Separating Linear Modalities

Jiaming Jiang and Harley Eades III

Abstract

TODO

1 Introduction

TODO [1]

2 Categorical Models

2.1 Lambek Categories

TODO: Define Lambek Categories

2.2 Lambek Categories with Weakening and Contraction

Definition 1. *A **Lambek category with weakening**, $(\mathcal{L}, w, \mathsf{weak})$, is a Lambek category equipped with a monoidal comonad (w, ε, δ), and a monoidal natural transformation* $\mathsf{weak}_A : wA \longrightarrow I$*. Furthermore,* weak *must be a coalgebra morphism. That is, the following digram must commute:*

$$\begin{tikzcd} wA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta_A"'] & I \arrow[d, "\mathsf{q}_I"] \\ w^2A \arrow[r, "w\mathsf{weak}_A"'] & wI \end{tikzcd}$$

Definition 2. *A **Lambek category with contraction**, $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$, is a Lambek category equipped with a monoidal comonad (c, ε, δ), and two monoidal natural transformations:*

$$\begin{aligned} \mathsf{contraL}_{A,B} &: cA \otimes B \longrightarrow (cA \otimes B) \otimes cA \\ \mathsf{contraR}_{A,B} &: B \otimes cA \longrightarrow cA \otimes (B \otimes cA) \end{aligned}$$

Furthermore, the following diagrams must commute:

$$\begin{tikzcd}
cA \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & cA \arrow[l, "\rho^{-1}_{cA}"'] \arrow[r, "\lambda^{-1}_{cA}"] & I \otimes cA \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(cA \otimes I) \otimes cA \arrow[rr, "\alpha_{cA,I,cA}"'] & & cA \otimes (I \otimes cA)
\end{tikzcd}$$

$$\begin{tikzcd}
cA \otimes cA \arrow[r, "id_{cA} \otimes \rho^{-1}_{cA}"] \arrow[dd, "\lambda^{-1}_{cA} \otimes id_{cA}"'] & cA \otimes (cA \otimes I) \arrow[r, "id_{cA} \otimes \mathsf{contraL}_{A,I}"] & cA \otimes ((cA \otimes I) \otimes cA) \arrow[d, "id_{cA} \otimes (\rho_{cA} \otimes id_{cA})"] \\
& & cA \otimes (cA \otimes cA) \\
(I \otimes cA) \otimes cA \arrow[r, "\mathsf{contraR}_{A,I} \otimes id_{cA}"'] & (cA \otimes (I \otimes cA)) \otimes cA \arrow[r, "(id_{cA} \otimes \lambda_{cA}) \otimes id_{cA}"'] & (cA \otimes cA) \otimes cA \arrow[u, "\alpha_{cA,cA,cA}"']
\end{tikzcd}$$

2.3 Lambek Categories with Exchange

Definition 3. *A **Lambek category with exchange**, $(\mathcal{L}, e, \mathsf{ex})$, is a Lambek category equipped with a monoidal comonad (e, ε, δ) on $\mathcal{L}$, and a monoidal natural transformation $\mathsf{ex}_{A,B} : eA \otimes eB \longrightarrow eB \otimes eA$. Furthermore, the following diagrams must commute:*

$$\begin{tikzcd}
eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] \arrow[dr, equal] & eB \otimes eA \arrow[d, "\mathsf{ex}_{B,A}"] \\
& eA \otimes eB
\end{tikzcd}
\qquad
\begin{tikzcd}
e^2A \otimes e^2A \arrow[r, "\mathsf{ex}_{eA,eB}"] \arrow[d, "\varepsilon_{eA} \otimes \varepsilon_{eB}"'] & e^2B \otimes e^2A \\
eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] & eB \otimes eA \arrow[u, "\delta_B \otimes \delta_A"']
\end{tikzcd}$$

Definition 4. *Suppose $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange. Then the **Eilenberg Moore category**, $\mathcal{L}^e$, of the comonad (e, ε, δ) has as objects all the e-coalgebras $(A, h_A : A \longrightarrow eA)$, and as morphisms all the coalgebra morphisms. We call h_A the action of the coalgebra. Furthermore, the following (action) diagrams must commute:*

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] & eA \arrow[d, "eh_A"] \\
eA \arrow[r, "\delta_A"'] & e^2A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "h_A"'] \arrow[dr, equal] & \\
eA \arrow[r, "\varepsilon_A"'] & A
\end{tikzcd}$$

Lemma 5 (The Eilenberg Moore Category is Symmetric Monoidal)**.** *Suppose $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange. Then the category $\mathcal{L}^e$ is symmetric monoidal.*

Proof. We must first define the unitors, the associator, and the symmetry. Then we show that they respect the symmetry monoidal coherence diagrams. Recall the definition of composition in the Eilenberg Moore category. Throughout this proof we will make use of the coalgebra (A, h_A), (B, h_B), and (C, h_C).

The tensor product of (A, h_A) and (B, h_b) is $(A \otimes B, q_{A,B} \circ (h_A \otimes h_B))$, and the unit of the tensor product is (I, q_I); both actions are easily shown to satisfies the action diagrams of the Eilenberg Moore category. The left and right unitors are $\lambda : I \otimes A \longrightarrow A$ and $\rho : A \otimes I \longrightarrow A$, because they are indeed coalgebra morphisms.

The respective diagram for the right unitor is as follows:

$$\begin{tikzcd}
A \otimes I \arrow[r, "h_A \otimes \mathsf{id}"] \arrow[d, "\rho"] & eA \otimes I \arrow[r, "\mathsf{id} \otimes \mathsf{q}_I"] \arrow[drr, "\lambda"] & eA \otimes eI \arrow[r, "\mathsf{q}_{A,I}"] & e(A \otimes I) \arrow[d, "e\rho"] \\
A \arrow[rrr, "h_A"] & & & eA
\end{tikzcd}$$

The top-left diagram commutes by naturality of ρ, the top-right diagram commutes by the fact that e is a monoidal functor, and the bottom diagram commutes by the action diagrams for the coalgebra (A, h_A). Showing the left unitor is a coalgebra morphism is similar.

The unitors are natural and isomorphisms, because they are essentially inherited from the underlying Lambek category.

The associator $\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ is also a coalgebra morphism. First, notice that:

$$\mathsf{q}_{A \otimes B, C} \circ ((\mathsf{q}_{A,B} \circ (h_A \otimes h_B)) \otimes h_c) = \mathsf{q}_{A \otimes B, C} \circ (\mathsf{q}_{A,B} \otimes \mathsf{id}) \circ ((h_A \otimes h_B) \otimes h_C)$$

where the left-hand side is the action of the coalgebra $(A \otimes B) \otimes C$. Similarly, the following is the action of the coalgebra $A \otimes (B \otimes C)$:

$$\mathsf{q}_{A, B \otimes C} \circ (h_A \otimes (\mathsf{q}_{B,C} \circ (h_B \otimes h_C))) = \mathsf{q}_{A, B \otimes C} \circ (\mathsf{id} \otimes \mathsf{q}_{B,C}) \circ (h_A \otimes (h_B \otimes h_C))$$

The following diagram must commute:

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "(h_A \otimes h_B) \otimes h_C"] \arrow[d, "\alpha"] & (eA \otimes eB) \otimes eC \arrow[r, "\mathsf{q} \otimes \mathsf{id}"] \arrow[d, "\alpha"] & e(A \otimes B) \otimes eC \arrow[r, "\mathsf{q}"] & e((A \otimes B) \otimes C) \arrow[d, "e\alpha"] \\
A \otimes (B \otimes C) \arrow[r, "h_A \otimes (h_B \otimes h_C)"] & eA \otimes (eB \otimes eC) \arrow[r, "\mathsf{id} \otimes \mathsf{q}"] & eA \otimes e(B \otimes C) \arrow[r, "\mathsf{q}"] & e(A \otimes (B \otimes C))
\end{tikzcd}
$$

The left diagram commutes by naturality of α, and the right diagram commutes because e is a monoidal functor.

