# Separating Linear Modalities

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Abstract

TODO

## 1 Introduction

TODO [?]

## 2 Categorical Models

## 2.1 Lambek Categories

TODO: Define Lambek Categories

## 2.2 Lambek Categories with Weakening and Contraction

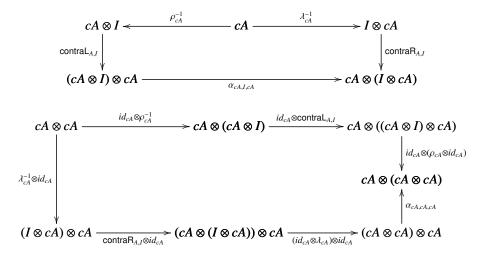
**Definition 1.** A Lambek category with weakening,  $(\mathcal{L}, w, \text{weak})$ , is a Lambek category equipped with a monoidal comonad  $(w, \varepsilon, \delta)$ , and a monoidal natural transformation  $\text{weak}_A : wA \longrightarrow I$ . Furthermore, weak must be a coalgebra morphism. That is, the following digram must commute:

$$wA \xrightarrow{\text{weak}_A} I$$
 $\delta_A \downarrow \qquad \qquad \downarrow q$ 
 $w^2A \xrightarrow{\text{wweak}_A} WI$ 

**Definition 2.** A Lambek category with contraction,  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$ , is a Lambek category equipped with a monoidal comonad  $(c, \varepsilon, \delta)$ , and two monoidal natural transformations:

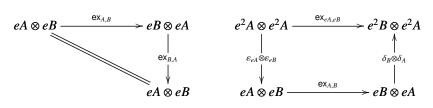
 $\operatorname{contraL}_{A,B}: cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$  $\operatorname{contraR}_{A,B}: B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$ 

Furthermore, the following diagrams must commute:



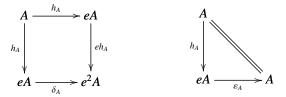
## 2.3 Lambek Categories with Exchange

**Definition 3.** A Lambek category with exchange,  $(\mathcal{L}, e, \Theta X)$ , is a Lambek category equipped with a monoidal comonad  $(e, \varepsilon, \delta)$  on  $\mathcal{L}$ , and a monoidal natural transformation  $\Theta X_{A,B} : eA \otimes eB \longrightarrow eB \otimes eA$ . Furthermore, the following diagrams must commute:



Define coalgebras and their morphisms

**Definition 4.** Suppose  $(\mathcal{L}, e, ex)$  is a Lambek category with exchange. Then the **Eilenberg Moore category**,  $\mathcal{L}^e$ , of the comonad  $(e, \varepsilon, \delta)$  has as objects all the e-coalgebras  $(A, h_A : A \longrightarrow eA)$ , and as morphisms all the coalgebra morphisms. We call  $h_A$  the action of the coalgebra. Furthermore, the following (action) diagrams must commute:

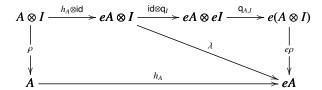


**Lemma 5** (The Eilenberg Moore Category is Symmetric Monoidal). *Suppose* ( $\mathcal{L}$ , e, ex) *is a Lambek category with exchange. Then the category*  $\mathcal{L}^e$  *is symmetric monoidal.* 

*Proof.* We must first define the unitors, the associator, and the symmetry. Then we show that they respect the symmetry monoidal coherence diagrams. Recall the definition of composition in the Eilenberg Moore category. Throughout this proof we will make use of the coalgebra  $(A, h_A)$ ,  $(B, h_B)$ , and  $(C, h_C)$ .

The tensor product of  $(A, h_A)$  and  $(B, h_b)$  is  $(A \otimes B, q_{A,B} \circ (h_A \otimes h_B))$ , and the unit of the tensor product is  $(I, q_I)$ ; both actions are easily shown to satisfies the action diagrams of the Eilenberg Moore category. The left and right unitors are  $\lambda : I \otimes A \longrightarrow A$  and  $\rho : A \otimes I \longrightarrow A$ , because they are indeed coalgebra morphisms.

The respective diagram for the right unitor is as follows:



The top-left diagram commutes by naturality of  $\rho$ , the top-right diagram commutes by the fact that e is a monoidal functor, and the bottom diagram commutes by the action diagrams for the coalgebra  $(A, h_A)$ . Showing the left unitor is a coalgebra morphism is similar.

The unitors are natural and isomorphisms, because they are essentially inherited from the underlying Lambek category.

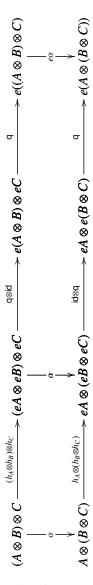
The associator  $\alpha: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$  is also a coalgebra morphism. First, notice that:

$$\mathsf{q}_{A\otimes B,C}\circ ((\mathsf{q}_{A,B}\circ (h_A\otimes h_B))\otimes h_C)=\mathsf{q}_{A\otimes B,C}\circ (\mathsf{q}_{A,B}\otimes \mathsf{id})\circ ((h_A\otimes h_B)\otimes h_C)$$

where the left-hand side is the action of the coalgebra  $(A \otimes B) \otimes C$ . Similarly, the following is the action of the coalgebra  $A \otimes (B \otimes C)$ :

$$\mathsf{q}_{A,B\otimes C}\circ (h_A\otimes (\mathsf{q}_{B,C}\circ (h_B\otimes h_C)))=\mathsf{q}_{A,B\otimes C}\circ (\mathsf{id}\otimes \mathsf{q}_{B,C})\circ (h_A\otimes (h_B\otimes h_C))$$

The following diagram must commute:



The left diagram commutes by naturality of  $\alpha$ , and the right diagram commutes because e is a monoidal functor.

Composition in  $\mathcal{L}^e$  is the same as  $\mathcal{L}$ , and thus, the monoidal coherence diagrams hold in  $\mathcal{L}^e$  as well. Thus,  $\mathcal{L}^e$  is monoidal. We now show that it is symmetric.

