

# On Linear Modalities for Exchange, Weakening, and Contraction

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## Abstract

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## 1 Introduction

TODO [1]

## 2 Categorical Models

### 2.1 Lambek Categories

► **Definition 1.** A **monoidal category**,  $(\mathcal{L}, \otimes, I, \lambda, \rho)$ , is a category,  $\mathcal{L}$ , equipped with a bifunctor,  $\otimes : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ , called the tensor product, a distinguished object  $I$  of  $\mathcal{L}$  called the unit, and three natural isomorphisms  $\lambda_A : I \otimes A \rightarrow A$ ,  $\rho_A : A \otimes I \rightarrow A$ , and  $\alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$  called the left and right unitors and the associator respectively. Finally, these are subject to the following coherence diagrams:

$$\begin{array}{ccccc}
 ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes \text{id}_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\
 \downarrow \alpha_{A \otimes B,C,D} & & & & \downarrow \text{id}_A \otimes \alpha_{B,C,D} \\
 (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} & & & A \otimes (B \otimes (C \otimes D))
 \end{array}$$
  

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\
 \downarrow \rho_A \otimes \text{id}_B & & \downarrow \text{id}_A \otimes \lambda_B \\
 A \otimes B & & A \otimes B
 \end{array}$$

► **Definition 2.** A **Lambek category** is a monoidal category  $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$  equipped with two bifunctors  $\multimap : \mathcal{L}^{\text{op}} \times \mathcal{L} \rightarrow \mathcal{L}$  and  $\multimap : \mathcal{L} \times \mathcal{L}^{\text{op}} \rightarrow \mathcal{L}$  that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\text{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \text{Hom}_{\mathcal{L}}(X, A \multimap B)$$

$$\text{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \text{Hom}_{\mathcal{L}}(X, B \multimap A)$$



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One might call Lambek categories biclosed monoidal categories, but we name them in homage to Lambek for his contributions to non-commutative linear logic.

► **Definition 3.** A monoidal category  $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$  is **symmetric** if there is a natural isomorphism  $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$  such that  $\beta_{B,A} \circ \beta_{A,B} = \text{id}_{A \otimes B}$  and the following commute:

$$\begin{array}{ccc}
 (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) \xrightarrow{\beta_{A,B \otimes C}} (B \otimes C) \otimes A \\
 \downarrow \beta_{A,B} \otimes \text{id}_C & & \downarrow \alpha_{B,C,A} \\
 (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) \xrightarrow{\text{id}_B \otimes \beta_{A,C}} B \otimes (C \otimes A)
 \end{array}
 \quad
 \begin{array}{ccc}
 I \otimes A & \xrightarrow{\beta_{I,A}} & A \otimes I \\
 \downarrow \lambda_A & & \downarrow \rho_A \\
 & A &
 \end{array}$$

► **Definition 4.** A symmetric monoidal category  $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \beta)$  is **closed** if it comes equipped with a bifunctor  $\multimap : \mathcal{L}^{\text{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$  that is right adjoint to the tensor product. That is, the following natural bijection  $\text{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \text{Hom}_{\mathcal{L}}(X, A \multimap B)$  holds.

## 2.2 Lambek Categories with Exchange

► **Definition 5.** A **Lambek category with exchange**,  $(\mathcal{L}, e, \text{ex})$ , is a Lambek category equipped with a monoidal comonad  $(e, \varepsilon, \delta)$  on  $\mathcal{L}$ , and a monoidal natural transformation  $\text{ex}_{A,B} : eA \otimes eB \longrightarrow eB \otimes eA$ . We require  $\text{ex}$  to be a coalgebra morphism, and the following diagrams must commute:

$$\begin{array}{ccc}
 e^2A \otimes e^2B & \xrightarrow{\text{ex}_{eA,eB}} & e^2B \otimes e^2A \\
 \downarrow q_{eA,eB} & & \downarrow q_{eB,eA} \\
 e(eA \otimes eB) & \xrightarrow{e\text{ex}_{A,B}} & e(eB \otimes eA)
 \end{array}
 \quad
 \begin{array}{ccc}
 eA \otimes eB & \xrightarrow{\text{ex}_{A,B}} & eB \otimes eA \\
 \downarrow \text{id} & & \downarrow \text{id} \\
 eA \otimes eB & \xrightarrow{\text{ex}_{B,A}} & eA \otimes eB
 \end{array}
 \quad
 \begin{array}{ccc}
 eI \otimes eA & \xrightarrow{\text{ex}_{I,A}} & eA \otimes eI \\
 \downarrow \varepsilon_I \otimes \text{id} & & \downarrow \text{id} \otimes \varepsilon_I \\
 I \otimes eA & & eA \otimes I \\
 \downarrow \lambda_{eA} & & \downarrow \rho_{eA} \\
 & eA &
 \end{array}$$

$$\begin{array}{ccc}
 (eA \otimes eB) \otimes eC & \xrightarrow{\alpha_{eA,eB,eC}} & eA \otimes (eB \otimes eC) \xrightarrow{\text{id}_{eA} \otimes (\delta_B \otimes \delta_C)} eA \otimes (e^2B \otimes e^2C) \xrightarrow{\text{id}_{eA} \otimes q_{eB,eC}} eA \otimes e(eB \otimes eC) \\
 \downarrow \text{ex}_{A,B} \otimes \text{id}_{eC} & & \downarrow \text{ex}_{eA,eB \otimes eC} \\
 (eB \otimes eA) \otimes eC & & e(eB \otimes eC) \otimes eA \\
 \downarrow \alpha_{eB,eA,eC} & & \downarrow \varepsilon_{eB \otimes eC} \otimes \text{id}_{eA} \\
 eB \otimes (eA \otimes eC) & \xrightarrow{\text{id}_{eB} \otimes \text{ex}_{A,C}} & eB \otimes (eC \otimes eA) \xleftarrow{\alpha_{eB,eC,eA}} (eB \otimes eC) \otimes eA
 \end{array}$$

Furthermore, for any coalgebra morphisms  $f : (eA, \delta) \longrightarrow (eB, \delta)$  and  $g : (eC, \delta) \longrightarrow (eD, \delta)$  between free coalgebras the following diagram must commute:

$$\begin{array}{ccc}
 eA \otimes eC & \xrightarrow{f \otimes g} & eB \otimes eD \\
 \downarrow \text{ex}_{A,C} & & \downarrow \text{ex}_{B,D} \\
 eC \otimes eA & \xrightarrow{g \otimes f} & eD \otimes eB
 \end{array}$$

The morphism  $q_{A,B} : eA \otimes eB \longrightarrow e(A \otimes B)$  makes  $(e, q)$  a monoidal functor.

The first diagram in the previous definition makes  $e : \mathcal{L} \longrightarrow \mathcal{L}$  a symmetric monoidal functor, and the second, third, and forth diagrams make the category of free coalgebras (the Kleisli category) symmetric monoidal.

► **Definition 6.** Suppose  $(\mathcal{L}, e, \text{ex})$  is a Lambek category with exchange. Then the **Eilenberg Moore category**,  $\mathcal{L}^e$ , of the comonad  $(e, \varepsilon, \delta)$  has as objects all the  $e$ -coalgebras  $(A, h_A : A \rightarrow eA)$ , and as morphisms all the coalgebra morphisms. We call  $h_A$  the action of the coalgebra. Furthermore, the following (action) diagrams must commute:

$$\begin{array}{ccc} A & \xrightarrow{h_A} & eA \\ h_A \downarrow & & \downarrow eh_A \\ eA & \xrightarrow{\delta_A} & e^2A \end{array} \quad \begin{array}{ccc} A & & \\ h_A \downarrow & \searrow & \\ eA & \xrightarrow{\varepsilon_A} & A \end{array}$$

► **Lemma 7** (The Eilenberg Moore Category of the comonad  $e$  is Monoidal). *The category  $\mathcal{L}^e$  is monoidal.*

**Proof.** For the complete proof see Appendix B.1.1. ◀

► **Lemma 8.** *In  $\mathcal{L}^e$  there is a natural transformation  $\beta_{A,B} : A \otimes B \rightarrow B \otimes A$ .*

**Proof.** We define  $\beta$  as follows:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{eX_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

For the proof that it is natural please see Appendix B.1.2. ◀

► **Corollary 9.** *For any coalgebras  $(A, h_A)$  and  $(B, h_B)$  the followings commute:*

$$\begin{array}{ccccc} A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{eX_{A,B}} & eB \otimes eA & \xrightarrow{\varepsilon_B \otimes \varepsilon_A} & B \otimes A \\ \parallel & & & & & & \downarrow h_B \otimes h_A \\ A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{eX_{A,B}} & eB \otimes eA & = & eB \otimes eA \\ & & eA \otimes eB & \xrightarrow{eX_{A,B}} & eB \otimes eA & & \\ & \downarrow q_{A,B} & & \downarrow q_{B,A} & & & \\ & e(A \otimes B) & \xrightarrow{e\beta_{A,B}} & e(B \otimes A) & & & \end{array}$$

**Proof.** For the complete proof please see Appendix B.1.3. ◀

► **Definition 10.** Given two parallel arrows  $f, g : B \rightarrow C$  in a category  $C$ , a **cofork** is a morphism  $c : A \rightarrow B$  such that the diagram  $A \xrightarrow{c} B \xrightarrow[f]{g} C$  commutes. That is,  $f \circ c = g \circ c$ .