Composition in $\mathcal{L}^e$ is the same as $\mathcal{L}$, and thus, the monoidal coherence diagrams hold in $\mathcal{L}^e$ as well. Thus, $\mathcal{L}^e$ is monoidal. We now show that it is symmetric.

The symmetry of $\mathcal{L}^e$ is defined as follows:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{ex_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

It turns out that $\beta_{B,A} \circ \beta_{A,B} = \mathsf{id}_{A \otimes B}$ which holds because the following diagram commutes:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[d, "h_A \otimes h_B"'] \arrow[dr, phantom, "(1)"] \arrow[rrrrdddd, bend left, equal] & eA \otimes eB \arrow[d, "\delta_A \otimes \delta_B"] \arrow[drr, phantom, "(8)"] & & & \\
eA \otimes eB \arrow[r, "eh_A \otimes eh_B"] \arrow[d, "\mathsf{ex}_{A,B}"'] \arrow[dr, phantom, "(2)"] & e^2A \otimes e^2B \arrow[d, "\mathsf{ex}_{eA,eB}"] \arrow[drr, equal] \arrow[dr, phantom, "(4)"] & & & \\
eB \otimes eA \arrow[r, "eh_B \otimes eh_A"] \arrow[dd, "\varepsilon_B \otimes \varepsilon_A"'] \arrow[ddr, phantom, "(3)"] & e^2B \otimes e^2A \arrow[rr, "\mathsf{ex}_{eB,eA}"] \arrow[dd, "\varepsilon_{eB} \otimes \varepsilon_{eA}"] \arrow[dr, phantom, "(5)"] & & e^2A \otimes e^2B \arrow[dd, "\varepsilon_{eA} \otimes \varepsilon_{eB}"] & \\
& & eA \otimes eB \arrow[ur, "\delta_A \otimes \delta_B"] \arrow[dr, equal] \arrow[d, phantom, "(6)"] \arrow[r, phantom, "(7)"] & & \\
B \otimes A \arrow[r, "h_B \otimes h_A"] & eB \otimes eA \arrow[ur, "\mathsf{ex}_{B,A}"] \arrow[rr, "\mathsf{ex}_{B,A}"] & & eA \otimes eB \arrow[r, "\varepsilon_A \otimes \varepsilon_B"] & A \otimes B
\end{tikzcd}
$$

Diagram one commutes by the action diagrams of the Eilenberg Moore category, diagram two commutes by naturality of ex, diagram three commutes by naturality of ε, diagram four and five commute by the coherence diagrams of ex, diagram six clearly commutes, diagram seven commutes because (e, ε, δ) is a comonad, and diagram eight commutes by both the action diagrams of the Eilenberg Moore category and the fact that (e, ε, δ) is a comonad.

At this point we must verify that β respects the coherence diagrams of a symmetric monoidal category; see Definition 18. Thus, we must show that each of the following diagrams hold:

Case

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"'] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}
$$

Case

$$
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}
$$

Case

$$\begin{tikzcd} \top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\ & A & \end{tikzcd}$$

□

Definition 6. *Let $(\mathcal{L}, e, \mathsf{ex})$ be a Lambek category with exchange. The **coKleisli Category of** e, $\mathcal{L}_e$, is a category with the same objects as $\mathcal{L}$. There is an arrow $\hat{f} : A \longrightarrow B$ in $\mathcal{L}_e$ if there is an arrow $f : eA \longrightarrow B$ in $\mathcal{L}$. The identity arrow $\hat{id_A} : A \longrightarrow A$ is the arrow $\varepsilon_A : eA \longrightarrow A$ in $\mathcal{L}$. Given $\hat{f} : A \longrightarrow B$ and $\hat{g} : B \longrightarrow C$ in $\mathcal{L}_e$, which are arrows $f : eA \longrightarrow B$ and $g : eB \longrightarrow C$ in $\mathcal{L}$, the composition $\hat{g} \circ \hat{f} : A \longrightarrow C$ is defined as $g \circ ef \circ \delta_A$.*

2.4 Linear Categories

Definition 7. *A **linear category**, $(\mathcal{L}, !, \mathsf{weak}, \mathsf{contra})$, is a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$ equipped with a symmetric monoidal comonad $(!, \varepsilon, \delta)$ with $\mathsf{q}_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $\mathsf{q}_I : I \longrightarrow !I$, and two monoidal natural transformations with components $\mathsf{weak}_A : !A \longrightarrow I$ and $\mathsf{contra}_A : !A \longrightarrow !A \otimes !A$, satisfying the following conditions:*

- *each $(!A, \mathsf{weak}_A, \mathsf{contra}_A)$ is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ \mathsf{contra}_A = \mathsf{contra}_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of $\mathcal{L}$;*

$$\begin{tikzcd} !A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & !A \otimes !A \arrow[r, "id_{!A} \otimes \mathsf{contra}_A"] & !A \otimes (!A \otimes !A) \\ !A \otimes !A \arrow[rr, "\mathsf{contra}_A \otimes id_{!A}"'] & & (!A \otimes !A) \otimes !A \arrow[u, "\alpha_{!A,!A,!A}"'] \end{tikzcd}$$

$$\begin{tikzcd} & !A \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\ I \otimes !A & !A \otimes !A \arrow[l, "\mathsf{weak}_A \otimes id_{!A}"] \arrow[r, "id_{!A} \otimes \mathsf{weak}_A"'] & !A \otimes I \end{tikzcd}$$

- *weak_A and contra_A are coalgebra morphisms, i.e. the following diagrams commute;*

$$\begin{tikzcd} !A \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\ !!A \arrow[r, "!\mathsf{weak}_A"'] & !I \end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[r, "\delta_A \otimes \delta_A"] & !!A \otimes !!A \arrow[d, "\mathsf{q}_{!A,!A}"] \\
!!A \arrow[rr, "!\mathsf{contra}_A"'] & & !(!A \otimes !A)
\end{tikzcd}$$

- *any coalgebra morphism $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ between free coalgebras preserve the comonoid structure given by* weak *and* contra*, i.e. the following diagrams commute.*

$$\begin{tikzcd}
!A \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & !B \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & !A \otimes !A \arrow[d, "f \otimes f"] \\
!B \arrow[r, "\mathsf{contra}_B"'] & !B \otimes !B
\end{tikzcd}$$

Definition 8. *Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction and $(\mathcal{L}, w, \mathsf{weak})$ is a Lambek category with weakening, we define a **distributive law** of c over w to be a natural transformation with components $dist_A : cwA \longrightarrow wcA$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
wA & & cwA \arrow[ll, "\varepsilon^c_{wA}"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "w\varepsilon^c_A"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cA & & cwA \arrow[ll, "c\varepsilon^w_A"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "\varepsilon^w_{cA}"] &
\end{tikzcd}$$

