Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

2 Categorical Models

2.1 Lambek Categories

TODO: Define Lambek Categories

2.2 Lambek Categories with Weakening and Contraction

Definition 1. *A **Lambek category with weakening**, $(\mathcal{L}, w, \mathsf{weak})$, is a Lambek category equipped with a monoidal comonad (w, ε, δ), and a monoidal natural transformation* $\mathsf{weak}_A : wA \longrightarrow I$*. Furthermore,* weak *must be a coalgebra morphism. That is, the following digram must commute:*

$$\begin{tikzcd}
wA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta_A"'] & I \arrow[d, "\mathsf{q}_I"] \\
w^2A \arrow[r, "w\mathsf{weak}_A"'] & wI
\end{tikzcd}$$

Definition 2. *A **Lambek category with contraction**, $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$, is a Lambek category equipped with a monoidal comonad (c, ε, δ), and two monoidal natural transformations:*

$$\begin{aligned}
\mathsf{contraL}_{A,B} &: cA \otimes B \longrightarrow (cA \otimes B) \otimes cA \\
\mathsf{contraR}_{A,B} &: B \otimes cA \longrightarrow cA \otimes (B \otimes cA)
\end{aligned}$$

Furthermore, the following diagrams must commute:

$$\begin{tikzcd}
cA \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & cA \arrow[l, "\rho^{-1}_{cA}"'] \arrow[r, "\lambda^{-1}_{cA}"] & I \otimes cA \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(cA \otimes I) \otimes cA \arrow[rr, "\alpha_{cA,I,cA}"'] & & cA \otimes (I \otimes cA)
\end{tikzcd}$$

$$\begin{tikzcd}
cA \otimes cA \arrow[r, "id_{cA}\otimes\rho^{-1}_{cA}"] \arrow[dd, "\lambda^{-1}_{cA}\otimes id_{cA}"'] & cA \otimes (cA \otimes I) \arrow[r, "id_{cA}\otimes\mathsf{contraL}_{A,I}"] & cA \otimes ((cA \otimes I) \otimes cA) \arrow[d, "id_{cA}\otimes(\rho_{cA}\otimes id_{cA})"] \\
 & & cA \otimes (cA \otimes cA) \\
(I \otimes cA) \otimes cA \arrow[r, "\mathsf{contraR}_{A,I}\otimes id_{cA}"'] & (cA \otimes (I \otimes cA)) \otimes cA \arrow[r, "(id_{cA}\otimes\lambda_{cA})\otimes id_{cA}"'] & (cA \otimes cA) \otimes cA \arrow[u, "\alpha_{cA,cA,cA}"']
\end{tikzcd}$$

2.3 Lambek Categories with Exchange

Definition 3. *A* ***Lambek category with exchange****, $(\mathcal{L}, e, \mathsf{ex})$, is a Lambek category equipped with a monoidal comonad (e, ε, δ) on $\mathcal{L}$, and a monoidal natural transformation $\mathsf{ex}_{A,B} : e(A \otimes B) \longrightarrow eB \otimes eA$. Furthermore, the following diagrams must commute:*

$$\begin{tikzcd}
e(A \otimes B) \arrow[rr, "\mathsf{ex}_{A,B}"] \arrow[d, equal] & & eB \otimes eA \arrow[d, "\mathsf{q}_{B,A}"] \\
e(A \otimes B) & eA \otimes eB \arrow[l, "\mathsf{q}_{A,B}"'] & e(B \otimes A) \arrow[l, "\mathsf{ex}_{B,A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
e(A \otimes B) \arrow[rr, "\delta_{A\otimes B}"] \arrow[d, "\mathsf{ex}_{A,B}"'] & & e^2(A \otimes B) \arrow[d, "e\mathsf{ex}_{A,B}"] \\
eB \otimes eA \arrow[r, "\delta_B\otimes\delta_A"] & e^2B \otimes e^2A \arrow[r, "\mathsf{q}_{eB,eA}"] & e(eB \otimes eA)
\end{tikzcd}$$

Note that the second diagram above shows that $\mathsf{ex}_{A,B}$ is a coalgebra morphism.

Definition 4. *Let $(\mathcal{L}, e, \mathsf{ex})$ be a Lambek category with exchange. The* ***co-Kleisli Category of*** *e, $\mathcal{L}_e$, is a category with the same objects as $\mathcal{L}$. There is an arrow $\hat{f} : A \longrightarrow B$ in $\mathcal{L}_e$ if there is an arrow $f : eA \longrightarrow B$ in $\mathcal{L}$. The identity arrow $\hat{id}_A : A \longrightarrow A$ is the arrow $\varepsilon_A : eA \longrightarrow A$ in $\mathcal{L}$. Given $\hat{f} : A \longrightarrow B$ and $\hat{g} : B \longrightarrow C$ in $\mathcal{L}_e$, which are arrows $f : eA \longrightarrow B$ and $g : eB \longrightarrow C$ in $\mathcal{L}$, the composition $\hat{g} \circ \hat{f} : A \longrightarrow C$ is defined as $g \circ ef \circ \delta_A$.*

Lemma 5 ($\mathcal{L}_e$ is symmetric)**.** *Suppose $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange. Then the co-Kleisli category $\mathcal{L}_e$ is symmetric monoidal. That is, there is a natural* *isomorphism $\gamma_{A,B} : e(A \otimes B) \longrightarrow B \otimes A$, and the following diagrams commute:*

$$
\begin{tikzcd}
e(A \otimes B) \arrow[r, "\hat{id}_{A \otimes B}"] \arrow[d, "\delta_{A \otimes B}"'] & A \otimes B \\
e^2(A \otimes B) \arrow[r, "e\gamma_{A,B}"'] & e(B \otimes A) \arrow[u, "\gamma_{B,A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
e(I \otimes A) \arrow[r, "\delta_{I \otimes A}"] \arrow[d, "\hat{\lambda}_A"'] & e^2(I \otimes A) \arrow[d, "e\gamma_{I,A}"] \\
A & e(A \otimes I) \arrow[l, "\hat{\rho}_A"]
\end{tikzcd}
$$

, where $\hat{id}$ is the identity morphism in $\mathcal{L}_e$, I is the identity object of $\mathcal{L}_e$, and $\hat{\lambda}$ and $\hat{\rho}$ are the left and right unitors of $\mathcal{L}_e$, respectively.

