On Linear Modalities for Exchange, Weakening, and Contraction

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Abstract

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1 Introduction

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2 Categorical Models

2.1 Lambek Categories

▶ **Definition 1.** A **monoidal category**, $(\mathcal{L}, \otimes, I, \lambda, \rho)$, is a category, $\mathcal{L}$, equipped with a bifunctor, $\otimes : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$, called the tensor product, a distinguished object I of $\mathcal{L}$ called the unit, and three natural isomorphisms $\lambda_A : I \otimes A \longrightarrow A$, $\rho_A : A \otimes I \longrightarrow A$, and $\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ called the left and right unitors and the associator respectively. Finally, these are subject to the following coherence diagrams:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B \otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "\mathsf{id}_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

▶ **Definition 2.** A **Lambek category** is a monoidal category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ equipped with two bifunctors $\rightharpoonup : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ and $\leftharpoonup : \mathcal{L} \times \mathcal{L}^{\mathsf{op}} \longrightarrow \mathcal{L}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

One might call Lambek categories biclosed monoidal categories, but we name them in homage to Lambek for his contributions to non-commutative linear logic.

▶ **Definition 3.** A monoidal category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ is **symmetric** if there is a natural isomorphism $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$ such that $\beta_{B,A} \circ \beta_{A,B} = \mathsf{id}_{A\otimes B}$ and the following commute:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B}\otimes \mathsf{id}_C"'] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes A \arrow[rr, "\beta_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

▶ **Definition 4.** A symmetric monoidal category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \beta)$ is **closed** if it comes equipped with a bifunctor $\multimap : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \multimap B)$ holds.

2.2 Lambek Categories with Weakening and Contraction

▶ **Definition 5.** A **Lambek category with weakening**, $(\mathcal{L}, w, \mathsf{weak})$, is a Lambek category equipped with a monoidal comonad (w, ε, δ), and a monoidal natural transformation $\mathsf{weak}_A : wA \longrightarrow I$. Furthermore, weak must be a coalgebra morphism. That is, the following digram must commute:

$$\begin{tikzcd}
wA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta_A"'] & I \arrow[d, "\mathsf{q}_I"] \\
w^2 A \arrow[r, "w\mathsf{weak}_A"'] & wI
\end{tikzcd}$$

▶ **Definition 6.** A **Lambek category with contraction**, $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$, is a Lambek category equipped with a monoidal comonad (c, ε, δ), and two monoidal natural transformations:

$$\mathsf{contraL}_{A,B} : cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$$
$$\mathsf{contraR}_{A,B} : B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$$

Furthermore, the following diagrams must commute:

$$\begin{tikzcd}
cA \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & cA \arrow[l, "\rho^{-1}_{cA}"'] \arrow[r, "\lambda^{-1}_{cA}"] & I \otimes cA \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(cA \otimes I) \otimes cA \arrow[rr, "\alpha_{cA,I,cA}"'] & & cA \otimes (I \otimes cA)
\end{tikzcd}$$

$$\begin{tikzcd}
cA \otimes cA \arrow[r, "id_{cA}\otimes \rho^{-1}_{cA}"] \arrow[dd, "\lambda^{-1}_{cA}\otimes id_{cA}"'] & cA \otimes (cA \otimes I) \arrow[r, "id_{cA}\otimes \mathsf{contraL}_{A,I}"] & cA \otimes ((cA \otimes I) \otimes cA) \arrow[d, "id_{cA}\otimes(\rho_{cA}\otimes id_{cA})"] \\
& & cA \otimes (cA \otimes cA) \\
(I \otimes cA) \otimes cA \arrow[r, "\mathsf{contraR}_{A,I}\otimes id_{cA}"'] & (cA \otimes (I \otimes cA)) \otimes cA \arrow[r, "(id_{cA}\otimes \lambda_{cA})\otimes id_{cA}"'] & (cA \otimes cA) \otimes cA \arrow[u, "\alpha_{cA,cA,cA}"']
\end{tikzcd}$$

2.3 Lambek Categories with Exchange

▶ **Definition 7.** A **Lambek category with exchange**, $(\mathcal{L}, e, \mathsf{ex})$, is a Lambek category equipped with a monoidal comonad (e, ε, δ) on $\mathcal{L}$, and a monoidal natural transformation $\mathsf{ex}_{A,B} : eA \otimes eB \longrightarrow eB \otimes eA$. We require ex to be a coalgebra morphism, and the following diagrams must commute:

$$\begin{tikzcd}
e^2A \otimes e^2B \arrow[r, "\mathsf{ex}_{eA,eB}"] \arrow[d, "\mathsf{q}_{eA,eB}"'] & e^2B \otimes e^2A \arrow[d, "\mathsf{q}_{eB,eA}"] \\
e(eA \otimes eB) \arrow[r, "e\mathsf{ex}_{A,B}"] & e(eB \otimes eA)
\end{tikzcd}
\qquad
\begin{tikzcd}
eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] \arrow[dr, equal] & eB \otimes eA \arrow[d, "\mathsf{ex}_{B,A}"] \\
& eA \otimes eB
\end{tikzcd}
\qquad
\begin{tikzcd}
eI \otimes eA \arrow[rr, "\mathsf{ex}_{I,A}"] \arrow[d, "\varepsilon_I \otimes \mathsf{id}"'] & & eA \otimes eI \arrow[d, "\mathsf{id} \otimes \varepsilon_I"] \\
I \otimes eA \arrow[dr, "\lambda_{eA}"'] & & eA \otimes I \arrow[dl, "\rho_{eA}"] \\
& eA &
\end{tikzcd}$$

$$\begin{tikzcd}
(eA \otimes eB) \otimes eC \arrow[r, "\alpha_{eA,eB,eC}"] \arrow[d, "\mathsf{ex}_{A,B} \otimes \mathsf{id}_{eC}"'] & eA \otimes (eB \otimes eC) \arrow[r, "\mathsf{id}_{eA} \otimes (\delta_B \otimes \delta_C)"] & eA \otimes (e^2B \otimes e^2C) \arrow[r, "\mathsf{id}_{eA} \otimes \mathsf{q}_{eB,eC}"] & eA \otimes e(eB \otimes eC) \arrow[d, "\mathsf{ex}_{eA,eB \otimes eC}"] \\
(eB \otimes eA) \otimes eC \arrow[d, "\alpha_{eB,eA,eC}"'] & & & e(eB \otimes eC) \otimes eA \arrow[d, "\varepsilon_{eB \otimes eC} \otimes \mathsf{id}_{eA}"] \\
eB \otimes (eA \otimes eC) \arrow[rr, "\mathsf{id}_{eB} \otimes \mathsf{ex}_{A,C}"] & & eB \otimes (eC \otimes eA) & (eB \otimes eC) \otimes eA \arrow[l, "\alpha_{eB,eC,eA}"']
\end{tikzcd}$$

Furthermore, for any coalgebra morphisms $f : (eA, \delta) \longrightarrow (eB, \delta)$ and $g : (eC, \delta) \longrightarrow (eD, \delta)$ between free coalgebras the following diagram must commute:

$$\begin{tikzcd}
eA \otimes eC \arrow[r, "f \otimes g"] \arrow[d, "\mathsf{ex}_{A,C}"'] & eB \otimes eD \arrow[d, "\mathsf{ex}_{B,D}"] \\
eC \otimes eA \arrow[r, "g \otimes f"] & eD \otimes eB
\end{tikzcd}$$

The morphism $\mathsf{q}_{A,B} : eA \otimes eB \longrightarrow e(A \otimes B)$ makes (e, q) a monoidal functor.

The first diagram in the previous definition makes $e : \mathcal{L} \longrightarrow \mathcal{L}$ a symmetric monoidal functor, and the second, third, and forth diagrams make the category of free coalgebras (the Kleisli category) symmetric monoidal.

▶ **Definition 8.** Suppose $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange. Then the **Eilenberg Moore category**, $\mathcal{L}^e$, of the comonad (e, ε, δ) has as objects all the e-coalgebras $(A, h_A : A \longrightarrow eA)$, and as morphisms all the coalgebra morphisms. We call h_A the action of the coalgebra. Furthermore, the following (action) diagrams must commute:

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] & eA \arrow[d, "eh_A"] \\
eA \arrow[r, "\delta_A"'] & e^2A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "h_A"'] \arrow[dr, equal] & \\
eA \arrow[r, "\varepsilon_A"'] & A
\end{tikzcd}$$

▶ **Lemma 9** (The Eilenberg Moore Category of the comonad e is Monoidal)**.** *The category $\mathcal{L}^e$ is monoidal.*

Proof. For the complete proof see Appendix B.1.1. ◀

▶ **Lemma 10.** *In $\mathcal{L}^e$ there is a natural transformation $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$.*

Proof. We define β as follows:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{ex_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

For the proof that it is natural please see Appendix B.1.2. ◀

▶ **Corollary 11.** *For any coalgebras (A, h_A) and (B, h_B) the followings commute:*

$$\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[d, equal] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] & eB \otimes eA \arrow[r, "\varepsilon_B \otimes \varepsilon_A"] & B \otimes A \arrow[d, "h_B \otimes h_A"] \\
A \otimes B \arrow[r, "h_A \otimes h_B"] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] & eB \otimes eA \arrow[r, equal] & eB \otimes eA
\end{tikzcd}$$

$$\begin{tikzcd}
eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] \arrow[d, "\mathsf{q}_{A,B}"'] & eB \otimes eA \arrow[d, "\mathsf{q}_{B,A}"] \\
e(A \otimes B) \arrow[r, "e\beta_{A,B}"'] & e(B \otimes A)
\end{tikzcd}$$

Proof. For the complete proof please see Appendix B.1.3. ◀

▶ **Definition 12.** Given two parallel arrows $f, g : B \longrightarrow C$ in a category C, a **cofork** is a morphism $c : A \longrightarrow B$ such that the diagram $A \xrightarrow{c} B \overset{f}{\underset{g}{\rightrightarrows}} C$ commutes. That is, $f \circ c = g \circ c$.