► **Lemma 11.** *The morphism  $eX_{A,B} \circ (h_A \otimes h_B)$  is a cofork of the morphisms  $(h_B \otimes h_A) \circ (\varepsilon_B \otimes \varepsilon_A)$  and  $(e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A)$ .*

**Proof.** This proof holds by straightforward equational reasoning. For the complete proof please see Appendix B.1.4. ◀

► **Lemma 12.** *In  $\mathcal{L}^e$ ,  $\beta$  is a coalgebra morphism.*

**Proof.** For the complete proof see Appendix B.1.5. ◀

► **Lemma 13** (The Eilenberg-Moore Category of the comonad  $e$  is Symmetric Monoidal). *The category  $\mathcal{L}^e$  is symmetric monoidal.*

**Proof.** For the complete proof please see Appendix B.1.6. ◀

### 2.3 Lambek Categories with Weakening

► **Definition 14.** A **Lambek category with weakening**,  $(\mathcal{L}, w, \text{weak})$ , is a Lambek category equipped with a monoidal comonad  $(w, \varepsilon, \delta)$ , and a monoidal natural transformation  $\text{weak}_A : wA \longrightarrow I$ . Furthermore,  $\text{weak}$  must be a coalgebra morphism. That is, the following diagram must commute:

$$\begin{array}{ccc} wA & \xrightarrow{\text{weak}_A} & I \\ \delta_A \downarrow & & \downarrow q_I \\ w^2A & \xrightarrow{w\text{weak}_A} & wI \end{array}$$

► **Definition 15.** Suppose  $(\mathcal{L}, w, \text{weak})$  is a Lambek category with weakening. Then the **Eilenberg Moore category**,  $\mathcal{L}^w$ , of the comonad  $(w, \varepsilon, \delta)$  has as objects all the  $w$ -coalgebras  $(A, h_A : A \longrightarrow wA)$ , and as morphisms all the coalgebra morphisms. We call  $h_A$  the action of the coalgebra. Furthermore, the following (action) diagrams must commute:

$$\begin{array}{ccc} A & \xrightarrow{h_A} & wA \\ h_A \downarrow & & \downarrow wh_A \\ wA & \xrightarrow{\delta_A} & w^2A \end{array} \quad \begin{array}{ccc} A & & \\ h_A \downarrow & \searrow & \\ wA & \xrightarrow{\varepsilon_A} & A \end{array}$$

► **Lemma 16** (The Eilenberg Moore Category of the comonad  $w$  is Monoidal). *The category  $\mathcal{L}^w$  is monoidal.*

**Proof.** The proof is similar with the proof of Lemma 7. ◀

### 2.4 Lambek Categories with Contraction

► **Definition 17.** A **Lambek category with contraction**,  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$ , is a Lambek category equipped with a monoidal comonad  $(c, \varepsilon, \delta)$ , and two monoidal natural transformations:

$$\begin{aligned} \text{contraL}_{A,B} : cA \otimes B &\longrightarrow (cA \otimes B) \otimes cA \\ \text{contraR}_{A,B} : B \otimes cA &\longrightarrow cA \otimes (B \otimes cA) \end{aligned}$$

Furthermore, the following diagrams must commute:

$$\begin{array}{ccccc} cA \otimes I & \xleftarrow{\rho_{cA}^{-1}} & cA & \xrightarrow{\lambda_{cA}^{-1}} & I \otimes cA \\ \text{contraL}_{A,I} \downarrow & & & & \downarrow \text{contraR}_{A,I} \\ (cA \otimes I) \otimes cA & \xrightarrow{\alpha_{cA,I,cA}} & & & cA \otimes (I \otimes cA) \\ \\ cA \otimes cA & \xrightarrow{id_{cA} \otimes \rho_{cA}^{-1}} & cA \otimes (cA \otimes I) & \xrightarrow{id_{cA} \otimes \text{contraL}_{A,I}} & cA \otimes ((cA \otimes I) \otimes cA) \\ \downarrow \lambda_{cA}^{-1} \otimes id_{cA} & & & & \downarrow id_{cA} \otimes (\rho_{cA} \otimes id_{cA}) \\ (I \otimes cA) \otimes cA & \xrightarrow{\text{contraR}_{A,I} \otimes id_{cA}} & (cA \otimes (I \otimes cA)) \otimes cA & \xrightarrow{(id_{cA} \otimes \lambda_{cA}) \otimes id_{cA}} & (cA \otimes cA) \otimes cA \\ & & & & \uparrow \alpha_{cA,cA,cA} \end{array}$$

► **Definition 18.** Suppose  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction. Then the **Eilenberg Moore category**,  $\mathcal{L}^c$ , of the comonad  $(c, \varepsilon, \delta)$  has as objects all the  $c$ -coalgebras  $(A, h_A : A \rightarrow cA)$ , and as morphisms all the coalgebra morphisms. We call  $h_A$  the action of the coalgebra. Furthermore, the following (action) diagrams must commute:

$$\begin{array}{ccc} A & \xrightarrow{h_A} & cA \\ h_A \downarrow & & \downarrow ch_A \\ cA & \xrightarrow{\delta_A} & c^2A \end{array} \quad \begin{array}{ccc} A & & \\ h_A \downarrow & \searrow & \\ cA & \xrightarrow{\varepsilon_A} & A \end{array}$$

► **Lemma 19** (The Eilenberg Moore Category of the comonad  $c$  is Monoidal). *The category  $\mathcal{L}^c$  is monoidal.*

**Proof.** The proof is similar with the proof of Lemma 7. ◀

## 2.5 Linear Categories

► **Definition 20.** A **linear category**,  $(\mathcal{L}, !, \text{weak}, \text{contra})$ , is a symmetric monoidal closed category  $(\mathcal{L}, I, \otimes, -\circ)$  equipped with a symmetric monoidal comonad  $(!, \varepsilon, \delta)$  with  $q_{A,B} : !A \otimes !B \rightarrow !(A \otimes B)$  and  $q_I : I \rightarrow !I$ , and two monoidal natural transformations with components  $\text{weak}_A : !A \rightarrow I$  and  $\text{contra}_A : !A \rightarrow !A \otimes !A$ , satisfying the following conditions:

- each  $(!A, \text{weak}_A, \text{contra}_A)$  is a commutative comonoid, i.e. the following diagrams commute and  $\beta \circ \text{contra}_A = \text{contra}_A$  where  $\beta_{B,C} : B \otimes C \rightarrow C \otimes B$  is the symmetry natural transformation of  $\mathcal{L}$ ;

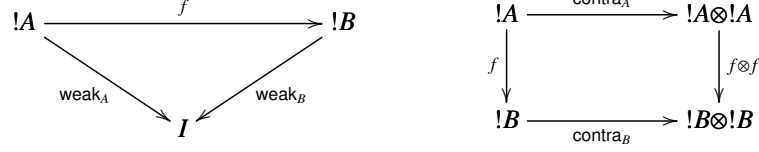
$$\begin{array}{ccccc} !A & \xrightarrow{\text{contra}_A} & !A \otimes !A & \xrightarrow{id_{!A} \otimes \text{contra}_A} & !A \otimes (!A \otimes !A) \\ \text{contra}_A \downarrow & & & & \uparrow \alpha_{!A, !A, !A} \\ !A \otimes !A & \xrightarrow{\text{contra}_A \otimes id_{!A}} & (!A \otimes !A) \otimes !A & & \\ & & \downarrow \text{contra}_A & & \\ & & !A & \xrightarrow{\lambda^{-1}} & I \otimes !A \\ & & \downarrow \text{contra}_A & & \downarrow \rho^{-1} \\ & & !A \otimes !A & \xrightarrow{id_{!A} \otimes \text{weak}_A} & !A \otimes I \end{array}$$

- $\text{weak}_A$  and  $\text{contra}_A$  are coalgebra morphisms, i.e. the following diagrams commute;

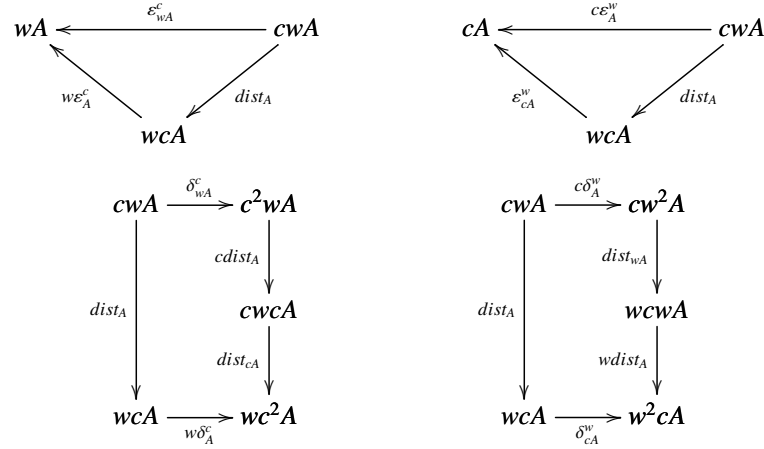
$$\begin{array}{ccc} !A & \xrightarrow{\text{weak}_A} & I \\ \delta_A \downarrow & & \downarrow q_I \\ !!A & \xrightarrow{! \text{weak}_A} & !I \end{array}$$

$$\begin{array}{ccccc} !A & \xrightarrow{\text{contra}_A} & !A \otimes !A & \xrightarrow{\delta_A \otimes \delta_A} & !!A \otimes !!A \\ \delta_A \downarrow & & & & \downarrow q_{!A, !A} \\ !!A & \xrightarrow{! \text{contra}_A} & !(A \otimes !A) & & \end{array}$$

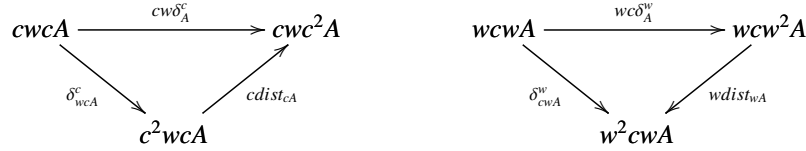
- any coalgebra morphism  $f : (!A, \delta_A) \rightarrow (!B, \delta_B)$  between free coalgebras preserve the comonoid structure given by  $\text{weak}$  and  $\text{contra}$ , i.e. the following diagrams commute.



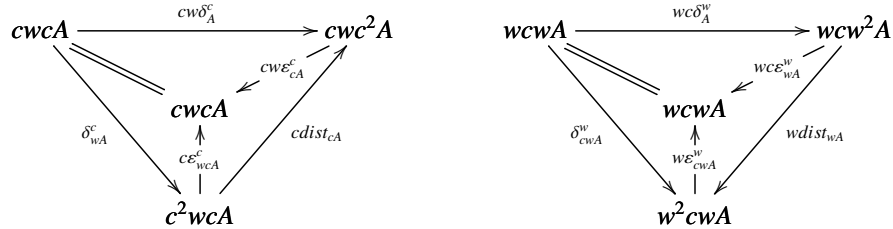
► **Definition 21.** Given two comonads  $(c, \varepsilon^c, \delta^c)$  and  $(w, \varepsilon^w, \delta^w)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction and  $(\mathcal{L}, w, \text{weak})$  is a Lambek category with weakening, we define a **distributive law** of  $c$  over  $w$  to be a natural transformation with components  $\text{dist}_A : cwA \rightarrow wcA$ , subject to the following coherence diagrams:



► **Lemma 22.** Given two comonads  $(c, \varepsilon^c, \delta^c)$  and  $(w, \varepsilon^w, \delta^w)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction and  $(\mathcal{L}, w, \text{weak})$  is a Lambek category with weakening, the following two diagrams commute:



**Proof.** The two diagrams above commute because the following ones commute by the distributive law and the comonad laws for  $c$  and  $w$ .



► **Lemma 23** (Composition of Weakening and Contraction). Suppose  $(\mathcal{L}, I, \otimes, w, \text{weak}^w, c, \text{contraL}, \text{contraR})$  is a Lambek category with weakening and contraction, where  $(w, \varepsilon^w, \delta^w)$  and  $(c, \varepsilon^c, \delta^c)$  are the respective monoidal comonads. Then the composition of  $c$  and  $w$  using the distributive law  $\text{dist}_A : cwA \rightarrow wcA$  is a monoidal comonad on  $\mathcal{L}$ .

