Non-Commutative Linear Logic in an Adjoint Model

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───── **Abstract** ───

TODO

1 Introduction

Linear logic is a well-known resource-sensitive logic. It has been used extensively to model attack trees. This paper concerns a non-commutative variant of linear logic and combines the non-commutative variant with Girard's linear logic []. We will only focus on the multiplicative (i.e. $\otimes$, $\multimap$) part of linear logic for simplicity. We construct the non-commutative variant by using a non-commutative tensor product $\rhd$ instead of the commutative $\otimes$, and two implications $\leftharpoonup$ and $\rightharpoonup$ for the two directions of $\multimap$.

We model the non-commutative linear logic categorically using an ajunction between a symmetric monoidal closed category and a Lambek category. Our categorial adjoint model has a similar structure as Benton's adjoint model [], in which the multiplicative part of intuitionistic linear logic (ILL) is modeled using an adjunction between a cartesian closed category and a symmetric monoidal closed category. On the other hand, Moggi [] uses monad models to map intuitionistic logic into ILL. As discussed in [**?**], Benton's adjoint models only gives rise to commutative monad models and the non-commutative part remained as an open problem. Therefore, by combining our adjoint models with Benton's, we would be able to get non-commutative monad models and thus non-commutative ILL.

The rest of the paper is organized as follows. Section 2 discusses existing approaches on constructing non-commutative linear logic. Section 3 contains the basic definitions in category theory that we will be using in our adjoint model. Familiar readers may skip this section. Section 4 contains the definition and essential properties of our adjoint model. Section 5 discusses the sequent calculus and natural deduction rules for our non-commutative linear logic. We prove that our sequent calculus has the property of cut-elimination and the natural deduction is strongly normalizing. Section 6 talks about the preliminary result after combing our non-commutative model with Benton's commutative model. Section 7 briefly mentions how our model could be used in attack trees and other areas. Section 8 concludes this paper with future work.

2 Related Work

Polakow and Pfenning discuseed Ordered Linear Logic (OLL) [], which combines intuitionistic, commutative linear and non-commutative linear logic, OLL contains sequents of the form $\Gamma, \Delta, \Omega \vdash$

A, where Γ is a multiset of intuitionistic assumptions, Δ is a multiset of commutative linear assumptions, and Ω is a list of non-commutative linear assumptions. OLL contains logical connectives from all three the logics. Therefore, our non-commutative adjoint model is a part of OLL and after combining with Benton's commutative adjoint model, we would get a simplification of OLL.

Greco and Palmigiano [] also presents a variant of the multiplicative fragment of non-commutative ILL. But they focus on proper display calculi while we use sequent calculi.

de Paiva and Eades [] also developed categorical models for the non-commutative ILL by adapting the Dialectica categorical models for linear logic.

3 Category Theory Basics

This section contains the basic definitions in category theory that we will be using in our adjoint model. Our model is based on special kinds of monoidal categories: Lambek categories and symmetric monoidal closed categories and Lambek categories, as defined in Definitions 2 and 3.

▶ **Definition 1.** A **monoidal category** $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ consists of

- a bifunctor $\triangleright : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- an object I, called the unit object;
- three natural isomorphisms α, λ, and ρ with components

$$\alpha_{A,B,C} : (A \triangleright B) \triangleright C \to A \triangleright (B \triangleright C)$$

$$\lambda_A : I \triangleright A \to A$$

$$\rho_A : A \triangleright I \to A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in $\mathcal{M}$:

$$
\begin{array}{ccccc}
((A \triangleright B) \triangleright C) \triangleright D & \xrightarrow{\alpha_{A,B,C} \triangleright id_D} & (A \triangleright (B \triangleright C)) \triangleright D & \xrightarrow{\alpha_{A,B\triangleright C,D}} & A \triangleright ((B \triangleright C) \triangleright D) \\
\downarrow{\alpha_{A\triangleright B,C,D}} & & & & \downarrow{id_A \triangleright \alpha_{B,C,D}} \\
(A \triangleright B) \triangleright (C \triangleright D) & & \xrightarrow{\alpha_{A,B,C\triangleright D}} & & A \triangleright (B \triangleright (C \triangleright D))
\end{array}
$$

$$
\begin{array}{ccc}
(A \triangleright I) \triangleright B & \xrightarrow{\alpha_{A,I,B}} & A \triangleright (I \triangleright B) \\
& {}_{\rho_A \triangleright id_B} \searrow \quad \swarrow {}_{id_A \triangleright \lambda_B} & \\
& A \triangleright B &
\end{array}
$$

▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\rightharpoonup : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and $\leftharpoonup : \mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \triangleright A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad\qquad \mathsf{Hom}_{\mathcal{L}}(A \triangleright X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

▶ **Definition 3.** A **symmetric monoidal category** (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\mathsf{ex}_{A,B} : A \otimes B \to B \otimes A$, called **exchange**,

such that the following diagrams commute:

We use $\rhd$ for non-symmetric monoidal categories while $\otimes$ for symmetric ones.

▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap: \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

The relation between SMMCs and Lambek categories are demonstrated in Lemma 5 and Corollary 6.

▶ **Lemma 5.** *Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$
\begin{array}{rcll}
Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\
& \cong & Hom[A \otimes C, B] & \text{By the exchange } \mathsf{ex}_{C,A} \\
& \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category}
\end{array}
$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◀

▶ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

The essential component in our non-commutative adjoint model is a monoidal adjunction, defined in Definitions 7-11.

▶ **Definition 7.** Let $(\mathcal{M}, \rhd, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \rhd', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from $\mathcal{M}$ to $\mathcal{M}'$ is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \rhd FB' \to F(A \rhd B)$, such that the following diagrams commute for any objects A, B, and C in $\mathcal{M}$:
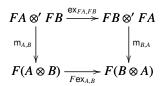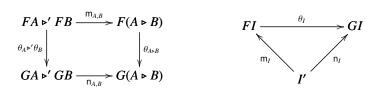
▶ **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{\;\mathsf{ex}_{FA,FB}\;} & FB \otimes' FA \\
\downarrow{\scriptstyle \mathsf{m}_{A,B}} & & \downarrow{\scriptstyle \mathsf{m}_{B,A}} \\
F(A \otimes B) & \xrightarrow[\;F\mathsf{ex}_{A,B}\;]{} & F(B \otimes A)
\end{array}
$$

▶ **Definition 9.** An **adjunction** between categories C and $\mathcal{D}$ consists of two functors $F : \mathcal{D} \to C$, called the **left adjoint**, and $G : C \to \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta : id_{\mathcal{D}} \to GF$, called the **unit**, and $\varepsilon : FG \to id_C$, called the **counit**, such that the following diagrams commute for any object A in C and B in $\mathcal{D}$:

$$
\begin{array}{ccc}
FB & \xrightarrow{\;F\eta_B\;} & FGFB \\
 & \searrow{\scriptstyle \varepsilon_{FB}} & \downarrow \\
 & & FB
\end{array}
\qquad
\begin{array}{ccc}
GA & \xrightarrow{\;\eta_{GA}\;} & GFGA \\
 & \searrow{\scriptstyle G\varepsilon_A} & \downarrow \\
 & & GA
\end{array}
$$

▶ **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ to a monoidal category $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$. A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, \mathsf{m}) \to (G, \mathsf{n})$ such that the following diagrams commute for any objects A and B in $\mathcal{M}$:

$$
\begin{array}{ccc}
FA \rhd' FB & \xrightarrow{\;\mathsf{m}_{A,B}\;} & F(A \rhd B) \\
\downarrow{\scriptstyle \theta_A \rhd' \theta_B} & & \downarrow{\scriptstyle \theta_{A \rhd B}} \\
GA \rhd' GB & \xrightarrow[\;\mathsf{n}_{A,B}\;]{} & G(A \rhd B)
\end{array}
\qquad
\begin{array}{ccc}
FI & \xrightarrow{\;\theta_I\;} & GI \\
{\scriptstyle \mathsf{m}_I}\nwarrow & & \nearrow{\scriptstyle \mathsf{n}_I} \\
 & I' &
\end{array}
$$

▶ **Definition 11.** Let $(\mathcal{M}, \rhd, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \rhd', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

In Moggi's monad model [], the monad is required to be strong, as defined in Definitions 12 and 13.

▶ **Definition 12.** Let C be a category. A **monad** on C consists of an endofunctor $T : C \to C$ together with two natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to T$, where id_C is the identity functor on C, such that the following diagrams commute:

$$
\begin{array}{ccc}
T^3 & \xrightarrow{\;T\mu\;} & T^2 \\
\downarrow{\scriptstyle \mu_T} & & \downarrow{\scriptstyle \mu} \\
T^2 & \xrightarrow[\;\mu\;]{} & T
\end{array}
\qquad
\begin{array}{ccc}
T & \xrightarrow{\;\eta_T\;} & T^2 \\
\downarrow{\scriptstyle T\eta} & \searrow & \downarrow{\scriptstyle \mu} \\
T^2 & \xrightarrow[\;\mu\;]{} & T
\end{array}
$$

▶ **Definition 13.** Let $(\mathcal{M}, \rhd, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on $\mathcal{M}$. T is a **strong monad** if there is natural transformation τ, called the **tensorial strength**, with components

$\tau_{A,B} : A \rhd TB \to T(A \rhd B)$ such that the following diagrams commute:

$$
\begin{array}{ccc}
I \rhd TA & \xrightarrow{\tau_{I,A}} & T(I \rhd A) \\
& & \\
{\scriptstyle \lambda_{TA}} \searrow & & \swarrow {\scriptstyle T\lambda_A} \\
& TA &
\end{array}
\qquad
\begin{array}{ccc}
A \rhd B & \xrightarrow{id_A \rhd \eta_B} & A \rhd TB \\
& & \\
{\scriptstyle \eta_{A \rhd B}} \searrow & & \swarrow {\scriptstyle \tau_{A,B}} \\
& T(A \rhd B) &
\end{array}
$$

$$
\begin{array}{ccc}
(A \rhd B) \rhd TC & \xrightarrow{\tau_{A \rhd B,C}} & T((A \rhd B) \rhd C) \\
{\scriptstyle \alpha_{A,B,TC}} \downarrow & & \downarrow {\scriptstyle T\alpha_{A,B,C}} \\
A \rhd (B \rhd TC) \xrightarrow{id_A \rhd \tau_{B,C}} A \rhd T(B \rhd C) & \xrightarrow{\tau_{A,B \rhd C}} & T(A \rhd (B \rhd C))
\end{array}
$$

$$
\begin{array}{ccc}
A \rhd T^2 B \xrightarrow{\tau_{A,TB}} T(A \rhd TB) & \xrightarrow{T\tau_{A,B}} & T^2(A \rhd B) \\
{\scriptstyle id_A \rhd \mu_B} \downarrow & & \downarrow {\scriptstyle \mu_{A \rhd B}} \\
A \rhd TB & \xrightarrow{\tau_{A,B}} & T(A \rhd B)
\end{array}
$$

▶ **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex, and (T, η, μ) be a strong monad on $\mathcal{M}$. Then there is a **"twisted" tensorial strength** $\tau'_{A,B} : TA \otimes B \to T(A \otimes B)$ defined as $\tau'_{A,B} = T\mathsf{ex} \circ \tau_{B,A} \circ \mathsf{ex}$. We can construct a pair of natural transformations Φ, Φ' with components $\Phi_{A,B}, \Phi'_{A,B} : TA \otimes TB \to T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A \otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A \otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.

▶ **Definition 15.** Let $\mathcal{L}$ be a category. A **comonad** on $\mathcal{L}$ consists of an endofunctor $S : \mathcal{L} \to \mathcal{L}$ together with two natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S^2 \to S$ such that the following diagrams commute:

$$
\begin{array}{ccc}
S & \xrightarrow{\delta} & S^2 \\
{\scriptstyle \delta} \downarrow & & \downarrow {\scriptstyle S\delta} \\
S^2 & \xrightarrow{\delta_S} & S^3
\end{array}
\qquad
\begin{array}{ccc}
S^2 & \xrightarrow{S\varepsilon} & S \\
{\scriptstyle \varepsilon_S} \downarrow & & \downarrow {\scriptstyle \delta} \\
S & \xrightarrow{\delta} & S^2
\end{array}
$$

4 Lambek Adjoint Model

Our adjoint model, Lambek Adjoint Model (LAM), has a similar structure as Benton's LNL model []. Benton's LNL model consists of a symmetric monoidal adjunction $F : C \dashv \mathcal{L} : G$ between a cartesian closed category C and a symmetric monoidal closed category $\mathcal{L}$. LAM consists of a monoidal adjunction between a symmetric monoidal closed category and a Lambek category.

▶ **Definition 16.** A **Lambek Adjoint Model (LAM)** $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ consists of

- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \rhd, I', \alpha', \lambda', \rho')$;
- a monoidal adjunction $F : C \dashv \mathcal{L} : G$ with unit $\eta_X : X \to GFX$ and counit $\varepsilon : FG \to id_{\mathcal{L}}$, where $(F : C \to \mathcal{L}, m)$ and $(G : \mathcal{L} \to C, n)$ are monoidal functors.

Thus, in LAM, the following four diagrams commute because η and ε are monoidal natural transformations:

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\;\;id_{X \otimes Y}\;\;} & X \otimes Y \\
{\scriptstyle \eta_X \otimes \eta_Y} \downarrow & & \downarrow {\scriptstyle \eta_{X \otimes Y}} \\
GFX \otimes GFY \xrightarrow{n_{FX,FY}} G(FX \otimes FY) & \xrightarrow{Gm_{X,Y}} & GF(X \otimes Y)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\;\eta_I\;} & GFI \\
\| & & \uparrow {\scriptstyle Gm_I} \\
I & \xrightarrow{\;n_{I'}\;} & GI'
\end{array}
$$

$$
\begin{array}{ccc}
FGA \otimes FGB \xrightarrow{m_{GA,GB}} F(GA \otimes GB) & \xrightarrow{Fn_{A,B}} & FG(A \otimes B) \\
{\scriptstyle \varepsilon_A \otimes \varepsilon_B} \downarrow & & \downarrow {\scriptstyle \varepsilon_{A \otimes B}} \\
A \otimes B & = & A \otimes B
\end{array}
\qquad
\begin{array}{ccc}
FGI' & \xrightarrow{\;\varepsilon_{I'}\;} & I' \\
{\scriptstyle Fn_{I'}} \uparrow & & \| \\
FI & \xleftarrow{\;m_I\;} & I'
\end{array}
$$

And the following two triangles commute because of the adjunction:

$$
\begin{array}{ccc}
FX & \xrightarrow{\;F\eta_X\;} & FGFX \\
 & \searrow{\scriptstyle \varepsilon_{FX}} & \downarrow \\
 & FX &
\end{array}
\qquad
\begin{array}{ccc}
GA & \xrightarrow{\;\eta_{GA}\;} & GFGA \\
 & \searrow{\scriptstyle G\varepsilon_A} & \downarrow \\
 & GA &
\end{array}
$$

Following the tradition, we use letters X, Y, Z for objects in $\mathcal{C}$ and A, B, C for objects in $\mathcal{L}$. The rest of this section proves essential properties of a LAM.