Lemma 9. *Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction and $(\mathcal{L}, w, \mathsf{weak})$ is a Lambek category with weakening, the following two diagrams commute:*

$$\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, "\delta^c_{wcA}"'] & & cwc^2A \\
& c^2wcA \arrow[ur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wdist_{wA}"] \\
& w^2cwA &
\end{tikzcd}$$

Proof. The two diagrams above commute because the following ones commute by the distributive law and the comonad laws for c and w.

$$\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, equal] \arrow[ddr, "\delta^c_{wA}"'] & & cwc^2A \arrow[dl, "cw\varepsilon^c_{cA}"'] \\
& cwcA & \\
& c^2wcA \arrow[u, "c\varepsilon^c_{wcA}"] \arrow[uur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, equal] \arrow[ddr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wc\varepsilon^w_{wA}"'] \arrow[ddl, "wdist_{wA}"] \\
& wcwA & \\
& w^2cwA \arrow[u, "w\varepsilon^w_{cwA}"] &
\end{tikzcd}$$

□

Lemma 10 (Composition of Weakening and Contraction)**.** *Suppose* $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}, \mathsf{contraR})$ *is a Lambek category with weakening and contraction, where* $(w, \varepsilon^w, \delta^w)$ *and* $(c, \varepsilon^c, \delta^c)$ *are the respective monoidal comonads. Then the composition of c and w using the distributive law* $dist_A : cwA \longrightarrow wcA$ *is a monoidal comonad on* $\mathcal{L}$*.*

Proof. For the complete proof see Appendix B.1. □

Definition 11. *A* ***Lambek category with*** *cw,* $(\mathcal{L}, cw, \mathsf{weak}^w, \mathsf{contraL}, \mathsf{contraR}, dist)$*, is a Lambek category with weakening and contraction, and a distributive law. Furthermore, the following coherence diagrams commute:*

$$\begin{tikzcd}
I \otimes cwA \arrow[r, "\lambda^{-1}_{I\otimes cwA}"] \arrow[d, "\mathsf{contraR}_{wA,I}"'] & I \otimes (I \otimes cwA) \\
cwA \otimes (I \otimes cwA) \arrow[r, "\varepsilon^c_{wA}\otimes id_{I\otimes cwA}"] & wA \otimes (I \otimes cwA) \arrow[u, "\mathsf{weak}^w_A\otimes id_{I\otimes cwA}"']
\end{tikzcd}$$

$$\begin{tikzcd}
cwA \otimes I \arrow[r, "\rho^{-1}_{cwA\otimes I}"] \arrow[d, "\mathsf{contraL}_{wA,I}"'] & (cwA \otimes I) \otimes I \\
(cwA \otimes I) \otimes cwA \arrow[r, "id_{cwA\otimes I}\otimes\varepsilon^c_{wA}"] & (cwA \otimes I) \otimes wA \arrow[u, "id_{cwA\otimes I}\otimes\mathsf{weak}^w_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[d, "\varepsilon^c_{wA}"'] & & cwB \arrow[d, "\varepsilon^c_{wB}"] \\
wA \arrow[r, "\mathsf{weak}^w_A"] & I & wB \arrow[l, "\mathsf{weak}^w_B"']
\end{tikzcd}$$

where $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ *is any coalgebra morphism between free coalgebras.*

Lemma 12. *Let* $(\mathcal{L}, cw, \mathsf{weak}^w, \mathsf{contraL}, \mathsf{contraR})$ *be a Lambek category with cw. Then the following conditions are satisfied:*

1. *There exist two natural transformations* $\mathsf{weak}_A : cwA \longrightarrow I$ *and* $\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$*.*
2. *Each* $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ *is a comonoid.*
3. weak_A *and* contra_A *are coalgebra morphisms.*
4. *Any coalgebra morphism* $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ *between free coalgebras preserves the comonoid structure given by* weak *and* contra*.*

Proof. We will only prove the first condition by defining weak and contra. For the complete proof see Appendix B.2. Each of weak and contracan be given two equivalent definitions. $\mathsf{weak}_A : cwA \longrightarrow I$ is defined as in the diagram below. The left triangle commutes by the definition of *dist* and the right triangle commutes by the definition of weak^w.

$$\begin{tikzcd}
 & wcA \arrow[d, "w\varepsilon^c_A"] \arrow[dr, "\mathsf{weak}^w_{cA}"] & \\
cwA \arrow[ur, "dist_A"] \arrow[r, "\varepsilon^c_{wA}"'] & wA \arrow[r, "\mathsf{weak}^w_A"'] & I
\end{tikzcd}$$

$\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$ is defined as below. The left part of the diagram commutes by the definitions of $\mathsf{contraL}$ and of $\mathsf{contraR}$, and the right part commutes because $\mathcal{L}$ is monoidal.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "\lambda^{-1}_{cwA}"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[dl, "\alpha_{cwA,I,cwA}"'] \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
I \otimes cwA \arrow[r, "\mathsf{contraR}_{wA,I}"'] & cwA \otimes (I \otimes cwA) \arrow[r, "id_{cwA} \otimes \lambda_{cwA}"'] & cwA \otimes cwA
\end{tikzcd}
$$

$\square$

Definition 13. *Given two comonads $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ on a category $\mathcal{L}$ such that $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ is a Lambek category with cw and $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange, we define a* ***distributive law for exchange*** *of cw over e to be a natural isomorphism with components $distEx_A : cweA \longrightarrow ecwA$, subject to the following coherence diagrams:*

$$
\begin{tikzcd}
eA & & cweA \arrow[ll, "\varepsilon^{cw}_{eA}"'] \arrow[dl, "distEx_A"] \\
& ecwA \arrow[ul, "e\varepsilon^{cw}_{A}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA & & cweA \arrow[ll, "cw\varepsilon^{e}_{A}"'] \arrow[dl, "distEx_A"] \\
& ecwA \arrow[ul, "\varepsilon^{e}_{cwA}"] &
\end{tikzcd}
$$

Lemma 14. *Given two comonads $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ on a category $\mathcal{L}$ such that $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ is a Lambek category with cw and $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange, the following two digrams also commute:*

$$
\begin{tikzcd}
cwecwA \arrow[rr, "cwe\delta^{cw}_{A}"] \arrow[dr, "\delta^{cw}_{cweA}"'] & & cwe(cw)^2A \\
& (cw)^2ecwA \arrow[ur, "cwdistEx_{cwA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
ecweA \arrow[rr, "ecw\delta^{e}_{A}"] \arrow[dr, "\delta^{e}_{cweA}"'] & & ecwe^2A \arrow[dl, "edistEx_{eA}"] \\
& e^2cweA &
\end{tikzcd}
$$

The proof is similar with the proof of Lemma 9 and we will not elaborate it here. Also, notice the difference between *dist* of *c* over *w* and *distEx* of *cw* over *e*. While *dist* is a natural transformation, *distEx* is a natural isomorphism.