▶ **Lemma 13.** *The morphism $\mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B)$ is a cofork of the morphisms $(\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\varepsilon_B \otimes \varepsilon_A)$ and $(e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A)$.*

Proof. This proof holds by straightforward equational reasoning. For the complete proof please see Appendix B.1.4. ◀

▶ **Lemma 14.** *In $\mathcal{L}^e$, β is a coalgebra morphism.*

Proof. For the complete proof see Appendix B.1.5. ◀

▶ **Lemma 15** (The Eilenberg-Moore Category of the comonad e is Symmetric Monoidal)**.** *The category $\mathcal{L}^e$ is symmetric monoidal.*

Proof. For the complete proof please see Appendix B.1.6. ◀

2.4 Linear Categories

▶ **Definition 16.** A **linear category**, $(\mathcal{L}, !, \mathsf{weak}, \mathsf{contra})$, is a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$ equipped with a symmetric monoidal comonad $(!, \varepsilon, \delta)$ with $\mathsf{q}_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $\mathsf{q}_I : I \longrightarrow !I$, and two monoidal natural transformations with components $\mathsf{weak}_A : !A \longrightarrow I$ and $\mathsf{contra}_A : !A \longrightarrow !A \otimes !A$, satisfying the following conditions:

- each $(!A, \mathsf{weak}_A, \mathsf{contra}_A)$ is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ \mathsf{contra}_A = \mathsf{contra}_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of $\mathcal{L}$;

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & !A \otimes !A \arrow[r, "id_{!A} \otimes \mathsf{contra}_A"] & !A \otimes (!A \otimes !A) \\
!A \otimes !A \arrow[rr, "\mathsf{contra}_A \otimes id_{!A}"'] & & (!A \otimes !A) \otimes !A \arrow[u, "\alpha_{!A,!A,!A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & !A \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes !A & !A \otimes !A \arrow[l, "\mathsf{weak}_A \otimes id_{!A}"] \arrow[r, "id_{!A} \otimes \mathsf{weak}_A"'] & !A \otimes I
\end{tikzcd}$$

- weak_A and contra_A are coalgebra morphisms, i.e. the following diagrams commute;

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
!!A \arrow[r, "!\mathsf{weak}_A"'] & !I
\end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[r, "\delta_A \otimes \delta_A"] & !!A \otimes !!A \arrow[d, "\mathsf{q}_{!A,!A}"] \\
!!A \arrow[rr, "!\mathsf{contra}_A"'] & & !(!A \otimes !A)
\end{tikzcd}$$

- any coalgebra morphism $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ between free coalgebras preserve the comonoid structure given by weak and contra, i.e. the following diagrams commute.

$$\begin{tikzcd}
!A \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & !B \arrow[dl, "\mathsf{weak}_B"] \\
 & I &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & !A \otimes !A \arrow[d, "f \otimes f"] \\
!B \arrow[r, "\mathsf{contra}_B"'] & !B \otimes !B
\end{tikzcd}$$

▶ **Definition 17.** Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction and $(\mathcal{L}, w, \mathsf{weak})$ is a Lambek category with weakening, we define a **distributive law** of c over w to be a natural transformation with components $dist_A : cwA \longrightarrow wcA$, subject to the following coherence diagrams:

$$\begin{tikzcd}
wA & & cwA \arrow[ll, "\varepsilon^c_{wA}"'] \arrow[dl, "dist_A"] \\
 & wcA \arrow[ul, "w\varepsilon^c_A"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cA & & cwA \arrow[ll, "c\varepsilon^w_A"'] \arrow[dl, "dist_A"] \\
 & wcA \arrow[ul, "\varepsilon^w_{cA}"] &
\end{tikzcd}$$

▶ **Lemma 18.** *Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction and $(\mathcal{L}, w, \mathsf{weak})$ is a Lambek category with weakening, the* *following two diagrams commute:*

$$
\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, "\delta^c_{wcA}"'] & & cwc^2A \\
& c^2wcA \arrow[ur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wdist_{wA}"] \\
& w^2cwA &
\end{tikzcd}
$$

Proof. The two diagrams above commute because the following ones commute by the distributive law and the comonad laws for c and w.

$$
\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, equal] \arrow[ddr, "\delta^c_{wA}"'] & & cwc^2A \arrow[dl, "cw\varepsilon^c_{cA}"'] \\
& cwcA & \\
& c^2wcA \arrow[u, "c\varepsilon^c_{wcA}"] \arrow[uur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, equal] \arrow[ddr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wc\varepsilon^w_{wA}"'] \arrow[ddl, "wdist_{wA}"] \\
& wcwA & \\
& w^2cwA \arrow[u, "w\varepsilon^w_{cwA}"] &
\end{tikzcd}
$$

◀

▶ **Lemma 19** (Composition of Weakening and Contraction)**.** *Suppose* $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}, \mathsf{contraR})$ *is a Lambek category with weakening and contraction, where* $(w, \varepsilon^w, \delta^w)$ *and* $(c, \varepsilon^c, \delta^c)$ *are the respective monoidal comonads. Then the composition of* c *and* w *using the distributive law* $dist_A : cwA \longrightarrow wcA$ *is a monoidal comonad on* $\mathcal{L}$.

Proof. For the complete proof see Appendix B.3. ◀

▶ **Definition 20.** A **Lambek category with** cw, $(\mathcal{L}, cw, \mathsf{weak}^w, \mathsf{contraL}, \mathsf{contraR}, dist)$, is a Lambek category with weakening and contraction, and a distributive law. Furthermore, the following coherence diagrams commute:

$$
\begin{tikzcd}
I \otimes cwA \arrow[r, "\lambda^{-1}_{I\otimes cwA}"] \arrow[d, "\mathsf{contraR}_{wA,I}"'] & I \otimes (I \otimes cwA) \\
cwA \otimes (I \otimes cwA) \arrow[r, "\varepsilon^c_{wA}\otimes id_{I\otimes cwA}"] & wA \otimes (I \otimes cwA) \arrow[u, "\mathsf{weak}^w_A\otimes id_{I\otimes cwA}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
cwA \otimes I \arrow[r, "\rho^{-1}_{cwA\otimes I}"] \arrow[d, "\mathsf{contraL}_{wA,I}"'] & (cwA \otimes I) \otimes I \\
(cwA \otimes I) \otimes cwA \arrow[r, "id_{cwA\otimes I}\otimes\varepsilon^c_{wA}"] & (cwA \otimes I) \otimes wA \arrow[u, "id_{cwA\otimes I}\otimes\mathsf{weak}^w_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[d, "\varepsilon^c_{wA}"'] & & cwB \arrow[d, "\varepsilon^c_{wB}"] \\
wA \arrow[r, "\mathsf{weak}^w_A"] & I & wB \arrow[l, "\mathsf{weak}^w_B"']
\end{tikzcd}
$$

where $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ is any coalgebra morphism between free coalgebras.

▶ **Lemma 21.** *Let* $(\mathcal{L}, cw, \mathsf{weak}^w, \mathsf{contraL}, \mathsf{contraR})$ *be a Lambek category with cw. Then the following conditions are satisfied:*

1. *There exist two natural transformations* $\mathsf{weak}_A : cwA \longrightarrow I$ *and* $\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$.
2. *Each* $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ *is a comonoid.*
3. weak_A *and* contra_A *are coalgebra morphisms.*
4. *Any coalgebra morphism* $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ *between free coalgebras preserves the comonoid structure given by* weak *and* contra.

Proof. We will only prove the first condition by defining weak and contra. For the complete proof see Appendix B.4. Each of weak and contracan be given two equivalent definitions. $\mathsf{weak}_A : cwA \longrightarrow I$ is defined as in the diagram below. The left triangle commutes by the definition of *dist* and the right triangle commutes by the definition of weak^w.

$$\begin{tikzcd}
 & wcA \arrow[d, dashed, "w\varepsilon^c_A"] \arrow[dr, "\mathsf{weak}^w_{cA}"] & \\
cwA \arrow[ur, "dist_A"] \arrow[r, "\varepsilon^c_{wA}"'] & wA \arrow[r, "\mathsf{weak}^w_A"'] & I
\end{tikzcd}$$

$\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$ is defined as below. The left part of the diagram commutes by the definitions of contraL and of contraR, and the right part commutes because $\mathcal{L}$ is monoidal.

$$\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "\lambda^{-1}_{cwA}"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[dl, "\alpha_{cwA,I,cwA}"'] \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
I \otimes cwA \arrow[r, "\mathsf{contraR}_{wA,I}"'] & cwA \otimes (I \otimes cwA) \arrow[r, "id_{cwA} \otimes \lambda_{cwA}"'] & cwA \otimes cwA
\end{tikzcd}$$

◀

► **Definition 22.** Given two comonads $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^e, \delta^e)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ is a Lambek category with cw and $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange, we define a **distributive law for exchange** of cw over e to be a natural isomorphism with components $distEx_A : cweA \longrightarrow ecwA$, subject to the following coherence diagrams:

$$\begin{tikzcd}
eA & & cweA \arrow[ll, "\varepsilon^{cw}_{eA}"'] \arrow[dl, "distEx_A"] \\
 & ecwA \arrow[ul, "e\varepsilon^{cw}_A"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA & & cweA \arrow[ll, "cw\varepsilon^e_A"'] \arrow[dl, "distEx_A"] \\
 & ecwA \arrow[ul, "\varepsilon^e_{cwA}"] &
\end{tikzcd}$$

► **Lemma 23.** *Given two comonads $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^e, \delta^e)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ is a Lambek category with cw and $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange, the following two digrams also commute:*

$$\begin{tikzcd}
cwecwA \arrow[rr, "cwe\delta^{cw}_A"] \arrow[dr, "\delta^{cw}_{cweA}"'] & & cwe(cw)^2A \\
 & (cw)^2ecwA \arrow[ur, "cwdistEx_{cwA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
ecweA \arrow[rr, "ecw\delta^e_A"] \arrow[dr, "\delta^e_{cweA}"'] & & ecwe^2A \arrow[dl, "edistEx_{eA}"] \\
 & e^2cweA &
\end{tikzcd}$$

The proof is similar with the proof of Lemma 18 and we will not elaborate it here. Also, notice the difference between *dist* of c over w and *distEx* of cw over e. While *dist* is a natural transformation, *distEx* is a natural isomorphism.