The symmetry of  $\mathcal{L}^e$  is defined as follows:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{ex_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

It turns out that  $\beta_{B,A} \circ \beta_{A,B} = \mathrm{id}_{A \otimes B}$  which holds because the following diagram com-

mutes:

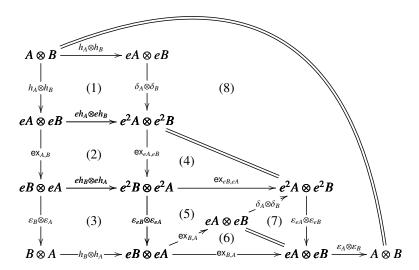
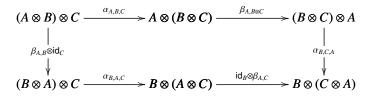


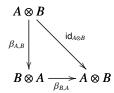
Diagram one commutes by the action diagrams of the Eilenberg Moore category, diagram two commutes by naturality of  $\varepsilon$ , diagram three commutes by naturality of  $\varepsilon$ , diagram four and five commute by the coherence diagrams of  $\varepsilon$ , diagram six clearly commutes, diagram seven commutes because  $(e, \varepsilon, \delta)$  is a comonad, and diagram eight commutes by both the action diagrams of the Eilenberg Moore category and the fact that  $(e, \varepsilon, \delta)$  is a comonad.

At this point we must verify that  $\beta$  respects the coherence diagrams of a symmetric monoidal category; see Definition ??. Thus, we must show that each of the following diagrams hold:

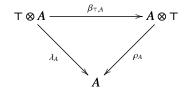
#### Case



Case

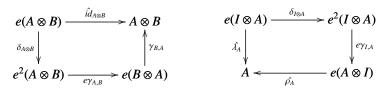


#### Case



**Definition 6.** Let  $(\mathcal{L}, e, \Theta X)$  be a Lambek category with exchange. The **coKleisli Category of** e,  $\mathcal{L}_e$ , is a category with the same objects as  $\mathcal{L}$ . There is an arrow  $\hat{f}: A \longrightarrow B$  in  $\mathcal{L}_e$  if there is an arrow  $f: eA \longrightarrow B$  in  $\mathcal{L}$ . The identity arrow  $i\hat{d}_A: A \longrightarrow A$  is the arrow  $\mathcal{E}_A: eA \longrightarrow A$  in  $\mathcal{L}$ . Given  $\hat{f}: A \longrightarrow B$  and  $\hat{g}: B \longrightarrow C$  in  $\mathcal{L}_e$ , which are arrows  $f: eA \longrightarrow B$  and  $g: eB \longrightarrow C$  in  $\mathcal{L}$ , the composition  $\hat{g} \circ \hat{f}: A \longrightarrow C$  is defined as  $g \circ ef \circ \delta_A$ .

**Lemma 7** ( $\mathcal{L}_e$  is symmetric). Suppose ( $\mathcal{L}, e, ex$ ) is a Lambek category with exchange. Then the co-Kleisli category  $\mathcal{L}_e$  is symmetric monoidal. That is, there is a natural isomorphism  $\gamma_{A,B}: e(A \otimes B) \longrightarrow B \otimes A$ , and the following diagrams commute: |||||||| HEAD



, where  $\hat{id}$  is the identity morphism in  $\mathcal{L}_e$ , I is the identity object of  $\mathcal{L}_e$ , and  $\hat{\lambda}$  and  $\hat{\rho}$  are the left and right unitors of  $\mathcal{L}_e$ , respectively.

Don't know how to express the two diagrams for monoidal. For example, the morphism  $id_A \otimes \lambda_A : A \otimes (I \otimes B) \to A \otimes B$  in  $\mathcal{L}$  is  $e(A \otimes (I \otimes B)) \to A \otimes B$  in  $\mathcal{L}_e$  and can be defined as  $(id_A \otimes \lambda_B) \circ \varepsilon_{A \otimes (I \otimes B)}$ . But I feel like it should use  $\hat{\lambda}$  in the definition.

#### ======

#### Fill in the diagrams

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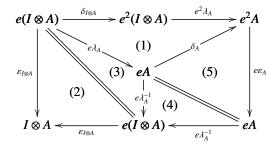
*Proof.* Let I be the identity object in  $\mathcal{L}$ . Then the identity object of  $\mathcal{L}_e$  is still I. The left and right unitors,  $\hat{\lambda}_A : I \otimes A \longrightarrow A$  and  $\hat{\rho}_A : A \otimes I \longrightarrow A$ , in  $\mathcal{L}_e$  are morphisms  $e(I \otimes A) \longrightarrow A$  and  $e(A \otimes I) \longrightarrow A$  in  $\mathcal{L}$ , respectively. Then we define  $\hat{\lambda}$  and  $\hat{\rho}$  as:

$$\hat{\lambda}_A = \varepsilon_A \circ e\lambda_A$$
$$\hat{\rho}_A = \varepsilon_A \circ e\rho_A,$$

where  $\lambda$  and  $\rho$  are the left and right unitors in  $\mathcal{L}$ , respectively. And we define their inverses as:

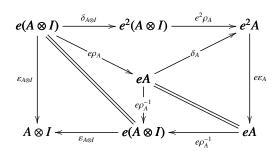
$$\hat{\lambda}_A^{-1} = \varepsilon_{I \otimes A} \circ e \lambda_A^{-1}$$
$$\hat{\rho}_A^{-1} = \varepsilon_{A \otimes I} \circ e \rho_A^{-1}$$

 $\hat{\lambda}$  is a nautral isomorphism with inverse  $\hat{\lambda}^{-1}$  because the following diagram chasing commutes:



(1) commutes by the naturality of  $\delta$ . (2), (3) and (4) commute trivially. And (5) commutes because e is a comonad.

Similarly,  $\hat{\rho}$  is a natural isomorphism with inverse  $\hat{\rho}^{-1}$  by the following diagram chasing:



The associator  $\hat{\alpha}_A : (A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$  in  $\mathcal{L}_e$  is the morphism  $e((A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$  in  $\mathcal{L}$ . We define  $\hat{\alpha}$  as:

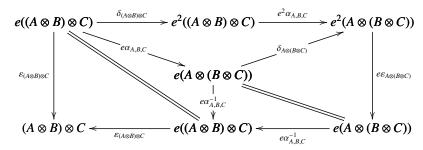
$$\hat{\alpha}_{A,B,C} = \varepsilon_{A\otimes (B\otimes C)} \circ e\alpha_{A,B,C},$$

where  $\alpha$  is the associator of  $\mathcal{L}$ . And its inverse is

$$\hat{\alpha}_{A,B,C}^{-1} = \varepsilon_{(A\otimes B)\otimes C} \circ e\alpha_{A,B,C}^{-1}$$

 $\hat{\alpha}$  is a natural isomorphism with inverse  $\hat{\alpha}^{-1}$  because the following diagram chasing

commutes:



Therefore,  $\mathcal{L}_e$  is a monoidal category.

