Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

1.1 Symmetric Monoidal Categories

Definition 1 *A **monoidal category** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned} \lambda_A &: \top \otimes A \longrightarrow A \\ \rho_A &: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{aligned}$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[rr, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A\otimes B,C,D}"] & & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B\otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C\otimes D}"] & & \\
A \otimes (B \otimes (C \otimes D)) & & A \otimes ((B \otimes C) \otimes D) \arrow[ll, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (\top \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Definition 2 *A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned} \lambda_A &: \top \otimes A \longrightarrow A \\ \rho_A &: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{aligned}$$

- *A symmetry natural transformation:*

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (\top \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 3 *A **symmetric monoidal closed category (SMCC)** is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Definition 4 *Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B,C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A,B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"]
\end{tikzcd}$$

Definition 5 *Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B,C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A,B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathrm{id}_{FA}"'] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathrm{id}_{FA} \otimes m_{\top_1}"'] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"'] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"] & F(B \otimes_1 A)
\end{tikzcd}$$

Definition 6 *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 7 *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***symmetric monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 8 *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$,* *are monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"]
\end{tikzcd}$$

Definition 9 *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***symmetric monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"]
\end{tikzcd}$$

Definition 10 *A* ***monoidal comonad*** *on a monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[rr, "\mathsf{m}_\top"] & & \top \\
& \top \arrow[ul, "\varepsilon_\top"] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

Definition 11 *A* ***symmetric monoidal comonad*** *on a symmetric monoidal category C is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[rr, "\mathsf{m}_\top"] & & \top \\
& \top \arrow[ul, "\varepsilon_\top"] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

1.2 LNL Model

Definition 12 *A **linear/non-linear (LNL) model** $(\mathcal{C}, \mathcal{L}, F, G)$ consists of*

- *a cartesian closed category $(\mathcal{C}, 1, \times, \longrightarrow)$*
- *a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,*
- *a pair of symmetric monoidal functors $(G, n) : \mathcal{L} \longrightarrow \mathcal{C}$ and $(F, m) : \mathcal{C} \longrightarrow \mathcal{L}$ that form a symmetric monoidal adjunction $\mathcal{C} : F \dashv G : \mathcal{L}$, subject to the following coherence conditions, where η and ε are the unit and counit of the adjunction and $A, B \in Ob(\mathcal{L})$, $X, Y \in Ob(\mathcal{C})$.*

$$\begin{tikzcd}
GA \arrow[rr, "\eta_{GA}"] \arrow[dr, equal] & & GFGA \arrow[dl, "G\varepsilon_A"] \\
& GA &
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[rr, "F\eta_X"] \arrow[dr, equal] & & FGFX \arrow[dl, "\varepsilon_{FX}"] \\
& FX &
\end{tikzcd}$$

$$\begin{tikzcd}
X \times Y \arrow[r, "\eta_X \times \eta_Y"] \arrow[d, "\eta_{X \times Y}"'] & GFX \times GFY \arrow[d, "\eta_{FX,FY}"] \\
GF(X \times Y) & G(FX \otimes GY) \arrow[l, "G\mathsf{m}_{X,Y}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FGA \otimes FGB \arrow[r, "\mathsf{m}_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes \varepsilon_B"'] & F(GA \times GB) \arrow[d, "F\mathsf{n}_{A,B}"] \\
A \otimes B & FG(A \otimes B) \arrow[l, "\eta_{A \otimes B}"']
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[rr, "\eta_1"] \arrow[dr, equal] & & GF(1) \arrow[dl, "G\mathsf{m}_{\top_1} \circ \mathsf{n}_I"] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
FG(I) \arrow[rr, "\varepsilon_I"] \arrow[dr, "F\mathsf{n}_I \circ \mathsf{m}_1"'] & & I \arrow[dl, equal] \\
& I &
\end{tikzcd}$$

Lemma 13 *Given a LNL model $(\mathcal{C}, \mathcal{L}, F, G)$, define $\mathsf{p}_{X,Y} : F(X \times Y) \longrightarrow FX \otimes FY$ as the composition $\varepsilon_{FX \otimes FY} \circ F\eta_{FX,FY} \circ F(\eta_X \times \eta_Y)$, and $\mathsf{p}_I : F1 \longrightarrow I$ as the composition $\varepsilon_I \circ F\mathsf{n}_I$. Then for $F \dashv G$, $\mathsf{m}_{X,Y}$ are components of a natural isomorphism with inverses $\mathsf{p}_{X,Y}$, and m_1 is an isomorphism with inverse p_I, i.e. $F(X) \otimes F(Y) \cong F(X \times Y)$, and $I \cong F(1)$.*

Lemma 14 *Given a LNL model $(\mathcal{C}, \mathcal{L}, F, G)$, the adjunction $F \dashv G$ induces a symmetric monoidal comonad $(!, \varepsilon, \delta)$ on $\mathcal{L}$, where $!$ represents FG, $\varepsilon : FG \longrightarrow 1$ is the counit of the adjunction, and $\delta : FG \longrightarrow FGFG$, i.e. $!$ is a symmetric monoidal functor and ε, δ are monoidal natural transformations.*

Lemma 15 *Given a LNL model $(\mathcal{C}, \mathcal{L}, F, G)$, F is a strong functor, i.e. F preserves the monoidal structure up to an isomorphism. And a strong functor induces a unique monoidal structure.*