► **Lemma 24.** *let $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^e, \delta^e)$ be two monoidal comonads on a Lambek category with cw and exchange $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$. Then the composition of cw and e using the distributive law for exchange $distEx_A : cweA \longrightarrow ecwA$ is a monoidal comonad $(cwe, \varepsilon, \delta)$ on $\mathcal{L}$.*

Proof. Suppose $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^e, \delta^e)$ are monoidal comonads, and $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with cw and exchange. Since by definition

$cw, e : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors, we know that their composition $cwe : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$
\begin{aligned}
\mathsf{q}_{A,B} &: cweA \otimes cweB \longrightarrow cwe(A \otimes B)\\
\mathsf{q}_{A,B} &= cw\mathsf{q}^{e}_{A,B} \circ \mathsf{q}^{cw}_{eA,eB}\\
\mathsf{q}_{I} &: I \longrightarrow cweI\\
\mathsf{q}_{I} &= cw\mathsf{q}^{e}_{I} \circ \mathsf{q}^{cw}_{I}
\end{aligned}
$$

Analogous to the proof of Lemma 19, each of ε and δ can be given two equivalent definitions:

$$
\begin{tikzcd}
cweA \arrow[r, "\varepsilon^{cw}_{eA}"] \arrow[d, "cw\varepsilon^{e}_{A}"'] & eA \arrow[d, "\varepsilon^{e}_{A}"] \\
cwA \arrow[r, "\varepsilon^{cw}_{A}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
cweA \arrow[r, "cw\delta^{e}_{A}"] \arrow[d, "\delta^{cw}_{eA}"'] & cwe^2A \arrow[r, "\delta^{cw}_{e^2A}"] \arrow[d, "\delta^{cw}_{e^2A}"'] & (cw)^2e^2A \arrow[d, "cwdist_{eA}"] \\
(cw)^2eA \arrow[r, "(cw)^2\delta^{e}_{A}"'] & (cw)^2e^2A \arrow[r, "cwdist_{eA}"'] & cwecweA
\end{tikzcd}
$$

And the comonad laws can be proved similarly, which we will not elaborate for simplicity. ◂

▸ **Definition 25.** Suppose $(\mathcal{L}, cwe, \mathsf{weak}, \mathsf{contra}, \mathsf{ex})$ is a Lambek category with contraction, weakening and exchange. Then the Eilenberg Moore category, $\mathcal{L}^{cwe}$, of the comonad $(cwe, \varepsilon, \delta)$ has as objects all the cwe-coalgebras $(A, h_A : A \longrightarrow cweA)$, and as morphisms all the coalgebra morphisms. Futhermore, the following (action) diagrams commute:

$$
\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] & eA \arrow[d, "eh_A"] \\
eA \arrow[r, "\delta_A"'] & e^2A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "h_A"'] \arrow[dr, equal] & \\
eA \arrow[r, "\varepsilon_A"'] & A
\end{tikzcd}
$$

▸ **Lemma 26** (The Eilenberg-Moore Category of the comonad cwe is a linear category)**.** *The category $\mathcal{L}^{cwe}$ is a linear category.*

Proof. **1.** $\mathcal{L}^{cwe}$ is a symmetric monoidal cloased category.
Similar as the proofs for Lemmas 9 and 15, $\mathcal{L}^{cwe}$ is symmetric monoidal. The symmetry $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$ is also defined similarly as:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{ex_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

2. $(cwe, \varepsilon, \delta)$ is a symmetric monoidal comonad.
Lemma 24 has shown that cwe is a monoidal comonad.
To prove it is also symmetric, we first define a natural transformation $\mathsf{ex}^{cwe}_{A,B} : cweA \otimes cweB \longrightarrow cweB \otimes cweA$ as follows:

$$\mathsf{ex}^{cwe}_{A,B} :=$$

$$cweA \otimes cweB \xrightarrow{distEx_A \otimes distEx_B} ecwA \otimes ecwB \xrightarrow{\mathsf{ex}_{cwA,cwB}} ecwB \otimes ecwA \xrightarrow{distEx_B^{-1} \otimes distEx_A^{-1}} cweB \otimes cweA$$

Then $\mathcal{L}^{cwe}$ is symmetric because the following diagram commutes

$$
\begin{tikzcd}
cweA \otimes cweB \arrow[r, "\mathsf{ex}^{cwe}_{A,B}"] \arrow[d, "\mathsf{q}_{A,B}"'] & cweB \otimes cweA \arrow[d, "\mathsf{q}_{B,A}"] \\
cwe(A \otimes B) \arrow[r, "cwe\beta_{A,B}"'] & cwe(B \otimes A)
\end{tikzcd}
$$

by the diagram chasing below:

$$\begin{tikzcd}
cweA \otimes cweB \arrow[rr, "distEx_A \otimes distEx_B"] \arrow[dr, "\varepsilon^{cw}_{eA} \otimes \varepsilon^{cw}_{eB}"] \arrow[ddd, "\mathsf{q}_{A,B}"'] & & ecwA \otimes ecwB \arrow[rr, "\mathsf{ex}_{cwA,cwB}"] \arrow[dl, "e\varepsilon^{cw}_{A} \otimes e\varepsilon^{cw}_{B}"'] & & ecwB \otimes ecwA \arrow[dl, "e\varepsilon^{cw}_{B} \otimes e\varepsilon^{cw}_{A}"'] \arrow[dd, "distEx_B^{-1} \otimes distEx_A^{-1}"] \\
& eA \otimes eB \arrow[rr, "\mathsf{ex}_{A,B}"] \arrow[d, "\mathsf{q}_{A,B}"'] & & eB \otimes eA \arrow[d, "\mathsf{q}_{B,A}"'] & \\
& e(A \otimes B) \arrow[rr, "e\beta_{A,B}"] & & e(B \otimes A) & cweB \otimes cweA \arrow[ul, "\varepsilon^{cw}_{eB} \otimes \varepsilon^{cw}_{eA}"'] \arrow[d, "\mathsf{q}_{B,A}"] \\
cwe(A \otimes B) \arrow[ur, "\varepsilon_{e(A \otimes B)^{cw}}"] \arrow[rrrr, "cwe\beta_{A,B}"'] & & & & cwe(B \otimes A) \arrow[ul, "\varepsilon^{cw}_{e(B \otimes A)}"']
\end{tikzcd}$$

The triangle on top and the triangle on the right commute by the definition of $distEx$. The quadrangle on the left and the one on the right commute because q is a natural transformation. The quadrangle on top and the one at the bottom commute by the naturality of ex and ε^{cw}, respectively. The square in the middle commutes by Corollary 11.

3. There are two monoidal natural transformations with components $\mathsf{weak}_A : cweA \longrightarrow I$ and $\mathsf{contra}_A : cweA \longrightarrow cweA \otimes cweA$.
   The natural transformations weak and contra_A are defined as:

$$\begin{aligned}
\mathsf{weak}_A &:= cweA \xrightarrow{\varepsilon^{c}_{weA}} weA \xrightarrow{\mathsf{weak}^{w}_{eA}} I \\
&:= cweA \xrightarrow{c\mathsf{weak}^{w}_{eA}} cI \xrightarrow{\varepsilon^{c}_{I}} I \\
\mathsf{contra}_A &:= cweA \xrightarrow{\mathsf{contra}^{cw}_{eA}} cweA \otimes cweA
\end{aligned}$$

   They are monoidal by definition.
4. Each $(cweA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a commutative comonoid.
   By Lemma 21, each $(cwA, \mathsf{weak}^{cw}, \mathsf{contra}^{cw})$ is a comonoid in the Lambek category with cw. Similarly, each $(cweA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid in $\mathcal{L}^{cwe}$. Since there is a symmetry β in $\mathcal{L}^{cwe}$, then $\beta_A \circ \mathsf{contra}_A = \mathsf{contra}_A$. So each such comonoid is commutative.
5. weak_A and contra_A are coalgebra morphisms, by similar proof as Lemma 21.
6. Any coalgebra morphism $f : (cweA, \delta_A) \longrightarrow (cweA, \delta_B)$ between free coalgebras preserve the comonoid structure given by weak and contra, by similar proof as Lemma 21.

