# Separating Linear Modalities

Jiaming Jiang and Harley Eades III

#### Abstract

TODO

## 1 Introduction

TODO [1]

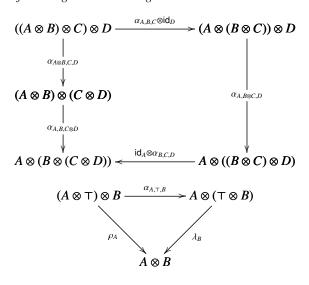
### 1.1 Symmetric Monoidal Categories

**Definition 1** A monoidal category is a category, M, with the following data:

- An object  $\top$  of  $\mathcal{M}$ ,
- A bi-functor  $\otimes$  :  $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ ,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• Subject to the following coherence diagrams:



**Definition 2** A symmetric monoidal category (SMC) is a category, M, with the following data:

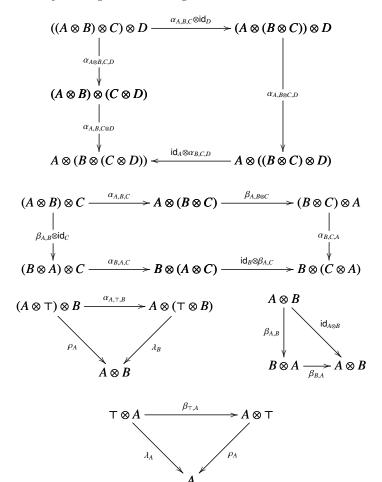
- An object  $\top$  of  $\mathcal{M}$ ,
- A bi-functor  $\otimes$  :  $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ ,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• A symmetry natural transformation:

$$\beta_{A,B}: A \otimes B \longrightarrow B \otimes A$$

• Subject to the following coherence diagrams:

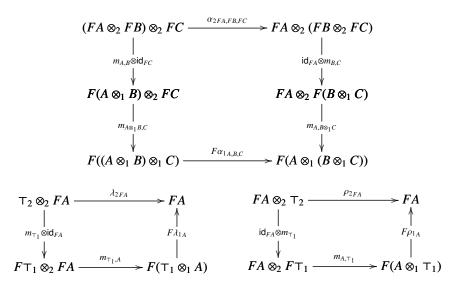


**Definition 3** A symmetric monoidal closed category (SMCC) is a symmetric monoidal category,  $(\mathcal{M}, \top, \otimes)$ , such that, for any object B of M, the functor  $-\otimes B : \mathcal{M} \longrightarrow \mathcal{M}$  has a specified right adjoint. Hence, for any objects A and C of M there is an object  $B \multimap C$  of M and a natural bijection:

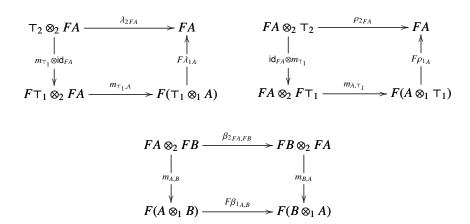
$$\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A \otimes B, C) \cong \operatorname{\mathsf{Hom}}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor  $\multimap: \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$  the internal hom of  $\mathcal{M}$ .

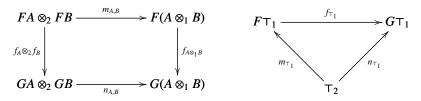
**Definition 4** Suppose we are given two monoidal categories  $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$ . Then a **monoidal functor** is a functor  $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ , a map  $m_{\top_1} : \top_2 \longrightarrow F \top_1$  and a natural transformation  $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$  subject to the following coherence conditions:



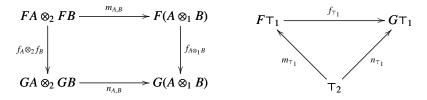
**Definition 5** Suppose we are given two symmetric monoidal closed categories  $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$ . Then a **symmetric monoidal** functor is a functor  $F: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ , a map  $m_{\top_1}: \top_2 \longrightarrow F \top_1$  and a natural transformation  $m_{A,B}: FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$  subject to the following coherence conditions:



**Definition 6** Suppose  $(\mathcal{M}_1, \top_1, \otimes_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2)$  are monoidal categories, and (F, m) and (G, n) are monoidal functors between  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Then a **monoidal natural transformation** is a natural transformation,  $f: F \longrightarrow G$ , subject to the following coherence diagrams:

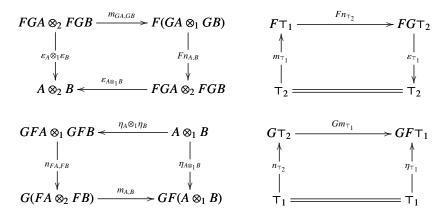


**Definition 7** Suppose  $(\mathcal{M}_1, \top_1, \otimes_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2)$  are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Then a symmetric monoidal natural transformation is a natural transformation,  $f: F \longrightarrow G$ , subject to the following coherence diagrams:

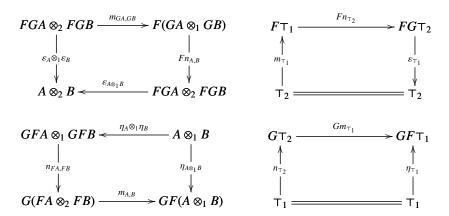


**Definition 8** Suppose  $(\mathcal{M}_1, \top_1, \otimes_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2)$  are monoidal categories, and (F, m) is a monoidal functor between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  and (G, n) is a monoidal functor between  $\mathcal{M}_2$  and  $\mathcal{M}_1$ . Then a **monoidal adjunction** is an ordinary adjunction  $\mathcal{M}_1: F \dashv G: \mathcal{M}_2$  such that the unit,  $\eta_A: A \to GFA$ , and the counit,  $\varepsilon_A: FGA \to A$ ,

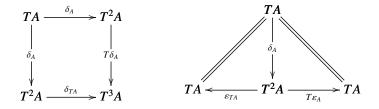
are monoidal natural transformations. Thus, the following diagrams must commute:



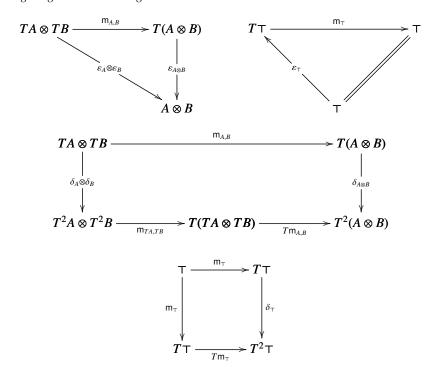
**Definition 9** Suppose  $(\mathcal{M}_1, \top_1, \otimes_1)$  and  $(\mathcal{M}_2, \top_2, \otimes_2)$  are SMCs, and (F, m) is a symmetric monoidal functor between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  and (G, n) is a symmetric monoidal functor between  $\mathcal{M}_2$  and  $\mathcal{M}_1$ . Then a **symmetric monoidal adjunction** is an ordinary adjunction  $\mathcal{M}_1: F \dashv G: \mathcal{M}_2$  such that the unit,  $\eta_A: A \to GFA$ , and the counit,  $\varepsilon_A: FGA \to A$ , are symmetric monoidal natural transformations. Thus, the following diagrams must commute:



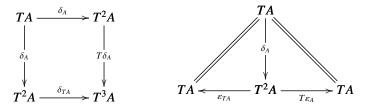
**Definition 10** A monoidal comonad on a monoidal category C is a triple  $(T, \varepsilon, \delta)$ , where  $(T, \mathsf{m})$  is a monoidal endofunctor on C,  $\varepsilon_A : TA \longrightarrow A$  and  $\delta_A : TA \to T^2A$  are monoidal natural transformations, which make the following diagrams commute:



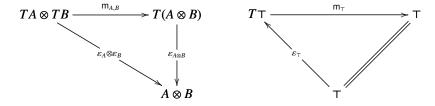
The assumption that  $\varepsilon$  and  $\delta$  are monoidal natural transformations amount to the following diagrams commuting:

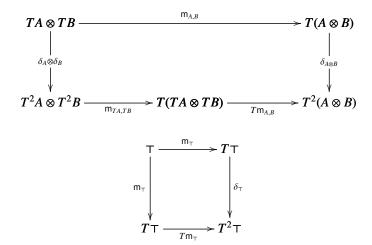


**Definition 11** A symmetric monoidal comonad on a symmetric monoidal category C is a triple  $(T, \varepsilon, \delta)$ , where (T, m) is a symmetric monoidal endofunctor on C,  $\varepsilon_A$ :  $TA \longrightarrow A$  and  $\delta_A : TA \to T^2A$  are symmetric monoidal natural transformations, which make the following diagrams commute:



The assumption that  $\varepsilon$  and  $\delta$  are symmetric monoidal natural transformations amount to the following diagrams commuting:

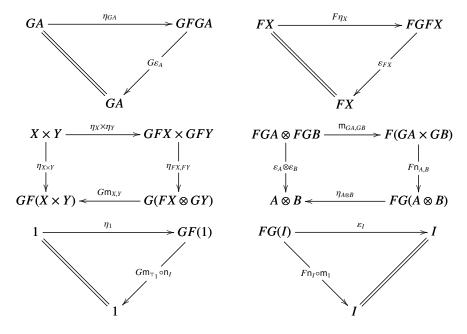




#### 1.2 LNL Model

**Definition 12** A linear/non-linear (LNL) model  $(C, \mathcal{L}, F, G)$  consists of

- a cartesian closed category  $(C, 1, \times, \longrightarrow)$
- a symmetric monoidal closed category  $(\mathcal{L}, I, \otimes, \multimap)$ ,
- a pair of symmetric monoidal functors  $(G, n): \mathcal{L} \longrightarrow \mathcal{C}$  and  $(F, m): \mathcal{C} \longrightarrow \mathcal{L}$  that form a symmetric monoidal adjunction  $\mathcal{C}: F \dashv G: \mathcal{L}$ , subject to the following coherence conditions, where  $\eta$  and  $\varepsilon$  are the unit and counit of the adjunction and  $A, B \in Ob(\mathcal{L}), X, Y \in Ob(\mathcal{C})$ .



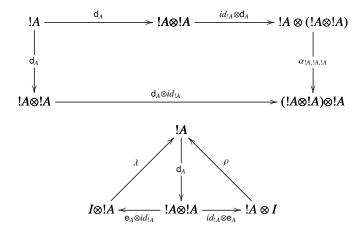
**Lemma 13** Given a LNL model  $(C, \mathcal{L}, F, G)$ , define  $p_{X,Y} : F(X \times Y) \longrightarrow FX \otimes FY$  as the composition  $\varepsilon_{FX \otimes FY} \circ F\eta_{FX,FY} \circ F(\eta_X \times \eta_Y)$ , and  $p_I : F1 \longrightarrow I$  as the composition  $\varepsilon_I \circ Fn_I$ . Then for  $F \dashv G$ ,  $m_{X,Y}$  are components of a natural isomorphism with inverses  $p_{X,Y}$ , and  $m_1$  is an isomorphism with inverse  $p_I$ , i.e.  $F(X) \otimes F(Y) \cong F(X \times Y)$ , and  $I \cong F(1)$ .

**Lemma 14** Given a LNL model  $(C, \mathcal{L}, F, G)$ , the adjunction  $F \dashv G$  induces a symmetric monoidal comonad  $(!, \varepsilon, \delta)$  on  $\mathcal{L}$ , where ! represents FG,  $\varepsilon : FG \longrightarrow 1$  is the counit of the adjunction, and  $\delta : FG \longrightarrow FGFG$ , i.e. ! is a symmetric monoidal functor and  $\varepsilon, \delta$  are monoidal natural transformations.

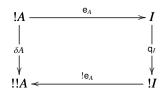
**Lemma 15** Given a LNL model  $(C, \mathcal{L}, F, G)$ , F is a strong functor, i.e. F preserves the monoidal structure up to an isomorphism. And a strong functor induces a unique monoidal structure.

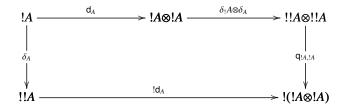
#### **Definition 16** A linear category is specified by

- a symmetric monoidal closed category  $(\mathcal{L}, I, \otimes, \multimap)$ ,
- a symmetric monoidal comonad  $(!, \epsilon, \delta)$  on  $\mathcal{L}$ , with  $q_{A,B} : !A \otimes !A \longrightarrow !(A \otimes B)$  and  $q_I : I \longrightarrow !I$ ;
- monoidal natural transformations on  $\mathcal{L}$  with components  $e_A : !A \longrightarrow I$  and  $d_A : !A \longrightarrow !A \otimes !A$ , s.t.
  - each (!A,  $e_A$ ,  $d_A$ ) is a commutative comonoid, i.e. the following diagrams commute and  $\beta \circ d_A = d_A$  where  $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$  is the symmetry natural transformation of  $\mathcal{L}$ ;

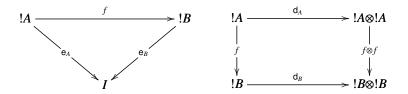


-  $e_A$  and  $d_A$  are coalgebra morphisms, i.e. the following diagrams commute;





- any coalgebra morphism  $f:(!A,\delta_A) \longrightarrow (!B,\delta_B)$  between free coalgebras preserve the comonoid structure given by e and d, i.e. the following diagrams commute.



**Theorem 17** Any LNL model is a linear category.

**Theorem 18** Any linear category gives rise to an LNL model, though it is not in general unique.

## 2 Related Work

**TODO** 

### 3 Conclusion

**TODO** 

### References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at http://research.microsoft.com/en-us/um/people/nick/mixed3.ps.