On Linear Based Intuitionistic Substructural Logics

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TODO

1998 ACM Subject Classification TODO

Keywords and phrases TODO

Digital Object Identifier 10.4230/LIPIcs...

- 1 Introduction
- 2 Main Ideas
- 3 Categorical Models
- ▶ **Definition 1.** A monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a category \mathcal{M} consists of
- a bifunctor \otimes : $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- **a** an object *I*, called the unit object;
- three natural isomorphisms α , λ , and ρ with components

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$\lambda_A: I \otimes A \to A$$

$$\rho_A: A \otimes I \to A$$

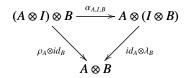
where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in \mathcal{M} :

$$((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha_{A,B,C} \otimes id_D} (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D)$$

$$\downarrow id_A \otimes \alpha_{B,C,D}$$

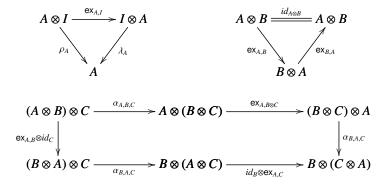
$$(A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C \otimes D}} A \otimes (B \otimes (C \otimes D))$$



▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ equipped with two bifunctors \rightarrow : $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and \leftarrow : $\mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\operatorname{\mathsf{Hom}}_f(X \otimes A, B) \cong \operatorname{\mathsf{Hom}}_f(X, A \rightharpoonup B)$$
 $\operatorname{\mathsf{Hom}}_f(A \otimes X, B) \cong \operatorname{\mathsf{Hom}}_f(X, B \leftharpoonup A)$

▶ **Definition 3.** A symmetric monoidal category (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $ex_{A,B} : A \otimes B \to B \otimes A$, called **exchange**, such that the following diagrams commute:



- ▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor \multimap : $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.
- ▶ **Lemma 5.** Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightarrow B) \cong (B \leftarrow A)$.

Proof. First, notice that for any object C we have

$$Hom[C, A \to B] \cong Hom[C \otimes A, B]$$
 \mathcal{L} is a Lambek category $\cong Hom[A \otimes C, B]$ By the exchange $ex_{C,A}$ $\cong Hom[C, B \leftarrow A]$ \mathcal{L} is a Lambek category

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma.

- ▶ Corollary 6. A Lambek category with exchange is symmetric monoidal closed.
- ▶ **Definition 7.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from \mathcal{M} to \mathcal{M}' is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \otimes FB' \to F(A \otimes B)$, such that the following diagrams commute for any objects A, B, and C in \mathcal{M} :

$$(FA \otimes' FB) \otimes' FC \xrightarrow{\alpha'_{FA,FB,FC}} FA \otimes' (FB \otimes' FC) \xrightarrow{id_{FA} \otimes' m_{A,B}} FA \otimes' F(B \otimes C)$$

$$\downarrow m_{A,B} \otimes' id_{FC} \downarrow \qquad \qquad \downarrow m_{A,B} \otimes C$$

$$F(A \otimes B) \otimes' FC \xrightarrow{m_{A \otimes B,C}} F((A \otimes B) \otimes C) \xrightarrow{F\alpha_{A,B,C}} F(A \otimes (B \otimes C))$$

$$I' \otimes' FA \xrightarrow{\alpha'_{FA}} FA \qquad \qquad FA \otimes' I' \xrightarrow{\rho'_{FA}} FA$$

$$\downarrow m_{I} \otimes id_{FA} \downarrow \qquad \qquad \uparrow F\alpha_{A,B} \otimes C$$

$$\downarrow m_{A,B} \otimes C$$

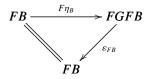
▶ **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories. A sym**metric monoidal functor** $F: \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$FA \otimes' FB \xrightarrow{\exp_{A,FB}} FB \otimes' FA$$

$$\downarrow^{\mathsf{m}_{A,B}} \qquad \qquad \downarrow^{\mathsf{m}_{B,A}}$$

$$F(A \otimes B) \xrightarrow{Fex_{A,B}} F(B \otimes A)$$

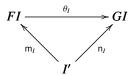
▶ **Definition 9.** An adjunction between categories C and D consists of two functors $F: \mathcal{D} \to C$, called the **left adjoint**, and $G: C \to \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta:id_{\mathcal{D}}\to GF$, called the **unit**, and $\varepsilon:FG\to id_{\mathcal{C}}$, called the **counit**, such that the following diagrams commute for any object A in C and B in D:





