Non-Commutative Linear Logic in an Adjoint Model

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Abstract

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1 Introduction

2 Related Work

3 Category Theory Basics

▶ **Definition 1.** A **monoidal category** $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ consists of

- a bifunctor $\triangleright : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- an object I, called the unit object;
- three natural isomorphisms α, λ, and ρ with components

$$\alpha_{A,B,C} : (A \triangleright B) \triangleright C \to A \triangleright (B \triangleright C)$$

$$\lambda_A : I \triangleright A \to A$$

$$\rho_A : A \triangleright I \to A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in $\mathcal{M}$:

$$\begin{tikzcd}
((A \triangleright B) \triangleright C) \triangleright D \arrow[r, "\alpha_{A,B,C} \triangleright id_D"] \arrow[d, "\alpha_{A \triangleright B,C,D}"'] & (A \triangleright (B \triangleright C)) \triangleright D \arrow[r, "\alpha_{A,B \triangleright C,D}"] & A \triangleright ((B \triangleright C) \triangleright D) \arrow[d, "id_A \triangleright \alpha_{B,C,D}"] \\
(A \triangleright B) \triangleright (C \triangleright D) \arrow[rr, "\alpha_{A,B,C \triangleright D}"'] & & A \triangleright (B \triangleright (C \triangleright D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \triangleright I) \triangleright B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \triangleright id_B"'] & & A \triangleright (I \triangleright B) \arrow[dl, "id_A \triangleright \lambda_B"] \\
& A \triangleright B &
\end{tikzcd}$$

▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\rightharpoonup : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and $\leftharpoonup : \mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \triangleright A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \triangleright X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

▶ **Definition 3.** A **symmetric monoidal category** (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\mathsf{ex}_{A,B} : A \otimes B \to B \otimes A$, called **exchange**, such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\mathsf{ex}_{A,I}"] \arrow[dr, "\rho_A"'] & & I \otimes A \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[rr, equal, "id_{A\otimes B}"] \arrow[dr, "\mathsf{ex}_{A,B}"'] & & A \otimes B \\
& B \otimes A \arrow[ur, "\mathsf{ex}_{B,A}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathsf{ex}_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\mathsf{ex}_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,A,C}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \mathsf{ex}_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

We use $\triangleright$ for non-symmetric monoidal categories while $\otimes$ for symmetric ones.

▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap$: $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

▶ **Lemma 5.** *Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$\begin{array}{rcll}
Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\
& \cong & Hom[A \otimes C, B] & \text{By the exchange } \mathsf{ex}_{C,A} \\
& \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category}
\end{array}$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◀

▶ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

▶ **Definition 7.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \triangleright', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from $\mathcal{M}$ to $\mathcal{M}'$ is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \triangleright FB' \to F(A \triangleright B)$, such that the following diagrams commute for any objects A, B, and C in $\mathcal{M}$:

$$\begin{tikzcd}
(FA \triangleright' FB) \triangleright' FC \arrow[r, "\alpha'_{FA,FB,FC}"] \arrow[d, "\mathsf{m}_{A,B} \triangleright' id_{FC}"'] & FA \triangleright' (FB \triangleright' FC) \arrow[r, "id_{FA} \triangleright' \mathsf{m}_{A,B}"] & FA \triangleright' F(B \triangleright C) \arrow[d, "\mathsf{m}_{A,B\triangleright C}"] \\
F(A \triangleright B) \triangleright' FC \arrow[r, "\mathsf{m}_{A\triangleright B,C}"'] & F((A \triangleright B) \triangleright C) \arrow[r, "F\alpha_{A,B,C}"'] & F(A \triangleright (B \triangleright C))
\end{tikzcd}$$

$$\begin{tikzcd}
I' \triangleright' FA \arrow[r, "\lambda'_{FA}"] \arrow[d, "\mathsf{m}_I \triangleright id_{FA}"'] & FA \\
FI \triangleright' FA \arrow[r, "\mathsf{m}_{I,A}"'] & F(I \triangleright A) \arrow[u, "F\lambda_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \triangleright' I' \arrow[r, "\rho'_{FA}"] \arrow[d, "\mathsf{id}_{FA} \triangleright \mathsf{m}_I"'] & FA \\
FA \triangleright' FI \arrow[r, "\mathsf{m}_{A,I}"'] & F(A \triangleright I) \arrow[u, "F\rho_A"']
\end{tikzcd}$$

▶ **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "\mathsf{ex}_{FA,FB}"] \arrow[d, "\mathsf{m}_{A,B}"'] & FB \otimes' FA \arrow[d, "\mathsf{m}_{B,A}"] \\
F(A \otimes B) \arrow[r, "F\mathsf{ex}_{A,B}"'] & F(B \otimes A)
\end{tikzcd}$$

▶ **Definition 9.** An **adjunction** between categories $\mathcal{C}$ and $\mathcal{D}$ consists of two functors $F : \mathcal{D} \to \mathcal{C}$, called the **left adjoint**, and $G : \mathcal{C} \to \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta : id_{\mathcal{D}} \to GF$, called the **unit**, and $\varepsilon : FG \to id_{\mathcal{C}}$, called the **counit**, such that the following diagrams commute for any object A in $\mathcal{C}$ and B in $\mathcal{D}$:

$$\begin{tikzcd}
FB \arrow[rr, "F\eta_B"] \arrow[dr, equal] & & FGFB \arrow[dl, "\varepsilon_{FB}"] \\
& FB &
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[rr, "\eta_{GA}"] \arrow[dr, equal] & & GFGA \arrow[dl, "G\varepsilon_A"] \\
& GA &
\end{tikzcd}$$

▶ **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ to a monoidal category $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$. A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, \mathsf{m}) \to (G, \mathsf{n})$ such that the following diagrams commute for any objects A and B in $\mathcal{M}$:

$$\begin{tikzcd}
FA \triangleright' FB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[d, "\theta_A \triangleright' \theta_B"'] & F(A \triangleright B) \arrow[d, "\theta_{A \triangleright B}"] \\
GA \triangleright' GB \arrow[r, "\mathsf{n}_{A,B}"'] & G(A \triangleright B)
\end{tikzcd}
\qquad
\begin{tikzcd}
FI \arrow[rr, "\theta_I"] & & GI \\
& I' \arrow[ul, "\mathsf{m}_I"] \arrow[ur, "\mathsf{n}_I"'] &
\end{tikzcd}$$

▶ **Definition 11.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \triangleright', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

▶ **Definition 12.** Let $\mathcal{C}$ be a category. A **monad** on $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta : id_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$, where $id_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$, such that the following diagrams commute:

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta_T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

▶ **Definition 13.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on $\mathcal{M}$. T is a **strong monad** if there is natural transformation τ, called the **tensorial strength**, with components

$\tau_{A,B} : A \triangleright TB \to T(A \triangleright B)$ such that the following diagrams commute:

$$\begin{tikzcd}
I \triangleright TA \arrow[rr, "\tau_{I,A}"] \arrow[dr, "\lambda_{TA}"'] & & T(I \triangleright A) \arrow[dl, "T\lambda_A"] \\
& TA &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \triangleright B \arrow[rr, "id_A \triangleright \eta_B"] \arrow[dr, "\eta_{A \triangleright B}"'] & & A \triangleright TB \arrow[dl, "\tau_{A,B}"] \\
& T(A \triangleright B) &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \triangleright B) \triangleright TC \arrow[rr, "\tau_{A \triangleright B,C}"] \arrow[d, "\alpha_{A,B,TC}"'] & & T((A \triangleright B) \triangleright C) \arrow[d, "T\alpha_{A,B,C}"] \\
A \triangleright (B \triangleright TC) \arrow[r, "id_A \triangleright \tau_{B,C}"] & A \triangleright T(B \triangleright C) \arrow[r, "\tau_{A,B \triangleright C}"] & T(A \triangleright (B \triangleright C))
\end{tikzcd}$$

$$\begin{tikzcd}
A \triangleright T^2 B \arrow[r, "\tau_{A,TB}"] \arrow[d, "id_A \triangleright \mu_B"'] & T(A \triangleright TB) \arrow[r, "T\tau_{A,B}"] & T^2(A \triangleright B) \arrow[d, "\mu_{A \triangleright B}"] \\
A \triangleright TB \arrow[rr, "\tau_{A,B}"'] & & T(A \triangleright B)
\end{tikzcd}$$

