Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

1.1 Symmetric Monoidal Categories

Definition 1. *A **monoidal category** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A \\
\rho_A &: A \otimes \top \longrightarrow A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (\top \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Definition 2. *A* ***symmetric monoidal category (SMC)*** *is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A \\
\rho_A &: A \otimes \top \longrightarrow A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

- *A symmetry natural transformation:*

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 3. *A **symmetric monoidal closed category (SMCC)** is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Definition 4. *Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2\,FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1\,A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2\,FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1, A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1\,A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2\,FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A, \top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1\,A}"]
\end{tikzcd}$$

Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

Definition 5. *Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2\,FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1\,A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"'] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"'] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"'] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"'] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"'] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"'] & F(B \otimes_1 A)
\end{tikzcd}$$

Definition 6. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 7. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***symmetric monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 8. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are* *monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 9. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***symmetric monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 10. *A* ***monoidal comonad*** *on a monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\
& \top &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A\otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

Definition 11. *A **symmetric monoidal comonad** on a symmetric monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\
& \top &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

1.2 Linear Category

Definition 12. *A **linear category**, $(\mathcal{L}, !, \mathsf{weak}, \mathsf{contra})$, is specified by*

- *a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,*
- *a symmetric monoidal comonad $(!, \varepsilon, \delta)$ on $\mathcal{L}$, with $\mathsf{q}_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $\mathsf{q}_I : I \longrightarrow !I$;*
- *monoidal natural transformations on $\mathcal{L}$ with components $\mathsf{weak}_A : !A \longrightarrow I$ and $\mathsf{contra}_A : !A \longrightarrow !A \otimes !A$, s.t.*
  - *each $(!A, \mathsf{weak}_A, \mathsf{contra}_A)$ is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ \mathsf{contra}_A = \mathsf{contra}_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of $\mathcal{L}$;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & !A \otimes !A \arrow[r, "id_{!A} \otimes \mathsf{contra}_A"] & !A \otimes (!A \otimes !A) \\
!A \otimes !A \arrow[rr, "\mathsf{contra}_A \otimes id_{!A}"'] & & (!A \otimes !A) \otimes !A \arrow[u, "\alpha_{!A,!A,!A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & !A \arrow[d, "\mathsf{contra}_A"] & \\
I \otimes !A \arrow[ur, "\lambda"] & !A \otimes !A \arrow[l, "\mathsf{weak}_A \otimes id_{!A}"] \arrow[r, "id_{!A} \otimes \mathsf{weak}_A"'] & !A \otimes I \arrow[ul, "\rho"']
\end{tikzcd}$$

– weak_A *and* contra_A *are coalgebra morphisms, i.e. the following diagrams commute;*

$$\begin{tikzcd} !A \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\ !!A \arrow[r, "!\mathsf{weak}_A"'] & !I \end{tikzcd}$$

$$\begin{tikzcd} !A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[r, "\delta_A \otimes \delta_A"] & !!A \otimes !!A \arrow[d, "\mathsf{q}_{!A,!A}"] \\ !!A \arrow[rr, "!\mathsf{contra}_A"'] & & !(!A \otimes !A) \end{tikzcd}$$

– *any coalgebra morphism* $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ *between free coalgebras preserve the comonoid structure given by* weak *and* contra*, i.e. the following diagrams commute.*

$$\begin{tikzcd} !A \arrow[rr, "!f"] \arrow[dr, "\mathsf{weak}_A"'] & & !B \arrow[dl, "\mathsf{weak}_B"] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} !A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & !A \otimes !A \arrow[d, "f \otimes f"] \\ !B \arrow[r, "\mathsf{contra}_B"'] & !B \otimes !B \end{tikzcd}$$

Definition 13. *A* ***linear category with weakening****,* $(\mathcal{L}, w, \mathsf{weak})$*, is specified by*

