Composing Monads and Comonads

Jiaming Jiang and Harley Eades III

Abstract

TODO

1 Compose Monads

Definition 1. *A **monad** on a category $\mathcal{C}$ is a triple (T, η, μ), where*

- *T is an endofunctor on $\mathcal{C}$ that associates to each morphism $A \longrightarrow B$ in $\mathcal{C}$ to a morphism $T(A) \longrightarrow T(B)$ in $\mathcal{C}$ s.t. $T(id_A) = id_{TA}$ and $T(f) \circ T(g) = T(f \circ g)$*

- *η is a natural transformation, i.e. $\eta \circ f = T(f) \circ \eta$, or the diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"] & B \arrow[d, "\eta_B"] \\ TA \arrow[r, "T(f)"] & TB \end{tikzcd}$$

- *μ is a natural transformation, i.e. $\mu \circ T^2(f) = T(f) \circ \mu$, or the diagram commutes:*

$$\begin{tikzcd} T^2A \arrow[r, "T^2(f)"] \arrow[d, "\mu_A"] & T^2B \arrow[d, "\mu_B"] \\ TA \arrow[r, "T(f)"] & TB \end{tikzcd}$$

- *$\mu \circ T\mu = \mu \circ \mu T$, i.e. the following diagram commutes:*

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "T"] & T \end{tikzcd}$$

- $\mu \circ \eta T = \mu \circ T\eta = id_T$, *i.e. the following diagrams commute:*

$$\begin{tikzcd} T \arrow[rr, equal] \arrow[dr, "T\eta"'] & & T \\ & T^2 \arrow[ur, "\mu"'] & \end{tikzcd} \qquad \begin{tikzcd} T \arrow[rr, equal] \arrow[dr, "\eta T"'] & & T \\ & T^2 \arrow[ur, "\mu"'] & \end{tikzcd}$$

Definition 2. *[1] Given two monads (M, η^M, μ^M) and (N, η^N, μ^N) on a category C, the* ***swap*** *morphism s associates to NMA for each object A in C an object MNA, i.e. $\mathsf{s}_A : NMA \longrightarrow MNA$, subject to the following conditions:*

- $\mathsf{s} \circ NM(f) = MN(f0 \circ \mathsf{s}$, *i.e.*

$$\begin{tikzcd} NMA \arrow[r, "NM(f)"] \arrow[d, "\mathsf{s}_A"] & NMB \arrow[d, "\mathsf{s}_B"] \\ MNA \arrow[r, "MN(f)"] & MNB \end{tikzcd}$$

- $\mathsf{s} \circ \eta^N M = M\eta^N$, *i.e.*

$$\begin{tikzcd} MA \arrow[rr, "(\eta^N M)_A"] \arrow[dr, "M\eta^N_A"'] & & NMA \arrow[dl, "\mathsf{s}_A"] \\ & MNA & \end{tikzcd}$$

- $\mathsf{s} \circ N\eta^M = M\eta^M N$, *i.e.*

$$\begin{tikzcd} NA \arrow[rr, "N\eta^M_A"] \arrow[dr, "(\eta^M N)_A"'] & & NMA \arrow[dl, "\mathsf{s}_A"] \\ & MNA & \end{tikzcd}$$

Definition 3. *Given two monads (M, η^M, μ^M) and (N, η^N, μ^N) on a category C, the* ***composition monad*** *(T, η, μ) of M and N is defined as follows:*

- *T maps each morphism $f : A \longrightarrow B$ in C to a morphism $T(f) = MN(f) : MNA \longrightarrow MNB$.*

- *$\eta_A : A \longrightarrow MNA$ and $\eta = \eta^M \circ \eta^N$.*

- *$\mu_A : MNMNA \longrightarrow MNA$ and $\mu = (M\mu^N) \circ (\mu^M N^2) \circ (M\mathsf{s}N)$.*

Theorem 4. *The composition monad T of M and N defined above is a monad on C.*

Proof.

- To show $T(id_A) = id_{TA}$:

$$\begin{aligned} T(id_A) &= MN(id_A) \\ &= M(id_{NA}) \\ &= id_{MNA} \\ &= id_{TA} \end{aligned}$$

- To show $T(f) \circ T(g) = T(f \circ g)$:

$$\begin{aligned} T(f) \circ T(g) &= MN(f) \circ MN(g) \\ &= M(N(f) \circ N(f)) \\ &= MN(f \circ g)) \\ &= T(f \circ g) \end{aligned}$$

- To show $\eta \circ f = T(f) \circ \eta$:

$$\begin{aligned} \eta \circ f &= \eta^M \circ \eta^N \circ f \\ &= \eta^M \circ N(f) \circ \eta^N \\ &= MN(f) \circ \eta^M \circ \eta^N \\ &= T(f) \circ \eta \end{aligned}$$