▸ **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex, and (T, η, μ) be a strong monad on $\mathcal{M}$. Then there is a **"twisted" tensorial strength** $\tau'_{A,B} : TA \otimes B \to T(A \otimes B)$ defined as $\tau'_{A,B} = T\mathsf{ex} \circ \tau_{B,A} \circ \mathsf{ex}$. We can construct a pair of natural transformations Φ, Φ' with components $\Phi_{A,B}, \Phi'_{A,B} : TA \otimes TB \to T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A \otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A \otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.

▸ **Definition 15.** Let $\mathcal{L}$ be a category. A **comonad** on $\mathcal{L}$ consists of an endofunctor $S : \mathcal{L} \to \mathcal{L}$ together with two natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S^2 \to S$ such that the following diagrams commute:

$$\begin{tikzcd}
S \arrow[r, "\delta"] \arrow[d, "\delta"'] & S^2 \arrow[d, "S\delta"] \\
S^2 \arrow[r, "\delta_S"'] & S^3
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2 \arrow[r, "S\varepsilon"] \arrow[d, "\varepsilon_S"'] & S \arrow[d, "\delta"] \\
S \arrow[r, "\delta"'] & S^2
\end{tikzcd}$$

4 An Adjoint Model

Our adjoint model, SMCC-Lambek model, has a similar structure as Benton's LNL model []. Benton's LNL model consists of a symmetric monoidal adjunction $F : C \dashv \mathcal{L} : G$ between a cartesian closed category C and a symmetric monoidal closed category $\mathcal{L}$.

▸ **Definition 16.** A **SMCC-Lambek model** consists of

- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$;
- a monoidal adjunction $F : C \dashv \mathcal{L} : G$, where $F : C \to \mathcal{L}$ and $G : \mathcal{L} \to C$ are monoidal functors.

Thus, in a SMCC-Lambek model, the following four diagrams commute because η and ε are monoidal natural transformations:

$$\begin{tikzcd}
X \otimes Y \arrow[rr, equal, "id_{X \otimes Y}"] \arrow[d, "\eta_X \otimes \eta_Y"'] & & X \otimes Y \arrow[d, "\eta_{X \otimes Y}"] \\
GFX \otimes GFY \arrow[r, "\mathsf{n}_{FX,FY}"] & G(FX \otimes FY) \arrow[r, "G\mathsf{m}_{X,Y}"] & GF(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\eta_I"] \arrow[d, equal] & GFI \\
I \arrow[r, "\mathsf{n}_{I'}"'] & GI' \arrow[u, "G\mathsf{m}_I"']
\end{tikzcd}$$

$$\begin{tikzcd}
FGA \otimes FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes \varepsilon_B"'] & F(GA \otimes GB) \arrow[r, "F\mathsf{n}_{A,B}"] & FG(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
A \otimes B \arrow[rr, equal] & & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
FGI' \arrow[r, "\varepsilon_{I'}"] & I' \arrow[d, equal] \\
FI \arrow[u, "F\mathsf{n}_{I'}"] & I' \arrow[l, "\mathsf{m}_I"]
\end{tikzcd}$$

And the following two diagrams commute because of the adjunction:

$$\begin{tikzcd}
FX \arrow[rr, "F\eta_X"] \arrow[dr, equal] & & FGFX \arrow[dl, "\varepsilon_{FX}"] \\
& FX &
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[rr, "\eta_{GA}"] \arrow[dr, equal] & & GFGA \arrow[dl, "G\varepsilon_A"] \\
& GA &
\end{tikzcd}$$

Following the tradition, we use letters X, Y, Z for objects in C and A, B, C for objects in $\mathcal{L}$. The following lemmas and theorems establish the essential properties of the monad and the comomad derived from the adjunction.

▶ **Lemma 17.** *The monad on the symmetric monoidal closed category C in a SMCC-Lambek model is monoidal.*

Proof. We define the monad T on the C in the adjunction of a SMCC-Lambek model as $T = GF$, and the two corresponding natural transformations $\eta : id_C \to T$ and $\mu : T^2 \to T$ as

$$\eta_X : X \to GFX \qquad\qquad \mu_X = G\varepsilon_{FX} : GFGFX \to GFX$$

where η is the unit and $\varepsilon : FG \to id_{\mathcal{L}}$ is the counit of the adjunction $F : C \dashv \mathcal{L} : G$. Since the adjunction is monoidal, then (F, m) and (G, n) are monoidal functors. Thus, we have

$$\mathsf{t}_{X,Y} = G\mathsf{m}_{X,Y} \circ \mathsf{n}_{FX,FY} : TX \otimes TY \to T(X \otimes Y) \qquad\qquad \mathsf{t}_I = G\mathsf{m}_I \circ \mathsf{n}_{I'} : I \to TI$$

The monad T being monoidal means

1. T is a monoidal functor, i.e. the folllowing diagrams commute:

$$\begin{tikzcd}
(TX \otimes TY) \otimes TZ \arrow[r, "\alpha_{TX,TY,TZ}"] \arrow[d, "\mathsf{t}_{X,Y} \otimes id_{TZ}"'] & TX \otimes (TY \otimes TZ) \arrow[r, "id_{TX} \otimes \mathsf{t}_{Y,Z}"] \arrow[d, phantom, "(1)"] & TX \otimes T(Y \otimes Z) \arrow[d, "\mathsf{t}_{X,Y \otimes Z}"] \\
T(X \otimes Y) \otimes TZ \arrow[r, "\mathsf{t}_{X \otimes Y,Z}"'] & T((X \otimes Y) \otimes Z) \arrow[r, "T\alpha_{X,Y,Z}"'] & T(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes TX \arrow[r, "\lambda_{TX}"] \arrow[d, "\mathsf{t}_I \otimes id_{TX}"'] \arrow[dr, phantom, "(2)"] & TX \\
TI \otimes TX \arrow[r, "\mathsf{t}_{I,X}"'] & T(I \otimes X) \arrow[u, "T\lambda_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \otimes I \arrow[r, "\rho_{TX}"] \arrow[d, "id_{TX} \otimes \mathsf{t}_I"'] \arrow[dr, phantom, "(3)"] & TX \\
TX \otimes TI \arrow[r, "\mathsf{t}_{X,I}"'] & T(X \otimes I) \arrow[u, "T\rho_X"']
\end{tikzcd}$$

We write GF instead of T in the proof for clarity.
By replacing $\mathsf{t}_{X,Y}$ with its definition, diagram (1) above commutes by the following commutative diagram, in which the two hexagons commute because G and F are monoidal functors, and the

two quadrilaterals commute by the naturality of n.