- *a monoidal category* $(\mathcal{L}, I, \otimes)$*,*
- *a monoidal comonad* (w, ε, δ) *on* $\mathcal{L}$ *with* $\mathsf{q}_{A,B} : wA \otimes wB \longrightarrow w(A \otimes B)$ *and* $\mathsf{q}_I : I \longrightarrow wI$*, and*
- *a monoidal natural transformation* weak *on* $\mathcal{L}$ *with components* $\mathsf{weak}_A : wA \longrightarrow I$ *s.t. the following diagrams commute:*

$$\begin{tikzcd} wA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\ wwA \arrow[r, "w\mathsf{weak}_A"'] & wI \end{tikzcd} \qquad \begin{tikzcd} wA \arrow[rr, "wf"] \arrow[dr, "\mathsf{weak}_A"'] & & wB \arrow[dl, "\mathsf{weak}_B"] \\ & I & \end{tikzcd}$$

Definition 14. *A* ***linear category with contraction****,* $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$*, is specified by*

- *a monoidal category* $(\mathcal{L}, I, \otimes)$*,*
- *a monoidal comonad* (c, ε, δ) *on* $\mathcal{L}$ *with* $\mathsf{q}_{A,B} : cA \otimes cB \longrightarrow c(A \otimes B)$ *and* $\mathsf{q}_I : I \longrightarrow cI$*, and*
- *monoidal natural transformations* $\mathsf{contraL}$ *and* $\mathsf{contraL}$ *on* $\mathcal{L}$ *with components* $\mathsf{contraL}_{A,B} : cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$ *and* $\mathsf{contraR}_{A,B} : B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$*, s.t. the following diagrams commutes:*

$$\begin{tikzcd}
cA \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & cA \arrow[l, "\rho^{-1}_{cA}"'] \arrow[r, "\lambda^{-1}_{cA}"] & I \otimes cA \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(cA \otimes I) \otimes cA \arrow[rr, "\alpha_{cA,I,cA}"'] & & cA \otimes (I \otimes cA)
\end{tikzcd}$$

The following two diagrams are added to prove $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid, corresponding to part of the first diagram in the definition of linear category.

$$\begin{tikzcd}
cA \otimes cA \arrow[r, "id_{cA} \otimes \rho^{-1}_{cA}"] \arrow[dd, "\lambda^{-1}_{cA} \otimes id_{cA}"'] & cA \otimes (cA \otimes I) \arrow[r, "id_{cA} \otimes \mathsf{contraL}_{A,I}"] & cA \otimes ((cA \otimes I) \otimes cA) \arrow[d, "id_{cA} \otimes (\rho_{cA} \otimes id_{cA})"] \\
& & cA \otimes (cA \otimes cA) \\
(I \otimes cA) \otimes cA \arrow[r, "\mathsf{contraR}_{A,I} \otimes id_{cA}"'] & (cA \otimes (I \otimes cA)) \otimes cA \arrow[r, "(id_{cA} \otimes \lambda_{cA}) \otimes id_{cA}"'] & (cA \otimes cA) \otimes cA \arrow[u, "\alpha_{cA,cA,cA}"']
\end{tikzcd}$$

$$\begin{tikzcd}
cA \otimes cA \arrow[r, "id_{cA} \otimes \lambda^{-1}_{cA}"] \arrow[dd, "\rho^{-1}_{cA} \otimes id_{cA}"'] & cA \otimes (I \otimes cA) \arrow[r, "id_{cA} \otimes \mathsf{contraR}_{A,I}"] & cA \otimes (cA \otimes (I \otimes cA)) \arrow[d, "id_{cA} \otimes (id_{cA} \otimes \lambda_{cA})"] \\
& & cA \otimes (cA \otimes cA) \\
(cA \otimes I) \otimes cA \arrow[r, "\mathsf{contraL}_{A,I} \otimes id_{cA}"'] & ((cA \otimes I) \otimes cA) \otimes cA \arrow[r, "(\rho_{cA} \otimes id_{cA}) \otimes id_{cA}"'] & (cA \otimes cA) \otimes cA \arrow[u, "\alpha_{cA,cA,cA}"']
\end{tikzcd}$$