**Proof.** For the complete proof see Appendix B.3. ◀

► **Definition 24.** A **Lambek category with  $cw$** ,  $(\mathcal{L}, cw, \text{weak}^w, \text{contraL}, \text{contraR}, \text{dist})$ , is a Lambek category with weakening and contraction, and a distributive law. Furthermore, the following coherence diagrams commute:

$$\begin{array}{ccc}
 I \otimes cwA & \xrightarrow{\lambda_{I \otimes cwA}^{-1}} & I \otimes (I \otimes cwA) \\
 \downarrow \text{contraR}_{wA, I} & & \uparrow \text{weak}_A^w \otimes id_{I \otimes cwA} \\
 cwA \otimes (I \otimes cwA) & \xrightarrow{\varepsilon_{wA}^c \otimes id_{I \otimes cwA}} & wA \otimes (I \otimes cwA)
 \end{array}$$
  

$$\begin{array}{ccc}
 cwA \otimes I & \xrightarrow{\rho_{cwA \otimes I}^{-1}} & (cwA \otimes I) \otimes I \\
 \downarrow \text{contraL}_{wA, I} & & \uparrow id_{cwA \otimes I} \otimes \text{weak}_A^w \\
 (cwA \otimes I) \otimes cwA & \xrightarrow{id_{cwA \otimes I} \otimes \varepsilon_{wA}^c} & (cwA \otimes I) \otimes wA
 \end{array}$$
  

$$\begin{array}{ccc}
 cwA & \xrightarrow{f} & cwB \\
 \downarrow \varepsilon_{wA}^c & & \downarrow \varepsilon_{wB}^c \\
 wA & \xrightarrow{\text{weak}_A^w} I & \xleftarrow{\text{weak}_B^w} wB
 \end{array}$$

where  $f : (cwA, \delta_A) \rightarrow (cwB, \delta_B)$  is any coalgebra morphism between free coalgebras.

► **Lemma 25.** Let  $(\mathcal{L}, cw, \text{weak}^w, \text{contraL}, \text{contraR})$  be a Lambek category with  $cw$ . Then the following conditions are satisfied:

1. There exist two natural transformations  $\text{weak}_A : cwA \rightarrow I$  and  $\text{contra}_A : cwA \rightarrow cwA \otimes cwA$ .
2. Each  $(cwA, \text{weak}_A, \text{contra}_A)$  is a comonoid.
3.  $\text{weak}_A$  and  $\text{contra}_A$  are coalgebra morphisms.
4. Any coalgebra morphism  $f : (cwA, \delta_A) \rightarrow (cwB, \delta_B)$  between free coalgebras preserves the comonoid structure given by  $\text{weak}$  and  $\text{contra}$ .

**Proof.** We will only prove the first condition by defining  $\text{weak}$  and  $\text{contra}$ . For the complete proof see Appendix B.4. Each of  $\text{weak}$  and  $\text{contra}$  can be given two equivalent definitions.  $\text{weak}_A : cwA \rightarrow I$  is defined as in the diagram below. The left triangle commutes by the definition of  $\text{dist}$  and the right triangle commutes by the definition of  $\text{weak}^w$ .

$$\begin{array}{ccccc}
 & & wcA & & \\
 & \nearrow \text{dist}_A & \downarrow w\varepsilon_A^c & \searrow \text{weak}_{cwA}^w & \\
 cwA & \xrightarrow{\varepsilon_{wA}^c} & wA & \xrightarrow{\text{weak}_A^w} & I
 \end{array}$$

$\text{contra}_A : cwA \rightarrow cwA \otimes cwA$  is defined as below. The left part of the diagram commutes by the definitions of  $\text{contraL}$  and of  $\text{contraR}$ , and the right part commutes because  $\mathcal{L}$  is monoidal.

$$\begin{array}{ccccc}
 cwA & \xrightarrow{\rho_{cwA}^{-1}} & cwA \otimes I & \xrightarrow{\text{contraL}_{wA, I}} & (cwA \otimes I) \otimes cwA \\
 \downarrow \lambda_{cwA}^{-1} & & & \swarrow \alpha_{cwA, I, cwA} & \downarrow \rho_{cwA} \otimes id_{cwA} \\
 I \otimes cwA & \xrightarrow{\text{contraR}_{wA, I}} & cwA \otimes (I \otimes cwA) & \xrightarrow{id_{cwA} \otimes \lambda_{cwA}} & cwA \otimes cwA
 \end{array}$$

► **Definition 26.** Given two comonads  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, cw, \text{weak}, \text{contra})$  is a Lambek category with  $cw$  and  $(\mathcal{L}, e, \text{ex})$  is a Lambek category with exchange, we define a **distributive law for exchange** of  $cw$  over  $e$  to be a natural isomorphism with components  $\text{distEx}_A : cweA \longrightarrow ecwA$ , subject to the following coherence diagrams:

$$\begin{array}{ccc} eA & \xleftarrow{\varepsilon_{eA}^{cw}} & cweA \\ & \swarrow e\varepsilon_A^{cw} & \searrow \text{distEx}_A \\ & ecwA & \end{array} \qquad \begin{array}{ccc} cwA & \xleftarrow{cw\varepsilon_A^e} & cweA \\ & \swarrow \varepsilon_{cwA}^e & \searrow \text{distEx}_A \\ & ecwA & \end{array}$$

► **Lemma 27.** Given two comonads  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, cw, \text{weak}, \text{contra})$  is a Lambek category with  $cw$  and  $(\mathcal{L}, e, \text{ex})$  is a Lambek category with exchange, the following two digrams also commute:

$$\begin{array}{ccc} cweA & \xrightarrow{cwe\delta_A^{cw}} & cwe(cw)^2A \\ \delta_{cweA}^{cw} \searrow & & \nearrow cwe\delta_{cwA}^{cw} \\ & (cw)^2ecwA & \end{array} \qquad \begin{array}{ccc} ecweA & \xrightarrow{ecw\delta_A^e} & ecwe^2A \\ \delta_{ecweA}^e \searrow & & \nearrow ecw\delta_{eA}^e \\ & e^2cweA & \end{array}$$

The proof is similar with the proof of Lemma 22 and we will not elaborate it here. Also, notice the difference between  $\text{dist}$  of  $c$  over  $w$  and  $\text{distEx}$  of  $cw$  over  $e$ . While  $\text{dist}$  is a natural transformation,  $\text{distEx}$  is a natural isomorphism.

► **Lemma 28.** let  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  be two monoidal comonads on a Lambek category with  $cw$  and exchange  $(\mathcal{L}, I, \otimes, cw, \text{weak}, \text{contra}, e, \text{ex})$ . Then the composition of  $cw$  and  $e$  using the distributive law for exchange  $\text{distEx}_A : cweA \longrightarrow ecwA$  is a monoidal comonad  $(cwe, \varepsilon, \delta)$  on  $\mathcal{L}$ .

**Proof.** Suppose  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  are monoidal comonads, and  $(\mathcal{L}, I, \otimes, cw, \text{weak}, \text{contra}, e, \text{ex})$  is a Lambek category with  $cw$  and exchange. Since by definition  $cw, e : \mathcal{L} \longrightarrow \mathcal{L}$  are monoidal functors, we know that their composition  $cwe : \mathcal{L} \longrightarrow \mathcal{L}$  is a monoidal functor:

$$q_{A,B} : cweA \otimes cweB \longrightarrow cwe(A \otimes B)$$

$$q_{A,B} = cwq_{A,B}^e \circ q_{eA,eB}^{cw}$$

$$q_I : I \longrightarrow cweI$$

$$q_I = cwq_I^e \circ q_I^{cw}$$

Analogous to the proof of Lemma 23, each of  $\varepsilon$  and  $\delta$  can be given two equivalent definitions:

$$\begin{array}{ccc} cweA & \xrightarrow{\varepsilon_{eA}^{cw}} & eA \\ \downarrow cw\varepsilon_A^e & & \downarrow \varepsilon_A^e \\ cwA & \xrightarrow{\varepsilon_A^{cw}} & A \end{array} \qquad \begin{array}{ccccc} cweA & \xrightarrow{cw\delta_A^e} & cwe^2A & \xrightarrow{\delta_{e^2A}^{cw}} & (cw)^2e^2A \\ \downarrow \delta_{eA}^{cw} & & \downarrow \delta_{e^2A}^{cw} & & \downarrow cwe\delta_{eA}^e \\ (cw)^2eA & \xrightarrow{(cw)^2\delta_A^e} & (cw)^2e^2A & \xrightarrow{cwe\delta_{eA}^e} & cwe^2cweA \end{array}$$

And the comonad laws can be proved similarly, which we will not elaborate for simplicity. ◀

► **Definition 29.** Suppose  $(\mathcal{L}, cwe, \text{weak}, \text{contra}, \text{ex})$  is a Lambek category with contraction, weakening and exchange. Then the Eilenberg Moore category,  $\mathcal{L}^{cwe}$ , of the comonad  $(cwe, \varepsilon, \delta)$  has as



objects all the  $cwe$ -coalgebras  $(A, h_A : A \rightarrow cweA)$ , and as morphisms all the coalgebra morphisms. Furthermore, the following (action) diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{h_A} & eA \\ h_A \downarrow & & \downarrow eh_A \\ eA & \xrightarrow{\delta_A} & e^2A \end{array} \quad \begin{array}{ccc} A & & \\ h_A \downarrow & \searrow & \\ eA & \xrightarrow{\varepsilon_A} & A \end{array}$$

► **Lemma 30** (The Eilenberg-Moore Category of the comonad  $cwe$  is a linear category). *The category  $\mathcal{L}^{cwe}$  is a linear category.*

**Proof.** 1.  $\mathcal{L}^{cwe}$  is a symmetric monoidal closed category.

Similar as the proofs for Lemmas 7 and 13,  $\mathcal{L}^{cwe}$  is symmetric monoidal. The symmetry  $\beta_{A,B} : A \otimes B \rightarrow B \otimes A$  is also defined similarly as:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{ex_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

2.  $(cwe, \varepsilon, \delta)$  is a symmetric monoidal comonad.

Lemma 28 has shown that  $cwe$  is a monoidal comonad.