4.1 An Isomorphism

Let $(\mathcal{C}, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM, where (F, m) and (G, n) are monoidal functors. Similarly as in Benton's LNL model, $m_{X,Y}$ are components of a natural isomorphism and m_I is an isomorphism. This is essential for deriving certain rules of our non-commutative linear logic, such as tensor elimination in natural deduction.
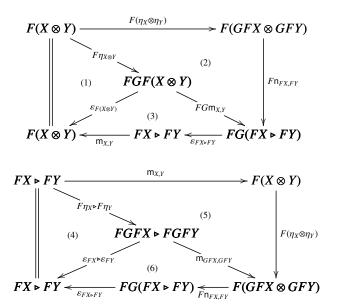
We define the inverses of $m_{X,Y} : FX \triangleright FY \to F(X \otimes Y)$ and $m_I : I' \to FI$ as:

$$
p_{X,Y} : F(X \otimes Y) \xrightarrow{F(\eta_X \otimes \eta_Y)} F(GFX \otimes GFY) \xrightarrow{Fn_{FX,FY}} FG(FX \triangleright FY) \xrightarrow{\varepsilon_{FX \triangleright FX}} FX \triangleright FY
$$

$$
p_I : FI \xrightarrow{Fn_{I'}} FGI' \xrightarrow{\varepsilon_{I'}} I'
$$

▶ **Theorem 17.** $m_{X,Y}$ *are components of a natural isomorphism and their inverses are* $p_{X,Y}$.

Proof. We need to show that $m_{X,Y} \circ p_{X,Y} = id_{F(X \otimes Y)}$ and $p_{X,Y} \circ m_{X,Y} = id_{FX \triangleright FX}$. The two equations hold because the following diagrams commute: (1)-adjunction; (2)-η is a monoidal natural transformation; (3)-naturality of ε; (4)-adjunction; (5)-naturality of m; (6)-ε is a monoidal natural

transformation.

$$
\begin{array}{ccc}
F(X \otimes Y) & \xrightarrow{\;F(\eta_X \otimes \eta_Y)\;} & F(GFX \otimes GFY) \\
\end{array}
$$

Diagram with vertices $F(X \otimes Y)$, $F(GFX \otimes GFY)$, $FGF(X \otimes Y)$, labels $F\eta_{X \otimes Y}$, (1), (2), $Fn_{FX,FY}$, $\varepsilon_{F(X \otimes Y)}$, (3), $FGm_{X,Y}$, $F(X \otimes Y) \xleftarrow{m_{X,Y}} FX \triangleright FY \xleftarrow{\varepsilon_{FX \triangleright FY}} FG(FX \triangleright FY)$.

Diagram with vertices $FX \triangleright FY \xrightarrow{m_{X,Y}} F(X \otimes Y)$, $FGFX \triangleright FGFY$, labels $F\eta_X \triangleright F\eta_Y$, (4), (5), $F(\eta_X \otimes \eta_Y)$, $\varepsilon_{FX} \triangleright \varepsilon_{FY}$, (6), $m_{GFX,GFY}$, $FX \triangleright FY \xleftarrow{\varepsilon_{FX \triangleright FY}} FG(FX \triangleright FY) \xleftarrow{Fn_{FX,FY}} F(GFX \otimes GFY)$.

◄

▶ **Theorem 18.** m_I *is an isomorphism and its inverse is* p_I.

Proof. This is equivalent to equations $m_I \circ p_I = id_{FI}$ and $p_I \circ m_I = id_{I'}$, equivalent to the following diagrams, which commute because ε is a monoidal natural transformation.

$$
\begin{array}{ccc}
FI & \xrightarrow{Fn_{I'}} & FGI' \\
\Big\| & & \Big\downarrow \varepsilon_{I'} \\
FI & \xleftarrow{m_I} & I'
\end{array}
\qquad\qquad
\begin{array}{ccc}
I' & \xrightarrow{m_I} & FI \\
\Big\| & & \Big\downarrow Fn_{I'} \\
I' & \xleftarrow{\varepsilon_{I'}} & FGI'
\end{array}
$$

◄

4.2   Monad on C

We first show that the monad on C in LAM is strong but non-commutative. In Benton's LNL model, the monad on the cartesian closed category is commutative.

▶ **Lemma 19.** *The monad on the symmetric monoidal closed category C in LAM is monoidal.*

Proof. Let $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM. We define the monad $(T, \eta : id_C \to T, \mu : T^2 \to T)$ on C as

$$
T = GF \qquad\qquad \eta_X : X \to GFX \qquad\qquad \mu_X = G\varepsilon_{FX} : GFGFX \to GFX
$$
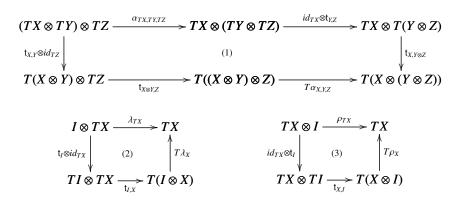
Since (F, m) and (G, n) are monoidal functors, we have

$$
t_{X,Y} = Gm_{X,Y} \circ n_{FX,FY} : TX \otimes TY \to T(X \otimes Y) \qquad\qquad t_I = Gm_I \circ n_{I'} : I \to TI
$$

The monad T being monoidal means:

1. T is a monoidal functor, i.e. the folllowing diagrams commute:

$$
\begin{array}{ccccc}
(TX \otimes TY) \otimes TZ & \xrightarrow{\alpha_{TX,TY,TZ}} & TX \otimes (TY \otimes TZ) & \xrightarrow{id_{TX}\otimes t_{Y,Z}} & TX \otimes T(Y \otimes Z) \\
{\scriptstyle t_{X,Y}\otimes id_{TZ}}\downarrow & & (1) & & \downarrow {\scriptstyle t_{X,Y\otimes Z}} \\
T(X \otimes Y) \otimes TZ & \xrightarrow{\quad t_{X\otimes Y,Z}\quad} & T((X \otimes Y) \otimes Z) & \xrightarrow{\quad T\alpha_{X,Y,Z}\quad} & T(X \otimes (Y \otimes Z))
\end{array}
$$

$$
\begin{array}{ccc}
I \otimes TX & \xrightarrow{\lambda_{TX}} & TX \\
{\scriptstyle t_I\otimes id_{TX}}\downarrow \quad (2) & & \uparrow {\scriptstyle T\lambda_X} \\
TI \otimes TX & \xrightarrow{\quad t_{I,X}\quad} & T(I \otimes X)
\end{array}
\qquad
\begin{array}{ccc}
TX \otimes I & \xrightarrow{\rho_{TX}} & TX \\
{\scriptstyle id_{TX}\otimes t_I}\downarrow \quad (3) & & \uparrow {\scriptstyle T\rho_X} \\
TX \otimes TI & \xrightarrow{\quad t_{X,I}\quad} & T(X \otimes I)
\end{array}
$$

We write GF instead of T in the proof for clarity.

By replacing $t_{X,Y}$ with its definition, diagram (1) above commutes by the following commutative diagram, in which the two hexagons commute because G and F are monoidal functors, and the two quadrilaterals commute by the naturality of n.

$$
\begin{array}{ccccc}
(GFX \otimes GFY) \otimes GFZ & \xrightarrow{\alpha_{GFX,GFY,GFZ}} & GFX \otimes (GFY \otimes GFZ) & \xrightarrow{id_{GFX}\otimes n_{FY,FZ}} & GFX \otimes G(FY \triangleright FZ) \\
{\scriptstyle n_{FX,FY}\otimes id_{GFZ}}\downarrow & & {\scriptstyle n_{FX,FY\triangleright FZ}} & & \downarrow {\scriptstyle id_{GFX}\otimes Gm_{Y,Z}} \\
G(FX \triangleright FY) \otimes GFZ & & G(FX \triangleright (FY \triangleright FZ)) & & GFX \otimes GF(Y \otimes Z) \\
{\scriptstyle Gm_{X,Y}\otimes id_{GFZ}}\downarrow & {\scriptstyle n_{FX\triangleright FY,FZ}} & {\scriptstyle G\alpha'_{FX,FY,FZ}} \quad {\scriptstyle G(id_{FX}\triangleright m_{Y,Z})} & & \downarrow {\scriptstyle n_{FX,F(Y\otimes Z)}} \\
GF(X \otimes Y) \otimes GFZ & & G((FX \triangleright FY) \triangleright FZ) & & G(FX \triangleright F(Y \otimes Z)) \\
{\scriptstyle n_{F(X\otimes Y),FZ}}\downarrow & {\scriptstyle G(m_{X,Y}\otimes id_{FZ})} & & & \downarrow {\scriptstyle Gm_{X,Y\otimes Z}} \\
G(F(X \otimes Y) \triangleright FZ) & \xrightarrow{\quad Gm_{X\otimes Y,Z}\quad} & GF((X \otimes Y) \otimes Z) & \xrightarrow{\quad GF\alpha_{X,Y,Z}\quad} & GF(X \otimes (Y \otimes Z))
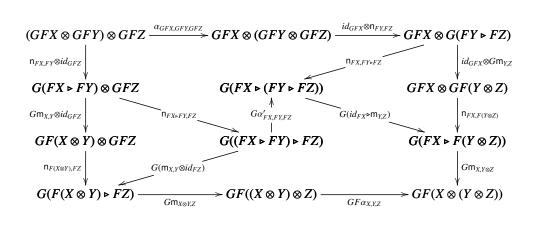\end{array}
$$

Diagram (2) commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.
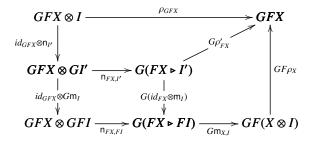
$$
\begin{array}{ccccc}
I \otimes GFX & & \xrightarrow{\lambda_{GFX}} & & GFX \\
{\scriptstyle n_{I'}\otimes id_{GFX}}\downarrow & & & {\scriptstyle G\lambda'_{FX}} & \uparrow \\
GI' \otimes GFX & \xrightarrow{n_{I',FX}} & G(I' \triangleright FX) & & \\
{\scriptstyle Gm_I\otimes id_{GFX}}\downarrow & & \downarrow {\scriptstyle G(m_I\triangleright id_{FX})} & & {\scriptstyle GF\lambda_X}\uparrow \\
GFI \otimes GFX & \xrightarrow{n_{FI,FX}} & G(FI \triangleright FX) & \xrightarrow{Gm_{I,X}} & GF(I \otimes X)
\end{array}
$$

Similarly, diagram (3) commutes as follows:

$$
\begin{array}{ccc}
GFX \otimes I & \xrightarrow{\ \rho_{GFX}\ } & GFX \\
{\scriptstyle id_{GFX}\otimes \mathsf{n}_{I'}}\downarrow & \nearrow{\scriptstyle G\rho'_{FX}} & \uparrow{\scriptstyle GF\rho_X} \\
GFX \otimes GI' \xrightarrow{\mathsf{n}_{FX,I'}} G(FX \rhd I') & & \\
\downarrow{\scriptstyle id_{GFX}\otimes G\mathsf{m}_I} \quad \downarrow{\scriptstyle G(id_{FX}\otimes \mathsf{m}_I)} & & \\
GFX \otimes GFI \xrightarrow{\mathsf{n}_{FX,FI}} G(FX \rhd FI) \xrightarrow{G\mathsf{m}_{X,I}} GF(X \otimes I) &
\end{array}
$$

2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal by definition and thus the following two diagrams commute.

$$
\begin{array}{ccc}
X \otimes Y & =\!=\!= & X \otimes Y \\
{\scriptstyle \eta_X\otimes\eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_{X\otimes Y}} \\
TX \otimes TY & \xrightarrow{\ \mathsf{t}_{X,Y}\ } & T(X \otimes Y)
\end{array}
\qquad\qquad
\begin{array}{ccc}
I & \xrightarrow{\ \eta_I\ } & TI \\
\| & & \nearrow{\scriptstyle \mathsf{t}_I} \\
& I &
\end{array}
$$

3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ. Thus the following diagrams commute.

$$
\begin{array}{ccc}
T^2X \otimes T^2Y \xrightarrow{\mathsf{t}_{TX,TY}} T(TX \otimes TY) \xrightarrow{T\mathsf{t}_{X,Y}} T^2(X \otimes Y) \\
{\scriptstyle \mu_X\otimes\mu_Y}\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \mu_{X\otimes Y}} \\
TX \otimes TY \xrightarrow{\qquad\ \mathsf{t}_{X,Y}\ \qquad} T(X \otimes Y)
\end{array}
\qquad
\begin{array}{ccc}
T^2I & \xrightarrow{\ \mu_I\ } & TI \\
{\scriptstyle T\mathsf{t}_I}\uparrow & & \uparrow{\scriptstyle \mathsf{t}_I} \\
TI & \xleftarrow{\ \mathsf{t}_I\ } & I
\end{array}
$$

◄

However, the monad is not symmetric becauase the following diagram does not commute, for the Lambek category $\mathcal{L}$ is not symmetric.

$$
\begin{array}{ccccc}
GFX \otimes GFY & \xrightarrow{\mathsf{ex}_{GFX,GFY}} & GFY \otimes GFX & \xrightarrow{\mathsf{n}_{FY,FX}} & G(FY \rhd FX) \\
{\scriptstyle \mathsf{n}_{FX,FY}}\downarrow & & & & \downarrow{\scriptstyle G\mathsf{m}_{Y,X}} \\
G(FX \rhd FY) & \xrightarrow{G\mathsf{m}_{X,Y}} & GF(X \otimes Y) & \xrightarrow{GF\mathsf{ex}_{X,Y}} & GF(Y \otimes X)
\end{array}
$$

► **Lemma 20.** *The monad on the symmetric monoidal closed category in LAM is strong.*