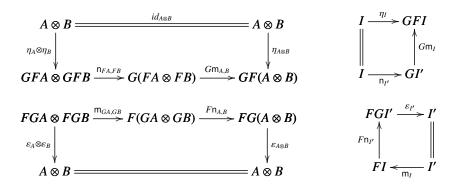
Definition 10. Let (F, m) and (G, n) be monoidal functors from a monoidal category \mathcal{M} to a monoidal category \mathcal{M}' . A monoidal natural transformation from (F, m) to (G, n) is a natural transformation $\theta: (F, m) \to (G, n)$ such that the following diagrams commute for any objects A and B in \mathcal{M} :





- ▶ **Definition 11.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}$ \mathcal{M}' and $G: \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F: \mathcal{M} \dashv \mathcal{M}': G$ is a **monoidal adjunction** if Fand G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.
- ▶ Definition 12. A SMCC-Lambek model consists of
- **a** symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$;
- \blacksquare a monoidal adjunction $F: C \dashv \mathcal{L}: G$, where $F: C \to \mathcal{L}$ and $G: \mathcal{L} \to C$ are monoidal functors.

Thus, in a SMCC-Lambek model, the following four diagrams commute because η and ε are monoidal natural transformations:



XX:4 On Linear Based Intuitionistic Substructural Logics

And the following two diagrams commute because of the adjunction:



▶ **Definition 13.** Let *C* be a category. A **monad** on *C* consists of an endofunctor $T: C \to C$ together with two natural transformations $\eta: id_C \to T$ and $\mu: T^2 \to id_C$, where id_C is the identity functor on *C*, such that the following diagrams commute:



▶ **Lemma 14.** The monad on the SMCC C in a SMCC-Lambek model is monoidal.

Proof. We define the monad T on the C in the adjunction of a SMCC-Lambek model as T = GF, and the two corresponding natural transformations $\eta: id_C \to T$ and $\mu: T^2 \to T$ are defined as

$$\eta: id_C \to GF$$

$$\mu = GF \varepsilon_A = \varepsilon_{GFA}: GFGF \to GF$$

where η is the unit and μ is the counit in the adjunction $F: C \dashv \mathcal{L}: G$, and (F, m) and (G, n) are monoidal functors.

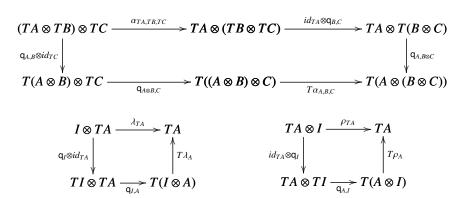
Thus, we have

$$\mathsf{q}_{A,B} = G\mathsf{m}_{A,B} \circ \mathsf{n}_{FA,FB} : TA \otimes TB \to T(A \otimes B)$$

$$\mathsf{q}_{I} = G\mathsf{m}_{I} \circ \mathsf{n}_{I'} : I \to TI$$

The monad T being monoidal means

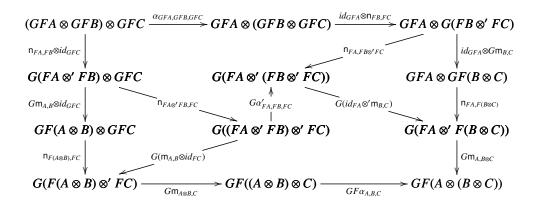
1. *T* is a monoidal functor i.e. the following diagrams commute:



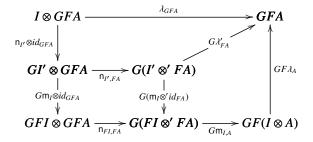
We write GF instead of T in the diagram chasings for clarity.

By replacing q with its definition, the first diagram above commutes by the following diagram

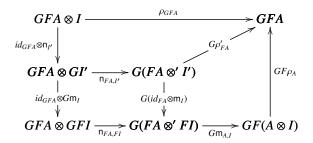
chasing, where the two hexagons commute because G and F are monoidal functors, and the two quadrilaterals commute by the naturality of n.



The first square above commutes by the following diagram chasing, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.



Similarly, the second square above commutes by the following diagram chasing:

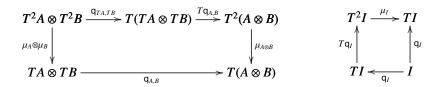


2. η is a monoidal natural transformation, i.e. the following diagrams commute. In fact, since η is the unit of the monoidal adjunction, η is monoidal and thus the following two diagrams commute.



