Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

1.1 Symmetric Monoidal Categories

Definition 1. *A **monoidal category** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned} \lambda_A &: \top \otimes A \longrightarrow A \\ \rho_A &: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{aligned}$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd} ((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\ (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\ A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"'] \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (\top \otimes B) \arrow[dl, "\lambda_B"] \\ & A \otimes B & \end{tikzcd}$$

Definition 2. *A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A\\
\rho_A &: A \otimes \top \longrightarrow A\\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

- *A symmetry natural transformation:*

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A"'] & & A \otimes (\top \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 3. *A **symmetric monoidal closed category (SMCC)** is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Definition 4. *Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2\,FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1\,A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2\,FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1, A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1\,A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2\,FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A, \top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1\,A}"]
\end{tikzcd}$$

Definition 5. *Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2\,FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1\,A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$
\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"'] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"'] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"'] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"] & F(B \otimes_1 A)
\end{tikzcd}
$$

Definition 6. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$
\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}
$$

Definition 7. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***symmetric monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$
\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}
$$

Definition 8. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are* *monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 9. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***symmetric monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 10. *A* ***monoidal comonad*** *on a monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[rr, "\mathsf{m}_\top"] & & \top \\
& \top \arrow[ul, "\varepsilon_\top"] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

Definition 11. *A **symmetric monoidal comonad** on a symmetric monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[rr, "\mathsf{m}_\top"] & & \top \\
& \top \arrow[ul, "\varepsilon_\top"] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2 A \otimes T^2 B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

1.2 Linear Category

Definition 12. *A **linear category**, $(\mathcal{L}, !, \mathsf{e}, \mathsf{d})$, is specified by*

- *a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,*
- *a symmetric monoidal comonad $((!, \mathsf{q}), \varepsilon, \delta)$ on $\mathcal{L}$, with $\mathsf{q}_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $\mathsf{q}_I : I \longrightarrow !I$;*
- *monoidal natural transformations on $\mathcal{L}$ with components $\mathsf{e}_A : !A \longrightarrow I$ and $\mathsf{d}_A : !A \longrightarrow !A \otimes !A$, s.t.*
  - *each $(!A, \mathsf{e}_A, \mathsf{d}_A)$ is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ \mathsf{d}_A = \mathsf{d}_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of $\mathcal{L}$;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{d}_A"] \arrow[d, "\mathsf{d}_A"'] & !A \otimes !A \arrow[r, "id_{!A} \otimes \mathsf{d}_A"] & !A \otimes (!A \otimes !A) \arrow[d, "\alpha_{!A,!A,!A}"] \\
!A \otimes !A \arrow[rr, "\mathsf{d}_A \otimes id_{!A}"] & & (!A \otimes !A) \otimes !A
\end{tikzcd}$$

$$\begin{tikzcd}
 & !A \arrow[d, "\mathsf{d}_A"] & \\
I \otimes !A \arrow[ur, "\lambda"] & !A \otimes !A \arrow[l, "\mathsf{e}_A \otimes id_{!A}"] \arrow[r, "id_{!A} \otimes \mathsf{e}_A"'] & !A \otimes I \arrow[ul, "\rho"']
\end{tikzcd}$$

– e_A *and* d_A *are coalgebra morphisms, i.e. the following diagrams commute;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{e}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
!!A & !I \arrow[l, "!\mathsf{e}_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{d}_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[r, "\delta_{!A} \otimes \delta_A"] & !!A \otimes !!A \arrow[d, "\mathsf{q}_{!A,!A}"] \\
!!A \arrow[rr, "!\mathsf{d}_A"] & & !(!A \otimes !A)
\end{tikzcd}$$

– *any coalgebra morphism* $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ *between free coalgebras preserve the comonoid structure given by* e *and* d*, i.e. the following diagrams commute.*

$$\begin{tikzcd}
!A \arrow[rr, "!f"] \arrow[dr, "\mathsf{e}_A"'] & & !B \arrow[dl, "\mathsf{e}_B"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathsf{d}_A"] \arrow[d, "f"'] & !A \otimes !A \arrow[d, "f \otimes f"] \\
!B \arrow[r, "\mathsf{d}_B"] & !B \otimes !B
\end{tikzcd}$$

Definition 13. *A **(modified) linear category with weakening**,* $(\mathcal{L}, w, \mathsf{e})$*, is specified by*