Proof. Let $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ be a LAM, where $(C, \otimes, I, \alpha, \lambda, \rho)$ is symmetric monoidal closed, $(\mathcal{L}, \rhd, I', \alpha', \lambda', \rho')$ is Lambek. In Lemma 19, we have proved that the monad $(T = GF, \eta, \mu)$ is monoidal with the natural transformation $\mathsf{t}_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ and the morphism $\mathsf{t}_I : I \to TI$. We define the tensorial strength $\tau_{X,Y} : X \otimes TY \to T(X \otimes Y)$ as $\tau_{X,Y} = \mathsf{t}_{X,Y} \circ (\eta_X \otimes id_{TY})$.
Since η is a monoidal natural transformation, we have $\eta_I = G\mathsf{m}_I \circ \mathsf{n}_{I'}$. Therefore $\eta_I = \mathsf{t}_I$. Thus the following diagram commutes because T is monoidal, where the composition $\mathsf{t}_{I,X} \circ (\mathsf{t}_I \otimes id_{TX})$ is the definition of $\tau_{I,X}$. So the first triangle in Defition 13 commutes.

$$
\begin{array}{ccc}
I \otimes TX & \xrightarrow{\mathsf{t}_I\otimes id_{TX}} & TI \otimes TX \\
{\scriptstyle \lambda_{TX}}\downarrow & & \downarrow{\scriptstyle \mathsf{t}_{I,X}} \\
TX & \xleftarrow{\ T\lambda_X\ } & T(I \otimes X)
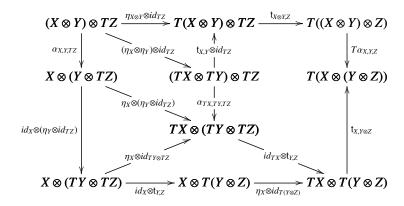\end{array}
$$

Similarly, by using the definition of τ, the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{id_X \otimes \eta_Y} & X \otimes TY \\
{\scriptstyle \eta_{X \otimes Y}} \downarrow & {\scriptstyle \eta_X \otimes \eta_Y} & \downarrow {\scriptstyle \eta_X \otimes id_{TY}} \\
T(X \otimes Y) & \xleftarrow{\;\;t_{X,Y}\;\;} & TX \otimes TY
\end{array}
$$

The first pentagon in the definition commutes by the following commutative diagrams, because η and α are natural transformations and T is monoidal:

$$
\begin{array}{ccccc}
(X \otimes Y) \otimes TZ & \xrightarrow{\eta_{X \otimes Y} \otimes id_{TZ}} & T(X \otimes Y) \otimes TZ & \xrightarrow{t_{X \otimes Y,Z}} & T((X \otimes Y) \otimes Z) \\
{\scriptstyle \alpha_{X,Y,TZ}} \downarrow & {\scriptstyle (\eta_X \otimes \eta_Y) \otimes id_{TZ}} \nearrow & \uparrow {\scriptstyle t_{X,Y} \otimes id_{TZ}} & & \downarrow {\scriptstyle T\alpha_{X,Y,Z}} \\
X \otimes (Y \otimes TZ) & & (TX \otimes TY) \otimes TZ & & T(X \otimes (Y \otimes Z)) \\
 & {\scriptstyle \eta_X \otimes (\eta_Y \otimes id_{TZ})} \searrow & \downarrow {\scriptstyle \alpha_{TX,TY,TZ}} & & \uparrow \\
{\scriptstyle id_X \otimes (\eta_Y \otimes id_{TZ})} \downarrow & & TX \otimes (TY \otimes TZ) & & \uparrow {\scriptstyle t_{X,Y \otimes Z}} \\
 & {\scriptstyle \eta_X \otimes id_{TY \otimes TZ}} \nearrow & & {\scriptstyle id_{TX} \otimes t_{Y,Z}} \searrow & \\
X \otimes (TY \otimes TZ) & \xrightarrow{id_X \otimes t_{Y,Z}} & X \otimes T(Y \otimes Z) & \xrightarrow{\eta_X \otimes id_{T(Y \otimes Z)}} & TX \otimes T(Y \otimes Z)
\end{array}
$$

The last diagram in the definition commtues by the following commutative diagram, because T is a monad, t is a natural transformation, and μ is a monoidal natural transformation:

$$
\begin{array}{ccccc}
X \otimes T^2 Y & \xrightarrow{\eta_X \otimes id_{T^2 Y}} TX \otimes T^2 Y & & \xrightarrow{\;\;t_{X,TY}\;\;} & T(X \otimes TY) \\
{\scriptstyle id_X \otimes \mu_Y} \downarrow & \parallel \quad {\scriptstyle T\eta_X \otimes id_{T^2Y}} \searrow & & & \downarrow {\scriptstyle T(\eta_X \otimes id_{TY})} \\
X \otimes TY \;{\scriptstyle id_{TX} \otimes \mu_Y}\; TX \otimes T^2 Y & \xleftarrow{\mu_X \otimes id_{T^2Y}} T^2 X \otimes T^2 Y & \xrightarrow{t_{TX,TY}} & T(TX \otimes TY) \\
{\scriptstyle \eta_X \otimes id_{TY}} \downarrow \; {\scriptstyle id_{TX} \otimes \mu_Y} \; {\scriptstyle \mu_X \otimes \mu_Y} & & & & \downarrow {\scriptstyle Tt_{X,Y}} \\
TX \otimes TY & \xrightarrow{t_{X,Y}} T(X \otimes Y) \xleftarrow{\qquad \mu_{X \otimes Y} \qquad} & & & T^2(X \otimes Y)
\end{array}
$$

◄

The following lemma is adopted from [].

▶ **Lemma 21.** *Let $\mathcal{M}$ be a symmetric monoidal category and T be a strong monad on $\mathcal{M}$. Then T is commutative iff it is symmetric.*

▶ **Theorem 22.** *The monad on the SMCC in LAM is strong but non-commutative.*

Proof. The proof is obvious. Based on Lemma 20 and Lemma 21, the monad is non-commutative.

◄

4.3 Comonad on $\mathcal{L}$

▶ **Lemma 23.** *The comonad on the Lambek category in a LAM is monoidal.*

Proof. We define the comonad $(S, \varepsilon : S \to id_{\mathcal{L}}, \delta : S \to S^2)$ on the Lambek category $\mathcal{L}$ as:

$$S = FG \qquad\qquad \varepsilon_A : SA \to A \qquad\qquad \delta_A = F\eta_{GA} : SA \to S^2A$$

Thus, we have natural transformation s and morphism s_I defined as:

$$\mathsf{s}_{A,B} = F\mathsf{n}_{A,B} \circ \mathsf{m}_{GA,GB} : SA \triangleright SB \to SA \triangleright SB \qquad\qquad \mathsf{s}_I = F\mathsf{n}_{I'} \circ \mathsf{m}_I : I' \to SI'$$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$
\begin{array}{ccc}
(SA \triangleright SB) \triangleright SC & \xrightarrow{\ \alpha'_{SA,SB,SC}\ } SA \triangleright (SB \triangleright SC) & \xrightarrow{\ id_{SA} \triangleright \mathsf{s}_{B,C}\ } SA \triangleright S(B \triangleright C) \\
\downarrow{\scriptstyle \mathsf{s}_{A,B} \triangleright id_{SC}} & & \downarrow{\scriptstyle \mathsf{s}_{A,B \triangleright C}} \\
S(A \triangleright B) \triangleright SC & \xrightarrow{\ \mathsf{s}_{A \triangleright B,C}\ } S((A \triangleright B) \triangleright C) \xrightarrow{\ S\alpha'_{A,B,C}\ } & S(A \triangleright (B \triangleright C))
\end{array}
$$

$$
\begin{array}{ccc}
I' \triangleright SA & \xrightarrow{\ \lambda'_{SA}\ } & SA \\
\downarrow{\scriptstyle \mathsf{s}_{I'} \triangleright id_{SA}} & & \uparrow{\scriptstyle S\lambda'_A} \\
SI' \triangleright SA & \xrightarrow[\ \mathsf{s}_{I',A}\ ]{} & S(I' \triangleright A)
\end{array}
\qquad
\begin{array}{ccc}
SA \triangleright I' & \xrightarrow{\ \rho'_{SA}\ } & SA \\
\downarrow{\scriptstyle id'_{SA} \triangleright \mathsf{s}_{I'}} & & \uparrow{\scriptstyle S\rho'_A} \\
SA \triangleright SI' & \xrightarrow[\ \mathsf{s}_{A,I'}\ ]{} & S(A \triangleright I')
\end{array}
$$

2. ε is a monoidal natural transformation:

$$
\begin{array}{ccc}
SA \triangleright SB & \xrightarrow{\ \mathsf{s}_{A,B}\ } & S(A \triangleright B) \\
\downarrow{\scriptstyle \varepsilon_A \triangleright \varepsilon_B} & & \downarrow{\scriptstyle \varepsilon_{A \triangleright B}} \\
A \triangleright B & =\!=\!=\!= & A \triangleright B
\end{array}
\qquad
\begin{array}{ccc}
SI' & \xrightarrow{\ \varepsilon_{I'}\ } & I' \\
& {\scriptstyle \mathsf{s}_{I'}}\nwarrow & \| \\
& I' &
\end{array}
$$

3. δ is a monoidal natural transformation:

$$
\begin{array}{ccc}
SA \triangleright SA & \xrightarrow{\quad\quad \mathsf{s}_{A,B}\quad\quad} & S(A \triangleright B) \\
\downarrow{\scriptstyle \delta_A \triangleright \delta_B} & & \downarrow{\scriptstyle \delta_{A \triangleright B}} \\
S^2A \triangleright S^2B & \xrightarrow[\ \mathsf{s}_{SA,SB}\ ]{} S(SA \triangleright SB) \xrightarrow[\ S\mathsf{s}_{A,B}\ ]{} & S^2(A \triangleright B)
\end{array}
\qquad
\begin{array}{ccc}
SI' & \xrightarrow{\ \delta_{I'}\ } & S^2I' \\
\uparrow{\scriptstyle \mathsf{s}_{I'}} & & \uparrow{\scriptstyle S\mathsf{s}_{I'}} \\
I' & \xrightarrow[\ \mathsf{s}_{I'}\ ]{} & SI'
\end{array}
$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma 19. We do not include the proof here for simplicity. ◄

We then show that the co-Eilenberg-Moore category of the comonad S is symmetric monoidal closed.

▶ **Definition 24.** Let (S, ε, δ) be a comonad on a category $\mathcal{L}$. Then the **co-Eilenberg-Moore category** $\mathcal{L}^S$ of the comonad has

■ as objects the S-coalgebras $(A, h_A : A \to SA)$, where A is an object in $\mathcal{L}$, s.t. the following diagrams commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\ h_A\ } & SA \\
\downarrow{\scriptstyle h_A} & & \downarrow{\scriptstyle \delta_A} \\
SA & \xrightarrow[\ Sh_A\ ]{} & S^2A
\end{array}
\qquad
\begin{array}{ccc}
& SA & \\
{\scriptstyle h_A}\nearrow & & \searrow{\scriptstyle \varepsilon_A} \\
A & =\!=\!=\!= & A
\end{array}
$$

- as morphisms the coalgebra morphisms, i.e. morphisms $f : (A, h_A) \to (B, h_B)$ between coalgebras s.t. the diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
h_A \downarrow & & \downarrow h_B \\
SA & \xrightarrow[\ Sf\ ]{} & SB
\end{array}
$$

▶ **Lemma 25.** *Given a LAM* $(C, \mathcal{L}, F, G, \eta, \varepsilon)$ *and the comonad S on $\mathcal{L}$, the co-Eilenberg-Moore category $\mathcal{L}^S$ has an exchange natural transformation* $\mathsf{ex}^S_{A,B} : A \triangleright B \to B \triangleright A$.

Proof. We define the exchange $\mathsf{ex}^S_{A,B} : A \triangleright B \to B \triangleright A$ as

$$
A \triangleright B \xrightarrow{h_A \triangleright h_B} FGA \triangleright FGB \xrightarrow{\mathsf{m}_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\mathsf{ex}_{GA,GB}} F(GB \otimes GA) \xrightarrow{F\mathsf{n}_{B,A}} FG(B \triangleright A) \xrightarrow{\varepsilon_{B \triangleright A}} B \triangleright A
$$

in which (F, m) and (G, n) are monoidal functors, and ex is the exchange for C. Then ex^S is a natural transformation because the following diagrams commute for morphisms $f : A \to A'$ and $g : B \to B'$:

$$
\begin{array}{ccccccccccc}
A \triangleright B & \xrightarrow{h_A \triangleright h_B} & FGA \triangleright FGB & \xrightarrow{\mathsf{m}_{GA,GB}} & F(GA \otimes GB) & \xrightarrow{F\mathsf{ex}_{A,B}} & F(GB \otimes GA) & \xrightarrow{F\mathsf{n}_{B,A}} & FG(B \triangleright A) & \xrightarrow{\varepsilon_{B \triangleright A}} & B \triangleright A \\
f \triangleright g \downarrow & & \downarrow FGf \triangleright FGg & & \downarrow F(Gf \otimes Gg) & & \downarrow F(Gg \otimes Gf) & & \downarrow FG(g \triangleright f) & & \downarrow g \triangleright f \\
A' \triangleright B' & \xrightarrow[h_{A'} \triangleright h_{B'}]{} & FGA' \triangleright FGB' & \xrightarrow[\mathsf{m}_{GA',GB'}]{} & F(GA' \otimes GB') & \xrightarrow[F\mathsf{ex}_{A',B'}]{} & F(GB' \otimes GA') & \xrightarrow[F\mathsf{n}_{B',A'}]{} & FG(B' \triangleright A') & \xrightarrow[\varepsilon_{B' \triangleright A'}]{} & B' \triangleright A'
\end{array}
$$

