1 NA Logic

...

2 AN Logic

...

3 AR Logic

...

4 AL Logic

...

5 ANA Logic

5.1 Examples

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Var}}
\UnaryInfC{$X \vdash_{\mathcal{A}} X$}
\AxiomC{}
\RightLabel{\textsc{Var}}
\UnaryInfC{$Y \vdash_{\mathcal{A}} Y$}
\RightLabel{\textsc{Tr}}
\BinaryInfC{$X, Y \vdash_{\mathcal{A}} X \trianglerighteq Y$}
\RightLabel{\textsc{Gr}}
\UnaryInfC{$X, Y \vdash_{\mathcal{L}} \mathsf{G}\,(X \trianglerighteq Y)$}
\RightLabel{\textsc{Gl}}
\UnaryInfC{$\mathsf{G}\,X, Y \vdash_{\mathcal{L}} \mathsf{G}\,(X \trianglerighteq Y)$}
\RightLabel{\textsc{Gl}}
\UnaryInfC{$\mathsf{G}\,X, \mathsf{G}\,Y \vdash_{\mathcal{L}} \mathsf{G}\,(X \trianglerighteq Y)$}
\RightLabel{TL}
\UnaryInfC{$\mathsf{G}\,X \triangleright \mathsf{G}\,Y \vdash_{\mathcal{L}} \mathsf{G}\,(X \trianglerighteq Y)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textsc{Var}}
\UnaryInfC{$X \vdash_{\mathcal{A}} X$}
\RightLabel{\textsc{Gr}}
\UnaryInfC{$X \vdash_{\mathcal{L}} \mathsf{G}\,X$}
\AxiomC{}
\RightLabel{\textsc{Var}}
\UnaryInfC{$Y \vdash_{\mathcal{A}} Y$}
\RightLabel{\textsc{Gr}}
\UnaryInfC{$Y \vdash_{\mathcal{L}} \mathsf{G}\,Y$}
\RightLabel{\textsc{Tr}}
\BinaryInfC{$X, Y \vdash_{\mathcal{L}} \mathsf{G}\,X \triangleright \mathsf{G}\,Y$}
\RightLabel{\textsc{Tl}}
\UnaryInfC{$X \trianglerighteq Y \vdash_{\mathcal{L}} \mathsf{G}\,X \triangleright \mathsf{G}\,Y$}
\RightLabel{\textsc{Gl}}
\UnaryInfC{$\mathsf{G}\,(X \trianglerighteq Y) \vdash_{\mathcal{L}} \mathsf{G}\,X \triangleright \mathsf{G}\,Y$}
\end{prooftree}
$$

Using the above two proofs with cut we can show that:

$$
\begin{gathered}
(\mathsf{G}\,\mathsf{F}\,A \triangleright \mathsf{G}\,\mathsf{F}\,B) \triangleright \mathsf{G}\,\mathsf{F}\,C \vdash_{\mathcal{L}} \mathsf{G}\,\mathsf{F}\,A \triangleright (\mathsf{G}\,\mathsf{F}\,B \triangleright \mathsf{G}\,\mathsf{F}\,C) \\
\text{if and only if} \\
\mathsf{G}\,((\mathsf{F}\,A \trianglerighteq \mathsf{F}\,B) \trianglerighteq \mathsf{F}\,C) \vdash_{\mathcal{L}} \mathsf{G}\,(\mathsf{F}\,A \trianglerighteq (\mathsf{F}\,B \trianglerighteq \mathsf{F}\,C)) \\
\text{if} \\
(\mathsf{F}\,A \trianglerighteq \mathsf{F}\,B) \trianglerighteq \mathsf{F}\,C \vdash_{\mathcal{A}} \mathsf{F}\,A \trianglerighteq (\mathsf{F}\,B \trianglerighteq \mathsf{F}\,C)
\end{gathered}
$$

Similarly, we have the following:

$$
\begin{gathered}
\mathsf{G}\,\mathsf{F}\,A \triangleright (\mathsf{G}\,\mathsf{F}\,B \triangleright \mathsf{G}\,\mathsf{F}\,C) \vdash_{\mathcal{L}} (\mathsf{G}\,\mathsf{F}\,A \triangleright \mathsf{G}\,\mathsf{F}\,B) \triangleright \mathsf{G}\,\mathsf{F}\,C \\
\text{if and only if} \\
\mathsf{G}\,(\mathsf{F}\,A \trianglerighteq (\mathsf{F}\,B \trianglerighteq \mathsf{F}\,C)) \vdash_{\mathcal{L}} \mathsf{G}\,((\mathsf{F}\,A \trianglerighteq \mathsf{F}\,B) \trianglerighteq \mathsf{F}\,C) \\
\text{if} \\
\mathsf{F}\,A \trianglerighteq (\mathsf{F}\,B \trianglerighteq \mathsf{F}\,C) \vdash_{\mathcal{A}} (\mathsf{F}\,A \trianglerighteq \mathsf{F}\,B) \trianglerighteq \mathsf{F}\,C
\end{gathered}
$$

5.2 Cut Elimination

...

