Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

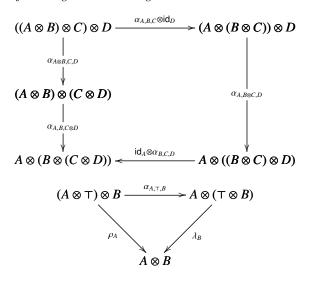
1.1 Symmetric Monoidal Categories

Definition 1. A monoidal category is a category, M, with the following data:

- An object \top of \mathcal{M} ,
- A bi-functor \otimes : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• Subject to the following coherence diagrams:



Definition 2. A symmetric monoidal category (SMC) is a category, M, with the following data:

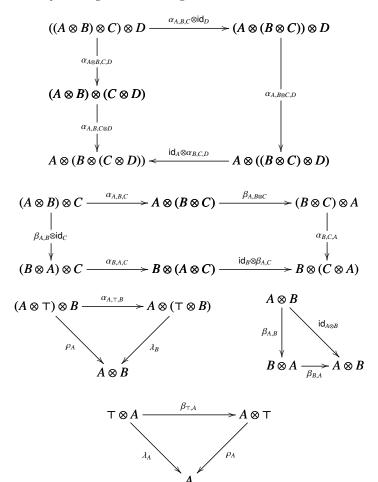
- An object \top of \mathcal{M} ,
- A bi-functor \otimes : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• A symmetry natural transformation:

$$\beta_{A,B}: A \otimes B \longrightarrow B \otimes A$$

• Subject to the following coherence diagrams:

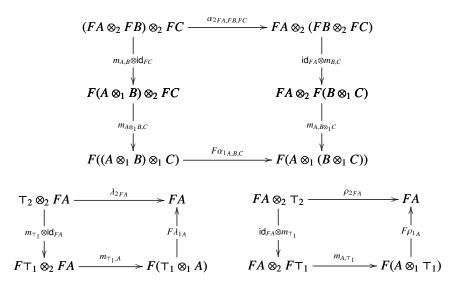


Definition 3. A symmetric monoidal closed category (SMCC) is a symmetric monoidal category, (M, \top, \otimes) , such that, for any object B of M, the functor $- \otimes B : M \longrightarrow M$ has a specified right adjoint. Hence, for any objects A and C of M there is an object $B \multimap C$ of M and a natural bijection:

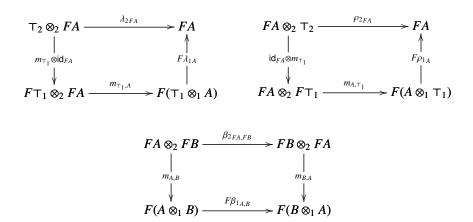
$$\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A \otimes B, C) \cong \operatorname{\mathsf{Hom}}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap: \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of \mathcal{M} .

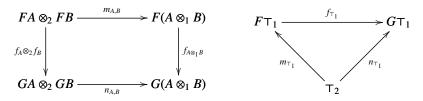
Definition 4. Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F \top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:



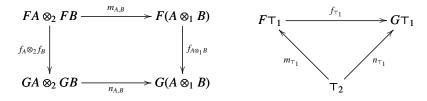
Definition 5. Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal** functor is a functor $F: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F \top_1$ and a natural transformation $m_{AB}: FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:



Definition 6. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **monoidal natural transformation** is a natural transformation, $f: F \longrightarrow G$, subject to the following coherence diagrams:

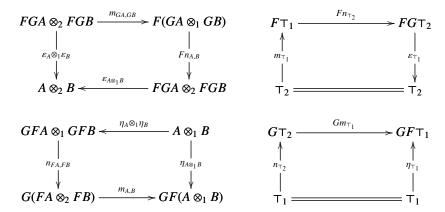


Definition 7. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **symmetric monoidal** natural transformation is a natural transformation, $f: F \longrightarrow G$, subject to the following coherence diagrams:

