Composition of Structural Rules using Adjoints

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1 The Algebras of Composed Comonads

Suppose (k, ε, δ) is a comonad on a category $\mathcal{L}$. Then it is well known that it can be decomposed into a the following adjunction:

$$\begin{tikzcd}
\mathcal{L}^k \arrow[rr, bend left, "U"] \arrow[rr, phantom, "\perp"] & & \mathcal{L} \arrow[ll, bend left, "F"] \arrow[loop above, "k"]
\end{tikzcd}$$

where $U : \mathcal{L}^k \longrightarrow \mathcal{L}$ is the forgetful functor, $F : \mathcal{L} \longrightarrow \mathcal{L}^k$ is the free functor, and $k = UF : \mathcal{L} \longrightarrow \mathcal{L}$.

Now suppose we have the following adjunctions:

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{L}_2 \arrow[ll, bend left, "G"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}_2 \arrow[rr, bend left, "H"] \arrow[rr, phantom, "\perp"] & & \mathcal{L}_1 \arrow[ll, bend left, "J"]
\end{tikzcd}$$

Then they can be composed into the adjunction:

$$\begin{tikzcd}
\mathcal{L}_3 \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{L}_2 \arrow[ll, bend left, "G"] \arrow[rr, bend left, "H"] \arrow[rr, phantom, "\perp"] & & \mathcal{L}_1 \arrow[ll, bend left, "J"]
\end{tikzcd}$$

Keep in mind that this gives rise to a comonad $HFGJ : \mathcal{L}_1 \longrightarrow \mathcal{L}_1$.

We are going to use these two facts to compose comonads using adjunctions. Suppose we have the comonads $(k_1, \varepsilon^1, \delta^1)$ and $(k_2, \varepsilon^2, \delta^2)$ both on a category $\mathcal{L}$ with a distributive law $\mathsf{dist} : k_2 k_1 \longrightarrow k_1 k_2$. Thus, making $k_2 k_1 : \mathcal{L} \longrightarrow \mathcal{L}$ a comonad. Then we can decompose k_1 into a an adjunction:

$$\begin{tikzcd}
\mathcal{L}^{k_1} \arrow[rr, bend left, "U_1"] \arrow[rr, phantom, "\perp"] & & \mathcal{L} \arrow[ll, bend left, "F_1"] \arrow[loop above, "k_1"] \arrow[loop below, "k_2"]
\end{tikzcd}$$

Here we know that $k_1 = U_1F_1 : \mathcal{L} \longrightarrow \mathcal{L}$, but we also know something about k_2. We can extend it to a comonad on $\mathcal{L}^{k_1}$.

First, we define the functor $\widetilde{k_2} : \mathcal{L}^{k_1} \longrightarrow \mathcal{L}^{k_1}$ to send objects (A, h_A) to (k_2A, h'_A), where $h'_A := k_2A \xrightarrow{k_2h_A} k_2k_1A \xrightarrow{\mathsf{dist}_A} k_1k_2A$, and to send morphisms $f : (A, h_A) \longrightarrow (B, h_B)$ to the morphism $k_2f : (k_2A, h'_A) \longrightarrow (k_2B, h'_B)$. We must show that $k_2f : k_2A \longrightarrow k_2B$ is a coalgebra morphism, but the following diagram commutes:

$$\begin{tikzcd}
k_2A \arrow[r, "k_2f"] \arrow[d, "k_2h_A"] & k_2B \arrow[d, "k_2h_B"] \\
k_2k_1A \arrow[r, "k_2k_1f"] \arrow[d, "\mathsf{dist}_A"] & k_2k_1B \arrow[d, "\mathsf{dist}_B"] \\
k_1k_2A \arrow[r, "k_1k_2f"] & k_1k_2B
\end{tikzcd}$$

The top diagram commutes because f is a coalgebra morphism and the bottom diagram commutes by naturality of dist. Since the morphism part of $\widetilde{k_2}$ is defined using the functor k_2 we know $\widetilde{k_2}$ will respect composition and identities.

We now must show that $\widetilde{k_2}$ is a comonad. The natural transformation $\widetilde{\varepsilon^2} : \widetilde{k_2} \longrightarrow \mathsf{Id}$ has components $\widetilde{\varepsilon^2}_{(A,h_A)} = \varepsilon^2_A : \widetilde{k_2}(A, h_A) \longrightarrow (A, h_A)$. We must show that ε^2_A is a coalgebra morphism between $\widetilde{k_2}(A, h_A) = (k_2A, k_2h_A; \mathsf{dist}_A)$ and (A, h_A), but this follows from the following diagram:

$$\begin{tikzcd}
k_2A \arrow[r, "\varepsilon^2_A"] \arrow[d, "k_2h_A"] & A \arrow[d, "h_A"] \\
k_2k_1A \arrow[r, "\varepsilon^2_{k_1A}"] \arrow[d, "\mathsf{dist}_A"] & k_1A \arrow[d, equal] \\
k_1k_2A \arrow[r, "k_1\varepsilon^2_A"] & k_1A
\end{tikzcd}$$

The top diagram commutes by naturality of ε^2 and the bottom diagram commutes by the conditions of the distributive law. Naturality for $\widetilde{\varepsilon^2}$ easily follows from the fact that it is defined to be ε_2.

