

Non-Commutative Linear Logic in an Adjoint Model

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Abstract

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1 Introduction

2 Related Work

3 Category Theory Basics

► **Definition 1.** A **monoidal category** $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ is a category \mathcal{M} consists of

- a bifunctor $\triangleright : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$, called the tensor product;
- an object I , called the unit object;
- three natural isomorphisms α , λ , and ρ with components

$$\alpha_{A,B,C} : (A \triangleright B) \triangleright C \rightarrow A \triangleright (B \triangleright C)$$

$$\lambda_A : I \triangleright A \rightarrow A$$

$$\rho_A : A \triangleright I \rightarrow A$$

where α is called associator, λ is left unitor, and ρ is right unitor,

such that the following diagrams commute for any objects A, B, C in \mathcal{M} :

$$\begin{array}{ccc} ((A \triangleright B) \triangleright C) \triangleright D & \xrightarrow{\alpha_{A,B,C} \triangleright id_D} & (A \triangleright (B \triangleright C)) \triangleright D \xrightarrow{\alpha_{A,B,C,D}} A \triangleright ((B \triangleright C) \triangleright D) \\ \downarrow \alpha_{A \triangleright B, C, D} & & \downarrow id_A \triangleright \alpha_{B, C, D} \\ (A \triangleright B) \triangleright (C \triangleright D) & \xrightarrow{\alpha_{A, B, C \triangleright D}} & A \triangleright (B \triangleright (C \triangleright D)) \end{array}$$

$$\begin{array}{ccc} (A \triangleright I) \triangleright B & \xrightarrow{\alpha_{A, I, B}} & A \triangleright (I \triangleright B) \\ \downarrow \rho_A \triangleright id_B & & \downarrow id_A \triangleright \lambda_B \\ A \triangleright B & & \end{array}$$

► **Definition 2.** A **Lambek category** (or a **biclosed monoidal category**) is a monoidal category $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ equipped with two bifunctors $\multimap : \mathcal{M}^{op} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\multimap : \mathcal{M} \times \mathcal{M}^{op} \rightarrow \mathcal{M}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:



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$$\text{Hom}_{\mathcal{L}}(X \triangleright A, B) \cong \text{Hom}_{\mathcal{L}}(X, A \multimap B)$$

$$\text{Hom}_{\mathcal{L}}(A \triangleright X, B) \cong \text{Hom}_{\mathcal{L}}(X, B \multimap A)$$

► **Definition 3.** A **symmetric monoidal category** (SMCC) is a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ together with a natural transformation with components $\text{ex}_{A,B} : A \otimes B \rightarrow B \otimes A$, called **exchange**, such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes I & \xrightarrow{\text{ex}_{A,I}} & I \otimes A \\ \rho_A \searrow & & \nearrow \lambda_A \\ & A & \end{array} \qquad \begin{array}{ccc} A \otimes B & \xrightarrow{id_{A \otimes B}} & A \otimes B \\ \text{ex}_{A,B} \searrow & & \nearrow \text{ex}_{B,A} \\ & B \otimes A & \end{array}$$

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\text{ex}_{A,B \otimes C}} & (B \otimes C) \otimes A \\ \downarrow \text{ex}_{A,B} \otimes id_C & & & & \downarrow \alpha_{B,A,C} \\ (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \text{ex}_{A,C}} & B \otimes (C \otimes A) \end{array}$$

We use \triangleright for non-symmetric monoidal categories while \otimes for symmetric ones.

► **Definition 4.** A **symmetric monoidal closed category** $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is a symmetric monoidal category equipped with a bifunctor $\multimap : \mathcal{M}^{op} \times \mathcal{M} \rightarrow \mathcal{M}$ that is right adjoint to the tensor product. That is, the following natural bijection $\text{Hom}_{\mathcal{M}}(X \otimes A, B) \cong \text{Hom}_{\mathcal{M}}(X, A \multimap B)$ holds.

► **Lemma 5.** Let A and B be two objects in a Lambek category with the exchange natural transformation. Then $(A \multimap B) \cong (B \multimap A)$.

Proof. First, notice that for any object C we have

$$\begin{aligned} \text{Hom}[C, A \multimap B] &\cong \text{Hom}[C \otimes A, B] && \mathcal{L} \text{ is a Lambek category} \\ &\cong \text{Hom}[A \otimes C, B] && \text{By the exchange } \text{ex}_{C,A} \\ &\cong \text{Hom}[C, B \multimap A] && \mathcal{L} \text{ is a Lambek category} \end{aligned}$$

Thus, $A \multimap B \cong B \multimap A$ by the Yoneda lemma. ◀

► **Corollary 6.** A Lambek category with exchange is symmetric monoidal closed.

► **Definition 7.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \triangleright', I', \alpha', \lambda', \rho')$ be monoidal categories. A **monoidal functor** (F, m) from \mathcal{M} to \mathcal{M}' is a functor $F : \mathcal{M} \rightarrow \mathcal{M}'$ together with a morphism $m_I : I' \rightarrow F(I)$ and a natural transformation $m_{A,B} : FA' \triangleright FB' \rightarrow F(A \triangleright B)$, such that the following diagrams commute for any objects A, B , and C in \mathcal{M} :

$$\begin{array}{ccccc} (FA \triangleright' FB) \triangleright' FC & \xrightarrow{\alpha'_{FA,FB,FC}} & FA \triangleright' (FB \triangleright' FC) & \xrightarrow{id_{FA} \triangleright' m_{A,B}} & FA \triangleright' F(B \triangleright C) \\ \downarrow m_{A,B} \triangleright' id_{FC} & & & & \downarrow m_{A,B \triangleright C} \\ F(A \triangleright B) \triangleright' FC & \xrightarrow{m_{A \triangleright B, C}} & F((A \triangleright B) \triangleright C) & \xrightarrow{F\alpha_{A,B,C}} & F(A \triangleright (B \triangleright C)) \end{array}$$

$$\begin{array}{ccc} I' \triangleright' FA & \xrightarrow{\lambda'_{FA}} & FA \\ \downarrow m_I \triangleright id_{FA} & & \uparrow F\lambda_A \\ FI \triangleright' FA & \xrightarrow{m_{I,A}} & F(I \triangleright A) \end{array} \qquad \begin{array}{ccc} FA \triangleright' I' & \xrightarrow{\rho'_{FA}} & FA \\ \downarrow id_{FA} \triangleright m_I & & \uparrow F\rho_A \\ FA \triangleright' FI & \xrightarrow{m_{A,I}} & F(A \triangleright I) \end{array}$$

► **Definition 8.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$ be symmetric monoidal categories. A **symmetric monoidal functor** $F : \mathcal{M} \rightarrow \mathcal{M}'$ is a monoidal functor (F, m) that satisfies the following coherence diagram:

$$\begin{array}{ccc} FA \otimes' FB & \xrightarrow{\theta_{FA, FB}} & FB \otimes' FA \\ \downarrow m_{A, B} & & \downarrow m_{B, A} \\ F(A \otimes B) & \xrightarrow{F\theta_{A, B}} & F(B \otimes A) \end{array}$$

