Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

2 Categorical Models

2.1 Lambek Categories

TODO: Define Lambek Categories

2.2 Lambek Categories with Weakening and Contraction

Definition 1. *A **Lambek category with weakening**, $(\mathcal{L}, w, \mathsf{weak})$, is specified by*

- *a monoidal category $(\mathcal{L}, I, \otimes)$,*
- *a monoidal comonad (w, ε, δ) on $\mathcal{L}$ with $\mathsf{q}_{A,B} : wA \otimes wB \longrightarrow w(A \otimes B)$ and $\mathsf{q}_I : I \longrightarrow wI$, and*
- *a monoidal natural transformation weak on $\mathcal{L}$ with components $\mathsf{weak}_A : wA \longrightarrow I$ s.t. the following diagrams commutes:*

$$\begin{tikzcd}
wA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
wwA \arrow[r, "w\mathsf{weak}_A"'] & wI
\end{tikzcd}$$

Definition 2. *A **Lambek category with contraction**, $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$, is specified by*

- *a monoidal category $(\mathcal{L}, I, \otimes)$,*

- *a monoidal comonad* (c, ε, δ) *on* $\mathcal{L}$ *with* $\mathsf{q}_{A,B} : cA \otimes cB \longrightarrow c(A \otimes B)$ *and* $\mathsf{q}_I : I \longrightarrow cI$*, and*

- *monoidal natural transformations* $\mathsf{contraL}$ *and* $\mathsf{contraR}$ *on* $\mathcal{L}$ *with components* $\mathsf{contraL}_{A,B} : cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$ *and* $\mathsf{contraR}_{A,B} : B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$*, s.t. the following diagrams commutes:*

$$\begin{tikzcd}
cA \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & cA \arrow[l, "\rho^{-1}_{cA}"'] \arrow[r, "\lambda^{-1}_{cA}"] & I \otimes cA \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(cA \otimes I) \otimes cA \arrow[rr, "\alpha_{cA,I,cA}"'] & & cA \otimes (I \otimes cA)
\end{tikzcd}$$

The following two diagrams are added to prove $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid, corresponding to part of the first diagram in the definition of linear category.

$$\begin{tikzcd}
cA \otimes cA \arrow[r, "id_{cA} \otimes \rho^{-1}_{cA}"] \arrow[dd, "\lambda^{-1}_{cA} \otimes id_{cA}"'] & cA \otimes (cA \otimes I) \arrow[r, "id_{cA} \otimes \mathsf{contraL}_{A,I}"] & cA \otimes ((cA \otimes I) \otimes cA) \arrow[d, "id_{cA} \otimes (\rho_{cA} \otimes id_{cA})"] \\
& & cA \otimes (cA \otimes cA) \\
(I \otimes cA) \otimes cA \arrow[r, "\mathsf{contraR}_{A,I} \otimes id_{cA}"'] & (cA \otimes (I \otimes cA)) \otimes cA \arrow[r, "(id_{cA} \otimes \lambda_{cA}) \otimes id_{cA}"'] & (cA \otimes cA) \otimes cA \arrow[u, "\alpha_{cA,cA,cA}"']
\end{tikzcd}$$

$$\begin{tikzcd}
cA \otimes cA \arrow[r, "id_{cA} \otimes \lambda^{-1}_{cA}"] \arrow[dd, "\rho^{-1}_{cA} \otimes id_{cA}"'] & cA \otimes (I \otimes cA) \arrow[r, "id_{cA} \otimes \mathsf{contraR}_{A,I}"] & cA \otimes (cA \otimes (I \otimes cA)) \arrow[d, "id_{cA} \otimes (id_{cA} \otimes \lambda_{cA})"] \\
& & cA \otimes (cA \otimes cA) \\
(cA \otimes I) \otimes cA \arrow[r, "\mathsf{contraL}_{A,I} \otimes id_{cA}"'] & ((cA \otimes I) \otimes cA) \otimes cA \arrow[r, "(\rho_{cA} \otimes id_{cA}) \otimes id_{cA}"'] & (cA \otimes cA) \otimes cA \arrow[u, "\alpha_{cA,cA,cA}"']
\end{tikzcd}$$

2.3 Lambek Categories with Exchange

Definition 3. *A* ***Lambek category with exchange****,* $(\mathcal{L}, e, \mathsf{ex})$*, is specified by*

- *a monoidal category* $(\mathcal{L}, I, \otimes)$*,*

- *a monoidal comonad* (e, ε, δ) *on* $\mathcal{L}$ *with* $\mathsf{q}_{A,B} : eA \otimes eB \longrightarrow e(A \otimes B)$ *and* $\mathsf{q}_I : I \longrightarrow eI$*, and*

- *a monoidal natural transformation* ex *on* $\mathcal{L}$ *with components* $\mathsf{ex}_{A,B} : e(A \otimes B) \longrightarrow eB \otimes eA$

subject to the following coherence condition:

$$\begin{tikzcd}
e(A\otimes B) \arrow[rr, "\mathsf{ex}_{A,B}"] \arrow[d, equal] & & eB\otimes eA \arrow[d, "\mathsf{q}_{B,A}"] \\
e(A\otimes B) & eA\otimes eB \arrow[l, "\mathsf{q}_{A,B}"'] & e(B\otimes A) \arrow[l, "\mathsf{ex}_{B,A}"']
\end{tikzcd}$$

and $(e(A\otimes B), \delta_{A\otimes B})$ and $(eB\otimes eA, \mathsf{q}_{eB,eA}\circ(\delta_B\otimes\delta_A))$ are coalgebras, i.e. the following diagram commutes:

$$\begin{tikzcd}
e(A\otimes B) \arrow[rr, "\delta_{A\otimes B}"] \arrow[d, "\mathsf{ex}_{A,B}"'] & & e^2(A\otimes B) \arrow[d, "e\mathsf{ex}_{A,B}"] \\
eB\otimes eA \arrow[r, "\delta_B\otimes\delta_A"] & e^2B\otimes e^2A \arrow[r, "\mathsf{q}_{eB,eA}"] & e(eB\otimes eA)
\end{tikzcd}$$

Lemma 4. *Let (e,ε,δ) be a monoidal comonad over a monoidal category $(\mathcal{L}, I, \otimes)$ such that $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange. Then in the co-Kleisli category of $\mathcal{L}$, there exists a natural isomorphism with components $\gamma_{A,B} : e(A\otimes B)\longrightarrow B\otimes A$.*