To prove it is also symmetric, we first define a natural transformation  $ex_{A,B}^{cwe} : cweA \otimes cweB \rightarrow cweB \otimes cweA$  as follows:

$$ex_{A,B}^{cwe} := cweA \otimes cweB \xrightarrow{distEx_A \otimes distEx_B} ecwA \otimes ecwB \xrightarrow{ex_{cwA,cwB}} ecwB \otimes ecwA \xrightarrow{distEx_B^{-1} \otimes distEx_A^{-1}} cweB \otimes cweA$$

Then  $\mathcal{L}^{cwe}$  is symmetric because the following diagram commutes

$$\begin{array}{ccc} cweA \otimes cweB & \xrightarrow{ex_{A,B}^{cwe}} & cweB \otimes cweA \\ q_{A,B} \downarrow & & \downarrow q_{B,A} \\ cwe(A \otimes B) & \xrightarrow{cwe\beta_{A,B}} & cwe(B \otimes A) \end{array}$$

by the diagram chasing below:

$$\begin{array}{ccccc} cweA \otimes cweB & \xrightarrow{distEx_A \otimes distEx_B} & ecwA \otimes ecwB & \xrightarrow{ex_{cwA,cwB}} & ecwB \otimes ecwA \\ \downarrow q_{A,B} & \swarrow \varepsilon_{eA}^{cw} \otimes \varepsilon_{eB}^{cw} & \swarrow e\varepsilon_A^{cw} \otimes e\varepsilon_B^{cw} & \swarrow e\varepsilon_B^{cw} \otimes e\varepsilon_A^{cw} & \downarrow distEx_B^{-1} \otimes distEx_A^{-1} \\ & eA \otimes eB & \xrightarrow{ex_{A,B}} & eB \otimes eA & \\ & \downarrow q_{A,B} & & \downarrow q_{B,A} & \\ & e(A \otimes B) & \xrightarrow{e\beta_{A,B}} & e(B \otimes A) & \\ \downarrow \varepsilon_{e(A \otimes B)}^{cw} & & & & \downarrow \varepsilon_{e(B \otimes A)}^{cw} \\ cwe(A \otimes B) & \xrightarrow{cwe\beta_{A,B}} & cwe(B \otimes A) & & cweB \otimes cweA \\ & & & & \downarrow q_{B,A} \end{array}$$

The triangle on top and the triangle on the right commute by the definition of  $distEx$ . The quadrangle on the left and the one on the right commute because  $q$  is a natural transformation. The quadrangle on top and the one at the bottom commute by the naturality of  $ex$  and  $\varepsilon^{cw}$ , respectively. The square in the middle commutes by Corollary 9.

3. There are two monoidal natural transformations with components  $\text{weak}_A : cweA \longrightarrow I$  and  $\text{contra}_A : cweA \longrightarrow cweA \otimes cweA$ .

The natural transformations  $\text{weak}$  and  $\text{contra}_A$  are defined as:

$$\begin{aligned} \text{weak}_A &:= cweA \xrightarrow{\varepsilon_{weA}^c} weA \xrightarrow{\text{weak}_{eA}^w} I \\ &:= cweA \xrightarrow{c\text{weak}_{eA}^w} cI \xrightarrow{\varepsilon_I^c} I \\ \text{contra}_A &:= cweA \xrightarrow{\text{contra}_{eA}^{cw}} cweA \otimes cweA \end{aligned}$$

They are monoidal by definition.

4. Each  $(cweA, \text{weak}_A, \text{contra}_A)$  is a commutative comonoid.  
 By Lemma 25, each  $(cwA, \text{weak}^{cw}, \text{contra}^{cw})$  is a comonoid in the Lambek category with  $cw$ . Similarly, each  $(cweA, \text{weak}_A, \text{contra}_A)$  is a comonoid in  $\mathcal{L}^{cwe}$ . Since there is a symmetry  $\beta$  in  $\mathcal{L}^{cwe}$ , then  $\beta_A \circ \text{contra}_A = \text{contra}_A$ . So each such comonoid is commutative.
5.  $\text{weak}_A$  and  $\text{contra}_A$  are coalgebra morphisms, by similar proof as Lemma 25.
6. Any coalgebra morphism  $f : (cweA, \delta_A) \longrightarrow (cweA, \delta_B)$  between free coalgebras preserve the comonoid structure given by  $\text{weak}$  and  $\text{contra}$ , by similar proof as Lemma 25.

◀

### 3 Related Work

TODO

### 4 Conclusion

TODO

## References

- 1 P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at <http://research.microsoft.com/en-us/um/people/nick/mixed3.ps>.

## A Appendix

### A.1 Basic Structures on Monoidal Categories

► **Definition 31.** Suppose we are given two monoidal categories  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Then a **monoidal functor** is a functor  $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ , a map  $m_{I_1} : I_2 \longrightarrow FI_1$  and a natural transformation  $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$  subject to the following coherence conditions:

$$\begin{array}{ccc} (FA \otimes_2 FB) \otimes_2 FC & \xrightarrow{\alpha_{2FA, FB, FC}} & FA \otimes_2 (FB \otimes_2 FC) \\ \downarrow m_{A,B} \otimes \text{id}_{FC} & & \downarrow \text{id}_{FA} \otimes m_{B,C} \\ F(A \otimes_1 B) \otimes_2 FC & & FA \otimes_2 F(B \otimes_1 C) \\ \downarrow m_{A \otimes_1 B, C} & & \downarrow m_{A, B \otimes_1 C} \\ F((A \otimes_1 B) \otimes_1 C) & \xrightarrow{F\alpha_{1A, B, C}} & F(A \otimes_1 (B \otimes_1 C)) \end{array}$$

$$\begin{array}{ccc}
I_2 \otimes_2 FA & \xrightarrow{\lambda_{2FA}} & FA \\
\downarrow m_{I_1} \otimes \text{id}_{FA} & & \uparrow F\lambda_{1A} \\
FI_1 \otimes_2 FA & \xrightarrow{m_{I_1, A}} & F(I_1 \otimes_1 A)
\end{array}
\quad
\begin{array}{ccc}
FA \otimes_2 I_2 & \xrightarrow{\rho_{2FA}} & FA \\
\downarrow \text{id}_{FA} \otimes m_{I_1} & & \uparrow F\rho_{1A} \\
FA \otimes_2 FI_1 & \xrightarrow{m_{A, I_1}} & F(A \otimes_1 I_1)
\end{array}$$

Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

► **Definition 32.** Suppose we are given two symmetric monoidal closed categories  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Then a **symmetric monoidal functor** is a monoidal functor  $(F, m) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$  subject to the following additional coherence condition:

$$\begin{array}{ccc}
FA \otimes_2 FB & \xrightarrow{\beta_{2FA, FB}} & FB \otimes_2 FA \\
\downarrow m_{A, B} & & \downarrow m_{B, A} \\
F(A \otimes_1 B) & \xrightarrow{F\beta_{1A, B}} & F(B \otimes_1 A)
\end{array}$$

► **Definition 33.** Suppose  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are monoidal categories, and  $(F, m)$  and  $(G, n)$  are monoidal functors between  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Then a **monoidal natural transformation** is a natural transformation,  $f : F \rightarrow G$ , subject to the following coherence diagrams:

$$\begin{array}{ccc}
FA \otimes_2 FB & \xrightarrow{m_{A, B}} & F(A \otimes_1 B) \\
\downarrow f_A \otimes f_B & & \downarrow f_{A \otimes_1 B} \\
GA \otimes_2 GB & \xrightarrow{n_{A, B}} & G(A \otimes_1 B)
\end{array}
\quad
\begin{array}{ccc}
FI_1 & \xrightarrow{f_{I_1}} & GI_1 \\
\swarrow m_{I_1} & & \searrow n_{I_1} \\
& I_2 &
\end{array}$$

► **Definition 34.** Suppose  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are monoidal categories, and  $(F, m)$  is a monoidal functor between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  and  $(G, n)$  is a monoidal functor between  $\mathcal{M}_2$  and  $\mathcal{M}_1$ . Then a **monoidal adjunction** is an ordinary adjunction  $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$  such that the unit,  $\eta_A : A \rightarrow GFA$ , and the counit,  $\varepsilon_A : FGA \rightarrow A$ , are monoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{array}{ccc}
FGA \otimes_2 FGB & \xrightarrow{m_{GA, GB}} & F(GA \otimes_1 GB) \\
\downarrow \varepsilon_A \otimes \varepsilon_B & & \downarrow F n_{A, B} \\
A \otimes_2 B & \xleftarrow{\varepsilon_{A \otimes_1 B}} & FGA \otimes_2 FGB
\end{array}
\quad
\begin{array}{ccc}
FI_1 & \xrightarrow{F n_{I_2}} & FGI_2 \\
\uparrow m_{I_1} & & \downarrow \varepsilon_{I_1} \\
I_2 & \xlongequal{\quad} & I_2
\end{array}$$

$$\begin{array}{ccc}
GFA \otimes_1 GFB & \xleftarrow{\eta_A \otimes \eta_B} & A \otimes_1 B \\
\downarrow n_{FA, FB} & & \downarrow \eta_{A \otimes_1 B} \\
G(FA \otimes_2 FB) & \xrightarrow{m_{A, B}} & GF(A \otimes_1 B)
\end{array}
\quad
\begin{array}{ccc}
GI_2 & \xrightarrow{G m_{I_1}} & GFI_1 \\
\uparrow n_{I_2} & & \uparrow \eta_{I_1} \\
I_1 & \xlongequal{\quad} & I_1
\end{array}$$

► **Definition 35.** A **monoidal comonad** on a monoidal category  $\mathcal{C}$  is a triple  $(T, \varepsilon, \delta)$ , where  $(T, m)$  is a monoidal endofunctor on  $\mathcal{C}$ ,  $\varepsilon_A : TA \rightarrow A$  and  $\delta_A : TA \rightarrow T^2A$  are monoidal natural transform-

ations, which make the following diagrams commute:

$$\begin{array}{ccc}
 TA & \xrightarrow{\delta_A} & T^2A \\
 \delta_A \downarrow & & \downarrow T\delta_A \\
 T^2A & \xrightarrow{\delta_{TA}} & T^3A
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & TA & & \\
 & \swarrow & \downarrow \delta_A & \searrow & \\
 TA & \xleftarrow{\varepsilon_{TA}} & T^2A & \xrightarrow{T\varepsilon_A} & TA
 \end{array}$$

The assumption that  $\varepsilon$  and  $\delta$  are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{array}{ccc}
 TA \otimes TB & \xrightarrow{m_{A,B}} & T(A \otimes B) \\
 \varepsilon_A \otimes \varepsilon_B \searrow & & \downarrow \varepsilon_{A \otimes B} \\
 & & A \otimes B
 \end{array}
 \qquad
 \begin{array}{ccc}
 TI & \xleftarrow{m_I} & I \\
 \varepsilon_I \searrow & & \downarrow \\
 & & I
 \end{array}$$
  

$$\begin{array}{ccc}
 TA \otimes TB & \xrightarrow{m_{A,B}} & T(A \otimes B) \\
 \delta_A \otimes \delta_B \downarrow & & \downarrow \delta_{A \otimes B} \\
 T^2A \otimes T^2B & \xrightarrow{m_{TA,TB}} & T(TA \otimes TB) \xrightarrow{Tm_{A,B}} T^2(A \otimes B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 I & \xrightarrow{m_I} & TI \\
 m_I \downarrow & & \downarrow \delta_I \\
 TI & \xrightarrow{Tm_I} & T^2I
 \end{array}$$

## B Proofs

### B.1 Lambek Categories with Exchange

#### B.1.1 Proof of The Eilenberg-Moore Category is Monoidal (Lemma 7)

We must first define the unitors, and then the associator. Then we show that they respect the symmetry monoidal coherence diagrams. Throughout this proof we will make use of the coalgebra  $(A, h_A)$ ,  $(B, h_B)$ , and  $(C, h_C)$ .