◀

▶ **Lemma 26.** *The following diagrams commute in the co-Eilenberg-Moore category $\mathcal{L}^S$:*

$$
\begin{array}{ccccc}
F((GA \otimes GB) \otimes GC) & \xrightarrow{F(\mathsf{n}_{A,B} \otimes id_{GC})} & F(G(A \triangleright B) \otimes GC) & \xrightarrow{F(\mathsf{ex}_{A,B} \otimes id_{GC})} & FG((A \triangleright B) \triangleright C) \\
F(\mathsf{ex}_{A,B} \otimes id_{GC}) \downarrow & & & & \downarrow \varepsilon_{(A \triangleright B) \triangleright C} \\
F(G(B \triangleright A) \otimes GC) & & & & (A \triangleright B) \triangleright C \\
F(\mathsf{n}_{B,A} \otimes id_{GC}) \downarrow & & & & \downarrow \mathsf{ex}^S_{A,B} \triangleright id_C \\
F(G(B \triangleright A) \otimes GC) & \xrightarrow{F\mathsf{n}_{B \triangleright A,C}} & FG((B \triangleright A) \triangleright C) & \xrightarrow{\varepsilon_{(B \triangleright A) \triangleright C}} & (B \triangleright A) \triangleright C
\end{array}
$$

$$
\begin{array}{ccccc}
F(GB \otimes (GC \otimes GA)) & \xrightarrow{F(id_{GB} \otimes \mathsf{n}_{C,A})} & F(GB \otimes G(C \triangleright A)) & \xrightarrow{F\mathsf{n}_{B,C \triangleright A}} & FG(B \triangleright (C \triangleright A)) \\
F(id_{GB} \otimes \mathsf{ex}_{C,A}) \downarrow & & & & \downarrow \varepsilon_{B \triangleright (C \triangleright A)} \\
F(GB \otimes (GA \otimes GC)) & & & & B \triangleright (C \triangleright A) \\
F(id_{GB} \otimes \mathsf{n}_{A,C}) \downarrow & & & & \downarrow id_A \triangleright \mathsf{ex}^S_{C,A} \\
F(GB \otimes G(A \triangleright C)) & \xrightarrow{F\mathsf{n}_{B,A \triangleright C}} & FG(B \triangleright (A \triangleright C)) & \xrightarrow{\varepsilon_{B \triangleright (A \triangleright C)}} & B \triangleright (A \triangleright C)
\end{array}
$$

Proof. We only write the proof for the first diagram. The proof for the second one is similar. (1), (2), (3)–naturality of m; (4)–F is monoidal; (5), (12)–ε is monoidal; (6), (7), (8), (9), (10)–obvious;

(11)–coalgebra.

$$
\begin{array}{ccccccc}
F(G(A \triangleright B) \otimes GC) & \xrightarrow{F\mathsf{n}_{A\triangleright B,C}} & FG((A \triangleright B) \triangleright C) & \xrightarrow{\varepsilon_{(A\triangleright B)\triangleright C}} & (A \triangleright B) \triangleright C & =\!=\!= & (A \triangleright B) \triangleright C
\end{array}
$$

(1)–(12) commutative diagram with labels: $F(\mathsf{n}_{A,B}\otimes id_{GC})$, $\mathsf{m}_{G(A\triangleright B),GC}$, (5), $\varepsilon_{A\triangleright B}\triangleright\varepsilon_C$, $\varepsilon_{A\triangleright B}\triangleright id_C$, (6), $(\varepsilon_A\triangleright\varepsilon_B)\triangleright id_C$, (11), $(h_A\triangleright h_B)\triangleright id_C$, $F((GA\otimes GB)\otimes GC)$, (1), $FG(A\triangleright B)\triangleright FGC$, $\xrightarrow{id\triangleright\varepsilon_C}$, $FG(A\triangleright B)\triangleright C$, (12), $(FGA \triangleright FGB)\triangleright C$, $F(\mathsf{ex}_{A,B}\otimes id_{GC})$, $\mathsf{m}_{GA\otimes GB,GC}$, $F\mathsf{n}_{A,B}\triangleright id_{FGC}$, (7), $F\mathsf{n}_{A,B}\otimes id_C$, $\mathsf{m}_{GA,GB}\triangleright id_C$, $F((GB\otimes GA)\otimes GC)$, (2), $F(GA\otimes GB)\triangleright FGC$, $\xrightarrow{id_{F(GA\otimes GB)}\triangleright\varepsilon_C}$, $F(GA\otimes GB)\triangleright C$, $F(\mathsf{n}_{B,A}\otimes id_{GC})$, (3), $\mathsf{m}_{GB\otimes GA,GC}$, $F\mathsf{ex}_{A,B}\triangleright id_{FGC}$, (8), $F\mathsf{ex}_{A,B}\triangleright id_C$, $F(G(B\triangleright A)\otimes GC)$, $\xleftarrow{\mathsf{m}_{G(B\triangleright A),GC}}$, $FG(B\triangleright A)\triangleright FGC$, $\xleftarrow{F\mathsf{n}_{B,A}\triangleright id}$, $F(GB\otimes GA)\triangleright FGC$, $\xrightarrow{id\triangleright\varepsilon_C}$, $F(GB\otimes GA)\triangleright C$, $F\mathsf{n}_{B\triangleright A,C}$, (4), $\varepsilon_{B\triangleright A}\triangleright\varepsilon_C$, (9), $id_{FG(B\triangleright A)}\triangleright\varepsilon_C$, (10), $F\mathsf{n}_{B,A}\triangleright id_C$, $FG((B\triangleright A)\triangleright C)$, $\xrightarrow{\varepsilon_{(B\triangleright A)\triangleright C}}$, $(B\triangleright A)\triangleright C$, $\xleftarrow{\varepsilon_{B\triangleright A}\triangleright id_C}$, $FG(B\triangleright A)\triangleright C$.

◄

▶ **Theorem 27.** *The co-Eilenberg-Moore category $\mathcal{L}^S$ of S is symmetric monoidal closed.*

Proof. Let $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ be the Lambek cateogry in a SMCC-Lambek model and S be the comonad on $\mathcal{L}$. Since $\mathcal{L}$ is a Lambek category, it is obvious that $\mathcal{L}$ is also Lambek. By Corollary 6, we only need to prove the exchange defined in Lemma 25 satisfies the three commutative diagrams in Definition 3.

The first triangle in Definition 3 commutes as follows: (1)–coalgebra; (2)–ε is monoidal; (3)–naturality of ρ; (4)–naturality of ε; (5)–naturality of m; (6)–F is monoidal; (7)–C is symmetric; (8)–naturality of ex; (9)–G is monoidal.

Second commutative diagram with labels: $A \triangleright I'$, $\xrightarrow{h_A\triangleright h_{I'}}$, $FGA \triangleright FGI'$, $\xrightarrow{\mathsf{m}_{GA,GI'}}$, $F(GA\otimes GI')$, (1), $id_{FGA}\triangleright\varepsilon_{I'}$, $id_{FGA}\triangleright F\mathsf{n}_{I'}$, (5), $F(id_{GA}\otimes\mathsf{n}_{I'})$, $FGA \triangleright I'$, (2), $FGA \triangleright FI$, $\xrightarrow{\mathsf{m}_{GA,I}}$, $F(GA\otimes I)$, ρ'_A, $\varepsilon_{FGA}\triangleright id_{I'}$, $id_{FGA}\triangleright\mathsf{m}_I$, (6), $A \triangleright I'$, $FGA \triangleright I'$, $F\rho_{GA}$, $F\mathsf{ex}_{GA,I}$, (8), $F\mathsf{ex}_{GA,GI'}$, (3), ρ'_{FGA}, (7), A, $\xleftarrow{\varepsilon_A}$, FGA, $\xleftarrow{F\lambda_{GA}}$, $F(I\otimes GA)$, λ'_A, (4), $FG\lambda'_A$, (9), $F(\mathsf{n}_{I'}\otimes id_{GA})$, $I' \triangleright A$, $\xleftarrow{\varepsilon_{I'\triangleright A}}$, $FG(I'\triangleright A)$, $\xleftarrow{F\mathsf{n}_{I',A}}$, $F(GI'\otimes GA)$.

The second triangle in the proof commutes as follows: (1) and (5)–coalgebra; (2) and (4)–ε is

monoidal; (3)–$\mathcal{C}$ is symmetric.

$$A \rhd B \xrightarrow{h_A \rhd h_B} FGA \rhd FGB \xrightarrow{m_{GA,GB}} F(GA \otimes GB) \xrightarrow{Fex_{A,B}} F(GB \otimes GA) \xrightarrow{Fn_{B,A}} FG(B \rhd A)$$

(1) $\varepsilon_A \rhd \varepsilon_B$ (2) (3) (4) $B \rhd A$ $\varepsilon_B \rhd \varepsilon_A$ (5) $\varepsilon_{B \rhd A}$

$$A \rhd B \xleftarrow{\varepsilon_{A \rhd B}} FG(A \rhd B) \xleftarrow{Fn_{A,B}} F(GA \otimes GB) \xleftarrow{Fex_{B,A}} F(GB \otimes GA) \xleftarrow{m_{GB,GA}} FGB \rhd FGA \xleftarrow{h_A \rhd h_A} B \rhd A$$

The third diagram commutes as follows: (1) and (7)–Lemma 26; (2)–naturality of α'; (3) and (8)–naturality of ε; (4), (6) and (12)–G is a monoidal functor; (5)–$\mathcal{C}$ is symmetrical monoidal closed; (9)–coalgebra; (10)–ε is a monoidal natural transformation; (11)–naturality of ex.

5   Non-Commutative Linear Logic

In a LAM, the SMCC $\mathcal{C}$ models the commutative linear logic and the Lambeck category $\mathcal{L}$ models the non-commutative variant. In Section 5.1, we will present the term assignment for sequent calculus of both sides and prove the cut elimination theorem. In Section 5.2, we present the term assignment for natural deduction of both sides and prove the logic is strongly normalizing.

A sequent in the commutative side is of the form $\Psi, \Phi \vdash_C t : X$. The types must be X, Y, Z, etc., which are objects in the SMCC C. Tye typing contexts are multisets. Suppose Ψ is the set $x_1 : X_1, x_2 : X_2, ..., x_m : X_m$ and Φ is the set $y_1 : Y_1, y_2 : Y_2, ..., y_n : Y_n$, then the categorical interpretation of the sequent is the morphism $(X_1 \otimes X_2 \otimes ... \otimes X_m) \otimes (Y_1 \otimes Y_2 \otimes ... \otimes Y_n) \to X$.

A sequent in the non-commutative side is of the form $\Gamma, \Delta \vdash_{\mathcal{L}} s : A$. The types must be A, B, C, etc., which are objects in the Lambek category $\mathcal{L}$. Tye typing contexts are lists instead of multisets. The typing contexts are mixed in the sense that they could include contexts from the commutative side. When a commutative context $\Psi = \{x_1 : X_1, ..., x_m : X_m\}$ is included, it is interpreted as the object $F(X_1 \otimes ... \otimes X_m)$. Therefore, the interpretation of the sequent $\Psi, \Gamma \vdash_{\mathcal{L}} s : A$, where Ψ is defined as above and Γ is the list $y_1 : A_1, ..., y_n : A_n$, is the morphism $F(X_1 \otimes ... \otimes X_m) \triangleright (A_1 \triangleright ... \triangleright A_n) \to A$.

For the commutative side, since the contexts are multisets, the following exchange rule is implicit in both sequent calculus and natural deduction:

$$\frac{\Phi, x : X, y : Y, \Psi \vdash_C t : Z}{\Phi, z : Y, w : X, \Psi \vdash_C \text{ex } w, z \text{ with } x, y \text{ in } t : Z} \quad \text{T\_BETA}$$

5.1   Sequent Calculus

The term assignment for sequent calculus of the commutative part of the model, i.e. the SMCC of the adjunction, is defined in Figure 1. And the term assignme for the non-commutative part, i.e. the Lambek category of the adjunction, is defined in Figure 2. We do not have the structural rules except for exchange because the calculus is for linear logic.

$$\frac{}{x : X \vdash_C x : X} \quad \text{T\_AX} \qquad \frac{\Psi \vdash_C t : X}{x : \text{UnitT}, \Psi \vdash_C \text{let } x : \text{UnitT be trivT in } t : X} \quad \text{T\_UNITL} \qquad \frac{}{\cdot \vdash_C \text{trivT} : \text{UnitT}} \quad \text{T\_UNITR}$$

$$\frac{\Phi, x : X, y : Y, \Psi \vdash_C t : Z}{\Phi, z : X \otimes Y, \Psi \vdash_C \text{let } z : X \otimes Y \text{ be } x \otimes y \text{ in } t : Z} \quad \text{T\_TENL} \qquad \frac{\Phi \vdash_C t_1 : X \quad \Psi \vdash_C t_2 : Y}{\Phi, \Psi \vdash_C t_1 \otimes t_2 : X \otimes Y} \quad \text{T\_TENR}$$

$$\frac{\Phi \vdash_C t_1 : X \quad \Psi_1, x : Y, \Psi_2 \vdash_C t_2 : Z}{\Psi_1, y : X \multimap Y, \Phi, \Psi_2 \vdash_C [\text{app } y\, t_1/x]t_2 : Z} \quad \text{T\_IMPL} \qquad \frac{\Phi, x : X \vdash_C t : Y}{\Phi \vdash_C \lambda x : X.t : X \multimap Y} \quad \text{T\_IMPR}$$

$$\frac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_C \text{G}s : \text{G}A} \quad \text{T\_GR} \qquad \frac{\Phi \vdash_C t_1 : X \quad \Psi_1, x : X, \Psi_2 \vdash_C t_2 : Y}{\Psi_1, \Phi, \Psi_2 \vdash_C [t_1/x]t_2 : Y} \quad \text{T\_CUT}$$

■ **Figure 1** Sequent Calculus: Commutative Part

Next, we prove cut elimination for the sequent calculus. We define the **degree $|X|$ (or $|A|$) of a commutative (or non-commutative) formula** to be the number of logical connectives $|X|$ plus 1. For instance, $|X \otimes Y| = |X| + |Y| + 1$. And the **degree of a cut rule** is the degree of the cut formula. The following key cases demonstrate how we can replace a cut with at most two cuts with lower degree. The **degree $|\Pi|$ of a proof** Π is the maximum of the degrees of all cut fules in the proof and $|\Pi| = 0$ if Π is cut-free. Finally, the **height $h(\Pi)$ of a proof** Π is the length of the longest path in the proof tree and the height of an axiom is 0.

We consider the following 11 key cases in proving cut elimination, each of which is a (R, L) pair for the same connective.

