On Linear Based Intuitionistic Substructural Logics

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Abstract
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1 Introduction

2 Main Ideas

3 Categorical Models

We develop a categorical framework in which many different intuitionistic substructural logics may be modeled. The locus of this framework is an adjunction. We initially take a monoidal category, $\mathcal{L}$, as a base, and then extend it with one or more structural morphisms – a morphism corresponding to a structural rule in logic – to obtain a second category $\hat{\mathcal{L}}$. Then we form a monoidal adjunction $\hat{\mathcal{L}} : F \dashv G : \mathcal{L}$ just as Benton [1] did for intuitionistic linear logic. Depending on which structural morphisms we add to $\hat{\mathcal{L}}$ we will obtain different models. In particular, each model will come endowed with a comonad on $\mathcal{L}$ which equips $\mathcal{L}$ with the ability to track the corresponding structural rule(s).

We will show that by adding the morphisms for either weakening, contraction, or exchange, to $\mathcal{L}$ will yield an adjoint model of non-commutative relevance logic/linear logic, non-commutative contraction logic/linear logic, and commutative/non-commutative linear logic. The latter model will come with a monoidal comonad $e : \mathcal{L} \longrightarrow \mathcal{L}$ such that there is a symmetry $\mathsf{ex}_{A,B} : eA \triangleright eB \longrightarrow eB \triangleright eA$, where $- \triangleright -$ denotes a non-commutative tensor product. In fact, this is the first adjoint model of the Lambek calculus with the exchange comonad.

At this point we will have adjoint models for each individual structural rule. What if we want more than one structural rule? There are a few different choices that one can choose from depending on the scenario. First, if $\hat{\mathcal{L}}$ contains more than one structural morphism, then $\mathcal{L}$ will have a single comonad that adds all of those structural morphisms to $\mathcal{L}$. For example, if $\hat{\mathcal{L}}$ contains weakening, contraction, and exchange, then $\hat{\mathcal{L}}$ is cartesian closed and $\mathcal{L}$ will have the usual $! : \mathcal{L} \longrightarrow \mathcal{L}$ comonad. The second scenario is when $\mathcal{L}$ also contains some structural morphisms. For example, if $\hat{\mathcal{L}}$ contains exchange and weakening and $\mathcal{L}$ contains exchange, then $\mathcal{L}$ will have a comonad, $r : \mathcal{L} \longrightarrow \mathcal{L}$, which combines linear logic with relevance logic. Thus, how we instantiate the two categories in the adjunction influences which logic one may model.

What if we want multiple comonads tracking different logics? In this scenario the different comonads would allow us to mix the different logics in interesting ways. Suppose $\mathcal{L}$ has no structural morphisms and $\mathcal{E}$ is $\mathcal{L}$ with exchange and $\mathcal{EW}$ is $\mathcal{E}$ with weakening. Then we can form two adjunctions $\mathcal{E} : F \dashv G : \mathcal{L}$ and $\mathcal{EW} : H \dashv J : \mathcal{E}$, but the categories $\mathcal{E}$ and $\mathcal{EW}$ have a structural morphism in common. So instead, we form the adjunction $\mathcal{EW} : H \dashv J : \mathcal{E} : F \dashv G : \mathcal{L}$. Thus, $\mathcal{L}$ has the exchange comonad $e = FG : \mathcal{L} \longrightarrow \mathcal{L}$ as well as the relevance logic comonad $r = FHJG : \mathcal{L} \longrightarrow \mathcal{L}$. Additionally, there is a comonad $w = JH : \mathcal{E} \longrightarrow \mathcal{E}$ adding weakening to $\mathcal{E}$. This idea is based on the amazing work of Mellies [4]. Throughout the remainder of this section we make these ideas precise.

3.1 Lambek Categories

The bases of all of our models will be what we call Lambek categories. These are named after Joachim Lambek to pay homage to his work on the Lambek calculus which can be seen as non-commutative intuitionistic linear logic [3]. Thus, each of our models have a very basic foundation.

Lambek categories are based on (non-symmetric) monoidal categories.