$$
\begin{tikzcd}
(GFX \otimes GFY) \otimes GFZ \arrow[r, "\alpha_{GFX,GFY,GFZ}"] \arrow[d, "\mathsf{n}_{FX,FY} \otimes id_{GFZ}"'] & GFX \otimes (GFY \otimes GFZ) \arrow[r, "id_{GFX} \otimes \mathsf{n}_{FY,FZ}"] & GFX \otimes G(FY \triangleright FZ) \arrow[d, "id_{GFX} \otimes G\mathsf{m}_{Y,Z}"] \arrow[dl, "\mathsf{n}_{FX,FY \triangleright FZ}"'] \\
G(FX \triangleright FY) \otimes GFZ \arrow[d, "G\mathsf{m}_{X,Y} \otimes id_{GFZ}"'] \arrow[dr, "\mathsf{n}_{FX \triangleright FY,FZ}"] & G(FX \triangleright (FY \triangleright FZ)) \arrow[dr, "G(id_{FX} \triangleright \mathsf{m}_{Y,Z})"] & GFX \otimes GF(Y \otimes Z) \arrow[d, "\mathsf{n}_{FX,F(Y \otimes Z)}"] \\
GF(X \otimes Y) \otimes GFZ \arrow[d, "\mathsf{n}_{F(X \otimes Y),FZ}"'] & G((FX \triangleright FY) \triangleright FZ) \arrow[u, "G\alpha'_{FX,FY,FZ}"] \arrow[dl, "G(\mathsf{m}_{X,Y} \otimes id_{FZ})"'] & G(FX \triangleright F(Y \otimes Z)) \arrow[d, "G\mathsf{m}_{X,Y \otimes Z}"] \\
G(F(X \otimes Y) \triangleright FZ) \arrow[r, "G\mathsf{m}_{X \otimes Y,Z}"'] & GF((X \otimes Y) \otimes Z) \arrow[r, "GF\alpha_{X,Y,Z}"'] & GF(X \otimes (Y \otimes Z))
\end{tikzcd}
$$

Diagram (2) commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n.

$$
\begin{tikzcd}
I \otimes GFX \arrow[rr, "\lambda_{GFX}"] \arrow[d, "\mathsf{n}_{I'} \otimes id_{GFX}"'] & & GFX \\
GI' \otimes GFX \arrow[r, "\mathsf{n}_{I',FX}"'] \arrow[d, "G\mathsf{m}_{I} \otimes id_{GFX}"'] & G(I' \triangleright FX) \arrow[ur, "G\lambda'_{FX}"] \arrow[d, "G(\mathsf{m}_{I} \triangleright id_{FX})"] & \\
GFI \otimes GFX \arrow[r, "\mathsf{n}_{FI,FX}"'] & G(FI \triangleright FX) \arrow[r, "G\mathsf{m}_{I,X}"'] & GF(I \otimes X) \arrow[uu, "GF\lambda_{X}"']
\end{tikzcd}
$$

Similarly, diagram (3) commutes as follows:

$$
\begin{tikzcd}
GFX \otimes I \arrow[rr, "\rho_{GFX}"] \arrow[d, "id_{GFX} \otimes \mathsf{n}_{I'}"'] & & GFX \\
GFX \otimes GI' \arrow[r, "\mathsf{n}_{FX,I'}"'] \arrow[d, "id_{GFX} \otimes G\mathsf{m}_{I}"'] & G(FX \triangleright I') \arrow[ur, "G\rho'_{FX}"] \arrow[d, "G(id_{FX} \otimes \mathsf{m}_{I})"] & \\
GFX \otimes GFI \arrow[r, "\mathsf{n}_{FX,FI}"'] & G(FX \triangleright FI) \arrow[r, "G\mathsf{m}_{X,I}"'] & GF(X \otimes I) \arrow[uu, "GF\rho_{X}"']
\end{tikzcd}
$$

2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal by definition and thus the following two diagrams commute.

$$
\begin{tikzcd}
X \otimes Y \arrow[r, equal] \arrow[d, "\eta_X \otimes \eta_Y"'] & X \otimes Y \arrow[d, "\eta_{X \otimes Y}"] \\
TX \otimes TY \arrow[r, "\mathsf{t}_{X,Y}"'] & T(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[rr, "\eta_I"] \arrow[dr, equal] & & TI \\
& I \arrow[ur, "\mathsf{t}_I"'] &
\end{tikzcd}
$$

3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ. Thus the following diagrams commute.

$$
\begin{tikzcd}
T^2X \otimes T^2Y \arrow[r, "\mathsf{t}_{TX,TY}"] \arrow[d, "\mu_X \otimes \mu_Y"'] & T(TX \otimes TY) \arrow[r, "T\mathsf{t}_{X,Y}"] & T^2(X \otimes Y) \arrow[d, "\mu_{X \otimes Y}"] \\
TX \otimes TY \arrow[rr, "\mathsf{t}_{X,Y}"'] & & T(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2I \arrow[r, "\mu_I"] & TI \\
TI \arrow[u, "T\mathsf{t}_I"] & I \arrow[l, "\mathsf{t}_I"] \arrow[u, "\mathsf{t}_I"']
\end{tikzcd}
$$

◀

However, the monad is not symmetric becauase the following diagram does not commute, for the lambek category $\mathcal{L}$ is not symmetric.

$$
\begin{tikzcd}
GFX \otimes GFY \arrow[r, "\mathsf{ex}_{GFX,GFY}"] \arrow[d, "\mathsf{n}_{FX,FY}"'] & GFY \otimes GFX \arrow[r, "\mathsf{n}_{FY,FX}"] & G(FY \triangleright FX) \arrow[d, "G\mathsf{m}_{Y,X}"] \\
G(FX \triangleright FY) \arrow[r, "G\mathsf{m}_{X,Y}"'] & GF(X \otimes Y) \arrow[r, "GF\mathsf{ex}_{X,Y}"'] & GF(Y \otimes X)
\end{tikzcd}
$$

▶ **Lemma 18.** *The monad on the symmetric monoidal closed category in a SMCC-Lambek model is strong.*

Proof. Let $F : \mathcal{C} \vdash \mathcal{L} : G$ be a SMCC-Lambek model, where $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is symmetric monoidal closed, $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ is a Lambek category, and (F, m) and (G, n) are monoidal functors. We have proved that the monad $(T = GF, \eta, \mu)$ is monoidal with the natural transformation $\mathsf{t}_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ and the morphism $\mathsf{t}_I : I \to TI$ defined as in Lemma 17.
We define the tensorial strength $\tau_{X,Y} : X \otimes TY \to T(X \otimes Y)$ as $\tau_{X,Y} = \mathsf{t}_{X,Y} \circ (\eta_X \otimes id_{TY})$.
Since η is a monoidal natural transformation, we have $\eta_I = G\mathsf{m}_I \circ \mathsf{n}_{I'}$. Therefore $\eta_I = \mathsf{t}_I$. Thus the following diagram commutes because T is monoidal, where the composition $\mathsf{t}_{I,X} \circ (\mathsf{t}_I \otimes id_{TX})$ is the definition of $\tau_{I,X}$. So the first triangle in Defition 13 commutes.

$$
\begin{tikzcd}
I \otimes TX \arrow[r, "\mathsf{t}_I \otimes id_{TX}"] \arrow[d, "\lambda_{TX}"'] & TI \otimes TX \arrow[d, "\mathsf{t}_{I,X}"] \\
TX & T(I \otimes X) \arrow[l, "T\lambda_X"]
\end{tikzcd}
$$

Similarly, by using the definition of τ, the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$
\begin{tikzcd}
X \otimes Y \arrow[r, "id_X \otimes \eta_Y"] \arrow[d, "\eta_{X \otimes Y}"'] \arrow[dr, "\eta_X \otimes \eta_Y"] & X \otimes TY \arrow[d, "\eta_X \otimes id_{TY}"] \\
T(X \otimes Y) & TX \otimes TY \arrow[l, "\mathsf{t}_{X,Y}"]
\end{tikzcd}
$$