- *a monoidal closed category* $(\mathcal{L}, I, \otimes)$*,*
- *a monoidal comonad* $((w, \mathsf{q}), \varepsilon, \delta)$ *on* $\mathcal{L}$ *with* $\mathsf{q}_{A,B} : wA \otimes wB \longrightarrow w(A \otimes B)$ *and* $\mathsf{q}_I : I \longrightarrow wI$*, and*
- *a monoidal natural transformation* e *on* $\mathcal{L}$ *with components* $\mathsf{e}_A : wA \longrightarrow 1$ *s.t. the following diagrams commute:*

$$\begin{tikzcd}
wA \arrow[r, "\mathsf{e}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
wwA & wI \arrow[l, "w\mathsf{e}_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
wA \arrow[rr, "wf"] \arrow[dr, "\mathsf{e}_A"'] & & wB \arrow[dl, "\mathsf{e}_B"] \\
& I &
\end{tikzcd}$$

Definition 14. *A **(modified) linear category with contraction**,* $(\mathcal{L}, c, \mathsf{d}^1, \mathsf{d}^2)$*, is specified by*

- *a monoidal closed category* $(\mathcal{L}, I, \otimes)$*,*
- *a monoidal comonad* $((c, \mathsf{q}), \varepsilon, \delta)$ *on* $\mathcal{L}$ *with* $\mathsf{q}_{A,B} : cA \otimes cB \longrightarrow c(A \otimes B)$ *and* $\mathsf{q}_I : I \longrightarrow cI$*, and*

- *monoidal natural transformations* d^1 *and* d^2 *on* $\mathcal{L}$ *with components* $\mathsf{d}^1_{A,B} : cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$ *and* $\mathsf{d}^2_{A,B} : B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$*, s.t. the following diagram commutes:*

$$\begin{tikzcd}
cA \otimes I \arrow[d, "\mathsf{d}^1_{A,I}"] & cA \arrow[l, "\rho^{-1}_{cA}"'] \arrow[r, "\lambda^{-1}_{cA}"] & I \otimes cA \arrow[d, "\mathsf{d}^2_{A,I}"] \\
(cA \otimes I) \otimes cA \arrow[rr, "\alpha_{cA,I,cA}"] & & cA \otimes (I \otimes cA)
\end{tikzcd}$$

Definition 15. *Given two comonads* $(c, \varepsilon^c, \delta^c)$ *and* $(w, \varepsilon^w, \delta^w)$ *on a category* $\mathcal{L}$*, define a natural transformation dist with components* $dist_A : cwA \longrightarrow wcA$*, subject to the following coherence diagrams:*

$$\begin{tikzcd}
cwA \arrow[r, "cw(f)"] \arrow[d, "dist_A"'] & cwB \arrow[d, "dist_B"] \\
wcA \arrow[r, "wc(f)"'] & wcB
\end{tikzcd}
\qquad
\begin{tikzcd}
wA & & cwA \arrow[ll, "\varepsilon^c_{wA}"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "w\varepsilon^c_A"] &
\end{tikzcd}$$

Definition 16. *Given a (modified) linear category with weakening* $(\mathcal{L}, w, \mathsf{e}^w)$ *and a (modified) linear category with contraction* $(\mathcal{L}, c, \mathsf{d}^{c1}, \mathsf{d}^{c2})$ *on the same monoidal closed category* $(\mathcal{L}, I, \otimes)$*, where* $((w, \mathsf{q}^w), \varepsilon^w, \delta^w)$ *and* $((c, \mathsf{q}^c), \varepsilon^c, \delta^c)$ *are corresponding comonads on* $\mathcal{L}$*, the composition of c and w is a comonad* $((cw, \mathsf{q}), \varepsilon, \delta)$ *on* $\mathcal{L}$*, where:*

- $\mathsf{q}_{A,B} : cwA \otimes cwB \longrightarrow cw(A \otimes B)$ *is defined as*

$$\begin{aligned}
\mathsf{q}_{A,B} &= c\mathsf{q}^w_{A,B} \circ \mathsf{q}^c_{wA,wB} \\
&= dist^{-1}_{A\otimes B} \circ w\mathsf{q}^c_{A,B} \circ \mathsf{q}^w_{cA,cB} \circ (dist_A \otimes dist_B)
\end{aligned}$$

  i.e. the following diagram commutes by the naturality of $\mathsf{q}^c_{A,B}$ *and* $\mathsf{q}^w_{A,B}$*:*

$$\begin{tikzcd}
cwA \otimes cwB \arrow[r, "\mathsf{q}^c_{wA,wB}"] \arrow[d, "dist_A \otimes dist_B"'] & c(wA \otimes wB) \arrow[r, "c\mathsf{q}^w_{A,B}"] & cw(A \otimes B) \\
wcA \otimes wcB \arrow[r, "\mathsf{q}^w_{cA,cB}"'] & w(cA \otimes cB) \arrow[r, "w\mathsf{q}^c_{A,B}"'] & wc(A \otimes B) \arrow[u, "dist^{-1}_{A\otimes B}"']
\end{tikzcd}$$