▶ **Definition 1.** A **monoidal category**, $(\mathcal{L}, \triangleright, I, \lambda, \rho)$, is a category, $\mathcal{L}$, equipped with a bifunctor, $\triangleright : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$, called the tensor product, a distinguished object I of $\mathcal{L}$ called the unit, and three natural isomorphisms $\lambda_A : I \triangleright A \longrightarrow A$, $\rho_A : A \triangleright I \longrightarrow A$, and $\alpha_{A,B,C} : (A \triangleright B) \triangleright C \longrightarrow A \triangleright (B \triangleright C)$ called the left and right unitors and the associator respectively. Finally, these are subject to the following coherence diagrams:

$$\begin{tikzcd}
((A \triangleright B) \triangleright C) \triangleright D \arrow[r, "\alpha_{A,B,C} \triangleright \mathsf{id}_D"] \arrow[d, "\alpha_{A \triangleright B,C,D}"'] & (A \triangleright (B \triangleright C)) \triangleright D \arrow[r, "\alpha_{A,B \triangleright C,D}"] & A \triangleright ((B \triangleright C) \triangleright D) \arrow[d, "\mathsf{id}_A \triangleright \alpha_{B,C,D}"] \\
(A \triangleright B) \triangleright (C \triangleright D) \arrow[rr, "\alpha_{A,B,C \triangleright D}"] & & A \triangleright (B \triangleright (C \triangleright D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \triangleright I) \triangleright B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \triangleright id_B"'] & & A \triangleright (I \triangleright B) \arrow[dl, "id_A \triangleright \lambda_B"] \\
& A \triangleright B &
\end{tikzcd}$$

A Lambek category adds closure to monoidal categories.

▶ **Definition 2.** A **Lambek category** is a monoidal category $(\mathcal{L}, \triangleright, I, \lambda, \rho, \alpha)$ equipped with two bifunctors $\rightharpoonup : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ and $\leftharpoonup : \mathcal{L} \times \mathcal{L}^{\mathsf{op}} \longrightarrow \mathcal{L}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \triangleright A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \triangleright X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

An alternative name for Lambek categories is biclosed monoidal categories.

If we add a symmetry to a Lambek category then we will obtain a symmetric monoidal closed category. The following two definitions and lemma capture this result.

▶ **Definition 3.** A monoidal category $(\mathcal{L}, \triangleright, I, \lambda, \rho, \alpha)$ is **symmetric** if there is a natural transformation $\mathsf{ex}_{A,B} : A \triangleright B \longrightarrow B \triangleright A$ such that $\mathsf{ex}_{B,A} \circ \mathsf{ex}_{A,B} = \mathsf{id}_{A \triangleright B}$ and the following commute:

$$\begin{tikzcd}
(A \triangleright B) \triangleright C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathsf{ex}_{A,B} \triangleright \mathsf{id}_C"'] & A \triangleright (B \triangleright C) \arrow[r, "\mathsf{ex}_{A,B \triangleright C}"] & (B \triangleright C) \triangleright A \arrow[d, "\alpha_{B,A,C}"] \\
(B \triangleright A) \triangleright C \arrow[r, "\alpha_{B,A,C}"] & B \triangleright (A \triangleright C) \arrow[r, "\mathsf{id}_B \triangleright \mathsf{ex}_{A,C}"] & B \triangleright (C \triangleright A)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \triangleright A \arrow[rr, "\mathsf{ex}_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \triangleright I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Throughout this paper when $- \triangleright -$ is symmetric we denote it by $- \otimes -$.

We call a symmetric Lambek category a Lambek category with exchange, because the symmetry models the exchange rule.

▶ **Definition 4.** A symmetric monoidal category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \beta)$ is **closed** if it comes equipped with a bifunctor $\multimap : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \multimap B)$ holds.

▶ **Lemma 5.** *Let A and B be two objects in a Lambek category with exchange. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$\begin{array}{rcll} Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\ & \cong & Hom[A \otimes C, B] & \text{By the symmetry } \mathsf{ex}_{C,A} \\ & \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category} \end{array}$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◀

▶ **Corollary 6.** *A Lambek category with exchange is symmetric monoidal closed.*

We will also be discussing two other structural rules: weakening and contraction. These are defined as follows.

▶ **Definition 7.** A **Lambek category with weakening**, $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \mathsf{weak})$, is a Lambek category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ equipped with a natural transformation $\mathsf{weak}_A : A \longrightarrow I$.

▶ **Definition 8.** A **Lambek category with contraction**, $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \mathsf{contraL}, \mathsf{contraR})$, is a Lambek category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ equipped with natural transformations:

$$\mathsf{contraL}_{A,B} : (A \otimes B) \longrightarrow (A \otimes B) \otimes A \qquad \mathsf{contraR}_{A,B} : (B \otimes A) \longrightarrow A \otimes (B \otimes A)$$