Proof. Suppose $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange. We define a natural transformation $\gamma_{A,B} : e(A\otimes B)\longrightarrow B\otimes A$ as

$$\gamma_{A,B} = (\varepsilon_B\otimes\varepsilon_A)\circ\mathsf{ex}_{A,B}.$$

γ is a natural isomorphism if the following diagram commutes:

$$\begin{tikzcd}
e(A\otimes B) \arrow[r, "\delta_{A\otimes B}"] \arrow[d, "\varepsilon_{A\otimes B}"'] & e^2(A\otimes B) \arrow[d, "e\gamma_{A,B}"] \\
A\otimes B & e(B\otimes A) \arrow[l, "\gamma_{B,A}"]
\end{tikzcd}$$

, which does commute by the diagram chasing below.

$$\begin{tikzcd}
e(A\otimes B) \arrow[rr, "\delta_{A\otimes B}"] \arrow[dr, "\mathsf{ex}_{A,B}"] \arrow[ddr, equal] \arrow[ddd, "\varepsilon_{A\otimes B}"'] & & e^2(A\otimes B) \arrow[rr, "e\mathsf{ex}_{A,B}"] \arrow[d, phantom, "(3)"] & & e(eB\otimes eA) \arrow[ddd, "e(\varepsilon_B\otimes\varepsilon_A)"] \\
 & eB\otimes eA \arrow[rr, "\delta_B\otimes\delta_A"] \arrow[drr, equal] & {} \arrow[d, phantom, "(4)"] & e^2B\otimes e^2A \arrow[ur, "\mathsf{q}_{eB,eA}"'] \arrow[d, "e\varepsilon_B\otimes e\varepsilon_A"] \arrow[r, phantom, "(5)"] & {} \\
 \arrow[r, phantom, "(1)"] & e(A\otimes B) & {} \arrow[d, phantom, "(2)"] & eB\otimes eA \arrow[dr, "\mathsf{q}_{B,A}"] & \\
A\otimes B & & eA\otimes eB \arrow[ll, "\varepsilon_A\otimes\varepsilon_B"] \arrow[ul, "\mathsf{q}_{A,B}"'] & & e(B\otimes A) \arrow[ll, "\mathsf{ex}_{B,A}"]
\end{tikzcd}$$

(1) commutes because e is monoidal. (2) and (3) commute by the definition of the Lambek category with exchange. (4) commutes because e is a comonad. And (5) commutes by the naturality of q. $\square$

2.4 Linear Categories

Definition 5. *A **linear category**, $(\mathcal{L}, !, \mathsf{weak}, \mathsf{contra})$, is specified by*

- *a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,*
- *a symmetric monoidal comonad $(!, \varepsilon, \delta)$ on $\mathcal{L}$, with $\mathsf{q}_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $\mathsf{q}_I : I \longrightarrow !I$;*
- *monoidal natural transformations on $\mathcal{L}$ with components $\mathsf{weak}_A : !A \longrightarrow I$ and $\mathsf{contra}_A : !A \longrightarrow !A \otimes !A$, s.t.*
  - *each $(!A, \mathsf{weak}_A, \mathsf{contra}_A)$ is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ \mathsf{contra}_A = \mathsf{contra}_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of $\mathcal{L}$;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & !A \otimes !A \arrow[r, "id_{!A} \otimes \mathsf{contra}_A"] & !A \otimes (!A \otimes !A) \\
!A \otimes !A \arrow[rr, "\mathsf{contra}_A \otimes id_{!A}"'] & & (!A \otimes !A) \otimes !A \arrow[u, "\alpha_{!A,!A,!A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & !A \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes !A & !A \otimes !A \arrow[l, "\mathsf{weak}_A \otimes id_{!A}"] \arrow[r, "id_{!A} \otimes \mathsf{weak}_A"'] & !A \otimes I
\end{tikzcd}$$

  - *weak_A and contra_A are coalgebra morphisms, i.e. the following diagrams commute;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
!!A \arrow[r, "!\mathsf{weak}_A"'] & !I
\end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[r, "\delta_A \otimes \delta_A"] & !!A \otimes !!A \arrow[d, "\mathsf{q}_{!A,!A}"] \\
!!A \arrow[rr, "!\mathsf{contra}_A"'] & & !(!A \otimes !A)
\end{tikzcd}$$

– *any coalgebra morphism $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ between free coalgebras preserve the comonoid structure given by* weak *and* contra*, i.e. the following diagrams commute.*

$$
\begin{tikzcd}
!A \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & !B \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & !A\otimes !A \arrow[d, "f\otimes f"] \\
!B \arrow[r, "\mathsf{contra}_B"'] & !B\otimes !B
\end{tikzcd}
$$

Definition 6. *Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction and $(\mathcal{L}, w, \mathsf{weak})$ is a Lambek category with weakening, we define a* ***distributive law*** *of c over w to be a natural transformation with components $dist_A : cwA \longrightarrow wcA$, subject to the following coherence diagrams:*

$$
\begin{tikzcd}
wA & & cwA \arrow[ll, "\varepsilon^c_{wA}"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "w\varepsilon^c_A"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cA & & cwA \arrow[ll, "c\varepsilon^w_A"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "\varepsilon^w_{cA}"] &
\end{tikzcd}
$$

By the definiiton of the distribute law *dist* and the comonad laws of c and w, the following two diagrams also commute:

$$
\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, "\delta^c_{wcA}"'] & & cwc^2A \\
& c^2wcA \arrow[ur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wdist_{wA}"] \\
& w^2cwA &
\end{tikzcd}
$$

shown by the diagram chasings below:

$$
\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, equal] \arrow[ddr, "\delta^c_{wA}"'] & & cwc^2A \arrow[dl, "cw\varepsilon^c_{cA}"'] \\
& cwcA & \\
& c^2wcA \arrow[u, "c\varepsilon^c_{wcA}"] \arrow[uur, "cdist_{cA}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, equal] \arrow[ddr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wc\varepsilon^w_{wA}"'] \arrow[ddl, "wdist_{wA}"] \\
& wcwA & \\
& w^2cwA \arrow[u, "w\varepsilon^w_{cwA}"] &
\end{tikzcd}
$$

Lemma 7. *Let $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ be two monoidal comonads on a Lambek category with weakening and contraction $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}, \mathsf{contraR})$. Then the composition of c and w using the distributive law $dist_A : cwA \longrightarrow wcA$ is a monoidal comonad $(cw, \varepsilon, \delta)$ on $\mathcal{L}$.*