Definition 15. *Given two comonads* $(c, \varepsilon^c, \delta^c)$ *and* $(w, \varepsilon^w, \delta^w)$ *on a category* $\mathcal{L}$*,* ***the distributive law*** *of* c *over* w *is a natural transformation with components* $dist_A : cwA \longrightarrow wcA$*, subject to the following coherence diagrams:*

$$\begin{tikzcd}
wA & & cwA \arrow[ll, "\varepsilon^c_{wA}"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "w\varepsilon^c_A"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cA & & cwA \arrow[ll, "c\varepsilon^w_A"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "\varepsilon^w_{cA}"] &
\end{tikzcd}$$

$$
\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, "\delta^c_{wcA}"'] & & cwc^2A \\
& c^2wcA \arrow[ur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wdist_{wA}"] \\
& w^2cwA &
\end{tikzcd}
$$

Lemma 16. *Let $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ be two monoidal comonads on a linear category with weakening and contraction $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}^c, \mathsf{contraR}^c)$. Then the composition of c and w using the distributive law $dist_A : cwA \longrightarrow wcA$ is a monoidal comonad $(cw, \varepsilon, \delta)$ on $\mathcal{L}$.*

Proof. Suppose $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ are monoidal comonads, and $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}^c, \mathsf{contraR}^c)$ is a linear category with weakening and contraction. Since by definition $w, c : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors we know that their composition $wc : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$\mathsf{q}_{A,B} : cwA \otimes cwB \longrightarrow cw(A \otimes B)$

$$\mathsf{q}_{A,B} = c\mathsf{q}^w_{A,B} \circ \mathsf{q}^c_{wA,wB}$$

and $\mathsf{q}_I : I \longrightarrow cwI$

$$\mathsf{q}_I = c\mathsf{q}^w_I \circ \mathsf{q}^c_I$$

We must now define both $\varepsilon_A : cwA \longrightarrow A$ and $\delta_A : cwA \longrightarrow cwcwA$, and then show that they are monoidal natural transformations subject to the comonad laws. Since we are composing two comonads each of ε and δ can be given two definitions, but they are equivalent:

- $\varepsilon_A : cwA \longrightarrow A$ is defined as in the diagram below, which commutes by the naturality of ε^c.

$$
\begin{tikzcd}
cwA \arrow[r, "\varepsilon^c_{wA}"] \arrow[d, "c\varepsilon^w_A"'] & wA \arrow[d, "\varepsilon^w_A"] \\
cA \arrow[r, "\varepsilon^c_A"'] & A
\end{tikzcd}
$$

- $\delta_A : cwA \longrightarrow cwcwA$ is defined as in the diagram:

$$
\begin{tikzcd}
cwA \arrow[r, "c\delta^w_A"] \arrow[d, "\delta^c_{wA}"'] & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, "\delta^c_{w^2A}"'] & c^2w^2A \arrow[d, "cdist_{wA}"] \\
c^2wA \arrow[r, "c^2\delta^w_A"'] & c^2w^2A \arrow[r, "cdist_{wA}"'] & cwcwA
\end{tikzcd}
$$

The left part of the diagram commutes by the naturality of δ^c and the right part commutes trivially.

The remainder of the proof shows that the comonad laws hold.

Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & cwcwA \arrow[d, "cw\delta_A"] \\
cwcwA \arrow[r, "\delta_{cwA}"'] & cwcwcwA
\end{tikzcd}
$$

The previous diagram commutes because the following one does.