◀

3 Related Work

TODO

4 Conclusion

TODO

References

1 P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.

A Appendix

A.1 Basic Structures on Monoidal Categories

▶ **Definition 27.** Suppose we are given two monoidal categories $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{I_1} : I_2 \longrightarrow FI_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:

$$
\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B,C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A,B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}
$$

$$
\begin{tikzcd}
I_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{I_1} \otimes \mathsf{id}_{FA}"] & FA \\
FI_1 \otimes_2 FA \arrow[r, "m_{I_1,A}"] & F(I_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 I_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{I_1}"] & FA \\
FA \otimes_2 FI_1 \arrow[r, "m_{A,I_1}"] & F(A \otimes_1 I_1) \arrow[u, "F\rho_{1A}"]
\end{tikzcd}
$$

Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

▶ **Definition 28.** Suppose we are given two symmetric monoidal closed categories $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **symmetric monoidal functor** is a monoidal functor $(F, m) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ subject to the following additional coherence condition:

$$
\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"'] & F(B \otimes_1 A)
\end{tikzcd}
$$

▶ **Definition 29.** Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a **monoidal natural transformation** is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:

$$
\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
FI_1 \arrow[rr, "f_{I_1}"] & & GI_1 \\
& I_2 \arrow[ul, "m_{I_1}"] \arrow[ur, "n_{I_1}"'] &
\end{tikzcd}
$$

▶ **Definition 30.** Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are monoidal categories, and (F, m) is a monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a **monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \rightarrow GFA$, and the counit, $\varepsilon_A : FGA \to A$, are monoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FI_1 \arrow[r, "Fn_{I_2}"] & FGI_2 \arrow[d, "\varepsilon_{I_1}"] \\
I_2 \arrow[u, "m_{I_1}"] \arrow[r, equal] & I_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
GI_2 \arrow[r, "Gm_{I_1}"] & GFI_1 \\
I_1 \arrow[u, "n_{I_2}"] \arrow[r, equal] & I_1 \arrow[u, "\eta_{I_1}"']
\end{tikzcd}$$

▶ **Definition 31.** A **monoidal comonad** on a monoidal category C is a triple (T, ε, δ), where (T, m) is a monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
 & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
TI \arrow[dr, "\varepsilon_I"'] & & I \arrow[ll, "\mathsf{m}_I"'] \arrow[dl, equal] \\
 & I &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\mathsf{m}_I"] \arrow[d, "\mathsf{m}_I"'] & TI \arrow[d, "\delta_I"] \\
TI \arrow[r, "T\mathsf{m}_I"'] & T^2I
\end{tikzcd}$$

B Proofs

B.1 Lambek Categories with Exchange

B.1.1 Proof of The Eilenberg-Moore Category is Monoidal (Lemma 9)

We must first define the unitors, and then the associator. Then we show that they respect the symmetry monoidal coherence diagrams. Throughout this proof we will make use of the coalgebra

(A, h_A), (B, h_B), and (C, h_C).

The tensor product of (A, h_A) and (B, h_b) is $(A \otimes B, q_{A,B} \circ (h_A \otimes h_B))$, and the unit of the tensor product is (I, q_I); both actions are easily shown to satisfies the action diagrams of the Eilenberg Moore category. The left and right unitors are $\lambda : I \otimes A \longrightarrow A$ and $\rho : A \otimes I \longrightarrow A$, because they are indeed coalgebra morphisms.

The respective diagram for the right unitor is as follows:

$$\begin{tikzcd}
A \otimes I \arrow[r, "h_A \otimes \mathsf{id}"] \arrow[d, "\rho"] & eA \otimes I \arrow[r, "\mathsf{id} \otimes \mathsf{q}_I"] \arrow[drr, "\rho"] & eA \otimes eI \arrow[r, "\mathsf{q}_{A,I}"] & e(A \otimes I) \arrow[d, "e\rho"] \\
A \arrow[rrr, "h_A"] & & & eA
\end{tikzcd}$$

The left diagram commutes by naturality of ρ, the right diagram commutes by the fact that e is a monoidal functor. Showing the left unitor is a coalgebra morphism is similar.

The unitors are natural and isomorphisms, because they are essentially inherited from the underlying Lambek category.

The associator $\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ is also a coalgebra morphism. First, notice that:

$$\mathsf{q}_{A \otimes B, C} \circ ((\mathsf{q}_{A,B} \circ (h_A \otimes h_B)) \otimes h_c) = \mathsf{q}_{A \otimes B, C} \circ (\mathsf{q}_{A,B} \otimes \mathsf{id}) \circ ((h_A \otimes h_B) \otimes h_C)$$

where the left-hand side is the action of the coalgebra $(A \otimes B) \otimes C$. Similarly, the following is the action of the coalgebra $A \otimes (B \otimes C)$:

$$\mathsf{q}_{A, B \otimes C} \circ (h_A \otimes (\mathsf{q}_{B,C} \circ (h_B \otimes h_C))) = \mathsf{q}_{A, B \otimes C} \circ (\mathsf{id} \otimes \mathsf{q}_{B,C}) \circ (h_A \otimes (h_B \otimes h_C))$$

The following diagram must commute:

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "(h_A \otimes h_B) \otimes h_C"] \arrow[d, "\alpha"] & (eA \otimes eB) \otimes eC \arrow[r, "\mathsf{q} \otimes \mathsf{id}"] \arrow[d, "\alpha"] & e(A \otimes B) \otimes eC \arrow[r, "\mathsf{q}"] & e((A \otimes B) \otimes C) \arrow[d, "e\alpha"] \\
A \otimes (B \otimes C) \arrow[r, "h_A \otimes (h_B \otimes h_C)"] & eA \otimes (eB \otimes eC) \arrow[r, "\mathsf{id} \otimes \mathsf{q}"] & eA \otimes e(B \otimes C) \arrow[r, "\mathsf{q}"] & e(A \otimes (B \otimes C))
\end{tikzcd}
$$

The left diagram commutes by naturality of α, and the right diagram commutes because e is a monoidal functor.

Composition in $\mathcal{L}^e$ is the same as $\mathcal{L}$, and thus, the monoidal coherence diagrams hold in $\mathcal{L}^e$ as well. Thus, $\mathcal{L}^e$ is monoidal. We now show that it is symmetric.

B.1.2 Proof of Lemma 10

We define β as follows:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{ex_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

Suppose $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ are two coalgebra morphisms. Then the following diagram shows that $\beta_{A,B}$ is a natural transformation:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[d, "f \otimes g"] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] \arrow[d, "ef \otimes eg"] & eB \otimes eA \arrow[r, "\varepsilon_B \otimes \varepsilon_A"] \arrow[d, "eg \otimes ef"] & B \otimes A \arrow[d, "g \otimes f"] \\
A' \otimes B' \arrow[r, "h_{A'} \otimes h_{B'}"] & eA' \otimes eB' \arrow[r, "\mathsf{ex}_{A',B'}"] & eB' \otimes eA' \arrow[r, "\varepsilon_{B'} \otimes \varepsilon_{A'}"] & B' \otimes A'
\end{tikzcd}
$$

The left diagram commutes because f and g are both coalgebra morphisms, the middle diagram commutes because $\mathsf{ex}_{A,B}$ is a natural transformation, and the right diagram commutes by naturality of ε.

B.1.3 Proof of Corollary 11

The first diagram commutes by the fact that the following diagram commutes:

$$
\begin{tikzcd}
A \otimes B \arrow[r, equal] \arrow[d, "h_A \otimes h_B"] & A \otimes B \arrow[r, equal] \arrow[d, "h_A \otimes h_B"] & A \otimes B \arrow[d, "h_A \otimes h_B"] \\
eA \otimes eB \arrow[r, equal] \arrow[dd, "\mathsf{ex}_{A,B}"] & eA \otimes eB \arrow[d, "h_{eA} \otimes h_{eB}"] & eA \otimes eB \arrow[ddd, "\mathsf{ex}_{A,B}"] \\
& e^2A \otimes e^2B \arrow[ul, "\varepsilon_{eA} \otimes \varepsilon_{eB}"] \arrow[d, "\mathsf{ex}_{eA,eB}"] & \\
eB \otimes eA & e^2B \otimes e^2A \arrow[l, "\varepsilon_{eB} \otimes \varepsilon_{eA}"] & \\
& B \otimes A \arrow[ul, "h_B \otimes h_A"] & eB \otimes eA \arrow[l, "\varepsilon_B \otimes \varepsilon_A"]
\end{tikzcd}
$$

The diagram on the right commutes because $\beta_{A,B}$ is a natural transformation, and the other diagrams commute either because $\mathcal{L}$ is a Lambek category with exchange, or by the action diagrams.