■ $(\text{T\_UNITR}, \text{T\_UNITL})$:

$$\frac{\dfrac{}{\cdot \vdash_C \text{trivT} : \text{UnitT}} \text{UNITR} \quad \dfrac{\Psi \vdash_C t : X}{x : \text{UnitT}, \Psi \vdash_C \text{let } x : \text{UnitT be trivT in } t : X} \text{UNITL}}{\Psi \vdash_C [\text{trivT}/x](\text{let } x : \text{UnitT be trivT in } t) : X} \text{CUT}$$

is transformed to

$$\frac{}{x : A \vdash_{\mathcal{L}} x : A} \text{ S\_AX} \qquad \frac{}{\cdot \vdash_{\mathcal{L}} \text{trivS} : \text{UnitS}} \text{ S\_UNITR} \qquad \frac{\Delta \vdash_{\mathcal{L}} s : A}{x : \text{UnitT}, \Delta \vdash_{\mathcal{L}} \text{let} x : \text{UnitT be trivT in } s : A} \text{ S\_UNITL1}$$

$$\frac{\Delta \vdash_{\mathcal{L}} s : A}{x : \text{UnitS}, \Delta \vdash_{\mathcal{L}} \text{let} x : \text{UnitS be trivS in } s : A} \text{ S\_UNITL2} \qquad \frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : Y, w : X, \Delta \vdash_{\mathcal{L}} \text{ex} w, z \text{ with } x, y \text{ in } s : A} \text{ S\_BETA}$$

$$\frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \otimes Y, \Delta \vdash_{\mathcal{L}} \text{let} z : X \otimes Y \text{ be } x \otimes y \text{ in } s : A} \text{ S\_TENL1} \qquad \frac{\Gamma, x : A, y : B, \Delta \vdash_{\mathcal{L}} s : C}{\Gamma, z : A \triangleright B, \Delta \vdash_{\mathcal{L}} \text{let} z : A \triangleright B \text{ be } x \triangleright y \text{ in } s : C} \text{ S\_TENL2}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Delta \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B} \text{ S\_TENR} \qquad \frac{\Phi \vdash_C t : X \quad \Gamma, x : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : X \multimap Y, \Phi, \Delta \vdash_{\mathcal{L}} [\text{app } y\, t/x] s : A} \text{ S\_IMPL}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta, x : B \vdash_{\mathcal{L}} s_2 : C}{\Delta, y : A \rightharpoonup B, \Gamma \vdash_{\mathcal{L}} [\text{app}_r\, y\, s_1/x] s_2 : C} \text{ S\_IMPRL} \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad x : B, \Delta \vdash_{\mathcal{L}} s_2 : C}{\Gamma, y : B \leftharpoonup A, \Delta \vdash_{\mathcal{L}} [\text{app}_l\, y\, s_1/x] s_2 : C} \text{ S\_IMPLL}$$

$$\frac{\Gamma, x : A \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A.s : A \rightharpoonup B} \text{ S\_IMPRR} \qquad \frac{x : A, \Gamma \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A.s : B \leftharpoonup A} \text{ S\_IMPLR} \qquad \frac{\Phi \vdash_C t : X}{\Phi \vdash_{\mathcal{L}} Ft : FX} \text{ S\_FR}$$

$$\frac{\Gamma, x : X, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : FX, \Delta \vdash_{\mathcal{L}} \text{let} y : FX \text{ be } F x \text{ in } s : A} \text{ S\_FL} \qquad \frac{\Gamma, x : A, \Delta \vdash_{\mathcal{L}} s : B}{\Gamma, y : GA, \Delta \vdash_{\mathcal{L}} \text{let} y : GA \text{ be } G x \text{ in } s : B} \text{ S\_GL}$$

$$\frac{\Phi \vdash_C t : X \quad \Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_1 \vdash_{\mathcal{L}} [t/x] s : A} \text{ S\_CUT1} \qquad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta_1, x : A, \Delta_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [s_1/x] s_2 : B} \text{ S\_CUT2}$$

■ **Figure 2** Sequent Calculus: Non-Commutative Part

$$\Psi \vdash_C t : X$$

■ (T_UNITR, S_UNITL1):

$$\frac{\dfrac{}{\cdot \vdash_C \text{trivT} : \text{UnitT}} \text{UNITR} \quad \dfrac{\Delta \vdash_{\mathcal{L}} s : A}{x : \text{UnitT}, \Delta \vdash_{\mathcal{L}} \text{let} x : \text{UnitT be trivT in } s : A} \text{UNITL}}{\Delta \vdash_{\mathcal{L}} [\text{trivT}/x](\text{let} x : \text{UnitT be trivT in } s) : A} \text{CUT}$$

is transformed to

$$\Delta \vdash_{\mathcal{L}} s : A$$

■ (T_TENR, T_TENL):

$$\frac{\dfrac{\Phi_1 \vdash_C t_1 : X \quad \Phi_2 \vdash_C t_2 : Y}{\Phi_1, \Phi_2 \vdash_C t_1 \otimes t_2 : X \otimes Y} \text{TENR} \quad \dfrac{\Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, z : X \otimes Y, \Psi_2 \vdash_C \text{let} z : X \otimes Y \text{ be } x \otimes y \text{ in } t_3 : Z} \text{TENL}}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_C [t_1 \otimes t_2/z](\text{let} z : X \otimes Y \text{ be } x \otimes y \text{ in } t_3) : Z} \text{CUT}$$

is transformed to

$$\frac{\dfrac{\Phi_1 \vdash_C t_1 : X \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, \Phi_1, y : Y, \Psi_2 \vdash_C [t_1/x] t_3 : Z} \text{CUT} \quad \Phi_2 \vdash_C t_2 : Y}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_C [t_2/y][t_1/x] t_3 : Z} \text{CUT}$$

■ (T_TENR, S_TENL1):

$$\frac{\dfrac{\Phi_1 \vdash_C t_1 : X \quad \Phi_2 \vdash_C t_2 : Y}{\Phi_1, \Phi_2 \vdash_C t_1 \otimes t_2 : X \otimes Y} \text{TENR} \quad \dfrac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \otimes Y, \Delta \vdash_{\mathcal{L}} \text{let} z : X \otimes Y \text{ be } x \otimes y \text{ in } s : A} \text{TENL1}}{\Gamma, \Phi_1, \Phi_2, \Delta \vdash_{\mathcal{L}} [t_1 \otimes t_2/z](\text{let} z : X \otimes Y \text{ be } x \otimes y \text{ in } s) : A} \text{CUT1}$$

is transformed to

$$\frac{\dfrac{\Phi_1 \vdash_C t_1 : X \quad \Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, \Phi_1, y : Y, \Delta \vdash_{\mathcal{L}} [t_1/x] s : A} \text{CUT1} \quad \Phi_2 \vdash_C t_2 : Y}{\Gamma, \Phi_1, \Phi_2, \Delta \vdash_{\mathcal{L}} [t_2/y][t_1/x] s : A} \text{CUT1}$$

■ (T_IMPR, T_IMPL)

$$\frac{\dfrac{\Phi_1, x : X \vdash_C t_1 : Y}{\Phi_1 \vdash_C \lambda x : X.t_1 : X \multimap Y} \text{IMPR} \quad \dfrac{\Phi_2 \vdash_C t_2 : X \quad \Psi_1, y : Y, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, z : X \multimap Y, \Phi_2, \Psi_2 \vdash_C [\text{app } z\, t_2/y] t_3 : Z} \text{IMPL}}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_C [(\lambda x : X.t_1)/z][\text{app } z\, t_2/y] t_3 : Z} \text{CUT}$$

is transformed to

$$\dfrac{\dfrac{\Phi_1, x : X \vdash_C t_1 : Y \qquad \Phi_2 \vdash_C t_2 : X}{\Phi_1, \Phi_2 \vdash_C [t_2/x]t_1 : Y} \ \text{CUT} \qquad \Psi_1, y : Y, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_C [([t_2/x]t_1)/y]t_3 : Z} \ \text{CUT}$$

- (S_UNITR, S_UNITL2):

$$\dfrac{\dfrac{}{\cdot \vdash_{\mathcal{L}} \text{trivS} : \text{UnitS}} \ \text{UNITR} \qquad \dfrac{\Delta \vdash_{\mathcal{L}} s : A}{x : \text{UnitS}, \Delta \vdash_{\mathcal{L}} \text{let } x : \text{UnitS be trivS in } s : A} \ \text{UNITL2}}{\Delta \vdash_{\mathcal{L}} [\text{trivS}/x](\text{let } x : \text{UnitS be trivS in } s) : A} \ \text{CUT2}$$

is transformed to

$$\Delta \vdash_{\mathcal{L}} s : A$$

- (S_TENR, S_TENL2):

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : B}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B} \ \text{TENR} \qquad \dfrac{\Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, z : A \triangleright B, \Delta_2 \vdash_{\mathcal{L}} \text{let } z : A \triangleright B \text{ be } x \triangleright y \text{ in } s : C} \ \text{TENL1}}{\Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} [s_1 \triangleright s_2/z](\text{let } z : A \triangleright B \text{ be } x \triangleright y \text{ in } s) : C} \ \text{CUT2}$$

is transformed to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, y : B, \Delta_2 \vdash_{\mathcal{L}} [s_1/x]s_3 : C} \ \text{CUT2} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} [s_2/y][s_1/x]s_3 : C} \ \text{CUT2}$$

- (S_IMPRR, S_IMPRL):

$$\dfrac{\dfrac{\Gamma, x : A \vdash_{\mathcal{L}} s_1 : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A.s_1 : A \to B} \ \text{IMPRR} \qquad \dfrac{\Delta_1 \vdash_{\mathcal{L}} s_2 : A \qquad \Delta_2, y : B \vdash_{\mathcal{L}} s_3 : C}{\Delta_2, z : A \to B, \Delta_1 \vdash_{\mathcal{L}} [\text{app}_r z s_2/y]s_3 : C} \ \text{IMPRL}}{\Delta_2, \Gamma, \Delta_1 \vdash_{\mathcal{L}} [(\lambda_r x : A.s_1)/z][\text{app}_r z s_2/y]s_3 : C} \ \text{CUT2}$$

is transformed to

$$\dfrac{\dfrac{\Gamma, x : A \vdash_{\mathcal{L}} s_1 : B \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta_1 \vdash_{\mathcal{L}} [s_2/x]s_1 : B} \ \text{CUT2} \qquad \Delta_2, y : B \vdash_{\mathcal{L}} s_3 : C}{\Delta_2, \Gamma, \Delta_1 \vdash_{\mathcal{L}} [([s_2/x]s_1)/y]s_3 : C} \ \text{CUT2}$$

- (S_IMPLR, S_IMPLL):

$$\dfrac{\dfrac{x : A, \Gamma \vdash_{\mathcal{L}} s_1 : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A.s_1 : B \leftarrow A} \ \text{IMPLR} \qquad \dfrac{\Delta_1 \vdash_{\mathcal{L}} s_2 : A \qquad y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, z : B \leftarrow A, \Delta_2 \vdash_{\mathcal{L}} [\text{app}_l z s_2/y]s_3 : C} \ \text{IMPLL}}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [(\lambda_l x : A.s_1)/z][\text{app}_l z s_2/y]s_3 : C} \ \text{CUT2}$$

is transformed to

$$\dfrac{\dfrac{x : A, \Gamma \vdash_{\mathcal{L}} s_1 : B \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A}{\Delta_1, \Gamma \vdash_{\mathcal{L}} [s_2/x]s_1 : B} \ \text{CUT2} \qquad y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [([s_2/x]s_1)/y]s_3 : C} \ \text{CUT2}$$

- (S_FR, S_FL):

$$\dfrac{\dfrac{\Phi \vdash_C t : X}{\Phi \vdash_{\mathcal{L}} Ft : FX} \ \text{FR} \qquad \dfrac{\Gamma, x : X, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : FX, \Delta \vdash_{\mathcal{L}} \text{let } y : FX \text{ be } Fx \text{ in } s : A} \ \text{FL}}{\Gamma, \Phi, \Delta \vdash_{\mathcal{L}} [Ft/y](\text{let } y : FX \text{ be } Fx \text{ in } s) : A} \ \text{CUT2}$$

is transformed to

$$\dfrac{\Phi \vdash_C t : X \qquad \Gamma, x : A, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, \Phi, \Delta \vdash_{\mathcal{L}} [t/x]s : A} \ \text{CUT1}$$

- (T_GR, S_GL):

$$\dfrac{\dfrac{\Phi \vdash_{\mathcal{L}} s_1 : A}{\Phi \vdash_C Gs_1 : GA} \ \text{GR} \qquad \dfrac{\Gamma, x : A, \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, y : GA, \Delta \vdash_{\mathcal{L}} \text{let } y : GA \text{ be } Gx \text{ in } s_2 : B} \ \text{GL}}{\Gamma, \Phi, \Delta \vdash_{\mathcal{L}} [Gs_1/y](\text{let } y : GA \text{ be } Gx \text{ in } s_2) : B} \ \text{CUT1}$$

is transformed to

$$\dfrac{\Phi \vdash_{\mathcal{L}} s_1 : A \qquad \Gamma, x : A, \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Phi, \Delta \vdash_{\mathcal{L}} [s_1/x]s_2 : B} \ \text{CUT}$$

Based on the key cases, given a formula (either commutative or non-commutative) L and proofs Π, Π', of sequents $M \vdash N$ and $M' \vdash N'$ respectively with degrees less than $|L|$, there is a proof of $M, N \vdash M', N'$ with degree less than $|L|$, s.t. all currences of formula L is removed. This can be proved by induction on $h(\Pi) + h(\Pi')$. Therefore, we have the result that given a proof of a sequent with degree $d > 0$, there is a proof of the same sequent. As a result, we have the cut elimination theorem.

▶ **Theorem 28** (Cut Elimination). *Let Π be a proof of a sequent $\Phi \vdash_C t : X$ or a sequent $\Gamma \vdash_{\mathcal{L}} s : A$ s.t. $|\Pi| > 0$. Then there is a cut-free proof of the same sequent.*

5.2 Natural Deduction

The term assignment for natural deduction of the commutative part of the model, i.e. the SMCC of the adjunction, is defined in Figure 3. And the term assignme for the non-commutative part, i.e. the Lambek category of the adjunction, is defined in Figure 4.

