A Family of Adjoint Models as the Foundation of Process Tree Based Threat Analysis

Harley Eades III[1] and Jiaming Jiang[2]

1 **Computer Science, Augusta University, Augusta, Georgia, USA**
`heades@augusta.edu`
2 **Computer Science, North Carolina State University, Raleigh, North Carolina, USA**
`jjiang13@ncsu.edu`

Abstract

TODO

1998 ACM Subject Classification TODO

Keywords and phrases TODO

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

TODO [1]

2 Categorical Models

We develop a categorical framework in which many different intuitionistic substructural logics may be modeled. The locus of this framework is an adjunction. We initially take a monoidal category, $\mathcal{L}$, as a base, and then extend it with one or more structural morphisms – a morphism corresponding to a structural rule in logic – to obtain a second category $\hat{\mathcal{L}}$. Then we form a monoidal adjunction $\hat{\mathcal{L}} : F \dashv G : \mathcal{L}$ just as Benton [1] did for intuitionistic linear logic. Depending on which structural morphisms we add to $\hat{\mathcal{L}}$ we will obtain different models. In particular, each model will come endowed with a comonad on $\mathcal{L}$ which equips $\mathcal{L}$ with the ability to track the corresponding structural rule(s).

We will show that by adding the morphisms for either weakening, contraction, or exchange, to $\mathcal{L}$ will yield an adjoint model of non-commutative relevance logic/linear logic, non-commutative contraction logic/linear logic, and commutative/non-commutative linear logic. The latter model will come with a monoidal comonad $e : \mathcal{L} \longrightarrow \mathcal{L}$ such that there is a symmetry $\mathsf{ex}_{A,B} : eA \triangleright eB \longrightarrow eB \triangleright eA$, where $- \triangleright -$ denotes a non-commutative tensor product. In fact, this is the first adjoint model of the Lambek calculus with the exchange comonad.

At this point we will have adjoint models for each individual structural rule. What if we want more than one structural rule? There are a few different choices that one can choose from depending on the scenario. First, if $\hat{\mathcal{L}}$ contains more than one structural morphism, then $\mathcal{L}$ will have a single comonad that adds all of those structural morphisms to $\mathcal{L}$. For example, if $\hat{\mathcal{L}}$ contains weakening, contraction, and exchange, then $\hat{\mathcal{L}}$ is cartesian closed and $\mathcal{L}$ will have the usual $! : \mathcal{L} \longrightarrow \mathcal{L}$ comonad. The second scenario is when $\mathcal{L}$ also contains some structural morphisms. For example, if $\hat{\mathcal{L}}$ contains exchange and weakening and $\mathcal{L}$ contains exchange, then $\mathcal{L}$ will have a comonad, $r : \mathcal{L} \longrightarrow \mathcal{L}$, which combines linear logic with relevance logic. Thus, how we instantiate the two categories in the adjunction influences which logic one may model.

What if we want multiple comonads tracking different logics? In this scenario the different comonads would allow us to mix the different logics in interesting ways. Suppose $\mathcal{L}$ has no structural

Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

morphisms and $\mathcal{E}$ is $\mathcal{L}$ with exchange and $\mathcal{EW}$ is $\mathcal{E}$ with weakening. Then we can form two adjunctions $\mathcal{E} : F \dashv G : \mathcal{L}$ and $\mathcal{EW} : H \dashv J : \mathcal{L}$, but the categories $\mathcal{E}$ and $\mathcal{EW}$ have a structural morphism in common. So instead, we form the adjunction $\mathcal{EW} : H \dashv J : \mathcal{E} : F \dashv G : \mathcal{L}$. Thus, $\mathcal{L}$ has the exchange comonad $e = FG : \mathcal{L} \longrightarrow \mathcal{L}$ as well as the relevance logic comonad $r = FHJG : \mathcal{L} \longrightarrow \mathcal{L}$. Additionally, there is a comonad $w = JH : \mathcal{E} \longrightarrow \mathcal{E}$ adding weakening to $\mathcal{E}$. This idea is based on the amazing work of Mellies [3]. Throughout the remainder of this section we make these ideas precise.

2.1 Lambek Categories

The bases of all of our models will be what we call Lambek categories. These are named after Joachim Lambek to pay homage to his work on the Lambek calculus which can be seen as non-commutative intuitionistic linear logic [2]. Thus, each of our models have a very basic foundation.

Lambek categories are based on (non-symmetric) monoidal categories.