Don't know how to express the two diagrams for monoidal. For example, the morphism $id_A \otimes \lambda_A : A \otimes (I \otimes B) \to A \otimes B$ in $\mathcal{L}$ is $e(A \otimes (I \otimes B)) \to A \otimes B$ in $\mathcal{L}_e$ and can be defined as $(id_A \otimes \lambda_B) \circ \varepsilon_{A \otimes (I \otimes B)}$. But I feel like it should use $\hat{\lambda}$ in the definition.

Proof. Let I be the identity object in $\mathcal{L}$. Then the identity object of $\mathcal{L}_e$ is still I. The left and right unitors, $\hat{\lambda}_A : I \otimes A \longrightarrow A$ and $\hat{\rho}_A : A \otimes I \longrightarrow A$, in $\mathcal{L}_e$ are morphisms $e(I \otimes A) \longrightarrow A$ and $e(A \otimes I) \longrightarrow A$ in $\mathcal{L}$, respectively. Then we define $\hat{\lambda}$ and $\hat{\rho}$ as:

$$
\begin{aligned}
\hat{\lambda}_A &= \varepsilon_A \circ e\lambda_A \\
\hat{\rho}_A &= \varepsilon_A \circ e\rho_A,
\end{aligned}
$$

where λ and ρ are the left and right unitors in $\mathcal{L}$, respectively. And we define their inverses as:

$$
\begin{aligned}
\hat{\lambda}_A^{-1} &= \varepsilon_{I \otimes A} \circ e\lambda_A^{-1} \\
\hat{\rho}_A^{-1} &= \varepsilon_{A \otimes I} \circ e\rho_A^{-1}
\end{aligned}
$$

$\hat{\lambda}$ is a nautral isomorphism with inverse $\hat{\lambda}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
e(I \otimes A) \arrow[r, "\delta_{I \otimes A}"] \arrow[d, "\varepsilon_{I \otimes A}"'] \arrow[dr, "e\lambda_A"] \arrow[ddr, equal] & e^2(I \otimes A) \arrow[r, "e^2\lambda_A"] \arrow[d, phantom, "(1)"] & e^2 A \arrow[dd, "e\varepsilon_A"] \\
(2) & eA \arrow[ur, "\delta_A"] \arrow[d, "e\lambda_A^{-1}"] \arrow[dr, equal] & (5) \\
I \otimes A & e(I \otimes A) \arrow[l, "\varepsilon_{I \otimes A}"] & eA \arrow[l, "e\lambda_A^{-1}"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ. (2), (3) and (4) commute trivially. And (5) commutes because e is a comonad.

Similarly, $\hat{\rho}$ is a natural isomorphism with inverse $\hat{\rho}^{-1}$ by the following diagram chasing:

$$
\begin{tikzcd}
e(A \otimes I) \arrow[r, "\delta_{A\otimes I}"] \arrow[d, "\varepsilon_{A\otimes I}"'] \arrow[dr, "e\rho_A"] \arrow[ddr, equal] & e^2(A \otimes I) \arrow[r, "e^2\rho_A"] & e^2A \arrow[dd, "e\varepsilon_A"] \\
 & eA \arrow[ur, "\delta_A"'] \arrow[d, "e\rho_A^{-1}"] \arrow[dr, equal] & \\
A \otimes I & e(A \otimes I) \arrow[l, "\varepsilon_{A\otimes I}"] & eA \arrow[l, "e\rho_A^{-1}"]
\end{tikzcd}
$$

The associator $\hat{\alpha}_A : (A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}_e$ is the morphism $e((A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}$. We define $\hat{\alpha}$ as:

$$\hat{\alpha}_{A,B,C} = \varepsilon_{A\otimes(B\otimes C)} \circ e\alpha_{A,B,C},$$

where α is the associator of $\mathcal{L}$. And its inverse is

$$\hat{\alpha}^{-1}_{A,B,C} = \varepsilon_{(A\otimes B)\otimes C} \circ e\alpha^{-1}_{A,B,C}$$

$\hat{\alpha}$ is a natural isomorphism with inverse $\hat{\alpha}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
e((A \otimes B) \otimes C) \arrow[r, "\delta_{(A\otimes B)\otimes C}"] \arrow[dd, "\varepsilon_{(A\otimes B)\otimes C}"'] \arrow[dr, "e\alpha_{A,B,C}"] \arrow[ddr, equal] & e^2((A \otimes B) \otimes C) \arrow[r, "e^2\alpha_{A,B,C}"] & e^2(A \otimes (B \otimes C)) \arrow[dd, "e\varepsilon_{A\otimes(B\otimes C)}"] \\
 & e(A \otimes (B \otimes C)) \arrow[ur, "\delta_{A\otimes(B\otimes C)}"'] \arrow[d, "e\alpha^{-1}_{A,B,C}"] \arrow[dr, equal] & \\
(A \otimes B) \otimes C & e((A \otimes B) \otimes C) \arrow[l, "\varepsilon_{(A\otimes B)\otimes C}"] & e(A \otimes (B \otimes C)) \arrow[l, "e\alpha^{-1}_{A,B,C}"]
\end{tikzcd}
$$

Therefore, $\mathcal{L}_e$ is a monoidal category.

Then, we define a natural transformation $\gamma_{A,B} : e(A \otimes B) \longrightarrow B \otimes A$ as

$$\gamma_{A,B} = (\varepsilon_B \otimes \varepsilon_A) \circ \mathsf{ex}_{A,B}.$$

Clearly, γ is natural because it is the composition of natural transformations. The following diagram shows γ is an isomorphism in $\mathcal{L}_e$:

$$
\begin{tikzcd}
e(A \otimes B) \arrow[r, "\delta_{A\otimes B}"] \arrow[d, "\varepsilon_{A\otimes B}"'] & e^2(A \otimes B) \arrow[d, "e\gamma_{A,B}"] \\
A \otimes B & e(B \otimes A) \arrow[l, "\gamma_{B,A}"]
\end{tikzcd}
$$