The tensor product of  $(A, h_A)$  and  $(B, h_B)$  is  $(A \otimes B, q_{A,B} \circ (h_A \otimes h_B))$ , and the unit of the tensor product is  $(I, q_I)$ ; both actions are easily shown to satisfy the action diagrams of the Eilenberg-Moore category. The left and right unitors are  $\lambda : I \otimes A \rightarrow A$  and  $\rho : A \otimes I \rightarrow A$ , because they are indeed coalgebra morphisms.

The respective diagram for the right unitor is as follows:

$$\begin{array}{ccccccc}
 A \otimes I & \xrightarrow{h_A \otimes \text{id}} & eA \otimes I & \xrightarrow{\text{id} \otimes q_I} & eA \otimes eI & \xrightarrow{q_{A,I}} & e(A \otimes I) \\
 \downarrow \rho & & & \searrow \rho & & & \downarrow e\rho \\
 A & \xrightarrow{h_A} & eA & & & & eA
 \end{array}$$

The left diagram commutes by naturality of  $\rho$ , the right diagram commutes by the fact that  $e$  is a monoidal functor. Showing the left unitor is a coalgebra morphism is similar.

The unitors are natural and isomorphisms, because they are essentially inherited from the underlying Lambek category.

The associator  $\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$  is also a coalgebra morphism. First, notice that:

$$q_{A \otimes B, C} \circ ((q_{A, B} \circ (h_A \otimes h_B)) \otimes h_C) = q_{A \otimes B, C} \circ (q_{A, B} \otimes \text{id}) \circ ((h_A \otimes h_B) \otimes h_C)$$

where the left-hand side is the action of the coalgebra  $(A \otimes B) \otimes C$ . Similarly, the following is the action of the coalgebra  $A \otimes (B \otimes C)$ :

$$q_{A, B \otimes C} \circ (h_A \otimes (q_{B, C} \circ (h_B \otimes h_C))) = q_{A, B \otimes C} \circ (\text{id} \otimes q_{B, C}) \circ (h_A \otimes (h_B \otimes h_C))$$

The following diagram must commute:

$$\begin{array}{ccc}
 (A \otimes B) \otimes C & \xrightarrow{(h_A \otimes h_B) \otimes h_C} & (eA \otimes eB) \otimes eC \\
 \downarrow \alpha & & \downarrow \alpha \\
 A \otimes (B \otimes C) & \xrightarrow{h_A \otimes (h_B \otimes h_C)} & eA \otimes (eB \otimes eC) \\
 \uparrow q & \uparrow \text{id} \otimes q & \uparrow q \\
 (A \otimes B) \otimes eC & \xrightarrow{q \otimes \text{id}} & e(A \otimes B) \otimes eC \\
 \downarrow e\alpha & & \downarrow e\alpha \\
 e((A \otimes B) \otimes C) & \xrightarrow{q} & e(A \otimes (B \otimes C))
 \end{array}$$

The left diagram commutes by naturality of  $\alpha$ , and the right diagram commutes because  $e$  is a monoidal functor.

Composition in  $\mathcal{L}^e$  is the same as  $\mathcal{L}$ , and thus, the monoidal coherence diagrams hold in  $\mathcal{L}^e$  as well. Thus,  $\mathcal{L}^e$  is monoidal. We now show that it is symmetric.

### B.1.2 Proof of Lemma 8

We define  $\beta$  as follows:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{ex_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

Suppose  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$  are two coalgebra morphisms. Then the following diagram shows that  $\beta_{A,B}$  is a natural transformation:

$$\begin{array}{ccccccc} A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{ex_{A,B}} & eB \otimes eA & \xrightarrow{\varepsilon_B \otimes \varepsilon_A} & B \otimes A \\ \downarrow f \otimes g & & \downarrow ef \otimes eg & & \downarrow eg \otimes ef & & \downarrow g \otimes f \\ A' \otimes B' & \xrightarrow{h_{A'} \otimes h_{B'}} & eA' \otimes eB' & \xrightarrow{ex_{A',B'}} & eB' \otimes eA' & \xrightarrow{\varepsilon_{B'} \otimes \varepsilon_{A'}} & B' \otimes A' \end{array}$$

The left diagram commutes because  $f$  and  $g$  are both coalgebra morphisms, the middle diagram commutes because  $ex_{A,B}$  is a natural transformation, and the right diagram commutes by naturality of  $\varepsilon$ .

### B.1.3 Proof of Corollary 9

The first diagram commutes by the fact that the following diagram commutes:

$$\begin{array}{ccccc} A \otimes B & \xlongequal{\quad} & A \otimes B & \xlongequal{\quad} & A \otimes B \\ \downarrow h_A \otimes h_B & & \downarrow h_A \otimes h_B & & \downarrow h_A \otimes h_B \\ eA \otimes eB & \xlongequal{\quad} & eA \otimes eB & & eA \otimes eB \\ & \nwarrow \varepsilon_{eA} \otimes \varepsilon_{eB} & \downarrow h_{eA} \otimes h_{eB} & & \\ & & e^2A \otimes e^2B & & \\ & \swarrow ex_{A,B} & \downarrow ex_{eA,eB} & & \downarrow ex_{A,B} \\ eB \otimes eA & \xleftarrow{\varepsilon_{eB} \otimes \varepsilon_{eA}} & e^2B \otimes e^2A & & eB \otimes eA \\ & \nwarrow h_B \otimes h_A & & & \downarrow \varepsilon_B \otimes \varepsilon_A \\ & & B \otimes A & \xleftarrow{\varepsilon_B \otimes \varepsilon_A} & eB \otimes eA \end{array}$$

The diagram on the right commutes because  $\beta_{A,B}$  is a natural transformation, and the other diagrams commute either because  $\mathcal{L}$  is a Lambek category with exchange, or by the action diagrams.

The second diagram commutes by the following:

$$\begin{array}{ccc}
 eA \otimes eB & \xrightarrow{\text{ex}_{A,B}} & eB \otimes eA \\
 \downarrow q_{A,B} & \swarrow e h_A \otimes e h_B & \nwarrow e h_B \otimes e h_A \\
 e^2 A \otimes e^2 B & \xrightarrow{\text{ex}_{eA,eB}} & e^2 B \otimes e^2 A \xrightarrow{e \varepsilon_B \otimes e \varepsilon_A} eB \otimes eA \\
 \downarrow q_{eA,eB} & & \downarrow q_{eB,eA} \\
 e(eA \otimes eB) & \xrightarrow{e \text{ex}_{A,B}} & e(eB \otimes eA) \\
 \downarrow e(h_A \otimes h_B) & & \downarrow e(\varepsilon_B \otimes \varepsilon_A) \\
 e(A \otimes B) & \xrightarrow{e\beta_{A,B}} & e(B \otimes A)
 \end{array}$$

The top quadrangle commutes by the naturality of  $\text{ex}$ . The triangle commutes by the definition  $\mathcal{L}^e$  (Definition 6). The left and right quadrangles commute by the naturality of  $q$ . And the quadrangle at the bottom is the definition of  $\beta$ .

### B.1.4 Proof of Lemma 11

We prove this by equational reasoning as follows:

$$\begin{aligned}
 & (h_B \otimes h_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) \\
 &= (h_B \otimes h_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ (h_B \otimes h_A) \circ \beta_{A,B} && \text{(Corollary 9)} \\
 &= (h_B \otimes h_A) \circ ((\varepsilon_B \circ h_B) \otimes (\varepsilon_A \circ h_A)) \circ \beta_{A,B} \\
 &= (h_B \otimes h_A) \circ (\text{id}_B \otimes \text{id}_A) \circ \beta_{A,B} && \text{(Action diagrams)} \\
 &= (h_B \otimes h_A) \circ \beta_{A,B} \\
 &= \text{ex}_{A,B} \circ (h_A \otimes h_B) && \text{(Corollary 9)} \\
 &= (\text{id}_B \otimes \text{id}_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) \\
 &= ((e\varepsilon_B \circ \delta_B) \otimes (e\varepsilon_A \circ \delta_A)) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) && \text{(Monoidal Comonad)} \\
 &= (e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B)
 \end{aligned}$$

Or simplified as:

$$\begin{aligned}
 & (h_B \otimes h_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) \\
 &= \text{ex}_{A,B} \circ (h_A \otimes h_B) && \text{(Corollary 9)} \\
 &= (\text{id}_B \otimes \text{id}_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) \\
 &= ((e\varepsilon_B \circ \delta_B) \otimes (e\varepsilon_A \circ \delta_A)) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) && \text{(Monoidal Comonad)} \\
 &= (e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B)
 \end{aligned}$$

Or by diagram chasing:

$$\begin{array}{ccccc}
 A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{\text{ex}_{A,B}} & eB \otimes eA \\
 \downarrow h_A \otimes h_B & & & & \downarrow \varepsilon_B \otimes \varepsilon_A \\
 eA \otimes eB & & & & B \otimes A \\
 \downarrow \text{ex}_{A,B} & & & & \downarrow h_B \otimes h_A \\
 eB \otimes eA & \xlongequal{\quad\quad\quad} & & & eB \otimes eA \\
 & \searrow \delta_B \otimes \delta_A & & \nearrow e\varepsilon_B \otimes e\varepsilon_B & \\
 & e^2 B \otimes e^2 A & & &
 \end{array}$$

The upper rectangle commutes by Corollary 9 and the lower triangle commutes because  $e$  is a comonad.

### B.1.5 Proof of Lemma 12

The proof follows from the commutativity of the following diagram:

$$\begin{array}{ccccccc}
 A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{\Theta_{X_{A,B}}} & eB \otimes eA & \xrightarrow{\varepsilon_B \otimes \varepsilon_A} & B \otimes A \\
 \downarrow h_A \otimes h_B & (1) & \downarrow \delta_A \otimes \delta_B & (2) & \downarrow \delta_B \otimes \delta_A & (3) & \downarrow h_B \otimes h_A \\
 eA \otimes eB & \xrightarrow{e h_A \otimes e h_B} & e^2 A \otimes e^2 B & \xrightarrow{\Theta_{X_{eA,eB}}} & e^2 B \otimes e^2 A & \xrightarrow{e \varepsilon_B \otimes e \varepsilon_A} & eA \otimes eB \\
 \downarrow q_{A,B} & (4) & \downarrow q_{eA,eB} & (5) & \downarrow q_{eB,eA} & (6) & \downarrow q_{B,A} \\
 e(A \otimes B) & \xrightarrow{e(h_A \otimes h_B)} & e(eA \otimes eB) & \xrightarrow{e \Theta_{X_{A,B}}} & e(eB \otimes eA) & \xrightarrow{e(\varepsilon_B \otimes \varepsilon_A)} & e(B \otimes A)
 \end{array}$$

Diagram one commutes by the action diagrams for the coalgebras  $(A, h_A)$  and  $(B, h_B)$ , diagram two commutes because  $\mathcal{L}$  is a Lambek category with exchange, diagram three does not commute, but holds by Lemma 11, diagram four and six commute by naturality of  $q$ , and diagram five commutes because  $\mathcal{L}$  is a Lambek category with exchange.