- $\mathsf{q}_I : I \longrightarrow cwI$ *is defined as*

$$\begin{aligned}
\mathsf{q}_I &= \mathsf{q}^c_{wI} \circ \mathsf{q}^w_I \\
&= c\mathsf{q}^w_I \circ \mathsf{q}^c_I
\end{aligned}$$

i.e.

$$
\begin{tikzcd}
 & I \arrow[dl, "\mathsf{q}^w_I"'] \arrow[dr, "\mathsf{q}^c_I"] & \\
wI \arrow[r, "\mathsf{q}^c_{wI}"'] & cwI \arrow[u, "\mathsf{e}_I"] & cI \arrow[l, "c\mathsf{q}^w_I"]
\end{tikzcd}
$$

- $\varepsilon_A : cwA \longrightarrow A$ *is defined as:*

$$
\begin{aligned}
\varepsilon_A &= \varepsilon^c_{wA} \circ \varepsilon^w_A \\
&= \varepsilon^c_A \circ c\varepsilon^w_A
\end{aligned}
$$

i.e. the following diagram commutes by the naturality of ε^c*:*

$$
\begin{tikzcd}
cwA \arrow[r, "\varepsilon^c_{wA}"] \arrow[d, "c\varepsilon^w_A"] & wA \arrow[d, "\varepsilon^w_A"] \\
cA \arrow[r, "\varepsilon^c_A"'] & A
\end{tikzcd}
$$

- $\delta_A : cwA \longrightarrow cwcwA$ *is defined as:*

$$
\begin{aligned}
\delta_A &= cdist_{wA} \circ c^2\delta^w_A \circ \delta^c_{wA} \\
&= cdist_{wA} \circ \delta^c_{w^2A} \circ c\delta^w_A
\end{aligned}
$$

i.e. the following diagram commutes by the naturality of δ^c *and equality:*

$$
\begin{tikzcd}
cwA \arrow[r, "c\delta^w_A"] \arrow[d, "\delta^c_{wA}"] & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, "\delta^c_{w^2A}"] & c^2w^2A \arrow[d, "cdist_{wA}"] \\
c^2wA \arrow[r, "c^2\delta^w_A"'] & c^2w^2A \arrow[r, "cdist_{wA}"'] & cwcwA
\end{tikzcd}
$$

- $\mathsf{e}_A : cwA \longrightarrow I$ *is defined as:*

$$
\begin{aligned}
\mathsf{e}_A &= \mathsf{e}^w_{cA} \circ dist_A \\
&= \mathsf{e}^w_A \circ \varepsilon^c_{wA}
\end{aligned}
$$

i.e. the following diagram commutes by the definitions of $dist$ and e^w:

$$\begin{tikzcd} & wcA \arrow[dl, "dist_A"'] \arrow[d, "w\varepsilon^c_A"] \arrow[dr, "\mathsf{e}^w_{cA}"] & \\ cwA \arrow[r, "\varepsilon^c_{wA}"'] & wA \arrow[r, "\mathsf{e}^w_A"'] & I \end{tikzcd}$$

- $\mathsf{d}_A : cwA \longrightarrow cwA \otimes cwA$ *is defined as*

$$\begin{aligned} \mathsf{d}_A &= (\rho_{cwA} \otimes id_{cwA}) \circ \mathsf{d}^{c1}_{wA,I} \circ \rho^{-1}_{cwA} \\ &= (id_{cwA} \otimes \lambda_{cwA}) \circ \mathsf{d}^{c2}_{wA,I} \circ \lambda^{-1}_{cwA} \end{aligned}$$

i.e. the following diagram commutes by the definitions of d^{c1} and d^{c2}:

$$\begin{tikzcd} cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "\lambda^{-1}_{cwA}"'] & cwA \otimes I \arrow[r, "\mathsf{d}^{c1}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[dl, "\alpha_{cwA,I,cwA}"'] \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\ I \otimes cwA \arrow[r, "\mathsf{d}^{c2}_{wA,I}"'] & cwA \otimes (I \otimes cwA) \arrow[r, "id_{cwA} \otimes \lambda_{cwA}"'] & cwA \otimes cwA \end{tikzcd}$$

Theorem 17. *The composition cw of comonads c and w on $\mathcal{L}$ defined in Definition 16 is a comonad on $\mathcal{L}$.*

Proof.