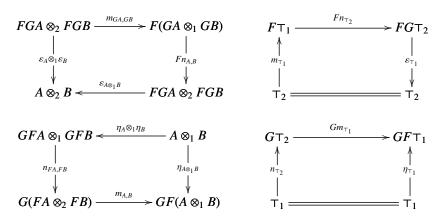


Definition 8. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **monoidal adjunction** is an ordinary adjunction \mathcal{M}_1 : $F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are

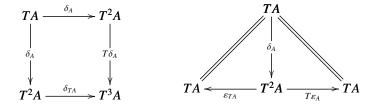
monoidal natural transformations. Thus, the following diagrams must commute:



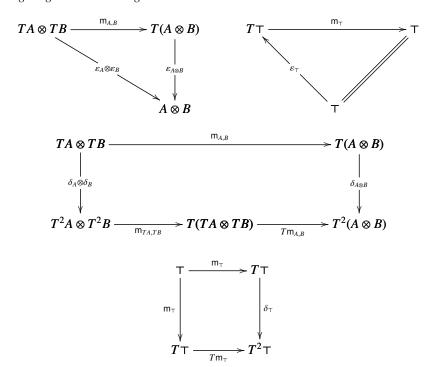
Definition 9. Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a symmetric monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1: F \dashv G: \mathcal{M}_2$ such that the unit, $\eta_A: A \to GFA$, and the counit, $\varepsilon_A: FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:



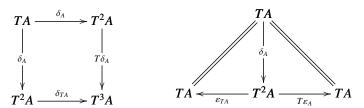
Definition 10. A monoidal comonad on a monoidal category C is a triple (T, ε, δ) , where (T, m) is a monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:



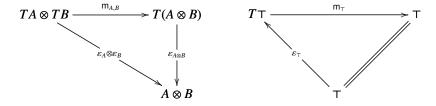
The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

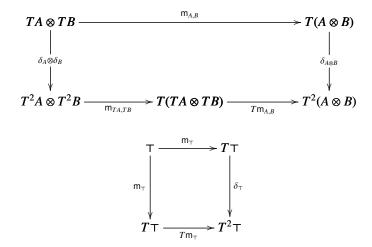


Definition 11. A symmetric monoidal comonad on a symmetric monoidal category C is a triple (T, ε, δ) , where (T, m) is a symmetric monoidal endofunctor on C, ε_A : $TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:



The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

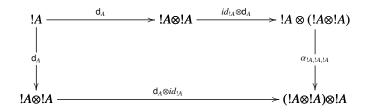


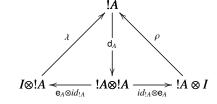


1.2 Linear Category

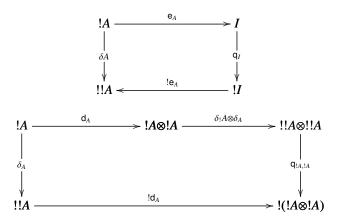
Definition 12. A linear category, $(\mathcal{L}, !, e, d)$, is specified by

- a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,
- a symmetric monoidal comonad $((!, q), \varepsilon, \delta)$ on \mathcal{L} , with $q_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $q_I : I \longrightarrow !I$;
- monoidal natural transformations on \mathcal{L} with components $e_A : !A \longrightarrow I$ and $d_A : !A \longrightarrow !A \otimes !A$, s.t.
 - each (!A, e_A , d_A) is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ d_A = d_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of \mathcal{L} ;

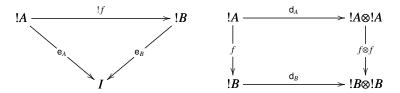




- e_A and d_A are coalgebra morphisms, i.e. the following diagrams commute;

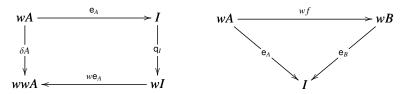


- any coalgebra morphism $f:(!A,\delta_A) \longrightarrow (!B,\delta_B)$ between free coalgebras preserve the comonoid structure given by Θ and G, i.e. the following diagrams commute.



