On Linear Based Intuitionistic Substructural Logics

Harley Eades III[1] and Jiaming Jiang[2]

1 **Computer Science, Augusta University, Augusta, Georgia, USA**
`heades@augusta.edu`

2 **Computer Science, North Carolina State University, Raleigh, North Carolina, USA**
`jjiang13@ncsu.edu`

Abstract

TODO

1998 ACM Subject Classification TODO

Keywords and phrases TODO

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

2 Main Ideas

3 Categorical Models

▶ **Definition 1.** A **monoidal category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ consists of

- a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- an object I, called the unit object;
- three natural isomorphisms α, λ, and ρ with components

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$\lambda_A : I \otimes A \to A$$

$$\rho_A : A \otimes I \to A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in $\mathcal{M}$:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes id_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B \otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "id_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

▸ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\rightharpoonup : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and $\leftharpoonup : \mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

▸ **Definition 3.** A **symmetric monoidal category** is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\mathsf{ex}_{A,B} : A \otimes B \to B \otimes A$, called exchange, such that the following diagrams commute:

$$\begin{tikzcd} A \otimes I \arrow[rr, "\mathsf{ex}_{A,I}"] \arrow[dr, "\rho_A"'] & & I \otimes A \arrow[dl, "\lambda_A"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[rr, equal, "id_{A\otimes B}"] \arrow[dr, "\mathsf{ex}_{A,B}"'] & & A \otimes B \\ & B \otimes A \arrow[ur, "\mathsf{ex}_{B,A}"'] & \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathsf{ex}_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\mathsf{ex}_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,A,C}"] \\ (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \mathsf{ex}_{A,C}"'] & B \otimes (C \otimes A) \end{tikzcd}$$

▸ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

▸ **Lemma 5.** *Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$\begin{array}{rcll} Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\ & \cong & Hom[A \otimes C, B] & \text{By the exchange } \mathsf{ex}_{C,A} \\ & \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category} \end{array}$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◂

▸ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

▸ **Definition 7.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from $\mathcal{M}$ to $\mathcal{M}'$ is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \otimes FB' \to F(A \otimes B)$, such that the following diagrams commute for any objects A, B, and C in $\mathcal{M}$:

$$\begin{tikzcd} (FA \otimes' FB) \otimes' FC \arrow[r, "\alpha'_{FA,FB,FC}"] \arrow[d, "\mathsf{m}_{A,B} \otimes' id_{FC}"'] & FA \otimes' (FB \otimes' FC) \arrow[r, "id_{FA} \otimes' \mathsf{m}_{A,B}"] & FA \otimes' F(B \otimes C) \arrow[d, "\mathsf{m}_{A,B\otimes C}"] \\ F(A \otimes B) \otimes' FC \arrow[r, "\mathsf{m}_{A\otimes B,C}"'] & F((A \otimes B) \otimes C) \arrow[r, "F\alpha_{A,B,C}"'] & F(A \otimes (B \otimes C)) \end{tikzcd}$$

$$\begin{tikzcd} I' \otimes' FA \arrow[r, "\lambda'_{FA}"] \arrow[d, "\mathsf{m}_I \otimes id_{FA}"'] & FA \\ FI \otimes' FA \arrow[r, "\mathsf{m}_{I,A}"'] & F(I \otimes A) \arrow[u, "F\lambda_A"'] \end{tikzcd} \qquad \begin{tikzcd} FA \otimes' I' \arrow[r, "\rho'_{FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes \mathsf{m}_I"'] & FA \\ FA \otimes' FI \arrow[r, "\mathsf{m}_{A,I}"'] & F(A \otimes I) \arrow[u, "F\rho_A"'] \end{tikzcd}$$

▸ **Definition 8.** Let (F, m) and (G, n) be monoidal functors from a monoidal category $\mathcal{M}$ to a monoidal category $\mathcal{M}'$. A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, \mathsf{m}) \to (G, \mathsf{n})$ such that the following diagrams commute for any objects A and B in $\mathcal{M}$:

$$\begin{tikzcd} FA \otimes' FB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[d, "\theta_A \otimes' \theta_B"'] & F(A \otimes B) \arrow[d, "\theta_{A\otimes B}"] \\ GA \otimes' GB \arrow[r, "\mathsf{n}_{A,B}"'] & G(A \otimes B) \end{tikzcd} \qquad \begin{tikzcd} fI \arrow[rr, "\theta_I"] & & GI \\ & I' \arrow[ul, "\mathsf{m}_I"] \arrow[ur, "\mathsf{n}_I"'] & \end{tikzcd}$$

▸ **Definition 9.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit $\eta : id_{\mathcal{M}} \to GF$ and the counit $\varepsilon : FG \to id_{\mathcal{M}'}$ are monoidal natural transformations, where $id_{\mathcal{M}}$ and $id_{\mathcal{M}'}$ are the identity functors on $\mathcal{M}$ and $\mathcal{M}'$ respectively.

▸ **Lemma 10.** *Let $(C, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal closed category and $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$ be a Lambek category. Since C and $\mathcal{L}$ are both monoidal, let $F : C \to \mathcal{L}$ and $G : \mathcal{L} \to C$ be monoidal functors. Then the adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is monoidal.*

Proof. Proving the adjunction being monoidal is equivalent to proving the unit η and the counit ε are monoidal natural transformations. First, we show that η and ε are natural transformations, i.e. the following diagrams commute:

◂

▸ **Lemma 11.** *Let C be a category. A* ***monad*** *on C consists of an endofunctor $T : C \to C$ together with two natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to id_C$, where id_C is the identity functor on C, such that the following diagrams commute:*

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

4 Logic

5 Applications

6 Related Work

TODO

7 Conclusion

TODO

A Appendix