Then, we define a natural transformation  $\gamma_{A,B}: e(A \otimes B) \longrightarrow B \otimes A$  as

$$\gamma_{A,B} = (\varepsilon_B \otimes \varepsilon_A) \circ \mathsf{ex}_{A,B}.$$

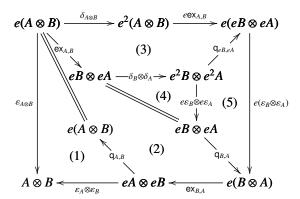
Clearly,  $\gamma$  is natural because it is the composition of natural transformations. The following diagram shows  $\gamma$  is an isomorphism in  $\mathcal{L}_e$ :

$$e(A \otimes B) \xrightarrow{\delta_{A \otimes B}} e^{2}(A \otimes B)$$

$$\downarrow^{\epsilon_{A \otimes B}} \downarrow \qquad \qquad \downarrow^{\epsilon_{Y_{A,B}}}$$

$$A \otimes B \xleftarrow{\gamma_{B,A}} e(B \otimes A)$$

The previous diagram commutes because the following one does:



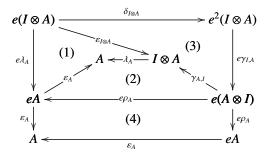
iiiiiii HEAD Diagram (1) commutes because e is monoidal, diagrams (2) and (3) commute by the definition of the Lambek category with exchange, diagram (4) commutes because e is a comonad, and diagram (5) commutes by the naturality of q.

Further, the following diagram also commutes

$$e(I \otimes A) \xrightarrow{\delta_{I \otimes A}} e^{2}(I \otimes A)$$

$$\downarrow e\gamma_{I,A} \\ A \xleftarrow{\rho_{A}} e(A \otimes I)$$

by the diagram chasing below:



in which (1) commutes by the naturality of  $\varepsilon$ , (3) commutes because  $\gamma$  is a natural isomorphism, and (4) commutes trivially.

Need (2)?

In conclusion,  $\mathcal{L}_e$  is symmetric monoidal. ====== Diagram (1) commutes because e is monoidal, diagrams (2) and (3) commute by the definition of the Lambek category with exchange, diagram (4) commutes because e is a comonad, and diagram (5) commutes by the naturality of q.

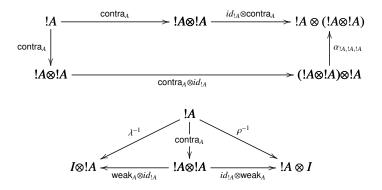
## prove the rest

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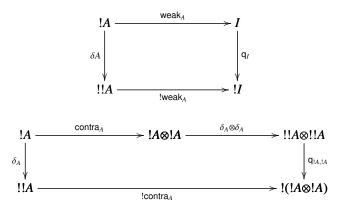
## 2.4 Linear Categories

**Definition 8.** A linear category,  $(\mathcal{L}, !, weak, contra)$ , is a symmetric monoidal closed category  $(\mathcal{L}, I, \otimes, \multimap)$  equipped with a symmetric monoidal comonad  $(!, \varepsilon, \delta)$  with  $q_{A,B}: !A \otimes !B \longrightarrow !(A \otimes B)$  and  $q_I: I \longrightarrow !I$ , and two monoidal natural transformations with components  $weak_A: !A \longrightarrow I$  and  $contra_A: !A \longrightarrow !A \otimes !A$ , satisfying the following conditions:

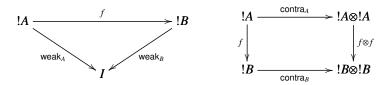
• each (!A, weak<sub>A</sub>, contra<sub>A</sub>) is a commutative comonoid, i.e. the following diagrams commute and  $\beta \circ \text{contra}_A = \text{contra}_A$  where  $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$  is the symmetry natural transformation of  $\mathcal{L}$ ;



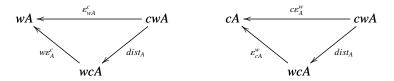
• Weak<sub>A</sub> and contra<sub>A</sub> are coalgebra morphisms, i.e. the following diagrams commute;



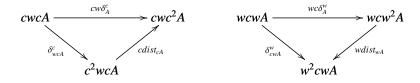
• any coalgebra morphism  $f:(!A,\delta_A) \longrightarrow (!B,\delta_B)$  between free coalgebras preserve the comonoid structure given by Weak and contra, i.e. the following diagrams commute.



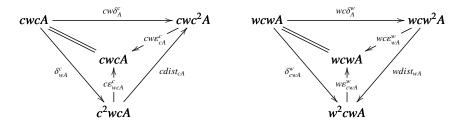
**Definition 9.** Given two comonads  $(c, \varepsilon^c, \delta^c)$  and  $(w, \varepsilon^w, \delta^w)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$  is a Lambek category with contraction and  $(\mathcal{L}, w, \mathsf{weak})$  is a Lambek category with weakening, we define a **distributive law** of c over w to be a natural transformation with components  $dist_A : cwA \longrightarrow wcA$ , subject to the following coherence diagrams:



**Lemma 10.** Given two comonads  $(c, \varepsilon^c, \delta^c)$  and  $(w, \varepsilon^w, \delta^w)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction and  $(\mathcal{L}, w, \text{weak})$  is a Lambek category with weakening, the following two diagrams commute:



*Proof.* IIIIIII HEAD The two diagrams above commute because the following ones commute by the distributive law and the comonad laws for c and w. ====== The two diagrams above commute because the following ones due:  $\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}$  origin/master

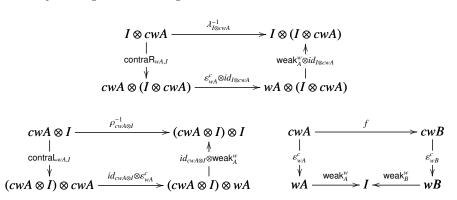


Lemma 11 (Composition of Weakening and Contraction). Suppose

 $(\mathcal{L}, I, \otimes, w, \text{weak}^w, c, \text{contraL}, \text{contraR})$  is a Lambek category with weakening and contraction, where  $(w, \varepsilon^w, \delta^w)$  and  $(c, \varepsilon^c, \delta^c)$  are the respective monoidal comonads. Then the composition of c and w using the distributive law  $dist_A : cwA \longrightarrow wcA$  is a monoidal comonad on  $\mathcal{L}$ .