Appendix

A Full Specification of NA Logic

$vars,\ n,\ a,\ x,\ y,\ z,\ w,\ m,\ o$			
$ivar,\ i,\ k,\ j,\ l$			
$const,\ b$			
$A,\ B,\ C$	$::=$		
	$\mid$	J	Unit
	$\mid$	$A \rhd B$	Non-associative Non-commutative tensor
	$\mid$	$A \rightharpoonup B$	Implication
Γ	$::=$		
	$\mid$	$\cdot$	
	$\mid$	A	
	$\mid$	Γ_1, Γ_2	
	$\mid$	(Γ)	
	$\mid$	Γ	

$\boxed{\Gamma \vdash_{\mathcal{L}} A}$

$$\frac{}{A \vdash_{\mathcal{L}} A}\ \textsc{L\_var}$$

$$\frac{}{\cdot \vdash_{\mathcal{L}} J}\ \textsc{L\_JR}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A}{\Gamma, J \vdash_{\mathcal{L}} A}\ \textsc{L\_JL}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad C \vdash_{\mathcal{L}} B}{\Gamma, C \vdash_{\mathcal{L}} A \rhd B}\ \textsc{L\_TR}$$

$$\frac{A, B, \Gamma \vdash_{\mathcal{L}} C}{A \rhd B, \Gamma \vdash_{\mathcal{L}} C}\ \textsc{L\_TL}$$

$$\frac{\Gamma, A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} A \rightharpoonup B}\ \textsc{L\_IRR}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad B \vdash_{\mathcal{L}} C}{\Gamma, A \rightharpoonup B \vdash_{\mathcal{L}} C}\ \textsc{L\_IRL}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} B}\ \textsc{L\_CutL}$$

$$\frac{C \vdash_{\mathcal{L}} A \quad \Gamma, A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} B}\ \textsc{L\_CutR}$$

B Full Specification of AN Logic

$vars$, n, a, x, y, z, w, m, o
$ivar$, i, k, j, l
$const$, b

$A,\ B,\ C$	$::=$		
	$\mid$	J	Unit
	$\mid$	$A \rhd B$	Non-associative Non-commutative tensor
	$\mid$	$B \leftharpoonup A$	Implication

Γ	$::=$	
	$\mid$	$\cdot$
	$\mid$	A
	$\mid$	Γ_1, Γ_2
	$\mid$	(Γ)
	$\mid$	Γ

$\boxed{\Gamma \vdash_{\mathcal{L}} A}$

$$\frac{}{A \vdash_{\mathcal{L}} A}\ \textsc{L\_var}$$

$$\frac{}{\cdot \vdash_{\mathcal{L}} J}\ \textsc{L\_JR}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A}{J, \Gamma \vdash_{\mathcal{L}} A}\ \textsc{L\_JL}$$

$$\frac{C \vdash_{\mathcal{L}} A \quad \Gamma \vdash_{\mathcal{L}} B}{C, \Gamma \vdash_{\mathcal{L}} A \rhd B}\ \textsc{L\_TR}$$

$$\frac{\Gamma, A, B \vdash_{\mathcal{L}} C}{\Gamma, A \rhd B \vdash_{\mathcal{L}} C}\ \textsc{L\_TL}$$

$$\frac{A, \Gamma \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} B \leftharpoonup A}\ \textsc{L\_IRR}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad B \vdash_{\mathcal{L}} C}{B \leftharpoonup A, \Gamma \vdash_{\mathcal{L}} C}\ \textsc{L\_IRL}$$

$$\frac{\Gamma_2 \vdash_{\mathcal{L}} A \quad \Gamma_1, A \vdash_{\mathcal{L}} B \quad |\Gamma_2| > 1}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} B}\ \textsc{L\_CutL}$$

$$\frac{\Gamma_2 \vdash_{\mathcal{L}} A \quad \Gamma_1, A, \Gamma_3 \vdash_{\mathcal{L}} B \quad |\Gamma_2| \leq 1}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\mathcal{L}} B}\ \textsc{L\_CutR}$$

C Full Specification of AR Logic

$vars$, n, a, x, y, z, w, m, o
$ivar$, i, k, j, l
$const$, b

$A,\ B,\ C$	$::=$		
	$\mid$	J	Unit
	$\mid$	$A \rhd B$	Non-associative Non-commutative tensor
	$\mid$	$B \leftharpoonup A$	Implication

