On Linear Based Intuitionistic Substructural Logics

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- 1 Introduction
- 2 Main Ideas
- 3 Categorical Models
- ▶ **Definition 1.** A monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a category \mathcal{M} consists of
- a bifunctor \otimes : $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the tensor product;
- **a** an object *I*, called the unit object;
- three natural isomorphisms α, λ , and ρ with components

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$\lambda_A: I \otimes A \to A$$

$$\rho_A: A \otimes I \to A$$

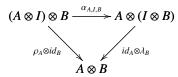
where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in \mathcal{M} :

$$((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha_{A,B,C} \otimes id_D} (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D)$$

$$\downarrow id_A \otimes \alpha_{B,C,D}$$

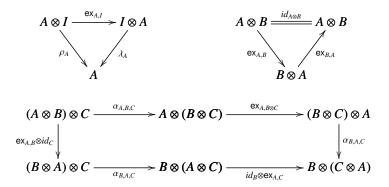
$$(A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C \otimes D}} A \otimes (B \otimes (C \otimes D))$$



▶ **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ equipped with two bifunctors \rightarrow : $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ and \leftarrow : $\mathcal{M} \times \mathcal{M}^{op} \to \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\operatorname{\mathsf{Hom}}_f(X \otimes A, B) \cong \operatorname{\mathsf{Hom}}_f(X, A \rightharpoonup B) \qquad \operatorname{\mathsf{Hom}}_f(A \otimes X, B) \cong \operatorname{\mathsf{Hom}}_f(X, B \leftharpoonup A)$$

▶ **Definition 3.** A **symmetric monoidal category** is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $ex_{A,B} : A \otimes B \to B \otimes A$, called exchange, such that the following diagrams commute:



- ▶ **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor \multimap : $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.
- ▶ **Lemma 5.** Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \rightarrow B) \cong (B \leftarrow A)$.

Proof. First, notice that for any object C we have

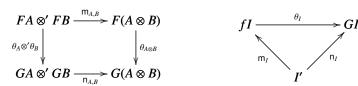
$$Hom[C, A \rightarrow B] \cong Hom[C \otimes A, B]$$
 \mathcal{L} is a Lambek category $\cong Hom[A \otimes C, B]$ By the exchange $ex_{C,A}$ $\cong Hom[C, B \leftarrow A]$ \mathcal{L} is a Lambek category

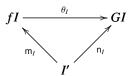
Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma.

- ▶ Corollary 6. A Lambek category with exchange is symmetric monoidal closed.
- ▶ **Definition 7.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from \mathcal{M} to \mathcal{M}' is a functor $F : \mathcal{M} \to \mathcal{M}'$ together with a morphism $\mathsf{m}_I : I' \to F(I)$ and a natural transformation $\mathsf{m}_{A,B} : FA' \otimes FB' \to F(A \otimes B)$, such that the following diagrams commute for any objects A, B, and C in \mathcal{M} :

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Definition 8. Let (F, m) and (G, n) be monoidal functors from a monoidal category \mathcal{M} to a monoidal category \mathcal{M}' . A monoidal natural transformation from (F, m) to (G, n) is a natural transformation $\theta: (F, m) \to (G, n)$ such that the following diagrams commute for any objects A and B in \mathcal{M} :





- ▶ **Definition 9.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \to \mathcal{M}$ \mathcal{M}' and $G: \mathcal{M}' \to \mathcal{M}$ be functors. The adjunction $F: \mathcal{M} + \mathcal{M}': G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit $\eta:id_{\mathcal{M}}\to GF$ and the counit $\varepsilon:FG\to id_{\mathcal{M}'}$ are monoidal natural transformations, where $id_{\mathcal{M}}$ and $id_{\mathcal{M}'}$ are the identity functors on \mathcal{M} and \mathcal{M}' respectively.
- ▶ **Lemma 10.** Let $(\mathcal{L}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal closed category and $(\mathcal{L}, \otimes', I', \alpha', \lambda', \rho')$ be a Lambek category. Since C and \mathcal{L} are both monoidal, let $F: C \to \mathcal{L}$ and $G: \mathcal{L} \to C$ be monoidal *functors. Then the adjunction* $F : \mathcal{M} \dashv \mathcal{M}' : G$ *is monoidal.*
- **Proof.** Proving the adjunction being monoidal is equivalent to proving the unit η and the counit ε are monoidal natural transformations. First, we show that η and ε are natural transformations, i.e. the following diagrams commute:
- ▶ **Lemma 11.** Let C be a category. A **monad** on C consists of an endofunctor $T: C \to C$ together with two natural transformations $\eta: id_C \to T$ and $\mu: T^2 \to id_C$, where id_C is the identity functor on C, such that the following diagrams commute:





- Logic
- **Applications**
- **Related Work**

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Conclusion

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Appendix