Proof. Suppose $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ are monoidal comonads, and $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with weakening and contraction. Since by definition $c, w : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors we know that their composition $cw : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$\begin{aligned}
\mathsf{q}_{A,B} &: cwA \otimes cwB \longrightarrow cw(A \otimes B)\\
\mathsf{q}_{A,B} &= c\mathsf{q}^w_{A,B} \circ \mathsf{q}^c_{wA,wB}\\
\mathsf{q}_I &: I \longrightarrow cwI\\
\mathsf{q}_I &= c\mathsf{q}^w_I \circ \mathsf{q}^c_I
\end{aligned}$$

We must now define both $\varepsilon_A : cwA \longrightarrow A$ and $\delta_A : cwA \longrightarrow cwcwA$, and then show that they are monoidal natural transformations subject to the comonad laws. Since we are composing two comonads each of ε and δ can be given two definitions, but they are equivalent:

- $\varepsilon_A : cwA \longrightarrow A$ is defined as in the diagram below, which commutes by the naturality of ε^c.

$$\begin{tikzcd}
cwA \arrow[r, "\varepsilon^c_{wA}"] \arrow[d, "c\varepsilon^w_A"'] & wA \arrow[d, "\varepsilon^w_A"] \\
cA \arrow[r, "\varepsilon^c_A"'] & A
\end{tikzcd}$$

- $\delta_A : cwA \longrightarrow cwcwA$ is defined as in the diagram:

$$\begin{tikzcd}
cwA \arrow[r, "c\delta^w_A"] \arrow[d, "\delta^c_{wA}"'] & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, "\delta^c_{w^2A}"] & c^2w^2A \arrow[d, "cdist_{wA}"] \\
c^2wA \arrow[r, "c^2\delta^w_A"'] & c^2w^2A \arrow[r, "cdist_{wA}"'] & cwcwA
\end{tikzcd}$$

  The left part of the diagram commutes by the naturality of δ^c and the right part commutes trivially.

The remainder of the proof shows that the comonad laws hold.

Case 1:

$$\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & cwcwA \arrow[d, "cw\delta_A"] \\
cwcwA \arrow[r, "\delta_{cwA}"'] & cwcwcwA
\end{tikzcd}$$

The previous diagram commutes because the following one does.

$$\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, phantom, "(1)"] & cwcwA \arrow[r, "cwc\delta^w_A"] \arrow[dl, equal] \arrow[d, "\delta^c_{wcwA}"] \arrow[dr, phantom, "(2)"] & cwcw^2A \arrow[r, "cw\delta^c_{w^2A}"] \arrow[d, "\delta^c_{wcw^2A}"] \arrow[dr, phantom, "(5)"] & cwc^2w^2A \arrow[dd, "cwcdist_{wA}"] \\
cwcwA \arrow[d, "c\delta^w_{cwA}"'] \arrow[dr, phantom, "(4)"] & c^2wcwA \arrow[r, "c^2wc\delta^w_A"] \arrow[dr, "c^2\delta^w_{cwA}"] \arrow[dr, phantom, "(3)"] & c^2wcw^2A \arrow[ur, "cdist_{cw^2A}"] \arrow[d, "c^2wdist_{wA}"] \arrow[dr, phantom, "(6)"] & \\
cw^2cwA \arrow[rr, "\delta^c_{w^2cwA}"'] & & c^2w^2cwA \arrow[r, "cdist_{wcwA}"'] & cwcwcwA
\end{tikzcd}$$

(1) commutes by equality and we will not expand δ_A for simplicity. (2) and (4) commutes by the naturality of δ^c. (3), (5) commutes by the conditions of $dist$. (6) commutes by the naturality of $dist$.

Case 2:

$$\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[dr, equal] & cwcwA \arrow[d, "cw\varepsilon_A"] \\
& cwA
\end{tikzcd}$$

The triangle commutes because of the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\delta^w_A"] \arrow[ddd, equal] \arrow[ddr, "\varepsilon^c_{wA}"'] & \arrow[d, phantom, "(1)"] & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, equal] \arrow[dr, phantom, "(2)"] & c^2w^2A \arrow[dl, "c\varepsilon^c_{w^2A}"'] \arrow[ddd, "cdist_{wA}"] \\
 & (3) & cw^2A \arrow[ull, "c\varepsilon^w_{wA}"'] \arrow[d, "\varepsilon^c_{w^2A}"] \arrow[r, phantom, "(4)"] & \\
(5) & wA \arrow[d, phantom, "(6)"] & w^2A \arrow[l, "\varepsilon^w_{wA}"'] \arrow[dr, phantom, "(7)"] & \\
cwA \arrow[ur, "\varepsilon^c_{wA}"] & & wcwA \arrow[ll, "\varepsilon^w_{cwA}"] \arrow[u, "w\varepsilon^c_{wA}"] & cwcwA \arrow[l, "\varepsilon^c_{wcwA}"] \arrow[uul, "cw\varepsilon^c_{wA}"']
\end{tikzcd}
$$

(1) commutes by the comonad law for w with components δ^w_A and ε^w_{wA}. (2) commutes by the comonad law for c with components $\delta^c_{w^2A}$ and $\varepsilon^c_{w^2A}$. (3) and (7) commute by the naturality of ε^c. (4) commutes by the condition of *dist*. (5) commutes trivially. And (6) commutes by the naturality of ε^w.

Case 3:

$$
\begin{tikzcd}
cwA \arrow[d, "\delta_A"'] \arrow[dr, equal] & \\
cwcwA \arrow[r, "\varepsilon_{cwA}"'] & cwA
\end{tikzcd}
$$

The previous triangle commutes because the following diagram chasing does.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\delta^w_A"] \arrow[drr, "\delta^c_{wA}"] \arrow[ddd, equal] & & cw^2A \arrow[rr, "\delta^c_{w^2A}"] \arrow[d, phantom, "(1)"] & & c^2w^2A \arrow[ddd, "cdist_{wA}"] \arrow[ddll, "c^2w\varepsilon^w_A"] \\
& (2) & c^2wA \arrow[urr, "c^2\delta^w_A"'] \arrow[d, equal] & (3) & \\
& (4) & c^2wA \arrow[uull, "c\varepsilon^c_{wA}"] \arrow[d, "cdist_A"] & (5) & \\
cwA & & cwcA \arrow[ll, "cw\varepsilon^c_A"] & & cwcwA \arrow[ll, "cwc\varepsilon^w_A"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ^c. (2) is the comonad law for c with components δ^c_{wA} and ε^c_{wA}. (3) is the comonad law for w with components δ^w_A and ε^w_A. (4) commutes by the condition of $dist$. And (5) commute by the naturality of $dist$.