*Proof.* For the complete proof see Appendix ??.

**Definition 12.** A Lambek category with cw,  $(\mathcal{L}, cw, weak^w, contraL, contraR, dist)$ , is a Lambek category with weakening and contraction, and a distributive law. Furthermore, the following coherence diagrams commute:



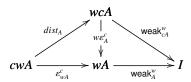
where  $f:(cwA, \delta_A) \longrightarrow (cwB, \delta_B)$  is any coalgebra morphism between free coalgebras.

**Lemma 13.** Let  $(\mathcal{L}, cw, weak^w, contraL, contraR)$  be a Lambek category with cw. Then the following conditions are satisfied:

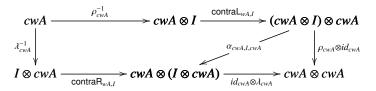
- 1. There exist two natural transformations  $weak_A : cwA \longrightarrow I$  and  $contra_A : cwA \longrightarrow cwA \otimes cwA$ .
- 2. Each (cwA, weak<sub>A</sub>, contra<sub>A</sub>) is a comonoid.

- 3. weak<sub>A</sub> and contra<sub>A</sub> are coalgebra morphisms.
- 4. Any coalgebra morphism  $f:(cwA, \delta_A) \longrightarrow (cwB, \delta_B)$  between free coalgebras preserves the comonoid structure given by weak and contra.

*Proof.* We will only prove the first condition by defining weak and contra. For the complete proof see Appendix ??. Each of weak and contracan be given two equivalent definitions. weak<sub>A</sub>:  $cwA \longrightarrow I$  is defined as in the diagram below. The left triangle commutes by the definition of *dist* and the right triangle commutes by the definition of weak<sup>w</sup>.

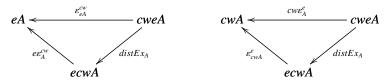


 $\mathsf{contra}_A : \mathit{cwA} \longrightarrow \mathit{cwA} \otimes \mathit{cwA}$  is defined as below. The left part of the diagram commutes by the definitions of  $\mathsf{contraL}$  and of  $\mathsf{contraR}$ , and the right part commutes because  $\mathcal L$  is monoidal.

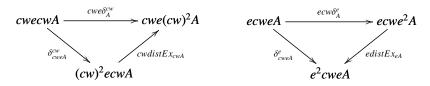


**Definition 14.** Given two comonads  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, cw, weak, contra)$  is a Lambek category with cw and  $(\mathcal{L}, e, ex)$  is a Lambek category with exchange, we define a **distributive law for exchange** of cw over e to be a natural isomorphism with components  $distEx_A$ :  $cweA \longrightarrow ecwA$ , subject to the following coherence diagrams:

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**Lemma 15.** Given two comonads  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^{e}, \delta^{e})$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, cw, \text{weak}, \text{contra})$  is a Lambek category with cw and  $(\mathcal{L}, e, \text{ex})$  is a Lambek category with exchange, the following two digrams also commute:



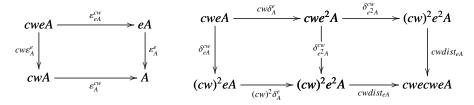
The proof is similar with the proof of Lemma  $\ref{Lemma}$  and we will not elaborate it here. Also, notice the difference between dist of c over w and distEx of cw over e. While dist is a natural transformation, distEx is a natural isomorphism.

**Lemma 16.** let  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  be two monoidal comonads on a Lambek category with cw and exchange  $(\mathcal{L}, I, \otimes, cw, weak, contra, e, ex)$ . Then the composition of cw and e using the distributive law for exchange dist $Ex_A$ :  $cweA \longrightarrow ecwA$  is a monoidal comonad  $(cwe, \varepsilon, \delta)$  on  $\mathcal{L}$ .

*Proof.* Suppose  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  are monoidal comonads, and  $(\mathcal{L}, I, \otimes, cw, \text{weak}, \text{contra}, e, \text{ex})$  is a Lambek category with cw and exchange. Since by definition  $cw, e: \mathcal{L} \longrightarrow \mathcal{L}$  are monoidal functors, we know that their composition  $cwe: \mathcal{L} \longrightarrow \mathcal{L}$  is a monoidal functor:

$$\begin{aligned} \mathsf{q}_{A,B} &: cweA \otimes cweB \longrightarrow cwe(A \otimes B) \\ \mathsf{q}_{A,B} &= cw\mathsf{q}_{A,B}^e \circ \mathsf{q}_{eA,eB}^{cw} \\ \mathsf{q}_I &: I \longrightarrow cweI \\ \mathsf{q}_I &= cw\mathsf{q}_I^e \circ \mathsf{q}_I^{cw} \end{aligned}$$

Analogous to the proof of Lemma ??, each of  $\varepsilon$  and  $\delta$  can be given two equivalent definitions:



And the comonad laws can be proved similarly, which we will not elaborate for simplicity.

**Lemma 17.** Let  $(cwe, \varepsilon, \delta)$  be a monoidal comonad over a monoidal category  $(\mathcal{L}, I, \otimes)$  such that  $(\mathcal{L}, I, \otimes, cw, weak, contra, e, ex)$  is a Lambek category with cw and exchange. Then the co-Kleisli category of  $\mathcal{L}$ ,  $\mathcal{L}_{cwe}$ , is a linear category.

*Proof.* The identity object of  $\mathcal{L}_{cwe}$  is still I.