The first pentagon in the definition commutes by the following commutative diagrams, because η are α natural transformations and T is monoidal:

$$
\begin{tikzcd}
(X \otimes Y) \otimes TZ \arrow[r, "\eta_{X \otimes Y} \otimes id_{TZ}"] \arrow[d, "\alpha_{X,Y,TZ}"'] \arrow[dr, "(\eta_X \otimes \eta_Y) \otimes id_{TZ}"] & T(X \otimes Y) \otimes TZ \arrow[r, "\mathsf{t}_{X \otimes Y,Z}"] & T((X \otimes Y) \otimes Z) \arrow[d, "T\alpha_{X,Y,Z}"] \\
X \otimes (Y \otimes TZ) \arrow[dr, "\eta_X \otimes (\eta_Y \otimes id_{TZ})"] \arrow[ddd, "id_X \otimes (\eta_Y \otimes id_{TZ})"'] & (TX \otimes TY) \otimes TZ \arrow[u, "\mathsf{t}_{X,Y} \otimes id_{TZ}"'] \arrow[d, "\alpha_{TX,TY,TZ}"] & T(X \otimes (Y \otimes Z)) \\
& TX \otimes (TY \otimes TZ) \arrow[ddr, "id_{TX} \otimes \mathsf{t}_{Y,Z}"] & \\
& & \\
X \otimes (TY \otimes TZ) \arrow[uur, "\eta_X \otimes id_{TY \otimes TZ}"] \arrow[r, "id_X \otimes \mathsf{t}_{Y,Z}"'] & X \otimes T(Y \otimes Z) \arrow[r, "\eta_X \otimes id_{T(Y \otimes Z)}"'] & TX \otimes T(Y \otimes Z) \arrow[uuu, "\mathsf{t}_{X,Y \otimes Z}"']
\end{tikzcd}
$$

The last diagram in the definition commtues by the following commutative diagram, because T is a monad, t is a natural transformation, and μ is a monoidal natural transformation:

$$\begin{tikzcd}
X \otimes T^2Y \arrow[r, "\eta_X \otimes id_{T^2Y}"] \arrow[d, "id_X \otimes \mu_Y"'] & TX \otimes T^2Y \arrow[rr, "\mathsf{t}_{X,TY}"] \arrow[d, equal] \arrow[dr, "T\eta_X \otimes id_{T^2Y}"] \arrow[ddl, "id_{TX} \otimes \mu_Y"'] & & T(X \otimes TY) \arrow[d, "T(\eta_X \otimes id_{TY})"] \\
X \otimes TY \arrow[d, "\eta_X \otimes id_{TY}"'] & TX \otimes T^2Y \arrow[dl, "id_{TX} \otimes \mu_Y"] & T^2X \otimes T^2Y \arrow[l, "\mu_X \otimes id_{T^2Y}"'] \arrow[r, "\mathsf{t}_{TX,TY}"] \arrow[dll, "\mu_X \otimes \mu_Y"] & T(TX \otimes TY) \arrow[d, "T\mathsf{t}_{X,Y}"] \\
TX \otimes TY \arrow[r, "\mathsf{t}_{X,Y}"'] & T(X \otimes Y) & & T^2(X \otimes Y) \arrow[ll, "\mu_{X \otimes Y}"]
\end{tikzcd}$$

◀

▶ **Lemma 19** ([?]). *Let $\mathcal{M}$ be a symmetric monoidal category and T be a strong monad on $\mathcal{M}$. Then T is symmetric iff it is commutative.*

▶ **Theorem 20.** *The monad on the SMCC in a SMCC-Lambek model is monoidal and non-commutative.*

▶ **Lemma 21.** *The comonad on the Lambek category in a SMCC-Lambek model is monoidal.*

Proof. We define the comonad S on the Lambek category $\mathcal{L}$ in the adjunction $F : C \vdash \mathcal{L} : G$ of a SMCC-Lambek model as $S = FG$. The two corresponding natural transformations $\varepsilon : S \to id_{\mathcal{L}}$ and $\delta : S \to S^2$ are defined as

$$\varepsilon_A : SA \to A \qquad\qquad \delta_A = F\eta_{GA} : SA \to S^2A$$

where ε is the counit and $\eta : id_{\mathcal{L}} \to GF$ is the unit of the adjunction, and (F, m) and (G, n) are monoidal functors. Thus, we have

$$\mathsf{s}_{A,B} = F\mathsf{n}_{A,B} \circ \mathsf{m}_{GA,GB} : SA \triangleright SB \to SA \triangleright SB \qquad\qquad \mathsf{s}_I = F\mathsf{n}_{I'} \circ \mathsf{m}_I : I' \to SI'$$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$\begin{tikzcd}
(SA \triangleright SB) \triangleright SC \arrow[r, "\alpha'_{SA,SB,SC}"] \arrow[d, "\mathsf{s}_{A,B} \triangleright id_{SC}"'] & SA \triangleright (SB \triangleright SC) \arrow[r, "id_{SA} \triangleright \mathsf{s}_{B,C}"] & SA \triangleright S(B \triangleright C) \arrow[d, "\mathsf{s}_{A,B \triangleright C}"] \\
S(A \triangleright B) \triangleright SC \arrow[r, "\mathsf{s}_{A \triangleright B,C}"'] & S((A \triangleright B) \triangleright C) \arrow[r, "S\alpha'_{A,B,C}"'] & S(A \triangleright (B \triangleright C))
\end{tikzcd}$$

$$\begin{tikzcd}
I' \triangleright SA \arrow[r, "\lambda'_{SA}"] \arrow[d, "\mathsf{s}_{I'} \triangleright id_{SA}"'] & SA \\
SI' \triangleright SA \arrow[r, "\mathsf{s}_{I',A}"'] & S(I' \triangleright A) \arrow[u, "S\lambda'_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
SA \triangleright I' \arrow[r, "\rho'_{SA}"] \arrow[d, "id'_{SA} \triangleright \mathsf{s}_{I'}"'] & SA \\
SA \triangleright SI' \arrow[r, "\mathsf{s}_{A,I'}"'] & S(A \triangleright I') \arrow[u, "S\rho'_A"']
\end{tikzcd}$$

2. ε is a monoidal natural transformation:

$$\begin{tikzcd}
SA \triangleright SB \arrow[r, "\mathsf{s}_{A,B}"] \arrow[d, "\varepsilon_A \triangleright \varepsilon_B"'] & S(A \triangleright B) \arrow[d, "\varepsilon_{A \triangleright B}"] \\
A \triangleright B \arrow[r, equal] & A \triangleright B
\end{tikzcd}
\qquad
\begin{tikzcd}
SI' \arrow[rr, "\varepsilon_{I'}"] & & I' \\
& I' \arrow[ul, "\mathsf{s}_{I'}"] \arrow[ur, equal] &
\end{tikzcd}$$

3. δ is a monoidal natural transformation:

$$\begin{tikzcd} SA \triangleright SA \arrow[rr, "\mathsf{s}_{A,B}"] \arrow[d, "\delta_A \triangleright \delta_B"'] & & S(A \triangleright B) \arrow[d, "\delta_{A\triangleright B}"] \\ S^2A \triangleright S^2B \arrow[r, "\mathsf{s}_{SA,SB}"] & S(SA \triangleright SB) \arrow[r, "S\mathsf{s}_{A,B}"] & S^2(A \triangleright B) \end{tikzcd} \qquad \begin{tikzcd} SI' \arrow[r, "\delta_{I'}"] & S^2I' \\ I' \arrow[u, "\mathsf{s}_{I'}"] \arrow[r, "\mathsf{s}_{I'}"'] & SI' \arrow[u, "S\mathsf{s}_{I'}"'] \end{tikzcd}$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma 17. We do not include the proof here for simplicity. ◀

The comonad S on the Lambek category $\mathcal{L}$ of the adjunction is clearly not symmetric because $\mathcal{L}$ is not. However, it is symmetric on the co-Eilenberg-Moore category of the comonad.