Furthermore, the following diagrams must commute:

$$\begin{tikzcd}
A \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & A \arrow[l, "\rho_A^{-1}"'] \arrow[r, "\lambda_A^{-1}"] & I \otimes A \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(A \otimes I) \otimes A \arrow[rr, "\alpha_{A,I,A}"'] & & A \otimes (I \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[r, "id_A \otimes \rho_A^{-1}"] \arrow[dd, "\lambda_A^{-1} \otimes id_A"'] & A \otimes (A \otimes I) \arrow[r, "id_A \otimes \mathsf{contraL}_{A,I}"] & A \otimes ((A \otimes I) \otimes A) \arrow[d, "id_A \otimes (\rho_A \otimes id_A)"] \\
 & & A \otimes (A \otimes A) \\
(I \otimes A) \otimes A \arrow[r, "\mathsf{contraR}_{A,I} \otimes id_A"'] & (A \otimes (I \otimes A)) \otimes A \arrow[r, "(id_A \otimes \lambda_A) \otimes id_A"'] & (A \otimes A) \otimes A \arrow[u, "\alpha_{A,A,A}"']
\end{tikzcd}$$

We call the morphisms:

$\mathsf{ex}_{A,B} : A \otimes B \longrightarrow B \otimes A$
$\mathsf{weak}_A : A \longrightarrow I$
$\mathsf{contraL}_{A,B} : (A \otimes B) \longrightarrow (A \otimes B) \otimes A$
$\mathsf{contraR}_{A,B} : (B \otimes A) \longrightarrow A \otimes (B \otimes A)$

structural morphisms, because they all model the various structural rules in intuitionistic logic.

3.2 Structural Adjoint Models

Now we turn to making our model precise.

▶ **Definition 9.** Suppose $\mathcal{L}_0, \ldots, \mathcal{L}_n$ is a family of Lambek categories with zero or more structural morphisms where $\mathcal{L}_0$ is a full subcategory of each $\mathcal{L}_i$ for $0 < i \leq n$. Then a **composition of structure**, $\overleftarrow{\mathcal{L}_n : F_{n-1} \dashv G_{n-1} : \mathcal{L}_{n-1}}$, is a composition of monoidal adjunctions:

$$\mathcal{L}_n : F_{n-1} \dashv G_{n-1} : \mathcal{L}_{n-1} : F_{n-2} \dashv G_{n-2} : \mathcal{L}_{n-2} : \cdots : \mathcal{L}_1 : F_0 \dashv G_0 : \mathcal{L}_0.$$

We call $\mathcal{L}_0$ the base of the composition.

This definition is an extension – or perhaps a simplification due to the isolation of exchange – of the models discussed by Melliés [4].

Our composition of structure subsumes Benton's [1] linear/non-linear model (LNL). Simply take the sequence of Lambek categories to be $\mathcal{L}_0$, a Lambek category with exchange, and $\mathcal{L}_1$, a Lambek category with weakening, contraction, and exchange, and thus, $\mathcal{L}_1$ is cartesian closed. However, our model is a lot more flexible and expressive.

▶ **Lemma 10** (Comonads in a Composition of Structure)**.** *Suppose $\overleftarrow{\mathcal{L}_n : F_{n-1} \dashv G_{n-1} : \mathcal{L}_{n-1}}$ is a composition of structure. Then there are the following comonads:*

- $(\mathcal{L}_0, F_0G_0, \varepsilon^0_0, \delta^0_0), \ldots, (\mathcal{L}_{n-1}, F_nG_n, \varepsilon^0_{n-1}, \delta^0_{n-1})$
- $(\mathcal{L}_0, F_0F_1G_1G_0, \varepsilon^1_0, \delta^1_0), \ldots, (\mathcal{L}_{n-1}, F_{n-1}F_nG_nG_{n-1}, \varepsilon^1_{n-1}, \delta^1_{n-1})$
- $(\mathcal{L}_0, F_0F_1F_2G_2G_1G_0, \varepsilon^2_0, \delta^2_0), \ldots, (\mathcal{L}_{n-1}, F_{n-2}F_{n-1}F_nG_nG_{n-1}G_{n-2}, \varepsilon^2_{n-1}, \delta^2_{n-1})$

  $\vdots$
- $(\mathcal{L}_0, F_0 \cdots F_nG_n \cdots G_0, \varepsilon^n_0, \delta^n_0)$

Proof. This proof easily follows from the well-known fact that adjoints induce comonads – as well as monads – and composition of adjoints. ◀

The previous lemma shows that a Lambek category $\mathcal{L}_i$ in the sequence is endowed with all of the structure found in each of the categories above it, but this structure is explicitly tracked using the various comonads. That is, the Eilenberg-Moore category of each of the comonads mentioned in the previous lemma has the corresponding structural rule as morphisms.