The second diagram commtues by the following:

$$
\begin{tikzcd}
eA \otimes eB \arrow[rrrr, "\mathsf{ex}_{A,B}"] \arrow[dr, "eh_A \otimes eh_B"] \arrow[ddd, "\mathsf{q}_{A,B}"] & & & & eB \otimes eA \arrow[dll, "eh_B \otimes eh_A"'] \arrow[dl, equal] \arrow[ddd, "\mathsf{q}_{B,A}"] \\
& e^2A \otimes e^2B \arrow[r, "\mathsf{ex}_{eA,eB}"] \arrow[d, "\mathsf{q}_{eA,eB}"] & e^2B \otimes e^2A \arrow[r, "e\varepsilon_B \otimes e\varepsilon_A"] \arrow[d, "\mathsf{q}_{eB,eA}"] & eB \otimes eA & \\
& e(eA \otimes eB) \arrow[r, "e\mathsf{ex}_{A,B}"] & e(eB \otimes eA) \arrow[drr, "e(\varepsilon_B \otimes \varepsilon_A)"] & & \\
e(A \otimes B) \arrow[ur, "e(h_A \otimes h_B)"] \arrow[rrrr, "e\beta_{A,B}"'] & & & & e(B \otimes A)
\end{tikzcd}
$$

The top quadrangle commutes by the naturality of ex. The triangle commutes by the definition $\mathcal{L}^e$ (Definition 8). The left and right quadrangles commute by the naturality of q. And the quadrangle at the bottom is the definition of β.

B.1.4 Proof of Lemma 13

We prove this by equational reasoning as follows:

$$
\begin{aligned}
&(\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) \\
&\quad = (\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ (\mathsf{h}_B \otimes \mathsf{h}_A) \circ \beta_{A,B} && \text{(Corollary 11)} \\
&\quad = (\mathsf{h}_B \otimes \mathsf{h}_A) \circ ((\varepsilon_B \circ \mathsf{h}_B) \otimes (\varepsilon_A \circ \mathsf{h}_A)) \circ \beta_{A,B} \\
&\quad = (\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\mathsf{id}_B \otimes \mathsf{id}_A) \circ \beta_{A,B} && \text{(Action diagrams)} \\
&\quad = (\mathsf{h}_B \otimes \mathsf{h}_A) \circ \beta_{A,B} \\
&\quad = \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) && \text{(Corollary 11)} \\
&\quad = (\mathsf{id}_B \otimes \mathsf{id}_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) \\
&\quad = ((e\varepsilon_B \circ \delta_B) \otimes (e\varepsilon_A \circ \delta_A)) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) && \text{(Monoidal Comonad)} \\
&\quad = (e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B)
\end{aligned}
$$

Or simplified as:

$$
\begin{aligned}
&(\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) \\
&\quad = \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) && \text{(Corollary 11)} \\
&\quad = (\mathsf{id}_B \otimes \mathsf{id}_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) \\
&\quad = ((e\varepsilon_B \circ \delta_B) \otimes (e\varepsilon_A \circ \delta_A)) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) && \text{(Monoidal Comonad)} \\
&\quad = (e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B)
\end{aligned}
$$

Or by diagram chasing:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[d, "h_A \otimes h_B"'] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] & eB \otimes eA \arrow[d, "\varepsilon_B \otimes \varepsilon_A"] \\
eA \otimes eB \arrow[d, "\mathsf{ex}_{A,B}"'] & & B \otimes A \arrow[d, "h_B \otimes h_A"] \\
eB \otimes eA \arrow[rr, equal] \arrow[dr, "\delta_B \otimes \delta_A"'] & & eB \otimes eA \\
& e^2B \otimes e^2A \arrow[ur, "e\varepsilon_B \otimes e\varepsilon_B"'] &
\end{tikzcd}
$$

The upper rectangle commutes by Corollary 11 and the lower triangle commutes because e is a comonad.

B.1.5 Proof of Lemma 14

The proof follows from the commutativity of the following diagram:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[d, "h_A \otimes h_B"'] \arrow[dr, phantom, "(1)"] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] \arrow[dr, phantom, "(2)"] & eB \otimes eA \arrow[r, "\varepsilon_B \otimes \varepsilon_A"] \arrow[d, "\delta_B \otimes \delta_A"'] \arrow[dr, phantom, "(3)"] & B \otimes A \arrow[d, "h_B \otimes h_A"'] \\
eA \otimes eB \arrow[r, "eh_A \otimes eh_B"] \arrow[d, "\mathsf{q}_{A,B}"'] \arrow[dr, phantom, "(4)"] & e^2A \otimes e^2B \arrow[r, "\mathsf{ex}_{eA,eB}"] \arrow[d, "\mathsf{q}_{eA,eB}"'] \arrow[dr, phantom, "(5)"] & e^2B \otimes e^2A \arrow[r, "e\varepsilon_B \otimes e\varepsilon_A"] \arrow[d, "\mathsf{q}_{eB,eA}"'] \arrow[dr, phantom, "(6)"] & eA \otimes eB \arrow[d, "\mathsf{q}_{B,A}"'] \\
e(A \otimes B) \arrow[r, "e(h_A \otimes h_B)"] & e(eA \otimes eB) \arrow[r, "e\mathsf{ex}_{A,B}"] & e(eB \otimes eA) \arrow[r, "e(\varepsilon_B \otimes \varepsilon_A)"] & e(B \otimes A)
\end{tikzcd}
$$

Diagram one commutes by the action diagrams for the coalgebras (A, h_A) and (B, h_B), diagram two commutes because $\mathcal{L}$ is a Lambek category with exchange, diagram three does not commute, but holds by Lemma 13, diagram four and six commute by naturality of q, and diagram five commutes because $\mathcal{L}$ is a Lambek category with exchange.

B.1.6 Proof of The Eilenberg-Moore Category is Symmetric Monoidal (Lemma 15)

The following diagram shows that $\beta_{B,A} \circ \beta_{A,B} = \mathsf{id}_{A\otimes B}$:

$$\begin{tikzcd}
A \otimes B \arrow[r, "h_A\otimes h_B"] \arrow[d, "h_A\otimes h_B"'] \arrow[dr, phantom, "(1)"] \arrow[dddrrr, equal, bend left=40] & eA \otimes eB \arrow[d, "\delta_A\otimes\delta_B"] \arrow[drr, phantom, "(6)"] & & \\
eA \otimes eB \arrow[r, "\delta_A\otimes\delta_B"] \arrow[d, "\mathsf{ex}_{A,B}"'] \arrow[dr, phantom, "(2)"] & e^2A \otimes e^2B \arrow[d, "\mathsf{ex}_{eA,eB}"] \arrow[dr, equal] \arrow[dr, phantom, "(4)"', near start] & & \\
eB \otimes eA \arrow[r, "\delta_B\otimes\delta_A"] \arrow[d, "\varepsilon_B\otimes\varepsilon_A"'] \arrow[dr, phantom, "(3)"] & e^2B \otimes e^2A \arrow[r, "\mathsf{ex}_{eB,eA}"] \arrow[d, "e\varepsilon_B\otimes e\varepsilon_A"] \arrow[dr, phantom, "(5)"] & e^2A \otimes e^2B \arrow[d, "e\varepsilon_A\otimes e\varepsilon_B"] & \\
B \otimes A \arrow[r, "h_B\otimes h_A"] & eB \otimes eA \arrow[r, "\mathsf{ex}_{B,A}"] & eA \otimes eB \arrow[r, "\varepsilon_A\otimes\varepsilon_B"] & A \otimes B
\end{tikzcd}$$

Diagram one trivially commutes, diagram two, four, and five commute because $\mathcal{L}$ is a Lambek category with exchange, diagram three does not commute, but holds by Lemma 13, diagrams six, seven, and eight commute by the fact that (e, ε, δ) is a comonad and the action diagrams of the Eilenberg Moore category.

At this point we must verify that β respects the coherence diagrams of a symmetric monoidal category; see Definition 3. Thus, we must show that each of the following diagrams hold:

Case

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B}\otimes\mathsf{id}_C"'] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B\otimes\beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

We can show that this diagram commutes, by reducing it to the corresponding diagram on free coalgebras which we know holds by the assumption that $\mathcal{L}$ is a Lambek category with exchange. This reduction is as follows (due to the size of the diagram it is broken up into three diagrams that can be straightforwardly composed):

Diagram 1:

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[rr, equal] \arrow[d, "(h_A \otimes h_B) \otimes \mathsf{id}"] & & (A \otimes B) \otimes C \arrow[d, "(h_A \otimes h_B) \otimes h_C"] \\
(eA \otimes eB) \otimes C \arrow[rr, "\mathsf{id} \otimes h_C"] \arrow[d, "\mathsf{ex} \otimes \mathsf{id}"] & & (eA \otimes eB) \otimes eC \arrow[d, "\mathsf{ex} \otimes \mathsf{id}"] \\
(eB \otimes eA) \otimes C \arrow[r, "(\delta \otimes \delta) \otimes \mathsf{id}"] \arrow[rr, bend left, "\mathsf{id} \otimes h_C"] \arrow[d, "(\varepsilon \otimes \varepsilon) \otimes \mathsf{id}"] \arrow[dr, phantom, "(2)"] & (e^2 B \otimes e^2 A) \otimes C \arrow[d, "(e\varepsilon \otimes e\varepsilon) \otimes \mathsf{id}"] \arrow[r, phantom, "(1)"] & (eB \otimes eA) \otimes eC \arrow[d, "\alpha"] \\
(B \otimes A) \otimes C \arrow[r, "(h_B \otimes h_A) \otimes \mathsf{id}"] \arrow[d, equal] & (eB \otimes eA) \otimes C \arrow[ur, "\mathsf{id} \otimes h_C"] \arrow[d, phantom, "(3)"] & eB \otimes (eA \otimes eC) \\
(B \otimes A) \otimes C \arrow[rr, "\alpha"] \arrow[d, equal] & & B \otimes (A \otimes C) \arrow[u, "h_B \otimes (h_A \otimes h_C)"] \arrow[d, "\mathsf{id} \otimes (h_A \otimes h_C)"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id} \otimes (h_A \otimes h_C)"] & B \otimes (eA \otimes eC)
\end{tikzcd}
$$

Diagram one commutes because (e, ε, δ) is a comonad, diagram two does not commute, but holds by Lemma 13, and diagram 3 commutes by naturality of α. All other diagrams commute trivially.