▶ **Definition 1.** A **monoidal category**, $(\mathcal{L}, \triangleright, I, \lambda, \rho)$, is a category, $\mathcal{L}$, equipped with a bifunctor, $\triangleright : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$, called the tensor product, a distinguished object I of $\mathcal{L}$ called the unit, and three natural isomorphisms $\lambda_A : I \triangleright A \longrightarrow A$, $\rho_A : A \triangleright I \longrightarrow A$, and $\alpha_{A,B,C} : A \triangleright (B \triangleright C) \longrightarrow (A \triangleright B) \triangleright C$ called the left and right unitors and the associator respectively. Finally, these are subject to the following coherence diagrams:

$$\begin{tikzcd}
((A \triangleright B) \triangleright C) \triangleright D \arrow[r, "\alpha_{A,B,C} \triangleright \mathsf{id}_D"] \arrow[d, "\alpha_{A \triangleright B,C,D}"'] & (A \triangleright (B \triangleright C)) \triangleright D \arrow[r, "\alpha_{A,B \triangleright C,D}"] & A \triangleright ((B \triangleright C) \triangleright D) \arrow[d, "\mathsf{id}_A \triangleright \alpha_{B,C,D}"] \\
(A \triangleright B) \triangleright (C \triangleright D) \arrow[rr, "\alpha_{A,B,C \triangleright D}"] & & A \triangleright (B \triangleright (C \triangleright D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \triangleright I) \triangleright B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \triangleright id_B"'] & & A \triangleright (I \triangleright B) \arrow[dl, "id_A \triangleright \lambda_B"] \\
& A \triangleright B &
\end{tikzcd}$$

A Lambek category adds closure to monoidal categories.

▶ **Definition 2.** A **Lambek category** is a monoidal category $(\mathcal{L}, \triangleright, I, \lambda, \rho, \alpha)$ equipped with two bifunctors $\rightharpoonup : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ and $\leftharpoonup : \mathcal{L} \times \mathcal{L}^{\mathsf{op}} \longrightarrow \mathcal{L}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \triangleright A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \triangleright X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

An alternative name for Lambek categories is biclosed monoidal categories.

If we add add a symmetry to a Lambek category then we will obtain a symmetric monoidal closed category. The following two definitions and lemma capture this result.

▶ **Definition 3.** A monoidal category $(\mathcal{L}, \triangleright, I, \lambda, \rho, \alpha)$ is **symmetric** if there is a natural transformation $\mathsf{ex}_{A,B} : A \triangleright B \longrightarrow B \triangleright A$ such that $\mathsf{ex}_{B,A} \circ \mathsf{ex}_{A,B} = \mathsf{id}_{A \triangleright B}$ and the following commute:

$$\begin{tikzcd}
(A \triangleright B) \triangleright C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathsf{ex}_{A,B} \triangleright \mathsf{id}_C"'] & A \triangleright (B \triangleright C) \arrow[r, "\mathsf{ex}_{A,B \triangleright C}"] & (B \triangleright C) \triangleright A \arrow[d, "\alpha_{B,A,C}"] \\
(B \triangleright A) \triangleright C \arrow[r, "\alpha_{B,A,C}"] & B \triangleright (A \triangleright C) \arrow[r, "\mathsf{id}_B \triangleright \mathsf{ex}_{A,C}"] & B \triangleright (C \triangleright A)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \triangleright A \arrow[rr, "\mathsf{ex}_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \triangleright I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Throughout this paper when $- \triangleright -$ is symmetric we denote it by $- \otimes -$.

We call a symmetric Lambek category a Lambek category with exchange, because the symmetric models the exchange rule.

▶ **Definition 4.** A symmetric monoidal category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \beta)$ is **closed** if it comes equipped with a bifunctor $\multimap : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \multimap B)$ holds.

▶ **Lemma 5.** *Let A and B be two objects in a Lambek category with exchange. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$\begin{array}{rcll} Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\ & \cong & Hom[A \otimes C, B] & \text{By the symmetry } \mathsf{ex}_{C,A} \\ & \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category} \end{array}$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◀

▶ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

We will also be discussing two other structural rules: weakening and contraction. These are defined as follows.

▶ **Definition 7.** A **Lambek category with weakening**, $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \mathsf{weak})$, is a Lambek category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ equipped with a natural transformation $\mathsf{weak}_A : A \longrightarrow I$.

▶ **Definition 8.** A **Lambek category with contraction**, $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \mathsf{contraL}, \mathsf{contraR})$, is a Lambek category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ equipped with natural transformations:

$$\mathsf{contraL}_{A,B} : (A \otimes B) \longrightarrow (A \otimes B) \otimes A \qquad \mathsf{contraR}_{A,B} : (B \otimes A) \longrightarrow A \otimes (B \otimes A)$$

Furthermore, the following diagrams must commute:

$$\begin{tikzcd}
A \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & A \arrow[l, "\rho_A^{-1}"'] \arrow[r, "\lambda_A^{-1}"] & I \otimes A \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(A \otimes I) \otimes A \arrow[rr, "\alpha_{A,I,A}"'] & & A \otimes (I \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[r, "id_A \otimes \rho_A^{-1}"] \arrow[dd, "\lambda_A^{-1} \otimes id_A"'] & A \otimes (A \otimes I) \arrow[r, "id_A \otimes \mathsf{contraL}_{A,I}"] & A \otimes ((A \otimes I) \otimes A) \arrow[d, "id_A \otimes (\rho_A \otimes id_A)"] \\
& & A \otimes (A \otimes A) \\
(I \otimes A) \otimes A \arrow[r, "\mathsf{contraR}_{A,I} \otimes id_A"'] & (A \otimes (I \otimes A)) \otimes A \arrow[r, "(id_A \otimes \lambda_A) \otimes id_A"'] & (A \otimes A) \otimes A \arrow[u, "\alpha_{A,A,A}"']
\end{tikzcd}$$