▶ **Lemma 11.** *Suppose $\overleftarrow{\mathcal{L}_n : F_{n-1} \dashv G_{n-1} : \mathcal{L}_{n-1}}$ is a composition of structure containing the adjunction $\mathcal{L}_j : F_{j-1} \dashv G_{j-1} : \mathcal{L}_{j-1} : \cdots : \mathcal{L}_{i+1} : F_i \dashv G_i : \mathcal{L}_i$ for $0 \leq i < n$ and $0 < j \leq n$. If $(\mathcal{L}_i, M, \varepsilon, \delta)$ is the comonad defined by $MA = F_i \cdots F_{j-1}G_{j-1} \cdots G_iA$, then the Eilenberg-Moore category, $\mathcal{L}^E_M$, contains every structural morphism in the categories $\mathcal{L}_i, \ldots, \mathcal{L}_j$.*

Proof. This result holds similarly to Benton's [1] proof that the Eilenberg-Moore category for the of-course comoand is cartesian closed. So we omit the details. ◀

Multiple composition of structure can have the same Eilenberg-Moore category associated with the comonad induced by the model. For example, the comonad associated with $\mathcal{L}_e : F \dashv G : \mathcal{L}_c$ is the exchange comonad given below, and hence, its Eilenberg-Moore category contains both contraction and exchange, which is a model of strict logic. However, $\mathcal{L}_c : F \dashv G : \mathcal{L}_e$ induces the contraction comonad, but its Eilenberg-Moore category is the same. The difference between these two models is whats being explicitly tracked. In the first model contraction is a first-class citizen, but exchange is being tracked by the comonad, while in the second example exchange is a first-class citizen and contraction is being explicitly tracked by the comonad.

3.3 Example Compositions of Structure

We give a number of example adjoint structures that are of interest to the research community.

Lambek Calculus with Exchange. The first is a model that reveals how to combine the Lambek Calculus with the exchange comonad and Girard's of-course comonad. This model is of interest to the linguistics community [**?**], because they often only want exchange in very controlled instances. Valeria de Paiva [**?**] was the first to show that this is possible using Dialectica Categories and Reedy's [**?**] model. However, she uses a comonad with the natural transformation $\mathsf{ex}_{A,B} : A \triangleright eB \longrightarrow eB \triangleright A$, but we feel this goes against the standard view of algebraic binary operations. In addition, while Dialectica categories are extremely useful, but rather complex, we are interested in simpler models. Thus, we prefer an adjoint model with a comonad which has the natural transformation $\mathsf{ex}_{A,B} : eA \triangleright eB \longrightarrow eB \triangleright eA$.

As we have said in the introduction there are many security applications where one must have both a commutative and a non-commutative tensor product within the same logic. For example, when reasoning about process trees in threat analysis.

▶ **Definition 12.** Suppose $\mathcal{L}_{ewc}$ is a Lambek category with exchange, weakening, and contraction, $\mathcal{L}_e$ is a Lambek category with exchange, and $\mathcal{L}$ is a Lambek category. Then a **LC adjoint model** is the composition of structure $\mathcal{L}_{ewc} : H \dashv J : \mathcal{L}_e : F \dashv G : \mathcal{L}$.

We must now show that $\mathcal{L}$ in a LC adjoint model has two comonads $e : \mathcal{L} \longrightarrow \mathcal{L}$ adding exchange to $\mathcal{L}$, and $! : \mathcal{L} \longrightarrow \mathcal{L}$ – Girard's of-course modality – adding weakening, contraction, and exchange. We first have the following corollary to Lemma 10.

▶ **Corollary 13.** *Suppose $\mathcal{L}_{ewc} : H \dashv J : \mathcal{L}_e : F \dashv G : \mathcal{L}$ is a LC adjoint model. Then there are comonads $(e : \mathcal{L} \longrightarrow \mathcal{L}, \varepsilon^e, \delta^e)$, $(! : \mathcal{L} \longrightarrow \mathcal{L}, \varepsilon^!, \delta^!)$, and $(!_e : \mathcal{L}_e \longrightarrow \mathcal{L}_e, \varepsilon^{!_e}, \delta^{!_e})$.*

Proof. We only show how each of the tuples are defined:

- $eA = FGA$, $\varepsilon^e_A : eA \longrightarrow A$ is the counit of the adjunction, and $\delta^e_A = F\eta^e_{GA} : eA \longrightarrow eeA$, where $\eta^e_A : A \longrightarrow GFA$ is the unit of the adjunction.
- $!_eA = HJA$, $\varepsilon^{!_e}_A : !_eA \longrightarrow A$ is the counit of the adjunction, and $\delta^{!_e}_A = H\eta^{!_e}_{JA} : !_eA \longrightarrow !_e!_eA$, where $\eta^{!_e}_A : A \longrightarrow JHA$ is the unit of the adjunction.
- $!A = FHJGA$, $\varepsilon^!_A : !A \longrightarrow A$ is the counit of the adjunction, and $\delta^!_A = FH\eta^!_{JGA} : !A \longrightarrow !!A$, where $\eta^!_A : A \longrightarrow JGFHA$ is the unit of the adjunction.