The previous diagram commutes because the following one does:

$$
\begin{tikzcd}
e(A \otimes B) \arrow[rr, "\delta_{A\otimes B}"] \arrow[dddd, "\varepsilon_{A\otimes B}"'] \arrow[dr, "\mathsf{ex}_{A,B}"] \arrow[ddr, equal] & & e^2(A \otimes B) \arrow[rr, "e\mathsf{ex}_{A,B}"] \arrow[d, phantom, "(3)"] & & e(eB \otimes eA) \arrow[dddd, "e(\varepsilon_B \otimes \varepsilon_A)"] \\
& eB \otimes eA \arrow[rr, "\delta_B \otimes \delta_A"] \arrow[ddrr, equal] & \arrow[dd, phantom, "(4)"] & e^2B \otimes e^2A \arrow[ur, "\mathsf{q}_{eB,eA}"'] \arrow[dd, "e\varepsilon_B \otimes e\varepsilon_A"] & \arrow[dd, phantom, "(5)"] \\
& & & & \\
& e(A \otimes B) \arrow[d, phantom, "(1)"] & \arrow[d, phantom, "(2)"] & eB \otimes eA \arrow[dr, "\mathsf{q}_{B,A}"] & \\
A \otimes B & & eA \otimes eB \arrow[ll, "\varepsilon_A \otimes \varepsilon_B"] \arrow[ul, "\mathsf{q}_{A,B}"'] & & e(B \otimes A) \arrow[ll, "\mathsf{ex}_{B,A}"]
\end{tikzcd}
$$

Diagram (1) commutes because e is monoidal, diagrams (2) and (3) commute by the definition of the Lambek category with exchange, diagram (4) commutes because e is a comonad, and diagram (5) commutes by the naturality of q.

Further, the following diagram also commutes

$$
\begin{tikzcd}
e(I \otimes A) \arrow[r, "\delta_{I\otimes A}"] \arrow[d, "\hat{\lambda}_A"'] & e^2(I \otimes A) \arrow[d, "e\gamma_{I,A}"] \\
A & e(A \otimes I) \arrow[l, "\hat{\rho}_A"]
\end{tikzcd}
$$

by the diagram chasing below:

$$
\begin{tikzcd}
e(I \otimes A) \arrow[rrr, "\delta_{I\otimes A}"] \arrow[dd, "e\lambda_A"'] \arrow[drr, "\varepsilon_{I\otimes A}"] & & & e^2(I \otimes A) \arrow[dd, "e\gamma_{I,A}"] \\
\arrow[r, phantom, "(1)"] & A \arrow[d, phantom, "(2)"] & I \otimes A \arrow[l, "\lambda_A"'] \arrow[r, phantom, "(3)"] & \\
eA \arrow[ur, "\varepsilon_A"] \arrow[d, "\varepsilon_A"'] & \arrow[d, phantom, "(4)"] & & e(A \otimes I) \arrow[lll, "e\rho_A"] \arrow[ul, "\gamma_{A,I}"'] \arrow[d, "e\rho_A"] \\
A & & & eA \arrow[lll, "\varepsilon_A"]
\end{tikzcd}
$$

in which (1) commutes by the naturality of ε, (3) commutes because γ is a natural isomorphism, and (4) commutes trivially. Need (2)?

In conclusion, $\mathcal{L}_e$ is symmetric monoidal. □

2.4 Linear Categories

Definition 6. *A **linear category**,* $(\mathcal{L}, !, \mathsf{weak}, \mathsf{contra})$*, is specified by*

- *a symmetric monoidal closed category* $(\mathcal{L}, I, \otimes, \multimap)$*,*

- *a symmetric monoidal comonad* $(!, \varepsilon, \delta)$ *on* $\mathcal{L}$*, with* $\mathsf{q}_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ *and* $\mathsf{q}_I : I \longrightarrow !I$*;*
- *monoidal natural transformations on* $\mathcal{L}$ *with components* $\mathsf{weak}_A : !A \longrightarrow I$ *and* $\mathsf{contra}_A : !A \longrightarrow !A \otimes !A$*, s.t.*
  - *each* $(!A, \mathsf{weak}_A, \mathsf{contra}_A)$ *is a commutative comonoid, i.e. the following diagrams commute and* $\beta \circ \mathsf{contra}_A = \mathsf{contra}_A$ *where* $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ *is the symmetry natural transformation of* $\mathcal{L}$*;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & !A \otimes !A \arrow[r, "id_{!A} \otimes \mathsf{contra}_A"] & !A \otimes (!A \otimes !A) \\
!A \otimes !A \arrow[rr, "\mathsf{contra}_A \otimes id_{!A}"'] & & (!A \otimes !A) \otimes !A \arrow[u, "\alpha_{!A,!A,!A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & !A \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes !A & !A \otimes !A \arrow[l, "\mathsf{weak}_A \otimes id_{!A}"] \arrow[r, "id_{!A} \otimes \mathsf{weak}_A"'] & !A \otimes I
\end{tikzcd}$$

  - weak_A *and* contra_A *are coalgebra morphisms, i.e. the following diagrams commute;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
!!A \arrow[r, "!\mathsf{weak}_A"'] & !I
\end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[r, "\delta_A \otimes \delta_A"] & !!A \otimes !!A \arrow[d, "\mathsf{q}_{!A,!A}"] \\
!!A \arrow[rr, "!\mathsf{contra}_A"'] & & !(!A \otimes !A)
\end{tikzcd}$$

  - *any coalgebra morphism* $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ *between free coalgebras preserve the comonoid structure given by* weak *and* contra*, i.e. the follow-*

ing diagrams commute.

$$\begin{tikzcd}
!A \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & !B \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & !A \otimes !A \arrow[d, "f \otimes f"] \\
!B \arrow[r, "\mathsf{contra}_B"'] & !B \otimes !B
\end{tikzcd}$$

Definition 7. *Given two comonads* $(c, \varepsilon^c, \delta^c)$ *and* $(w, \varepsilon^w, \delta^w)$ *on a category* $\mathcal{L}$ *such that* $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ *is a Lambek category with contraction and* $(\mathcal{L}, w, \mathsf{weak})$ *is a Lambek category with weakening, we define a **distributive law** of* c *over* w *to be a natural transformation with components* $dist_A : cwA \longrightarrow wcA$*, subject to the following coherence diagrams:*

$$\begin{tikzcd}
wA & & cwA \arrow[ll, "\varepsilon^c_{wA}"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "w\varepsilon^c_A"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cA & & cwA \arrow[ll, "c\varepsilon^w_A"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "\varepsilon^w_{cA}"] &
\end{tikzcd}$$

Lemma 8. *Given two comonads* $(c, \varepsilon^c, \delta^c)$ *and* $(w, \varepsilon^w, \delta^w)$ *on a category* $\mathcal{L}$ *such that* $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ *is a Lambek category with contraction and* $(\mathcal{L}, w, \mathsf{weak})$ *is a Lambek category with weakening, the following two diagrams commute:*

$$\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, "\delta^c_{wcA}"'] & & cwc^2A \\
& c^2wcA \arrow[ur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wdist_{wA}"] \\
& w^2cwA &
\end{tikzcd}$$