► **Definition 9.** An **adjunction** between categories \mathcal{C} and \mathcal{D} consists of two functors $F : \mathcal{D} \rightarrow \mathcal{C}$, called the **left adjoint**, and $G : \mathcal{C} \rightarrow \mathcal{D}$, called the **right adjoint**, and two natural transformations $\eta : id_{\mathcal{D}} \rightarrow GF$, called the **unit**, and $\varepsilon : FG \rightarrow id_{\mathcal{C}}$, called the **counit**, such that the following diagrams commute for any object A in \mathcal{C} and B in \mathcal{D} :

$$\begin{array}{ccc} FB & \xrightarrow{F\eta_B} & FGFB \\ \parallel & \searrow \varepsilon_{FB} & \nearrow \\ FB & & FB \end{array} \quad \begin{array}{ccc} GA & \xrightarrow{\eta_{GA}} & GFGA \\ \parallel & \searrow G\varepsilon_A & \nearrow \\ GA & & GA \end{array}$$

► **Definition 10.** Let (F, m) and (G, n) be monoidal functors from a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ to a monoidal category $(\mathcal{M}', \otimes', I', \alpha', \lambda', \rho')$. A **monoidal natural transformation** from (F, m) to (G, n) is a natural transformation $\theta : (F, m) \rightarrow (G, n)$ such that the following diagrams commute for any objects A and B in \mathcal{M} :

$$\begin{array}{ccc} FA \triangleright' FB & \xrightarrow{m_{A, B}} & F(A \triangleright B) \\ \downarrow \theta_A \triangleright' \theta_B & & \downarrow \theta_{A \triangleright B} \\ GA \triangleright' GB & \xrightarrow{n_{A, B}} & G(A \triangleright B) \end{array} \quad \begin{array}{ccc} FI & \xrightarrow{\theta_I} & GI \\ \downarrow m_I & \searrow n_I & \nearrow \\ I' & & I' \end{array}$$

► **Definition 11.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ and $(\mathcal{M}', \triangleright', I', \alpha', \lambda', \rho')$ be monoidal categories, $F : \mathcal{M} \rightarrow \mathcal{M}'$ and $G : \mathcal{M}' \rightarrow \mathcal{M}$ be functors. The adjunction $F : \mathcal{M} \dashv \mathcal{M}' : G$ is a **monoidal adjunction** if F and G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

► **Definition 12.** Let \mathcal{C} be a category. A **monad** on \mathcal{C} consists of an endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ together with two natural transformations $\eta : id_{\mathcal{C}} \rightarrow T$ and $\mu : T^2 \rightarrow T$, where $id_{\mathcal{C}}$ is the identity functor on \mathcal{C} , such that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow \mu_T & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} \quad \begin{array}{ccc} T & \xrightarrow{\eta_T} & T^2 \\ \downarrow T\eta & \searrow \mu & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

► **Definition 13.** Let $(\mathcal{M}, \triangleright, I, \alpha, \lambda, \rho)$ be a monoidal category and (T, η, μ) be a monad on \mathcal{M} . T is a **strong monad** if there is natural transformation τ , called the **tensorial strength**, with components

$\tau_{A,B} : A \triangleright TB \rightarrow T(A \triangleright B)$ such that the following diagrams commute:

$$\begin{array}{ccc}
 I \triangleright TA & \xrightarrow{\tau_{I,A}} & T(I \triangleright A) \\
 \searrow \lambda_{TA} & & \swarrow T\lambda_A \\
 & TA &
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \triangleright B & \xrightarrow{id_A \triangleright \eta_B} & A \triangleright TB \\
 \searrow \eta_{A \triangleright B} & & \swarrow \tau_{A,B} \\
 & T(A \triangleright B) &
 \end{array}$$

$$\begin{array}{ccc}
 (A \triangleright B) \triangleright TC & \xrightarrow{\tau_{A \triangleright B, C}} & T((A \triangleright B) \triangleright C) \\
 \downarrow \alpha_{A,B,TC} & & \downarrow T\alpha_{A,B,C} \\
 A \triangleright (B \triangleright TC) & \xrightarrow{id_A \triangleright \tau_{B,C}} A \triangleright T(B \triangleright C) \xrightarrow{\tau_{A, B \triangleright C}} & T(A \triangleright (B \triangleright C))
 \end{array}$$

$$\begin{array}{ccc}
 A \triangleright T^2 B & \xrightarrow{\tau_{A, TB}} T(A \triangleright TB) \xrightarrow{T\tau_{A,B}} & T^2(A \triangleright B) \\
 \downarrow id_A \triangleright \mu_B & & \downarrow \mu_{A \triangleright B} \\
 A \triangleright TB & \xrightarrow{\tau_{A,B}} & T(A \triangleright B)
 \end{array}$$

► **Definition 14.** Let $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ be a symmetric monoidal category with exchange ex , and (T, η, μ) be a strong monad on \mathcal{M} . Then there is a “twisted” tensorial strength $\tau'_{A,B} : TA \otimes B \rightarrow T(A \otimes B)$ defined as $\tau'_{A,B} = T\text{ex} \circ \tau_{B,A} \circ \text{ex}$. We can construct a pair of natural transformations Φ, Φ' with components $\Phi_{A,B}, \Phi'_{A,B} : TA \otimes TB \rightarrow T(A \otimes B)$ defined as $\Phi_{A,B} = \mu_{A \otimes B} \circ T\tau'_{A,B} \circ \tau_{TA,B}$ and $\Phi'_{A,B} = \mu_{A \otimes B} \circ T\tau_{A,B} \circ \tau'_{A,TB}$. If $\Phi = \Phi'$, then the monad T is **commutative**.

► **Definition 15.** Let \mathcal{L} be a category. A **comonad** on \mathcal{L} consists of an endofunctor $S : \mathcal{L} \rightarrow \mathcal{L}$ together with two natural transformations $\varepsilon : S \rightarrow id_{\mathcal{L}}$ and $\delta : S^2 \rightarrow S$ such that the following diagrams commute:

$$\begin{array}{ccc}
 S & \xrightarrow{\delta} & S^2 \\
 \delta \downarrow & & \downarrow S\delta \\
 S^2 & \xrightarrow{\delta_S} & S^3
 \end{array}
 \qquad
 \begin{array}{ccc}
 S^2 & \xrightarrow{S\varepsilon} & S \\
 \varepsilon_S \downarrow & & \downarrow \delta \\
 S & \xrightarrow{\delta} & S^2
 \end{array}$$

4 An Adjoint Model

Our adjoint model, SMCC-Lambek model, has a similar structure as Benton’s LNL model [1]. Benton’s LNL model consists of a symmetric monoidal adjunction $F : C \dashv \mathcal{L} : G$ between a cartesian closed category C and a symmetric monoidal closed category \mathcal{L} .

► **Definition 16.** A **SMCC-Lambek model** consists of

- a symmetric monoidal closed category $(C, \otimes, I, \alpha, \lambda, \rho)$;
- a Lambek category $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$;
- a monoidal adjunction $F : C \dashv \mathcal{L} : G$, where $F : C \rightarrow \mathcal{L}$ and $G : \mathcal{L} \rightarrow C$ are monoidal functors.