We call the morphisms:

$\mathsf{ex}_{A,B} : A \otimes B \longrightarrow B \otimes A$
$\mathsf{weak}_A : A \longrightarrow I$
$\mathsf{contraL}_{A,B} : (A \otimes B) \longrightarrow (A \otimes B) \otimes A$
$\mathsf{contraR}_{A,B} : (B \otimes A) \longrightarrow A \otimes (B \otimes A)$

structural morphisms, because they all model the various structural rules in intuitionistic logic.

2.2 Structural Adjoint Models

Now we turn to making our model precise.

▶ **Definition 9.** Suppose $\mathcal{L}_0, \ldots, \mathcal{L}_n$ is a family of Lambek categories with zero or more structural morphisms where $\mathcal{L}_i$ is a full subcategory of $\mathcal{L}_{i+1}$. Then a **structural adjoint model**, $\overrightarrow{\mathcal{L}_n : F_n \dashv G_n : \mathcal{L}_{n-1}}$, is a composition of monoidal adjunctions:

$$\mathcal{L}_n : F_n \dashv G_n : \mathcal{L}_{n-1} : F_{n-1} \dashv G_{n-1} : \mathcal{L}_{n-2} : \cdots : \mathcal{L}_1 : F \dashv G : \mathcal{L}_0.$$

This definition is an extension – or perhaps a simplification due to the isolation of exchange – of the models discussed by Melliés [3].

Our structural adjoint model subsumes Benton's [1] linear/non-linear model (LNL). Simply take the sequence of Lambek categories to be $\mathcal{L}_0$, a Lambek category with exchange, and $\mathcal{L}_1$, a Lambek category with weakening, contraction, and exchange, and thus, $\mathcal{L}_1$ is cartesian closed. However, our model is a lot more flexible and expressive.

▶ **Lemma 10** (Comonads in a Structural Adjoint Model)**.** *Suppose $\overrightarrow{\mathcal{L}_n : F_n \dashv G_n : \mathcal{L}_{n-1}}$ is a structural adjoint model. Then there are the following comonads:*

- $(\mathcal{L}_0, F_0G_0, \varepsilon^0, \delta^0), \ldots, (\mathcal{L}_{n-1}, F_nG_n, \varepsilon^n, \delta^n)$
- $(\mathcal{L}_0, F_0F_1G_1G_0, \varepsilon^0, \delta^0), \ldots, (\mathcal{L}_{n-1}, F_{n-1}F_nG_nG_{n-1}, \varepsilon^n, \delta^n)$
- $(\mathcal{L}_0, F_0F_1F_2G_2G_1G_0, \varepsilon^0, \delta^0), \ldots, (\mathcal{L}_{n-1}, F_{n-2}F_{n-1}F_nG_nG_{n-1}G_{n-2}, \varepsilon^n, \delta^n)$

  $\vdots$
- $(\mathcal{L}_0, F_0 \cdots F_nG_n \cdots G_0, \varepsilon^0, \delta^0)$

Proof. This proof easily follows from the well-known fact that adjoints induce comonads – as well as monads – and composition of adjoints. ◀

The previous lemma shows that a Lambek category $\mathcal{L}_i$ in the sequence is endowed with all of the structure found in each of the categories above it, but this structure is explicitly tracked using the various comonads. That is, the Eilenberg-Moore category of each of the comonads mentioned in the previous lemma has the corresponding structural rule as morphisms.

▶ **Lemma 11.** *Suppose $\mathcal{L}_i : F_i \dashv G_i : \mathcal{L}_{i-1}$ is a monoidal adjunction in the structural adjoint model $\overrightarrow{\mathcal{L}_n : F_n \dashv G_n : \mathcal{L}_{n-1}}$. Then the Eilenberg-Moore category, $\mathcal{L}^E_{i-1}$, contains all of the structural morphisms from both $\mathcal{L}_{i-1}$ and $\mathcal{L}_i$.*

2.3 Example Structural Adjoint Models

Exchange.

Prove the Eilenberg-Moore category is symmetric monoidal.

Contraction.

Prove the Klesli category has contractions.

Contraction and Exchange.

Prove the Klesli category has weakening.

3 Related Work

TODO

4 Conclusion

TODO

References

1 P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.

2 Joachim Lambek. The mathematics of sentence structure. *Journal of Symbolic Logic*, 33(4):627–628, 1968.

3 Paul-André Melliès. Comparing hierarchies of types in models of linear logic. *Information and Computation*, 189(2):202 – 234, 2004. `doi:http://dx.doi.org/10.1016/j.ic.2003.10.003`.

A Appendix