The natural transformation $\widetilde{\delta^2} : \widetilde{k_2} \longrightarrow \widetilde{k_2}\widetilde{k_2}$ has components

$$\widetilde{\delta^2}_{(A,h_A)} = \delta^2_A : \widetilde{k_2}(A, h_A) \longrightarrow \widetilde{k_2}\widetilde{k_2}(A, h_A).$$

Just as above we must show that $\delta^2_A : k_2A \longrightarrow k_2^2A$ is a coalgebra morphism between $\widetilde{k_2}(A, h_A) = (k_2A, k_2h_A; \mathsf{dist}_A)$ and $\widetilde{k_2}\widetilde{k_2}(A, h_A) = (k_2^2A, k^2h_A; k_2\mathsf{dist}_A; \mathsf{dist}_{k_2A})$, but this follows from the following diagram:

$$\begin{tikzcd}
k_2A \arrow[r, "\delta^2_A"] \arrow[d, "k_2h_A"] & k_2^2A \arrow[d, "k_2^2h_A"] \\
k_2k_1A \arrow[r, "\delta^2_{k_1A}"] \arrow[dd, "\mathsf{dist}_A"] & k_2^2k_1A \arrow[d, "k_2\mathsf{dist}_A"] \\
 & k_2k_1k_2A \arrow[d, "\mathsf{dist}_{k_2A}"] \\
k_1k_2A \arrow[r, "k_1\delta^2_A"] & k_1k_2^2A
\end{tikzcd}$$

The top diagram commutes by naturality of δ^2 and the bottom diagram commutes by the conditions of the distributive law.

It is now easy to see that $\widetilde{\varepsilon^2}$ and $\widetilde{\delta^2}$ make $\widetilde{k_2}$ a comonad on $\mathcal{L}^{k_1}$, because the conditions of a comonad will be inherited from the fact that ε^2 and δ^2 define a comonad on $\mathcal{L}$.

At this point we have arrived at the following situation:

$$\begin{tikzcd}
\mathcal{L}^{k_1} \arrow[r, bend left, "U_1"] \arrow[r, phantom, "\perp"] \arrow[loop below, "\widetilde{k_2}"] & \mathcal{L} \arrow[l, bend left, "F_1"] \arrow[loop above, "k_1"] \arrow[loop below, "k_2"]
\end{tikzcd}$$

Since we have a comonad $\widetilde{k_2} : \mathcal{L}^{k_1} \longrightarrow \mathcal{L}^{k_1}$ we can form the following adjunction:

$$\begin{tikzcd}
(\mathcal{L}^{k_1})^{k_2} \arrow[r, bend left, "U_2"] \arrow[r, phantom, "\perp"] & \mathcal{L}^{k_1} \arrow[l, bend left, "F_2"] \arrow[loop below, "\widetilde{k_2}"]
\end{tikzcd}$$

The functor $F_2(A, h_A) = (\widetilde{k_2}(A, h_A), \widetilde{\delta^2}_{(A,h_A)})$ is the free functor, and $U_2(A, h_A) = A$ is the forgetful functor. Thus, we can think of $(\mathcal{L}^{k_1})^{k_2}$ as the world with all the structure of $\mathcal{L}$ extended with all of the structure k_1 brings and all the structure k_2 brings. That is, $(\mathcal{L}^{k_1})^{k_2}$ is the algebras of $k_2k_1 : \mathcal{L} \longrightarrow \mathcal{L}$.

We can see that the previous two adjunctions compose:

$$\begin{tikzcd}
(\mathcal{L}^{k_1})^{k_2} \arrow[r, bend left, "U_2"] \arrow[r, phantom, "\perp"] & \mathcal{L}^{k_1} \arrow[l, bend left, "F_2"] \arrow[loop below, "\widetilde{k_2}"] \arrow[r, bend left, "U_1"] \arrow[r, phantom, "\perp"] & \mathcal{L} \arrow[l, bend left, "F_1"] \arrow[loop above, "k_1"] \arrow[loop below, "k_2"]
\end{tikzcd}$$

Thus, we have a comonad $U_1U_2F_2F_1 : \mathcal{L} \longrightarrow \mathcal{L}$. Chasing an object through this comonad yields the following:

$$\begin{array}{lll}
U_1U_2F_2F_1A & = & U_1U_2F_2(k_1A, \delta^1_A) \\
& = & U_1U_2(\widetilde{k_2}((k_1A, \delta^1_A)), \widetilde{\delta^2}_{(k_1A,\delta^1_A)}) \\
& = & U_1U_2((k_2k_1A, k_2\delta^1_A; \mathsf{dist}_{k_1A}), \widetilde{\delta^2}_{(k_1A,\delta^1_A)}) \\
& = & U_1(k_2k_1A, k_2\delta^1_A; \mathsf{dist}_{k_1A}) \\
& = & k_2k_1A
\end{array}$$

Therefore, the above adjunction gives back the composition $k_2k_1 : \mathcal{L} \longrightarrow \mathcal{L}$.