Γ	$::=$	
	$\mid$	$\cdot$
	$\mid$	A

| | Γ_1, Γ_2
| | (Γ)
| | Γ

$\boxed{\Gamma \vdash_{\mathcal{L}} A}$

$$\frac{}{A \vdash_{\mathcal{L}} A} \quad \text{L\_VAR}$$

$$\frac{}{\cdot \vdash_{\mathcal{L}} J} \quad \text{L\_JR}$$

$$\frac{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} A}{\Gamma_1, J, \Gamma_2 \vdash_{\mathcal{L}} A} \quad \text{L\_JL}$$

$$\frac{\Gamma_1 \vdash_{\mathcal{L}} A \quad \Gamma_2 \vdash_{\mathcal{L}} B}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} A \rhd B} \quad \text{L\_TR}$$

$$\frac{\Gamma, A, B \vdash_{\mathcal{L}} C}{\Gamma, A \rhd B \vdash_{\mathcal{L}} C} \quad \text{L\_TL}$$

$$\frac{A, \Gamma \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} B \leftharpoondown A} \quad \text{L\_IRR}$$

$$\frac{\Gamma_2 \vdash_{\mathcal{L}} A \quad \Gamma_1, B, \Gamma_3 \vdash_{\mathcal{L}} C}{\Gamma_1, B \leftharpoondown A, \Gamma_2, \Gamma_3 \vdash_{\mathcal{L}} C} \quad \text{L\_IRL}$$

$$\frac{\Gamma_2 \vdash_{\mathcal{L}} A \quad \Gamma_1, A, \Gamma_3 \vdash_{\mathcal{L}} B}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\mathcal{L}} B} \quad \text{L\_CUT}$$

D Full Specification of AL Logic

vars, n, a, x, y, z, w, m, o
ivar, i, k, j, l
const, b

A, B, C	::=		
	|	J	Unit
	|	$A \rhd B$	Non-associative Non-commutative tensor
	|	$A \rightharpoonup B$	Implication

Γ	::=	
	|	$\cdot$
	|	A
	|	Γ_1, Γ_2
	|	(Γ)
	|	Γ

$\boxed{\Gamma \vdash_{\mathcal{L}} A}$

$$\frac{}{A \vdash_{\mathcal{L}} A} \quad \text{L\_VAR}$$

$$\frac{}{\cdot \vdash_{\mathcal{L}} J} \quad \text{L\_JR}$$

$$\frac{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} A}{\Gamma_1, J, \Gamma_2 \vdash_{\mathcal{L}} A} \quad \text{L\_JL}$$

$$\frac{\Gamma_1 \vdash_{\mathcal{L}} A \quad \Gamma_2 \vdash_{\mathcal{L}} B}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} A \rhd B} \quad \text{L\_TR}$$

$$\frac{A, B, \Gamma \vdash_{\mathcal{L}} C}{\Gamma, A \rhd B \vdash_{\mathcal{L}} C} \quad \text{L\_TL}$$

$$\frac{\Gamma, A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} A \rightharpoonup B} \quad \text{L\_IRR}$$

$$\frac{\Gamma_2 \vdash_{\mathcal{L}} A \quad \Gamma_1, B, \Gamma_3 \vdash_{\mathcal{L}} C}{\Gamma_1, A \rightharpoonup B, \Gamma_2, \Gamma_3 \vdash_{\mathcal{L}} C} \quad \text{L\_IRL}$$

$$\frac{\Gamma_2 \vdash_{\mathcal{L}} A \quad \Gamma_1, A, \Gamma_3 \vdash_{\mathcal{L}} B}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\mathcal{L}} B} \quad \text{L\_CUT}$$

E Full Specification of ANA Logic

vars, n, a, x, y, z, w, m, o
ivar, i, k, j, l
const, b

$X,\ Y,\ Z$	::=		
	$\mid$	I	Unit
	$\mid$	$X \trianglerighteq Y$	Associative Non-commutative tensor
	$\mid$	$X \rightharpoonup Y$	Implication
	$\mid$	$\mathsf{F}\,A$	Right adjoint
$A,\ B,\ C$	::=		
	$\mid$	J	Unit
	$\mid$	$A \rhd B$	Non-associative Non-commutative tensor
	$\mid$	$A \rightharpoonup B$	Implication
	$\mid$	$\mathsf{G}\,X$	Right adjoint
Γ	::=		
	$\mid$	$\cdot$	
	$\mid$	A	
	$\mid$	X	
	$\mid$	Γ_1, Γ_2	
	$\mid$	(Γ)	
	$\mid$	Γ	
Δ	::=		
	$\mid$	$\cdot$	
	$\mid$	X	
	$\mid$	Δ_1, Δ_2	
	$\mid$	(Δ)	
	$\mid$	Δ	