- cw is an endofunctor on $\mathcal{L}$, i.e. it preserves the identity I and compositions of morphisms:

$$\begin{aligned} cwid_A &= cid_{wA} \\ &= id_{cwA} \end{aligned}$$

For any morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ in $\mathcal{L}$,

$$\begin{aligned} cw(f \circ g) &= c(wf \circ wg) \\ &= cwf \circ cwg \end{aligned}$$

- $\mathsf{q}_{A,B} : cwA \otimes cwB \longrightarrow cw(A \otimes B)$ is natural by the naturality of $\mathsf{q}^c_{A,B}$ and $\mathsf{q}^w_{A,B}$:

$$\begin{tikzcd} cwA \otimes cwB \arrow[r, "\mathsf{q}^c_{wA,wB}"] \arrow[d, "cwf \otimes cwg"'] & c(wA \otimes wB) \arrow[r, "c\mathsf{q}^w_{A,B}"] \arrow[d, "c(wf \otimes wg)"] & cw(A \otimes B) \arrow[d, "cw(f \otimes g)"] \\ cwA' \otimes cwB' \arrow[r, "\mathsf{q}^c_{wA',wB'}"'] & c(wA' \otimes wB') \arrow[r, "c\mathsf{q}^w_{A',B'}"'] & cw(A' \otimes B') \end{tikzcd}$$

- $\mathsf{q}_I : I \longrightarrow cwI$ is natural by definition;
- $\varepsilon_A : cwA \longrightarrow A$ is natural by the naturality of ε^c and of ε^w:

$$
\begin{tikzcd}
cwA \arrow[r, "c\varepsilon^c_{wA}"] \arrow[d, "cwf"'] & wA \arrow[r, "\varepsilon^w_A"] \arrow[d, "wf"] & A \arrow[d, "f"] \\
cwB \arrow[r, "c\varepsilon^c_{wB}"'] & wB \arrow[r, "\varepsilon^w_B"'] & B
\end{tikzcd}
$$

- $\delta_A : cwA \longrightarrow cwcwA$ is natural by the naturality of δ^c and of δ^w and the definition of $dist$:

$$
\begin{tikzcd}
cwA \arrow[r, "\delta^c_{wA}"] \arrow[d, "cwf"'] & c^2wA \arrow[r, "c^2\delta^w_A"] \arrow[d, "c^2wf"] & c^2w^2A \arrow[r, "cdist_{wA}"] \arrow[d, "c^2w^2f"] & cwcwA \arrow[d, "cwcwf"] \\
cwB \arrow[r, "\delta^c_{wB}"'] & c^2wB \arrow[r, "c^2\delta^w_A"'] & c^2w^2B \arrow[r, "cdist_{wB}"'] & cwcwB
\end{tikzcd}
$$

- $cw\delta_A \circ \delta_A = \delta_{cwA} \circ \delta_A$, by the definition of δ_A, the naturality of δ^c, and definition of $dist$. The diagram proves $cw\delta_A = \delta_{cwA}$:

$$
\begin{tikzcd}
cwcwA \arrow[rr, "cwc\delta^w_A"] \arrow[dr, "\delta^c_{wcwA}"] \arrow[dd, "c\delta^w_{cwA}"'] & & cwcw^2A \arrow[rr, "cw\delta^c_{w^2A}"] \arrow[d, "\delta^c_{wcw^2A}"] & & cwc^2w^2A \\
& c^2wcwA \arrow[r, "c^2wc\delta^w_A"'] \arrow[dr, "c^2\delta^w_{cwA}"'] & c^2wcw^2A \arrow[urr, "cdist_{cw^2A}"'] & & \\
cw^2cwA \arrow[rr, "\delta^c_{w^2cwA}"'] & & c^2w^2cwA \arrow[u, "c^2wdist_{wA}"'] \arrow[rr, "cdist_{wcwA}"'] & & cwcwcwA \arrow[uu, "cwcdist_{wA}"']
\end{tikzcd}
$$

– $cw\varepsilon_A \circ \delta_A = \varepsilon_{cwA} \circ \delta_A = id_{cwA}$:

$$
\begin{tikzcd}
cwA \arrow[r, "c\delta^w_A"] \arrow[d, "c\delta^w_A"'] \arrow[dddrrr, equal] & cw^2A \arrow[r, "\delta^c_{w^2A}"] & c^2w^2A \arrow[r, "cdist_{wA}"] & cwcwA \arrow[dd, "cwc\varepsilon^w_A"'] \\
cw^2A \arrow[d, "\delta^c_{w^2A}"'] & & & \\
c^2w^2A \arrow[d, "cdist_{wA}"'] & & & cwcA \arrow[d, "cw\varepsilon^c_A"'] \\
cwcwA \arrow[r, "c\delta^w_{cwA}"] & c^2wA \arrow[rr, "\delta^c_{cwA}"] & & cwA
\end{tikzcd}
$$

$\square$

2 Related Work

TODO

3 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.