Notice that this result only works because we have a distributive law! Otherwise we may not be able to define $\widetilde{k_2}$. However, this result reveals a means that will allow us to abandon distributive laws in favor of adjunctions in the spirit of Benton's LNL models.

2 Combining Structural Rules

The above result tells us something important about combining comonads, that we should be using adjunctions, because they compose.

In this section we show how to model the Lambek Calculus without association or exchange with three comonads, one that adds back in association, one that adds back in exchange, and one the composes the two which will allow both to be used.

First, we define our base category.

Definition 1. *A **Lambek category** is a category $(\mathcal{L}, \otimes, I, \lambda, \rho)$ where $\otimes : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ is a bifunctor, I is a distinguished object called the unit, and two natural isomorphisms $\lambda_A : I \otimes A \longrightarrow A$ called the left unitor and $\rho_A : A \otimes I \longrightarrow A$ called the right unitor.*

We can equivalently phrase a Lambek category to be a monoidal category without the associator, and hence, all of the diagrams disappear as well.

Definition 2. *A Lambek category is **biclosed** if it is equipped with two bifunctors $\rightharpoonup: \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ and $\leftharpoonup: \mathcal{L} \times \mathcal{L}^{\mathsf{op}} \longrightarrow \mathcal{L}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:*

$$\begin{array}{c}
\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \\
\mathsf{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)
\end{array}$$

Next we extend Lambek categories with structural rules.

Definition 3. *A **Lambek category with exchange** is a Lambek category $(\mathcal{L}, \otimes, \lambda, \rho)$ equipped with a natural transformation $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$. We require the following diagrams to commute:*

$$\begin{tikzcd}
A \otimes B \arrow[r, "\beta_{A,B}"] \arrow[dr, equal] & B \otimes A \arrow[d, "\beta_{B,A}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes A \arrow[rr, "\beta_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 4. *A* ***Lambek category with association*** *is a Lambek category* $(\mathcal{L}, \otimes, \lambda, \rho)$ *equipped with a natural isomorphism* $\alpha_{A,B} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$*. We require the following diagrams to commute:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B \otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "\mathsf{id}_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

In other words, a Lambek category with association is a monoidal category.

At this point we define our model that equips a Lambek category with three comonads for association, exchange, and their combination.

Definition 5. *An* ***adjoint model for association and exchange or AE model****,* $(\mathcal{L}, L, P, \mathcal{E}, F, G, \mathcal{A}, H, J, \mathcal{M})$*, consists of the following:*

- $\mathcal{L}$ *is a Lambek category,*
- $\mathcal{E}$ *is a Lambek category with exchange,*
- $\mathcal{A}$ *is a Lambek category with association,*
- $\mathcal{M}$ *is a Lambek category with association and exchange,*
- $L : \mathcal{L} \longrightarrow \mathcal{E}$ *and* $P : \mathcal{E} \longrightarrow \mathcal{L}$ *are monoidal functors,*
- $F : \mathcal{L} \longrightarrow \mathcal{A}$ *and* $G : \mathcal{A} \longrightarrow \mathcal{L}$ *are monoidal functors,*
- $H : \mathcal{A} \longrightarrow \mathcal{M}$ *and* $J : \mathcal{M} \longrightarrow \mathcal{A}$ *are monoidal functors, and*
- $P \dashv L$*,* $G \dashv F$ *and* $J \dashv H$ *are monoidal adjunctions.*

Lemma 6 (Composition in the Adjoint Model)**.** *Suppose* $(\mathcal{L}, F, G, \mathcal{A}, H, J, \mathcal{M})$ *is an AE model. Then following holds:*

a. There is an monoidal comonad $a = GF : \mathcal{L} \longrightarrow \mathcal{L}$

b. There is an monoidal comonad $e = PL : \mathcal{L} \longrightarrow \mathcal{L}$

c. There is an monoidal comonad $\widetilde{e} = JH : \mathcal{A} \longrightarrow \mathcal{A}$

d. There is a monoidal adjunction $\mathcal{M} : GJ \dashv FH : \mathcal{L}$

e. There is a monoidal comonad $m = GJHF : \mathcal{L} \longrightarrow \mathcal{L}$

References