### B.1.6 Proof of The Eilenberg-Moore Category is Symmetric Monoidal (Lemma 13)

The following diagram shows that  $\beta_{B,A} \circ \beta_{A,B} = \text{id}_{A \otimes B}$ :

$$\begin{array}{ccccccc}
 A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & & & & \\
 \downarrow h_A \otimes h_B & (1) & \downarrow \delta_A \otimes \delta_B & (6) & & & \\
 eA \otimes eB & \xrightarrow{\delta_A \otimes \delta_B} & e^2 A \otimes e^2 B & & & & \\
 \downarrow \Theta_{X_{A,B}} & (2) & \downarrow \Theta_{X_{eA,eB}} & (4) & & & \\
 eB \otimes eA & \xrightarrow{\delta_B \otimes \delta_A} & e^2 B \otimes e^2 A & \xrightarrow{\Theta_{X_{eB,eA}}} & e^2 A \otimes e^2 B & & \\
 \downarrow \varepsilon_B \otimes \varepsilon_A & (3) & \downarrow e \varepsilon_B \otimes e \varepsilon_A & (5) & \downarrow e \varepsilon_A \otimes e \varepsilon_B & & \\
 B \otimes A & \xrightarrow{h_B \otimes h_A} & eB \otimes eA & \xrightarrow{\Theta_{X_{B,A}}} & eA \otimes eB & \xrightarrow{\varepsilon_A \otimes \varepsilon_B} & A \otimes B
 \end{array}$$

Diagram one trivially commutes, diagram two, four, and five commute because  $\mathcal{L}$  is a Lambek category with exchange, diagram three does not commute, but holds by Lemma 11, diagrams six, seven, and eight commute by the fact that  $(e, \varepsilon, \delta)$  is a comonad and the action diagrams of the Eilenberg Moore category.

At this point we must verify that  $\beta$  respects the coherence diagrams of a symmetric monoidal category; see Definition 3. Thus, we must show that each of the following diagrams hold:

$$\begin{array}{ccccc}
 (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\beta_{A,B \otimes C}} & (B \otimes C) \otimes A \\
 \downarrow \beta_{A,B} \otimes \text{id}_C & & & & \downarrow \alpha_{B,C,A} \\
 (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{\text{id}_B \otimes \beta_{A,C}} & B \otimes (C \otimes A)
 \end{array}$$

Case



We can show that this diagram commutes, by reducing it to the corresponding diagram on free coalgebras which we know holds by the assumption that  $\mathcal{L}$  is a Lambek category with exchange. This reduction is as follows (due to the size of the diagram it is broken up into three diagrams that can be straightforwardly composed):

Diagram 1:

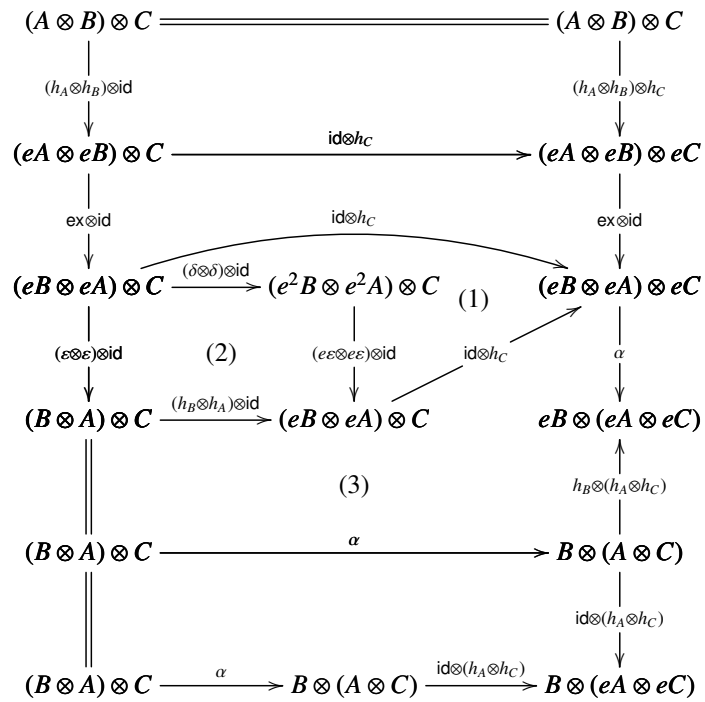


Diagram one commutes because  $(e, \varepsilon, \delta)$  is a comonad, diagram two does not commute, but holds by Lemma 11, and diagram 3 commutes by naturality of  $\alpha$ . All other diagrams commute trivially.

Diagram 2:

$$\begin{array}{c}
 \begin{array}{c}
 (A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C) \xrightarrow{h_A \otimes (h_B \otimes h_C)} eA \otimes (eB \otimes eC) \xrightarrow{\text{id} \otimes q} eA \otimes e(B \otimes C) \xrightarrow{\text{ex}} e(B \otimes C) \otimes eA \\
 \downarrow (h_A \otimes h_B) \otimes h_C \quad (1) \quad \downarrow h_A \otimes (h_B \otimes h_C) \quad (2) \quad \downarrow \text{id} \otimes (e h_B \otimes e h_C) \quad (3) \quad \downarrow \text{id} \otimes (h_B \otimes h_C) \quad (4) \quad \downarrow e(h_B \otimes h_C) \otimes \text{id} \\
 (eA \otimes eB) \otimes eC \xrightarrow{\alpha} eA \otimes (eB \otimes eC) \xrightarrow{\text{id} \otimes (\delta \otimes \delta)} eA \otimes (e^2 B \otimes e^2 C) \xrightarrow{\text{id} \otimes q} eA \otimes e(eB \otimes eC) \xrightarrow{\text{ex}} e(eB \otimes eC) \otimes eA \\
 \downarrow \text{ex} \otimes \text{id} \quad (5) \quad \downarrow \alpha \quad \downarrow \alpha \quad \downarrow \alpha \\
 (eB \otimes eA) \otimes eC \xrightarrow{\text{id} \otimes \text{ex}} eB \otimes (eA \otimes eC) \xrightarrow{\text{id} \otimes \text{ex}} B \otimes (eA \otimes eC) \xrightarrow{\text{id} \otimes (h_A \otimes h_C)} B \otimes (eA \otimes eC) \xrightarrow{\text{id} \otimes \text{ex}} B \otimes (eC \otimes eA) \xrightarrow{h_B \otimes \text{id}} eB \otimes (eC \otimes eA) \\
 \downarrow h_B \otimes (h_A \otimes h_C) \quad \downarrow \text{id} \otimes (h_A \otimes h_C) \quad \downarrow \text{id} \otimes (h_A \otimes h_C) \quad \downarrow h_B \otimes \text{id} \\
 B \otimes (A \otimes C) \xrightarrow{\text{id} \otimes (h_A \otimes h_C)} B \otimes (eA \otimes eC) \xrightarrow{\text{id} \otimes \text{ex}} B \otimes (eC \otimes eA) \xrightarrow{h_B \otimes \text{id}} eB \otimes (eC \otimes eA)
 \end{array}
 \end{array}$$

Diagram one commutes by naturality of  $\alpha$ , diagram two commutes by the action diagrams, diagram three commutes by naturality of  $q$ , diagram four commutes by naturality of  $\text{ex}$ , and diagram five commutes because  $\mathcal{L}$  is a Lambek category with exchange. All other diagrams trivially commute.

Diagram 3:

$$\begin{array}{ccc}
e(B \otimes C) \otimes eA & \xrightarrow{\varepsilon \otimes \varepsilon} & (B \otimes C) \otimes A \\
\downarrow e(h_B \otimes h_C) \otimes \text{id} & & \downarrow (h_B \otimes h_C) \otimes \text{id} \\
e(eB \otimes eC) \otimes eA & \xrightarrow{\varepsilon \otimes \varepsilon} & (eB \otimes eC) \otimes A \\
\downarrow \varepsilon \otimes \text{id} & & \downarrow (\varepsilon \otimes \varepsilon) \otimes \text{id} \\
(eB \otimes eC) \otimes eA & \xrightarrow{(\varepsilon \otimes \varepsilon) \otimes \varepsilon} & (B \otimes C) \otimes A \\
\downarrow \alpha & & \downarrow \alpha \\
eB \otimes (eC \otimes eA) & \xrightarrow{\varepsilon \otimes (\varepsilon \otimes \varepsilon)} & B \otimes (C \otimes A) \\
\parallel & & \parallel \\
eB \otimes (eC \otimes eA) & & B \otimes (C \otimes A) \\
\parallel & & \parallel \\
eB \otimes (eC \otimes eA) & \xrightarrow{\varepsilon \otimes (\varepsilon \otimes \varepsilon)} & B \otimes (C \otimes A)
\end{array}$$

The top most diagram commutes by naturality of  $\varepsilon$  and the middle diagram commutes by naturality of  $\alpha$ . All other diagrams trivially commute.

**Case**

$$\begin{array}{ccc}
I \otimes A & \xrightarrow{\beta_{I,A}} & A \otimes I \\
& \searrow \lambda_A & \swarrow \rho_A \\
& A &
\end{array}$$

Just as we did for the previous case we reduce this diagram down to the corresponding one on free coaglesbras that we know holds by the assumption that  $\mathcal{L}$  is a Lambek category with exchange. This case follows from the following commutative diagram:

$$\begin{array}{ccccccc}
I \otimes A & \xrightarrow{h_I \otimes h_A} & eI \otimes eA & \xrightarrow{\text{ex}} & eA \otimes eI & \xrightarrow{\varepsilon \otimes \varepsilon} & A \otimes I \\
\downarrow \lambda & \searrow \text{id} \otimes h_A & \downarrow \varepsilon \otimes \text{id} & & \downarrow \text{id} \otimes \varepsilon & \swarrow \varepsilon \otimes \text{id} & \downarrow \rho \\
& & I \otimes eA & & eA \otimes I & & \\
& & \downarrow \lambda & & \downarrow \rho & & \\
A & \xrightarrow{h_A} & eA & \xlongequal{\quad} & eA & \xrightarrow{\varepsilon} & A
\end{array}$$

The left most triangle commutes by the action diagrams and the lower diagram commutes by naturality of  $\lambda$ . Similarly, the right most lower diagram commutes by naturality of  $\rho$ . The middle diagram commutes because  $\mathcal{L}$  is a Lambek category with exchange. All other diagrams trivially commute.

