

Composing Monads and Comonads

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Abstract

TODO

1 Compose Monads

Definition 1. A *monad* on a category C is a triple (T, η, μ) , where

- T is an endofunctor on C that associates to each morphism $A \longrightarrow B$ in C to a morphism $T(A) \longrightarrow T(B)$ in C s.t. $T(id_A) = id_{TA}$ and $T(f) \circ T(g) = T(f \circ g)$
- η is a natural transformation, i.e. $\eta \circ f = T(f) \circ \eta$, or the diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \eta_A \downarrow & & \downarrow \eta_B \\ TA & \xrightarrow{T(f)} & TB \end{array}$$

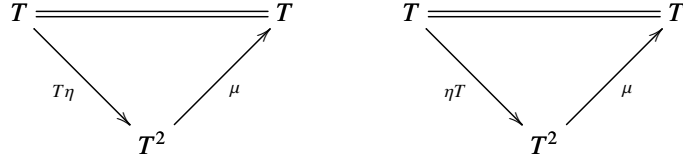
- μ is a natural transformation, i.e. $\mu \circ T^2(f) = T(f) \circ \mu$, or the diagram commutes:

$$\begin{array}{ccc} T^2A & \xrightarrow{T^2(f)} & T^2B \\ \mu_A \downarrow & & \downarrow \mu_B \\ TA & \xrightarrow{T(f)} & TB \end{array}$$

- $\mu \circ T\mu = \mu \circ \mu T$, i.e. the following diagram commutes:

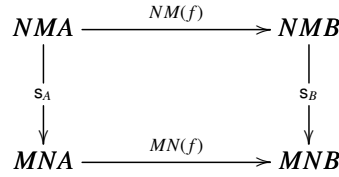
$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \mu T \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow{T} & T \end{array}$$

- $\mu \circ \eta T = \mu \circ T\eta = id_T$, i.e. the following diagrams commute:

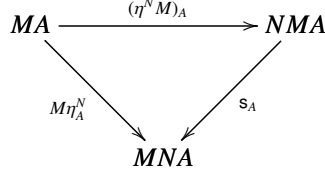


Definition 2. [1] Given two monads (M, η^M, μ^M) and (N, η^N, μ^N) on a category \mathcal{C} , the *swap* morphism \mathbf{s} associates to NMA for each object A in \mathcal{C} an object MNA , i.e. $\mathbf{s}_A : NMA \longrightarrow MNA$, subject to the following conditions:

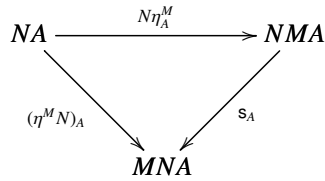
- $\mathbf{s} \circ NM(f) = MN(f) \circ \mathbf{s}$, i.e.



- $\mathbf{s} \circ \eta^N M = M\eta^N$, i.e.



- $\mathbf{s} \circ N\eta^M = M\eta^M N$, i.e.



Definition 3. Given two monads (M, η^M, μ^M) and (N, η^N, μ^N) on a category \mathcal{C} , the *composition monad* (T, η, μ) of M and N is defined as follows:

- T maps each morphism $f : A \longrightarrow B$ in \mathcal{C} to a morphism $T(f) = MN(f) : MNA \longrightarrow MNB$.
- $\eta_A : A \longrightarrow MNA$ and $\eta = \eta^M \circ \eta^N$.
- $\mu_A : MNMNA \longrightarrow MNA$ and $\mu = (M\mu^N) \circ (\mu^M N^2) \circ (MsN)$.

Theorem 4. *The composition monad T of M and N defined above is a monad on \mathcal{C} .*

Proof. • To show $T(id_A) = id_{TA}$:

$$\begin{aligned} T(id_A) &= MN(id_A) \\ &= M(id_{NA}) \\ &= id_{MNA} \\ &= id_{TA} \end{aligned}$$

• To show $T(f) \circ T(g) = T(f \circ g)$:

$$\begin{aligned} T(f) \circ T(g) &= MN(f) \circ MN(g) \\ &= M(N(f) \circ N(g)) \\ &= MN(f \circ g) \\ &= T(f \circ g) \end{aligned}$$

• To show $\eta \circ f = T(f) \circ \eta$:

$$\begin{aligned} \eta \circ f &= \eta^M \circ \eta^N \circ f \\ &= \eta^M \circ N(f) \circ \eta^N \\ &= MN(f) \circ \eta^M \circ \eta^N \\ &= T(f) \circ \eta \end{aligned}$$