3. μ is a monoidal natural transformation, i.e. the following diagrams commute. Since $\mu = \varepsilon_{GFA}$

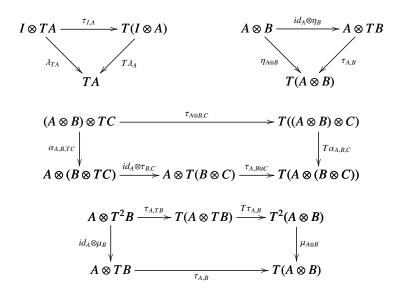
and ε is monoidal, so is μ . Thus the following diagrams commute.



However, the monad T we get from the SMCC-Lambek model is not symmetric because the following diagram does not commute:



▶ **Definition 15.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on \mathcal{M} . T is a **strong monad** if there is natural transformation τ , called the **tensorial strength**, with components $\tau_{A,B}: A \otimes TB \to T(A \otimes B)$ such that the following diagrams commute:



▶ **Lemma 16.** The monad on the SMCC in a SMCC-Lambek model is strong.

Proof. Let $F: C \vdash \mathcal{L}: G$ be a SMCC-Lambek model, where $(C, \otimes, I, \alpha, \lambda, \rho)$ is an SMCC, $(\mathcal{M}, \otimes', I', \alpha', \lambda', \rho')$ is a Lambek category, and (F, m) and (G, n) are monoidal functors. Let (T, η, μ) be the monad on C where T = GF. We have proved that T is monoidal with the natural transformation $\mathsf{q}_{A,B}: TA \otimes TB \to T(A \otimes B)$ and the morphism $\mathsf{q}_I: I \to TI$ defined as in Lemma \ref{log} ?. We define the tensorial strength $\ref{log}_{A,B}: A \otimes TB \to T(A \otimes B)$ as $\ref{log}_{A,B} = \mathsf{q}_{A,B} \circ \eta_A \otimes id_{TB}$. Since \ref{log}_I is a monoidal natural transformation, we have $\ref{log}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'}$. Therefore $\ref{log}_I = \mathsf{q}_I$. Thus the following diagram commutes because T is monoidal, where the composition $\ref{log}_{I,A} \circ \mathsf{q}_I \otimes id_{TA}$ is the

definition of $\tau_{I,A}$. So the first triangle in Defition ?? commutes.

$$I \otimes TA \xrightarrow{\mathbf{q}_{I} \otimes id_{TA}} TI \otimes TA$$

$$\downarrow^{\mathbf{q}_{I,A}} \qquad \qquad \downarrow^{\mathbf{q}_{I,A}}$$

$$TA \xleftarrow{T\lambda_{A}} T(I \otimes A)$$

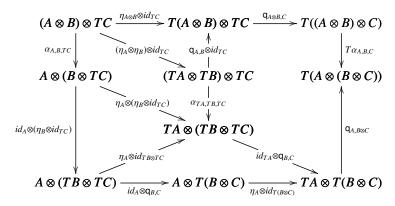
Similarly, by using the definition of τ , the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$A \otimes B \xrightarrow{id_A \otimes \eta_B} A \otimes TB$$

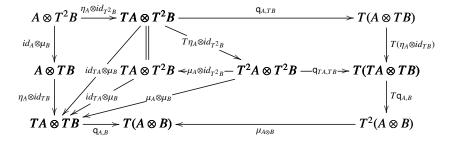
$$\downarrow^{\eta_A \otimes \eta_B} \qquad \downarrow^{\eta_A \otimes id_{TB}}$$

$$T(A \otimes B) \xrightarrow[\mathsf{Q}_{A,B}]{} TA \otimes TB$$

The first pentagon in the definition commutes by the following diagram chasing, because η are α natural transformations and T is monoidal:



The last diagram in the definition commtues by the following diagram chasing, because T is a monad, q is a natural transformation, and μ is a monoidal natural transformation:



- ▶ **Definition 17.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex, and (T, η, μ) be a strong monad on \mathcal{M} . Then there is a "**twisted**" **tensorial strength** $\tau'_{A,B}: TA \otimes B \to T(A \otimes B)$ defined as $\tau'_{A,B} = T$ ex $\circ \tau_{B,A} \circ$ ex. We can construct a pair of natural transformations Φ , Φ' with components $\Phi_{A,B}, \Phi'_{A,B}: TA \otimes TB \to T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A\otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A\otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.
- ▶ **Lemma 18.** Let \mathcal{M} be a symmetric monoidal category and T be a strong monad on \mathcal{M} . Then T is a symmetric monoidal functor iff it is commutative.
- ▶ Theorem 19. The monad on the SMCC in a SMCC-Lambek model is not commutative.

XX:8 On Linear Based Intuitionistic Substructural Logics

- 4 Logic
- 5 Applications
- 6 Related Work

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7 Conclusion

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A Appendix