Lemma 15. *let $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ be two monoidal comonads on a Lambek category with cw and exchange $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$. Then the composition of cw and e using the distributive law for exchange $distEx_A : cweA \longrightarrow ecwA$ is a monoidal comonad $(cwe, \varepsilon, \delta)$ on $\mathcal{L}$.*

Proof. Suppose $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ are monoidal comonads, and $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with *cw* and exchange. Since by definition $cw, e : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors, we know that their composition

$cwe : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$
\begin{aligned}
\mathsf{q}_{A,B} &: cweA \otimes cweB \longrightarrow cwe(A \otimes B) \\
\mathsf{q}_{A,B} &= cw\mathsf{q}^{e}_{A,B} \circ \mathsf{q}^{cw}_{eA,eB} \\
\mathsf{q}_{I} &: I \longrightarrow cweI \\
\mathsf{q}_{I} &= cw\mathsf{q}^{e}_{I} \circ \mathsf{q}^{cw}_{I}
\end{aligned}
$$

Analogous to the proof of Lemma 10, each of ε and δ can be given two equivalent definitions:

$$
\begin{tikzcd}
cweA \arrow[r, "\varepsilon^{cw}_{eA}"] \arrow[d, "cw\varepsilon^{e}_{A}"'] & eA \arrow[d, "\varepsilon^{e}_{A}"] \\
cwA \arrow[r, "\varepsilon^{cw}_{A}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
cweA \arrow[r, "cw\delta^{e}_{A}"] \arrow[d, "\delta^{cw}_{eA}"'] & cwe^2A \arrow[r, "\delta^{cw}_{e^2A}"] \arrow[d, "\delta^{cw}_{e^2A}"] & (cw)^2e^2A \arrow[d, "cwdist_{eA}"] \\
(cw)^2eA \arrow[r, "(cw)^2\delta^{e}_{A}"'] & (cw)^2e^2A \arrow[r, "cwdist_{eA}"'] & cwecweA
\end{tikzcd}
$$

And the comonad laws can be proved similarly, which we will not elaborate for simplicity. $\square$

Lemma 16. *Let $(cwe, \varepsilon, \delta)$ be a monoidal comonad over a monoidal category $(\mathcal{L}, I, \otimes)$ such that $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with cw and exchange. Then the co-Kleisli category of $\mathcal{L}$, $\mathcal{L}_{cwe}$, is a linear category.*

changed to liner category. Finish the proof when lemma 5 is proved.

Proof. The identity object of $\mathcal{L}_{cwe}$ is still I.

The left and right unitors, $\hat{\lambda}_A : I \otimes A \longrightarrow A$ and $\hat{\rho}_A : A \otimes I \longrightarrow A$, in $\mathcal{L}_{cwe}$ are morphisms $cwe(I \otimes A) \longrightarrow A$ and $cwe(A \otimes I) \longrightarrow A$ in $\mathcal{L}$, respectively. Then we define $\hat{\lambda}$ and $\hat{\rho}$ as:

$$
\begin{aligned}
\hat{\lambda}_A &= \varepsilon_A \circ cwe\lambda_A \\
\hat{\rho}_A &= \varepsilon_A \circ cwe\rho_A,
\end{aligned}
$$

where λ and ρ are the left and right unitors in $\mathcal{L}$, respectively. And we define their inverses as:

$$
\begin{aligned}
\hat{\lambda}^{-1}_A &= \varepsilon_{I\otimes A} \circ cwe\lambda^{-1}_A \\
\hat{\rho}^{-1}_A &= \varepsilon_{A\otimes I} \circ cwe\rho^{-1}_A
\end{aligned}
$$

$\hat{\lambda}$ is a nautral isomorphism with inverse $\hat{\lambda}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
cwe(I \otimes A) \arrow[r, "\delta_{I\otimes A}"] \arrow[dr, "cwe\lambda_A"] \arrow[dd, "\varepsilon_{I\otimes A}"'] \arrow[ddr, equal] & (cwe)^2(I \otimes A) \arrow[r, "(cwe)^2\lambda_A"] \arrow[d, phantom, "(1)"] & (cwe)^2A \arrow[dd, "cwe\varepsilon_A"] \\
(2) & (3)\ cweA \arrow[ur, "\delta_A"] \arrow[d, "cwe\lambda^{-1}_A"] \arrow[dr, equal] \arrow[r, phantom, "(5)"] & {} \\
I \otimes A & cwe(I \otimes A) \arrow[l, "\varepsilon_{I\otimes A}"] \arrow[r, phantom, "(4)"] & cweA \arrow[l, "cwe\lambda^{-1}_A"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ. (2), (3) and (4) commute trivially. And (5) commutes because cwe is a comonad.

Similarly, $\hat{\rho}$ is a natural isomorphism with inverse $\hat{\rho}^{-1}$ by the following diagram chasing:

$$\begin{tikzcd}
cwe(A \otimes I) \arrow[r, "\delta_{A\otimes I}"] \arrow[dd, "\varepsilon_{A\otimes I}"'] \arrow[dr, "cwe\rho_A"] \arrow[ddr, equal] & (cwe)^2(A \otimes I) \arrow[r, "(cwe)^2\rho_A"] & (cwe)^2 A \arrow[dd, "cwe\varepsilon_A"] \\
 & cweA \arrow[ur, "\delta_A"] \arrow[d, "cwe\rho_A^{-1}"] \arrow[dr, equal] & \\
A \otimes I & cwe(A \otimes I) \arrow[l, "\varepsilon_{A\otimes I}"] & cweA \arrow[l, "cwe\rho_A^{-1}"]
\end{tikzcd}$$

The associator $\hat{\alpha}_A : (A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}_{cwe}$ is the morphism $cwe((A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}$. We define $\hat{\alpha}$ as:

$$\hat{\alpha}_{A,B,C} = \varepsilon_{A\otimes(B\otimes C)} \circ cwe\alpha_{A,B,C},$$

where α is the associator of $\mathcal{L}$. And its inverse is

$$\hat{\alpha}^{-1}_{A,B,C} = \varepsilon_{(A\otimes B)\otimes C} \circ cwe\alpha^{-1}_{A,B,C}$$

$\hat{\alpha}$ is a natural isomorphism with inverse $\hat{\alpha}^{-1}$ because the following diagram chasing commutes:

$$\begin{tikzcd}
cwe((A \otimes B) \otimes C) \arrow[r, "\delta_{(A\otimes B)\otimes C}"] \arrow[dd, "\varepsilon_{(A\otimes B)\otimes C}"'] \arrow[dr, "cwe\alpha_{A,B,C}"] \arrow[ddr, equal] & (cwe)^2((A \otimes B) \otimes C) \arrow[r, "(cwe)^2\alpha_{A,B,C}"] & (cwe)^2(A \otimes (B \otimes C)) \arrow[dd, "cwe\varepsilon_{A\otimes(B\otimes C)}"] \\
 & cwe(A \otimes (B \otimes C)) \arrow[ur, "\delta_{A\otimes(B\otimes C)}"] \arrow[d, "cwe\alpha^{-1}_{A,B,C}"] \arrow[dr, equal] & \\
(A \otimes B) \otimes C & cwe((A \otimes B) \otimes C) \arrow[l, "\varepsilon_{(A\otimes B)\otimes C}"] & cwe(A \otimes (B \otimes C)) \arrow[l, "cwe\alpha^{-1}_{A,B,C}"]
\end{tikzcd}$$

Therefore, $\mathcal{L}_{cwe}$ is a monoidal category.