- To show $\mu \circ T^2(f) = T(f) \circ \mu$:

$$\begin{aligned} \mu \circ T^2(f) &= M\mu^N \circ \mu^M N^2 \circ M\mathsf{s}N \circ MNMN(f) \\ &= M\mu^N \circ \mu^M N^2 \circ M^2N^2(f) \circ M\mathsf{s}N \\ &= M\mu^N \circ MN^2(f) \circ \mu^M N^2 \circ M\mathsf{s}N \\ &= MN(f) \circ M\mu^N \circ \mu^M N^2 \circ M\mathsf{s}N \\ &= T(f) \circ \mu \end{aligned}$$

- To show $\mu \circ T\mu = \mu \circ \mu T$:
- To show $\mu \circ \eta T = \mu \circ T\eta$:

□

2 Compose Comonads

Definition 5. *A **comonad** on a category C is a triple (T, ε, δ), where*

- *T is an endofunctor on C that associates to each morphism $A \longrightarrow B$ in C to a morphism $T(A) \longrightarrow T(B)$ in C s.t. $T(id_A) = id_{TA}$ and $T(f) \circ T(g) = T(f \circ g)$*

- ε *is a natural transformation, i.e.* $\varepsilon \circ T(f) = f \circ \varepsilon$*, or the diagram commutes:*

$$\begin{tikzcd} TA \arrow[r, "T(f)"] \arrow[d, "\varepsilon_A"] & TB \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

- δ *is a natural transformation, i.e.* $\delta \circ T(f) = T^2(f) \circ \delta$*, or the diagram commutes:*

$$\begin{tikzcd} TA \arrow[r, "T(f)"] \arrow[d, "\delta_A"] & TB \arrow[d, "\delta_B"] \\ T^2A \arrow[r, "T^2(f)"] & T^2B \end{tikzcd}$$

- $\delta T \circ \delta = T\delta \circ \delta$*, i.e. the following diagram commutes:*

$$\begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"] & T^2 \arrow[d, "\delta T"] \\ T^2 \arrow[r, "T\delta"] & T^3 \end{tikzcd}$$

- $\varepsilon T \circ \delta = T\varepsilon \circ \delta = id_T$*, i.e. the following diagrams commute:*

$$\begin{tikzcd} T \arrow[rr, equal] \arrow[dr, "\delta"'] & & T \\ & T^2 \arrow[ur, "\varepsilon T"'] & \end{tikzcd} \qquad \begin{tikzcd} T \arrow[rr, equal] \arrow[dr, "\delta"'] & & T \\ & T^2 \arrow[ur, "T\varepsilon"'] & \end{tikzcd}$$

Definition 6. *Given two comonads* (M, η^M, μ^M) *and* (N, η^N, μ^N) *on a category* C*, the* ***co-swap*** *morphism* s' *associates to* MNA *for each object* A *in* C *an object* NMA*, i.e.* $\mathsf{s}'_A : MNA \longrightarrow NMA$*, subject to the following conditions:*

- $\mathsf{s}' \circ MN(f) = NM(f) \circ \mathsf{s}'$*, i.e.*

$$\begin{tikzcd} MNA \arrow[r, "MN(f)"] \arrow[d, "\mathsf{s}'_A"] & MNB \arrow[d, "\mathsf{s}'_B"] \\ NMA \arrow[r, "NM(f)"] & NMB \end{tikzcd}$$

- $N\varepsilon^M \circ \mathsf{s}' = \varepsilon^M N$, *i.e.*

$$\begin{tikzcd}
NA & & MNA \arrow[ll, "(\varepsilon^M N)_A"'] \\
& NMA \arrow[ul, "N\varepsilon^M_A"] \arrow[ur, "\mathsf{s}'_A"'] &
\end{tikzcd}$$

- $\mathsf{s}' \circ \varepsilon^N M = M\varepsilon^N$, *i.e.*

$$\begin{tikzcd}
MA & & MNA \arrow[ll, "M\varepsilon^N_A"'] \\
& NMA \arrow[ul, "(\varepsilon^N M)_A"] \arrow[ur, "\mathsf{s}'_A"'] &
\end{tikzcd}$$

Definition 7. *Given two comonads $(M, \varepsilon^M, \delta^M)$ and $(N, \varepsilon^N, \delta^N)$ on a category $\mathcal{C}$, the **composition comonad** (T, ε, δ) of M and N is defined as follows:*

- *T maps each morphism $f : A \longrightarrow B$ in $\mathcal{C}$ to a morphism $T(f) = MN(f) : MNA \longrightarrow MNB$.*
- *$\varepsilon_A : MNA \longrightarrow A$ and $\varepsilon = \varepsilon^N \circ \varepsilon^M$.*
- *$\delta_A : MNA \longrightarrow MNMNA$ and $\delta = (Ms'N) \circ (M^2\delta^N) \circ (\delta^M N)$.*

3 related work

todo

4 conclusion

todo

References

[1] Mark P Jones and Luc Duponcheel. Composing monads. Technical report, Citeseer, 1993.