On Linear Modalities for Exchange, Weakening, and Contraction

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Introduction. In 1987 Girard introduced a new logic [2] that is based on the property that every hypothesis must be used exactly once. He called this logic, linear logic. This means that the structural rules weakening and contraction are not admissible. However, to recover the ability to allow weakening and contraction without sacrificing linearity completely Girard introduced the of-course modality, denoted by $!A$, which is defined by the following rules:

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma_1, !A, !A, \Gamma_2 \vdash B}{\Gamma_1, !A, \Gamma_2 \vdash B} \qquad \frac{!\Gamma \vdash B}{!\Gamma \vdash !B} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$$

The first two rules allow formulas under the of-course modality to be weakened into the context and the second rules allows repeated hypotheses to be contracted. Now the last two rules imply that the of-course modality is a comonad.

Linear logic is also commutative, that is, it contains the exchange structural rule:

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, B, A, \Gamma_2 \vdash C}$$

Non-commutative formulations of linear logic go back to Lambek [4] which happened to be before Girard. Lambek used a non-commutative tensor product to model sentence structure in linguistics. Today non-commutative linear logic has seen many applications in computer science and linguistics. However, some have posed the question of whether an exchange modality can be added to non-commutative linear logic in the same way that Girard added weakening and contraction.

We propose a new logic that has the ability to model several substructural logics. The base of this logic is in non-commutative linear logic. Then we split the of-course modality into two separate modalities: wA and cA. The former models weakening and the latter contraction. Then using a distributive law we show how to compose these two modalities in such a way that the logic has the ability to model logics with both weakening and contraction. In addition to these two modalities we add a third, eA, which adds exchange. In fact, under eA we recover full intuitionistic linear logic. Finally, we show how to compose all three of these modalities to obtain that the ability to recover the of-course modality.

Categorical model. Following Bierman [1] we first show how to construct a categorical model in terms of monoidal categories. Suppose $(M, I, \otimes)$ is a monoidal category, where $\otimes : M \times M \longrightarrow M$ is the tensor product with I its unit. Then equip $(M, I, \otimes)$ with two comonads $(w, \varepsilon_w, \delta_w)$ and $(c, \varepsilon_c, \delta_c)$ where $w : M \longrightarrow M$ and $c : M \longrightarrow M$ are endofunctors, and $\varepsilon_w : wA \longrightarrow A$, $\varepsilon_c : cA \longrightarrow A$, $\delta_w : w^2A \longrightarrow wA$ and $\delta_c : c^2A \longrightarrow cA$ are all natural transformations. Then we add the natural transformations $\mathsf{weak}_A : wA \longrightarrow I$, $\mathsf{contraL}_{A,B} : cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$ and $\mathsf{contraR}_{A,B} : B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$ where the first models weakening and the last two model contraction. All of these are subject to several coherence diagrams; omitted for brevity. By adding a distributive law (natural transformation), $\mathsf{dist}_A : cwA \longrightarrow wcA$, we can construct another comonad $(wc, \varepsilon_{wc}, \delta_{wc})$ which models both weakening and contraction. Keep in mind that the Eilenberg-Moore category for $wc : M \longrightarrow M$ is not cartesian or even symmetric monoidal, because the exchange rule defined in terms of weakening and contraction does not have the necessary properties. We add a third comonad $(e, \varepsilon_e, \delta_e)$ with the natural isomorphism $\mathsf{ex}_{A,B} : eA \otimes eB \longrightarrow eB \otimes eA$; subject to several coherence diagrams. The Eilenberg-Moore category for this comonad is indeed symmetric monoidal. Thus, by adding another distributive law (natural isomorphism) $\mathsf{dist}_A : ecwA \longrightarrow wceA$ we can construct the comonad $(wce, \varepsilon_{wce}, \delta_{wce})$ whose Eilenberg-Moore category is cartesian. In this setting the two contractions blend together to form the usual $\mathsf{contra}_A : A \longrightarrow A \otimes A$.

Logic and type system. By exploiting the beautiful Curry-Howard-Lambek correspondence we can form a logic and type system using the previous model. The base of the logic is the Lambek Calculus (non-commutative linear logic) with the various comonads defined in the same way the of-course modality is defined. For example, the rules for the exchange modality are defined as follows:

$$\frac{\Gamma_1, eA, eB, \Gamma_2 \vdash C}{\Gamma_1, eB, eA, \Gamma_2 \vdash C} \qquad \frac{e\Gamma \vdash B}{e\Gamma \vdash eB} \qquad \frac{\Gamma, A \vdash B}{\Gamma, eA \vdash B}$$

At this point we are forced to ask a question, which of the distributive laws do we choose to add? There are many given three comonads. We choose the necessary laws to be able to model several different substructural logics: $\mathsf{dist}_1 : cwA \longrightarrow wcA$, $\mathsf{dist}_2 : weA \longrightarrow ewA$, $\mathsf{dist}_3 : ceA \longrightarrow ecA$, and $\mathsf{dist}_4 : cewA \longrightarrow ewcA$. These distributive laws allow for the encoding of the Lambek calculus with weakening, contraction, and exchange, affine logic, strict logic, and full intuitionistic linear logic. The type system is left for future work.

Conclusion. The main application we have in mind for the logic we describe above is to use it has a logical foundation of the graphical model of threat analysis known as attack trees [3]. These types of models require several different commutative and non-commutative tensor products within the same logic. We also believe that this type of system will be of interest to the linear logic community, because it allows for the encoding of several different forms of commutative and non-commutative substructural logics.

References

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