The symmetry, $\hat{\beta}_{A,B} : A \otimes B \longrightarrow B \otimes A$, in $\mathcal{L}_{cwe}$ is the morphism $cwe(A \otimes B) \longrightarrow B \otimes A$ in $\mathcal{L}$, which is defined as:

$$\hat{\beta}_{A,B} = \varepsilon^{cw}_{B\otimes A} \circ cw\gamma_{A,B},$$

where $\varepsilon^{cw}_A : cwA \longrightarrow A$ is a natural transformation associated with the comonad cw, and γ is the natural isomorphism defined in Lemma **??**. Then its inverse is

$$\hat{\beta}^{-1}_{A,B} = \varepsilon^{cw}_{A\otimes B} \circ cw\gamma_{B,A}$$

$\hat{\beta}$ is a natural isomorphism with inverse $\hat{\beta}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
A \otimes B & cw(A \otimes B) \arrow[l, "\varepsilon^{cw}_{A\otimes B}"'] \arrow[dl, equals] \arrow[d, "\delta^{cw}_{A\otimes B}"] & & cwe(A \otimes B) \arrow[ll, "cw\varepsilon^{e}_{A\otimes B}"'] \arrow[d, "\delta^{cw}_{e(A\otimes B)}"] \\
cw(A \otimes B) \arrow[u, "\varepsilon^{cw}_{A\otimes B}"] & (cw)^2(A \otimes B) \arrow[l, "cw\varepsilon^{cw}_{A\otimes B}"] & & (cw)^2e(A \otimes B) \arrow[ll, "(cw)^2\varepsilon^{e}_{A\otimes B}"] \arrow[d, "(cw)^2\delta^{e}_{A\otimes B}"'] \\
 & (cw)^2e(B \otimes A) \arrow[u, "(cw)^2\gamma_{B,A}"] \arrow[d, equals] \arrow[ddl, "cw\varepsilon^{cw}_{e(B\otimes A)}"'] & (cw)^2e^2(A \otimes B) \arrow[l, "(cw)^2e\gamma_{A,B}"] \arrow[r, equals] & (cw)^2e^2(A \otimes B) \arrow[d, "cwdistEx_{e(A\otimes B)}"] \\
 & (cw)^2e(B \otimes A) \arrow[drr, "cwdistEx_{B\otimes A}"'] & & (cwe)^2(A \otimes B) \arrow[ul, "cwdistEx^{-1}_{e(A\otimes B)}"'] \arrow[d, "cwecw\gamma_{A,B}"] \\
cwe(B \otimes A) \arrow[uuu, "cw\gamma_{B,A}"] & & & cwecw(B \otimes A) \arrow[lll, "cwe\varepsilon^{cw}_{B\otimes A}"] \arrow[uull, "cwdistEx^{-1}_{B\otimes A}"]
\arrow[from=1-1, to=2-2, phantom, "(1)"]
\arrow[from=2-1, to=1-2, phantom, "(2)"]
\arrow[from=1-2, to=2-4, phantom, "(3)"]
\arrow[from=2-1, to=5-2, phantom, "(4)"]
\arrow[from=2-2, to=3-3, phantom, "(5)"]
\arrow[from=4-2, to=5-1, phantom, "(6)"]
\arrow[from=3-2, to=5-4, phantom, "(7)"]
\arrow[from=3-3, to=5-4, phantom, "(8)"]
\arrow[from=3-3, to=4-4, phantom, "(9)"]
\end{tikzcd}
$$

(1), (7) and (9) commute trivially. (2) is the comonad law for cw. (3) commutes by the naturality of δ^{cw}. (4) commutes by the naturality of ε^{cw}. (5) commutes because γ is a natural isomorphism (Lemma **??**). (6) is the definition of $distEx$. (8) is the naturality of $distEx$.

In conclusion, $\mathcal{L}_{cwe}$ is a symmetric monoidal category. □

3 Related Work

TODO

4 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.

A Appendix

A.1 Symmetric Monoidal Categories

Definition 17. *A* ***monoidal category*** *is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}\lambda_A &: \top \otimes A \longrightarrow A\\ \rho_A &: A \otimes \top \longrightarrow A\\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)\end{aligned}$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[rr, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & & \\
A \otimes (B \otimes (C \otimes D)) & & A \otimes ((B \otimes C) \otimes D) \arrow[ll, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Definition 18. *A* ***symmetric monoidal category (SMC)*** *is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}\lambda_A &: \top \otimes A \longrightarrow A\\ \rho_A &: A \otimes \top \longrightarrow A\\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)\end{aligned}$$

- *A symmetry natural isomorphism:*

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 19. *A* ***monoidal biclosed category*** *is a monoidal category $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, each of the functors $-\otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ and $B \otimes - : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any object A and C of $\mathcal{M}$, there are two objects $C \leftharpoonup B$ and $B \rightharpoonup C$ of $\mathcal{M}$ and two natural bijections:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, C \leftharpoonup B)$$
$$\mathsf{Hom}_{\mathcal{M}}(B \otimes A, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \rightharpoonup C)$$

Definition 20. *A* ***symmetric monoidal closed category (SMCC)*** *is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$*

has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap$: $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Definition 21. *Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a* ***monoidal functor*** *is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[rr, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"'] & & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"'] & & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[rr, "F\alpha_{1A,B,C}"] & & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"'] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"'] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"']
\end{tikzcd}$$

Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

Definition 22. *Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a* ***symmetric monoidal functor*** *is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[rr, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"'] & & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"'] & & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[rr, "F\alpha_{1A,B,C}"] & & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"'] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1, A}"'] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"'] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A, \top_1}"'] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"'] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"'] & F(B \otimes_1 A)
\end{tikzcd}$$

Definition 23. *Suppose* $(\mathcal{M}_1, \top_1, \otimes_1)$ *and* $(\mathcal{M}_2, \top_2, \otimes_2)$ *are monoidal categories, and* (F, m) *and* (G, n) *are monoidal functors between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*. Then a* ***monoidal natural transformation*** *is a natural transformation,* $f : F \longrightarrow G$*, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 24. *Suppose* $(\mathcal{M}_1, \top_1, \otimes_1)$ *and* $(\mathcal{M}_2, \top_2, \otimes_2)$ *are SMCs, and* (F, m) *and* (G, n) *are symmetric monoidal functors between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*. Then a* ***symmetric monoidal natural transformation*** *is a natural transformation,* $f : F \longrightarrow G$*, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 25. *Suppose* $(\mathcal{M}_1, \top_1, \otimes_1)$ *and* $(\mathcal{M}_2, \top_2, \otimes_2)$ *are monoidal categories, and* (F, m) *is a monoidal functor between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *and* (G, n) *is a monoidal functor between* $\mathcal{M}_2$ *and* $\mathcal{M}_1$*. Then a* ***monoidal adjunction*** *is an ordinary adjunction* $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ *such that the unit,* $\eta_A : A \to GFA$*, and the counit,* $\varepsilon_A : FGA \to A$*, are* *monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 26. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***symmetric monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 27. *A* ***monoidal comonad*** *on a monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd} TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\ & A \otimes B \end{tikzcd} \qquad \begin{tikzcd} T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\ & \top & \end{tikzcd}$$

$$\begin{tikzcd} TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A\otimes B}"] \\ T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B) \end{tikzcd}$$

$$\begin{tikzcd} \top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\ T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top \end{tikzcd}$$