Diagram 2:

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\alpha"] \arrow[d, "(h_A\otimes h_B)\otimes h_C"'] \arrow[dr, phantom, "(1)"] & A\otimes(B\otimes C) \arrow[r, "h_A\otimes(h_B\otimes h_C)"] \arrow[d, "h_A\otimes(h_B\otimes h_C)"] \arrow[dr, phantom, "(2)"] & eA\otimes(eB\otimes eC) \arrow[r, "\mathsf{id}\otimes\mathsf{q}"] \arrow[d, "\mathsf{id}\otimes(eh_B\otimes eh_C)"] \arrow[dr, phantom, "(3)"] & eA\otimes e(B\otimes C) \arrow[r, "\mathsf{ex}"] \arrow[d, "\mathsf{id}\otimes e(h_B\otimes h_C)"] \arrow[dr, phantom, "(4)"] & e(B\otimes C)\otimes eA \arrow[d, "e(h_B\otimes h_C)\otimes\mathsf{id}"] \\
(eA\otimes eB)\otimes eC \arrow[r, "\alpha"] \arrow[d, "\mathsf{ex}\otimes\mathsf{id}"'] & eA\otimes(eB\otimes eC) \arrow[r, "\mathsf{id}\otimes(\delta\otimes\delta)"] & eA\otimes(e^2B\otimes e^2C) \arrow[r, "\mathsf{id}\otimes\mathsf{q}"] & eA\otimes e(eB\otimes eC) \arrow[r, "\mathsf{ex}"] & e(eB\otimes eC)\otimes eA \arrow[d, "\varepsilon\otimes\mathsf{id}"] \\
(eB\otimes eA)\otimes eC \arrow[d, "\alpha"'] & & (5) & & (eB\otimes eC)\otimes eA \arrow[d, "\alpha"] \\
eB\otimes(eA\otimes eC) \arrow[rrrr, "\mathsf{id}\otimes\mathsf{ex}"] & & & & eB\otimes(eC\otimes eA) \arrow[d, equal] \\
B\otimes(A\otimes C) \arrow[u, "h_B\otimes(h_A\otimes h_C)"] \arrow[r, "\mathsf{id}\otimes(h_A\otimes h_C)"] \arrow[d, "\mathsf{id}\otimes(h_A\otimes h_C)"'] & B\otimes(eA\otimes eC) \arrow[r, "\mathsf{id}\otimes\mathsf{ex}"] & B\otimes(eC\otimes eA) \arrow[rr, "h_B\otimes\mathsf{id}"] \arrow[d, equal] & & eB\otimes(eC\otimes eA) \arrow[d, equal] \\
B\otimes(eA\otimes eC) \arrow[rr, "\mathsf{id}\otimes\mathsf{ex}"] & & B\otimes(eC\otimes eA) \arrow[rr, "h_B\otimes\mathsf{id}"] & & eB\otimes(eC\otimes eA)
\end{tikzcd}$$

Diagram one commutes by naturality of α, diagram two commutes by the action diagrams, diagram three commutes by naturality of q, diagram four commutes by naturality of ex, and diagram five commutes because $\mathcal{L}$ is a Lambek category with exchange. All other diagrams trivially commute.

Diagram 3:

$$
\begin{tikzcd}
e(B \otimes C) \otimes eA \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "e(h_B\otimes h_C)\otimes\mathsf{id}"'] & (B \otimes C) \otimes A \arrow[d, "(h_B\otimes h_C)\otimes\mathsf{id}"] \\
e(eB \otimes eC) \otimes eA \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\varepsilon\otimes\mathsf{id}"'] & (eB \otimes eC) \otimes A \arrow[d, "(\varepsilon\otimes\varepsilon)\otimes\mathsf{id}"] \\
(eB \otimes eC) \otimes eA \arrow[r, "(\varepsilon\otimes\varepsilon)\otimes\varepsilon"] \arrow[d, "\alpha"'] & (B \otimes C) \otimes A \arrow[d, "\alpha"] \\
eB \otimes (eC \otimes eA) \arrow[r, "\varepsilon\otimes(\varepsilon\otimes\varepsilon)"] \arrow[d, equal] & B \otimes (C \otimes A) \arrow[dd, equal] \\
eB \otimes (eC \otimes eA) \arrow[d, equal] & \\
eB \otimes (eC \otimes eA) \arrow[r, "\varepsilon\otimes(\varepsilon\otimes\varepsilon)"] & B \otimes (C \otimes A)
\end{tikzcd}
$$

The top most diagram commutes by naturality of ε and the middle diagram commutes by naturality of α. All other diagrams trivially commute.

Case

$$
\begin{tikzcd}
I \otimes A \arrow[rr, "\beta_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}
$$

Just as we did for the previous case we reduce this diagram down to the corresponding one on free coaglebras that we know holds by the assumption that $\mathcal{L}$ is a Lambek category with exchange. This case follows from the following commutative diagram:

$$
\begin{tikzcd}
I \otimes A \arrow[r, "h_I\otimes h_A"] \arrow[dr, "\mathsf{id}\otimes h_A"'] \arrow[dd, "\lambda"'] & eI \otimes eA \arrow[r, "\mathsf{ex}"] \arrow[d, "\varepsilon\otimes\mathsf{id}"] & eA \otimes eI \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mathsf{id}\otimes\varepsilon"] & A \otimes I \arrow[dl, "\varepsilon\otimes\mathsf{id}"'] \arrow[dd, "\rho"] \\
& I \otimes eA \arrow[d, "\lambda"] & eA \otimes I \arrow[d, "\rho"] & \\
A \arrow[r, "h_A"] & eA \arrow[r, equal] & eA \arrow[r, "\varepsilon"] & A
\end{tikzcd}
$$

The left most triangle commutes by the action diagrams and the lower diagram commutes by naturality of λ. Similarly, the right most lower diagram commutes by naturality of ρ. The middle diagram commutes because $\mathcal{L}$ is a Lambek category with exchange. All other diagrams trivially commute.

B.2 Weakening and Contraction

B.3 Proof of Composition of Weakening and Contraction (Lemma 19)

Since by definition $w : \mathcal{L} \longrightarrow \mathcal{L}$ and $c : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors we know that their composition $cw : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$\begin{aligned}
&\mathsf{q}_{A,B} : cwA \otimes cwB \longrightarrow cw(A \otimes B)\\
&\mathsf{q}_{A,B} = c\mathsf{q}^{w}_{A,B} \circ \mathsf{q}^{c}_{wA,wB}\\
&\mathsf{q}_{I} : I \longrightarrow cwI\\
&\mathsf{q}_{I} = c\mathsf{q}^{w}_{I} \circ \mathsf{q}^{c}_{I}
\end{aligned}$$

We must now define both $\varepsilon_A : cwA \longrightarrow A$ and $\delta_A : cwA \longrightarrow cwcwA$, and then show that they are monoidal natural transformations subject to the comonad laws. Since we are composing two comonads each of ε and δ can be given two definitions, but they are equivalent:

- $\varepsilon_A : cwA \longrightarrow A$ is defined as in the diagram below, which commutes by the naturality of ε^c.

$$\begin{tikzcd}
cwA \arrow[r, "\varepsilon^{c}_{wA}"] \arrow[d, "c\varepsilon^{w}_{A}"'] & wA \arrow[d, "\varepsilon^{w}_{A}"] \\
cA \arrow[r, "\varepsilon^{c}_{A}"'] & A
\end{tikzcd}$$

- $\delta_A : cwA \longrightarrow cwcwA$ is defined as in the diagram:

$$\begin{tikzcd}
cwA \arrow[r, "c\delta^{w}_{A}"] \arrow[d, "\delta^{c}_{wA}"'] & cw^2A \arrow[r, "\delta^{c}_{w^2A}"] \arrow[d, "\delta^{c}_{w^2A}"] & c^2w^2A \arrow[d, "cdist_{wA}"] \\
c^2wA \arrow[r, "c^2\delta^{w}_{A}"'] & c^2w^2A \arrow[r, "cdist_{wA}"'] & cwcwA
\end{tikzcd}$$

The left part of the diagram commutes by the naturality of δ^c and the right part commutes trivially.

The remainder of the proof shows that the comonad laws hold.

Case 1:

$$\begin{tikzcd}
cwA \arrow[r, "\delta_{A}"] \arrow[d, "\delta_{A}"'] & cwcwA \arrow[d, "cw\delta_{A}"] \\
cwcwA \arrow[r, "\delta_{cwA}"'] & cwcwcwA
\end{tikzcd}$$

The previous diagram commutes because the following one does.

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, phantom, "(1)"] & cwcwA \arrow[r, "cwc\delta^w_A"] \arrow[d, "\delta^c_{wcwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwcw^2A \arrow[r, "cw\delta^c_{w^2A}"] \arrow[d, "\delta^c_{wcw^2A}"] \arrow[dr, phantom, "(5)"] & cwc^2w^2A \arrow[dd, "cwcdist_{wA}"] \\
cwcwA \arrow[d, "c\delta^w_{cwA}"'] \arrow[dr, phantom, "(4)"] & c^2wcwA \arrow[r, "c^2wc\delta^w_A"] \arrow[dr, "c^2\delta^w_{cwA}"] & c^2wcw^2A \arrow[ur, "cdist_{cw^2A}"] \arrow[d, "c^2wdist_{wA}"] \arrow[dr, phantom, "(6)"] \arrow[dl, phantom, "(3)"] & \\
cw^2cwA \arrow[rr, "\delta^c_{w^2cwA}"'] & & c^2w^2cwA \arrow[r, "cdist_{wcwA}"'] & cwcwcwA
\end{tikzcd}
$$

(1) commutes by equality and we will not expand δ_A for simplicity. (2) and (4) commutes by the naturality of δ^c. (3), (5) commutes by the conditions of $dist$. (6) commutes by the naturality of $dist$.