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, phantom, "(1)"] & cwcwA \arrow[r, "cwc\delta^w_A"] \arrow[d, "\delta^c_{wcwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwcw^2A \arrow[r, "cw\delta^c_{w^2A}"] \arrow[d, "\delta^c_{wcw^2A}"] \arrow[dr, phantom, "(5)"] & cwc^2w^2A \arrow[dd, "cwcdist_{wA}"] \\
cwcwA \arrow[d, "c\delta^w_{cwA}"'] \arrow[r, phantom, "(4)"] & c^2wcwA \arrow[r, "c^2wc\delta^w_A"] \arrow[dr, "c^2\delta^w_{cwA}"] \arrow[r, phantom, "(3)"] & c^2wcw^2A \arrow[ur, "cdist_{cw^2A}"'] \arrow[d, "c^2wdist_{wA}"] \arrow[r, phantom, "(6)"] & \\
cw^2cwA \arrow[rr, "\delta^c_{w^2cwA}"'] & & c^2w^2cwA \arrow[r, "cdist_{wcwA}"'] & cwcwcwA
\end{tikzcd}
$$

(1) commutes by equality and we will not expand δ_A for simplicity. (2) and (4) commutes by the naturality of δ^c. (3), (5) and (6) commute by the conditions of $dist$.

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[dr, equal] & cwcwA \arrow[d, "cw\varepsilon_A"] \\
& cwA
\end{tikzcd}
$$

The triangle commutes because of the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\delta^w_A"] \arrow[ddd, equal] \arrow[ddr, "\varepsilon^c_{wA}"'] \arrow[drr, phantom, "(1)"] & & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, equal] \arrow[dr, phantom, "(2)"] & c^2w^2A \arrow[dl, "c\varepsilon^c_{w^2A}"'] \arrow[ddd, "cdist_{wA}"] \\
 & & cw^2A \arrow[ull, "c\varepsilon^w_{wA}"'] \arrow[d, "\varepsilon^c_{w^2A}"] \arrow[dl, phantom, "(3)"] \arrow[ddr, phantom, "(4)"] & \\
 & wA \arrow[l, phantom, "(5)"] & w^2A \arrow[l, "\varepsilon^w_{wA}"'] \arrow[dr, phantom, "(7)"] & \\
cwA \arrow[uur, "\varepsilon^c_{wA}"] \arrow[urr, phantom, "(6)"] & & wcwA \arrow[ll, "\varepsilon^w_{cwA}"] \arrow[u, "w\varepsilon^c_{wA}"] & cwcwA \arrow[l, "\varepsilon^c_{wcwA}"] \arrow[uul, "cw\varepsilon^c_{wA}"']
\end{tikzcd}
$$

(1) commutes by the comonad law for w with components δ^w_A and ε^w_{wA}. (2) commutes by the comonad law for c with components $\delta^c_{w^2A}$ and $\varepsilon^c_{w^2A}$. (3) and (7) commute by the naturality of ε^c. (6) commutes by the naturality of ε^w. And (7) is equality.

Case 3:

$$
\begin{tikzcd}
cwA \arrow[d, "\delta_A"'] \arrow[dr, equal] & \\
cwcwA \arrow[r, "\varepsilon_{cwA}"'] & cwA
\end{tikzcd}
$$

The previous triangle commutes because the following diagram chasing does.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\delta^w_A"] \arrow[drr, "\delta^c_{wA}"] \arrow[ddd, equal] & & cw^2A \arrow[rr, "\delta^c_{w^2A}"] \arrow[d, phantom, "(1)"] & & c^2w^2A \arrow[ddll, "c^2w\varepsilon^w_A"] \arrow[ddd, "cdist_{wA}"] \\
 & \arrow[r, phantom, "(2)"] & c^2wA \arrow[urr, "c^2\delta^w_A"'] \arrow[d, equal] \arrow[r, phantom, "(3)"] & & \\
 & \arrow[d, phantom, "(4)"] & c^2wA \arrow[uull, "c\delta^c_{wA}"] \arrow[d, "cdist_A"] & \arrow[d, phantom, "(5)"] & \\
cwA & & cwcA \arrow[ll, "cw\varepsilon^c_A"] & & cwcwA \arrow[ll, "cwc\varepsilon^w_A"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ^c. (2) is the comonad law for c with components δ^c_{wA} and ε^c_{wA}. (3) is the comonad law for w with components δ^w_A and ε^w_A. (3) and (4) commute by the definition of *dist*.