Thus, in a SMCC-Lambek model, the following four diagrams commute because η and ε are monoidal natural transformations:

$$\begin{array}{ccc}
 X \otimes Y & \xrightarrow{id_{X \otimes Y}} & X \otimes Y \\
 \eta_X \otimes \eta_Y \downarrow & & \downarrow \eta_{X \otimes Y} \\
 GFX \otimes GFY & \xrightarrow{\eta_{FX, FY}} G(FX \otimes FY) \xrightarrow{Gm_{X,Y}} & GF(X \otimes Y)
 \end{array}
 \qquad
 \begin{array}{ccc}
 I & \xrightarrow{\eta_I} & GFI \\
 \parallel & & \uparrow Gm_I \\
 I & \xrightarrow{\eta_{I'}} & GI'
 \end{array}$$

$$\begin{array}{ccc}
 FGA \otimes FGB & \xrightarrow{m_{GA,GB}} & F(GA \otimes GB) \xrightarrow{F\eta_{A,B}} FG(A \otimes B) \\
 \downarrow \varepsilon_A \otimes \varepsilon_B & & \downarrow \varepsilon_{A \otimes B} \\
 A \otimes B & \xlongequal{\quad} & A \otimes B
 \end{array}
 \quad
 \begin{array}{ccc}
 FGI' & \xrightarrow{\varepsilon_{I'}} & I' \\
 \uparrow F\eta_{I'} & & \parallel \\
 FI & \xleftarrow{m_I} & I'
 \end{array}$$

And the following two diagrams commute because of the adjunction:

$$\begin{array}{ccc}
 FX & \xrightarrow{F\eta_X} & FGFX \\
 \parallel & \searrow \varepsilon_{FX} & \\
 FX & &
 \end{array}
 \quad
 \begin{array}{ccc}
 GA & \xrightarrow{\eta_{GA}} & GFGA \\
 \parallel & \searrow G\varepsilon_A & \\
 GA & &
 \end{array}$$

Following the tradition, we use letters X, Y, Z for objects in \mathcal{C} and A, B, C for objects in \mathcal{L} . The following lemmas and theorems establish the essential properties of the monad and the comonad derived from the adjunction.

► **Lemma 17.** *The monad on the symmetric monoidal closed category \mathcal{C} in a SMCC-Lambek model is monoidal.*

Proof. We define the monad T on the \mathcal{C} in the adjunction of a SMCC-Lambek model as $T = GF$, and the two corresponding natural transformations $\eta : id_{\mathcal{C}} \rightarrow T$ and $\mu : T^2 \rightarrow T$ as

$$\eta_X : X \rightarrow GFX \quad \mu_X = G\varepsilon_{FX} : GFGFX \rightarrow GFX$$

where η is the unit and $\varepsilon : FG \rightarrow id_{\mathcal{L}}$ is the counit of the adjunction $F : \mathcal{C} \dashv \mathcal{L} : G$. Since the adjunction is monoidal, then (F, m) and (G, n) are monoidal functors. Thus, we have

$$t_{X,Y} = Gm_{X,Y} \circ n_{FX,FY} : TX \otimes TY \rightarrow T(X \otimes Y) \quad t_I = Gm_I \circ n_{I'} : I \rightarrow TI$$

The monad T being monoidal means

1. T is a monoidal functor, i.e. the following diagrams commute:

$$\begin{array}{ccccc}
 (TX \otimes TY) \otimes TZ & \xrightarrow{\alpha_{TX,TY,TZ}} & TX \otimes (TY \otimes TZ) & \xrightarrow{id_{TX} \otimes t_{YZ}} & TX \otimes T(Y \otimes Z) \\
 \downarrow t_{X,Y} \otimes id_{TZ} & & (1) & & \downarrow t_{X,Y \otimes Z} \\
 T(X \otimes Y) \otimes TZ & \xrightarrow{t_{X \otimes Y, Z}} & T((X \otimes Y) \otimes Z) & \xrightarrow{T\alpha_{X,Y,Z}} & T(X \otimes (Y \otimes Z))
 \end{array}$$

$$\begin{array}{ccc}
 I \otimes TX & \xrightarrow{\lambda_{TX}} & TX \\
 \downarrow t_I \otimes id_{TX} & & \uparrow T\lambda_X \\
 TI \otimes TX & \xrightarrow{t_{I,X}} & T(I \otimes X)
 \end{array}
 \quad
 \begin{array}{ccc}
 TX \otimes I & \xrightarrow{\rho_{TX}} & TX \\
 \downarrow id_{TX} \otimes t_I & & \uparrow T\rho_X \\
 TX \otimes TI & \xrightarrow{t_{X,I}} & T(X \otimes I)
 \end{array}$$

We write GF instead of T in the proof for clarity.

By replacing $t_{X,Y}$ with its definition, diagram (1) above commutes by the following commutative diagram, in which the two hexagons commute because G and F are monoidal functors, and the

two quadrilaterals commute by the naturality of n .

$$\begin{array}{ccccc}
 (GFX \otimes GFY) \otimes GFZ & \xrightarrow{\alpha_{GFX,GFY,GFZ}} & GFX \otimes (GFY \otimes GFZ) & \xrightarrow{id_{GFX} \otimes n_{FY,FZ}} & GFX \otimes G(FY \triangleright FZ) \\
 \downarrow n_{FX,FY} \otimes id_{GFZ} & & \downarrow n_{FX,FY \triangleright FZ} & & \downarrow id_{GFX} \otimes Gm_{YZ} \\
 G(FX \triangleright FY) \otimes GFZ & & G(FX \triangleright (FY \triangleright FZ)) & & GFX \otimes GF(Y \otimes Z) \\
 \downarrow Gm_{X,Y} \otimes id_{GFZ} & \swarrow n_{FX \triangleright FY, FZ} & \uparrow G\alpha'_{FX,FY,FZ} & \searrow G(id_{FX} \triangleright m_{YZ}) & \downarrow n_{FX, F(Y \otimes Z)} \\
 GF(X \otimes Y) \otimes GFZ & & G((FX \triangleright FY) \triangleright FZ) & & G(FX \triangleright F(Y \otimes Z)) \\
 \downarrow n_{F(X \otimes Y), FZ} & \swarrow G(m_{X,Y} \otimes id_{FZ}) & \downarrow G\alpha'_{FX,FY,FZ} & \searrow Gm_{X,Y \otimes Z} & \downarrow Gm_{X,Y \otimes Z} \\
 G(F(X \otimes Y) \triangleright FZ) & \xrightarrow{Gm_{X \otimes Y, Z}} & GF((X \otimes Y) \otimes Z) & \xrightarrow{GF\alpha_{X,Y,Z}} & GF(X \otimes (Y \otimes Z))
 \end{array}$$

Diagram (2) commutes by the following commutative diagrams, in which the top quadrilateral commutes because G is monoidal, the right quadrilateral commutes because F is monoidal, and the left square commutes by the naturality of n .

$$\begin{array}{ccc}
 I \otimes GFX & \xrightarrow{\lambda_{GFX}} & GFX \\
 \downarrow n_{I'} \otimes id_{GFX} & & \downarrow G\lambda'_{FX} \\
 G'I' \otimes GFX & \xrightarrow{n_{I',FX}} & G(I' \triangleright FX) \\
 \downarrow Gm_I \otimes id_{GFX} & & \downarrow G(m_I \triangleright id_{FX}) \\
 GF'I' \otimes GFX & \xrightarrow{n_{FI,FX}} & G(FI \triangleright FX) \\
 & \searrow Gm_{I,X} & \uparrow GF\lambda_X \\
 & & GF(I \otimes X)
 \end{array}$$

Similarly, diagram (3) commutes as follows:

$$\begin{array}{ccc}
 GFX \otimes I & \xrightarrow{\rho_{GFX}} & GFX \\
 \downarrow id_{GFX} \otimes n_{I'} & & \downarrow G\rho'_{FX} \\
 GFX \otimes G'I' & \xrightarrow{n_{FX,I'}} & G(FX \triangleright I') \\
 \downarrow id_{GFX} \otimes Gm_I & & \downarrow G(id_{FX} \otimes m_I) \\
 GFX \otimes GF'I' & \xrightarrow{n_{FX,FI}} & G(FX \triangleright FI) \\
 & \searrow Gm_{X,I} & \uparrow GF\rho_X \\
 & & GF(X \otimes I)
 \end{array}$$