▶ **Definition 22.** Let (S, ε, δ) be a comonad on a category $\mathcal{L}$. Then the **co-Eilenberg-Moore category $\mathcal{L}^S$** of the comonad has

- as objects the S-coalgebras $(A, h_A : A \to SA)$, where A is an object in $\mathcal{L}$, s.t. the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "h_A"] \arrow[d, "h_A"'] & SA \arrow[d, "\delta_A"] \\ SA \arrow[r, "Sh_A"'] & S^2A \end{tikzcd} \qquad \begin{tikzcd} & SA \arrow[dr, "\varepsilon_A"] & \\ A \arrow[ur, "h_A"] \arrow[rr, equal] & & A \end{tikzcd}$$

- as morphisms the coalgebra morphisms, i.e. morphisms $f : (A, h_A) \to (B, h_B)$ between coalgebras s.t. the diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h_A"'] & B \arrow[d, "h_B"] \\ SA \arrow[r, "Sf"'] & SB \end{tikzcd}$$

▶ **Lemma 23.** *Given a SMCC-Lambek model $F : \mathcal{C} \dashv \mathcal{L} : G$ and the comonad S on $\mathcal{L}$, the co-Eilenberg-Moore category $\mathcal{L}^S$ of has an exchange natural transformation $\mathsf{ex}^S_{A,B} : A \triangleright B \to B \triangleright A$.*

Proof. We define the exchange $\mathsf{ex}^S_{A,B} : A \triangleright B \to B \triangleright A$ as

$$A \triangleright B \xrightarrow{h_A \triangleright h_B} FGA \triangleright FGB \xrightarrow{\mathsf{m}_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\mathsf{ex}_{GA,GB}} F(GB \otimes GA) \xrightarrow{F\mathsf{n}_{B,A}} FG(B \triangleright A) \xrightarrow{\varepsilon_{B\triangleright A}} B \triangleright A$$

in which (F, m) and (G, n) are monoidal functors, and ex is the exchange for $\mathcal{C}$. ex^S is a natural transformation because the following diagrams commute for morphisms $f : A \to A'$ and $g : B \to B'$:

$$\begin{tikzcd} A \triangleright B \arrow[r, "h_A \triangleright h_B"] \arrow[d, "f \triangleright g"'] & FGA \triangleright FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[d, "FGf \triangleright FGg"] & F(GA \otimes GB) \arrow[r, "F\mathsf{ex}_{A,B}"] \arrow[d, "F(Gf \otimes Gg)"] & F(GB \otimes GA) \arrow[r, "F\mathsf{n}_{B,A}"] \arrow[d, "F(Gg \otimes Gf)"] & FG(B \triangleright A) \arrow[r, "\varepsilon_{B\triangleright A}"] \arrow[d, "FG(g \triangleright f)"] & B \triangleright A \arrow[d, "g \triangleright f"] \\ A' \triangleright B' \arrow[r, "h_{A'} \triangleright h_{B'}"'] & FGA' \triangleright FGB' \arrow[r, "\mathsf{m}_{GA',GB'}"'] & F(GA' \otimes GB') \arrow[r, "F\mathsf{ex}_{A',B'}"'] & F(GB' \otimes GA') \arrow[r, "F\mathsf{n}_{B',A'}"'] & FG(B' \triangleright A') \arrow[r, "\varepsilon_{B'\triangleright A'}"'] & B' \triangleright A' \end{tikzcd}$$

◀

▶ **Lemma 24.** *The following diagrams commute in the co-Eilenberg-Moore category $\mathcal{L}^S$:*

$$
\begin{tikzcd}
F((GA\otimes GB)\otimes GC) \arrow[r, "F(\mathsf{n}_{A,B}\otimes id_{GC})"] \arrow[d, "F(\mathsf{ex}_{A,B}\otimes id_{GC})"'] & F(G(A\triangleright B)\otimes GC) \arrow[r, "F(\mathsf{ex}_{A,B}\otimes id_{GC})"] & FG((A\triangleright B)\triangleright C) \arrow[d, "\varepsilon_{(A\triangleright B)\triangleright C}"] \\
F(G(B\triangleright A)\otimes GC) \arrow[d, "F(\mathsf{n}_{B,A}\otimes id_{GC})"'] & & (A\triangleright B)\triangleright C \arrow[d, "\mathsf{ex}^S_{A,B}\triangleright id_C"] \\
F(G(B\triangleright A)\otimes GC) \arrow[r, "F\mathsf{n}_{B\triangleright A,C}"'] & FG((B\triangleright A)\triangleright C) \arrow[r, "\varepsilon_{(B\triangleright A)\triangleright C}"'] & (B\triangleright A)\triangleright C
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(GB\otimes (GC\otimes GA)) \arrow[r, "F(id_{GB}\otimes\mathsf{n}_{C,A})"] \arrow[d, "F(id_{GB}\otimes\mathsf{ex}_{C,A})"'] & F(GB\otimes G(C\triangleright A)) \arrow[r, "F\mathsf{n}_{B,C\triangleright A}"] & FG(B\triangleright (C\triangleright A)) \arrow[d, "\varepsilon_{B\triangleright(C\triangleright A)}"] \\
F(GB\otimes (GA\otimes GC)) \arrow[d, "F(id_{GB}\otimes\mathsf{n}_{A,C})"'] & & B\triangleright (C\triangleright A) \arrow[d, "id_A\triangleright\mathsf{ex}^S_{C,A}"] \\
F(GB\otimes G(A\triangleright C)) \arrow[r, "F\mathsf{n}_{B,A\triangleright C}"'] & FG(B\triangleright (A\triangleright C)) \arrow[r, "\varepsilon_{B\triangleright(A\triangleright C)}"'] & B\triangleright (A\triangleright C)
\end{tikzcd}
$$

Proof. We only write the proof for the first diagram. The proof for the second one is similar. (1), (2), (3)–naturality of m; (4)–F is monoidal; (5), (12)–ε is monoidal; (6), (7), (8), (9), (10)–obvious; (11)–coalgebra.