Definition 28. *A* ***symmetric monoidal comonad*** *on a symmetric monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd} TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\ T^2A \arrow[r, "\delta_{TA}"'] & T^3A \end{tikzcd} \qquad \begin{tikzcd} & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\ TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA \end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd} TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\ & A \otimes B \end{tikzcd} \qquad \begin{tikzcd} T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\ & \top & \end{tikzcd}$$

$$
\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}
$$

B Proofs

B.1 Proof of Composition of Weakening and Contraction (Lemma 10)

Since by definition $w : \mathcal{L} \longrightarrow \mathcal{L}$ and $c : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors we know that their composition $cw : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$
\begin{aligned}
&\mathsf{q}_{A,B} : cwA \otimes cwB \longrightarrow cw(A \otimes B) \\
&\mathsf{q}_{A,B} = c\mathsf{q}^w_{A,B} \circ \mathsf{q}^c_{wA,wB} \\
&\mathsf{q}_I : I \longrightarrow cwI \\
&\mathsf{q}_I = c\mathsf{q}^w_I \circ \mathsf{q}^c_I
\end{aligned}
$$

We must now define both $\varepsilon_A : cwA \longrightarrow A$ and $\delta_A : cwA \longrightarrow cwcwA$, and then show that they are monoidal natural transformations subject to the comonad laws. Since we are composing two comonads each of ε and δ can be given two definitions, but they are equivalent:

- $\varepsilon_A : cwA \longrightarrow A$ is defined as in the diagram below, which commutes by the naturality of ε^c.

$$
\begin{tikzcd}
cwA \arrow[r, "\varepsilon^c_{wA}"] \arrow[d, "c\varepsilon^w_A"'] & wA \arrow[d, "\varepsilon^w_A"] \\
cA \arrow[r, "\varepsilon^c_A"'] & A
\end{tikzcd}
$$

- $\delta_A : cwA \longrightarrow cwcwA$ is defined as in the diagram:

$$\begin{tikzcd}
cwA \arrow[r, "c\delta^w_A"] \arrow[d, "\delta^c_{wA}"'] & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, "\delta^c_{w^2A}"] & c^2w^2A \arrow[d, "cdist_{wA}"] \\
c^2wA \arrow[r, "c^2\delta^w_A"'] & c^2w^2A \arrow[r, "cdist_{wA}"'] & cwcwA
\end{tikzcd}$$

  The left part of the diagram commutes by the naturality of δ^c and the right part commutes trivially.

The remainder of the proof shows that the comonad laws hold.

Case 1:

$$\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & cwcwA \arrow[d, "cw\delta_A"] \\
cwcwA \arrow[r, "\delta_{cwA}"'] & cwcwcwA
\end{tikzcd}$$

The previous diagram commutes because the following one does.

$$\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, phantom, "(1)", near start] & cwcwA \arrow[r, "cwc\delta^w_A"] \arrow[d, "\delta^c_{wcwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwcw^2A \arrow[r, "cw\delta^c_{w^2A}"] \arrow[d, "\delta^c_{wcw^2A}"] \arrow[dr, phantom, "(5)", near start] & cwc^2w^2A \arrow[dd, "cwcdist_{wA}"] \\
cwcwA \arrow[d, "c\delta^w_{cwA}"'] \arrow[r, phantom, "(4)", shift right=8ex] & c^2wcwA \arrow[r, "c^2wc\delta^w_A"] \arrow[dr, "c^2\delta^w_{cwA}"] \arrow[r, phantom, "(3)", shift right=4ex] & c^2wcw^2A \arrow[ur, "cdist_{cw^2A}"] \arrow[d, "c^2wdist_{wA}"] \arrow[r, phantom, "(6)", shift right=8ex] & \\
cw^2cwA \arrow[rr, "\delta^c_{w^2cwA}"'] & & c^2w^2cwA \arrow[r, "cdist_{wcwA}"'] & cwcwcwA
\end{tikzcd}$$

(1) commutes by equality and we will not expand δ_A for simplicity. (2) and (4) commutes by the naturality of δ^c. (3), (5) commutes by the conditions of $dist$. (6) commutes by the naturality of $dist$.

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[dr, equal] & cwcwA \arrow[d, "cw\varepsilon_A"] \\
 & cwA
\end{tikzcd}
$$

The triangle commutes because of the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[rrrr, "c\delta^w_A"] \arrow[dddddd, equal] \arrow[ddddrr, "\varepsilon^c_{wA}"] & & & & cw^2A \arrow[rr, "\delta^c_{w^2A}"] \arrow[dd, equal] & & c^2w^2A \arrow[ddll, "c\varepsilon^c_{w^2A}"'] \arrow[dddddd, "cdist_{wA}"] \\
 & & & (1) & & (2) & \\
 & & & & cw^2A \arrow[uullll, "c\varepsilon^w_{wA}"'] \arrow[dd, "\varepsilon^c_{w^2A}"] & & \\
 & & & (3) & & (4) & \\
 & (5) & wA & & w^2A \arrow[ll, "\varepsilon^w_{wA}"'] & & \\
 & & & (6) & & (7) & \\
cwA \arrow[uurr, "\varepsilon^c_{wA}"] & & & & wcwA \arrow[uu, "w\varepsilon^c_{wA}"] \arrow[llll, "\varepsilon^w_{cwA}"] & & cwcwA \arrow[ll, "\varepsilon^c_{wcwA}"] \arrow[uuuull, "cw\varepsilon^c_{wA}"]
\end{tikzcd}
$$

(1) commutes by the comonad law for w with components δ^w_A and ε^w_{wA}. (2) commutes by the comonad law for c with components $\delta^c_{w^2A}$ and $\varepsilon^c_{w^2A}$. (3) and (7) commute by the naturality of ε^c. (4) commutes by the condition of $dist$. (5) commutes trivially. And (6) commutes by the naturality of ε^w.

Case 3:

$$
\begin{tikzcd}
cwA \arrow[d, "\delta_A"'] \arrow[dr, equal] & \\
cwcwA \arrow[r, "\varepsilon_{cwA}"'] & cwA
\end{tikzcd}
$$

The previous triangle commutes because the following diagram chasing does.

$$
\begin{tikzcd}
cwA \arrow[r, "c\delta^w_A"] \arrow[dr, "\delta^c_{wA}"] \arrow[ddd, equal] & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, phantom, "(1)"] & c^2w^2A \arrow[ddd, "cdist_{wA}"] \arrow[ddl, "c^2w\varepsilon^w_A"] \\
& c^2wA \arrow[ur, "c^2\delta^w_A"] \arrow[d, equal] \arrow[l, phantom, "(2)"] \arrow[r, phantom, "(3)"] & \\
& c^2wA \arrow[uul, "c\varepsilon^c_{wA}"] \arrow[d, "cdist_A"] \arrow[l, phantom, "(4)"] \arrow[r, phantom, "(5)"] & \\
cwA & cwcA \arrow[l, "cw\varepsilon^c_A"] & cwcwA \arrow[l, "cwc\varepsilon^w_A"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ^c. (2) is the comonad law for c with components δ^c_{wA} and ε^c_{wA}. (3) is the comonad law for w with components δ^w_A and ε^w_A. (4) commutes by the condition of $dist$. And (5) commute by the naturality of $dist$.