2. η is a monoidal natural transformation. In fact, since η is the unit of the monoidal adjunction, η is monoidal by definition and thus the following two diagrams commute.

$$\begin{array}{ccc}
 X \otimes Y & \xlongequal{\quad} & X \otimes Y \\
 \downarrow \eta_X \otimes \eta_Y & & \downarrow \eta_{X \otimes Y} \\
 TX \otimes TY & \xrightarrow{t_{X,Y}} & T(X \otimes Y)
 \end{array}
 \quad
 \begin{array}{ccc}
 I & \xrightarrow{\eta_I} & TI \\
 \parallel & \nearrow t_I & \\
 I & &
 \end{array}$$

3. μ is a monoidal natural transformation. It is obvious that since $\mu = G\varepsilon_{FA}$ and ε is monoidal, so is μ . Thus the following diagrams commute.

$$\begin{array}{ccc}
 T^2X \otimes T^2Y & \xrightarrow{t_{TX,TY}} & T(TX \otimes TY) \xrightarrow{Tt_{X,Y}} T^2(X \otimes Y) \\
 \downarrow \mu_X \otimes \mu_Y & & \downarrow \mu_{X \otimes Y} \\
 TX \otimes TY & \xrightarrow{t_{X,Y}} & T(X \otimes Y)
 \end{array}
 \quad
 \begin{array}{ccc}
 T^2I & \xrightarrow{\mu_I} & TI \\
 \uparrow Tt_I & & \uparrow t_I \\
 TI & \xleftarrow{t_I} & I
 \end{array}$$

However, the monad is not symmetric because the following diagram does not commute, for the lambek category \mathcal{L} is not symmetric.

$$\begin{array}{ccccc}
 GFX \otimes GFY & \xrightarrow{\Theta_{GFX,GFY}} & GFY \otimes GFX & \xrightarrow{\eta_{FY,FX}} & G(FY \triangleright FX) \\
 \downarrow \eta_{FX,FY} & & & & \downarrow Gm_{Y,X} \\
 G(FX \triangleright FY) & \xrightarrow{Gm_{X,Y}} & GF(X \otimes Y) & \xrightarrow{GF\Theta_{X,Y}} & GF(Y \otimes X)
 \end{array}$$

► **Lemma 18.** *The monad on the symmetric monoidal closed category in a SMCC-Lambek model is strong.*

Proof. Let $F : C \vdash \mathcal{L} : G$ be a SMCC-Lambek model, where $(C, \otimes, I, \alpha, \lambda, \rho)$ is symmetric monoidal closed, $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ is a Lambek category, and (F, m) and (G, n) are monoidal functors. We have proved that the monad $(T = GF, \eta, \mu)$ is monoidal with the natural transformation $t_{X,Y} : TX \otimes TY \rightarrow T(X \otimes Y)$ and the morphism $t_I : I \rightarrow TI$ defined as in Lemma 17.

We define the tensorial strength $\tau_{X,Y} : X \otimes TY \rightarrow T(X \otimes Y)$ as $\tau_{X,Y} = t_{X,Y} \circ (\eta_X \otimes id_{TY})$.

Since η is a monoidal natural transformation, we have $\eta_I = Gm_I \circ \eta_{I'}$. Therefore $\eta_I = t_I$. Thus the following diagram commutes because T is monoidal, where the composition $t_{I,X} \circ (t_I \otimes id_{TX})$ is the definition of $\tau_{I,X}$. So the first triangle in Definition 13 commutes.

$$\begin{array}{ccc}
 I \otimes TX & \xrightarrow{t_I \otimes id_{TX}} & TI \otimes TX \\
 \downarrow \lambda_{TX} & & \downarrow t_{I,X} \\
 TX & \xleftarrow{T\lambda_X} & T(I \otimes X)
 \end{array}$$

Similarly, by using the definition of τ , the the second triangle in the definition is equivalent to the following diagram, which commutes because η is a monoidal natural transformation:

$$\begin{array}{ccc}
 X \otimes Y & \xrightarrow{id_X \otimes \eta_Y} & X \otimes TY \\
 \downarrow \eta_{X \otimes Y} & \searrow \eta_X \otimes \eta_Y & \downarrow \eta_X \otimes id_{TY} \\
 T(X \otimes Y) & \xleftarrow{t_{X,Y}} & TX \otimes TY
 \end{array}$$

The first pentagon in the definition commutes by the following commutative diagrams, because η are α natural transformations and T is monoidal:

$$\begin{array}{ccccc}
 (X \otimes Y) \otimes TZ & \xrightarrow{\eta_{X \otimes Y} \otimes id_{TZ}} & T(X \otimes Y) \otimes TZ & \xrightarrow{t_{X \otimes Y, Z}} & T((X \otimes Y) \otimes Z) \\
 \downarrow \alpha_{X,Y,TZ} & \searrow (\eta_X \otimes \eta_Y) \otimes id_{TZ} & \uparrow t_{X,Y} \otimes id_{TZ} & & \downarrow T\alpha_{X,Y,Z} \\
 X \otimes (Y \otimes TZ) & & (TX \otimes TY) \otimes TZ & & T(X \otimes (Y \otimes Z)) \\
 \downarrow id_X \otimes (\eta_Y \otimes id_{TZ}) & \searrow \eta_X \otimes (\eta_Y \otimes id_{TZ}) & \downarrow \alpha_{TX,TY,TZ} & & \uparrow t_{X,Y \otimes Z} \\
 X \otimes (TY \otimes TZ) & & TX \otimes (TY \otimes TZ) & & \\
 \downarrow id_X \otimes t_{Y,Z} & \searrow \eta_X \otimes id_{TY \otimes TZ} & \downarrow id_{TX} \otimes t_{Y,Z} & & \\
 X \otimes T(Y \otimes Z) & \xrightarrow{id_X \otimes t_{Y,Z}} & X \otimes T(Y \otimes Z) & \xrightarrow{\eta_X \otimes id_{T(Y \otimes Z)}} & TX \otimes T(Y \otimes Z)
 \end{array}$$

The last diagram in the definition commutes by the following commutative diagram, because T is a monad, \mathfrak{t} is a natural transformation, and μ is a monoidal natural transformation:

$$\begin{array}{ccccc}
 X \otimes T^2 Y & \xrightarrow{\eta_X \otimes id_{T^2 Y}} & TX \otimes T^2 Y & \xrightarrow{\mathfrak{t}_{X, TY}} & T(X \otimes TY) \\
 id_X \otimes \mu_Y \downarrow & & \parallel & \swarrow T\eta_X \otimes id_{T^2 Y} & \downarrow T(\eta_X \otimes id_{TY}) \\
 X \otimes TY & \xrightarrow{id_{TX} \otimes \mu_Y} & TX \otimes T^2 Y & \xleftarrow{\mu_X \otimes id_{T^2 Y}} & T^2 X \otimes T^2 Y \xrightarrow{\mathfrak{t}_{TX, TY}} T(TX \otimes TY) \\
 \eta_X \otimes id_{TY} \downarrow & \swarrow id_{TX} \otimes \mu_Y & \swarrow \mu_X \otimes \mu_Y & & \downarrow T\mathfrak{t}_{X, Y} \\
 TX \otimes TY & \xrightarrow{\mathfrak{t}_{X, Y}} & T(X \otimes Y) & \xleftarrow{\mu_{X \otimes Y}} & T^2(X \otimes Y)
 \end{array}$$