Figure 3 Natural Deduction: Commutative Part

Figure 4 Natural Deduction: Non-Commutative Part

We could derive exchange comonadically as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{y_0 : GB \vdash_C y_0 : GB}^{\text{ID}}}{y_0 : GB \vdash_{\mathcal{L}} Fy_0 : FGB}^{\text{FI}} \quad \dfrac{\overline{x_0 : GA \vdash_C x_0 : GA}^{\text{ID}}}{x_0 : GA \vdash_{\mathcal{L}} Fx_0 : FGA}^{\text{FI}}}{y_0 : GB, x_0 : GA \vdash_{\mathcal{L}} Fy_0 \triangleright Fx_0 : FGB \triangleright FGA}^{\text{FE}}}{\vdots}}{\vdots}$$

We also have the three cut rules derivable in the natural deduction: (NOTE: Don't know how to prove the third one S_cut2.)

$$\dfrac{\Phi \vdash_C t_1 : X \quad \Psi_1, x : X, \Psi_2 \vdash_C t_2 : Y}{\Psi_1, \Phi, \Psi_2 \vdash_C [t_1/x]t_2 : Y}~\text{T\_CUT} \qquad \dfrac{\Phi \vdash_C t : X \quad \Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_1 \vdash_{\mathcal{L}} [t/x]s : A}~\text{S\_CUT1} \qquad \dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta_1, x : A, \Delta_2 \vdash_{\mathcal{L}} s_2 : B}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [s_1/x]s_2 : B}~\text{S\_CUT2}$$

5.2.1 One Step β-Reduction

We define the normalization procedure by considering the following pairs of introduction and elimination rules:

- (T_UNITI, T_UNITE):

$$\dfrac{\overline{\cdot \vdash_C \text{trivT} : \text{UnitT}}^{\text{UNITI}} \quad \Phi \vdash_C t : X}{\Phi \vdash_C \text{let trivT} : \text{UnitT be trivT in } t : X}~\text{UNITE}$$

normalizes to

$$\Phi \vdash_C t : X$$

- (T_UNITI, S_UNITE1):

$$\dfrac{\overline{\cdot \vdash_C \text{trivT} : \text{UnitT}}^{\text{UNITI}} \quad \Delta \vdash_{\mathcal{L}} s : A}{\Delta \vdash_{\mathcal{L}} \text{let trivT} : \text{UnitT be trivT in } s : A}~\text{UNITE2}$$

normalizes to

$$\Delta \vdash_{\mathcal{L}} s : A$$

- (T_TENI, T_TENE):

$$\dfrac{\dfrac{\Phi_1 \vdash_C t_1 : X \quad \Phi_2 \vdash_C t_2 : Y}{\Phi_1, \Phi_2 \vdash_C t_1 \otimes t_2 : X \otimes Y}^{\text{TENI}} \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_C \text{let } t_1 \otimes t_2 : X \otimes Y \text{ be } x \otimes y \text{ in } t_3 : Z}~\text{TENE}$$

normalizes to

$$\dfrac{\Phi_1 \vdash_C t_1 : X \quad \dfrac{\Phi_2 \vdash_C t_2 : Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, x : X, \Phi_2, \Psi_2 \vdash_C [t_2/y]t_3 : Z}^{\text{CUT}}}{\Psi_1, \Phi_1, \Phi_2, \Psi_2 \vdash_C [t_1/x][t_2/y]t_3 : Z}~\text{CUT}$$

- (T_TENI, S_TENE1):

$$\dfrac{\dfrac{\Phi \vdash_C t_1 : X \quad \Psi \vdash_C t_2 : Y}{\Phi, \Psi \vdash_C t_1 \otimes t_2 : X \otimes Y}^{\text{TENI}} \quad \Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, \Phi, \Psi, \Delta \vdash_{\mathcal{L}} \text{let } t_1 \otimes t_2 : X \otimes Y \text{ be } x \otimes y \text{ in } s : A}~\text{TENE1}$$

normalizes to

$$\dfrac{\Phi \vdash_C t_1 : X \quad \dfrac{\Psi \vdash_C t_2 : Y \quad \Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, x : X, \Psi, \Delta \vdash_{\mathcal{L}} [t_2/y]s : A}^{\text{CUT2}}}{\Gamma, \Phi, \Psi, \Delta \vdash_{\mathcal{L}} [t_1/x][t_2/y]s : A}~\text{CUT2}$$

- (T_IMPI, T_IMPE):

$$\dfrac{\dfrac{\Phi, x : X \vdash_C t_1 : Y}{\Phi \vdash_C \lambda x : X.t_1 : X \multimap Y}\ \text{\tiny IMPI} \qquad \Psi \vdash_C t_2 : X}{\Phi, \Psi \vdash_C \mathsf{app}\,(\lambda x : X.t_1)\,t_2 : Y}\ \text{\tiny IMPE}$$

normalizes to

$$\dfrac{\Phi, x : X \vdash_C t_1 : Y \qquad \Psi \vdash_C t_2 : X}{\Phi, \Psi \vdash_C [t_2/x]t_1 : Y}\ \text{\tiny CUT}$$

- (S_UNITI, S_UNITE2):

$$\dfrac{\dfrac{}{\cdot \vdash_{\mathcal{L}} \mathsf{trivS} : \mathsf{UnitS}}\ \text{\tiny UNITI} \qquad \Delta \vdash_{\mathcal{L}} s : A}{\Delta \vdash_{\mathcal{L}} \mathsf{let\,trivS} : \mathsf{UnitS\,be\,trivS\,in}\ s : A}\ \text{\tiny UNITE2}$$

normalizes to

$$\Delta \vdash_{\mathcal{L}} s : A$$

- (S_TENI, S_TENE2):

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : B}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B}\ \text{\tiny TENI} \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\ s_1 \triangleright s_2 : A \triangleright B\ \mathsf{be}\ x \triangleright y\ \mathsf{in}\ s_3 : C}\ \text{\tiny TENE2}$$

normalizes to

$$\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \qquad \dfrac{\Gamma_2 \vdash_{\mathcal{L}} s_2 : B \qquad \Delta_1, x : X, y : Y, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, x : X, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} [s_2/y]s_3 : C}\ \text{\tiny CUT2}}{\Delta_1, \Gamma_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} [s_1/x][s_2/y]s_3 : C}\ \text{\tiny CUT2}$$

- (S_IMPRI, S_IMPRE):

$$\dfrac{\dfrac{\Gamma, x : A \vdash_{\mathcal{L}} s_1 : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A.s_1 : A \rightharpoonup B}\ \text{\tiny IMPRI} \qquad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{app}_r\,(\lambda_r x : A.s_1)\,s_2 : B}\ \text{\tiny UNITE2}$$

normalizes to

$$\dfrac{\Gamma, x : A \vdash_{\mathcal{L}} s_1 : B \qquad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} [s_2/x]s_1 : B}\ \text{\tiny CUT2}$$

- (S_IMPLI, S_IMPLE):

$$\dfrac{\dfrac{x : A, \Gamma \vdash_{\mathcal{L}} s_1 : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A.s_1 : B \leftharpoonup A}\ \text{\tiny IMPLI} \qquad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta, \Gamma \vdash_{\mathcal{L}} \mathsf{app}_l\,(\lambda_l x : A.s_1)\,s_2 : B}\ \text{\tiny UNITE2}$$

normalizes to

$$\dfrac{x : A, \Gamma \vdash_{\mathcal{L}} s_1 : B \qquad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta, \Gamma \vdash_{\mathcal{L}} [s_2/x]s_1 : B}\ \text{\tiny CUT2}$$

- (S_FI, S_FE):

$$\dfrac{\dfrac{\Phi \vdash_C y : X}{\Phi \vdash_{\mathcal{L}} Fy : FX}\ \text{\tiny FI} \qquad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Phi, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\ Fx : FX\ \mathsf{be}\ Fy\ \mathsf{in}\ s : A}\ \text{\tiny FE}$$

normalizes to

$$\dfrac{\Phi \vdash_C y : X \qquad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Phi, \Delta_2 \vdash_{\mathcal{L}} [y/x]s : A}\ \text{\tiny CUT1}$$

- (T_GI, S_GE):

$$\dfrac{\dfrac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_C Gs : GA}\ \text{\tiny GI}}{\Phi \vdash_{\mathcal{L}} \mathsf{derelict}\ Gs : A}\ \text{\tiny GE}$$

normalizes to

$$\Phi \vdash_{\mathcal{L}} s : A$$

▶ **Theorem 29** (Normalization). *For a cut-free deduction Π, there is a deduction which is in normal form.*

Proof. By induction on the structure of Π.   ◀

5.2.2 Commuting Conversions

- Commutation of UnitT$_E$:

  - (T_unitE, T_unitE):

$$\cfrac{\cfrac{\Phi_1 \vdash_C t_1 : \text{UnitT} \qquad \Phi_2 \vdash_C t_2 : \text{UnitT}}{\Phi_2, \Phi_1 \vdash_C \text{let}\, t_2 : \text{UnitT be trivT in}\, t_1 : \text{UnitT}}\;_{\text{unitE}} \qquad \Phi_3 \vdash_C t_3 : X}{\Phi_2, \Phi_1, \Phi_3 \vdash_C \text{let}\,(\text{let}\, t_2 : \text{UnitT be trivT in}\, t_1) : \text{UnitT be trivT in}\, t_3 : X}\;_{\text{unitE}}$$

    commutes to

$$\cfrac{\cfrac{\Phi_1 \vdash_C t_1 : \text{UnitT} \qquad \Phi_3 \vdash_C t_3 : X}{\Phi_1, \Phi_3 \vdash_C \text{let}\, t_1 : \text{UnitT be trivT in}\, t_3 : X}\;_{\text{unitE}} \qquad \Phi_2 \vdash_C t_2 : \text{UnitT}}{\Phi_2, \Phi_1, \Phi_3 \vdash_C \text{let}\, t_2 : \text{UnitT be trivT in}\,(\text{let}\, t_1 : \text{UnitT be trivT in}\, t_3) : X}\;_{\text{unitE}}$$

  - (T_unitE, T_tenE) need multiple exchanges at the end:

$$\cfrac{\cfrac{\Phi_1 \vdash_C t_1 : X \otimes Y \qquad \Phi_2 \vdash_C t_2 : \text{UnitT}}{\Phi_2, \Phi_1 \vdash_C \text{let}\, t_2 : \text{UnitT be trivT in}\, t_1 : X \otimes Y}\;_{\text{unitE}} \qquad \Psi_1, x:X, y:Y, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, \Phi_2, \Phi_1, \Psi_2 \vdash_C \text{let}\,(\text{let}\, t_2 : \text{UnitT be trivT in}\, t_1) : X \otimes Y \,\text{be}\, x \otimes y \,\text{in}\, t_3 : Z}\;_{\text{tenE}}$$

    commutes to

$$\cfrac{\cfrac{\Phi_1 \vdash_C t_1 : X \otimes Y \qquad \Psi_1, x:X, y:Y, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, \Phi_1, \Psi_2 \vdash_C \text{let}\, t_1 : X \otimes Y \,\text{be}\, x \otimes y \,\text{in}\, t_3 : Z}\;_{\text{tenE}} \qquad \Phi_2 \vdash_C t_2 : \text{UnitT}}{\Phi_2, \Psi_1, \Phi_1, \Psi_2 \vdash_C \text{let}\, t_2 : \text{UnitT be trivT in}\,(\text{let}\, t_1 : X \otimes Y \,\text{be}\, x \otimes y \,\text{in}\, t_3) : Z}\;_{\text{unitE}}$$

  - (T_unitE, T_impE):

$$\cfrac{\cfrac{\Phi_1 \vdash_C t_1 : X \multimap Y \qquad \Phi_2 \vdash_C t_2 : \text{UnitT}}{\Phi_2, \Phi_1 \vdash_C \text{let}\, t_2 : \text{UnitT be trivT in}\, t_1 : X \multimap Y}\;_{\text{tenE}} \qquad \Phi_3 \vdash_C t_3 : X}{\Phi_2, \Phi_1, \Phi_3 \vdash_C \text{app}\,(\text{let}\, t_2 : \text{UnitT be trivT in}\, t_1)\, t_3 : Y}\;_{\text{unitE}}$$

    commutes to

$$\cfrac{\cfrac{\Phi_1 \vdash_C t_1 : X \multimap Y \qquad \Phi_3 \vdash_C t_3 : X}{\Phi_1, \Phi_3 \vdash_C \text{app}\, t_1\, t_3 : Y}\;_{\text{unitE}} \qquad \Phi_2 \vdash_C t_2 : \text{UnitT}}{\Phi_2, \Phi_1, \Phi_3 \vdash_C \text{let}\, t_2 : \text{UnitT be trivT in}\,(\text{app}\, t_1\, t_3) : Y}\;_{\text{tenE}}$$

- Commutation of $\otimes_E$:

  - (T_tenE, T_unitE):

$$\cfrac{\cfrac{\Phi_1, x:X, y:Y, \Phi_2 \vdash_C t_1 : \text{UnitT} \qquad \Psi_1 \vdash_C t_2 : X \otimes Y}{\Phi_1, \Psi_1, \Phi_2 \vdash_C \text{let}\, t_2 : X \otimes Y \,\text{be}\, x \otimes y \,\text{in}\, t_1 : \text{UnitT}}\;_{\text{tenE}} \qquad \Psi_2 \vdash_C t_3 : Z}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_C \text{let}\,(\text{let}\, t_2 : X \otimes Y \,\text{be}\, x \otimes y \,\text{in}\, t_1) : \text{UnitT be trivT in}\, t_3 : Z}\;_{\text{unitE}}$$

    commutes to

$$\cfrac{\cfrac{\Phi_1, x:X, y:Y, \Phi_2 \vdash_C t_1 : \text{UnitT} \qquad \Psi_2 \vdash_C t_3 : Z}{\Phi_1, x:X, y:Y, \Phi_2, \Psi_2 \vdash_C \text{let}\, t_1 : \text{UnitT be trivT in}\, t_3 : Z}\;_{\text{unitE}} \qquad \Psi_1 \vdash_C t_2 : X \otimes Y}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_C \text{let}\, t_2 : X \otimes Y \,\text{be}\, x \otimes y \,\text{in}\,(\text{let}\, t_1 : \text{UnitT be trivT in}\, t_3) : Z}\;_{\text{tenE}}$$