□

Definition 8. *A **Lambek category with** cw, $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$, is specified by*

- *a monoidal category $(\mathcal{L}, I, \otimes)$;*
- *a monoidal comonad $(c, \varepsilon^c, \delta^c)$ with monoidal natural transformations* $\mathsf{contraL}$ *and* $\mathsf{contraR}$ *on $\mathcal{L}$ s.t. $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction;*
- *a monoidal comonad $(w, \varepsilon^w, \delta^w)$ with a monoidal natural transformation* weak *on $\mathcal{L}$ s.t. $(\mathcal{L}, w, \mathsf{weak}^w)$ is a Lambek category with weakening;*
- *a natural transformation with components $dist_A : cwA \longrightarrow wcA$;*

where cw is the composite monoidal comonad cw defined in the proof of Lemma 7, s.t. the following additional coherence diagrams commute:

$$
\begin{tikzcd}
I \otimes cwA \arrow[r, "\lambda^{-1}_{I\otimes cwA}"] \arrow[d, "\mathsf{contraR}_{wA,I}"'] & I \otimes (I \otimes cwA) \\
cwA \otimes (I \otimes cwA) \arrow[r, "\varepsilon^c_{wA}\otimes id_{I\otimes cwA}"'] & wA \otimes (I \otimes cwA) \arrow[u, "\mathsf{weak}^w_A\otimes id_{I\otimes cwA}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
cwA \otimes I \arrow[r, "\rho^{-1}_{cwA\otimes I}"] \arrow[d, "\mathsf{contraL}_{wA,I}"'] & (cwA \otimes I) \otimes I \\
(cwA \otimes I) \otimes cwA \arrow[r, "id_{cwA\otimes I}\otimes\varepsilon^c_{wA}"'] & (cwA \otimes I) \otimes wA \arrow[u, "id_{cwA\otimes I}\otimes\mathsf{weak}^w_A"']
\end{tikzcd}
$$

$$\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & cwB \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}$$

for any coalgebra morphism $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ between free coalgebras.

Lemma 9. *Let $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ be a Lambek category with cw. Then the monoidal natural transformations* weak *and* contra *satisfy the following three conditions:*

Delete this lemma?

1. *Each $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid.*
2. weak_A *and* contra_A *are coalgebra morphisms.*
3. *Any coalgebra morphism $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ between free coalgebras preserves the comonoid structure given by* weak *and* contra.

Proof. We first define weak and contra. Each of them can also be given two equivalent definitions:

- $\mathsf{weak}_A : cwA \longrightarrow I$ is defined as in the diagram below. The left triangle commutes by the definition of *dist* and the right triangle commutes by the definition of weak^w.

$$\begin{tikzcd}
& wcA \arrow[d, "w\varepsilon^c_A"] \arrow[dr, "\mathsf{weak}^w_{cA}"] & \\
cwA \arrow[ur, "dist_A"] \arrow[r, "\varepsilon^c_{wA}"'] & wA \arrow[r, "\mathsf{weak}^w_A"'] & I
\end{tikzcd}$$

- $\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$ is defined as below. The left part of the diagram commutes by the definitions of contraL and of contraR, and the right part commutes because $\mathcal{L}$ is monoidal.

$$\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "\lambda^{-1}_{cwA}"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[dl, "\alpha_{cwA,I,cwA}"'] \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
I \otimes cwA \arrow[r, "\mathsf{contraR}_{wA,I}"'] & cwA \otimes (I \otimes cwA) \arrow[r, "id_{cwA} \otimes \lambda_{cwA}"'] & cwA \otimes cwA
\end{tikzcd}$$

Then we show each condition is satisfied.

1. Each $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid.

Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\mathsf{contra}_A"] & cwA \otimes (cwA \otimes cwA) \\
cwA \otimes cwA \arrow[rr, "\mathsf{contra}_A\otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}
$$

The previous diagram commutes by the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] \arrow[dr, phantom, "(1)"] & cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\rho^{-1}_{cwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwA \otimes (cwA \otimes I) \arrow[d, "id_{cwA}\otimes\mathsf{contraL}_{wA,I}"] \\
cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\lambda^{-1}_{cwA}"] \arrow[d, "\rho^{-1}_{cwA}\otimes id_{cwA}"'] & cwA \otimes (I \otimes cwA) \arrow[d, "id_{cwA}\otimes\mathsf{contraR}_{wA,I}"] & cwA \otimes ((cwA \otimes I) \otimes cwA) \arrow[d, "id_{cwA}\otimes(\rho_{cwA}\otimes id_{cwA})"] \arrow[dl, phantom, "(3)"] \\
(cwA \otimes I) \otimes cwA \arrow[d, "\mathsf{contraL}_{wA,I}\otimes id_{cwA}"'] \arrow[dr, phantom, "(4)"] & cwA \otimes (cwA \otimes (I \otimes cwA)) \arrow[r, "id_{cwA}\otimes(id_{cwA}\otimes\lambda_{cwA})"] \arrow[ur, "id_{cwA}\otimes\alpha_{cwA,I,cwA}"] & cwA \otimes (cwA \otimes cwA) \\
((cwA \otimes I) \otimes cwA) \otimes cwA \arrow[rr, "(\rho_{cwA}\otimes id_{cwA})\otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}
$$

(1) commutes trivially and we would not expand contra for simplicity. (2) and (4) commute because $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction. (3) commutes because $\mathcal{L}$ is monoidal.

Case 2:

$$
\begin{tikzcd}
& cwA \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes cwA & cwA \otimes cwA \arrow[l, "\mathsf{weak}_A\otimes id_{cwA}"] \arrow[r, "id_{cwA}\otimes\mathsf{weak}_A"'] & cwA \otimes I
\end{tikzcd}
$$

The diagram above commutes by the following diagram chasing.