Definition 13. A (modified) linear category with weakening, (\mathcal{L}, w, e) , is specified by

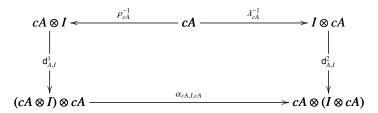
- a monoidal closed category $(\mathcal{L}, I, \otimes)$,
- a monoidal comonad $((w, q), \varepsilon, \delta)$ on \mathcal{L} with $q_{A,B} : wA \otimes wB \longrightarrow w(A \otimes B)$ and $q_I : I \longrightarrow wI$, and
- a monoidal natural transformation e on \mathcal{L} with components $e_A: wA \longrightarrow 1$ s.t. the following diagrams commute:



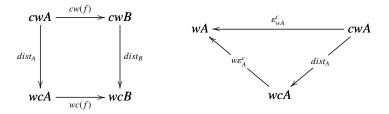
Definition 14. A (modified) linear category with contraction, $(\mathcal{L}, c, d^1, d^2)$, is specified by

- a monoidal closed category $(\mathcal{L}, I, \otimes)$,
- a monoidal comonad $((c,q), \varepsilon, \delta)$ on \mathcal{L} with $q_{A,B}: cA \otimes cB \longrightarrow c(A \otimes B)$ and $q_I: I \longrightarrow cI$, and

• monoidal natural transformations d^1 and d^2 on \mathcal{L} with components $d^1_{A,B}: cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$ and $d^2_{A,B}: B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$, s.t. the following diagram commutes:



Definition 15. Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category \mathcal{L} , define a natural transformation dist with components $dist_A : cwA \longrightarrow wcA$, subject to the following coherence diagrams:



Definition 16. Given a (modified) linear category with weakening (\mathcal{L}, w, e^w) and a (modified) linear category with contraction $(\mathcal{L}, c, d^{c1}, d^{c2})$ on the same monoidal closed category $(\mathcal{L}, I, \otimes)$, where $((w, q^w), \varepsilon^w, \delta^w)$ and $((c, q^c), \varepsilon^c, \delta^c)$ are corresponding comonads on \mathcal{L} , the composition of c and w is a comonad $((cw, q), \varepsilon, \delta)$ on \mathcal{L} , where:

• $q_{A,B} : cwA \otimes cwB \longrightarrow cw(A \otimes B)$ is defined as

$$\mathbf{q}_{A,B} = c\mathbf{q}_{A,B}^{w} \circ \mathbf{q}_{wA,wB}^{c}$$

$$= dist_{A\otimes B}^{-1} \circ w\mathbf{q}_{A,B}^{c} \circ \mathbf{q}_{cA,cB}^{w} \circ (dist_{A} \otimes dist_{B})$$

i.e. the following diagram commutes by the naturality of $q_{A,B}^c$ and $q_{A,B}^w$:

$$cwA \otimes cwB \xrightarrow{q_{wA,wB}^c} c(wA \otimes wB) \xrightarrow{cq_{A,B}^w} cw(A \otimes B)$$

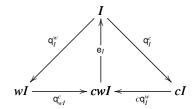
$$\downarrow dist_A \otimes dist_B \qquad \downarrow dist_{A \otimes B}^{-1}$$

$$\downarrow wcA \otimes wcB \xrightarrow{q_{wA,cB}^w} w(cA \otimes cB) \xrightarrow{wq_{A,B}^c} wc(A \otimes B)$$

• $q_I: I \longrightarrow cwI$ is defined as

$$\mathbf{q}_I = \mathbf{q}_{wI}^c \circ \mathbf{q}_I^w$$
$$= c\mathbf{q}_I^w \circ \mathbf{q}_I^w$$

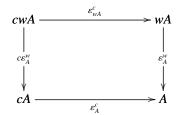
i.e.