◀

As a corollary to Lemma 11 we show that the Eilenberg-Moore categories contain the required structure.

▶ **Corollary 14.** *Suppose $\mathcal{L}_{ewc} : H \dashv J : \mathcal{L}_e : F \dashv G : \mathcal{L}$ is a LC adjoint model. Then the Eilenberg-Moore Categories associated with the comonads $(e : \mathcal{L} \longrightarrow \mathcal{L}, \varepsilon^e, \delta^e)$, $(! : \mathcal{L} \longrightarrow \mathcal{L}, \varepsilon^!, \delta^!)$, and $(!_e : \mathcal{L}_e \longrightarrow \mathcal{L}_e, \varepsilon^{!_e}, \delta^{!_e})$ have the structure: $\mathcal{L}^E_e$ is symmetric monoidal, and $\mathcal{L}^E_!$ and $\mathcal{L}^E_{!_e}$ are cartesian closed.*

Proof.
- $\mathcal{L}^E_{!_e}$ **is cartesian closed**: The proof that $\mathcal{L}^E_{!_e}$ is cartesian closed follows from Benton [1], because $\mathcal{L}_e$ is symmetric monoidal, and $\mathcal{L}_{ewc}$ is cartesian closed, and hence $\mathcal{L}_{ewc} : H \dashv J : \mathcal{L}_e$ is a LNL model.
- $\mathcal{L}^E_e$ **is symmetric monoidal**: The Eilenberg-Moore category, $\mathcal{L}^E_e$, of the comonad $(e, \epsilon^e, \delta^e)$ has as objects all the e-coalgebras $(A, h_A : A \longrightarrow eA)$, and as morphisms all the coalgebra morphisms

such that the following (action) diagrams commute:

$$
\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] & eA \arrow[d, "eh_A"] \\
eA \arrow[r, "\delta_A"'] & e^2A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "h_A"'] \arrow[dr, equal] & \\
eA \arrow[r, "\varepsilon_A"'] & A
\end{tikzcd}
$$

We first prove $\mathcal{L}_e^E$ is monoidal. The tensor product of (A, h_A) and (B, h_B) is $(A \otimes B, \mathsf{q}_{A,B} \circ (h_A \otimes h_B)) : A \otimes B \longrightarrow e(A \otimes B)$ and the unit of the tensor product is $(I, \mathsf{q}_I : I \longrightarrow eI)$. Obviously, they both satisfy the action diagrams above. The left unitor is $\hat{\lambda} : I \otimes A \longrightarrow A$, which is the morphism $I \otimes A \longrightarrow eA$ in $\mathcal{L}$ defined as $e\lambda_A \circ \mathsf{q}_{I,A} \circ (\mathsf{q}_I \otimes id_{eA}) \circ (id_I \otimes h_A)$, where λ is the left unitor in $\mathcal{L}$. The right unitor is $\hat{\rho} : A \otimes I \longrightarrow A$, which is the morphism $A \otimes I \longrightarrow A$ in $\mathcal{L}$ defined as $e\rho_A \circ \mathsf{q}_{A,I} \circ (id_{eA} \otimes \mathsf{q}_I) \circ (h_A \otimes id_I)$, where ρ is the right unitor in $\mathcal{L}$. And the associator is $\hat{\alpha} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$, which is the morphism $(A \otimes B) \otimes C \longrightarrow e(A \otimes (B \otimes C))$ in $\mathcal{L}$ defined as $e\mathsf{ex}_{A,B} \circ \mathsf{q}_{A,B} \circ (id_{eA} \otimes h_B) \circ (h_A \otimes id_B)$, where α is the associator in $\mathcal{L}$.
The left and right unitors are $\lambda : I \otimes A \longrightarrow A$ and $\rho : A \otimes I \longrightarrow A$, and the associator is $\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$. These left and right unitors and the associator are coalgebra morphisms because the following diagrams commute:

$$
\begin{tikzcd}
I \otimes A \arrow[r, "id_I \otimes h_A"] \arrow[d, "\lambda_A"'] & I \otimes eA \arrow[r, "\mathsf{q}_I \otimes id_{eA}"] \arrow[drr, "\lambda_{eA}"] & eI \otimes eA \arrow[r, "\mathsf{q}_{I,A}"] & e(I \otimes A) \arrow[d, "e\lambda_A"] \\
A \arrow[rrr, "h_A"] & & & eA
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes I \arrow[r, "h_A \otimes id_I"] \arrow[d, "\rho_A"'] & eA \otimes I \arrow[r, "id_{eA} \otimes \mathsf{q}_I"] \arrow[drr, "\rho_{eA}"] & eA \otimes eI \arrow[r, "\mathsf{q}_{A,I}"] & e(A \otimes I) \arrow[d, "e\rho_A"] \\
A \arrow[rrr, "h_A"] & & & eA
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "(h_A \otimes h_B) \otimes h_C"] \arrow[d, "\alpha_{A,B,C}"'] & (eA \otimes eB) \otimes eC \arrow[r, "\mathsf{q}_{A,B} \otimes id_{eC}"] \arrow[d, "\alpha_{eA,eB,eC}"] & e(A \otimes B) \otimes eC \arrow[r, "\mathsf{q}_{A \otimes B,C}"] & e((A \otimes B) \otimes C) \arrow[d, "e\alpha_{A,B,C}"] \\
A \otimes (B \otimes C) \arrow[r, "h_A \otimes (h_B \otimes h_C)"'] & eA \otimes (eB \otimes eC) \arrow[r, "id_{eA} \otimes \mathsf{q}_{B,C}"] & eA \otimes e(B \otimes C) \arrow[r, "\mathsf{q}_{A,B \otimes C}"'] & e(A \otimes (B \otimes C))
\end{tikzcd}
$$

Composition in $\mathcal{L}_e^E$ is the same as in $\mathcal{L}_e$, and thus, the monoidal coherence diagrams hold in $\mathcal{L}_e^E$ as well. Thus, $\mathcal{L}_e^E$ is monoidal. We now show that it is symmetric.
The symmetry is $\mathsf{ex}_{A,B} : A \otimes B \longrightarrow B \otimes A$, which is a coalgebra morphism by the following commutative diagram:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes id_B"] \arrow[d, "\mathsf{ex}_{A,B}"'] & eA \otimes B \arrow[r, "id_{eA} \otimes h_B"] \arrow[d, "\mathsf{ex}_{eA,B}"] & eA \otimes eB \arrow[r, "\mathsf{q}_{A,B}"] \arrow[d, "\mathsf{ex}_{eA,eB}"] & e(A \otimes B) \arrow[d, "e\mathsf{ex}_{A,B}"] \\
B \otimes A \arrow[r, "h_B \otimes id_A"'] & eB \otimes A \arrow[r, "id_{eB} \otimes h_A"'] & eB \otimes eA \arrow[r, "\mathsf{q}_{B,A}"'] & e(B \otimes A)
\end{tikzcd}
$$

Similarly, the coherence diagrams for symmetry commute. This concludes the proof that $\mathcal{L}_e^E$ is symmetric monoidal.

- **$\mathcal{L}_!^E$ is cartesian closed**: Similar with the proof that $\mathcal{L}_e^E$ is symmetric monoidal, $\mathcal{L}_!^E$ is also symmetric monoidal, with tensor product $(A \otimes B, \mathsf{q}_{A,B} \circ (h_A \otimes h_A : A \otimes B \longrightarrow ewc(A \otimes B))$, the unit $(I, \mathsf{q}_I : I \longrightarrow !I)$, the right unitor $\hat{\lambda} : I \otimes A \longrightarrow A$, the right unitor $\hat{\rho} : A \otimes I \longrightarrow A$, and the associator $\hat{\alpha} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$.
We know there is a pair of adjoint functors $K \dashv L$ between $\mathcal{L}_{ewc}^E$ and $\mathcal{L}_{ewc}$, where $L : \mathcal{L}_{ewc} \longrightarrow \mathcal{L}_{ewc}^E$ sends every object A to $(ewcA, \delta_A^{ewc})$ and $K : \mathcal{L}_{ewc}^E \longrightarrow \mathcal{L}_{ewc}$ sends every object (A, h_A) to A. Thus,

$$\begin{array}{rcll} Hom_{\mathcal{L}^E_{ewc}}[(A \otimes B, h_{A\otimes B}), (ewcC, \delta^{ewc}_C)] & \cong & Hom_{\mathcal{L}_{ewc}}[A \otimes B, C] & \text{By the adjunction } K \dashv L \\ & \cong & Hom_{\mathcal{L}_{ewc}}[A, B \rightharpoonup C] & \mathcal{L} \text{ is a Lambek category} \\ & \cong & Hom_{\mathcal{L}^E_{ewc}}[(A, h_A), (ewc(B \rightharpoonup C), \delta^{ewc}_{B\rightharpoonup C})] & \text{By the adjunction } K \dashv L \end{array}$$