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B.2 Proof of Conditions of Lambek category with cw (Lemma 12)

1. As shown in the paper.
2. Each $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid.

Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & cwA \otimes cwA \arrow[r, "id_{cwA} \otimes \mathsf{contra}_A"] & cwA \otimes (cwA \otimes cwA) \\
cwA \otimes cwA \arrow[rr, "\mathsf{contra}_A \otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}
$$

The previous diagram commutes by the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] \arrow[dr, phantom, "(1)"] & cwA \otimes cwA \arrow[r, "id_{cwA} \otimes \rho^{-1}_{cwA}"] \arrow[dl, equal] & cwA \otimes (cwA \otimes I) \arrow[d, "id_{cwA} \otimes \mathsf{contraL}_{wA,I}"] \\
cwA \otimes cwA \arrow[r, "id_{cwA} \otimes \lambda^{-1}_{cwA}"] \arrow[d, "\rho^{-1}_{cwA} \otimes id_{cwA}"'] & cwA \otimes (I \otimes cwA) \arrow[d, "id_{cwA} \otimes \mathsf{contraR}_{wA,I}"] \arrow[ur, phantom, "(2)"] & cwA \otimes ((cwA \otimes I) \otimes cwA) \arrow[d, "id_{cwA} \otimes (\rho_{cwA} \otimes id_{cwA})"] \\
(cwA \otimes I) \otimes cwA \arrow[d, "\mathsf{contraL}_{wA,I} \otimes id_{cwA}"'] & cwA \otimes (cwA \otimes (I \otimes cwA)) \arrow[r, "id_{cwA} \otimes (id_{cwA} \otimes \lambda_{cwA})"] \arrow[ur, "id_{cwA} \otimes \alpha_{cwA,I,cwA}"] \arrow[d, phantom, "(4)"] & cwA \otimes (cwA \otimes cwA) \arrow[ul, phantom, "(3)"] \\
((cwA \otimes I) \otimes cwA) \otimes cwA \arrow[rr, "(\rho_{cwA} \otimes id_{cwA}) \otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}
$$

(1) commutes trivially and we would not expand contra for simplicity. (2) and (4) commute because $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction. (3) commutes because $\mathcal{L}$ is monoidal.

Case 2:

$$
\begin{tikzcd}
 & cwA \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes cwA & cwA \otimes cwA \arrow[l, "\mathsf{weak}_A \otimes id_{cwA}"] \arrow[r, "id_{cwA} \otimes \mathsf{weak}_A"'] & cwA \otimes I
\end{tikzcd}
$$

The diagram above commutes by the following diagram chasing.

$$
\begin{tikzcd}
I \otimes cwA & & \arrow[d, phantom, "(2)"] & & wA \otimes cwA \arrow[llll, "\mathsf{weak}^w_A \otimes id_{cwA}"'] \\
\arrow[dr, phantom, "(1)"] & I \otimes (I \otimes cwA) \arrow[ul, "id_I \otimes \lambda_{cwA}"'] & \arrow[d, phantom, "(4)"] & wA \otimes (I \otimes cwA) \arrow[ll, "\mathsf{weak}^w_A \otimes id_{I \otimes cwA}"'] \arrow[ur, "id_{wA} \otimes \lambda_{cwA}"] & \arrow[dl, phantom, "(3)"] \\
 & I \otimes cwA \arrow[u, "\lambda^{-1}_{I \otimes cwA}"] \arrow[rr, "\mathsf{contraR}_{wA,I}"] & {} & cwA \otimes (I \otimes cwA) \arrow[u, "\varepsilon^c_{wA} \otimes id_{I \otimes cwA}"'] \arrow[dr, "id_{cwA} \otimes \lambda_{cwA}"] & \\
cwA \arrow[uuu, "\lambda^{-1}_{cwA}"] \arrow[ur, "\lambda^{-1}_{cwA}"] \arrow[dr, "\rho^{-1}_{cwA}"'] \arrow[ddd, "\rho^{-1}_{cwA}"'] & & (5) & & cwA \otimes cwA \arrow[uuu, "\varepsilon^c_{wA} \otimes cwA"'] \arrow[ddd, "id_{cwA} \otimes \varepsilon^c_{wA}"] \\
 & cwA \otimes I \arrow[d, "\rho^{-1}_{cwA}"] \arrow[rr, "\mathsf{contraL}_{wA,I}"] & \arrow[d, phantom, "(9)"] & (cwA \otimes I) \otimes cwA \arrow[ur, "\rho_{cwA} \otimes id_{cwA}"'] \arrow[d, "id_{cwA \otimes I} \otimes \varepsilon^c_{wA}"'] & \\
\arrow[ur, phantom, "(6)"] & (cwA \otimes I) \otimes I \arrow[dl, "\rho_{cwA} \otimes id_I"'] & \arrow[d, phantom, "(7)"] & (cwA \otimes I) \otimes wA \arrow[ll, "id_{cwA \otimes I} \otimes \mathsf{weak}^w_A"'] \arrow[dr, "\rho_{cwA} \otimes id_{wA}"] & \arrow[ul, phantom, "(8)"] \\
cwA \otimes I & & {} & & cwA \otimes wA \arrow[llll, "id_{cwA} \otimes \mathsf{weak}_A"]
\end{tikzcd}
$$

(1), (2) and (3) commute by the functionality of λ. (6), (7) and (8) commute by the functionality of ρ. (4) and (9) are conditions of the Lambek category with cw. And (5) is the definition of contra.

3. weak and contra are coalgebra morphisms.

Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
cwcwA \arrow[r, "cw\mathsf{weak}_A"'] & cwI
\end{tikzcd}
$$

The previous diagram commutes by the diagram below. (1) commutes by the naturality of δ^c. (2) commutes by the condition of $dist_{wA}$. (3), (5) and (6) commute because c is a monoidal comonad. (4) commutes because $(\mathcal{L}, w, \mathsf{weak}^w)$ is a Lambek category with weakening. (7) commutes because c and w are monoidal comonads.

$$\begin{tikzcd}
cwA \arrow[rr, "c\mathsf{weak}^w_A"] \arrow[d, "\delta^c_{wA}"'] \arrow[ddrr, "c\delta^w_A"] & & cI \arrow[rr, "\epsilon^c_I"] \arrow[dddrr, "c\mathsf{q}^w_I"] \arrow[dr, phantom, "(5)", near start] & & I \arrow[dl, equal] \arrow[dd, "\mathsf{q}^c_I"] \\
c^2wA \arrow[d, "c^2\delta^w_A"'] & (1) & (4) & I \arrow[ul, "\mathsf{q}^c_I"] & (6) \\
c^2w^2A \arrow[d, "cdist_{wA}"'] \arrow[drr, "c\varepsilon^c_{wA}"] \arrow[dr, phantom, "(2)"] & (3) & cw^2A \arrow[ll, "\delta^c_{w^2A}"'] \arrow[d, equal] & (7) & cI \arrow[ul, "\varepsilon^c_I"] \arrow[d, "c\mathsf{q}^w_I"] \\
cwcwA \arrow[rr, "cw\varepsilon^c_{wA}"'] & & cw^2A & & cwI \arrow[ll, "cw\mathsf{weak}^w_A"]
\end{tikzcd}$$

Case 2:

$$\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & cwA \otimes cwA \arrow[r, "\delta_A \otimes \delta_A"] & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}_{cwA,cwA}"] \\
cwcwA \arrow[rr, "cw\mathsf{contra}_A"'] & & cw(cwA \otimes cwA)
\end{tikzcd}$$