□

Lemma 17. *Let* $(c, \varepsilon^c, \delta^c)$ *and* $(w, \varepsilon^w, \delta^w)$ *be two monoidal comonads on a linear category with weakening and contraction* $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}^c, \mathsf{contraR}^c)$*, and* $(cw, \varepsilon, \delta)$ *be the monoidal comonad on* $\mathcal{L}$ *by composing c and w using the distributive law* $dist_A : cwA \longrightarrow wcA$*. The monoidal natural transformations* weak *and* contra *satisfy the following conditions:*

1. *Each* $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ *is a comonoid.*
2. weak_A *and* contra_A *are coalgebra morphisms.*
3. *Any coalgebra morphism* $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ *between free coalgebras preserves the comonoid structure given by* weak *and* contra.

Proof. We first define weak and contra. Each of them can also be given two equivalent definitions:

- $\mathsf{weak}_A : cwA \longrightarrow I$ is defined as in the diagram below. The left triangle commutes by the definition of *dist* and the right triangle commutes by the definition of weak^w.

$$
\begin{tikzcd}
 & wcA \arrow[d, "w\varepsilon^c_A"] \arrow[dr, "\mathsf{weak}^w_{cA}"] & \\
cwA \arrow[ur, "dist_A"] \arrow[r, "\varepsilon^c_{wA}"'] & wA \arrow[r, "\mathsf{weak}^w_A"'] & I
\end{tikzcd}
$$

- $\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$ is defined as below. The left part of the diagram commutes by the definitions of $\mathsf{contraL}^c$ and of $\mathsf{contraR}^c$, and the right part commutes because $\mathcal{L}$ is monoidal.

$$\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "\lambda^{-1}_{cwA}"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}^c_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[dl, "\alpha_{cwA,I,cwA}"'] \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
I \otimes cwA \arrow[r, "\mathsf{contraR}^c_{wA,I}"'] & cwA \otimes (I \otimes cwA) \arrow[r, "id_{cwA} \otimes \lambda_{cwA}"'] & cwA \otimes cwA
\end{tikzcd}$$

Then we show each condition is satisfied.

1. Each $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid, i.e. Diagrams 1 and 2 commute.

$$\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & cwA \otimes cwA \arrow[r, "id_{cwA} \otimes \mathsf{contra}_A"] & cwA \otimes (cwA \otimes cwA) \\
cwA \otimes cwA \arrow[rr, "\mathsf{contra}_A \otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}$$

Diagram 1: Commutative comonoid 1

Diagram 1 commutes by the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] \arrow[dr, phantom, "(1)"] & cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\rho^{-1}_{cwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwA \otimes (cwA \otimes I) \arrow[d, "id_{cwA}\otimes\mathsf{contraL}^c_{wA,I}"] \\
cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\lambda^{-1}_{cwA}"] \arrow[d, "\rho^{-1}_{cwA}\otimes id_{cwA}"'] & cwA \otimes (I \otimes cwA) \arrow[d, "id_{cwA}\otimes\mathsf{contraR}^c_{wA,I}"] & cwA \otimes ((cwA \otimes I) \otimes cwA) \arrow[d, "id_{cwA}\otimes(\rho_{cwA}\otimes id_{cwA})"] \arrow[dl, phantom, "(3)"] \\
(cwA \otimes I) \otimes cwA \arrow[d, "\mathsf{contraL}^c_{wA,I}\otimes id_{cwA}"'] & cwA \otimes (cwA \otimes (I \otimes cwA)) \arrow[r, "id_{cwA}\otimes(id_{cwA}\otimes\lambda_{cwA})"] \arrow[ur, "id_{cwA}\otimes\alpha_{cwA,I,cwA}"] \arrow[d, phantom, "(4)"] & cwA \otimes (cwA \otimes cwA) \\
((cwA \otimes I) \otimes cwA) \otimes cwA \arrow[rr, "(\rho_{cwA}\otimes id_{cwA})\otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}
$$