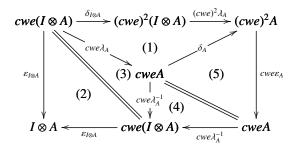
The left and right unitors,  $\hat{\lambda}_A: I \otimes A \longrightarrow A$  and  $\hat{\rho}_A: A \otimes I \longrightarrow A$ , in  $\mathcal{L}_{cwe}$  are morphisms  $cwe(I \otimes A) \longrightarrow A$  and  $cwe(A \otimes I) \longrightarrow A$  in  $\mathcal{L}$ , respectively. Then we define  $\hat{\lambda}$  and  $\hat{\rho}$  as:

$$\hat{\lambda}_A = \varepsilon_A \circ cwe\lambda_A$$
$$\hat{\rho}_A = \varepsilon_A \circ cwe\rho_A,$$

where  $\lambda$  and  $\rho$  are the left and right unitors in  $\mathcal{L}$ , respectively. And we define their inverses as:

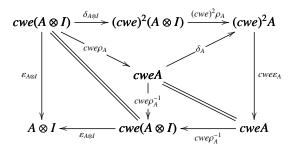
$$\hat{\lambda}_{A}^{-1} = \varepsilon_{I \otimes A} \circ cwe \lambda_{A}^{-1}$$
$$\hat{\rho}_{A}^{-1} = \varepsilon_{A \otimes I} \circ cwe \rho_{A}^{-1}$$

changed to liner category. Finish the proof when lemma 5 is proved.  $\hat{\lambda}$  is a nautral isomorphism with inverse  $\hat{\lambda}^{-1}$  because the following diagram chasing commutes:



(1) commutes by the naturality of  $\delta$ . (2), (3) and (4) commute trivially. And (5) commutes because *cwe* is a comonad.

Similarly,  $\hat{\rho}$  is a natural isomorphism with inverse  $\hat{\rho}^{-1}$  by the following diagram chasing:



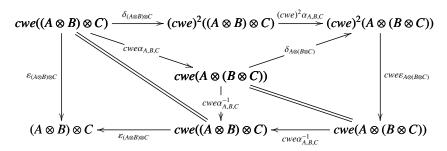
The associator  $\hat{\alpha}_A$ :  $(A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$  in  $\mathcal{L}_{cwe}$  is the morphism  $cwe((A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$  in  $\mathcal{L}$ . We define  $\hat{\alpha}$  as:

$$\hat{\alpha}_{A,B,C} = \varepsilon_{A\otimes(B\otimes C)} \circ cwe\alpha_{A,B,C},$$

where  $\alpha$  is the associator of  $\mathcal{L}$ . And its inverse is

$$\hat{\alpha}_{A,B,C}^{-1} = \varepsilon_{(A \otimes B) \otimes C} \circ cwe\alpha_{A,B,C}^{-1}$$

 $\hat{\alpha}$  is a natural isomorphism with inverse  $\hat{\alpha}^{-1}$  because the following diagram chasing commutes:



Therefore,  $\mathcal{L}_{cwe}$  is a monoidal category.

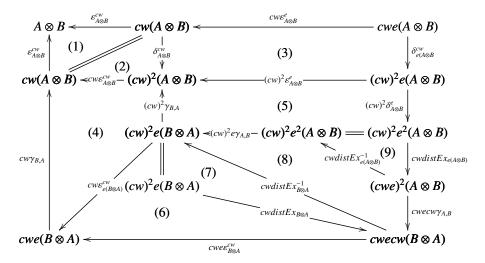
The symmetry,  $\hat{\beta}_{A,B}: A \otimes B \longrightarrow B \otimes A$ , in  $\mathcal{L}_{cwe}$  is the morphism  $cwe(A \otimes B) \longrightarrow B \otimes A$  in  $\mathcal{L}$ , which is defined as:

$$\hat{\beta}_{A,B} = \varepsilon_{B \otimes A}^{cw} \circ cw \gamma_{A,B},$$

where  $\varepsilon_A^{cw}: cwA \longrightarrow A$  is a natural transformation associated with the comonad cw, and  $\gamma$  is the natural isomorphism defined in Lemma ??. Then its inverse is

$$\hat{\beta}_{A,B}^{-1} = \varepsilon_{A \otimes B}^{cw} \circ cw \gamma_{B,A}$$

 $\hat{\beta}$  is a natural isomorphism with inverse  $\hat{\beta}^{-1}$  because the following diagram chasing commutes:



(1), (7) and (9) commute trivially. (2) is the comonad law for cw. (3) commutes by the naturality of  $\delta^{cw}$ . (4) commutes by the naturality of  $\varepsilon^{cw}$ . (5) commutes because  $\gamma$  is a natural isomorphism (Lemma ??). (6) is the definition of distEx. (8) is the naturality of distEx.

In conclusion,  $\mathcal{L}_{cwe}$  is a symmetric monoidal category.

## 3 Related Work

**TODO** 

## 4 Conclusion

**TODO** 

## A Appendix

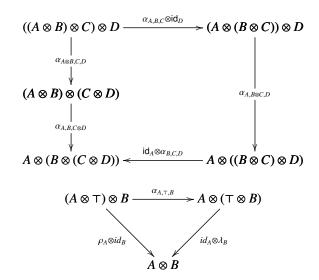
## A.1 Symmetric Monoidal Categories

**Definition 18.** A monoidal category is a category, M, with the following data:

- An object  $\top$  of  $\mathcal{M}$ ,
- A bi-functor  $\otimes$  :  $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ ,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• Subject to the following coherence diagrams:



**Definition 19.** A symmetric monoidal category (SMC) is a category, M, with the following data:

- An object  $\top$  of M,
- A bi-functor  $\otimes$  :  $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ ,
- The following natural isomorphisms:

$$\lambda_A : \top \otimes A \longrightarrow A$$

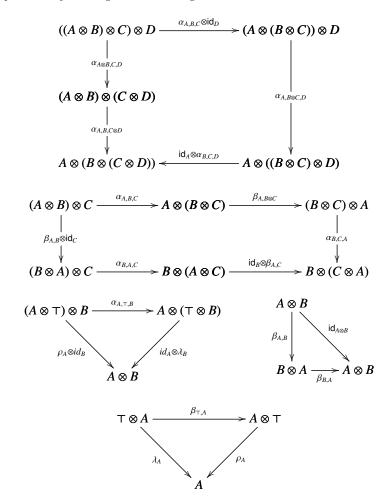
$$\rho_A : A \otimes \top \longrightarrow A$$

