Separating Linear Modalities

Jiaming Jiang and Harley Eades III

Abstract

TODO

1 Introduction

TODO [1]

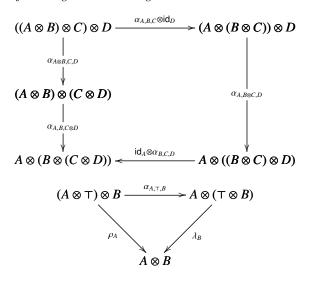
1.1 Symmetric Monoidal Categories

Definition 1. A monoidal category is a category, M, with the following data:

- An object \top of \mathcal{M} ,
- A bi-functor \otimes : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• Subject to the following coherence diagrams:



Definition 2. A symmetric monoidal category (SMC) is a category, M, with the following data:

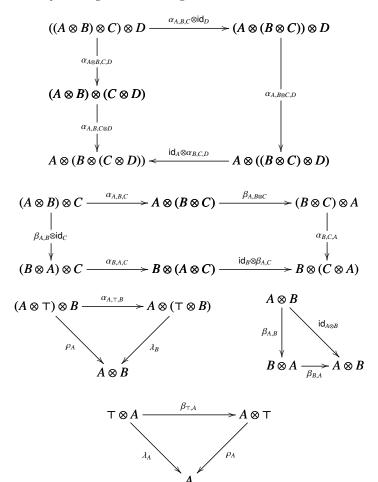
- An object \top of \mathcal{M} ,
- A bi-functor \otimes : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• A symmetry natural transformation:

$$\beta_{A,B}: A \otimes B \longrightarrow B \otimes A$$

• Subject to the following coherence diagrams:

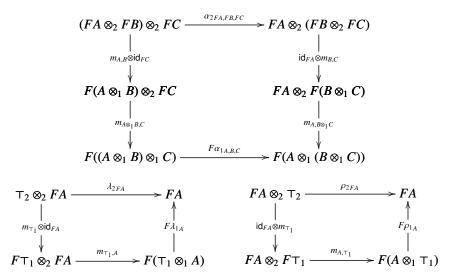


Definition 3. A symmetric monoidal closed category (SMCC) is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of M, the functor $-\otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of M there is an object $B \multimap C$ of M and a natural bijection:

$$\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A \otimes B, C) \cong \operatorname{\mathsf{Hom}}_{\mathcal{M}}(A, B \multimap C)$$

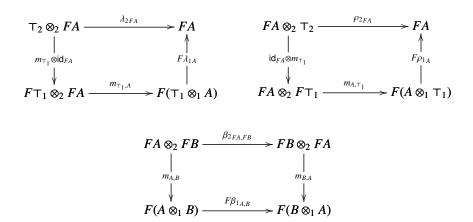
We call the functor \multimap : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of \mathcal{M} .

Definition 4. Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1}: \top_2 \longrightarrow F \top_1$ and a natural transformation $m_{A,B}: FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:

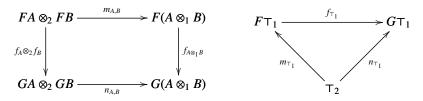


Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

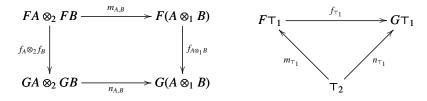
Definition 5. Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal** functor is a functor $F: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1}: \top_2 \longrightarrow F \top_1$ and a natural transformation $m_{A,B}: FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:



Definition 6. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **monoidal natural transformation** is a natural transformation, $f: F \longrightarrow G$, subject to the following coherence diagrams:

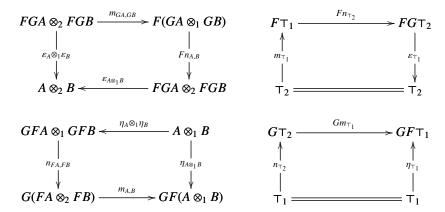


Definition 7. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **symmetric monoidal** natural transformation is a natural transformation, $f: F \longrightarrow G$, subject to the following coherence diagrams:

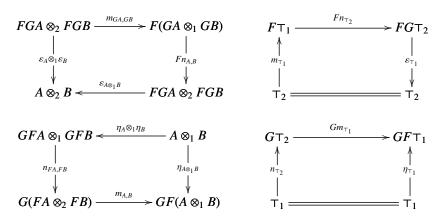


Definition 8. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **monoidal adjunction** is an ordinary adjunction \mathcal{M}_1 : $F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are

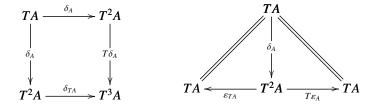
monoidal natural transformations. Thus, the following diagrams must commute:



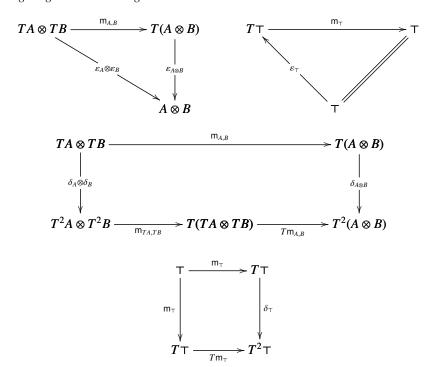
Definition 9. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a symmetric monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1: F \dashv G: \mathcal{M}_2$ such that the unit, $\eta_A: A \to GFA$, and the counit, $\varepsilon_A: FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:



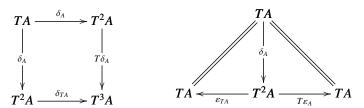
Definition 10. A monoidal comonad on a monoidal category C is a triple (T, ε, δ) , where (T, m) is a monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:



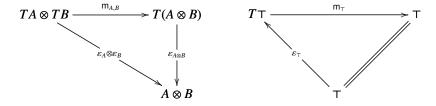
The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

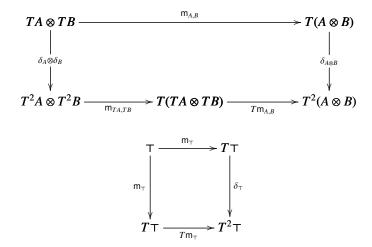


Definition 11. A symmetric monoidal comonad on a symmetric monoidal category C is a triple (T, ε, δ) , where (T, m) is a symmetric monoidal endofunctor on C, ε_A : $TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:



The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

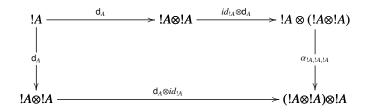


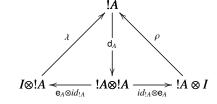


1.2 Linear Category

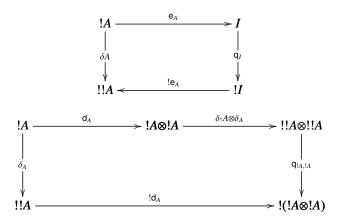
Definition 12. A linear category, $(\mathcal{L}, !, e, d)$, is specified by

- a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,
- a symmetric monoidal comonad $((!, q), \varepsilon, \delta)$ on \mathcal{L} , with $q_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $q_I : I \longrightarrow !I$;
- monoidal natural transformations on \mathcal{L} with components $e_A : !A \longrightarrow I$ and $d_A : !A \longrightarrow !A \otimes !A$, s.t.
 - each (!A, e_A , d_A) is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ d_A = d_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of \mathcal{L} ;

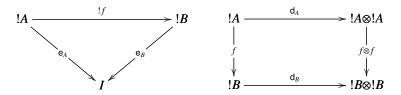




- e_A and d_A are coalgebra morphisms, i.e. the following diagrams commute;

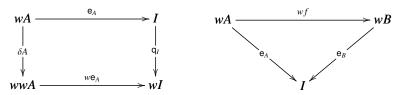


- any coalgebra morphism $f:(!A, \delta_A) \longrightarrow (!B, \delta_B)$ between free coalgebras preserve the comonoid structure given by e and d, i.e. the following diagrams commute.



Definition 13. A (modified) linear category with weakening, (\mathcal{L}, w, e) , is specified by

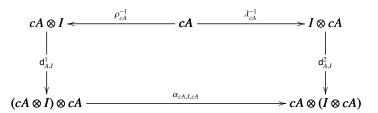
- a monoidal closed category $(\mathcal{L}, I, \otimes)$,
- a monoidal comonad $((w, q), \varepsilon, \delta)$ on \mathcal{L} with $q_{A,B} : wA \otimes wB \longrightarrow w(A \otimes B)$ and $q_I : I \longrightarrow wI$, and
- a monoidal natural transformation e on \mathcal{L} with components $e_A: wA \longrightarrow I$ s.t. the following diagrams commute:



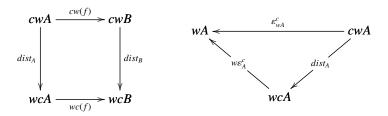
Definition 14. A (modified) linear category with contraction, $(\mathcal{L}, c, d^1, d^2)$, is specified by

- a monoidal closed category $(\mathcal{L}, I, \otimes)$,
- a monoidal comonad $((c,q), \varepsilon, \delta)$ on \mathcal{L} with $q_{A,B}: cA \otimes cB \longrightarrow c(A \otimes B)$ and $q_I: I \longrightarrow cI$, and