Proof. The two diagrams above commute because the following ones due:

$$\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, equal] \arrow[ddr, "\delta^c_{wA}"'] & & cwc^2A \arrow[dl, "cw\varepsilon^c_{cA}"'] \\
& cwcA & \\
& c^2wcA \arrow[u, "c\varepsilon^c_{wcA}"] \arrow[uur, "cdist_{cA}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, equal] \arrow[ddr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wc\varepsilon^w_{wA}"'] \arrow[ddl, "wdist_{wA}"] \\
& wcwA & \\
& w^2cwA \arrow[u, "w\varepsilon^w_{cwA}"] &
\end{tikzcd}$$

They both commute by the distribute law and the comonad laws of c and w. $\square$

Lemma 9 (Composition of Weakening and Contraction)**.** *Suppose* $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}, \mathsf{contraR})$ *is a Lambek category with weakening and contraction, where* $(w, \varepsilon^w, \delta^w)$ *and* $(c, \varepsilon^c, \delta^c)$ *are the respective monoidal comonads. Then the composition of c and w using the distributive law* $dist_A : cwA \longrightarrow wcA$ *is a monoidal comonad on* $\mathcal{L}$.

Proof. For the complete proof see Appendix **??**. $\square$

Definition 10. *A* ***Lambek category with*** *cw,* $(\mathcal{L}, cw, \mathsf{weak}^w, \mathsf{contraL}, \mathsf{contraR}, dist)$*, is a Lambek category with weakening and contraction, and a distributive law. Furthermore, the following coherence diagrams commute:*

$$\begin{tikzcd}
I \otimes cwA \arrow[r, "\lambda^{-1}_{I\otimes cwA}"] \arrow[d, "\mathsf{contraR}_{wA,I}"'] & I \otimes (I \otimes cwA) \\
cwA \otimes (I \otimes cwA) \arrow[r, "\varepsilon^c_{wA} \otimes id_{I\otimes cwA}"] & wA \otimes (I \otimes cwA) \arrow[u, "\mathsf{weak}^w_A \otimes id_{I\otimes cwA}"']
\end{tikzcd}$$

$$\begin{tikzcd}
cwA \otimes I \arrow[r, "\rho^{-1}_{cwA\otimes I}"] \arrow[d, "\mathsf{contraL}_{wA,I}"'] & (cwA \otimes I) \otimes I \\
(cwA \otimes I) \otimes cwA \arrow[r, "id_{cwA\otimes I} \otimes \varepsilon^c_{wA}"] & (cwA \otimes I) \otimes wA \arrow[u, "id_{cwA\otimes I} \otimes \mathsf{weak}^w_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[d, "\varepsilon^c_{wA}"'] & & cwB \arrow[d, "\varepsilon^c_{wB}"] \\
wA \arrow[r, "\mathsf{weak}^w_A"] & I & wB \arrow[l, "\mathsf{weak}^w_B"']
\end{tikzcd}$$

where $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ *is any coalgebra morphism between free coalgebras.*

Lemma 11. *Let* $(\mathcal{L}, cw, \mathsf{weak}^w, \mathsf{contraL}, \mathsf{contraR})$ *be a Lambek category with cw. Then the following three conditions are satisfied:*

1. *There exist two natural transformations* $\mathsf{weak}_A : cwA \longrightarrow I$ *and* $\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$.
2. *Each* $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ *is a comonoid.*
3. weak_A *and* contra_A *are coalgebra morphisms.*
4. *Any coalgebra morphism* $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ *between free coalgebras preserves the comonoid structure given by* weak *and* contra.

Proof. 1. Each of weak and contra can be given two equivalent definitions. $\mathsf{weak}_A : cwA \longrightarrow I$ is defined as in the diagram below. The left triangle commutes by the definition of $dist$ and the right triangle commutes by the definition of weak^w.

$$\begin{tikzcd}
& wcA \arrow[d, "w\varepsilon^c_A"] \arrow[dr, "\mathsf{weak}^w_{cA}"] & \\
cwA \arrow[ur, "dist_A"] \arrow[r, "\varepsilon^c_{wA}"'] & wA \arrow[r, "\mathsf{weak}^w_A"'] & I
\end{tikzcd}$$

$\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$ is defined as below. The left part of the diagram commutes by the definitions of $\mathsf{contraL}$ and of $\mathsf{contraR}$, and the right part commutes because $\mathcal{L}$ is monoidal.

$$\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "\lambda^{-1}_{cwA}"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[dl, "\alpha_{cwA,I,cwA}"'] \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
I \otimes cwA \arrow[r, "\mathsf{contraR}_{wA,I}"'] & cwA \otimes (I \otimes cwA) \arrow[r, "id_{cwA} \otimes \lambda_{cwA}"'] & cwA \otimes cwA
\end{tikzcd}$$

2. Each $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid.

Case 1:

$$\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & cwA \otimes cwA \arrow[r, "id_{cwA} \otimes \mathsf{contra}_A"] & cwA \otimes (cwA \otimes cwA) \\
cwA \otimes cwA \arrow[rr, "\mathsf{contra}_A \otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}$$

The previous diagram commutes by the following diagram chasing.

$$\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] \arrow[dr, phantom, "(1)"] & cwA \otimes cwA \arrow[r, "id_{cwA} \otimes \rho^{-1}_{cwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwA \otimes (cwA \otimes I) \arrow[d, "id_{cwA} \otimes \mathsf{contraL}_{wA,I}"] \\
cwA \otimes cwA \arrow[r, "id_{cwA} \otimes \lambda^{-1}_{cwA}"] \arrow[d, "\rho^{-1}_{cwA} \otimes id_{cwA}"'] & cwA \otimes (I \otimes cwA) \arrow[d, "id_{cwA} \otimes \mathsf{contraR}_{wA,I}"] \arrow[dr, phantom, "(3)"] & cwA \otimes ((cwA \otimes I) \otimes cwA) \arrow[d, "id_{cwA} \otimes (\rho_{cwA} \otimes id_{cwA})"] \\
(cwA \otimes I) \otimes cwA \arrow[d, "\mathsf{contraL}_{wA,I} \otimes id_{cwA}"'] & cwA \otimes (cwA \otimes (I \otimes cwA)) \arrow[r, "id_{cwA} \otimes (id_{cwA} \otimes \lambda_{cwA})"] \arrow[ur, "id_{cwA} \otimes \alpha_{cwA,I,cwA}"] \arrow[d, phantom, "(4)"] & cwA \otimes (cwA \otimes cwA) \\
((cwA \otimes I) \otimes cwA) \otimes cwA \arrow[rr, "(\rho_{cwA} \otimes id_{cwA}) \otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}$$

(1) commutes trivially and we would not expand contra for simplicity. (2) and (4) commute because $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction. (3) commutes because $\mathcal{L}$ is monoidal.

