FX_n \to A$, returns
$Gs \circ \mathsf{Gm}^{-1} \circ \eta : X_1 \otimes ... \otimes X_n \to GF(X_1 \otimes ... \otimes X_n) \to G(FX_1 \otimes' ... \otimes' FX_n) \to GA$

S rules: in the Lambek category of the adjunction model

S_identity: $id_A : A \to A$

S_unitI:

S_unitE:

5.2 Normaalization and Reduction

6 Applications

7 Related Work

TODO

8 Conclusion

TODO

A Appendix