• $\varepsilon_A : cwA \longrightarrow A$ is defined as:

$$\varepsilon_A = \varepsilon_{wA}^c \circ \varepsilon_A^w$$
$$= \varepsilon_A^c \circ c \varepsilon_A^w$$

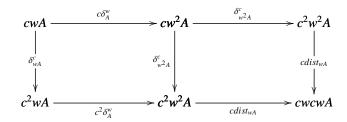
i.e. the following diagram commutes by the naturality of ε^c :



• $\delta_A : cwA \longrightarrow cwcwA$ is defined as:

$$\delta_A = cdist_{wA} \circ c^2 \delta_A^w \circ \delta_{wA}^c$$
$$= cdist_{wA} \circ \delta_{w^2A}^c \circ c\delta_A^w$$

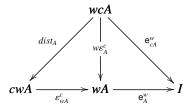
i.e. the following diagram commutes by the naturality of δ^c and equality:



• $e_A : cwA \longrightarrow I$ is defined as:

$$\mathbf{e}_{A} = \mathbf{e}_{cA}^{w} \circ dist_{A}$$
$$= \mathbf{e}_{A}^{w} \circ \varepsilon_{wA}^{c}$$

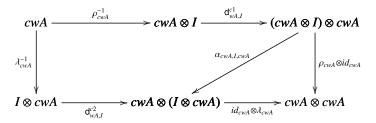
i.e. the following diagram commutes by the definitions of dist and e^{w} :



• $d_A : cwA \longrightarrow cwA \otimes cwA$ is defined as

$$\mathbf{d}_{A} = (\rho_{cwA} \otimes id_{cwA}) \circ \mathbf{d}_{wA,I}^{c1} \circ \rho_{cwA}^{-1}$$
$$= (id_{cwA} \otimes \lambda_{cwA}) \circ \mathbf{d}_{wA,I}^{c2} \circ \lambda_{cwA}^{-1}$$

i.e. the following diagram commutes by the definitions of d^{c1} and d^{c2} :



Theorem 17. The composition cw of comonads c and w on \mathcal{L} defined in Definition 16 is a comonad on \mathcal{L} .

Proof. – cw is an endofunctor on \mathcal{L} , i.e. it preserves the identity I and compositions of morphisms:

$$cwid_A = cid_{wA}$$

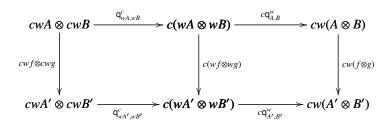
= id_{cwA}

For any morphisms $f: A \longrightarrow B$ and $g: B \longrightarrow C$ in \mathcal{L} ,

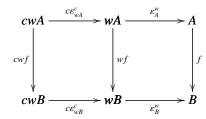
$$cw(f \circ g) = c(wf \circ wg)$$

= $cwf \circ cwg$

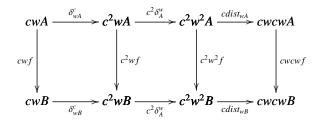
- $q_{A,B}$: $cwA \otimes cwB \longrightarrow cw(A \otimes B)$ is natural by the naturality of $q_{A,B}^c$ and $q_{A,B}^w$:



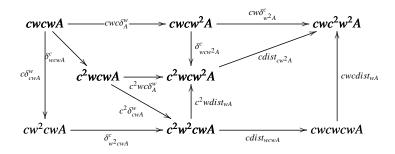
- $q_I: I \longrightarrow cwI$ is natural by definition;
- $\varepsilon_A : cwA \longrightarrow A$ is natural by the naturality of ε^c and of ε^w :



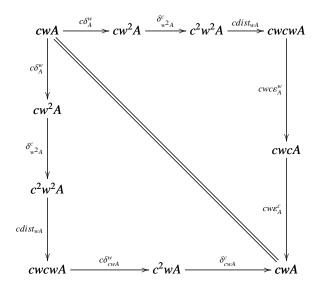
- δ_A : $cwA \longrightarrow cwcwA$ is natural by the naturality of δ^c and of δ^w and the definition of dist:



- $cw\delta_A \circ \delta_A = \delta_{cwA} \circ \delta_A$, by the definition of δ_A , the naturality of δ^c , and definition of *dist*. The diagram proves $cw\delta_A = \delta_{cwA}$:



 $- cw\varepsilon_A \circ \delta_A = \varepsilon_{cwA} \circ \delta_A = id_{cwA}:$



2 Related Work

TODO

3 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at http://research.microsoft.com/en-us/um/people/nick/mixed3.ps.