► **Lemma 19** ([?]). *Let \mathcal{M} be a symmetric monoidal category and T be a strong monad on \mathcal{M} . Then T is symmetric iff it is commutative.*

► **Theorem 20.** *The monad on the SMCC in a SMCC-Lambek model is monoidal and non-commutative.*

► **Lemma 21.** *The comonad on the Lambek category in a SMCC-Lambek model is monoidal.*

Proof. We define the comonad S on the Lambek category \mathcal{L} in the adjunction $F : \mathcal{C} \vdash \mathcal{L} : G$ of a SMCC-Lambek model as $S = FG$. The two corresponding natural transformations $\varepsilon : S \rightarrow id_{\mathcal{L}}$ and $\delta : S \rightarrow S^2$ are defined as

$$\varepsilon_A : SA \rightarrow A \quad \delta_A = F\eta_{GA} : SA \rightarrow S^2 A$$

where ε is the counit and $\eta : id_{\mathcal{L}} \rightarrow GF$ is the unit of the adjunction, and (F, m) and (G, n) are monoidal functors. Thus, we have

$$s_{A,B} = F\eta_{A,B} \circ m_{GA,GB} : SA \triangleright SB \rightarrow SA \triangleright SB \quad s_I = F\eta_{I'} \circ m_I : I' \rightarrow SI'$$

The comonad S being monoidal means

1. S is a monoidal functor, i.e. the following diagrams commute:

$$\begin{array}{ccccc}
 (SA \triangleright SB) \triangleright SC & \xrightarrow{\alpha'_{SA,SB,SC}} & SA \triangleright (SB \triangleright SC) & \xrightarrow{id_{SA} \triangleright s_{B,C}} & SA \triangleright S(B \triangleright C) \\
 s_{A,B} \triangleright id_{SC} \downarrow & & & & \downarrow s_{A,B \triangleright C} \\
 S(A \triangleright B) \triangleright SC & \xrightarrow{s_{A \triangleright B, C}} & S((A \triangleright B) \triangleright C) & \xrightarrow{S\alpha'_{A,B,C}} & S(A \triangleright (B \triangleright C))
 \end{array}$$

$$\begin{array}{ccc}
 I' \triangleright SA & \xrightarrow{\lambda'_{SA}} & SA \\
 s_{I'} \triangleright id_{SA} \downarrow & & \uparrow S\lambda'_A \\
 SI' \triangleright SA & \xrightarrow{s_{I', A}} & S(I' \triangleright A)
 \end{array}$$

$$\begin{array}{ccc}
 SA \triangleright I' & \xrightarrow{\rho'_{SA}} & SA \\
 id'_{SA} \triangleright s_{I'} \downarrow & & \uparrow S\rho'_A \\
 SA \triangleright SI' & \xrightarrow{s_{A, I'}} & S(A \triangleright I')
 \end{array}$$

2. ε is a monoidal natural transformation:

$$\begin{array}{ccc}
 SA \triangleright SB & \xrightarrow{s_{A,B}} & S(A \triangleright B) \\
 \varepsilon_A \triangleright \varepsilon_B \downarrow & & \downarrow \varepsilon_{A \triangleright B} \\
 A \triangleright B & \xrightarrow{=} & A \triangleright B
 \end{array}$$

$$\begin{array}{ccc}
 SI' & \xrightarrow{\varepsilon_{I'}} & I' \\
 s_{I'} \swarrow & & \searrow \\
 & I' &
 \end{array}$$

3. δ is a monoidal natural transformation:

$$\begin{array}{ccc}
 SA \triangleright SA & \xrightarrow{S_{A,B}} & S(A \triangleright B) \\
 \delta_{A \triangleright B} \downarrow & & \downarrow \delta_{A \triangleright B} \\
 S^2 A \triangleright S^2 B & \xrightarrow{S_{SA,SB}} S(SA \triangleright SB) \xrightarrow{S_{A,B}} S^2(A \triangleright B) &
 \end{array}
 \quad
 \begin{array}{ccc}
 SI' & \xrightarrow{\delta_{I'}} & S^2 I' \\
 SI' \uparrow & & \uparrow SI' \\
 I' & \xrightarrow{S_{I'}} & SI'
 \end{array}$$

The proof for the commutativity of the diagrams are similar as the proof in Lemma 17. We do not include the proof here for simplicity. \blacktriangleleft

The comonad S on the Lambek category \mathcal{L} of the adjunction is clearly not symmetric because \mathcal{L} is not. However, it is symmetric on the co-Eilenberg-Moore category of the comonad.

► **Definition 22.** Let (S, ε, δ) be a comonad on a category \mathcal{L} . Then the **co-Eilenberg-Moore category** \mathcal{L}^S of the comonad has

- as objects the S -coalgebras $(A, h_A : A \rightarrow SA)$, where A is an object in \mathcal{L} , s.t. the following diagrams commute:

$$\begin{array}{ccc}
 A & \xrightarrow{h_A} & SA \\
 h_A \downarrow & & \downarrow \delta_A \\
 SA & \xrightarrow{Sh_A} & S^2 A
 \end{array}
 \quad
 \begin{array}{ccc}
 & SA & \\
 h_A \nearrow & & \searrow \varepsilon_A \\
 A & \xlongequal{\quad} & A
 \end{array}$$

- as morphisms the coalgebra morphisms, i.e. morphisms $f : (A, h_A) \rightarrow (B, h_B)$ between coalgebras s.t. the diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 h_A \downarrow & & \downarrow h_B \\
 SA & \xrightarrow{Sf} & SB
 \end{array}$$

► **Lemma 23.** Given a SMCC-Lambek model $F : \mathcal{C} \dashv \mathcal{L} : G$ and the comonad S on \mathcal{L} , the co-Eilenberg-Moore category \mathcal{L}^S of has an exchange natural transformation $\text{ex}_{A,B}^S : A \triangleright B \rightarrow B \triangleright A$.

Proof. We define the exchange $\text{ex}_{A,B}^S : A \triangleright B \rightarrow B \triangleright A$ as

$$A \triangleright B \xrightarrow{h_A \triangleright h_B} FGA \triangleright FGB \xrightarrow{m_{GA,GB}} F(GA \otimes GB) \xrightarrow{F\text{ex}_{GA,GB}} F(GB \otimes GA) \xrightarrow{F\eta_{B,A}} FG(B \triangleright A) \xrightarrow{\varepsilon_{B \triangleright A}} B \triangleright A$$

in which (F, m) and (G, η) are monoidal functors, and ex is the exchange for \mathcal{C} . ex^S is a natural transformation because the following diagrams commute for morphisms $f : A \rightarrow A'$ and $g : B \rightarrow B'$:

$$\begin{array}{ccccccc}
 A \triangleright B & \xrightarrow{h_A \triangleright h_B} & FGA \triangleright FGB & \xrightarrow{m_{GA,GB}} & F(GA \otimes GB) & \xrightarrow{F\text{ex}_{A,B}} & F(GB \otimes GA) \xrightarrow{F\eta_{B,A}} FG(B \triangleright A) \xrightarrow{\varepsilon_{B \triangleright A}} B \triangleright A \\
 f \triangleright g \downarrow & & \downarrow FGf \triangleright FGg & & \downarrow F(Gf \otimes Gg) & & \downarrow F(Gg \otimes Gf) \\
 A' \triangleright B' & \xrightarrow{h_{A'} \triangleright h_{B'}} & FGA' \triangleright FGB' & \xrightarrow{m_{GA',GB'}} & F(GA' \otimes GB') & \xrightarrow{F\text{ex}_{A',B'}} & F(GB' \otimes GA') \xrightarrow{F\eta_{B',A'}} FG(B' \triangleright A') \xrightarrow{\varepsilon_{B' \triangleright A'}} B' \triangleright A'
 \end{array}$$