## B.2 Weakening and Contraction

### B.3 Proof of Composition of Weakening and Contraction (Lemma 23)

Since by definition  $w : \mathcal{L} \rightarrow \mathcal{L}$  and  $c : \mathcal{L} \rightarrow \mathcal{L}$  are monoidal functors we know that their composition  $cw : \mathcal{L} \rightarrow \mathcal{L}$  is a monoidal functor:

$$\begin{aligned} q_{A,B} &: cwA \otimes cwB \rightarrow cw(A \otimes B) \\ q_{A,B} &= cq_{A,B}^w \circ q_{wA,wB}^c \\ q_I &: I \rightarrow cwI \\ q_I &= cq_I^w \circ q_I^c \end{aligned}$$

We must now define both  $\varepsilon_A : cwA \rightarrow A$  and  $\delta_A : cwA \rightarrow cwcwA$ , and then show that they are monoidal natural transformations subject to the comonad laws. Since we are composing two comonads each of  $\varepsilon$  and  $\delta$  can be given two definitions, but they are equivalent:

- $\varepsilon_A : cwA \rightarrow A$  is defined as in the diagram below, which commutes by the naturality of  $\varepsilon^c$ .

$$\begin{array}{ccc} cwA & \xrightarrow{\varepsilon_{wA}^c} & wA \\ \downarrow c\varepsilon_A^w & & \downarrow \varepsilon_A^w \\ cA & \xrightarrow{\varepsilon_A^c} & A \end{array}$$

- $\delta_A : cwA \rightarrow cwcwA$  is defined as in the diagram:

$$\begin{array}{ccccc} cwA & \xrightarrow{c\delta_A^w} & cw^2A & \xrightarrow{\delta_{w^2A}^c} & c^2w^2A \\ \downarrow \delta_{wA}^c & & \downarrow \delta_{w^2A}^c & & \downarrow cdist_{wA} \\ c^2wA & \xrightarrow{c^2\delta_A^w} & c^2w^2A & \xrightarrow{cdist_{wA}} & cwcwA \end{array}$$

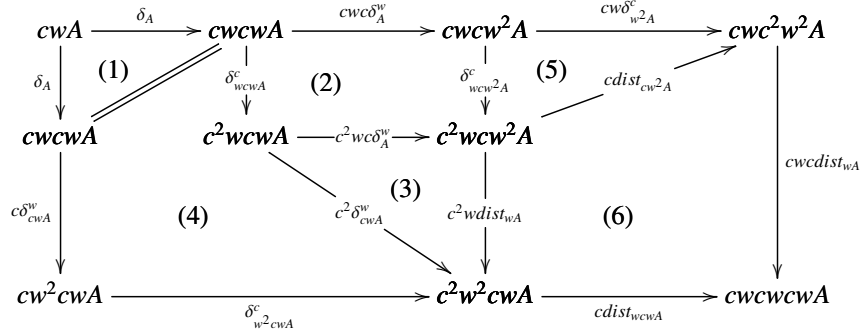
The left part of the diagram commutes by the naturality of  $\delta^c$  and the right part commutes trivially.

The remainder of the proof shows that the comonad laws hold.

**Case 1:**

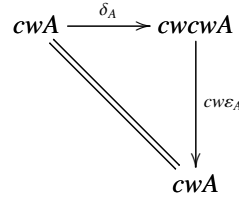
$$\begin{array}{ccc} cwA & \xrightarrow{\delta_A} & cwcwA \\ \downarrow \delta_A & & \downarrow cw\delta_A \\ cwcwA & \xrightarrow{\delta_{cwA}} & cwcwcwA \end{array}$$

The previous diagram commutes because the following one does.

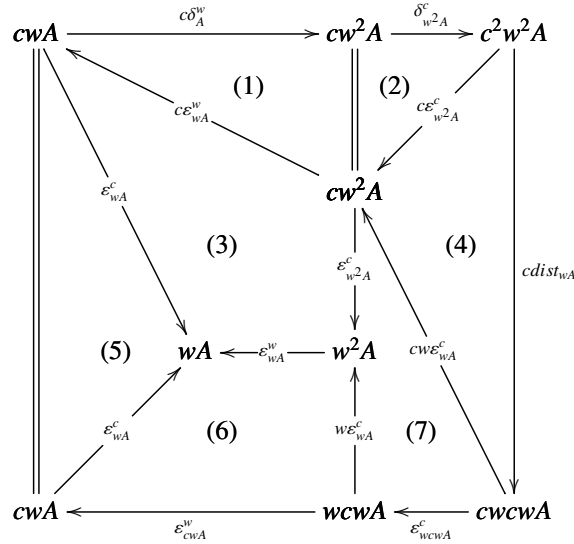


(1) commutes by equality and we will not expand  $\delta_A$  for simplicity. (2) and (4) commutes by the naturality of  $\delta^c$ . (3), (5) commutes by the conditions of  $dist$ . (6) commutes by the naturality of  $dist$ .

**Case 2:**



The triangle commutes because of the following diagram chasing.



(1) commutes by the comonad law for  $w$  with components  $\delta_A^w$  and  $\epsilon_{wA}^w$ . (2) commutes by the comonad law for  $c$  with components  $\delta_{w^2A}^c$  and  $\epsilon_{w^2A}^c$ . (3) and (7) commute by the naturality of  $\epsilon^c$ . (4) commutes by the condition of  $dist$ . (5) commutes trivially. And (6) commutes by the naturality of  $\epsilon^w$ .

Case 3:

$$\begin{array}{ccc}
 & cwA & \\
 \delta_A \downarrow & \searrow & \\
 cwcwA & \xrightarrow{\varepsilon_{cwA}} & cwA
 \end{array}$$

The previous triangle commutes because the following diagram chasing does.

$$\begin{array}{ccccc}
 cwA & \xrightarrow{c\delta_A^w} & cw^2A & \xrightarrow{\delta_{w^2A}^c} & c^2w^2A \\
 \parallel & \searrow \delta_{wA}^c & \downarrow c\varepsilon_{wA}^c & \nearrow c^2\delta_A^w & \downarrow cdist_{wA} \\
 & (2) & c^2wA & (3) & \\
 & \nearrow c\varepsilon_{wA}^c & \parallel & \searrow c^2w\varepsilon_A^w & \\
 & (4) & c^2wA & (5) & \\
 & \downarrow cdist_A & & & \\
 cwA & \xleftarrow{cW\varepsilon_A^c} & cwcA & \xleftarrow{cwc\varepsilon_A^w} & cwcwA
 \end{array}$$

(1) commutes by the naturality of  $\delta^c$ . (2) is the comonad law for  $c$  with components  $\delta_{wA}^c$  and  $\varepsilon_{wA}^c$ . (3) is the comonad law for  $w$  with components  $\delta_A^w$  and  $\varepsilon_A^w$ . (4) commutes by the condition of  $dist$ . And (5) commute by the naturality of  $dist$ .

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#### B.4 Proof of Conditions of Lambek category with $cw$ (Lemma 25)

1. As shown in the paper.
2. Each  $(cwA, \text{weak}_A, \text{contra}_A)$  is a comonoid.

Case 1:

$$\begin{array}{ccccc}
 cwA & \xrightarrow{\text{contra}_A} & cwA \otimes cwA & \xrightarrow{id_{cwA} \otimes \text{contra}_A} & cwA \otimes (cwA \otimes cwA) \\
 \text{contra}_A \downarrow & & & & \uparrow \alpha_{cwA, cwA, cwA} \\
 cwA \otimes cwA & \xrightarrow{\text{contra}_A \otimes id_{cwA}} & (cwA \otimes cwA) \otimes cwA & & 
 \end{array}$$

The previous diagram commutes by the following diagram chasing.

$$\begin{array}{ccccc}
cwA & \xrightarrow{\text{contra}_A} & cwA \otimes cwA & \xrightarrow{id_{cwA} \otimes \rho_{cwA}^{-1}} & cwA \otimes (cwA \otimes I) \\
\downarrow \text{contra}_A & \nearrow (1) & \downarrow id_{cwA} \otimes \lambda_{cwA}^{-1} & \nearrow (2) & \downarrow id_{cwA} \otimes \text{contra}_{wA,I} \\
cwA \otimes cwA & \xrightarrow{id_{cwA} \otimes \lambda_{cwA}^{-1}} & cwA \otimes (I \otimes cwA) & & cwA \otimes ((cwA \otimes I) \otimes cwA) \\
\downarrow \rho_{cwA}^{-1} \otimes id_{cwA} & & \downarrow id_{cwA} \otimes \text{contra}_{wA,I} & \nearrow id_{cwA} \otimes \alpha_{cwA,I,cwA} & \downarrow id_{cwA} \otimes (\rho_{cwA} \otimes id_{cwA}) \\
(cwA \otimes I) \otimes cwA & & cwA \otimes (cwA \otimes (I \otimes cwA)) & \xrightarrow{id_{cwA} \otimes (id_{cwA} \otimes \lambda_{cwA})} & cwA \otimes (cwA \otimes cwA) \\
\downarrow \text{contra}_{wA,I} \otimes id_{cwA} & & \downarrow id_{cwA} \otimes \alpha_{cwA,cwA,cwA} & & \uparrow \alpha_{cwA,cwA,cwA} \\
((cwA \otimes I) \otimes cwA) \otimes cwA & \xrightarrow{(\rho_{cwA} \otimes id_{cwA}) \otimes id_{cwA}} & & & (cwA \otimes cwA) \otimes cwA
\end{array}$$

(4)

(1) commutes trivially and we would not expand  $\text{contra}$  for simplicity. (2) and (4) commute because  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction. (3) commutes because  $\mathcal{L}$  is monoidal.

**Case 2:**

$$\begin{array}{ccccc}
& & cwA & & \\
& \swarrow \lambda^{-1} & \downarrow \text{contra}_A & \searrow \rho^{-1} & \\
I \otimes cwA & \xleftarrow{\text{weak}_A \otimes id_{cwA}} & cwA \otimes cwA & \xrightarrow{id_{cwA} \otimes \text{weak}_A} & cwA \otimes I
\end{array}$$

The diagram above commutes by the following diagram chasing.

$$\begin{array}{ccccc}
I \otimes cwA & \xleftarrow{\text{weak}_A^w \otimes id_{cwA}} & & & wA \otimes cwA \\
\uparrow id_I \otimes \lambda_{cwA} & & (2) & & \uparrow id_{wA} \otimes \lambda_{cwA} \\
I \otimes (I \otimes cwA) & \xleftarrow{\text{weak}_A^w \otimes id_{I \otimes cwA}} & & & wA \otimes (I \otimes cwA) \\
\uparrow \lambda_{I \otimes cwA}^{-1} & & (1) & & \uparrow \varepsilon_{wA}^c \otimes id_{I \otimes cwA} \\
I \otimes cwA & \xrightarrow{\text{contraR}_{wA,I}} & cwA \otimes (I \otimes cwA) & & \\
\downarrow \lambda_{cwA}^{-1} & & (5) & & \downarrow id_{cwA} \otimes \lambda_{cwA} \\
cwA & & & & cwA \otimes cwA \\
\downarrow \rho_{cwA}^{-1} & & (6) & & \downarrow \rho_{cwA} \otimes id_{cwA} \\
cwA \otimes I & \xrightarrow{\text{contraL}_{wA,I}} & (cwA \otimes I) \otimes cwA & & \\
\downarrow \rho_{cwA}^{-1} & & (9) & & \downarrow id_{cwA \otimes I} \otimes \varepsilon_{wA}^c \\
(cwA \otimes I) \otimes I & \xleftarrow{id_{cwA \otimes I} \otimes \text{weak}_A^w} & (cwA \otimes I) \otimes wA & & \\
\downarrow \rho_{cwA} \otimes id_I & & (7) & & \downarrow \rho_{cwA} \otimes id_{wA} \\
cwA \otimes I & \xleftarrow{id_{cwA} \otimes \text{weak}_A} & & & cwA \otimes wA
\end{array}$$

(8)

(1), (2) and (3) commute by the functionality of  $\lambda$ . (6), (7) and (8) commute by the functionality of  $\rho$ . (4) and (9) are conditions of the Lambek category with  $cw$ . And (5) is the definition of  $\text{contra}$ .