and

$$\begin{array}{rcll} Hom_{\mathcal{L}^E_{ewc}}[(A \otimes B, h_{A\otimes B}), (ewcC, \delta^{ewc}_C)] & \cong & Hom_{\mathcal{L}^E_{ewc}}[(B \otimes A, h_{B\otimes A}), (ewcC, \delta^{ewc}_C)] & \mathcal{L}^E_{ewc} \text{ is symmetric monoidal} \\ & \cong & Hom_{\mathcal{L}_{ewc}}[B \otimes A, C] & \text{By the adjunction } K \dashv L \\ & \cong & Hom_{\mathcal{L}_{ewc}}[A, C \leftharpoonup B] & \mathcal{L} \text{ is a Lambek category} \\ & \cong & Hom_{\mathcal{L}^E_{ewc}}[(A, h_A), (ewc(C \leftharpoonup B), \delta^{ewc}_{C\leftharpoonup B})] & \text{By the adjunction } K \dashv L \end{array}$$

Therefore, $Hom_{\mathcal{L}^E_{ewc}}[(A, h_A), (ewc(B \rightharpoonup C), \delta^{ewc}_{B\rightharpoonup C})] \cong Hom_{\mathcal{L}^E_{ewc}}[(A, h_A), (ewc(C \leftharpoonup B), \delta^{ewc}_{C\leftharpoonup B})]$. So $\mathcal{L}^E_{ewc}$ is closed.
Finally, $\mathcal{L}^E_{cwe}$ is cartesian closed with the tensor product cartesian. The projectsion are defined as $\pi_1 = \rho_A \circ (id_A \otimes \mathsf{weak}^E_B) \circ (id_A \otimes h_B) : A \otimes B \longrightarrow A$ and $\pi_2 = \lambda_B \circ (\mathsf{weak}^E_A \otimes id_B) \circ (h_A \otimes id_B) : A \otimes B \longrightarrow B$, where weak^E is the natural transformation lifted from the weakening structural morphism in $\mathcal{L}_{ewc}$.
There, $\mathcal{L}^E_{ewc}$ is cartesian closed.

◀

The previous result shows that $\mathcal{L}$ has the following structural morphisms:

$$\mathsf{ex}_{A,B} : eA \triangleright eB \longrightarrow eB \triangleright eA \qquad \mathsf{weak}_A : !A \longrightarrow I \qquad \mathsf{contra}_A : !A \triangleright !A \longrightarrow !A$$

In addition, the category $\mathcal{L}_e$ has the following structural morphisms:

$$\mathsf{ex}_{A,B} : A \otimes B \longrightarrow B \otimes A \qquad \mathsf{weak}_A : !_e A \longrightarrow I \qquad \mathsf{contra}_A : !_e A \otimes !_e A \longrightarrow !_e A$$

However, notice that exchange is a first class citizen, and hence, is not tracked by a comonad.

There are a number of additional models with similar features to the LC adjoint model. Suppose $r_1, r_2 \in \{w, c\}$. Then the set of additional models of the Lambek calculus with exchange and the of-course modality can be modeled by the compositions of structure defined as follows:

$$\mathcal{L}_{r_2} : K \dashv M : \mathcal{L}_{r_1} : H \dashv J : \mathcal{L}_e : F \dashv G : \mathcal{L}$$

Here we have decomposed $\mathcal{L}_{ewc}$ into an adjunction $\mathcal{L}_{r_2} : K \dashv M : \mathcal{L}_{r_1}$ where the Eilenberg-Moore category of the comonad induced by this adjunction is $\mathcal{L}_{r_1 r_2}$, and thus, when this adjunction is composed with the adjunction $\mathcal{L}_{r_1} : H \dashv J : \mathcal{L}_e : F \dashv G : \mathcal{L}$ it yields a comonad whose Eilenberg-Moore category $\mathcal{L}_{er_1 r_2}$ is isomorphic to $\mathcal{L}_{ewc}$.

However, each of these models are in fact different. They each track different structural rules in different ways. The LC adjoint model induces only two comonads, but the models just introduced induces six different comonads. Thus, depending what one wants to track and in which category determines which model is best.

Affine Logic. Affine logic has applications in verification of security protocols [2]. Compositions of structure provide a means to combine intuitionistic linear logic with affine logic.