• To show $\mu \circ T^2(f) = T(f) \circ \mu$:

$$\begin{aligned} \mu \circ T^2(f) &= M\mu^N \circ \mu^M N^2 \circ M\mathbf{s}N \circ MNMN(f) \\ &= M\mu^N \circ \mu^M N^2 \circ M^2 N^2(f) \circ M\mathbf{s}N \\ &= M\mu^N \circ MN^2(f) \circ \mu^M N^2 \circ M\mathbf{s}N \\ &= MN(f) \circ M\mu^N \circ \mu^M N^2 \circ M\mathbf{s}N \\ &= T(f) \circ \mu \end{aligned}$$

• To show $\mu \circ T\mu = \mu \circ \mu T$:

• To show $\mu \circ \eta T = \mu \circ T\eta$:

□

2 Compose Comonads

Definition 5. *A comonad on a category \mathcal{C} is a triple (T, ε, δ) , where*

- *T is an endofunctor on \mathcal{C} that associates to each morphism $A \longrightarrow B$ in \mathcal{C} to a morphism $T(A) \longrightarrow T(B)$ in \mathcal{C} s.t. $T(id_A) = id_{TA}$ and $T(f) \circ T(g) = T(f \circ g)$*

- ε is a natural transformation, i.e. $\varepsilon \circ T(f) = f \circ \varepsilon$, or the diagram commutes:

$$\begin{array}{ccc} TA & \xrightarrow{T(f)} & TB \\ \varepsilon_A \downarrow & & \downarrow \varepsilon_B \\ A & \xrightarrow{f} & B \end{array}$$

- δ is a natural transformation, i.e. $\delta \circ T(f) = T^2(f) \circ \delta$, or the diagram commutes:

$$\begin{array}{ccc} TA & \xrightarrow{T(f)} & TB \\ \delta_A \downarrow & & \downarrow \delta_B \\ T^2A & \xrightarrow{T^2(f)} & T^2B \end{array}$$

- $\delta T \circ \delta = T\delta \circ \delta$, i.e. the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\delta} & T^2 \\ \delta \downarrow & & \downarrow \delta T \\ T^2 & \xrightarrow{T\delta} & T^3 \end{array}$$

- $\varepsilon T \circ \delta = T\varepsilon \circ \delta = id_T$, i.e. the following diagrams commute:

$$\begin{array}{ccc} T & \xrightarrow{id} & T \\ \delta \searrow & & \nearrow \varepsilon T \\ & T^2 & \end{array} \quad \begin{array}{ccc} T & \xrightarrow{id} & T \\ \delta \searrow & & \nearrow T\varepsilon \\ & T^2 & \end{array}$$

Definition 6. Given two comonads (M, η^M, μ^M) and (N, η^N, μ^N) on a category \mathcal{C} , the **co-swap** morphism \mathbf{s}' associates to MNA for each object A in \mathcal{C} an object NMA , i.e. $\mathbf{s}'_A : MNA \longrightarrow NMA$, subject to the following conditions:

- $\mathbf{s}' \circ MN(f) = NM(f) \circ \mathbf{s}'$, i.e.

$$\begin{array}{ccc} MNA & \xrightarrow{MN(f)} & MNB \\ \mathbf{s}'_A \downarrow & & \downarrow \mathbf{s}'_B \\ NMA & \xrightarrow{NM(f)} & NMB \end{array}$$

- $N\varepsilon^M \circ s' = \varepsilon^M N$, i.e.

$$\begin{array}{ccc}
 NA & \xleftarrow{(\varepsilon^M N)_A} & MNA \\
 & \nwarrow N\varepsilon_A^M \quad \nearrow s'_A & \\
 & NMA &
 \end{array}$$

- $s' \circ \varepsilon^N M = M\varepsilon^N$, i.e.

$$\begin{array}{ccc}
 MA & \xleftarrow{M\varepsilon_A^N} & MNA \\
 & \nwarrow (\varepsilon^N M)_A \quad \nearrow s'_A & \\
 & NMA &
 \end{array}$$

Definition 7. Given two comonads $(M, \varepsilon^M, \delta^M)$ and $(N, \varepsilon^N, \delta^N)$ on a category C , the **composition comonad** (T, ε, δ) of M and N is defined as follows:

- T maps each morphism $f : A \longrightarrow B$ in C to a morphism $T(f) = MN(f) : MNA \longrightarrow MNB$.
- $\varepsilon_A : MNA \longrightarrow A$ and $\varepsilon = \varepsilon^N \circ \varepsilon^M$.
- $\delta_A : MNA \longrightarrow MNMNA$ and $\delta = (Ms'N) \circ (M^2\delta^N) \circ (\delta^M N)$.

3 related work

todo

4 conclusion

todo

References

- [1] Mark P Jones and Luc Duponcheel. Composing monads. Technical report, Cite-seer, 1993.