3. weak and contra are coalgebra morphisms.

**Case 1:**

$$\begin{array}{ccc}
 cwA & \xrightarrow{\text{weak}_A} & I \\
 \delta_A \downarrow & & \downarrow q_I \\
 cwcwA & \xrightarrow{cw\text{weak}_A} & cwI
 \end{array}$$

The previous diagram commutes by the diagram below. (1) commutes by the naturality of  $\delta^c$ . (2) commutes by the condition of  $\text{dist}_{wA}$ . (3), (5) and (6) commute because  $c$  is a monoidal comonad. (4) commutes because  $(\mathcal{L}, w, \text{weak}^w)$  is a Lambek category with weakening. (7) commutes because  $c$  and  $w$  are monoidal comonads.

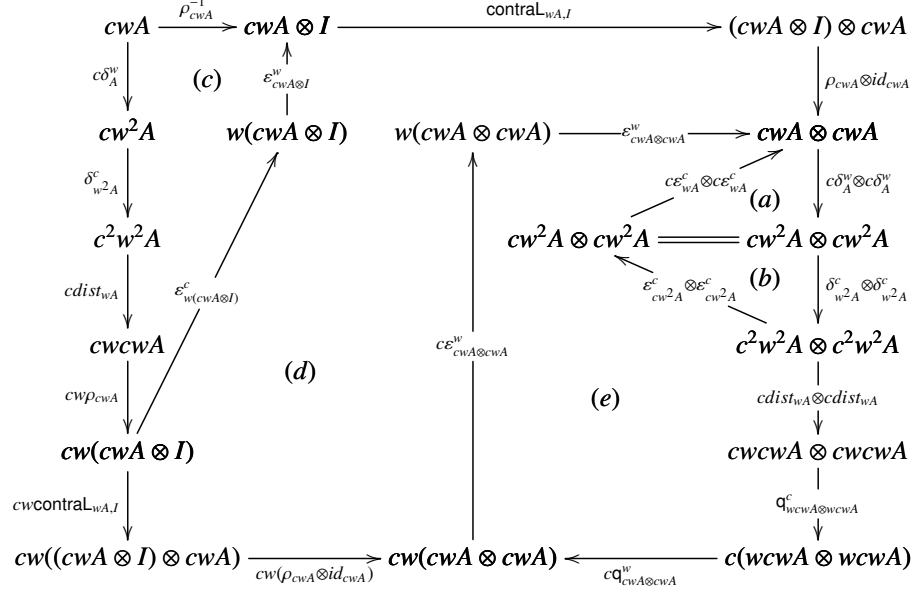
$$\begin{array}{ccccc}
 cwA & \xrightarrow{cw\text{weak}_A^w} & cI & \xrightarrow{\epsilon_I^c} & I \\
 \delta_{wA}^c \downarrow & \searrow c\delta_A^w & \swarrow q_I^c & \text{(5)} & \downarrow q_I^c \\
 c^2wA & & & & I \\
 c^2\delta_A^w \downarrow & \text{(1)} & \text{(4)} & & \text{(6)} \\
 c^2w^2A & \xleftarrow{\delta_{w^2A}^c} & cw^2A & & cI \\
 c\text{dist}_{wA} \downarrow & \text{(2)} & \text{(3)} & & \downarrow q_I^w \\
 cwcwA & \xrightarrow{cw\epsilon_{wA}^c} & cw^2A & \xleftarrow{cw\text{weak}_A^w} & cwI
 \end{array}$$

**Case 2:**

$$\begin{array}{ccccc}
 cwA & \xrightarrow{\text{contra}_A} & cwA \otimes cwA & \xrightarrow{\delta_A \otimes \delta_A} & cwcwA \otimes cwcwA \\
 \delta_A \downarrow & & & & \downarrow q_{cwA, cwA} \\
 cwcwA & \xrightarrow{cw\text{contra}_A} & & & cw(cwA \otimes cwA)
 \end{array}$$

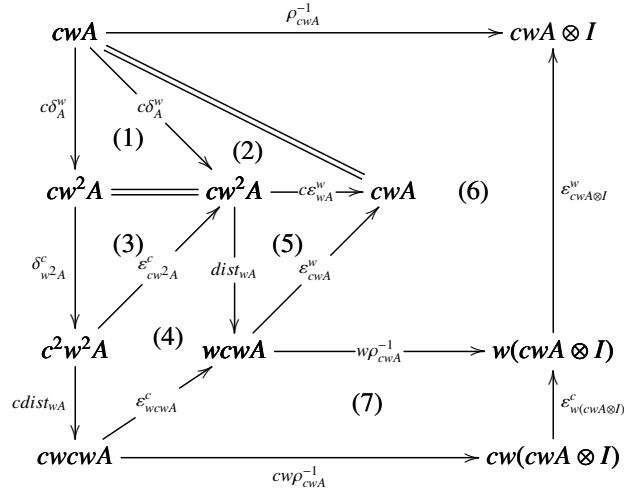
To prove the previous diagram commute, we first expand it, Then we divide it into five parts as shown below, and prove each part commutes.





Part (a) and (b) are comonad laws.

Part (c) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for  $w$ . (3) is the comonad law for  $c$ . (4) commutes by the naturality of  $\varepsilon^c$ . (5) is one of the conditions for  $dist_{wA}$ . (6) commutes by the naturality of  $\varepsilon^w$ . And (7) commutes by the naturality of  $\varepsilon^c$ .



Part (d) commutes by the following diagram chase. The upper two squares both commute by

the naturality of  $\varepsilon^w$ , and the lower two squares commute by the naturality of  $\varepsilon^c$ .

$$\begin{array}{ccccc}
 cwA \otimes I & \xrightarrow{\text{contraL}_{wA,I}} & (cwA \otimes I) \otimes cwA & \xrightarrow{\rho_{cwA} \otimes id_{cwA}} & cwA \otimes cwA \\
 \uparrow \varepsilon_{cwA \otimes I}^w & & \uparrow \varepsilon_{(cwA \otimes I) \otimes cwA}^w & & \uparrow \varepsilon_{cwA \otimes cwA}^w \\
 w(cwA \otimes I) & \xrightarrow{w\text{contraL}_{wA,I}} & w((cwA \otimes I) \otimes cwA) & \xrightarrow{w(\rho_{cwA} \otimes id_{cwA})} & w(cwA \otimes cwA) \\
 \uparrow \varepsilon_{w(cwA \otimes I)}^c & & \uparrow \varepsilon_{w((cwA \otimes I) \otimes cwA)}^c & & \uparrow \varepsilon_{w(cwA \otimes cwA)}^c \\
 cw(cwA \otimes I) & \xrightarrow{cw\text{contraL}_{wA,I}} & cw((cwA \otimes I) \otimes cwA) & \xrightarrow{cw(\rho_{cwA} \otimes id_{cwA})} & cw(cwA \otimes cwA)
 \end{array}$$

Part (e) commutes by the following diagram. (1) commutes by the condition of  $dist_{wA}$ . (2) and (4) commute by the naturality of  $\varepsilon^c$ . (3) and (5) commute because  $w$  and  $c$  are monoidal comonads.

$$\begin{array}{ccccc}
 cwA \otimes cwA & \xleftarrow{c\varepsilon_{wA}^w \otimes c\varepsilon_{wA}^w} & cw^2A \otimes cw^2A & \xleftarrow{\varepsilon_{cw^2A}^c \otimes \varepsilon_{cw^2A}^c} & c^2w^2A \otimes c^2w^2A \\
 \uparrow \varepsilon_{cwA \otimes cwA}^w & \swarrow \varepsilon_{cwA}^w \otimes \varepsilon_{cwA}^w & \downarrow dist_{wA} & \searrow \varepsilon_{cw^2A}^c \otimes \varepsilon_{cw^2A}^c & \downarrow cdist_{wA} \otimes cdist_{wA} \\
 & (1) & & (2) & \\
 & & w cwA \otimes w cwA & \xleftarrow{\varepsilon_{w cwA}^c \otimes \varepsilon_{w cwA}^c} & c w cwA \otimes c w cwA \\
 & \swarrow q_{cwA, cwA}^w & \downarrow & \searrow \varepsilon_{w cwA \otimes w cwA}^c & \downarrow q_{w cwA \otimes w cwA}^c \\
 w(cwA \otimes cwA) & \xleftarrow{\varepsilon_{w(cwA \otimes cwA)}^c} & cw(cwA \otimes cwA) & \xleftarrow{c q_{cwA \otimes cwA}^c} & c(w cwA \otimes w cwA) \\
 & & (4) & & (5)
 \end{array}$$

4. Any coalgebra morphism  $f : (cwA, \delta_A) \rightarrow (cwB, \delta_B)$  between free coalgebras preserves the comonoid structure given by  $weak$  and  $contra$ .

**Case 1:** This coherence diagram is given in the definition of the Lambek category with  $cw$ .

$$\begin{array}{ccc}
 cwA & \xrightarrow{f} & cwB \\
 & \searrow weak_A & \swarrow weak_B \\
 & I &
 \end{array}$$

**Case 2:**

$$\begin{array}{ccc}
 cwA & \xrightarrow{\text{contra}_A} & cwA \otimes cwA \\
 f \downarrow & & \downarrow f \otimes f \\
 cwB & \xrightarrow{\text{contra}_B} & cwB \otimes cwB
 \end{array}$$

The square commutes by the diagram chasing below, which commutes by the naturality of  $\rho$  and  $\text{contraL}$ .

$$\begin{array}{ccccccc}
 cwA & \xrightarrow{\rho_{cwA}^{-1}} & cwA \otimes I & \xrightarrow{\text{contraL}_{wA,I}} & (cwA \otimes I) \otimes cwA & \xrightarrow{\rho_{cwA} \otimes id_{cwA}} & cwA \otimes cwA \\
 \downarrow cw f & & \downarrow cw f \otimes id_I & & \downarrow (cw f \otimes id_I) \otimes cw f & & \downarrow cw f \otimes cw f \\
 cwB & \xrightarrow{\rho_{cwB}^{-1}} & cwB \otimes I & \xrightarrow{\text{contraL}_{wB,I}} & (cwB \otimes I) \otimes cwB & \xrightarrow{\rho_{cwB} \otimes id_{cwB}} & cwB \otimes cwB
 \end{array}$$

===== >>>> origin/master