► **Definition 15.** Suppose $\mathcal{L}_{ew}$ is a Lambek category with exchange and weakening, and $\mathcal{L}_e$ is a Lambek category with exchange. Then an **affine adjoint model** is the composition of structure $\mathcal{L}_{ew} : F \dashv G : \mathcal{L}_e$.

Just as above this model comes with a comonad $(w : \mathcal{L}_e \longrightarrow \mathcal{L}_e, \varepsilon^w, \delta^w)$ defined in the same way as $!_e A$ from above. This then equips $\mathcal{L}_e$ with the natural transformation $\mathsf{weak}_A : wA \longrightarrow I$. Finally, the Eilenberg-Moore category $\mathcal{L}^E_w$ models intuitionistic affine logic.

In this example we made both categories symmetric, but this is not strictly necessary. One could also model non-commutative affine logic as well. In fact, commutativity – as well as all of the other structural rules – is optional in every logic we discuss in this paper.

Strict Logic. Similarly to affine logic we can model strict logic – sometimes referred to as contraction logic – as well.

▶ **Definition 16.** Suppose $\mathcal{L}_{ec}$ is a Lambek category with exchange and contraction, and $\mathcal{L}_e$ is a Lambek category with exchange. Then a **strict adjoint model** is the composition of structure $\mathcal{L}_{ec} : F \dashv G : \mathcal{L}_e$.

Just as above this model comes with a comonad $(c : \mathcal{L}_e \longrightarrow \mathcal{L}_e, \varepsilon^c, \delta^c)$ defined in the same way as $!_e A$ and wA from above. This then equips $\mathcal{L}_e$ with the natural transformation $\mathsf{contra}_A : cA \otimes cA \longrightarrow cA$. Finally, the Eilenberg-Moore category $\mathcal{L}_c^E$ models intuitionistic strict logic.

3.4 Resource Adjoint Models

The main idea behind a composition of structure is that when we chain multiple Lambek categories together we can think of it as composing several comonads together resulting in a comonad which embodies the structure of each category in the sequence of adjunctions. How do we add several different types of comonads to a base category which do not arise as a composition of adjunctions?

As an example, suppose $\mathcal{L}$ is a Lambek category, and we want to track exchange, affine logic, and strict logic as different comonads in $\mathcal{L}$. Then a model of this logic would be the following set of compositions of structure:

$$\mathcal{L}_w : F_1 \dashv G_1 : \mathcal{L}_e : F_0 \dashv G_0 : \mathcal{L}$$
$$\mathcal{L}_{ec} : F_2 \dashv G_2 : \mathcal{L}$$

The first composition of structure endows $\mathcal{L}$ with two comonads that track exchange and affine logic – by composing the exchange comonad with the weakening comonad – respectively. The second adjunction adds in the comonad that tracks strict logic, because $\mathcal{L}_{ec}$ contains both exchange and contraction. Again, there are a few other sets of compositions of structure that will also allow one to model this situation, which is best depends on ones needs. The key point is that by combining multiple compositions of structure on the same base we increase the diversity of the kinds of resources that can be tracked.

We have now arrived at a new model of resource-conscious logics based on compositions of structure.

▶ **Definition 17.** Suppose $\mathcal{L}_0$ is a Lambek category called the base of the model. Then a **resource adjoint model**, $\{\overleftarrow{\mathcal{L}_n : F_{n-1} \dashv G_{n-1} : \mathcal{L}_{n-1}}\}$, is a set of compositions of structure whose bases are all $\mathcal{L}_0$.

This model not only subsumes previous models like LNL models – and hence Linear Categories – but also suggests new and interesting combinations of existing logics that have not been considered before.

4 Resource Adjoint Logics

We now turn to how adjoint resources models correspond to various substructural logics. The definition of the model along with Benton's [1] LNL adjoint logic suggests a clear design of each adjoint resource logic.

Every resource adjoint logic begins with the base logic consisting of the Lambek calculus potentially extended with one or more of the structural rules: exchange, weakening, or contraction. Then we connect this base logic with each other logic corresponding to the categories in a composition of structure. Each of these logics are then connected by syntactic versions of the adjoint functors.

Now we give several example logics with increasing complexity, and then conclude with a discussion for a general system for resource tracking. We begin with the most basic resource adjoint logic.

4.1 Lambek Calculus with Itself

4.2 Lambek Calculus with Exchange

4.3 Lambek Calculus with Exchange and Commutative Affine Logic

4.4 Linear Logic with Intuitionistic Logic, Affine Logic, and Contraction Logic

4.5 Complete Resource Tracking

5 Applications

6 Related Work

TODO

7 Conclusion

TODO

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A Appendix