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

• A symmetry natural isomorphism:

$$\beta_{A,B}: A \otimes B \longrightarrow B \otimes A$$

• Subject to the following coherence diagrams:



**Definition 20.** A monoidal biclosed category is a monoidal category  $(\mathcal{M}, \top, \otimes)$ , such that, for any object B of  $\mathcal{M}$ , each of the functors  $-\otimes B: \mathcal{M} \longrightarrow \mathcal{M}$  and  $B \otimes -: \mathcal{M} \longrightarrow \mathcal{M}$  has a specified right adjoint. Hence, for any object A and C of  $\mathcal{M}$ , there are two objects  $C \hookrightarrow B$  and  $B \hookrightarrow C$  of  $\mathcal{M}$  and two natural bijections:

$$\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A\otimes B,C)\cong\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A,C\leftharpoonup B)$$
  
 $\operatorname{\mathsf{Hom}}_{\mathcal{M}}(B\otimes A,C)\cong\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A,B\rightharpoonup C)$ 

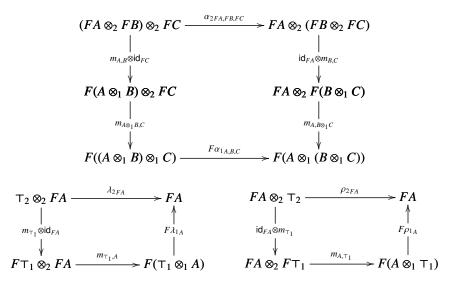
**Definition 21.** A symmetric monoidal closed category (SMCC) is a symmetric monoidal category,  $(\mathcal{M}, \top, \otimes)$ , such that, for any object B of  $\mathcal{M}$ , the functor  $-\otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ 

has a specified right adjoint. Hence, for any objects A and C of M there is an object  $B \multimap C$  of M and a natural bijection:

$$\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A \otimes B, C) \cong \operatorname{\mathsf{Hom}}_{\mathcal{M}}(A, B \multimap C)$$

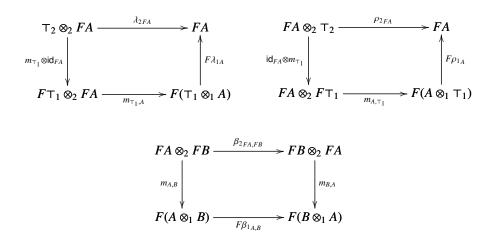
We call the functor  $\multimap$ :  $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$  the internal hom of  $\mathcal{M}$ .

**Definition 22.** Suppose we are given two monoidal categories  $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$ . Then a **monoidal functor** is a functor  $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ , a map  $m_{\top_1} : \top_2 \longrightarrow F \top_1$  and a natural transformation  $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$  subject to the following coherence conditions:

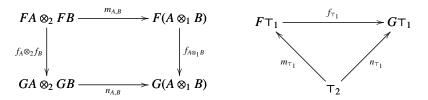


Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

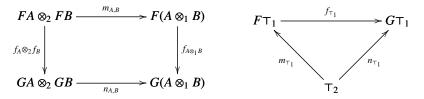
**Definition 23.** Suppose we are given two symmetric monoidal closed categories  $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$ . Then a **symmetric monoidal** functor is a functor  $F: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ , a map  $m_{\top_1} : \top_2 \longrightarrow F \top_1$  and a natural transformation  $m_{A,B}: FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$  subject to the following coherence conditions:



**Definition 24.** Suppose  $(M_1, \top_1, \otimes_1)$  and  $(M_2, \top_2, \otimes_2)$  are monoidal categories, and (F, m) and (G, n) are monoidal functors between  $M_1$  and  $M_2$ . Then a **monoidal natural transformation** is a natural transformation,  $f: F \longrightarrow G$ , subject to the following coherence diagrams:

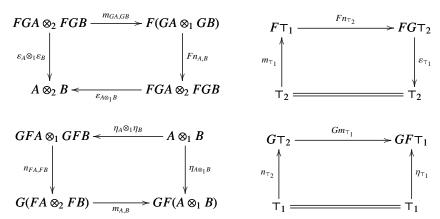


**Definition 25.** Suppose  $(\mathcal{M}_1, \top_1, \otimes_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2)$  are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Then a symmetric monoidal natural transformation is a natural transformation,  $f: F \longrightarrow G$ , subject to the following coherence diagrams:

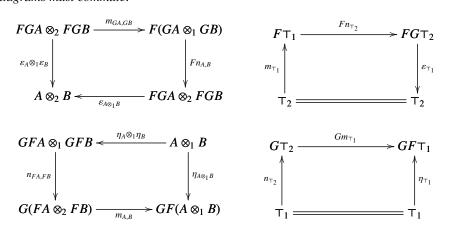


**Definition 26.** Suppose  $(M_1, \top_1, \otimes_1)$  and  $(M_2, \top_2, \otimes_2)$  are monoidal categories, and (F, m) is a monoidal functor between  $M_1$  and  $M_2$  and (G, n) is a monoidal functor between  $M_2$  and  $M_1$ . Then a **monoidal adjunction** is an ordinary adjunction  $M_1$ :  $F \dashv G : M_2$  such that the unit,  $\eta_A : A \to GFA$ , and the counit,  $\varepsilon_A : FGA \to A$ , are

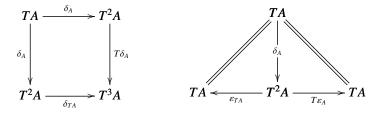
monoidal natural transformations. Thus, the following diagrams must commute:



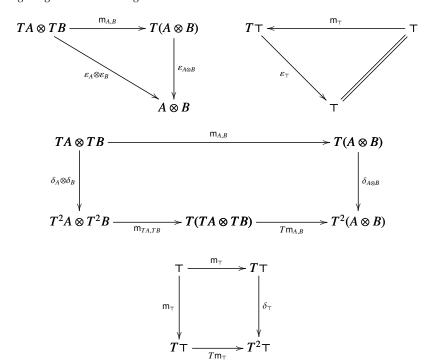
**Definition 27.** Suppose  $(M_1, \top_1, \otimes_1)$  and  $(M_2, \top_2, \otimes_2)$  are SMCs, and (F, m) is a symmetric monoidal functor between  $M_1$  and  $M_2$  and (G, n) is a symmetric monoidal functor between  $M_2$  and  $M_1$ . Then a **symmetric monoidal adjunction** is an ordinary adjunction  $M_1: F \dashv G: M_2$  such that the unit,  $\eta_A: A \to GFA$ , and the counit,  $\varepsilon_A: FGA \to A$ , are symmetric monoidal natural transformations. Thus, the following diagrams must commute:



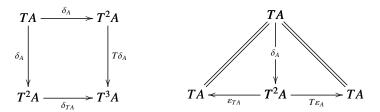
**Definition 28.** A monoidal comonad on a monoidal category C is a triple  $(T, \varepsilon, \delta)$ , where (T, m) is a monoidal endofunctor on C,  $\varepsilon_A : TA \longrightarrow A$  and  $\delta_A : TA \to T^2A$  are monoidal natural transformations, which make the following diagrams commute:



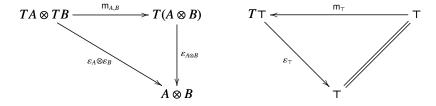
The assumption that  $\varepsilon$  and  $\delta$  are monoidal natural transformations amount to the following diagrams commuting:

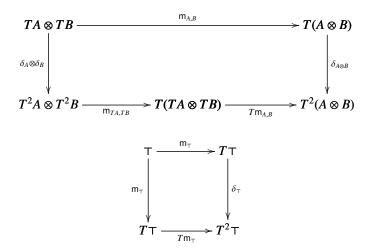


**Definition 29.** A symmetric monoidal comonad on a symmetric monoidal category C is a triple  $(T, \varepsilon, \delta)$ , where (T, m) is a symmetric monoidal endofunctor on C,  $\varepsilon_A$ :  $TA \longrightarrow A$  and  $\delta_A : TA \to T^2A$  are symmetric monoidal natural transformations, which make the following diagrams commute:



The assumption that  $\varepsilon$  and  $\delta$  are symmetric monoidal natural transformations amount to the following diagrams commuting:





## **B** Proofs

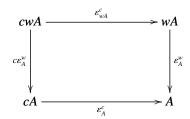
## **B.1** Proof of Composition of Weakening and Contraction (Lemma ??)

Since by definition  $w: \mathcal{L} \longrightarrow \mathcal{L}$  and  $c: \mathcal{L} \longrightarrow \mathcal{L}$  are monoidal functors we know that their composition  $cw: \mathcal{L} \longrightarrow \mathcal{L}$  is a monoidal functor:

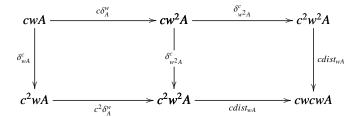
$$\begin{aligned} \mathsf{q}_{A,B} &: cwA \otimes cwB \longrightarrow cw(A \otimes B) \\ \mathsf{q}_{A,B} &= c\mathsf{q}_{A,B}^w \circ \mathsf{q}_{wA,wB}^c \\ \mathsf{q}_I &: I \longrightarrow cwI \\ \mathsf{q}_I &= c\mathsf{q}_I^w \circ \mathsf{q}_I^c \end{aligned}$$

We must now define both  $\varepsilon_A : cwA \longrightarrow A$  and  $\delta_A : cwA \longrightarrow cwcwA$ , and then show that they are monoidal natural transformations subject to the comonad laws. Since we are composing two comonads each of  $\varepsilon$  and  $\delta$  can be given two definitions, but they are equivalent:

•  $\varepsilon_A$ :  $cwA \longrightarrow A$  is defined as in the diagram below, which commutes by the naturality of  $\varepsilon^c$ .



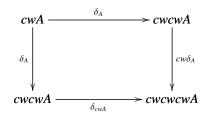
•  $\delta_A : cwA \longrightarrow cwcwA$  is defined as in the diagram:



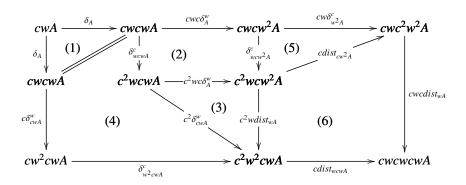
The left part of the diagram commutes by the naturality of  $\delta^c$  and the right part commutes trivially.

The remainder of the proof shows that the comonad laws hold.

#### Case 1:

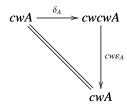


The previous diagram commutes because the following one does.

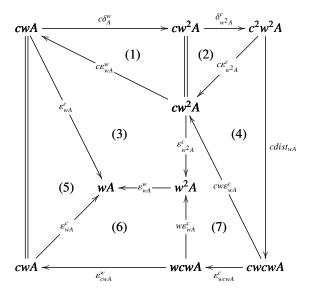


(1) commutes by equality and we will not expand  $\delta_A$  for simplicity. (2) and (4) commutes by the naturality of  $\delta^c$ . (3), (5) commutes by the conditions of *dist*. (6) commutes by the naturality of *dist*.

### Case 2:

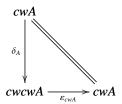


The triangle commutes because of the following diagram chasing.

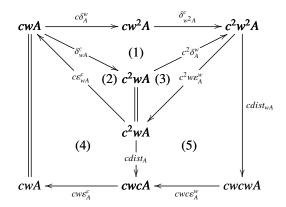


(1) commutes by the comonad law for w with components  $\delta_A^w$  and  $\varepsilon_{wA}^w$ . (2) commutes by the comonad law for c with components  $\delta_{w^2A}^c$  and  $\varepsilon_{w^2A}^c$ . (3) and (7) commute by the naturality of  $\varepsilon^c$ . (4) commutes by the condition of dist. (5) commutes trivially. And (6) commutes by the naturality of  $\varepsilon^w$ .

#### Case 3:



The previous triangle commutes because the following diagram chasing does.



(1) commutes by the naturality of  $\delta^c$ . (2) is the comonad law for c with components  $\delta^c_{wA}$  and  $\varepsilon^c_{wA}$ . (3) is the comonad law for w with components  $\delta^w_A$  and  $\varepsilon^w_A$ . (4) commutes by the condition of dist. And (5) commute by the naturality of dist.