Case 2:

$$
\begin{tikzcd}
 & cwA \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes cwA & cwA \otimes cwA \arrow[l, "\mathsf{weak}_A \otimes id_{cwA}"] \arrow[r, "id_{cwA} \otimes \mathsf{weak}_A"'] & cwA \otimes I
\end{tikzcd}
$$

The diagram above commutes by the following diagram chasing.

$$
\begin{tikzcd}
I \otimes cwA & & \arrow[d, phantom, "(2)"] & & wA \otimes cwA \arrow[llll, "\mathsf{weak}^w_A \otimes id_{cwA}"'] \\
\arrow[dr, phantom, "(1)"] & I \otimes (I \otimes cwA) \arrow[ul, "id_I \otimes \lambda_{cwA}"'] & \arrow[d, phantom, "(4)"] & wA \otimes (I \otimes cwA) \arrow[ll, "\mathsf{weak}^w_A \otimes id_{I \otimes cwA}"'] \arrow[ur, "id_{wA} \otimes \lambda_{cwA}"] \arrow[dr, phantom, "(3)"] & \\
 & I \otimes cwA \arrow[u, "\lambda^{-1}_{I \otimes cwA}"] \arrow[rr, "\mathsf{contraR}_{wA,I}"] & & cwA \otimes (I \otimes cwA) \arrow[u, "\varepsilon^c_{wA} \otimes id_{I \otimes cwA}"'] \arrow[dr, "id_{cwA} \otimes \lambda_{cwA}"] & \\
cwA \arrow[uuu, "\lambda^{-1}_{cwA}"] \arrow[ur, "\lambda^{-1}_{cwA}"] \arrow[dr, "\rho^{-1}_{cwA}"'] \arrow[ddd, "\rho^{-1}_{cwA}"'] \arrow[rrrr, phantom, "(5)"] & & & & cwA \otimes cwA \arrow[uuu, "\varepsilon^c_{wA} \otimes cwA"'] \arrow[ddd, "id_{cwA} \otimes \varepsilon^c_{wA}"] \\
\arrow[dr, phantom, "(6)"] & cwA \otimes I \arrow[d, "\rho^{-1}_{cwA}"] \arrow[rr, "\mathsf{contraL}_{wA,I}"] & \arrow[d, phantom, "(9)"] & (cwA \otimes I) \otimes cwA \arrow[ur, "\rho_{cwA} \otimes id_{cwA}"'] \arrow[d, "id_{cwA \otimes I} \otimes \varepsilon^c_{wA}"'] \arrow[dr, phantom, "(8)"] & \\
 & (cwA \otimes I) \otimes I \arrow[dl, "\rho_{cwA} \otimes id_I"] & \arrow[d, phantom, "(7)"] & (cwA \otimes I) \otimes wA \arrow[ll, "id_{cwA \otimes I} \otimes \mathsf{weak}^w_A"'] \arrow[dr, "\rho_{cwA} \otimes id_{wA}"'] & \\
cwA \otimes I & & & & cwA \otimes wA \arrow[llll, "id_{cwA} \otimes \mathsf{weak}_A"]
\end{tikzcd}
$$

(1), (2) and (3) commute by the functionality of λ. (6), (7) and (8) commute by the functionality of ρ. (4) and (9) are conditions of the Lambek category with cw. And (5) is the definition of contra.

3. weak and contra are coalgebra morphisms.

Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
cwcwA \arrow[r, "cw\mathsf{weak}_A"'] & cwI
\end{tikzcd}
$$

The previous diagram commutes by the diagram below. (1) commutes by the naturality of δ^c. (2) commutes by the condition of $dist_{wA}$. (3), (5) and (6) commute because c is a monoidal comonad. (4) commutes because $(\mathcal{L}, w, \mathsf{weak}^w)$ is a Lambek category with weakening. (7) commutes because c and w are monoidal comonads.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\mathsf{weak}^w_A"] \arrow[d, "\delta^c_{wA}"'] \arrow[ddrr, "c\delta^w_A"] & & cI \arrow[rr, "\epsilon^c_I"] \arrow[dddrr, "c\mathsf{q}^w_I"'] & & I \arrow[dl, equal] \arrow[dd, "\mathsf{q}^c_I"] \\
c^2wA \arrow[d, "c^2\delta^w_A"'] \arrow[r, phantom, "(1)"] & & \arrow[r, phantom, "(5)"] & I \arrow[ul, "\mathsf{q}^c_I"'] \arrow[r, phantom, "(6)"] & \\
c^2w^2A \arrow[d, "cdist_{wA}"'] \arrow[drr, "c\varepsilon^c_{wA}"] \arrow[r, phantom, "(2)"] & & cw^2A \arrow[ll, "\delta^c_{w^2A}"'] \arrow[d, equal] \arrow[u, phantom, "(4)"] \arrow[l, phantom, "(3)"] & \arrow[r, phantom, "(7)"] & cI \arrow[ul, "\varepsilon^c_I"'] \arrow[d, "c\mathsf{q}^w_I"] \\
cwcwA \arrow[rr, "cw\varepsilon^c_{wA}"'] & & cw^2A & & cwI \arrow[ll, "cw\mathsf{weak}^w_A"]
\end{tikzcd}
$$

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & cwA \otimes cwA \arrow[r, "\delta_A \otimes \delta_A"] & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}_{cwA,cwA}"] \\
cwcwA \arrow[rr, "cw\mathsf{contra}_A"'] & & cw(cwA \otimes cwA)
\end{tikzcd}
$$