$$
\begin{tikzcd}
F(G(A\triangleright B)\otimes GC) \arrow[r, "F\mathsf{n}_{A\triangleright B,C}"] & FG((A\triangleright B)\triangleright C) \arrow[r, "\varepsilon_{(A\triangleright B)\triangleright C}"] \arrow[d, phantom, "(5)"] & (A\triangleright B)\triangleright C \arrow[r, equal] \arrow[d, phantom, "(6)"] & (A\triangleright B)\triangleright C \arrow[d, "(h_A\triangleright h_B)\triangleright id_C"] \arrow[dl, phantom, "(11)"] \\
F((GA\otimes GB)\otimes GC) \arrow[u, "F(\mathsf{n}_{A,B}\otimes id_{GC})"] \arrow[d, "F(\mathsf{ex}_{A,B}\otimes id_{GC})"'] \arrow[r, phantom, "(1)"] & FG(A\triangleright B)\triangleright FGC \arrow[ul, "\mathsf{m}_{G(A\triangleright B),GC}"'] \arrow[ur, "\varepsilon_{A\triangleright B}\triangleright\varepsilon_C"] \arrow[r, "id\triangleright\varepsilon_C"] & FG(A\triangleright B)\triangleright C \arrow[u, "\varepsilon_{A\triangleright B}\triangleright id_C"] \arrow[d, phantom, "(7)"] \arrow[r, phantom, "(12)"] & (FGA\triangleright FGB)\triangleright C \arrow[ul, "(\varepsilon_A\triangleright\varepsilon_B)\triangleright id_C"'] \arrow[d, "\mathsf{m}_{GA,GB}\triangleright id_C"] \\
F((GB\otimes GA)\otimes GC) \arrow[d, "F(\mathsf{n}_{B,A}\otimes id_{GC})"'] \arrow[r, phantom, "(2)"] & F(GA\otimes GB)\triangleright FGC \arrow[ul, "\mathsf{m}_{GA\otimes GB,GC}"'] \arrow[u, "F\mathsf{n}_{A,B}\triangleright id_{FGC}"] \arrow[rr, "id_{F(GA\otimes GB)}\triangleright\varepsilon_C"] \arrow[dr, "F\mathsf{ex}_{A,B}\triangleright id_{FGC}"] \arrow[dl, phantom, "(3)"] & & F(GA\otimes GB)\triangleright C \arrow[ul, "F\mathsf{n}_{A,B}\otimes id_C"'] \arrow[d, "F\mathsf{ex}_{A,B}\triangleright id_C"] \arrow[dl, phantom, "(8)"] \\
F(G(B\triangleright A)\otimes GC) \arrow[d, "F\mathsf{n}_{B\triangleright A,C}"'] \arrow[dr, phantom, "(4)"] & FG(B\triangleright A)\triangleright FGC \arrow[l, "\mathsf{m}_{G(B\triangleright A),GC}"'] \arrow[d, "\varepsilon_{B\triangleright A}\triangleright\varepsilon_C"] \arrow[drr, "id_{FG(B\triangleright A)}\triangleright\varepsilon_C"] \arrow[r, phantom, "(9)"] & F(GB\otimes GA)\triangleright FGC \arrow[ull, "\mathsf{m}_{GB\otimes GA,GC}"'] \arrow[l, "F\mathsf{n}_{B,A}\triangleright id"] \arrow[r, "id\triangleright\varepsilon_C"] & F(GB\otimes GA)\triangleright C \arrow[d, "F\mathsf{n}_{B,A}\triangleright id_C"] \arrow[dl, phantom, "(10)"] \\
FG((B\triangleright A)\triangleright C) \arrow[r, "\varepsilon_{(B\triangleright A)\triangleright C}"'] & (B\triangleright A)\triangleright C & & FG(B\triangleright A)\triangleright C \arrow[ll, "\varepsilon_{B\triangleright A}\triangleright id_C"]
\end{tikzcd}
$$

◀

▶ **Theorem 25.** *The co-Eilenberg-Moore category $\mathcal{L}^S$ of S is symmetric monoidal closed.*

Proof. Let $(\mathcal{L},\triangleright,I',\alpha',\lambda',\rho')$ be the Lambek cateogry in a SMCC-Lambek model and S be the comonad on $\mathcal{L}$. Since $\mathcal{L}$ is a Lambek category, it is obvious that $\mathcal{L}$ is also Lambek. By Corollary 6, we only need to prove the exchange defined in Lemma 23 satisfies the three commutative diagrams in Definition 3.

The first triangle in Definition 3 commutes as follows: (1)–coalgebra; (2)–ε is monoidal; (3)–naturality of ρ; (4)–naturality of ε; (5)–naturality of m; (6)–F is monoidal; (7)–C is symmetric;

(8)–naturality of ex; (9)–G is monoidal.

$$
\begin{tikzcd}
A \triangleright I' \arrow[rr, "h_A \triangleright h_{I'}"] \arrow[ddd, "\rho'_A"'] \arrow[ddr, equal] & {} \arrow[d, phantom, "(1)"] & FGA \triangleright FGI' \arrow[rr, "\mathsf{m}_{GA,GI'}"] \arrow[dl, "id_{FGA} \triangleright \varepsilon_{I'}"'] & {} \arrow[d, phantom, "(5)"] & F(GA \otimes GI') \arrow[dddd, "F\mathsf{ex}_{GA,GI'}"] \\
 & FGA \triangleright I' \arrow[d, "\varepsilon_{FGA} \triangleright id_{I'}"] \arrow[dr, equal] \arrow[r, phantom, "(2)"] & FGA \triangleright FI \arrow[u, "id_{FGA} \triangleright F\mathsf{n}_{I'}"'] \arrow[r, "\mathsf{m}_{GA,I}"] \arrow[dr, phantom, "(6)"] & F(GA \otimes I) \arrow[ur, "F(id_{GA} \otimes \mathsf{n}_{I'})"'] \arrow[ddl, "F\rho_{GA}"'] \arrow[dd, "F\mathsf{ex}_{GA,I}"] & \\
 & A \triangleright I' \arrow[d, phantom, "(3)"] & FGA \triangleright I' \arrow[u, "id_{FGA} \triangleright \mathsf{m}_I"'] \arrow[d, "\rho'_{FGA}"] \arrow[dr, phantom, "(7)"] & {} \arrow[r, phantom, "(8)"] & {} \\
A \arrow[d, "\lambda'_A"'] & {} \arrow[d, phantom, "(4)"] & FGA \arrow[ll, "\varepsilon_A"'] & F(I \otimes GA) \arrow[l, "F\lambda_{GA}"'] \arrow[dr, "F(\mathsf{n}_{I'} \otimes id_{GA})"] \arrow[d, phantom, "(9)"] & \\
I' \triangleright A & {} & FG(I' \triangleright A) \arrow[ll, "\varepsilon_{I' \triangleright A}"] \arrow[u, "FG\lambda'_A"'] & {} & F(GI' \otimes GA) \arrow[ll, "F\mathsf{n}_{I',A}"]
\end{tikzcd}
$$

The second triangle in the proof commutes as follows: (1) and (5)–coalgebra; (2) and (4)–ε is monoidal; (3)–C is symmetric.

$$
\begin{tikzcd}
A \triangleright B \arrow[r, "h_A \triangleright h_B"] \arrow[dd, equal] \arrow[dr, phantom, "(1)"] & FGA \triangleright FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[ddl, "\varepsilon_A \triangleright \varepsilon_B"'] & F(GA \otimes GB) \arrow[r, "F\mathsf{ex}_{A,B}"] \arrow[dd, equal] & F(GB \otimes GA) \arrow[rr, "F\mathsf{n}_{B,A}"] \arrow[dd, equal] & & FG(B \triangleright A) \arrow[dd, "\varepsilon_{B \triangleright A}"] \\
 & {} \arrow[r, phantom, "(2)"] & {} \arrow[r, phantom, "(3)"] & {} \arrow[r, phantom, "(4)"] & B \triangleright A \arrow[dr, equal] \arrow[dr, phantom, "(5)"'] & \\
A \triangleright B & FG(A \triangleright B) \arrow[l, "\varepsilon_{A \triangleright B}"] & F(GA \otimes GB) \arrow[l, "F\mathsf{n}_{A,B}"] & F(GB \otimes GA) \arrow[l, "F\mathsf{ex}_{B,A}"] & FGB \triangleright FGA \arrow[l, "\mathsf{m}_{GB,GA}"] \arrow[u, "\varepsilon_B \triangleright \varepsilon_A"] & B \triangleright A \arrow[l, "h_A \triangleright h_A"]
\end{tikzcd}
$$

The third diagram commutes as follows, which uses Lemma **??**.