To prove the previous diagram commute, we first expand it, Then we divide it into five parts as shown belovee, and prove each part commutes.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, phantom, "(c)"] & cwA \otimes I \arrow[rrr, "\mathsf{contraL}_{wA,I}"] & & & (cwA \otimes I) \otimes cwA \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
cw^2A \arrow[d, "\delta^c_{w^2A}"'] & w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA \otimes I}"'] \arrow[ddd, phantom, "(d)"] & w(cwA \otimes cwA) \arrow[rr, "\varepsilon^w_{cwA \otimes cwA}"] & & cwA \otimes cwA \arrow[d, "c\delta^w_A \otimes c\delta^w_A"] \arrow[dl, phantom, "(a)"] \\
c^2w^2A \arrow[d, "cdist_{wA}"'] & & & cw^2A \otimes cw^2A \arrow[ur, "c\varepsilon^c_{wA} \otimes c\varepsilon^c_{wA}"] \arrow[r, equal] \arrow[ddd, phantom, "(e)"] & cw^2A \otimes cw^2A \arrow[d, "\delta^c_{w^2A} \otimes \delta^c_{w^2A}"] \arrow[dl, phantom, "(b)"] \\
cwcwA \arrow[d, "cw\rho_{cwA}"'] & & & & c^2w^2A \otimes c^2w^2A \arrow[ul, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
cw(cwA \otimes I) \arrow[d, "cw\mathsf{contraL}_{wA,I}"'] \arrow[uuur, "\varepsilon^c_{w(cwA \otimes I)}"] & & & & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}^c_{wcwA \otimes wcwA}"] \\
cw((cwA \otimes I) \otimes cwA) \arrow[rr, "cw(\rho_{cwA} \otimes id_{cwA})"'] & & cw(cwA \otimes cwA) \arrow[uuuu, "c\varepsilon^w_{cwA \otimes cwA}"] & & c(wcwA \otimes wcwA) \arrow[ll, "c\mathsf{q}^w_{cwA \otimes cwA}"]
\end{tikzcd}
$$

Part (a) and (b) are comonad laws.

Part (c) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for w. (3) is the comonad law for c. (4) commutes by the naturality of ε^c. (5) is one of the conditions for $dist_{wA}$. (6) commutes by the naturality of ε^w. And (7) commutes by the naturality of ε^c.

$$
\begin{tikzcd}
cwA \arrow[rrr, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, "c\delta^w_A"] \arrow[drr, equal] \arrow[dr, phantom, "(1)"'] & & & cwA \otimes I \\
cw^2A \arrow[r, equal] \arrow[d, "\delta^c_{w^2A}"'] \arrow[dr, phantom, "(3)"] & cw^2A \arrow[r, "c\varepsilon^w_{wA}"] \arrow[d, "dist_{wA}"] \arrow[ur, phantom, "(2)"] \arrow[dr, phantom, "(5)"] & cwA \arrow[r, phantom, "(6)"] & {} \\
c^2w^2A \arrow[ur, "\varepsilon^c_{cw^2A}"] \arrow[d, "cdist_{wA}"'] \arrow[r, phantom, "(4)"] & wcwA \arrow[ur, "\varepsilon^w_{cwA}"'] \arrow[rr, "w\rho^{-1}_{cwA}"] \arrow[drr, phantom, "(7)"] & & w(cwA \otimes I) \arrow[uu, "\varepsilon^w_{cwA \otimes I}"'] \\
cwcwA \arrow[ur, "\varepsilon^c_{wcwA}"] \arrow[rrr, "cw\rho^{-1}_{cwA}"'] & & & cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA \otimes I)}"']
\end{tikzcd}
$$

Part (d) commutes by the following diagram chase. The upper two squares both commute by the naturality of ε^w, and the lower two squares commute by the naturality of ε^c.

$$
\begin{tikzcd}
cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] & cwA \otimes cwA \\
w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA \otimes I}"] \arrow[r, "w\mathsf{contraL}_{wA,I}"] & w((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^w_{(cwA \otimes I) \otimes cwA}"] \arrow[r, "w(\rho_{cwA} \otimes id_{cwA})"] & w(cwA \otimes cwA) \arrow[u, "\varepsilon^w_{cwA \otimes cwA}"] \\
cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA \otimes I)}"] \arrow[r, "cw\mathsf{contraL}_{wA,I}"'] & cw((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^c_{w((cwA \otimes I) \otimes A)}"] \arrow[r, "cw(\rho_{cwA} \otimes id_{cwA})"'] & cw(cwA \otimes cwA) \arrow[u, "\varepsilon^c_{w(cwA \otimes cwA)}"']
\end{tikzcd}
$$

Part (e) commutes by the following diagram. (1) commutes by the condition of $dist_{wA}$. (2) and (4) commute by the naturality of ε^c. (3) and (5) commute because w and c are monoidal comonads.

$$
\begin{tikzcd}
cwA \otimes cwA & cw^2A \otimes cw^2A \arrow[l, "c\varepsilon^w_{wA} \otimes c\varepsilon^w_{wA}"'] \arrow[d, "dist_{wA} \otimes dist_{wA}"] \arrow[dl, phantom, "(1)"] \arrow[dr, phantom, "(2)"] & c^2w^2A \otimes c^2w^2A \arrow[l, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"'] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
\arrow[r, phantom, "(3)"] & wcwA \otimes wcwA \arrow[ul, "\varepsilon^w_{cwA} \otimes \varepsilon^w_{cwA}"'] \arrow[dl, "\mathsf{q}^w_{cwA,cwA}"] \arrow[d, phantom, "(4)"] & cwcwA \otimes cwcwA \arrow[l, "\varepsilon^c_{wcwA} \otimes \varepsilon^c_{wcwA}"'] \arrow[d, "\mathsf{q}^c_{wcwA \otimes wcwA}"] \arrow[dl, phantom, "(5)"] \\
w(cwA \otimes cwA) \arrow[uu, "\varepsilon^w_{cwA \otimes cwA}"] & cw(cwA \otimes cwA) \arrow[l, "\varepsilon^c_{w(cwA \otimes cwA)}"] & c(wcwA \otimes wcwA) \arrow[l, "c\mathsf{q}_{cwA \otimes cwA}"] \arrow[ul, "\varepsilon^c_{wcwA \otimes wcwA}"']
\end{tikzcd}
$$

4. Any coalgebra morphism $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ between free coalgebras preserves the comonoid structure given by weak and contra.

   Case 1: This coherence diagram is given in the definition of the Lambek category with cw.

$$
\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & cwB \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
$$

   Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & cwA \otimes cwA \arrow[d, "f \otimes f"] \\
cwB \arrow[r, "\mathsf{contra}_B"'] & cwB \otimes cwB
\end{tikzcd}
$$

   The square commutes by the diagram chasing below, which commutes by the naturality of ρ and contraL.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "cwf"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] \arrow[d, "cwf \otimes id_I"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] \arrow[d, "(cwf \otimes id_I) \otimes cwf"] & cwA \otimes cwA \arrow[d, "cwf \otimes cwf"] \\
cwB \arrow[r, "\rho^{-1}_{cwB}"'] & cwB \otimes I \arrow[r, "\mathsf{contraL}_{wB,I}"'] & (cwB \otimes I) \otimes cwB \arrow[r, "\rho_{cwB} \otimes id_{cwB}"'] & cwB \otimes cwB
\end{tikzcd}
$$

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