$\boxed{\Delta \vdash_{\mathcal{A}} X}$

$$\frac{}{X \vdash_{\mathcal{A}} X} \quad \text{A\_VAR}$$

$$\frac{}{\cdot \vdash_{\mathcal{A}} I} \quad \text{A\_IR}$$

$$\frac{\Delta \vdash_{\mathcal{A}} X}{\Delta, I \vdash_{\mathcal{A}} X} \quad \text{A\_IL}$$

$$\frac{\Delta_1 \vdash_{\mathcal{A}} X \quad \Delta_2 \vdash_{\mathcal{A}} Y}{\Delta_1, \Delta_2 \vdash_{\mathcal{A}} X \trianglerighteq Y} \quad \text{A\_TR}$$

$$\frac{\Delta_1, X, Y, \Delta_2 \vdash_{\mathcal{A}} Z}{\Delta_1, X \trianglerighteq Y, \Delta_2 \vdash_{\mathcal{A}} Z} \quad \text{A\_TL}$$

$$\frac{\Delta, X \vdash_{\mathcal{A}} Y}{\Delta \vdash_{\mathcal{A}} X \rightharpoonup Y} \quad \text{A\_IRR}$$

$$\frac{\Delta_2 \vdash_{\mathcal{A}} X \quad \Delta_1, Y, \Delta_3 \vdash_{\mathcal{A}} Z}{\Delta_1, X \rightharpoonup Y, \Delta_2, \Delta_3 \vdash_{\mathcal{A}} Z} \quad \text{A\_IRL}$$

$$\frac{\Delta_2 \vdash_{\mathcal{A}} X \quad \Delta_1, X, \Delta_3 \vdash_{\mathcal{A}} Y}{\Delta_1, \Delta_2, \Delta_3 \vdash_{\mathcal{A}} Y} \quad \text{A\_CUT}$$

$$\frac{\Delta \vdash_{\mathcal{L}} A}{\Delta \vdash_{\mathcal{A}} \mathsf{F}\, A} \quad \text{A\_FR}$$

$\boxed{\Gamma \vdash_{\mathcal{L}} A}$

$$\frac{}{A \vdash_{\mathcal{L}} A} \quad \text{L\_VAR}$$

$$\frac{}{\cdot \vdash_{\mathcal{L}} J} \quad \text{L\_JR}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A}{\Gamma, J \vdash_{\mathcal{L}} A} \quad \text{L\_JL}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A}{\Gamma, I \vdash_{\mathcal{L}} A} \quad \text{L\_IL}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad C \vdash_{\mathcal{L}} B}{\Gamma, C \vdash_{\mathcal{L}} A \triangleright B} \quad \text{L\_TR}$$

$$\frac{A, B \vdash_{\mathcal{L}} C}{A \triangleright B \vdash_{\mathcal{L}} C} \quad \text{L\_TL}$$

$$\frac{\Delta_1, X, Y, \Delta_2 \vdash_{\mathcal{L}} C}{\Delta_1, X \trianglerighteq Y, \Delta_2 \vdash_{\mathcal{L}} C} \quad \text{L\_ATL}$$

$$\frac{\Gamma, A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} A \rightharpoonup B} \quad \text{L\_IRR}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad B \vdash_{\mathcal{L}} C}{\Gamma, A \rightharpoonup B \vdash_{\mathcal{L}} C} \quad \text{L\_IRL}$$

$$\frac{\Delta_2 \vdash_{\mathcal{A}} X \quad \Delta_1, X, \Delta_3 \vdash_{\mathcal{L}} A}{\Delta_1, X \rightharpoonup Y, \Delta_2, \Delta_3 \vdash_{\mathcal{L}} A} \quad \text{L\_AIRL}$$

$$\frac{\Delta_2 \vdash_{\mathcal{A}} X \quad \Delta_1, X, \Delta_3 \vdash_{\mathcal{L}} A}{\Delta_1, \Delta_2, \Delta_3 \vdash_{\mathcal{L}} A} \quad \text{L\_ACUT}$$

$$\frac{\Gamma \vdash_{\mathcal{L}} A \quad A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} B} \quad \text{L\_CUT}$$

$$\frac{\Delta \vdash_{\mathcal{A}} X}{\Delta \vdash_{\mathcal{L}} \mathsf{G}\, X} \quad \text{L\_GR}$$

$$\frac{\Gamma_1, X, \Gamma_2 \vdash_{\mathcal{L}} A}{\Gamma_1, \mathsf{G}\, X, \Gamma_2 \vdash_{\mathcal{L}} A} \quad \text{L\_G}\textsc{l}$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash_{\mathcal{L}} B}{\Gamma_1, \mathsf{F}\, A, \Gamma_2 \vdash_{\mathcal{L}} B} \quad \text{L\_F}\textsc{l}$$