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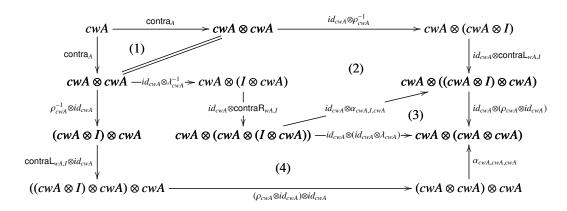
## **B.2** Proof of Conditions of Lambek category with cw (Lemma ??)

- 1. As shown in the paper.
- 2. Each  $(cwA, weak_A, contra_A)$  is a comonoid.

#### Case 1:

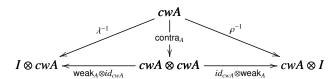


The previous diagram commutes by the following diagram chasing.

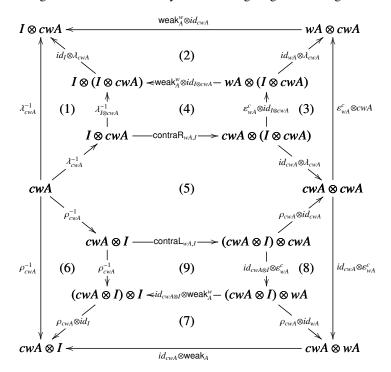


(1) commutes trivially and we would not expand contra for simplicity. (2) and (4) commute because  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction. (3) commutes because  $\mathcal{L}$  is monoidal.

### Case 2:



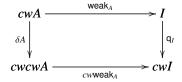
The diagram above commutes by the following diagram chasing.



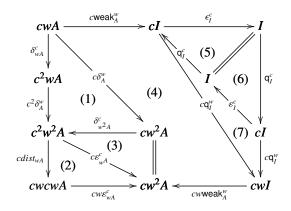
(1), (2) and (3) commute by the functionality of  $\lambda$ . (6), (7) and (8) commute by the functionality of  $\rho$ . (4) and (9) are conditions of the Lambek category with cw. And (5) is the definition of contra.

## 3. weak and contra are coalgebra morphisms.

### Case 1:



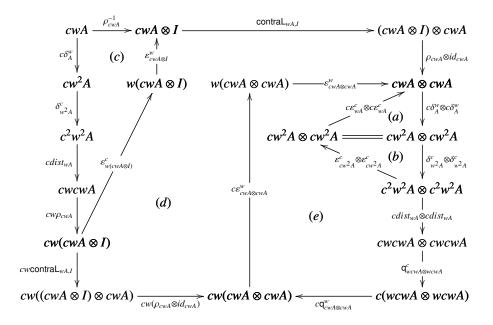
The previous diagram commutes by the diagram below. (1) commutes by the naturality of  $\delta^c$ . (2) commutes by the condition of  $dist_{wA}$ . (3), (5) and (6) commute because c is a monoidal comonad. (4) commutes because  $(\mathcal{L}, w, \mathbf{weak}^w)$  is a Lambek category with weakening. (7) commutes because c and w are monoidal comonads.



### Case 2:

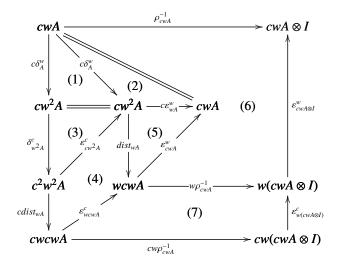


To prove the previous diagram commute, we first expand it, Then we divide it into five parts as shown belovee, and prove each part commutes.



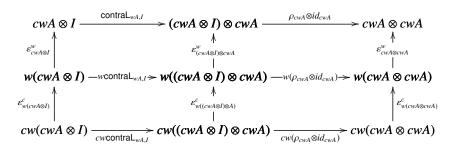
Part (a) and (b) are comonad laws.

Part (c) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for w. (3) is the comonad law for c. (4) commutes by the naturality of  $\varepsilon^c$ . (5) is one of the conditions for  $dist_{wA}$ . (6) commutes by the naturality of  $\varepsilon^w$ . And (7) commutes by the naturality of  $\varepsilon^c$ .

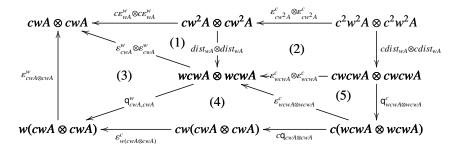


Part (d) commutes by the following diagram chase. The upper two squares both commute by the naturality of  $\varepsilon^w$ , and the lower two squares commute

by the naturality of  $\varepsilon^c$ .

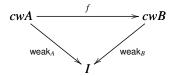


Part (e) commutes by the following diagram. (1) commutes by the condition of  $dist_{wA}$ . (2) and (4) commute by the naturality of  $\varepsilon^c$ . (3) and (5) commute because w and c are monoidal comonads.



4. Any coalgebra morphism  $f:(cwA, \delta_A) \longrightarrow (cwB, \delta_B)$  between free coalgebras preserves the comonoid structure given by weak and contra.

Case 1: This coherence diagram is given in the definition of the Lambek category with cw.



#### Case 2:

$$cwA \xrightarrow{\text{contra}_A} cwA \otimes cwA$$

$$f \downarrow \qquad \qquad \downarrow f \otimes f$$

$$cwB \xrightarrow{\text{contra}_B} cwB \otimes cwB$$

The square commutes by the diagram chasing below, which commutes by the naturality of  $\rho$  and contral.

$$cwA \xrightarrow{\rho_{cwA}^{-1}} cwA \otimes I \xrightarrow{contraL_{wA,I}} (cwA \otimes I) \otimes cwA \xrightarrow{\rho_{cwA} \otimes id_{cwA}} cwA \otimes cwA$$

$$cwf \downarrow cwf \otimes id_{I} \qquad (cwf \otimes id_{I}) \otimes cwf \qquad cwf \otimes cwf$$

$$cwB \xrightarrow{\rho_{cwB}^{-1}} cwB \otimes I \xrightarrow{contraL_{wB,I}} (cwB \otimes I) \otimes cwB \xrightarrow{\rho_{cwB} \otimes id_{cwB}} cwB \otimes cwB$$

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