\blacktriangleleft

► **Lemma 24.** *The following diagrams commute in the co-Eilenberg-Moore category \mathcal{L}^S :*

$$\begin{array}{ccccc}
 F((GA \otimes GB) \otimes GC) & \xrightarrow{F(\eta_{A,B} \otimes id_{GC})} & F(G(A \triangleright B) \otimes GC) & \xrightarrow{F(\epsilon_{A,B} \otimes id_{GC})} & FG((A \triangleright B) \triangleright C) \\
 \downarrow F(\epsilon_{A,B} \otimes id_{GC}) & & & & \downarrow \epsilon_{(A \triangleright B) \triangleright C} \\
 F(G(B \triangleright A) \otimes GC) & & & & (A \triangleright B) \triangleright C \\
 \downarrow F(\eta_{B,A} \otimes id_{GC}) & & & & \downarrow \epsilon_{A,B}^S \triangleright id_C \\
 F(G(B \triangleright A) \otimes GC) & \xrightarrow{F\eta_{B \triangleright A, C}} & FG((B \triangleright A) \triangleright C) & \xrightarrow{\epsilon_{(B \triangleright A) \triangleright C}} & (B \triangleright A) \triangleright C
 \end{array}$$

$$\begin{array}{ccccc}
 F(GB \otimes (GC \otimes GA)) & \xrightarrow{F(id_{GB} \otimes \eta_{C,A})} & F(GB \otimes G(C \triangleright A)) & \xrightarrow{F\eta_{B, C \triangleright A}} & FG(B \triangleright (C \triangleright A)) \\
 \downarrow F(id_{GB} \otimes \epsilon_{C,A}) & & & & \downarrow \epsilon_{B \triangleright (C \triangleright A)} \\
 F(GB \otimes (GA \otimes GC)) & & & & B \triangleright (C \triangleright A) \\
 \downarrow F(id_{GB} \otimes \eta_{A,C}) & & & & \downarrow id_{A \triangleright \epsilon_{C,A}^S} \\
 F(GB \otimes G(A \triangleright C)) & \xrightarrow{F\eta_{B, A \triangleright C}} & FG(B \triangleright (A \triangleright C)) & \xrightarrow{\epsilon_{B \triangleright (A \triangleright C)}} & B \triangleright (A \triangleright C)
 \end{array}$$

Proof. We only write the proof for the first diagram. The proof for the second one is similar. (1), (2), (3)–naturality of m ; (4)– F is monoidal; (5), (12)– ϵ is monoidal; (6), (7), (8), (9), (10)–obvious; (11)–coalgebra.

$$\begin{array}{ccccccc}
 F(G(A \triangleright B) \otimes GC) & \xrightarrow{F\eta_{A \triangleright B, C}} & FG((A \triangleright B) \triangleright C) & \xrightarrow{\epsilon_{(A \triangleright B) \triangleright C}} & (A \triangleright B) \triangleright C & \xlongequal{\quad} & (A \triangleright B) \triangleright C \\
 \uparrow F(\eta_{A,B} \otimes id_{GC}) & \nwarrow m_{G(A \triangleright B), GC} & \nwarrow (5) & \nwarrow \epsilon_{A \triangleright B \triangleright C} & \nwarrow \epsilon_{A \triangleright B \triangleright id_C} & \nwarrow (\epsilon_{A \triangleright B} \triangleright id_C) & \nwarrow (11) \\
 F((GA \otimes GB) \otimes GC) & \xrightarrow{(1)} & FG(A \triangleright B) \triangleright FGC & \xrightarrow{id \triangleright \epsilon_C} & FG(A \triangleright B) \triangleright C & \xrightarrow{(12)} & (FGA \triangleright FGB) \triangleright C \\
 \downarrow F(\epsilon_{A,B} \otimes id_{GC}) & \nwarrow m_{GA \otimes GB, GC} & \nwarrow F\eta_{A,B} \triangleright id_{FGC} & \nwarrow (7) & \nwarrow F\eta_{A,B} \triangleright id_C & \nwarrow m_{GA, GB \triangleright id_C} & \nwarrow (8) \\
 F((GB \otimes GA) \otimes GC) & \xrightarrow{(2)} & F(GA \otimes GB) \triangleright FGC & \xrightarrow{id \triangleright (F(GA \otimes GB) \triangleright \epsilon_C)} & F(GA \otimes GB) \triangleright C & \xrightarrow{F\epsilon_{A,B} \triangleright id_C} & F(GB \otimes GA) \triangleright C \\
 \downarrow F(\eta_{B,A} \otimes id_{GC}) & \nwarrow m_{GB \otimes GA, GC} & \nwarrow F\epsilon_{A,B} \triangleright id_{FGC} & \nwarrow (3) & \nwarrow m_{GB \otimes GA, GC} & \nwarrow F\epsilon_{A,B} \triangleright id_C & \nwarrow (10) \\
 F(G(B \triangleright A) \otimes GC) & \xleftarrow{m_{G(B \triangleright A), GC}} & FG(B \triangleright A) \triangleright FGC & \xleftarrow{F\eta_{B,A} \triangleright id} & F(GB \otimes GA) \triangleright FGC & \xrightarrow{id \triangleright \epsilon_C} & F(GB \otimes GA) \triangleright C \\
 \downarrow F\eta_{B \triangleright A, C} & \nwarrow (4) & \nwarrow \epsilon_{B \triangleright A \triangleright C} & \nwarrow (9) & \nwarrow id_{FG(B \triangleright A) \triangleright \epsilon_C} & \nwarrow (10) & \nwarrow F\eta_{B,A} \triangleright id_C \\
 FG((B \triangleright A) \triangleright C) & \xrightarrow{\epsilon_{(B \triangleright A) \triangleright C}} & (B \triangleright A) \triangleright C & \xleftarrow{\epsilon_{B \triangleright A} \triangleright id_C} & FG(B \triangleright A) \triangleright C & &
 \end{array}$$

► **Theorem 25.** *The co-Eilenberg-Moore category \mathcal{L}^S of S is symmetric monoidal closed.*

Proof. Let $(\mathcal{L}, \triangleright, I', \alpha', \lambda', \rho')$ be the Lambek category in a SMCC-Lambek model and S be the comonad on \mathcal{L} . Since \mathcal{L} is a Lambek category, it is obvious that \mathcal{L} is also Lambek. By Corollary 6, we only need to prove the exchange defined in Lemma 23 satisfies the three commutative diagrams in Definition 3.