$$
\begin{tikzcd}
I \otimes cwA & & & & & & wA \otimes cwA \arrow[llllll, "\mathsf{weak}^w_A \otimes id_{cwA}"'] \arrow[from=dddddd, "\varepsilon^c_{wA} \otimes cwA"'] \\
& & & (2) & & & \\
& & I \otimes (I \otimes cwA) \arrow[uull, "id_I \otimes \lambda_{cwA}"] & & wA \otimes (I \otimes cwA) \arrow[ll, "\mathsf{weak}^w_A \otimes id_{I \otimes cwA}"'] \arrow[uurr, "id_{wA} \otimes \lambda_{cwA}"'] & & \\
& (1) & & (4) & & (3) & \\
& & I \otimes cwA \arrow[uu, "\lambda^{-1}_{I \otimes cwA}"] \arrow[rr, "\mathsf{contraR}_{wA,I}"] & & cwA \otimes (I \otimes cwA) \arrow[uu, "\varepsilon^c_{wA} \otimes id_{I \otimes cwA}"'] \arrow[ddrr, "id_{cwA} \otimes \lambda_{cwA}"] & & \\
& & & & & & \\
cwA \arrow[uuuuuu, "\lambda^{-1}_{cwA}"] \arrow[uurr, "\lambda^{-1}_{cwA}"'] \arrow[ddrr, "\rho^{-1}_{cwA}"] \arrow[dddddd, "\rho^{-1}_{cwA}"'] & & & (5) & & & cwA \otimes cwA \arrow[dddddd, "id_{cwA} \otimes \varepsilon^c_{wA}"] \\
& & & & & & \\
& & cwA \otimes I \arrow[rr, "\mathsf{contraL}_{wA,I}"] \arrow[dd, "\rho^{-1}_{cwA}"] & & (cwA \otimes I) \otimes cwA \arrow[uurr, "\rho_{cwA} \otimes id_{cwA}"'] \arrow[dd, "id_{cwA \otimes I} \otimes \varepsilon^c_{wA}"'] & & \\
& (6) & & (9) & & (8) & \\
& & (cwA \otimes I) \otimes I \arrow[ddll, "\rho_{cwA} \otimes id_I"'] & & (cwA \otimes I) \otimes wA \arrow[ll, "id_{cwA \otimes I} \otimes \mathsf{weak}^w_A"'] \arrow[ddrr, "\rho_{cwA} \otimes id_{wA}"] & & \\
& & & (7) & & & \\
cwA \otimes I & & & & & & cwA \otimes wA \arrow[llllll, "id_{cwA} \otimes \mathsf{weak}_A"]
\end{tikzcd}
$$

(1), (2) and (3) commute by the functionality of λ. (6), (7) and (8) commute by the functionality of ρ. (4) and (9) are conditions of the Lambek category with cw. And (5) is the definition of contra.

- weak and contra are coalgebra morphisms.

  Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
cwcwA \arrow[r, "cw\mathsf{weak}_A"'] & cwI
\end{tikzcd}
$$

  The previous diagram commutes by the diagram below. (1) commutes by the naturality of δ^c. (2) commutes by the condition of $dist_{wA}$. (3), (5) and (6) commute because c is a monoidal comonad. (4) commutes because $(\mathcal{L}, w, \mathsf{weak}^w)$ is a Lambek category with weakening. (7) commutes because c and w are monoidal comonads.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\mathsf{weak}^w_A"] \arrow[d, "\delta^c_{wA}"'] \arrow[ddrr, "c\delta^w_A"] & & cI \arrow[rr, "\epsilon^c_I"] \arrow[dddrr, "c\mathsf{q}^w_I"'] & & I \arrow[dl, equal] \arrow[dd, "\mathsf{q}^c_I"] \\
c^2wA \arrow[d, "c^2\delta^w_A"] \arrow[r, phantom, "(1)"] & & \arrow[r, phantom, "(4)"] & I \arrow[ul, "\mathsf{q}^c_I"'] \arrow[r, phantom, "(6)"] & \\
c^2w^2A \arrow[d, "cdist_{wA}"'] \arrow[drr, "c\varepsilon^c_{wA}"] \arrow[r, phantom, "(2)\ (3)"] & & cw^2A \arrow[ll, "\delta^c_{w^2A}"'] \arrow[d, equal] & \arrow[r, phantom, "(7)"] & cI \arrow[ul, "\varepsilon^c_I"'] \arrow[d, "c\mathsf{q}^w_I"] \\
cwcwA \arrow[rr, "cw\varepsilon^c_{wA}"'] & & cw^2A & & cwI \arrow[ll, "cw\mathsf{weak}^w_A"]
\end{tikzcd}
$$

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & cwA \otimes cwA \arrow[r, "\delta_A \otimes \delta_A"] & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}_{cwA,cwA}"] \\
cwcwA \arrow[rr, "cw\mathsf{contra}_A"'] & & cw(cwA \otimes cwA)
\end{tikzcd}
$$

To prove the previous diagram commute, we first expand it, Then we divide it into five parts as shown belovee, and prove each part commutes.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, phantom, "(c)"] & cwA \otimes I \arrow[rrr, "\mathsf{contraL}_{wA,I}"] & & & (cwA \otimes I) \otimes cwA \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
cw^2A \arrow[d, "\delta^c_{w^2A}"'] & w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA\otimes I}"'] & w(cwA \otimes cwA) \arrow[rr, "\varepsilon^w_{cwA\otimes cwA}"] & & cwA \otimes cwA \arrow[d, "c\delta^w_A \otimes c\delta^w_A"] \arrow[dl, phantom, "(a)"] \\
c^2w^2A \arrow[d, "cdist_{wA}"'] & & & cw^2A \otimes cw^2A \arrow[ur, "c\varepsilon^c_{wA} \otimes c\varepsilon^c_{wA}"] \arrow[r, equal] & cw^2A \otimes cw^2A \arrow[d, "\delta^c_{w^2A} \otimes \delta^c_{w^2A}"] \arrow[dl, phantom, "(b)"] \\
cwcwA \arrow[d, "cw\rho_{cwA}"'] & \arrow[d, phantom, "(d)"] & & & c^2w^2A \otimes c^2w^2A \arrow[ul, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"'] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
cw(cwA \otimes I) \arrow[d, "cw\mathsf{contraL}_{wA,I}"'] \arrow[uuur, "\varepsilon^c_{w(cwA\otimes I)}"] & & & \arrow[phantom, "(e)"] & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}^c_{wcwA\otimes wcwA}"] \\
cw((cwA \otimes I) \otimes cwA) \arrow[rr, "cw(\rho_{cwA}\otimes id_{cwA})"'] & & cw(cwA \otimes cwA) \arrow[uuuu, "c\varepsilon^w_{cwA\otimes cwA}"] & & c(wcwA \otimes wcwA) \arrow[ll, "c\mathsf{q}^w_{cwA\otimes cwA}"]
\end{tikzcd}
$$

Part (a) and (b) are comonad laws.