In the diagram chasing above, (1) commutes trivially and we would not expand contra for simplicity. (2) and (4) commute because $(\mathcal{L}, c, \mathsf{contraL}^c, \mathsf{contraR}^c)$ is a linear category with contraction. (3) commutes because $\mathcal{L}$ is monoidal.

$$
\begin{tikzcd}
& cwA \arrow[d, "\mathsf{contra}_A"] & \\
I \otimes cwA \arrow[ur, "\lambda"] & cwA \otimes cwA \arrow[l, "\mathsf{weak}_A\otimes id_{cwA}"] \arrow[r, "id_{cwA}\otimes\mathsf{weak}_A"'] & cwA \otimes I \arrow[ul, "\rho"']
\end{tikzcd}
$$

Diagram 2: Commutative Comonoid 2

- weak and contra are coalgebra morphisms, i.e. Diagrams 3 and 4 commute.

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
cwcwA \arrow[r, "cw\mathsf{weak}_A"'] & cwI
\end{tikzcd}
$$

Diagram 3: Coalgebra Morphism 1

We show that Diagram 3 commutes by the diagram below. (1) commutes by the naturality of δ^c. (2) commutes by the condition of $dist_{wA}$. (3), (5) and (6) commute because c is a monoidal comonad. (4) commutes because $(\mathcal{L}, w, \mathsf{weak}^w)$ is a linear category with weakening. (7) commutes because c and w are monoidal comonads.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\mathsf{weak}^w_A"] \arrow[d, "\delta^c_{wA}"'] \arrow[ddrr, "c\delta^w_A"] & & cI \arrow[rr, "\epsilon^c_I"] \arrow[dddrr, "c\mathsf{q}^w_I"'] & & I \arrow[dl, equal] \arrow[dd, "\mathsf{q}^c_I"] \\
c^2wA \arrow[d, "c^2\delta^w_A"'] & (1) & (4) & I \arrow[ul, "\mathsf{q}^c_I"'] & (6) \\
c^2w^2A \arrow[d, "cdist_{wA}"'] \arrow[drr, "c\varepsilon^c_{wA}"] & (2) & cw^2A \arrow[ll, "\delta^c_{w^2A}"'] \arrow[d, equal] & (7) & cI \arrow[ul, "\varepsilon^c_I"'] \arrow[d, "c\mathsf{q}^w_I"] \\
cwcwA \arrow[rr, "cw\varepsilon^c_{wA}"'] & & cw^2A & & cwI \arrow[ll, "cw\mathsf{weak}^w_A"]
\end{tikzcd}
$$

(3) (5)

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & cwA \otimes cwA \arrow[r, "\delta_A \otimes \delta_A"] & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}_{cwA,cwA}"] \\
cwcwA \arrow[rr, "cw\mathsf{contra}_A"'] & & cw(cwA \otimes cwA)
\end{tikzcd}
$$