• monoidal natural transformations d^1 and d^2 on \mathcal{L} with components $d^1_{A,B}: cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$ and $d^2_{A,B}: B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$, s.t. the following diagram commutes:

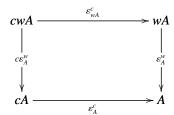


Definition 15. Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category \mathcal{L} , **a distributive law** of c over w is a natural transformation with components dist_A: $cwA \longrightarrow wcA$, subject to the following coherence diagrams:

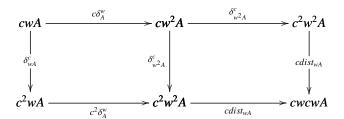


Lemma 16. Let $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ be two monoidal comonads and on a monoidal closed category $(\mathcal{L}, I, \otimes)$ such that (\mathcal{L}, w, e^w) is a (modified) linear category with weakening and $(\mathcal{L}, c, d^{c_1}, d^{c_2})$ is a (modified) linear category with contraction. Then the composition of c and w using the distributive law dist $_A: cwA \longrightarrow wcA$ for any object A in \mathcal{L} is a monoidal comonad $(cw, \varepsilon, \delta)$ on \mathcal{L} , where for any objects A, B in \mathcal{L} :

- $q_{A,B} : cwA \otimes cwB \longrightarrow cw(A \otimes B)$ is defined as: $q_{A,B} = cq_{A,B}^w \circ q_{wA,wB}^c$, and $q_I : I \longrightarrow cwI$ is defined as: $q_I = cq_I^w \circ q_I^c$.
- ε_A : $cwA \longrightarrow A$ is defined as in the diagram below, which commutes by the naturality of ε^c .

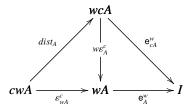


• $\delta_A : cwA \longrightarrow cwcwA$ is defined as in the diagram:



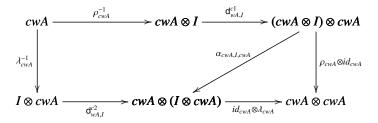
The left part of the diagram commutes by the naturality of δ^c and the right part commutes by equality.

• $e_A : cwA \longrightarrow I$ is defined as in



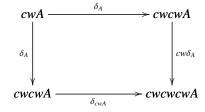
The left triangle commutes by the definition of dist and the right triangle commutes by the definition of e^w .

• $d_A : cwA \longrightarrow cwA \otimes cwA$ is defined as in

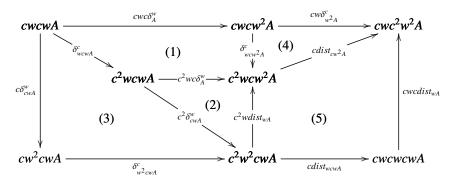


The left part of the diagram commutes by the definitions of d^{c1} and of d^{c2} , and the right part commutes because \mathcal{L} is monoidal.

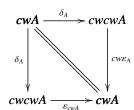
• The comonad law $cw\delta_A \circ \delta_A = \delta_{cwA} \circ \delta_A$, expressed in the diagram below,



is satisfied by the following diagram chasing. (1) and (3) commutes by the naturality of δ^c . (2), (4) and (5) commute by the conditions of dist.

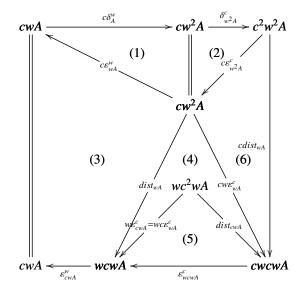


• The comonad law $cw\varepsilon_A \circ \delta_A = \varepsilon_{cwA} \circ \delta_A = id_{cwA}$, expressed in the diagram below,

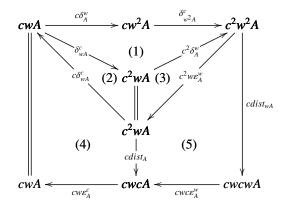


is satisfied by the following two diagram chasings below.

The left triangle is expanded in the following diagram chasing. (1) commutes by the comonad law for w with components δ^w_A and ε^w_{wA} . (2) commutes by the comonad law for c with components $\delta^c_{w^2A}$ and $\varepsilon^c_{w^2A}$. (3), (4), (5) and (6) commute by the definition of dist.



The right triangle is expanded in the following diagram chasing. (1) commutes by the naturality of δ^c . (2) is the comonad law for c with components δ^c_{wA} and ε^c_{wA} . (3) is the comonad law for w with components δ^w_A and ε^w_A . (3) and (4) commute by the definition of dist.



2 Related Work

TODO

3 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at http://research.microsoft.com/en-us/um/people/nick/mixed3.ps.