Part (c) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for w. (3) is the comonad law for c. (4) commutes by the naturality of ε^c. (5) is one of the conditions for $dist_{wA}$. (6) commutes by the naturality of ε^w. And (7) commutes by the naturality of ε^c.

$$\begin{tikzcd}
cwA \arrow[rrr, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, "c\delta^w_A"] \arrow[drr, equal] & & & cwA \otimes I \\
cw^2A \arrow[r, equal] \arrow[d, "\delta^c_{w^2A}"'] \arrow[ur, phantom, "(1)"] & cw^2A \arrow[r, "c\varepsilon^w_{wA}"] \arrow[d, "dist_{wA}"] \arrow[ur, phantom, "(2)"] & cwA \arrow[r, phantom, "(6)"] & \\
c^2w^2A \arrow[ur, "\varepsilon^c_{cw^2A}"] \arrow[d, "cdist_{wA}"'] \arrow[ur, phantom, "(3)"] & wcwA \arrow[ur, "\varepsilon^w_{cwA}"'] \arrow[rr, "w\rho^{-1}_{cwA}"] \arrow[ur, phantom, "(5)"] & & w(cwA \otimes I) \arrow[uu, "\varepsilon^w_{cwA\otimes I}"'] \\
cwcwA \arrow[ur, "\varepsilon^c_{wcwA}"] \arrow[rrr, "cw\rho^{-1}_{cwA}"'] \arrow[ur, phantom, "(4)"] \arrow[urrr, phantom, "(7)"] & & & cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA\otimes I)}"']
\end{tikzcd}$$

Part (d) commutes by the following diagram chase. The upper two squares both commute by the naturality of ε^w, and the lower two squares commute by the naturality of ε^c.

$$\begin{tikzcd}
cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] & cwA \otimes cwA \\
w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA\otimes I}"] \arrow[r, "w\mathsf{contraL}_{wA,I}"] & w((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^w_{(cwA\otimes I)\otimes cwA}"] \arrow[r, "w(\rho_{cwA} \otimes id_{cwA})"] & w(cwA \otimes cwA) \arrow[u, "\varepsilon^w_{cwA\otimes cwA}"] \\
cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA\otimes I)}"] \arrow[r, "cw\mathsf{contraL}_{wA,I}"'] & cw((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^c_{w((cwA\otimes I)\otimes A)}"] \arrow[r, "cw(\rho_{cwA} \otimes id_{cwA})"'] & cw(cwA \otimes cwA) \arrow[u, "\varepsilon^c_{w(cwA\otimes cwA)}"']
\end{tikzcd}$$

Part (e) commutes by the following diagram. (1) commutes by the condition of $dist_{wA}$. (2) and (4) commute by the naturality of ε^c. (3) and (5) commute because w and c are monoidal comonads.

$$
\begin{tikzcd}
cwA \otimes cwA & cw^2A \otimes cw^2A \arrow[l, "c\varepsilon^w_{wA} \otimes c\varepsilon^w_{wA}"'] \arrow[d, "dist_{wA} \otimes dist_{wA}"] \arrow[dr, phantom, "(2)"] & c^2w^2A \otimes c^2w^2A \arrow[l, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"'] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
(3) & wcwA \otimes wcwA \arrow[ul, "\varepsilon^w_{cwA} \otimes \varepsilon^w_{cwA}"] \arrow[dl, "\mathsf{q}^w_{cwA,cwA}"'] \arrow[d, phantom, "(4)"] & cwcwA \otimes cwcwA \arrow[l, "\varepsilon^c_{wcwA} \otimes \varepsilon^c_{wcwA}"'] \arrow[d, "\mathsf{q}^c_{wcwA \otimes wcwA}"] \\
w(cwA \otimes cwA) \arrow[uu, "\varepsilon^w_{cwA \otimes cwA}"] & cw(cwA \otimes cwA) \arrow[l, "\varepsilon^c_{w(cwA \otimes cwA)}"] & c(wcwA \otimes wcwA) \arrow[l, "c\mathsf{q}_{cwA \otimes cwA}"] \arrow[ul, "\varepsilon^c_{wcwA \otimes wcwA}"']
\end{tikzcd}
$$

- Any coalgebra morphism $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ between free coalgebras preserves the comonoid structure given by weak and contra.

  Case 1: This coherence diagram is given in the definition of the Lambek category with cw.

$$
\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & cwB \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
$$

  Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & cwA \otimes cwA \arrow[d, "f \otimes f"] \\
cwB \arrow[r, "\mathsf{contra}_B"'] & cwB \otimes cwB
\end{tikzcd}
$$

  The square commutes by the diagram chasing below, which commutes by the naturality of ρ and $\mathsf{contraL}$.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "cwf"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] \arrow[d, "cwf \otimes id_I"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] \arrow[d, "(cwf \otimes id_I) \otimes cwf"] & cwA \otimes cwA \arrow[d, "cwf \otimes cwf"] \\
cwB \arrow[r, "\rho^{-1}_{cwB}"'] & cwB \otimes I \arrow[r, "\mathsf{contraL}_{wB,I}"'] & (cwB \otimes I) \otimes cwB \arrow[r, "\rho_{cwB} \otimes id_{cwB}"'] & cwB \otimes cwB
\end{tikzcd}
$$

$\square$

Definition 10. *Given two comonads $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ on a category $\mathcal{L}$ such that $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ is a Lambek category with cw and $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange, we define a* ***distributive law for exchange*** *of cw over e to be a natural isomorphism with components $distEx_A : cweA \longrightarrow ecwA$, subject to the following coherence diagrams:*

$$
\begin{tikzcd}
eA & & cweA \arrow[ll, "\varepsilon^{cw}_{eA}"'] \arrow[dl, "distEx_A"] \\
& ecwA \arrow[ul, "e\varepsilon^{cw}_{A}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA & & cweA \arrow[ll, "cw\varepsilon^{e}_{A}"'] \arrow[dl, "distEx_A"] \\
& ecwA \arrow[ul, "\varepsilon^{e}_{cwA}"] &
\end{tikzcd}
$$

Same as the distributive law $dist$, the following digrams also commute:

$$
\begin{tikzcd}
cwecwA \arrow[rr, "cwe\delta^{cw}_{A}"] \arrow[dr, "\delta^{cw}_{cweA}"'] & & cwe(cw)^2A \\
& (cw)^2ecwA \arrow[ur, "cwdistEx_{cwA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
ecweA \arrow[rr, "ecw\delta^{e}_{A}"] \arrow[dr, "\delta^{e}_{cweA}"'] & & ecwe^2A \arrow[dl, "edistEx_{eA}"] \\
& e^2cweA &
\end{tikzcd}
$$

Notice the difference between $dist$ of c over w and $distEx$ of cw over e. While $dist$ is a natural transformation, $distEx$ is a natural isomorphism.