Diagram 4: Coalgebra Morphism 2

To prove Diagram 4 commutes, we first expand it, Then we divide it into three parts and prove each part commutes, as shown below.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^{w}_{A}"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}^{c}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] & cwA \otimes cwA \arrow[d, "\delta^{c}_{wA}\delta^{c}_{wA}"] \\
cw^2A \arrow[dd, "\delta^{c}_{w^2A}"'] & & & c^2wA \otimes c^2wA \arrow[d, "c^2\delta^{w}_{A} \otimes c^2\delta^{w}_{A}"] \\
 & w(cwA \otimes I) \arrow[uu, "\varepsilon^{w}_{cwA \otimes I}"'] \arrow[l, phantom, "(a)"] \arrow[r, phantom, "(b)"] & w(cwA \otimes cwA) \arrow[uur, "\varepsilon^{w}_{cwA \otimes cwA}"] \arrow[r, phantom, "(c)"] & c^2w^2A \otimes c^2w^2A \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
c^2w^2A \arrow[dd, "cdist_{wA}"'] & & & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}^{c}_{wcwA,wcwA}"] \\
 & & & c(wcwA \otimes wcwA) \arrow[d, "c\mathsf{q}^{w}_{cwA,cwA}"] \\
cwcwA \arrow[r, "cw\rho^{-1}_{cwA}"'] & cw(cwA \otimes I) \arrow[uuu, "\varepsilon^{c}_{w(cwA \otimes I)}"'] \arrow[r, "cw\mathsf{contraL}^{c}_{wA,I}"'] & cw((cwA \otimes I) \otimes cwA) \arrow[r, "cw(\rho_{cwA} \otimes id_{cwA})"'] & cw(cwA \otimes cwA) \arrow[uuul, "\varepsilon^{c}_{w(cwA \otimes cwA)}"']
\end{tikzcd}
$$

Part (a) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for w. (3) is the comonad law for c. (4) commutes by the naturality of ε^{c}. (5) is one of the conditions for $dist_{wA}$. (6) commutes by the naturality of

ε^w. And (7) commutes by the naturality of ε^c.

$$\begin{tikzcd}
cwA \arrow[rrr, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, "c\delta^w_A"] \arrow[drr, equal] & & & cwA \otimes I \\
cw^2A \arrow[r, equal] \arrow[d, "\delta^c_{w^2A}"'] \arrow[ur, phantom, "(1)"] & cw^2A \arrow[r, "c\varepsilon^w_{wA}"] \arrow[d, "dist_{wA}"] & cwA \arrow[ul, phantom, "(2)"] \arrow[r, phantom, "(6)"] & \\
c^2w^2A \arrow[ur, "\varepsilon^c_{cw^2A}"] \arrow[d, "cdist_{wA}"'] \arrow[u, phantom, "(3)"] & wcwA \arrow[ur, "\varepsilon^w_{cwA}"] \arrow[rr, "w\rho^{-1}_{cwA}"] \arrow[u, phantom, "(5)"] \arrow[drr, phantom, "(7)"] & & w(cwA \otimes I) \arrow[uu, "\varepsilon^w_{cwA\otimes I}"'] \\
cwcwA \arrow[ur, "\varepsilon^c_{wcwA}"] \arrow[rrr, "cw\rho^{-1}_{cwA}"'] \arrow[u, phantom, "(4)"] & & & cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA\otimes I)}"']
\end{tikzcd}$$

Part (b) commutes by the following diagram chase. The upper two squares both commute by the naturality of ε^w, and the lower two squares commute by the naturality of ε^c.

$$\begin{tikzcd}
cwA \otimes I \arrow[r, "\mathsf{contraL}^c_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA}\otimes id_{cwA}"] & cwA \otimes cwA \\
w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA\otimes I}"] \arrow[r, "w\mathsf{contraL}^c_{wA,I}"] & w((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^w_{(cwA\otimes I)\otimes cwA}"] \arrow[r, "w(\rho_{cwA}\otimes id_{cwA})"] & w(cwA \otimes cwA) \arrow[u, "\varepsilon^w_{cwA\otimes cwA}"] \\
cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA\otimes I)}"] \arrow[r, "cw\mathsf{contraL}^c_{wA,I}"'] & cw((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^c_{w((cwA\otimes I)\otimes A)}"] \arrow[r, "cw(\rho_{cwA}\otimes id_{cwA})"'] & cw(cwA \otimes cwA) \arrow[u, "\varepsilon^c_{w(cwA\otimes cwA)}"']
\end{tikzcd}$$

Part (c) commutes by the following diagram. (1) and (2) are the comonad law. (3) commutes by the condition of $dist_{wA}$. (4) and (6) commute by the naturality of ε^c. (5) and (7) commute because w and c are monoidal comonads.