To prove the previous diagram commute, we first expand it, Then we divide it into five parts as shown belovee, and prove each part commutes.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, phantom, "(c)"] & cwA \otimes I \arrow[rrr, "\mathsf{contraL}_{wA,I}"] & & & (cwA \otimes I) \otimes cwA \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
cw^2A \arrow[d, "\delta^c_{w^2A}"'] & w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA\otimes I}"'] & w(cwA \otimes cwA) \arrow[rr, "\varepsilon^w_{cwA\otimes cwA}"] & & cwA \otimes cwA \arrow[d, "c\delta^w_A \otimes c\delta^w_A"] \arrow[dl, phantom, "(a)"] \\
c^2w^2A \arrow[d, "cdist_{wA}"'] & & & cw^2A \otimes cw^2A \arrow[ur, "c\varepsilon^c_{wA} \otimes c\varepsilon^c_{wA}"] \arrow[r, equal] & cw^2A \otimes cw^2A \arrow[d, "\delta^c_{w^2A} \otimes \delta^c_{w^2A}"] \arrow[dl, phantom, "(b)"] \\
cwcwA \arrow[d, "cw\rho_{cwA}"'] \arrow[rr, phantom, "(d)"] & & & & c^2w^2A \otimes c^2w^2A \arrow[ul, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
cw(cwA \otimes I) \arrow[uuur, "\varepsilon^c_{w(cwA\otimes I)}"] \arrow[d, "cw\mathsf{contraL}_{wA,I}"'] & & & & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}^c_{wcwA\otimes wcwA}"] \arrow[ll, phantom, "(e)"] \\
cw((cwA \otimes I) \otimes cwA) \arrow[rr, "cw(\rho_{cwA} \otimes id_{cwA})"'] & & cw(cwA \otimes cwA) \arrow[uuuu, "c\varepsilon^w_{cwA\otimes cwA}"] & & c(wcwA \otimes wcwA) \arrow[ll, "c\mathsf{q}^w_{cwA\otimes cwA}"]
\end{tikzcd}
$$

Part (a) and (b) are comonad laws.

Part (c) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for w. (3) is the comonad law for c. (4) commutes by the naturality of ε^c. (5) is one of the conditions for $dist_{wA}$. (6) commutes by the naturality of ε^w. And (7) commutes by the naturality of ε^c.

$$
\begin{tikzcd}
cwA \arrow[rrr, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, "c\delta^w_A"] \arrow[drr, equal] & & & cwA \otimes I \\
cw^2A \arrow[r, equal] \arrow[d, "\delta^c_{w^2A}"'] \arrow[u, phantom, "(1)"] & cw^2A \arrow[r, "c\varepsilon^w_{wA}"] \arrow[d, "dist_{wA}"] \arrow[u, phantom, "(2)"] & cwA \arrow[r, phantom, "(6)"] & \\
c^2w^2A \arrow[d, "cdist_{wA}"'] \arrow[ur, "\varepsilon^c_{cw^2A}"] \arrow[u, phantom, "(3)"] \arrow[r, phantom, "(4)"] & wcwA \arrow[ur, "\varepsilon^w_{cwA}"] \arrow[rr, "w\rho^{-1}_{cwA}"] \arrow[u, phantom, "(5)"] & & w(cwA \otimes I) \arrow[uu, "\varepsilon^w_{cwA\otimes I}"'] \\
cwcwA \arrow[ur, "\varepsilon^c_{wcwA}"] \arrow[rrr, "cw\rho^{-1}_{cwA}"'] & & (7) & cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA\otimes I)}"']
\end{tikzcd}
$$

Part (d) commutes by the following diagram chase. The upper two squares both commute by the naturality of ε^w, and the lower two squares commute by the naturality of ε^c.

$$\begin{tikzcd}
cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] & cwA \otimes cwA \\
w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA \otimes I}"] \arrow[r, "w\mathsf{contraL}_{wA,I}"] & w((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^w_{(cwA \otimes I) \otimes cwA}"] \arrow[r, "w(\rho_{cwA} \otimes id_{cwA})"] & w(cwA \otimes cwA) \arrow[u, "\varepsilon^w_{cwA \otimes cwA}"] \\
cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA \otimes I)}"] \arrow[r, "cw\mathsf{contraL}_{wA,I}"'] & cw((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^c_{w((cwA \otimes I) \otimes A)}"] \arrow[r, "cw(\rho_{cwA} \otimes id_{cwA})"'] & cw(cwA \otimes cwA) \arrow[u, "\varepsilon^c_{w(cwA \otimes cwA)}"']
\end{tikzcd}$$

Part (e) commutes by the following diagram. (1) commutes by the condition of $dist_{wA}$. (2) and (4) commute by the naturality of ε^c. (3) and (5) commute because w and c are monoidal comonads.

$$\begin{tikzcd}
cwA \otimes cwA & cw^2A \otimes cw^2A \arrow[l, "c\varepsilon^w_{wA} \otimes c\varepsilon^w_{wA}"'] \arrow[d, "dist_{wA} \otimes dist_{wA}"] \arrow[dr, phantom, "(2)"] & c^2w^2A \otimes c^2w^2A \arrow[l, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"'] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
\arrow[r, phantom, "(3)"] & wcwA \otimes wcwA \arrow[ul, "\varepsilon^w_{cwA} \otimes \varepsilon^w_{cwA}", "(1)"'] \arrow[dl, "\mathsf{q}^w_{cwA,cwA}"'] \arrow[d, phantom, "(4)"] & cwcwA \otimes cwcwA \arrow[l, "\varepsilon^c_{wcwA} \otimes \varepsilon^c_{wcwA}"'] \arrow[d, "\mathsf{q}^c_{wcwA \otimes wcwA}", "(5)"'] \\
w(cwA \otimes cwA) \arrow[uu, "\varepsilon^w_{cwA \otimes cwA}"] & cw(cwA \otimes cwA) \arrow[l, "\varepsilon^c_{w(cwA \otimes cwA)}"] & c(wcwA \otimes wcwA) \arrow[l, "c\mathsf{q}_{cwA \otimes cwA}"] \arrow[ul, "\varepsilon^c_{wcwA \otimes wcwA}"]
\end{tikzcd}$$

4. Any coalgebra morphism $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ between free coalgebras preserves the comonoid structure given by weak and contra.