Lemma 11. *let $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ be two monoidal comonads on a Lambek category with cw and exchange $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$. Then the composition of cw and e using the distributive law for exchange $distEx_A : cweA \longrightarrow ecwA$ is a monoidal comonad $(cwe, \varepsilon, \delta)$ on $\mathcal{L}$.*

Proof. Suppose $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ are monoidal comonads, and $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with cw and exchange. Since by definition $cw, e : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors, we know that their composition $cwe : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$
\begin{aligned}
\mathsf{q}_{A,B} &: cweA \otimes cweB \longrightarrow cwe(A \otimes B) \\
\mathsf{q}_{A,B} &= cw\mathsf{q}^{e}_{A,B} \circ \mathsf{q}^{cw}_{eA,eB} \\
\mathsf{q}_{I} &: I \longrightarrow cweI \\
\mathsf{q}_{I} &= cw\mathsf{q}^{e}_{I} \circ \mathsf{q}^{cw}_{I}
\end{aligned}
$$

Analogous to the proof of Lemma 7, each of ε and δ can be given two equivalent definitions:

$$
\begin{tikzcd}
cweA \arrow[r, "\varepsilon^{cw}_{eA}"] \arrow[d, "cw\varepsilon^{e}_{A}"'] & eA \arrow[d, "\varepsilon^{e}_{A}"] \\
cwA \arrow[r, "\varepsilon^{cw}_{A}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
cweA \arrow[r, "cw\delta^{e}_{A}"] \arrow[d, "\delta^{cw}_{eA}"'] & cwe^2A \arrow[r, "\delta^{cw}_{e^2A}"] \arrow[d, "\delta^{cw}_{e^2A}"] & (cw)^2e^2A \arrow[d, "cwdist_{eA}"] \\
(cw)^2eA \arrow[r, "(cw)^2\delta^{e}_{A}"'] & (cw)^2e^2A \arrow[r, "cwdist_{eA}"'] & cwecweA
\end{tikzcd}
$$

And the comonad laws can be proved similarly, which we will not elaborate for simplicity. ◻

Lemma 12. *Let $(cwe, \varepsilon, \delta)$ be a monoidal comonad over a monoidal category $(\mathcal{L}, I, \otimes)$ such that $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with cw and exchange. Then the co-Kleisli category of $\mathcal{L}$, $\mathcal{L}_{cwe}$, is a symmetric monoidal category.*

Proof. The identity object of $\mathcal{L}_{cwe}$ is still I.

The left and right unitors, $\hat{\lambda}_A : I \otimes A \longrightarrow A$ and $\hat{\rho}_A : A \otimes I \longrightarrow A$, in $\mathcal{L}_{cwe}$ are morphisms $cwe(I \otimes A) \longrightarrow A$ and $cwe(A \otimes I) \longrightarrow A$ in $\mathcal{L}$, respectively. Then we define $\hat{\lambda}$ and $\hat{\rho}$ as:

$$
\begin{aligned}
\hat{\lambda}_A &= \varepsilon_A \circ cwe\lambda_A \\
\hat{\rho}_A &= \varepsilon_A \circ cwe\rho_A,
\end{aligned}
$$

where λ and ρ are the left and right unitors in $\mathcal{L}$, respectively. And we define their inverses as:

$$
\begin{aligned}
\hat{\lambda}_A^{-1} &= \varepsilon_{I\otimes A} \circ cwe\lambda_A^{-1} \\
\hat{\rho}_A^{-1} &= \varepsilon_{A\otimes I} \circ cwe\rho_A^{-1}
\end{aligned}
$$

$\hat{\lambda}$ is a nautral isomorphism with inverse $\hat{\lambda}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
cwe(I \otimes A) \arrow[r, "\delta_{I\otimes A}"] \arrow[dr, "cwe\lambda_A"] \arrow[dd, "\varepsilon_{I\otimes A}"'] \arrow[ddr, equal] & (cwe)^2(I \otimes A) \arrow[r, "(cwe)^2\lambda_A"] \arrow[d, phantom, "(1)"] & (cwe)^2 A \arrow[dd, "cwe\varepsilon_A"] \\
\arrow[r, phantom, "(2)"] & cweA \arrow[ur, "\delta_A"'] \arrow[d, "cwe\lambda_A^{-1}"] \arrow[dr, equal] \arrow[r, phantom, "(5)"] \arrow[l, phantom, "(3)"] \arrow[dr, phantom, "(4)"'] & {} \\
I \otimes A & cwe(I \otimes A) \arrow[l, "\varepsilon_{I\otimes A}"] & cweA \arrow[l, "cwe\lambda_A^{-1}"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ. (2), (3) and (4) commute trivially. And (5) commutes because *cwe* is a comonad.

Similarly, $\hat{\rho}$ is a natural isomorphism with inverse $\hat{\rho}^{-1}$ by the following diagram chasing:

$$
\begin{tikzcd}
cwe(A \otimes I) \arrow[r, "\delta_{A\otimes I}"] \arrow[dr, "cwe\rho_A"] \arrow[dd, "\varepsilon_{A\otimes I}"'] \arrow[ddr, equal] & (cwe)^2(A \otimes I) \arrow[r, "(cwe)^2\rho_A"] & (cwe)^2 A \arrow[dd, "cwe\varepsilon_A"] \\
 & cweA \arrow[ur, "\delta_A"'] \arrow[d, "cwe\rho_A^{-1}"] \arrow[dr, equal] & \\
A \otimes I & cwe(A \otimes I) \arrow[l, "\varepsilon_{A\otimes I}"] & cweA \arrow[l, "cwe\rho_A^{-1}"]
\end{tikzcd}
$$

The associator $\hat{\alpha}_A : (A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}_{cwe}$ is the morphism $cwe((A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}$. We define $\hat{\alpha}$ as:

$$
\hat{\alpha}_{A,B,C} = \varepsilon_{A\otimes(B\otimes C)} \circ cwe\alpha_{A,B,C},
$$

where α is the associator of $\mathcal{L}$. And its inverse is

$$\hat{\alpha}^{-1}_{A,B,C} = \varepsilon_{(A\otimes B)\otimes C} \circ cwe\alpha^{-1}_{A,B,C}$$

$\hat{\alpha}$ is a natural isomorphism with inverse $\hat{\alpha}^{-1}$ because the following diagram chasing commutes:

$$\begin{tikzcd}
cwe((A\otimes B)\otimes C) \arrow[r, "\delta_{(A\otimes B)\otimes C}"] \arrow[dr, "cwe\alpha_{A,B,C}"] \arrow[dd, "\varepsilon_{(A\otimes B)\otimes C}"'] \arrow[ddr, equals] & (cwe)^2((A\otimes B)\otimes C) \arrow[r, "(cwe)^2\alpha_{A,B,C}"] & (cwe)^2(A\otimes(B\otimes C)) \arrow[dd, "cwe\varepsilon_{A\otimes(B\otimes C)}"] \\
 & cwe(A\otimes(B\otimes C)) \arrow[ur, "\delta_{A\otimes(B\otimes C)}"] \arrow[d, "cwe\alpha^{-1}_{A,B,C}"] \arrow[dr, equals] & \\
(A\otimes B)\otimes C & cwe((A\otimes B)\otimes C) \arrow[l, "\varepsilon_{(A\otimes B)\otimes C}"] & cwe(A\otimes(B\otimes C)) \arrow[l, "cwe\alpha^{-1}_{A,B,C}"]
\end{tikzcd}$$

Therefore, $\mathcal{L}_{cwe}$ is a monoidal category.