The first triangle in Definition 3 commutes as follows: (1)–coalgebra; (2)– ϵ is monoidal; (3)–naturality of ρ ; (4)–naturality of ϵ ; (5)–naturality of m ; (6)– F is monoidal; (7)– C is symmetric;

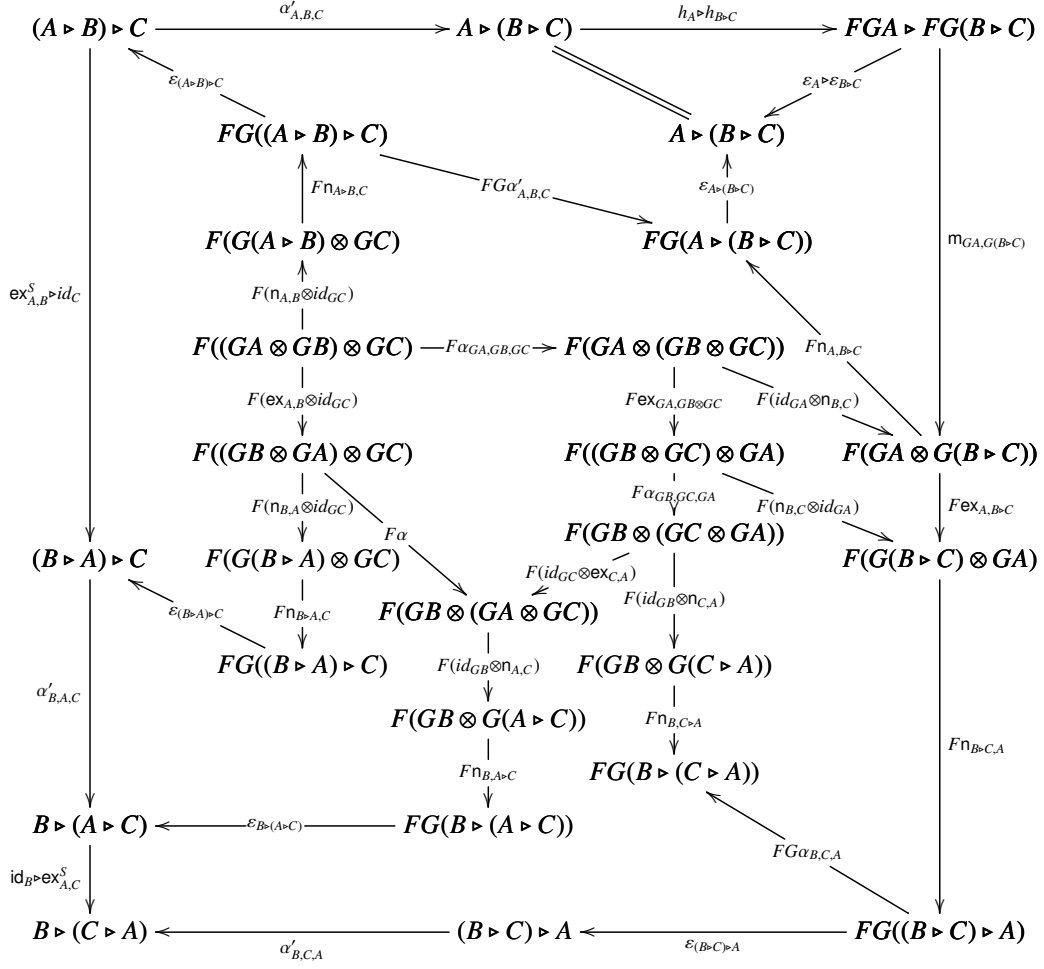
The second triangle in the proof commutes as follows: (1) and (5)–coalgebra; (2) and (4)– ε is monoidal; (3)– C is symmetric.

[illegible]

The second triangle in the proof commutes as follows: (1) and (5)–coalgebra; (2) and (4)– ε is monoidal; (3)– C is symmetric.

$$\begin{array}{ccccccc}
A \triangleright B & \xrightarrow{h_A \triangleright h_B} & FGA \triangleright FGB & \xrightarrow{m_{GA,GB}} & F(GA \otimes GB) & \xrightarrow{Fex_{A,B}} & F(GB \otimes GA) & \xrightarrow{F\eta_{B,A}} & FG(B \triangleright A) \\
\parallel & \swarrow \scriptstyle \varepsilon_A \triangleright \varepsilon_B & & & \parallel & & \parallel & & \downarrow \scriptstyle \varepsilon_{B \triangleright A} \\
(1) & & (2) & & (3) & & (4) & & (5) \\
A \triangleright B & \xleftarrow{\varepsilon_{A \triangleright B}} & FG(A \triangleright B) & \xleftarrow{F\eta_{A,B}} & F(GA \otimes GB) & \xleftarrow{Fex_{B,A}} & F(GB \otimes GA) & \xleftarrow{m_{GB,GA}} & FGB \triangleright FGA & \xleftarrow{h_A \triangleright h_B} & B \triangleright A
\end{array}$$

The third diagram commutes as follows, which uses Lemma ??.



5 Non-Commutative Linear Logic

5.1 Term Assignment for Sequent Calculus

5.2 Term Assignment for Natural Deduction

The term assignment for natural deduction of the non-commutative part of the model, i.e. the SMCC of the adjunction, is defined in Figure 1. And the term assignme for the commutative part, i.e. the Lambek category of the adjunction, is defined in Figure 2. Ψ and Φ are contexts for the non-commutative part and they are lists. Γ and Δ are contexts for the commutative part and they are multisets, therefore the following exchange rules are implicit.

$$\frac{\Phi, x : X, y : Y, \Psi \vdash_C t : Z}{\Phi, z : Y, w : X, \Psi \vdash_C \text{ex } w, z \text{ with } x, y \text{ in } t : Z} \quad \text{T_BETA} \qquad \frac{\Gamma, x : X, y : Y, \Delta \vdash_{\mathcal{L}} s : A}{\Gamma, z : Y, w : X, \Delta \vdash_{\mathcal{L}} \text{ex } w, z \text{ with } x, y \text{ in } s : A} \quad \text{S_BETA}$$

$$\begin{array}{c}
\frac{}{x : X \vdash_C x : X} \text{ T_ID} \quad \frac{}{\vdash_C \text{trivT} : \text{UnitT}} \text{ T_UNITI} \quad \frac{\Phi \vdash_C t_1 : \text{UnitT} \quad \Psi \vdash_C t_2 : Y}{\Phi, \Psi \vdash_C \text{let } t_1 : \text{UnitT be } \text{trivT in } t_2 : Y} \text{ T_UNITE} \\
\\
\frac{\Phi \vdash_C t_1 : X \quad \Psi \vdash_C t_2 : Y}{\Phi, \Psi \vdash_C t_1 \otimes t_2 : X \otimes Y} \text{ T_TENI} \quad \frac{\Phi \vdash_C t_1 : X \otimes Y \quad \Psi_1, x : X, y : Y, \Psi_2 \vdash_C t_2 : Z}{\Psi_1, \Phi, \Psi_2 \vdash_C \text{let } t_1 : X \otimes Y \text{ be } x \otimes y \text{ in } t_2 : Z} \text{ T_TENE} \\
\\
\frac{\Phi, x : X \vdash_C t : Y}{\Phi \vdash_C \lambda x : X. t : X \multimap Y} \text{ T_IMPI} \quad \frac{\Phi \vdash_C t_1 : X \multimap Y \quad \Psi \vdash_C t_2 : X}{\Phi, \Psi \vdash_C \text{app } t_1 t_2 : Y} \text{ T_IMPE} \quad \frac{\Phi \vdash_{\mathcal{L}} s : A}{\Phi \vdash_C \text{Gs} : \text{GA}} \text{ T_GI}
\end{array}$$

