1 Examples

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash_{\mathcal{A}} X}\,\textsc{Var} \qquad \dfrac{}{Y \vdash_{\mathcal{A}} Y}\,\textsc{Var}}{X, Y \vdash_{\mathcal{A}} X \unrhd Y}\,\textsc{Tr}}{X, Y \vdash_{\mathcal{L}} \mathsf{G}\,(X \unrhd Y)}\,\textsc{Gr}}{\mathsf{G}\,X, Y \vdash_{\mathcal{L}} \mathsf{G}\,(X \unrhd Y)}\,\textsc{Gl}}{\mathsf{G}\,X, \mathsf{G}\,Y \vdash_{\mathcal{L}} \mathsf{G}\,(X \unrhd Y)}\,\textsc{Gl}}{\mathsf{G}\,X \rhd \mathsf{G}\,Y \vdash_{\mathcal{L}} \mathsf{G}\,(X \unrhd Y)}\,\text{TL}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash_{\mathcal{A}} X}\,\textsc{Var}}{X \vdash_{\mathcal{L}} \mathsf{G}\,X}\,\textsc{Gr} \qquad \dfrac{\dfrac{}{Y \vdash_{\mathcal{A}} Y}\,\textsc{Var}}{Y \vdash_{\mathcal{L}} \mathsf{G}\,Y}\,\textsc{Gr}}{X, Y \vdash_{\mathcal{L}} \mathsf{G}\,X \rhd \mathsf{G}\,Y}\,\textsc{Tr}}{X \unrhd Y \vdash_{\mathcal{L}} \mathsf{G}\,X \rhd \mathsf{G}\,Y}\,\textsc{Tl}}{\mathsf{G}\,(X \unrhd Y) \vdash_{\mathcal{L}} \mathsf{G}\,X \rhd \mathsf{G}\,Y}\,\textsc{Gl}$$

Using the above two proofs with cut we can show that:

$$(\mathsf{G}\,\mathsf{F}\,A \rhd \mathsf{G}\,\mathsf{F}\,B) \rhd \mathsf{G}\,\mathsf{F}\,C \vdash_{\mathcal{L}} \mathsf{G}\,\mathsf{F}\,A \rhd (\mathsf{G}\,\mathsf{F}\,B \rhd \mathsf{G}\,\mathsf{F}\,C)$$
$$\text{if and only if}$$
$$\mathsf{G}\,((\mathsf{F}\,A \unrhd \mathsf{F}\,B) \unrhd \mathsf{F}\,C) \vdash_{\mathcal{L}} \mathsf{G}\,(\mathsf{F}\,A \unrhd (\mathsf{F}\,B \unrhd \mathsf{F}\,C))$$
$$\text{if and only if}$$
$$(\mathsf{F}\,A \unrhd \mathsf{F}\,B) \unrhd \mathsf{F}\,C \vdash_{\mathcal{A}} \mathsf{F}\,A \unrhd (\mathsf{F}\,B \unrhd \mathsf{F}\,C)$$

Similarly, we have the following:

$$\mathsf{G}\,\mathsf{F}\,A \rhd (\mathsf{G}\,\mathsf{F}\,B \rhd \mathsf{G}\,\mathsf{F}\,C) \vdash_{\mathcal{L}} (\mathsf{G}\,\mathsf{F}\,A \rhd \mathsf{G}\,\mathsf{F}\,B) \rhd \mathsf{G}\,\mathsf{F}\,C$$
$$\text{if and only if}$$
$$\mathsf{G}\,(\mathsf{F}\,A \unrhd (\mathsf{F}\,B \unrhd \mathsf{F}\,C)) \vdash_{\mathcal{L}} \mathsf{G}\,((\mathsf{F}\,A \unrhd \mathsf{F}\,B) \unrhd \mathsf{F}\,C)$$
$$\text{if and only if}$$
$$\mathsf{F}\,A \unrhd (\mathsf{F}\,B \unrhd \mathsf{F}\,C) \vdash_{\mathcal{A}} (\mathsf{F}\,A \unrhd \mathsf{F}\,B) \unrhd \mathsf{F}\,C$$

Appendix

A Full Specification

vars, n, a, x, y, z, w, m, o
ivar, i, k, j, l
const, b

$X,\ Y,\ Z$	::=		
	|	I	Unit
	|	$X \unrhd Y$	Associative Non-commutative tensor
	|	$X \rightharpoonup Y$	Implication
	|	$\mathsf{F}\,A$	Right adjoint
$A,\ B,\ C$	::=		
	|	J	Unit
	|	$A \rhd B$	Non-associative Non-commutative tensor
	|	$A \rightharpoonup B$	Implication
	|	$\mathsf{G}\,X$	Right adjoint

$$
\begin{array}{lcl}
\Gamma & ::= & \\
& | & \cdot \\
& | & A \\
& | & X \\
& | & \Gamma_1, \Gamma_2 \\
& | & (\Gamma) \\
& | & \Gamma
\end{array}
$$

$$
\begin{array}{lcl}
\Delta & ::= & \\
& | & \cdot \\
& | & X \\
& | & \Delta_1, \Delta_2 \\
& | & (\Delta) \\
& | & \Delta
\end{array}
$$