   Case 1: This coherence diagram is given in the definition of the Lambek category with cw.

$$\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & cwB \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}$$

   Case 2:

$$\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & cwA \otimes cwA \arrow[d, "f \otimes f"] \\
cwB \arrow[r, "\mathsf{contra}_B"'] & cwB \otimes cwB
\end{tikzcd}$$

   The square commutes by the diagram chasing below, which commutes by the naturality of ρ and $\mathsf{contraL}$.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "cwf"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] \arrow[d, "cwf \otimes id_I"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] \arrow[d, "(cwf \otimes id_I) \otimes cwf"] & cwA \otimes cwA \arrow[d, "cwf \otimes cwf"] \\
cwB \arrow[r, "\rho^{-1}_{cwB}"'] & cwB \otimes I \arrow[r, "\mathsf{contraL}_{wB,I}"'] & (cwB \otimes I) \otimes cwB \arrow[r, "\rho_{cwB} \otimes id_{cwB}"'] & cwB \otimes cwB
\end{tikzcd}
$$

□

Definition 12. *Given two comonads* $(cw, \varepsilon^{cw}, \delta^{cw})$ *and* $(e, \varepsilon^{e}, \delta^{e})$ *on a category* $\mathcal{L}$ *such that* $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ *is a Lambek category with cw and* $(\mathcal{L}, e, \mathsf{ex})$ *is a Lambek category with exchange, we define a* ***distributive law for exchange*** *of cw over e to be a natural isomorphism with components* $distEx_A : cweA \longrightarrow ecwA$*, subject to the following coherence diagrams:*

$$
\begin{tikzcd}
eA & & cweA \arrow[ll, "\varepsilon^{cw}_{eA}"'] \arrow[dl, "distEx_A"] \\
& ecwA \arrow[ul, "e\varepsilon^{cw}_{A}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA & & cweA \arrow[ll, "cw\varepsilon^{e}_{A}"'] \arrow[dl, "distEx_A"] \\
& ecwA \arrow[ul, "\varepsilon^{e}_{cwA}"] &
\end{tikzcd}
$$

Lemma 13. *Given two comonads* $(cw, \varepsilon^{cw}, \delta^{cw})$ *and* $(e, \varepsilon^{e}, \delta^{e})$ *on a category* $\mathcal{L}$ *such that* $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ *is a Lambek category with cw and* $(\mathcal{L}, e, \mathsf{ex})$ *is a Lambek category with exchange, the following two digrams also commute:*

$$
\begin{tikzcd}
cwecwA \arrow[rr, "cwe\delta^{cw}_{A}"] \arrow[dr, "\delta^{cw}_{cweA}"'] & & cwe(cw)^2A \\
& (cw)^2ecwA \arrow[ur, "cwdistEx_{cwA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
ecweA \arrow[rr, "ecw\delta^{e}_{A}"] \arrow[dr, "\delta^{e}_{cweA}"'] & & ecwe^2A \arrow[dl, "edistEx_{eA}"] \\
& e^2cweA &
\end{tikzcd}
$$

The proof is similar with the proof of Lemma 8 and we will not elaborate it here. Also, notice the difference between *dist* of *c* over *w* and *distEx* of *cw* over *e*. While *dist* is a natural transformation, *distEx* is a natural isomorphism.

Lemma 14. *let* $(cw, \varepsilon^{cw}, \delta^{cw})$ *and* $(e, \varepsilon^{e}, \delta^{e})$ *be two monoidal comonads on a Lambek category with cw and exchange* $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$*. Then the composition of cw and e using the distributive law for exchange* $distEx_A : cweA \longrightarrow ecwA$ *is a monoidal comonad* $(cwe, \varepsilon, \delta)$ *on* $\mathcal{L}$*.*

Proof. Suppose $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ are monoidal comonads, and $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with *cw* and exchange. Since by definition $cw, e : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors, we know that their composition

$cwe : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$
\begin{aligned}
\mathsf{q}_{A,B} &: cweA \otimes cweB \longrightarrow cwe(A \otimes B)\\
\mathsf{q}_{A,B} &= cw\mathsf{q}^{e}_{A,B} \circ \mathsf{q}^{cw}_{eA,eB}\\
\mathsf{q}_{I} &: I \longrightarrow cweI\\
\mathsf{q}_{I} &= cw\mathsf{q}^{e}_{I} \circ \mathsf{q}^{cw}_{I}
\end{aligned}
$$

Analogous to the proof of Lemma 9, each of ε and δ can be given two equivalent definitions:

$$
\begin{tikzcd}
cweA \arrow[r, "\varepsilon^{cw}_{eA}"] \arrow[d, "cw\varepsilon^{e}_{A}"'] & eA \arrow[d, "\varepsilon^{e}_{A}"] \\
cwA \arrow[r, "\varepsilon^{cw}_{A}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
cweA \arrow[r, "cw\delta^{e}_{A}"] \arrow[d, "\delta^{cw}_{eA}"'] & cwe^2A \arrow[r, "\delta^{cw}_{e^2A}"] \arrow[d, "\delta^{cw}_{e^2A}"] & (cw)^2e^2A \arrow[d, "cwdist_{eA}"] \\
(cw)^2eA \arrow[r, "(cw)^2\delta^{e}_{A}"'] & (cw)^2e^2A \arrow[r, "cwdist_{eA}"'] & cwecweA
\end{tikzcd}
$$

And the comonad laws can be proved similarly, which we will not elaborate for simplicity. ◻

Lemma 15. *Let $(cwe, \varepsilon, \delta)$ be a monoidal comonad over a monoidal category $(\mathcal{L}, I, \otimes)$ such that $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with cw and exchange. Then the co-Kleisli category of $\mathcal{L}$, $\mathcal{L}_{cwe}$, is a symmetric monoidal category.*

Proof. The identity object of $\mathcal{L}_{cwe}$ is still I.

The left and right unitors, $\hat{\lambda}_A : I \otimes A \longrightarrow A$ and $\hat{\rho}_A : A \otimes I \longrightarrow A$, in $\mathcal{L}_{cwe}$ are morphisms $cwe(I \otimes A) \longrightarrow A$ and $cwe(A \otimes I) \longrightarrow A$ in $\mathcal{L}$, respectively. Then we define $\hat{\lambda}$ and $\hat{\rho}$ as:

$$
\begin{aligned}
\hat{\lambda}_A &= \varepsilon_A \circ cwe\lambda_A\\
\hat{\rho}_A &= \varepsilon_A \circ cwe\rho_A,
\end{aligned}
$$

where λ and ρ are the left and right unitors in $\mathcal{L}$, respectively. And we define their inverses as:

$$
\begin{aligned}
\hat{\lambda}^{-1}_A &= \varepsilon_{I\otimes A} \circ cwe\lambda^{-1}_A\\
\hat{\rho}^{-1}_A &= \varepsilon_{A\otimes I} \circ cwe\rho^{-1}_A
\end{aligned}
$$

$\hat{\lambda}$ is a nautral isomorphism with inverse $\hat{\lambda}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
cwe(I \otimes A) \arrow[r, "\delta_{I\otimes A}"] \arrow[dr, "cwe\lambda_A"] \arrow[dd, "\varepsilon_{I\otimes A}"'] \arrow[ddr, equal] & (cwe)^2(I \otimes A) \arrow[r, "(cwe)^2\lambda_A"] \arrow[d, phantom, "(1)"] & (cwe)^2A \arrow[dd, "cwe\varepsilon_A"] \\
\arrow[r, phantom, "(2)"] & (3)\ cweA \arrow[ur, "\delta_A"'] \arrow[d, "cwe\lambda^{-1}_A"] \arrow[dr, equal] \arrow[r, phantom, "(5)"] & {} \\
I \otimes A & cwe(I \otimes A) \arrow[l, "\varepsilon_{I\otimes A}"] \arrow[ur, phantom, "(4)"] & cweA \arrow[l, "cwe\lambda^{-1}_A"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ. (2), (3) and (4) commute trivially. And (5) commutes because cwe is a comonad.