  - (T_tenE, T_tenE):

$$\cfrac{\cfrac{\Phi_1, x:X_2, y:Y_2, \Phi_2 \vdash_C t_1 : X_1 \otimes Y_1 \qquad \Psi \vdash_C t_2 : X_2 \otimes Y_2}{\Phi_1, \Psi, \Phi_2 \vdash_C \text{let}\, t_2 : X_2 \otimes Y_2 \,\text{be}\, x \otimes y \,\text{in}\, t_1 : X_1 \otimes Y_1}\;_{\text{tenE}} \qquad \Psi_1, w:X_1, z:Y_1, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, \Phi_1, \Psi, \Phi_2, \Psi_2 \vdash_C \text{let}\,(\text{let}\, t_2 : X_2 \otimes Y_2 \,\text{be}\, x \otimes y \,\text{in}\, t_1) : X_1 \otimes Y_1 \,\text{be}\, w \otimes z \,\text{in}\, t_3 : Z}\;_{\text{tenE}}$$

    commutes to

$$\cfrac{\cfrac{\Phi_1, x:X_2, y:Y_2, \Phi_2 \vdash_C t_1 : X_1 \otimes Y_1 \qquad \Psi_1, w:X_1, z:Y_1, \Psi_2 \vdash_C t_3 : Z}{\Psi_1, \Phi_1, x:X_2, y:Y_2, \Phi_2, \Psi_2 \vdash_C \text{let}\, t_1 : X_1 \otimes Y_1 \,\text{be}\, w \otimes z \,\text{in}\, t_3 : Z}\;_{\text{tenE}} \qquad \Psi \vdash_C t_2 : X_2 \otimes Y_2}{\Psi_1, \Phi_1, \Psi, \Phi_2, \Psi_2 \vdash_C \text{let}\, t_2 : X_2 \otimes Y_2 \,\text{be}\, x \otimes y \,\text{in}\,(\text{let}\, t_1 : X_1 \otimes Y_1 \,\text{be}\, w \otimes z \,\text{in}\, t_3) : Z}\;_{\text{tenE}}$$

  - (T_tenE, T_impE):

$$\cfrac{\cfrac{\Phi_1, x:X_2, y:Y_2, \Phi_2 \vdash_C t_1 : X_1 \multimap Y_1 \qquad \Psi_1 \vdash_C t_2 : X_2 \otimes Y_2}{\Phi_1, \Psi_1, \Phi_2 \vdash_C \text{let}\, t_2 : X_2 \otimes Y_2 \,\text{be}\, x \otimes y \,\text{in}\, t_1 : X_1 \multimap Y_1}\;_{\text{impE}} \qquad \Psi_2 \vdash_C t_3 : X_1}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_C \text{app}\,(\text{let}\, t_2 : X_2 \otimes Y_2 \,\text{be}\, x \otimes y \,\text{in}\, t_1)\, t_3 : Y_1}\;_{\text{tenE}}$$

    commutes to

$$\cfrac{\cfrac{\Phi_1, x:X_2, y:Y_2, \Phi_2 \vdash_C t_1 : X_1 \multimap Y_1 \qquad \Psi_2 \vdash_C t_3 : X_1}{\Phi_1, x:X_2, y:Y_2, \Phi_2, \Psi_2 \vdash_C \text{app}\, t_1\, t_3 : Y_1}\;_{\text{tenE}} \qquad \Psi_1 \vdash_C t_2 : X_2 \otimes Y_2}{\Phi_1, \Psi_1, \Phi_2, \Psi_2 \vdash_C \text{let}\, t_2 : X_2 \otimes Y_2 \,\text{be}\, x \otimes y \,\text{in}\,(\text{app}\, t_1\, t_3) : Y_1}\;_{\text{impE}}$$

- Commutation of $\multimap_E$:

  - (T_impE, T_unitE):

$$\cfrac{\cfrac{\Phi_1 \vdash_C t_1 : \text{UnitT} \qquad \Phi_2 \vdash_C t_2 : \text{UnitT} \multimap \text{UnitT}}{\Phi_2, \Phi_1 \vdash_C \text{app}\, t_2\, t_1 : \text{UnitT}}\;_{\text{impE}} \qquad \Phi_3 \vdash_C t_3 : \text{UnitT}}{\Phi_2, \Phi_1, \Phi_3 \vdash_C \text{let}\,(\text{app}\, t_2\, t_1) : \text{UnitT be trivT in}\, t_3 : \text{UnitT}}\;_{\text{tenE}}$$

commutes to

$$\dfrac{\dfrac{\Phi_1 \vdash_C t_1 : \mathsf{UnitT} \qquad \Phi_3 \vdash_C t_3 : \mathsf{UnitT}}{\Phi_1, \Phi_3 \vdash_C \mathsf{let}\, t_1 : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, t_3 : \mathsf{UnitT}}\ \text{\tiny TENE} \qquad \Phi_2 \vdash_C t_2 : \mathsf{UnitT} \multimap \mathsf{UnitT}}{\Phi_2, \Phi_1, \Phi_3 \vdash_C \mathsf{app}\, t_2\, (\mathsf{let}\, t_1 : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, t_3) : \mathsf{UnitT}}\ \text{\tiny IMPE}$$

- (T_ImpE, T_tenE): ?
- (T_ImpE, T_ImpE): ?

- Commutation of $\rhd_E$:

  - (S_unitE2, S_unitE2):

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{UnitS}}{\Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_1 : \mathsf{UnitS}}\ \text{\tiny UNITE} \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_1) : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_3 : A}\ \text{\tiny UNITE}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_3 : A}\ \text{\tiny UNITE} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{UnitS}}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, (\mathsf{let}\, s_1 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_3) : A}\ \text{\tiny UNITE}$$

  - (S_unitE2, S_tenE2): Does NOT commute

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \rhd B \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{UnitS}}{\Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_1 : A \rhd B}\ \text{\tiny UNITE} \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_2, \Gamma_1, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_1) : A \rhd B\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, s_3 : C}\ \text{\tiny TENE2}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \rhd B \qquad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \rhd B\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, s_3 : C}\ \text{\tiny TENE2} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{UnitS}}{\Gamma_2, \Delta_1, \Gamma_1, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, (\mathsf{let}\, s_1 : A \rhd B\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, s_3) : C}\ \text{\tiny UNITE}$$

  - (S_unitE2, S_impRE):

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{UnitS}}{\Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_1 : A \rightharpoonup B}\ \text{\tiny UNITE} \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{app}_r\, (\mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_1)\, s_3 : B}\ \text{\tiny IMPRE}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_3 : B}\ \text{\tiny IMPRE} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{UnitS}}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, (\mathsf{app}_r\, s_1\, s_3) : B}\ \text{\tiny UNITE}$$

  - (S_unitE2, S_impLE): Does NOT commute.

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : B \leftharpoonup A \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{UnitS}}{\Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_1 : B \leftharpoonup A}\ \text{\tiny UNITE} \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_3, \Gamma_2, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{app}_l\, (\mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_1)\, s_3 : B}\ \text{\tiny IMPRE}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1 \vdash_{\mathcal{L}} s_1 : B \leftharpoonup A \qquad \Gamma_3 \vdash_{\mathcal{L}} s_3 : A}{\Gamma_3, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_3 : B}\ \text{\tiny IMPRE} \qquad \Gamma_2 \vdash_{\mathcal{L}} s_2 : \mathsf{UnitS}}{\Gamma_2, \Gamma_3, \Gamma_1 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, (\mathsf{app}_l\, s_1\, s_3) : B}\ \text{\tiny UNITE}$$

  - (S_tenE2, S_unitE2):

$$\dfrac{\dfrac{\Gamma_1, x : A, y : B, \Gamma_2 \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A \rhd B}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A \rhd B\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, s_1 : \mathsf{UnitS}}\ \text{\tiny TENE2} \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, s_2 : A \rhd B\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, s_1) : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_3 : C}\ \text{\tiny UNITE2}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1, x : A, y : B, \Gamma_2 \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Gamma_1, x : A, y : B, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_3 : C}\ \text{\tiny UNITE2} \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A \rhd B}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A \rhd B\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, (\mathsf{let}\, s_1 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_3) : C}\ \text{\tiny TENE2}$$

  - (S_tenE2, S_tenE2):

$$\dfrac{\dfrac{\Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A_1 \rhd B_1 \qquad \Gamma \vdash_{\mathcal{L}} s_2 : A_2 \rhd B_2}{\Gamma_1, \Gamma, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \rhd B_2\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, s_1 : A_1 \rhd B_1}\ \text{\tiny TENE2} \qquad \Delta_1, w : A_1, z : B_1, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, \Gamma, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, s_2 : A_2 \rhd B_2\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, s_1) : A_1 \rhd B_1\, \mathsf{be}\, w \rhd z\, \mathsf{in}\, s_3 : C}\ \text{\tiny TENE2}$$

commutes to

$$\dfrac{\dfrac{\Gamma_1, x : A_2, y : B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A_1 \rhd B_1 \qquad \Delta_1, w : A_1, z : B_1, \Delta_2 \vdash_{\mathcal{L}} s_3 : C}{\Delta_1, \Gamma_1, x : A_2, y : B_2, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A_1 \rhd B_1\, \mathsf{be}\, w \rhd z\, \mathsf{in}\, s_3 : C}\ \text{\tiny TENE2} \qquad \Gamma \vdash_{\mathcal{L}} s_2 : A_2 \rhd B_2}{\Delta_1, \Gamma_1, \Gamma, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \rhd B_2\, \mathsf{be}\, x \rhd y\, \mathsf{in}\, (\mathsf{let}\, s_1 : A_1 \rhd B_1\, \mathsf{be}\, w \rhd z\, \mathsf{in}\, s_3) : C}\ \text{\tiny TENE2}$$

  - (S_tenE2, S_impRE):

$$\cfrac{\cfrac{\Gamma_1, x:A_2, y:B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A_1 \to B_1 \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2 \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, s_1 : A_1 \to B_1} \ \text{TenE2} \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : A_1}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{app}_r (\mathsf{let}\, s_2 : A_2 \triangleright B_2 \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, s_1)\, s_3 : B_1} \ \text{ImprE}$$

commutes to

$$\cfrac{\cfrac{\Gamma_1, x:A_2, y:B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A_1 \to B_1 \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : A_1}{\Gamma_1, x:A_2, y:B_2, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_3 : B_1} \ \text{ImprE} \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2 \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, (\mathsf{app}_r\, s_1\, s_3) : B_1} \ \text{TenE2}$$

- (S_TenE2, S_ImplE):

$$\cfrac{\cfrac{\Gamma_1, x:A_2, y:B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : B_1 \leftarrow A_1 \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2 \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, s_1 : B_1 \leftarrow A_1} \ \text{TenE2} \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : A_1}{\Delta_2, \Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{app}_l (\mathsf{let}\, s_2 : A_2 \triangleright B_2 \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, s_1)\, s_3 : B_1} \ \text{ImplE}$$

commutes to

$$\cfrac{\cfrac{\Gamma_1, x:A_2, y:B_2, \Gamma_2 \vdash_{\mathcal{L}} s_1 : B_1 \leftarrow A_1 \qquad \Delta_2 \vdash_{\mathcal{L}} s_3 : A_1}{\Delta_2, \Gamma_1, x:A_2, y:B_2, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_3 : B_1} \ \text{ImplE} \qquad \Delta_1 \vdash_{\mathcal{L}} s_2 : A_2 \triangleright B_2}{\Delta_2, \Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_2 : A_2 \triangleright B_2 \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, (\mathsf{app}_l\, s_1\, s_3) : B_1} \ \text{TenE2}$$

- Commutation of F_E:

  - (S_FE, S_UnitE2):

$$\cfrac{\cfrac{\Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \qquad \Delta_1 \vdash_{\mathcal{L}} y : FX}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s_1 : \mathsf{UnitS}} \ \text{FE} \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s_1) : \mathsf{UnitS} \,\mathsf{be}\, \mathsf{trivS} \,\mathsf{in}\, s_2 : A} \ \text{UnitE2}$$

commutes to

$$\cfrac{\cfrac{\Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Gamma_1, x:X, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{UnitS} \,\mathsf{be}\, \mathsf{trivS} \,\mathsf{in}\, s_2 : A} \ \text{UnitE2} \qquad \Delta_1 \vdash_{\mathcal{L}} y : FX}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, (\mathsf{let}\, s_1 : \mathsf{UnitS} \,\mathsf{be}\, \mathsf{trivS} \,\mathsf{in}\, s_2) : A} \ \text{FE}$$

  - (S_FE, S_TenE2):

$$\cfrac{\cfrac{\Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \Delta \vdash_{\mathcal{L}} y : FX}{\Gamma_1, \Delta, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s_1 : A \triangleright B} \ \text{FE} \qquad \Delta_1, x:A, y:B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma_1, \Delta, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, (\mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s_1) : A \triangleright B \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, s_2 : C} \ \text{TenE2}$$

commutes to

$$\cfrac{\cfrac{\Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \triangleright B \qquad \Delta_1, x:A, y:B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma_1, x:X, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, s_2 : C} \ \text{TenE2} \qquad \Delta \vdash_{\mathcal{L}} y : FX}{\Delta_1, \Gamma_1, \Delta, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, (\mathsf{let}\, s_1 : A \triangleright B \,\mathsf{be}\, x \triangleright y \,\mathsf{in}\, s_2) : C} \ \text{FE}$$

  - (S_FE, S_ImprE):

$$\cfrac{\cfrac{\Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \to B \qquad \Delta_1 \vdash_{\mathcal{L}} y : FX}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s_1 : A \to B} \ \text{FE} \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{app}_r (\mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s_1)\, s_2 : B} \ \text{ImprE}$$

commutes to

$$\cfrac{\cfrac{\Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \to B \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Gamma_1, x:X, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_2 : B} \ \text{ImprE} \qquad \Delta_1 \vdash_{\mathcal{L}} y : FX}{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, (\mathsf{app}_r\, s_1\, s_2) : B} \ \text{FE}$$

  - (S_FE, S_ImplE):

$$\cfrac{\cfrac{\Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \leftarrow B \qquad \Delta_1 \vdash_{\mathcal{L}} y : FX}{\Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s_1 : A \leftarrow B} \ \text{FE} \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Delta_2, \Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{app}_l (\mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s_1)\, s_2 : B} \ \text{ImplE}$$

commutes to

$$\cfrac{\cfrac{\Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} s_1 : A \leftarrow B \qquad \Delta_2 \vdash_{\mathcal{L}} s_2 : A}{\Delta_2, \Gamma_1, x:X, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_2 : B} \ \text{ImplE} \qquad \Delta_1 \vdash_{\mathcal{L}} y : FX}{\Delta_2, \Gamma_1, \Delta_1, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, (\mathsf{app}_l\, s_1\, s_2) : B} \ \text{FE}$$

5.3   Mappings Between Sequent Calculus and Natural Deduction

Function $S : ND \to SE$ maps a proof in the natural deduction to a proof of the same sequent in the sequent calculus. The function is defined as follows:

- The axioms map to axioms.
- Introduction rules map to right rules.
- Elimination rules map to combinations of left rules with cuts:

  - T_UNITE:

$$\frac{\Phi \vdash_C t_1 : \mathsf{UnitT} \quad \Psi \vdash_C t_2 : Y}{\Phi, \Psi \vdash_C \mathsf{let}\, t_1 : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, t_2 : Y} \; \text{T\_UNITE}$$

    maps to

$$\frac{\Phi \vdash_C t_1 : \mathsf{UnitT} \quad \dfrac{\Psi \vdash_C t_2 : Y}{x : \mathsf{UnitT}, \Psi \vdash_C \mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, t_2 : Y}\; \text{UNITL}}{\Phi, \Psi \vdash_C [t_1/x](\mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, t_2) : Y} \; \text{CUT}$$

  - T_TENE:

$$\frac{\Phi \vdash_C t_1 : X \otimes Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_2 : Z}{\Psi_1, \Phi, \Psi_2 \vdash_C \mathsf{let}\, t_1 : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2 : Z} \; \text{T\_TENE}$$

    maps to

$$\frac{\Phi \vdash_C t_1 : X \otimes Y \quad \dfrac{\Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_2 : Z}{\Psi_1, z : X \otimes Y, \Psi_2 \vdash_C \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2 : Z}\; \text{UNITL}}{\Psi_1, \Phi, \Psi_2 \vdash_C [t_1/z](\mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, t_2) : Z} \; \text{CUT}$$

  - T_IMPE:

$$\frac{\Phi \vdash_C t_1 : X \multimap Y \quad \Psi \vdash_C t_2 : X}{\Phi, \Psi \vdash_C \mathsf{app}\, t_1\, t_2 : Y} \; \text{T\_IMPE}$$

    maps to

$$\frac{\Phi \vdash_C t_1 : X \multimap Y \quad \dfrac{\Psi \vdash_C t_2 : X \quad x : Y \vdash_C x : Y}{y : X \multimap Y, \Psi \vdash_C [\mathsf{app}\, y\, t_2/x]x : Y}\; \text{UNITL}}{\Phi, \Psi \vdash_C [t_1/y][\mathsf{app}\, y\, t_2/x]x : Y} \; \text{CUT}$$

  - S_UNITE1:

$$\frac{\Phi \vdash_C t : \mathsf{UnitT} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, s : A} \; \text{S\_UNITE1}$$

    maps to

$$\frac{\Phi \vdash_C t : \mathsf{UnitT} \quad \dfrac{\Gamma \vdash_{\mathcal{L}} s : A}{x : \mathsf{UnitT}, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, s : A}\; \text{UNITL1}}{\Phi, \Psi \vdash_{\mathcal{L}} [t/x](\mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, s) : A} \; \text{CUT1}$$

  - S_UNITE2:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_2 : A} \; \text{S\_UNITE2}$$

    maps to

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \quad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A}{x : \mathsf{UnitS}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_2 : A}\; \text{UNITL2}}{\Gamma, \Delta \vdash_{\mathcal{L}} [s_1/x](\mathsf{let}\, x : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s_2) : A} \; \text{CUT2}$$

  - S_TENE1:

$$\frac{\Phi \vdash_C t : X \otimes Y \quad \Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, t : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A} \; \text{S\_TENE1}$$

    maps to

$$\frac{\Phi \vdash_C t : X \otimes Y \quad \dfrac{\Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, z : X \otimes Y, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A}\; \text{TENL1}}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} [t/z](\mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s) : A} \; \text{CUT1}$$

  - S_TENE2:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C} \; \text{S\_TENE2}$$

    maps to

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \dfrac{\Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, z : A \triangleright B, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2 : C}\; \text{TENL2}}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [s_1/z](\mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s_2) : C} \; \text{CUT2}$$

- S_IMPrE: (NOT SURE)

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \to B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{app}_r \, s_1 \, s_2 : B} \; \text{S\_IMPrE}$$

maps to

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \to B \quad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A \quad x : B \vdash_{\mathcal{L}} x : B}{y : A \to B, \Delta \vdash_{\mathcal{L}} [\mathsf{app}_r \, y \, s_2/x] x : B} \; \text{IMPrL}}{\Gamma, \Delta \vdash_{\mathcal{L}} [s_1/y][\mathsf{app}_r \, y \, s_2/x] x : B} \; \text{CUT2}$$

- S_IMPlE: (NOT SURE)

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftarrow A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta, \Gamma \vdash_{\mathcal{L}} \mathsf{app}_l \, s_1 \, s_2 : B} \; \text{S\_IMPlE}$$

maps to

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftarrow A \quad \dfrac{\Delta \vdash_{\mathcal{L}} s_2 : A \quad x : B \vdash_{\mathcal{L}} x : B}{\Delta, y : B \leftarrow A \vdash_{\mathcal{L}} [\mathsf{app}_l \, y \, s_2/x] x : B} \; \text{IMPlL}}{\Delta, \Gamma \vdash_{\mathcal{L}} [s_1/y][\mathsf{app}_l \, y \, s_2/x] x : B} \; \text{CUT2}$$

- S_FE:

$$\frac{\Gamma \vdash_{\mathcal{L}} y : FX \quad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, Fx : FX \,\mathsf{be}\, y \,\mathsf{in}\, s : A} \; \text{S\_FE}$$

maps to

$$\frac{\Gamma \vdash_{\mathcal{L}} y : FX \quad \dfrac{\Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, z : FX, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, z : FX \,\mathsf{be}\, Fx \,\mathsf{in}\, s : A} \; \text{FL}}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} [y/z](\mathsf{let}\, y : FX \,\mathsf{be}\, Fx \,\mathsf{in}\, s) : A} \; \text{CUT2}$$

- S_GE:

$$\frac{\Phi \vdash_C t : GA}{\Phi \vdash_{\mathcal{L}} \mathsf{derelict}\, t : A} \; \text{S\_GE}$$

maps to

$$\frac{\dfrac{x : A \vdash_{\mathcal{L}} x : A}{y : GA \vdash_{\mathcal{L}} \mathsf{let}\, y : GA \,\mathsf{be}\, G\,x \,\mathsf{in}\, x : A} \; \text{GL} \quad \Phi \vdash_C t : GA}{\Phi \vdash_{\mathcal{L}} [t/y](\mathsf{let}\, y : GA \,\mathsf{be}\, G\,x \,\mathsf{in}\, x) : A} \; \text{CUT1}$$

Function $N : SE \to ND$ maps a proof in the sequent calculus to a proof of the same sequent in the natural deduction. The function is defined as follows:

- Axioms map to axioms.
- Instances of cut rules map to the admissible substitution rules.
- Right rules map to introductions.
- Left rules map to eliminations modulo some structural fiddling.

  - T_UNITL:

$$\frac{\Psi \vdash_C t : X}{x : \mathsf{UnitT}, \Psi \vdash_C \mathsf{let}\, x : \mathsf{UnitT} \,\mathsf{be}\, \mathsf{trivT} \,\mathsf{in}\, t : X} \; \text{T\_UNITL}$$

maps to

$$\frac{x : \mathsf{UnitT} \vdash_C x : \mathsf{UnitT} \quad \Psi \vdash_C t : X}{x : \mathsf{UnitT}, \Psi \vdash_C \mathsf{let}\, x : \mathsf{UnitT} \,\mathsf{be}\, \mathsf{trivT} \,\mathsf{in}\, t : X} \; \text{UNITE}$$

  - T_TENL:

$$\frac{\Phi, x : X, y : Y, \Psi \vdash_C t : Z}{\Phi, z : X \otimes Y, \Psi \vdash_C \mathsf{let}\, z : X \otimes Y \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t : Z} \; \text{T\_TENL}$$

maps to

$$\frac{z : X \otimes Y \vdash_C z : X \otimes Y \quad \Phi, x : X, y : Y, \Psi \vdash_C t : Z}{\Phi, z : X \otimes Y, \Psi \vdash_C \mathsf{let}\, z : X \otimes Y \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, t : Z} \; \text{TENE}$$

  - T_IMPL:

$$\frac{\Phi \vdash_C t_1 : X \quad \Psi_1, x : Y, \Psi_2 \vdash_C t_2 : Z}{\Psi_1, y : X \multimap Y, \Phi, \Psi_2 \vdash_C [\mathsf{app}\, y \, t_1/x] t_2 : Z} \; \text{T\_IMPL}$$

maps to

$$\frac{\dfrac{z : X \multimap Y \vdash_C z : X \multimap Y \quad \Phi \vdash_C t_1 : X}{z : X \multimap Y, \Phi \vdash_C \mathsf{app}\, z \, t_1 : Y} \; \text{IMPE} \quad \Psi_1, x : Y, \Psi_2 \vdash_C t_2 : Z}{\Psi_1, z : X \multimap Y, \Phi, \Psi_2 \vdash_C [\mathsf{app}\, z \, t_2/x] t_2 : Z} \; \text{CUT1}$$

  - S_UNITL1:

$$\frac{\Delta \vdash_{\mathcal{L}} s : A}{x : \mathsf{UnitT}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, s : A} \; \text{S\_unitL1}$$

maps to

$$\frac{x : \mathsf{UnitT} \vdash_{\mathcal{C}} x : \mathsf{UnitT} \qquad \Delta \vdash_{\mathcal{L}} s : A}{x : \mathsf{UnitT}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitT}\, \mathsf{be}\, \mathsf{trivT}\, \mathsf{in}\, s : A} \; \text{unitE1}$$

- S_unitL2:

$$\frac{\Delta \vdash_{\mathcal{L}} s : A}{x : \mathsf{UnitS}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s : A} \; \text{S\_unitL2}$$

maps to

$$\frac{x : \mathsf{UnitS} \vdash_{\mathcal{L}} x : \mathsf{UnitS} \qquad \Delta \vdash_{\mathcal{L}} s : A}{x : \mathsf{UnitS}, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, x : \mathsf{UnitS}\, \mathsf{be}\, \mathsf{trivS}\, \mathsf{in}\, s : A} \; \text{unitE2}$$

- S_tenL1:

$$\frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \otimes Y, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A} \; \text{S\_tenL1}$$

maps to

$$\frac{z : X \otimes Y \vdash_{\mathcal{C}} z : X \otimes Y \qquad \Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \otimes Y, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : X \otimes Y\, \mathsf{be}\, x \otimes y\, \mathsf{in}\, s : A} \; \text{tenE1}$$

- S_tenL2:

$$\frac{\Gamma, x : A, y : B, \Delta \vdash_{\mathcal{L}} s : C}{\Gamma, z : A \triangleright B, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s : C} \; \text{S\_tenL2}$$

maps to

$$\frac{z : A \triangleright B \vdash_{\mathcal{L}} z : A \triangleright B \qquad \Gamma, x : A, y : B, \Delta \vdash_{\mathcal{L}} s : C}{\Gamma, z : A \triangleright B, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, z : A \triangleright B\, \mathsf{be}\, x \triangleright y\, \mathsf{in}\, s : C} \; \text{tenE2}$$

- S_impL:

$$\frac{\Phi \vdash_{\mathcal{C}} t : X \qquad \Gamma, x : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : X \multimap Y, \Phi, \Delta \vdash_{\mathcal{L}} [\mathsf{app}\, y\, t/x]s : A} \; \text{S\_impL}$$

maps to

$$\frac{\dfrac{z : X \multimap Y \vdash_{\mathcal{C}} z : X \multimap Y \qquad \Phi \vdash_{\mathcal{C}} t : X}{z : X \multimap Y, \Phi \vdash_{\mathcal{C}} \mathsf{app}\, z\, t : Y} \; \text{impE} \qquad \Gamma, x : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : X \multimap Y, \Phi, \Delta \vdash_{\mathcal{L}} [\mathsf{app}\, z\, t/x]s : A} \; \text{cut1}$$

- S_imprL:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \qquad \Delta, x : B \vdash_{\mathcal{L}} s_2 : C}{\Delta, y : A \rightarrow B, \Gamma \vdash_{\mathcal{L}} [\mathsf{app}_r\, y\, s_1/x]s_2 : C} \; \text{S\_imprL}$$

maps to

$$\frac{\dfrac{z : A \rightarrow B \vdash_{\mathcal{L}} z : A \rightarrow B \qquad \Gamma \vdash_{\mathcal{L}} s_1 : A}{z : A \rightarrow B, \Gamma \vdash_{\mathcal{L}} \mathsf{app}_r\, z\, s_1 : B} \; \text{imprE} \qquad \Delta, x : B \vdash_{\mathcal{L}} s_2 : C}{\Delta, z : A \rightarrow B, \Gamma \vdash_{\mathcal{L}} [\mathsf{app}_r\, z\, s_1/x]s_2 : C} \; \text{cut2}$$

- S_implL:

$$\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \qquad x : B, \Delta \vdash_{\mathcal{L}} s_2 : C}{\Gamma, y : B \leftarrow A, \Delta \vdash_{\mathcal{L}} [\mathsf{app}_l\, y\, s_1/x]s_2 : C} \; \text{S\_implL}$$

maps to

$$\frac{\dfrac{z : B \leftarrow A \vdash_{\mathcal{L}} z : B \leftarrow A \qquad \Gamma \vdash_{\mathcal{L}} s_1 : A}{\Gamma, z : B \leftarrow A \vdash_{\mathcal{L}} \mathsf{app}_l\, z\, s_1 : B} \; \text{implE} \qquad x : B, \Delta \vdash_{\mathcal{L}} s_2 : C}{\Gamma, z : B \leftarrow A, \Delta \vdash_{\mathcal{L}} [\mathsf{app}_l\, z\, s_1/x]s_2 : C} \; \text{cut2}$$

- S_FL:

$$\frac{\Gamma, x : X, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, y : FX, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : FX\, \mathsf{be}\, F\, x\, \mathsf{in}\, s : A} \; \text{S\_FL}$$

maps to

$$\frac{z : FX \vdash_{\mathcal{L}} z : FX \qquad \Gamma, x : X, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : FX, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, F\, x : FX\, \mathsf{be}\, z\, \mathsf{in}\, s : A} \; \text{FE}$$

- S_GL:

$$\frac{\Gamma, x : A, \Delta \vdash_{\mathcal{L}} s : B}{\Gamma, y : GA, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, y : GA\, \mathsf{be}\, G\, x\, \mathsf{in}\, s : B} \; \text{S\_GL}$$

maps to

$$\frac{\dfrac{y : GA \vdash_{\mathcal{C}} y : GA}{y : GA \vdash_{\mathcal{L}} \mathsf{derelict}\, y : A} \; \text{GE} \qquad \Gamma, x : A, \Delta \vdash_{\mathcal{L}} s : B}{\Gamma, y : GA, \Delta \vdash_{\mathcal{L}} [\mathsf{derelict}\, y/x]s : B} \; \text{cut2}$$

6 Combining with Benton's Adjoint Model

7 Applications

8 Conclusion

TODO

A Appendix