$$\begin{tikzcd}
(A\triangleright B)\triangleright C \arrow[rr, "\alpha'_{A,B,C}"] \arrow[dddddd, "\mathsf{ex}^S_{A,B}\triangleright id_C"'] & & A\triangleright(B\triangleright C) \arrow[rr, "h_A\triangleright h_{B\triangleright C}"] \arrow[dr, equal] & & FGA\triangleright FG(B\triangleright C) \arrow[dl, "\varepsilon_A\triangleright\varepsilon_{B\triangleright C}"'] \arrow[dddd, "\mathsf{m}_{GA,G(B\triangleright C)}"] \\
& FG((A\triangleright B)\triangleright C) \arrow[ul, "\varepsilon_{(A\triangleright B)\triangleright C}"] \arrow[drr, "FG\alpha'_{A,B,C}"] & & A\triangleright(B\triangleright C) & \\
& F(G(A\triangleright B)\otimes GC) \arrow[u, "F\mathsf{n}_{A\triangleright B,C}"] & & FG(A\triangleright(B\triangleright C)) \arrow[u, "\varepsilon_{A\triangleright(B\triangleright C)}"] & \\
& F((GA\otimes GB)\otimes GC) \arrow[u, "F(\mathsf{n}_{A,B}\otimes id_{GC})"] \arrow[rr, "F\alpha_{GA,GB,GC}"] \arrow[d, "F(\mathsf{ex}_{A,B}\otimes id_{GC})"] & & F(GA\otimes(GB\otimes GC)) \arrow[d, "F\mathsf{ex}_{GA,GB\otimes GC}"] \arrow[dr, "F(id_{GA}\otimes\mathsf{n}_{B,C})"] & \\
& F((GB\otimes GA)\otimes GC) \arrow[dd, "F(\mathsf{n}_{B,A}\otimes id_{GC})"] \arrow[dddr, "F\alpha"] & & F((GB\otimes GC)\otimes GA) \arrow[d, "F\alpha_{GB,GC,GA}"] \arrow[ddr, "F(\mathsf{n}_{B,C}\otimes id_{GA})"] & F(GA\otimes G(B\triangleright C)) \arrow[uul, "F\mathsf{n}_{A,B\triangleright C}"'] \arrow[dd, "F\mathsf{ex}_{A,B\triangleright C}"] \\
& & & F(GB\otimes(GC\otimes GA)) \arrow[ddl, "F(id_{GC}\otimes\mathsf{ex}_{C,A})"'] \arrow[ddd, "F(id_{GB}\otimes\mathsf{n}_{C,A})"] & \\
(B\triangleright A)\triangleright C \arrow[ddddd, "\alpha'_{B,A,C}"'] & F(G(B\triangleright A)\otimes GC) \arrow[dd, "F\mathsf{n}_{B\triangleright A,C}"] & & & F(G(B\triangleright C)\otimes GA) \arrow[dddddd, "F\mathsf{n}_{B\triangleright C,A}"] \\
& & F(GB\otimes(GA\otimes GC)) \arrow[dd, "F(id_{GB}\otimes\mathsf{n}_{A,C})"] & & \\
& FG((B\triangleright A)\triangleright C) \arrow[uul, "\varepsilon_{(B\triangleright A)\triangleright C}"] & & F(GB\otimes G(C\triangleright A)) \arrow[dd, "F\mathsf{n}_{B,C\triangleright A}"] & \\
& & F(GB\otimes G(A\triangleright C)) \arrow[dd, "F\mathsf{n}_{B,A\triangleright C}"] & & \\
& & & FG(B\triangleright(C\triangleright A)) & \\
B\triangleright(A\triangleright C) \arrow[d, "\mathsf{id}_B\triangleright\mathsf{ex}^S_{A,C}"'] & & FG(B\triangleright(A\triangleright C)) \arrow[ll, "\varepsilon_{B\triangleright(A\triangleright C)}"'] & & \\
B\triangleright(C\triangleright A) & & (B\triangleright C)\triangleright A \arrow[ll, "\alpha'_{B,C,A}"] & & FG((B\triangleright C)\triangleright A) \arrow[ll, "\varepsilon_{(B\triangleright C)\triangleright A}"] \arrow[uul, "FG\alpha_{B,C,A}"']
\end{tikzcd}$$

◀

5 Non-Commutative Linear Logic

5.1 Term Assignment for Sequent Calculus

5.2 Term Assignment for Natural Deduction

The term assignment for natural deduction of the non-commutative part of the model, i.e. the SMCC of the adjunction, is defined in Figure 1. And the term assignme for the commutative part, i.e. the Lambek category of the adjunction, is defined in Figure 2. Ψ and Φ are contexts for the non-commutative part and they are lists. Γ and Δ are contexts for the commutative part and they are multisets, therefore the following exchange rules are implicit.

$$\frac{\Phi, x : X, y : Y, \Psi \vdash_{\mathcal{C}} t : Z}{\Phi, z : Y, w : X, \Psi \vdash_{\mathcal{C}} \mathsf{ex}\ w, z\ \mathsf{with}\ x, y\ \mathsf{in}\ t : Z}\ \textsc{T\_beta} \qquad \frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : Y, w : X, \Delta \vdash_{\mathcal{L}} \mathsf{ex}\ w, z\ \mathsf{with}\ x, y\ \mathsf{in}\ s : A}\ \textsc{S\_beta}$$

$$
\dfrac{}{x : X \vdash_{\mathcal{C}} x : X}\ \text{T\_ID}
\qquad
\dfrac{}{\cdot \vdash_{\mathcal{C}} \mathsf{trivT} : \mathsf{UnitT}}\ \text{T\_UNITI}
\qquad
\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : \mathsf{UnitT} \quad \Psi \vdash_{\mathcal{C}} t_2 : Y}{\Phi, \Psi \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : \mathsf{UnitT}\,\mathsf{be}\,\mathsf{trivT}\,\mathsf{in}\, t_2 : Y}\ \text{T\_UNITE}
$$

$$
\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \quad \Psi \vdash_{\mathcal{C}} t_2 : Y}{\Phi, \Psi \vdash_{\mathcal{C}} t_1 \otimes t_2 : X \otimes Y}\ \text{T\_TENI}
\qquad
\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \otimes Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_{\mathcal{C}} t_2 : Z}{\Psi_1, \Phi, \Psi_2 \vdash_{\mathcal{C}} \mathsf{let}\, t_1 : X \otimes Y\,\mathsf{be}\, x \otimes y\,\mathsf{in}\, t_2 : Z}\ \text{T\_TENE}
$$

$$
\dfrac{\Phi, x : X \vdash_{\mathcal{C}} t : Y}{\Phi \vdash_{\mathcal{C}} \lambda x : X.t : X \multimap Y}\ \text{T\_IMPI}
\qquad
\dfrac{\Phi \vdash_{\mathcal{C}} t_1 : X \multimap Y \quad \Psi \vdash_{\mathcal{C}} t_2 : X}{\Phi, \Psi \vdash_{\mathcal{C}} \mathsf{app}\, t_1\, t_2 : Y}\ \text{T\_IMPE}
\qquad
\dfrac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_{\mathcal{C}} \mathsf{G}s : \mathsf{G}A}\ \text{T\_GI}
$$