The symmetry, $\hat{\beta}_{A,B} : A\otimes B \longrightarrow B\otimes A$, in $\mathcal{L}_{cwe}$ is the morphism $cwe(A\otimes B)\longrightarrow B\otimes A$ in $\mathcal{L}$, which is defined as:

$$\hat{\beta}_{A,B} = \varepsilon^{cw}_{B\otimes A} \circ cw\gamma_{A,B},$$

where $\varepsilon^{cw}_A : cwA \longrightarrow A$ is a natural transformation associated with the comonad cw, and γ is the natural isomorphism defined in Lemma 4. Then its inverse is

$$\hat{\beta}^{-1}_{A,B} = \varepsilon^{cw}_{A\otimes B} \circ cw\gamma_{B,A}$$

$\hat{\beta}$ is a natural isomorphism with inverse $\hat{\beta}^{-1}$ because the following diagram chasing commutes:

$$\begin{tikzcd}
A\otimes B \arrow[dr, phantom, "(1)" near start] & cw(A\otimes B) \arrow[l, "\varepsilon^{cw}_{A\otimes B}"'] \arrow[dl, equals] \arrow[d, "\delta^{cw}_{A\otimes B}"] \arrow[drr, phantom, "(3)"] & & cwe(A\otimes B) \arrow[ll, "cw\varepsilon^{e}_{A\otimes B}"'] \arrow[d, "\delta^{cw}_{e(A\otimes B)}"] \\
cw(A\otimes B) \arrow[u, "\varepsilon^{cw}_{A\otimes B}"] \arrow[r, phantom, "(2)" above] \arrow[ddr, phantom, "(4)"] & (cw)^2(A\otimes B) \arrow[l, "cw\varepsilon^{cw}_{A\otimes B}"] \arrow[drr, phantom, "(5)"] & & (cw)^2e(A\otimes B) \arrow[ll, "(cw)^2\varepsilon^{e}_{A\otimes B}"] \arrow[d, "(cw)^2\delta^{e}_{A\otimes B}"] \\
 & (cw)^2e(B\otimes A) \arrow[u, "(cw)^2\gamma_{B,A}"] \arrow[d, equals] \arrow[ddl, "cw\varepsilon^{cw}_{e(B\otimes A)}"'] & (cw)^2e^2(A\otimes B) \arrow[l, "(cw)^2e\gamma_{A,B}"] \arrow[r, equals] \arrow[d, phantom, "(8)"] \arrow[dr, phantom, "(9)"] & (cw)^2e^2(A\otimes B) \arrow[d, "cwdistEx_{e(A\otimes B)}"] \\
 & (cw)^2e(B\otimes A) \arrow[drr, "cwdistEx_{B\otimes A}"] \arrow[d, phantom, "(6)"] \arrow[r, phantom, "(7)"] & & (cwe)^2(A\otimes B) \arrow[ul, "cwdistEx^{-1}_{e(A\otimes B)}"] \arrow[d, "cwecw\gamma_{A,B}"] \\
cwe(B\otimes A) \arrow[uuu, "cw\gamma_{B,A}"] & & & cwecw(B\otimes A) \arrow[lll, "cwe\varepsilon^{cw}_{B\otimes A}"] \arrow[uull, "cwdistEx^{-1}_{B\otimes A}"']
\end{tikzcd}$$

(1), (7) and (9) commute trivially. (2) is the comonad law for cw. (3) commutes by the naturality of δ^{cw}. (4) commutes by the naturality of ε^{cw}. (5) commutes because γ is a natural isomorphism (Lemma 4). (6) is the definition of $distEx$. (8) is the naturality of $distEx$.

In conclusion, $\mathcal{L}_{cwe}$ is a symmetric monoidal category. $\square$

3 Related Work

TODO

4 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.

A Appendix

A.1 Symmetric Monoidal Categories

Definition 13. *A **monoidal category** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A \\
\rho_A &: A \otimes \top \longrightarrow A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Definition 14. *A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A \\
\rho_A &: A \otimes \top \longrightarrow A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

- *A symmetry natural isomorphism:*

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 15. *A* ***monoidal biclosed category*** *is a monoidal category $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, each of the functors $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ and $B \otimes - : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any object A and C of $\mathcal{M}$, there are two objects $C \leftharpoonup B$ and $B \rightharpoonup C$ of $\mathcal{M}$ and two natural bijections:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, C \leftharpoonup B)$$
$$\mathsf{Hom}_{\mathcal{M}}(B \otimes A, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \rightharpoonup C)$$

Definition 16. *A* ***symmetric monoidal closed category (SMCC)*** *is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Definition 17. *Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a* ***monoidal functor*** *is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2\,FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"'] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"'] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1\,A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1}\otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA}\otimes m_{\top_1}"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"]
\end{tikzcd}$$

Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

Definition 18. *Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B}\otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA}\otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A\otimes_1 B,C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A,B\otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1}\otimes \mathsf{id}_{FA}"'] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"'] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA}\otimes m_{\top_1}"'] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"'] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"'] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"'] & F(B \otimes_1 A)
\end{tikzcd}$$

Definition 19. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **monoidal natural transformation** is a natural transformation, $f : F \longrightarrow G$, subject to the following* *coherence diagrams:*

$$
\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}
$$

Definition 20. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***symmetric monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$
\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}
$$

Definition 21. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are monoidal natural transformations. Thus, the following diagrams must commute:*

$$
\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}
$$

Definition 22. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***symmetric monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations. Thus, the following* *diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 23. *A* ***monoidal comonad*** *on a monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2 A$ are monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2 A \arrow[d, "T\delta_A"] \\
T^2 A \arrow[r, "\delta_{TA}"'] & T^3 A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2 A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
 & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\
 & \top &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

Definition 24. *A **symmetric monoidal comonad** on a symmetric monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
 & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\
 & \top &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$