■ Figure 1 Commutative Part

$$\begin{array}{c}
\frac{}{x : A \vdash_{\mathcal{L}} x : A} \text{ S_ID} \quad \frac{}{\vdash_{\mathcal{L}} \text{trivS} : \text{UnitS}} \text{ S_UNITI} \quad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : \text{UnitS} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \text{let } s_1 : \text{UnitS be } \text{trivS in } s_2 : A} \text{ S_UNITE1} \\
\\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : \text{UnitS} \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Delta, \Gamma \vdash_{\mathcal{L}} \text{let } s_1 : \text{UnitS be } \text{trivS in } s_2 : A} \text{ S_UNITE2} \quad \frac{\Phi \vdash_C t : \text{UnitT} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Phi, \Gamma \vdash_{\mathcal{L}} \text{let } t : \text{UnitT be } \text{trivT in } s : A} \text{ S_UNITE3} \\
\\
\frac{\Phi \vdash_C t : \text{UnitT} \quad \Gamma \vdash_{\mathcal{L}} s : A}{\Gamma, \Phi \vdash_{\mathcal{L}} \text{let } t : \text{UnitT be } \text{trivT in } s : A} \text{ S_UNITE4} \quad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \quad \Delta \vdash_{\mathcal{L}} s_2 : B}{\Gamma, \Delta \vdash_{\mathcal{L}} s_1 \triangleright s_2 : A \triangleright B} \text{ S_TENI} \\
\\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \triangleright B \quad \Delta_1, x : A, y : B, \Delta_2 \vdash_{\mathcal{L}} s_2 : C}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \text{let } s_1 : A \triangleright B \text{ be } x \triangleright y \text{ in } s_2 : C} \text{ S_TENE1} \quad \frac{\Phi \vdash_C t : X \otimes Y \quad \Gamma_1, x : X, y : Y, \Gamma_2 \vdash_{\mathcal{L}} s : A}{\Gamma_1, \Phi, \Gamma_2 \vdash_{\mathcal{L}} \text{let } t : X \otimes Y \text{ be } x \otimes y \text{ in } s : A} \text{ S_TENE2} \\
\\
\frac{\Gamma, x : A \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda x : A. s : A \multimap B} \text{ S_IMPRI} \quad \frac{\Gamma \vdash_{\mathcal{L}} s_1 : A \multimap B \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \text{app}_r s_1 s_2 : B} \text{ S_IMPRE} \quad \frac{x : A, \Gamma \vdash_{\mathcal{L}} s : B}{\Gamma \vdash_{\mathcal{L}} \lambda_l x : A. s : B \multimap A} \text{ S_IMPLI} \\
\\
\frac{\Gamma \vdash_{\mathcal{L}} s_1 : B \multimap A \quad \Delta \vdash_{\mathcal{L}} s_2 : A}{\Gamma, \Delta \vdash_{\mathcal{L}} \text{app}_l s_1 s_2 : B} \text{ S_IMPLE} \quad \frac{\Phi \vdash_C t : X}{\Phi \vdash_{\mathcal{L}} \text{Ft} : \text{FX}} \text{ S_FI} \\
\\
\frac{\Gamma \vdash_{\mathcal{L}} y : \text{FX} \quad \Delta_1, x : X, \Delta_2 \vdash_{\mathcal{L}} s : A}{\Delta_1, \Gamma, \Delta_2 \vdash_{\mathcal{L}} \text{let } \text{Fx} : \text{FX be } y \text{ in } s : A} \text{ S_FE} \quad \frac{\Phi \vdash_C t : \text{GA}}{\Phi \vdash_{\mathcal{L}} \text{derelect } t : A} \text{ S_GE}
\end{array}$$

■ Figure 2 Non-Commutative Part

$$\begin{array}{c}
\frac{}{y_0 : \text{GB} \vdash_C y_0 : \text{GB}} \text{ ID} \quad \frac{}{x_0 : \text{GA} \vdash_C x_0 : \text{GA}} \text{ ID} \\
\frac{y_0 : \text{GB} \vdash_{\mathcal{L}} \text{Fy}_0 : \text{FGB}}{y_0 : \text{GB}, x_0 : \text{GA} \vdash_{\mathcal{L}} \text{Fy}_0 \triangleright \text{Fx}_0 : \text{FGB} \triangleright \text{FGA}} \text{ FI} \quad \frac{x_0 : \text{GA} \vdash_{\mathcal{L}} \text{Fx}_0 : \text{FGA}}{y_0 : \text{GB}, x_0 : \text{GA} \vdash_{\mathcal{L}} \text{Fy}_0 \triangleright \text{Fx}_0 : \text{FGB} \triangleright \text{FGA}} \text{ FI} \\
\frac{}{y_0 : \text{GB}, x_0 : \text{GA} \vdash_{\mathcal{L}} \text{Fy}_0 \triangleright \text{Fx}_0 : \text{FGB} \triangleright \text{FGA}} \text{ FE} \\
\frac{}{x_2 : \text{FGA} \vdash_{\mathcal{L}} x_2 : \text{FGA}} \text{ ID} \quad \frac{x_1 : \text{GA}, y_1 : \text{GB} \vdash_{\mathcal{L}} \text{ex } y_1, x_1 \text{ with } y_0, x_0 \text{ in } (\text{Fy}_0 \triangleright \text{Fx}_0) : \text{FGB} \triangleright \text{FGA}}{x_1 : \text{GA}, y_2 : \text{FGB} \vdash_{\mathcal{L}} \text{let } \text{Fy}_1 : \text{FGB be } y_2 \text{ in } (\text{ex } y_1, x_1 \text{ with } y_0, x_0 \text{ in } (\text{Fy}_0 \triangleright \text{Fx}_0)) : \text{FGB} \triangleright \text{FGA}} \text{ BETA} \\
\frac{}{z : \text{FGA} \triangleright \text{FGB} \vdash_{\mathcal{L}} z : \text{FGA} \triangleright \text{FGB}} \text{ ID} \quad \frac{x_2 : \text{FGA}, y_2 : \text{FGB} \vdash_{\mathcal{L}} \text{let } \text{Fx}_1 : \text{FGA be } x_2 \text{ in } (\text{let } \text{Fy}_1 : \text{FGB be } y_2 \text{ in } (\text{ex } y_1, x_1 \text{ with } y_0, x_0 \text{ in } (\text{Fy}_0 \triangleright \text{Fx}_0)))) : \text{FGB} \triangleright \text{FGA}}{z : \text{FGA} \triangleright \text{FGB} \vdash_{\mathcal{L}} \text{let } z : \text{FGA} \triangleright \text{FGB be } x_2 \triangleright y_2 \text{ in } (\text{let } \text{Fx}_1 : \text{FGA be } x_2 \text{ in } (\text{let } \text{Fy}_1 : \text{FGB be } y_2 \text{ in } (\text{ex } y_1, x_1 \text{ with } y_0, x_0 \text{ in } (\text{Fy}_0 \triangleright \text{Fx}_0)))) : \text{FGB} \triangleright \text{FGA}} \text{ FE} \\
\frac{}{\vdash_{\mathcal{L}} \lambda x : \text{FGA} \triangleright \text{FGB. let } z : \text{FGA} \triangleright \text{FGB be } x_2 \triangleright y_2 \text{ in } (\text{let } \text{Fx}_1 : \text{FGA be } x_2 \text{ in } (\text{let } \text{Fy}_1 : \text{FGB be } y_2 \text{ in } (\text{ex } y_1, x_1 \text{ with } y_0, x_0 \text{ in } (\text{Fy}_0 \triangleright \text{Fx}_0)))) : (\text{FGA} \triangleright \text{FGB}) \multimap (\text{FGB} \triangleright \text{FGA})} \text{ IMPRI}
\end{array}$$

6 Applications

7 Related Work

TODO

8 Conclusion

TODO

A Appendix