Figure 1 Commutative Part

$$
\dfrac{}{x : A \vdash_{\mathcal{L}} x : A}\ \text{S\_ID}
\qquad
\dfrac{}{\cdot \vdash_{\mathcal{L}} \mathsf{trivS} : \mathsf{UnitS}}\ \text{S\_UNITI}
\qquad
\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{UnitS}\,\mathsf{be}\,\mathsf{trivS}\,\mathsf{in}\, s_2 : A}\ \text{S\_UNITE1}
$$

$$
\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : \mathsf{UnitS} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : \mathsf{UnitS}\,\mathsf{be}\,\mathsf{trivS}\,\mathsf{in}\, s_2 : A}\ \text{S\_UNITE2}
\qquad
\dfrac{\Phi \vdash_{\mathcal{C}} t : \mathsf{UnitT} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi, \Gamma \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{UnitT}\,\mathsf{be}\,\mathsf{trivT}\,\mathsf{in}\, s : A}\ \text{S\_UNITE3}
$$

$$
\dfrac{\Phi \vdash_{\mathcal{C}} t : \mathsf{UnitT} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Gamma, \Phi \vdash_{\mathcal{L}} \mathsf{let}\, t : \mathsf{UnitT}\,\mathsf{be}\,\mathsf{trivT}\,\mathsf{in}\, s : A}\ \text{S\_UNITE4}
\qquad
\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Delta \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B}\ \text{S\_TENI}
$$

$$
\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\, s_1 : A \triangleright B\,\mathsf{be}\, x \triangleright y\,\mathsf{in}\, s_2 : C}\ \text{S\_TENE1}
\qquad
\dfrac{\Phi \vdash_{\mathcal{C}} t : X \otimes Y \quad \Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} \mathsf{let}\, t : X \otimes Y\,\mathsf{be}\, x \otimes y\,\mathsf{in}\, s : A}\ \text{S\_TENE2}
$$

$$
\dfrac{\Gamma, x : A \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_r x : A.s : A \rightharpoonup B}\ \text{S\_IMPRI}
\qquad
\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : A \rightharpoonup B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{app}_r\, s_1\, s_2 : B}\ \text{S\_IMPRE}
\qquad
\dfrac{x : A, \Gamma \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A.s : B \leftharpoonup A}\ \text{S\_IMPLI}
$$

$$
\dfrac{\Gamma \vdash_{\mathcal{L}} s_1 : B \leftharpoonup A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{app}_l\, s_1\, s_2 : B}\ \text{S\_IMPLE}
\qquad
\dfrac{\Phi \vdash_{\mathcal{C}} t : X}{\Phi \vdash_{\mathcal{L}} \mathsf{F}t : \mathsf{F}X}\ \text{S\_FI}
$$

$$
\dfrac{\Gamma \vdash_{\mathcal{L}} y : \mathsf{F}X \quad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \mathsf{let}\,\mathsf{F}x : \mathsf{F}X\,\mathsf{be}\, y\,\mathsf{in}\, s : A}\ \text{S\_FE}
\qquad
\dfrac{\Phi \vdash_{\mathcal{C}} t : \mathsf{G}A}{\Phi \vdash_{\mathcal{L}} \mathsf{derelict}\, t : A}\ \text{S\_GE}
$$

Figure 2 Non-Commutative Part

$$
\dfrac{\dfrac{\dfrac{}{z : \mathsf{FG}A \triangleright \mathsf{FG}B \vdash_{\mathcal{L}} z : \mathsf{FG}A \triangleright \mathsf{FG}B}\ \text{ID} \quad \dfrac{\dfrac{}{x_2 : \mathsf{FG}A \vdash_{\mathcal{L}} x_2 : \mathsf{FG}A}\ \text{ID} \quad \dfrac{\dfrac{}{y_2 : \mathsf{FG}B \vdash_{\mathcal{L}} y_2 : \mathsf{FG}B}\ \text{ID} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{y_0 : \mathsf{G}B \vdash_{\mathcal{C}} y_0 : \mathsf{G}B}\ \text{ID}}{y_0 : \mathsf{G}B \vdash_{\mathcal{L}} \mathsf{F}y_0 : \mathsf{FG}B}\ \text{FI} \quad \dfrac{\dfrac{}{x_0 : \mathsf{G}A \vdash_{\mathcal{C}} x_0 : \mathsf{G}A}\ \text{ID}}{x_0 : \mathsf{G}A \vdash_{\mathcal{L}} \mathsf{F}x_0 : \mathsf{FG}A}\ \text{FI}}{y_0 : \mathsf{G}B, x_0 : \mathsf{G}A \vdash_{\mathcal{L}} \mathsf{F}y_0 \triangleright \mathsf{F}x_0 : \mathsf{FG}B \triangleright \mathsf{FG}A}\ \text{FE}}{x_1 : \mathsf{G}A, y_1 : \mathsf{G}B \vdash_{\mathcal{L}} \mathsf{ex}\, y_1, x_1\,\mathsf{with}\, y_0, x_0\,\mathsf{in}\,(\mathsf{F}y_0 \triangleright \mathsf{F}x_0) : \mathsf{FG}B \triangleright \mathsf{FG}A}\ \text{BETA}}{x_1 : \mathsf{G}A, y_2 : \mathsf{FG}B \vdash_{\mathcal{L}} \mathsf{let}\,\mathsf{F}y_1 : \mathsf{FG}B\,\mathsf{be}\, y_2\,\mathsf{in}\,(\mathsf{ex}\, y_1, x_1\,\mathsf{with}\, y_0, x_0\,\mathsf{in}\,(\mathsf{F}y_0 \triangleright \mathsf{F}x_0)) : \mathsf{FG}B \triangleright \mathsf{FG}A}\ \text{FE}}{x_2 : \mathsf{FG}A, y_2 : \mathsf{FG}B \vdash_{\mathcal{L}} \mathsf{let}\,\mathsf{F}x_1 : \mathsf{FG}A\,\mathsf{be}\, x_2\,\mathsf{in}\,(\mathsf{let}\,\mathsf{F}y_1 : \mathsf{FG}B\,\mathsf{be}\, y_2\,\mathsf{in}\,(\mathsf{ex}\, y_1, x_1\,\mathsf{with}\, y_0, x_0\,\mathsf{in}\,(\mathsf{F}y_0 \triangleright \mathsf{F}x_0))) : \mathsf{FG}B \triangleright \mathsf{FG}A}\ \text{FE}}{z : \mathsf{FG}A \triangleright \mathsf{FG}B \vdash_{\mathcal{L}} \mathsf{let}\, z : \mathsf{FG}A \triangleright \mathsf{FG}B\,\mathsf{be}\, x_2 \triangleright y_2\,\mathsf{in}\,(\mathsf{let}\,\mathsf{F}x_1 : \mathsf{FG}A\,\mathsf{be}\, x_2\,\mathsf{in}\,(\mathsf{let}\,\mathsf{F}y_1 : \mathsf{FG}B\,\mathsf{be}\, y_2\,\mathsf{in}\,(\mathsf{ex}\, y_1, x_1\,\mathsf{with}\, y_0, x_0\,\mathsf{in}\,(\mathsf{F}y_0 \triangleright \mathsf{F}x_0)))) : \mathsf{FG}B \triangleright \mathsf{FG}A}\ \text{TENE2}}{\cdot \vdash_{\mathcal{L}} \lambda_r z : \mathsf{FG}A \triangleright \mathsf{FG}B.\mathsf{let}\, z : \mathsf{FG}A \triangleright \mathsf{FG}B\,\mathsf{be}\, x_2 \triangleright y_2\,\mathsf{in}\,(\mathsf{let}\,\mathsf{F}x_1 : \mathsf{FG}A\,\mathsf{be}\, x_2\,\mathsf{in}\,(\mathsf{let}\,\mathsf{F}y_1 : \mathsf{FG}B\,\mathsf{be}\, y_2\,\mathsf{in}\,(\mathsf{ex}\, y_1, x_1\,\mathsf{with}\, y_0, x_0\,\mathsf{in}\,(\mathsf{F}y_0 \triangleright \mathsf{F}x_0)))) : (\mathsf{FG}A \triangleright \mathsf{FG}B) \rightharpoonup (\mathsf{FG}B \triangleright \mathsf{FG}A)}\ \text{IMPRI}
$$

6 Applications

7 Related Work

TODO

8 Conclusion

TODO

A Appendix