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[dr, equal] & cwcwA \arrow[d, "cw\varepsilon_A"] \\
& cwA
\end{tikzcd}
$$

The triangle commutes because of the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\delta^w_A"] \arrow[ddr, "\varepsilon^c_{wA}"] \arrow[ddd, equal] \arrow[drr, phantom, "(1)"] & & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, equal] \arrow[dr, phantom, "(2)"] & c^2w^2A \arrow[dl, "c\varepsilon^c_{w^2A}"] \arrow[ddd, "cdist_{wA}"] \\
& & cw^2A \arrow[ull, "c\varepsilon^w_{wA}"] \arrow[d, "\varepsilon^c_{w^2A}"] \arrow[dl, phantom, "(3)"] \arrow[ddr, phantom, "(4)"] & \\
\arrow[r, phantom, "(5)"] & wA & w^2A \arrow[l, "\varepsilon^w_{wA}"'] & \\
cwA \arrow[ur, "\varepsilon^c_{wA}"] \arrow[urr, phantom, "(6)"] & & wcwA \arrow[ll, "\varepsilon^w_{cwA}"] \arrow[u, "w\varepsilon^c_{wA}"] \arrow[ur, phantom, "(7)"] & cwcwA \arrow[l, "\varepsilon^c_{wcwA}"] \arrow[uul, "cw\varepsilon^c_{wA}"']
\end{tikzcd}
$$

(1) commutes by the comonad law for w with components δ^w_A and ε^w_{wA}. (2) commutes by the comonad law for c with components $\delta^c_{w^2A}$ and $\varepsilon^c_{w^2A}$. (3) and (7) commute by the naturality of ε^c. (4) commutes by the condition of $dist$. (5) commutes trivially. And (6) commutes by the naturality of ε^w.

Case 3:

$$\begin{tikzcd}
cwA \arrow[d, "\delta_A"'] \arrow[dr, equal] & \\
cwcwA \arrow[r, "\varepsilon_{cwA}"'] & cwA
\end{tikzcd}$$

The previous triangle commutes because the following diagram chasing does.

$$\begin{tikzcd}
cwA \arrow[r, "c\delta^w_A"] \arrow[dr, "\delta^c_{wA}"] \arrow[ddd, equal] & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, phantom, "(1)"] & c^2w^2A \arrow[ddd, "cdist_{wA}"] \arrow[ddl, "c^2w\varepsilon^w_A"] \\
\arrow[r, phantom, "(2)"] & c^2wA \arrow[ur, "c^2\delta^w_A"] \arrow[d, equal] \arrow[r, phantom, "(3)"] & {} \\
\arrow[r, phantom, "(4)"] & c^2wA \arrow[uul, "c\varepsilon^c_{wA}"] \arrow[d, "cdist_A"] \arrow[r, phantom, "(5)"] & {} \\
cwA & cwcA \arrow[l, "cw\varepsilon^c_A"] & cwcwA \arrow[l, "cwc\varepsilon^w_A"]
\end{tikzcd}$$

(1) commutes by the naturality of δ^c. (2) is the comonad law for c with components δ^c_{wA} and ε^c_{wA}. (3) is the comonad law for w with components δ^w_A and ε^w_A. (4) commutes by the condition of $dist$. And (5) commute by the naturality of $dist$.

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B.4 Proof of Conditions of Lambek category with cw (Lemma 21)

1. As shown in the paper.
2. Each $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid.

Case 1:

$$\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & cwA \otimes cwA \arrow[r, "id_{cwA} \otimes \mathsf{contra}_A"] & cwA \otimes (cwA \otimes cwA) \\
cwA \otimes cwA \arrow[rr, "\mathsf{contra}_A \otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}$$

The previous diagram commutes by the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] \arrow[dr, phantom, "(1)"] & cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\rho^{-1}_{cwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwA \otimes (cwA \otimes I) \arrow[d, "id_{cwA}\otimes\mathsf{contraL}_{wA,I}"] \\
cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\lambda^{-1}_{cwA}"] \arrow[d, "\rho^{-1}_{cwA}\otimes id_{cwA}"'] & cwA \otimes (I \otimes cwA) \arrow[d, "id_{cwA}\otimes\mathsf{contraR}_{wA,I}"] \arrow[dr, phantom, "(3)"] & cwA \otimes ((cwA \otimes I) \otimes cwA) \arrow[d, "id_{cwA}\otimes(\rho_{cwA}\otimes id_{cwA})"] \\
(cwA \otimes I) \otimes cwA \arrow[d, "\mathsf{contraL}_{wA,I}\otimes id_{cwA}"'] \arrow[dr, phantom, "(4)"] & cwA \otimes (cwA \otimes (I \otimes cwA)) \arrow[r, "id_{cwA}\otimes(id_{cwA}\otimes\lambda_{cwA})"] \arrow[ur, "id_{cwA}\otimes\alpha_{cwA,I,cwA}"] & cwA \otimes (cwA \otimes cwA) \\
((cwA \otimes I) \otimes cwA) \otimes cwA \arrow[rr, "(\rho_{cwA}\otimes id_{cwA})\otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}
$$

(1) commutes trivially and we would not expand contra for simplicity. (2) and (4) commute because $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction. (3) commutes because $\mathcal{L}$ is monoidal.

Case 2:

$$
\begin{tikzcd}
& cwA \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes cwA & cwA \otimes cwA \arrow[l, "\mathsf{weak}_A\otimes id_{cwA}"] \arrow[r, "id_{cwA}\otimes\mathsf{weak}_A"'] & cwA \otimes I
\end{tikzcd}
$$

The diagram above commutes by the following diagram chasing.

$$
\begin{tikzcd}
I \otimes cwA & & & wA \otimes cwA \arrow[lll, "\mathsf{weak}^w_A\otimes id_{cwA}"'] \\
& I \otimes (I \otimes cwA) \arrow[ul, "id_I\otimes\lambda_{cwA}"'] \arrow[r, phantom, "(4)"] & wA \otimes (I \otimes cwA) \arrow[l, "\mathsf{weak}^w_A\otimes id_{I\otimes cwA}"'] \arrow[ur, "id_{wA}\otimes\lambda_{cwA}"] \arrow[u, phantom, "(2)"] & \\
& I \otimes cwA \arrow[u, "\lambda^{-1}_{I\otimes cwA}"] \arrow[r, "\mathsf{contraR}_{wA,I}"] \arrow[l, phantom, "(1)"] & cwA \otimes (I \otimes cwA) \arrow[u, "\varepsilon^c_{wA}\otimes id_{I\otimes cwA}"'] \arrow[dr, "id_{cwA}\otimes\lambda_{cwA}"] \arrow[r, phantom, "(3)"] & {} \\
cwA \arrow[uuu, "\lambda^{-1}_{cwA}"] \arrow[ur, "\lambda^{-1}_{cwA}"] \arrow[dr, "\rho^{-1}_{cwA}"'] \arrow[ddd, "\rho^{-1}_{cwA}"'] & & (5) & cwA \otimes cwA \arrow[uuu, "\varepsilon^c_{wA}\otimes cwA"'] \arrow[ddd, "id_{cwA}\otimes\varepsilon^c_{wA}"] \\
& cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] \arrow[d, "\rho^{-1}_{cwA}"] \arrow[l, phantom, "(6)"] & (cwA \otimes I) \otimes cwA \arrow[ur, "\rho_{cwA}\otimes id_{cwA}"'] \arrow[d, "id_{cwA\otimes I}\otimes\varepsilon^c_{wA}"'] \arrow[dl, phantom, "(9)"] \arrow[r, phantom, "(8)"] & {} \\
& (cwA \otimes I) \otimes I \arrow[dl, "\rho_{cwA}\otimes id_I"] \arrow[r, phantom, "(7)"] & (cwA \otimes I) \otimes wA \arrow[l, "id_{cwA\otimes I}\otimes\mathsf{weak}^w_A"'] \arrow[dr, "\rho_{cwA}\otimes id_{wA}"] & \\
cwA \otimes I & & & cwA \otimes wA \arrow[lll, "id_{cwA}\otimes\mathsf{weak}_A"]
\end{tikzcd}
$$

(1), (2) and (3) commute by the functionality of λ. (6), (7) and (8) commute by the functionality of ρ. (4) and (9) are conditions of the Lambek category with cw. And (5) is the definition of contra.

3. weak and contra are coalgebra morphisms.