$\boxed{\Delta \vdash_{\mathcal{A}} X}$

$$\frac{}{X \vdash_{\mathcal{A}} X} \quad \text{A\_VAR}$$

$$\frac{}{\cdot \vdash_{\mathcal{A}} I} \quad \text{A\_IR}$$

$$\frac{\Delta \vdash_{\mathcal{A}} X}{\Delta, I \vdash_{\mathcal{A}} X} \quad \text{A\_IL}$$

$$\frac{\Delta_1 \vdash_{\mathcal{A}} X \quad \Delta_2 \vdash_{\mathcal{A}} Y}{\Delta_1, \Delta_2 \vdash_{\mathcal{A}} X \trianglerighteq Y} \quad \text{A\_TR}$$

$$\frac{\Delta_1, X, Y, \Delta_2 \vdash_{\mathcal{A}} Z}{\Delta_1, X \trianglerighteq Y, \Delta_2 \vdash_{\mathcal{A}} Z} \quad \text{A\_TL}$$

$$\frac{\Delta, X \vdash_{\mathcal{A}} Y}{\Delta \vdash_{\mathcal{A}} X \rightharpoonup Y} \quad \text{A\_IRR}$$

$$\frac{\Delta_2 \vdash_{\mathcal{A}} X \quad \Delta_1, Y, \Delta_3 \vdash_{\mathcal{A}} Z}{\Delta_1, X \rightharpoonup Y, \Delta_2, \Delta_3 \vdash_{\mathcal{A}} Z} \quad \text{A\_IRL}$$

$$\frac{\Delta_2 \vdash_{\mathcal{A}} X \quad \Delta_1, X, \Delta_3 \vdash_{\mathcal{A}} Y}{\Delta_1, \Delta_2, \Delta_3 \vdash_{\mathcal{A}} Y} \quad \text{A\_CUT}$$

$$\frac{\Delta \vdash_{\mathcal{L}} A}{\Delta \vdash_{\mathcal{A}} \mathsf{F}\, A} \quad \text{A\_FR}$$

$\boxed{\Gamma \vdash_{\mathcal{L}} A}$

$$\frac{}{A \vdash_{\mathcal{L}} A} \quad \text{L\_VAR}$$

$$\frac{}{\cdot \vdash_{\mathcal{L}} J} \quad \text{L\_JR}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A}{\Gamma, J \vdash_{\mathcal{L}} A} \quad \text{L\_JL}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A}{\Gamma, I \vdash_{\mathcal{L}} A} \quad \text{L\_IL}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad C \vdash_{\mathcal{L}} B}{\Gamma, C \vdash_{\mathcal{L}} A \triangleright B} \quad \text{L\_TR}$$

$$\frac{A, B \vdash_{\mathcal{L}} C}{A \triangleright B \vdash_{\mathcal{L}} C} \quad \text{L\_T}\textsc{l}$$

$$\frac{\Delta_1, X, Y, \Delta_2 \vdash_{\mathcal{L}} C}{\Delta_1, X \trianglerighteq Y, \Delta_2 \vdash_{\mathcal{L}} C} \quad \text{L\_AT}\textsc{l}$$

$$\frac{\Gamma, A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} A \rightharpoonup B} \quad \text{L\_IR}\textsc{r}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad B \vdash_{\mathcal{L}} C}{\Gamma, A \rightharpoonup B \vdash_{\mathcal{L}} C} \quad \text{L\_IR}\textsc{l}$$

$$\frac{\Delta_2 \vdash_{\mathcal{A}} X \quad \Delta_1, X, \Delta_3 \vdash_{\mathcal{L}} A}{\Delta_1, X \rightharpoonup Y, \Delta_2, \Delta_3 \vdash_{\mathcal{L}} A} \quad \text{L\_AIR}\textsc{l}$$

$$\frac{\Delta_2 \vdash_{\mathcal{A}} X \quad \Delta_1, X, \Delta_3 \vdash_{\mathcal{L}} A}{\Delta_1, \Delta_2, \Delta_3 \vdash_{\mathcal{L}} A} \quad \text{L\_AC}\textsc{ut}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} B} \quad \text{L\_C}\textsc{ut}$$

$$\frac{\Delta \vdash_{\mathcal{A}} X}{\Delta \vdash_{\mathcal{L}} \mathsf{G}\, X} \quad \text{L\_G}\textsc{r}$$

$$\frac{\Gamma_1, X, \Gamma_2 \vdash_{\mathcal{L}} A}{\Gamma_1, \mathsf{G}\, X, \Gamma_2 \vdash_{\mathcal{L}} A} \quad \text{L\_G}\textsc{l}$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash_{\mathcal{L}} B}{\Gamma_1, \mathsf{F}\, A, \Gamma_2 \vdash_{\mathcal{L}} B} \quad \text{L\_F}\textsc{l}$$