$$
\begin{tikzcd}
cwA \otimes cwA \arrow[rr, "c\delta^w_A \otimes c\delta^w_A"] & & cw^2A \otimes cw^2A \arrow[rr, "\delta^c_{w^2A} \otimes \delta^c_{w^2A}"] \arrow[d, equal] \arrow[dl, phantom, "(1)"] \arrow[dr, phantom, "(2)"] & & c^2w^2A \otimes c^2w^2A \arrow[dll, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"'] \arrow[dd, "cdist_{wA} \otimes cdist_{wA}"] \\
& & cw^2A \otimes cw^2A \arrow[ull, "c\varepsilon^w_{wA} \otimes c\varepsilon^w_{wA}"'] \arrow[d, "dist_{wA} \otimes dist_{wA}"] \arrow[dl, phantom, "(3)"] \arrow[dr, phantom, "(4)"] & & \\
& (5) & wcwA \otimes wcwA \arrow[uull, "\varepsilon^w_{cwA} \otimes \varepsilon^w_{cwA}"] \arrow[dll, "\mathsf{q}^w_{cwA,cwA}"'] \arrow[d, phantom, "(6)"] & (7) & cwcwA \otimes cwcwA \arrow[ll, "\varepsilon^c_{wcwA} \otimes \varepsilon^c_{wcwA}"'] \arrow[d, "\mathsf{q}^c_{wcwA,wcwA}"] \\
w(cwA \otimes cwA) \arrow[uuu, "\varepsilon^w_{cwA \otimes cwA}"] & & cw(cwA \otimes cwA) \arrow[ll, "\varepsilon^c_{w(cwA \otimes cwA)}"] & & c(wcwA \otimes wcwA) \arrow[ll, "c\mathsf{q}^w_{cwA,cwA}"] \arrow[ull, "\varepsilon^c_{wcwA \otimes wcwA}"']
\end{tikzcd}
$$

- Any coalgebra morphism $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ between free coalgebras preserve the comonoid structure given by weak and contra, i.e. the following two diagrams commute.

$$
\begin{tikzcd}
cwA \arrow[rr, "cwf"] \arrow[dr, "\mathsf{weak}_A"'] & & cwB \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & cwA \otimes cwA \arrow[d, "f \otimes f"] \\
cwB \arrow[r, "\mathsf{contra}_B"'] & cwB \otimes cwB
\end{tikzcd}
$$

The triangle on the left commutes by the diagram chasing below, which commutes by the definition of weak^w.

$$
\begin{tikzcd}
cwA \arrow[rr, "cwf"] \arrow[dr, "c\mathsf{weak}^w_A"] \arrow[dd, "\varepsilon^c_{wA}"'] & & cwB \arrow[dl, "c\mathsf{weak}^w_B"'] \arrow[dd, "\varepsilon^c_{wB}"] \\
& cI \arrow[d, "\varepsilon^c_I"] & \\
wA \arrow[r, "\mathsf{weak}^w_A"'] & I & wB \arrow[l, "\mathsf{weak}^w_B"]
\end{tikzcd}
$$

The square on the left commutes by the diagram chasing below, which commutes by the naturality of ρ and $\mathsf{contraL}^c$.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "cwf"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}^c_{wA,I}"] \arrow[d, "cwf \otimes id_I"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] \arrow[d, "(cwf \otimes id_I) \otimes cwf"] & cwA \otimes cwA \arrow[d, "cwf \otimes cwf"] \\
cwB \arrow[r, "\rho^{-1}_{cwB}"'] & cwB \otimes I \arrow[r, "\mathsf{contraL}^c_{wB,I}"'] & (cwB \otimes I) \otimes cwB \arrow[r, "\rho_{cwB} \otimes id_{cwB}"'] & cwB \otimes cwB
\end{tikzcd}
$$

$\square$

2 Related Work

TODO

3 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.