Case 1:

$$\begin{tikzcd}
cwA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
cwcwA \arrow[r, "cw\mathsf{weak}_A"'] & cwI
\end{tikzcd}$$

The previous diagram commutes by the diagram below. (1) commutes by the naturality of δ^c. (2) commutes by the condition of $dist_{wA}$. (3), (5) and (6) commute because c is a monoidal comonad. (4) commutes because $(\mathcal{L}, w, \mathsf{weak}^w)$ is a Lambek category with weakening. (7) commutes because c and w are monoidal comonads.

$$\begin{tikzcd}
cwA \arrow[r, "c\mathsf{weak}^w_A"] \arrow[d, "\delta^c_{wA}"'] \arrow[ddr, "c\delta^w_A"] & cI \arrow[rr, "\epsilon^c_I"] \arrow[dddrr, "c\mathsf{q}^w_I"'] \arrow[dr, phantom, "(5)"] & & I \arrow[dl, equal] \arrow[dd, "\mathsf{q}^c_I"] \\
c^2wA \arrow[d, "c^2\delta^w_A"'] \arrow[r, phantom, "(1)"] & (4) & I \arrow[ul, "\mathsf{q}^c_I"'] \arrow[r, phantom, "(6)"] & {} \\
c^2w^2A \arrow[d, "cdist_{wA}"'] \arrow[dr, "c\varepsilon^c_{wA}"] \arrow[r, phantom, "(3)", shift right=4] & cw^2A \arrow[l, "\delta^c_{w^2A}"'] \arrow[d, equal] & (7) & cI \arrow[ul, "\varepsilon^c_I"'] \arrow[d, "c\mathsf{q}^w_I"] \\
cwcwA \arrow[r, "cw\varepsilon^c_{wA}"'] \arrow[ur, phantom, "(2)"] & cw^2A & & cwI \arrow[ll, "cw\mathsf{weak}^w_A"]
\end{tikzcd}$$

Case 2:

$$\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & cwA \otimes cwA \arrow[r, "\delta_A \otimes \delta_A"] & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}_{cwA,cwA}"] \\
cwcwA \arrow[rr, "cw\mathsf{contra}_A"'] & & cw(cwA \otimes cwA)
\end{tikzcd}$$

To prove the previous diagram commute, we first expand it, Then we divide it into five parts as shown belovee, and prove each part commutes.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^{w}_{A}"'] \arrow[dr, phantom, "(c)"] & cwA \otimes I \arrow[rrr, "\mathsf{contraL}_{wA,I}"] & & & (cwA \otimes I) \otimes cwA \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
cw^2A \arrow[d, "\delta^{c}_{w^2A}"'] & w(cwA \otimes I) \arrow[u, "\varepsilon^{w}_{cwA \otimes I}"] & w(cwA \otimes cwA) \arrow[rr, "\varepsilon^{w}_{cwA \otimes cwA}"] & & cwA \otimes cwA \arrow[d, "c\delta^{w}_{A} \otimes c\delta^{w}_{A}"] \arrow[dl, phantom, "(a)"] \\
c^2w^2A \arrow[d, "cdist_{wA}"'] & & & cw^2A \otimes cw^2A \arrow[ur, "c\varepsilon^{c}_{wA} \otimes c\varepsilon^{c}_{wA}"] \arrow[r, equal] & cw^2A \otimes cw^2A \arrow[d, "\delta^{c}_{w^2A} \otimes \delta^{c}_{w^2A}"] \arrow[dl, phantom, "(b)"] \\
cwcwA \arrow[d, "cw\rho_{cwA}"'] \arrow[r, phantom, "(d)"] & & & \arrow[r, phantom, "(e)"] & c^2w^2A \otimes c^2w^2A \arrow[ul, "\varepsilon^{c}_{cw^2A} \otimes \varepsilon^{c}_{cw^2A}"] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
cw(cwA \otimes I) \arrow[uuur, "\varepsilon^{c}_{w(cwA \otimes I)}"] \arrow[d, "cw\mathsf{contraL}_{wA,I}"'] & & & & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}^{c}_{wcwA \otimes wcwA}"] \\
cw((cwA \otimes I) \otimes cwA) \arrow[rr, "cw(\rho_{cwA} \otimes id_{cwA})"'] & & cw(cwA \otimes cwA) \arrow[uuuu, "c\varepsilon^{w}_{cwA \otimes cwA}"] & & c(wcwA \otimes wcwA) \arrow[ll, "c\mathsf{q}^{w}_{cwA \otimes cwA}"]
\end{tikzcd}
$$

Part (a) and (b) are comonad laws.

Part (c) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for w. (3) is the comonad law for c. (4) commutes by the naturality of ε^{c}. (5) is one of the conditions for $dist_{wA}$. (6) commutes by the naturality of ε^{w}. And (7) commutes by the naturality of ε^{c}.

$$
\begin{tikzcd}
cwA \arrow[rrr, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^{w}_{A}"'] \arrow[dr, "c\delta^{w}_{A}"] \arrow[drr, equal] & & & cwA \otimes I \\
cw^2A \arrow[r, equal] \arrow[d, "\delta^{c}_{w^2A}"'] \arrow[ur, phantom, "(1)"] & cw^2A \arrow[r, "c\varepsilon^{w}_{wA}"] \arrow[d, "dist_{wA}"] \arrow[u, phantom, "(2)"] & cwA \arrow[r, phantom, "(6)"] & \\
c^2w^2A \arrow[ur, "\varepsilon^{c}_{cw^2A}"] \arrow[d, "cdist_{wA}"'] \arrow[u, phantom, "(3)"] \arrow[r, phantom, "(4)"] & wcwA \arrow[ur, "\varepsilon^{w}_{cwA}"] \arrow[rr, "w\rho^{-1}_{cwA}"] \arrow[u, phantom, "(5)"] & & w(cwA \otimes I) \arrow[uu, "\varepsilon^{w}_{cwA \otimes I}"'] \\
cwcwA \arrow[ur, "\varepsilon^{c}_{wcwA}"] \arrow[rrr, "cw\rho^{-1}_{cwA}"'] & & \arrow[u, phantom, "(7)"] & cw(cwA \otimes I) \arrow[u, "\varepsilon^{c}_{w(cwA \otimes I)}"']
\end{tikzcd}
$$

Part (d) commutes by the following diagram chase. The upper two squares both commute by the naturality of ε^w, and the lower two squares commute by the naturality of ε^c.

$$
\begin{tikzcd}
cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] & cwA \otimes cwA \\
w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA\otimes I}"] \arrow[r, "w\mathsf{contraL}_{wA,I}"] & w((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^w_{(cwA\otimes I)\otimes cwA}"] \arrow[r, "w(\rho_{cwA} \otimes id_{cwA})"] & w(cwA \otimes cwA) \arrow[u, "\varepsilon^w_{cwA\otimes cwA}"'] \\
cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA\otimes I)}"] \arrow[r, "cw\mathsf{contraL}_{wA,I}"'] & cw((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^c_{w((cwA\otimes I)\otimes A)}"] \arrow[r, "cw(\rho_{cwA} \otimes id_{cwA})"'] & cw(cwA \otimes cwA) \arrow[u, "\varepsilon^c_{w(cwA\otimes cwA)}"']
\end{tikzcd}
$$

Part (e) commutes by the following diagram. (1) commutes by the condition of $dist_{wA}$. (2) and (4) commute by the naturality of ε^c. (3) and (5) commute because w and c are monoidal comonads.

$$
\begin{tikzcd}
cwA \otimes cwA & cw^2A \otimes cw^2A \arrow[l, "c\varepsilon^w_{wA} \otimes c\varepsilon^w_{wA}"'] \arrow[d, "dist_{wA} \otimes dist_{wA}"] \arrow[dl, phantom, "(1)"] \arrow[dr, phantom, "(2)"] & c^2w^2A \otimes c^2w^2A \arrow[l, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"'] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
(3) & wcwA \otimes wcwA \arrow[ul, "\varepsilon^w_{cwA} \otimes \varepsilon^w_{cwA}"] \arrow[dl, "\mathsf{q}^w_{cwA,cwA}"] \arrow[d, phantom, "(4)"] & cwcwA \otimes cwcwA \arrow[l, "\varepsilon^c_{wcwA} \otimes \varepsilon^c_{wcwA}"'] \arrow[d, "\mathsf{q}^c_{wcwA\otimes wcwA}"] \arrow[dl, phantom, "(5)"] \\
w(cwA \otimes cwA) \arrow[uu, "\varepsilon^w_{cwA\otimes cwA}"] & cw(cwA \otimes cwA) \arrow[l, "\varepsilon^c_{w(cwA\otimes cwA)}"] & c(wcwA \otimes wcwA) \arrow[l, "c\mathsf{q}_{cwA\otimes cwA}"] \arrow[ul, "\varepsilon^c_{wcwA\otimes wcwA}"]
\end{tikzcd}
$$

4. Any coalgebra morphism $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ between free coalgebras preserves the comonoid structure given by weak and contra.

Case 1: This coherence diagram is given in the definition of the Lambek category with cw.

$$
\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & cwB \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
$$

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & cwA \otimes cwA \arrow[d, "f\otimes f"] \\
cwB \arrow[r, "\mathsf{contra}_B"'] & cwB \otimes cwB
\end{tikzcd}
$$

The square commutes by the diagram chasing below, which commutes by the naturality of ρ and $\mathsf{contraL}$.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "cwf"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] \arrow[d, "cwf\otimes id_I"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA}\otimes id_{cwA}"] \arrow[d, "(cwf\otimes id_I)\otimes cwf"] & cwA \otimes cwA \arrow[d, "cwf\otimes cwf"] \\
cwB \arrow[r, "\rho^{-1}_{cwB}"'] & cwB \otimes I \arrow[r, "\mathsf{contraL}_{wB,I}"'] & (cwB \otimes I) \otimes cwB \arrow[r, "\rho_{cwB}\otimes id_{cwB}"'] & cwB \otimes cwB
\end{tikzcd}
$$

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