Definition 16 *A **linear category** is specified by*

- *a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,*
- *a symmetric monoidal comonad $(!, \varepsilon, \delta)$ on $\mathcal{L}$, with $\mathsf{q}_{A,B} : !A \otimes !A \longrightarrow !(A \otimes B)$ and $\mathsf{q}_I : I \longrightarrow !I$;*
- *monoidal natural transformations on $\mathcal{L}$ with components $\mathsf{e}_A : !A \longrightarrow I$ and $\mathsf{d}_A : !A \longrightarrow !A \otimes !A$, s.t.*
  - *each $(!A, \mathsf{e}_A, \mathsf{d}_A)$ is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ \mathsf{d}_A = \mathsf{d}_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of $\mathcal{L}$;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{d}_A"] \arrow[d, "\mathsf{d}_A"'] & !A \otimes !A \arrow[r, "id_{!A} \otimes \mathsf{d}_A"] & !A \otimes (!A \otimes !A) \arrow[d, "\alpha_{!A,!A,!A}"] \\
!A \otimes !A \arrow[rr, "\mathsf{d}_A \otimes id_{!A}"] & & (!A \otimes !A) \otimes !A
\end{tikzcd}$$

$$\begin{tikzcd}
 & !A \arrow[d, "\mathsf{d}_A"] & \\
I \otimes !A \arrow[ur, "\lambda"] & !A \otimes !A \arrow[l, "\mathsf{e}_A \otimes id_{!A}"] \arrow[r, "id_{!A} \otimes \mathsf{e}_A"'] & !A \otimes I \arrow[ul, "\rho"']
\end{tikzcd}$$

  - *e_A and d_A are coalgebra morphisms, i.e. the following diagrams commute;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{e}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
!!A & !I \arrow[l, "!\mathsf{e}_A"]
\end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{d}_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[r, "\delta_{!A} \otimes \delta_A"] & !!A \otimes !!A \arrow[d, "\mathsf{q}_{!A,!A}"] \\
!!A \arrow[rr, "!\mathsf{d}_A"] & & !(!A \otimes !A)
\end{tikzcd}$$

– *any coalgebra morphism $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ between free coalgebras preserve the comonoid structure given by e and d, i.e. the following diagrams commute.*

$$\begin{tikzcd}
!A \arrow[rr, "f"] \arrow[dr, "\mathsf{e}_A"'] & & !B \arrow[dl, "\mathsf{e}_B"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathsf{d}_A"] \arrow[d, "f"'] & !A \otimes !A \arrow[d, "f \otimes f"] \\
!B \arrow[r, "\mathsf{d}_B"] & !B \otimes !B
\end{tikzcd}$$

Theorem 17 *$(!, \varepsilon, \delta)$ induces a symmetric monoidal adjunction $(\mathcal{L}^!, \otimes, I) : F \dashv G : (\mathcal{L}, \otimes, I)$, where $\mathcal{L}$ is the category of Eilenberg-Moore coalgebras.*

Theorem 18 *Any LNL model is a linear category.*

Theorem 19 *Any linear category gives rise to an LNL model, though it is not in general unique.*

Definition 20 *A* ***Lafont Category*** *is a SMCC $(\mathcal{L}, \otimes, I)$ in which for every object A of $\mathcal{L}$, there exists a commutative comonoid $(!A, \mathsf{d}_A, \mathsf{e}_A)$ and a morphism $\varepsilon : !A \longrightarrow A$ s.t. for all commutative comonoid (X, d, e) and for all $f : X \longrightarrow A$, there exists a unique comonoid morphism $f^+ : (X, d, e) \longrightarrow (!A, \mathsf{d}_A, \mathsf{e}_A)$ s.t. the following diagram commutes:*

$$\begin{tikzcd}
!A \arrow[rr, "\varepsilon_A"] & & A \\
& X \arrow[ul, "f^+"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Alternative definition*: a SMCC $(\mathcal{L}, \otimes, I)$ in which the forgetful functor $U : Comon(\mathcal{L}, \otimes, I) \longrightarrow \mathcal{L}$ has a right adjoint $!$, where $Comon(\mathcal{L}, \otimes, I)$ is the category of commutative comonoids in $\mathcal{L}$.*

Theorem 21 *$Comon(\mathcal{L}, \otimes, I)$ is cartesion. hence, every Lafont category defines a LNL model.*

Definition 22 *A* ***Seely Category*** *is a SMCC $(\mathcal{L}, \otimes, I)$ with products (&) and a terminal object $\top$, together with*

- *a comonad $(!, \delta, \varepsilon)$, where $\delta_A : !A \longrightarrow !!A$ and $\varepsilon_A : !A \longrightarrow A$, and*

- *two natural isomorphisms* $\mathsf{m}_{A,B} : !A \otimes !B \cong !(A \& B)$ *and* $\mathsf{m}_\top : I \cong !T$ *making* $(!, m) : (\mathcal{L}, \&, \top) \longrightarrow (\mathcal{L}, \otimes, I)$ *a symmetric monoidal functor s.t. the following diagram commutes:*

$$\begin{tikzcd}
!A \otimes !A \arrow[r, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & !(A \& B) \arrow[r, "\delta_{A \otimes B}"] & !!(A \& B) \arrow[d, "!(!\pi_1, !\pi_2)"] \\
!!A \otimes !!B \arrow[rr, "\mathsf{m}_{!A,!B}"] & & !(!A \& !B)
\end{tikzcd}$$

Theorem 23 *The comonad* $(!, \delta, \varepsilon)$ *on* $\mathcal{L}$ *generates an adjunction* $\mathcal{L}_! : L \dashv M : \mathcal{L}$ *between* $\mathcal{L}$ *and the (co-)Kleisli category* $\mathcal{L}_!$ *associated to the comonad. Thus, a Seely category defines a LNL model.*

2 Related Work

TODO

3 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.