Similarly, $\hat{\rho}$ is a natural isomorphism with inverse $\hat{\rho}^{-1}$ by the following diagram chasing:

$$
\begin{tikzcd}
cwe(A \otimes I) \arrow[rr, "\delta_{A\otimes I}"] \arrow[dd, "\varepsilon_{A\otimes I}"'] \arrow[drr, "cwe\rho_A"] \arrow[ddrr, equal] & & (cwe)^2(A \otimes I) \arrow[rr, "(cwe)^2\rho_A"] & & (cwe)^2A \arrow[dd, "cwe\varepsilon_A"] \\
& & cweA \arrow[urr, "\delta_A"] \arrow[d, "cwe\rho_A^{-1}"] \arrow[drr, equal] & & \\
A \otimes I & & cwe(A \otimes I) \arrow[ll, "\varepsilon_{A\otimes I}"] & & cweA \arrow[ll, "cwe\rho_A^{-1}"]
\end{tikzcd}
$$

The associator $\hat{\alpha}_A : (A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}_{cwe}$ is the morphism $cwe((A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}$. We define $\hat{\alpha}$ as:

$$\hat{\alpha}_{A,B,C} = \varepsilon_{A\otimes(B\otimes C)} \circ cwe\alpha_{A,B,C},$$

where α is the associator of $\mathcal{L}$. And its inverse is

$$\hat{\alpha}^{-1}_{A,B,C} = \varepsilon_{(A\otimes B)\otimes C} \circ cwe\alpha^{-1}_{A,B,C}$$

$\hat{\alpha}$ is a natural isomorphism with inverse $\hat{\alpha}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
cwe((A \otimes B) \otimes C) \arrow[rr, "\delta_{(A\otimes B)\otimes C}"] \arrow[dd, "\varepsilon_{(A\otimes B)\otimes C}"'] \arrow[drr, "cwe\alpha_{A,B,C}"] \arrow[ddrr, equal] & & (cwe)^2((A \otimes B) \otimes C) \arrow[rr, "(cwe)^2\alpha_{A,B,C}"] & & (cwe)^2(A \otimes (B \otimes C)) \arrow[dd, "cwe\varepsilon_{A\otimes(B\otimes C)}"] \\
& & cwe(A \otimes (B \otimes C)) \arrow[urr, "\delta_{A\otimes(B\otimes C)}"] \arrow[d, "cwe\alpha^{-1}_{A,B,C}"] \arrow[drr, equal] & & \\
(A \otimes B) \otimes C & & cwe((A \otimes B) \otimes C) \arrow[ll, "\varepsilon_{(A\otimes B)\otimes C}"] & & cwe(A \otimes (B \otimes C)) \arrow[ll, "cwe\alpha^{-1}_{A,B,C}"]
\end{tikzcd}
$$

Therefore, $\mathcal{L}_{cwe}$ is a monoidal category.

The symmetry, $\hat{\beta}_{A,B} : A \otimes B \longrightarrow B \otimes A$, in $\mathcal{L}_{cwe}$ is the morphism $cwe(A \otimes B) \longrightarrow B \otimes A$ in $\mathcal{L}$, which is defined as:

$$\hat{\beta}_{A,B} = \varepsilon^{cw}_{B\otimes A} \circ cw\gamma_{A,B},$$

where $\varepsilon^{cw}_A : cwA \longrightarrow A$ is a natural transformation associated with the comonad cw, and γ is the natural isomorphism defined in Lemma **??**. Then its inverse is

$$\hat{\beta}^{-1}_{A,B} = \varepsilon^{cw}_{A\otimes B} \circ cw\gamma_{B,A}$$

$\hat{\beta}$ is a natural isomorphism with inverse $\hat{\beta}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
A \otimes B \arrow[dr, phantom, "(1)"] & cw(A \otimes B) \arrow[l, "\varepsilon^{cw}_{A\otimes B}"'] \arrow[dl, equal] \arrow[d, "\delta^{cw}_{A\otimes B}"] \arrow[drr, phantom, "(3)"] & & cwe(A \otimes B) \arrow[ll, "cw\varepsilon^{e}_{A\otimes B}"'] \arrow[d, "\delta^{cw}_{e(A\otimes B)}"] \\
cw(A \otimes B) \arrow[u, "\varepsilon^{cw}_{A\otimes B}"] \arrow[r, phantom, "(2)"] & (cw)^2(A \otimes B) \arrow[l, "cw\varepsilon^{cw}_{A\otimes B}"'] \arrow[drr, phantom, "(5)"] & & (cw)^2e(A \otimes B) \arrow[ll, "(cw)^2\varepsilon^{e}_{A\otimes B}"'] \arrow[d, "(cw)^2\delta^{e}_{A\otimes B}"'] \\
\arrow[r, phantom, "(4)"] & (cw)^2e(B \otimes A) \arrow[u, "(cw)^2\gamma_{B,A}"] \arrow[ddl, "cw\varepsilon^{cw}_{e(B\otimes A)}"'] \arrow[d, equal] \arrow[r, phantom, "(7)"] & (cw)^2e^2(A \otimes B) \arrow[l, "(cw)^2e\gamma_{A,B}"'] \arrow[r, equal] \arrow[d, phantom, "(8)"] & (cw)^2e^2(A \otimes B) \arrow[d, "cwdistEx_{e(A\otimes B)}"] \arrow[dl, phantom, "(9)"] \\
 & (cw)^2e(B \otimes A) \arrow[drr, "cwdistEx_{B\otimes A}"] \arrow[d, phantom, "(6)"] & & (cwe)^2(A \otimes B) \arrow[ul, "cwdistEx^{-1}_{e(A\otimes B)}"'] \arrow[d, "cwecw\gamma_{A,B}"] \\
cwe(B \otimes A) \arrow[uuu, "cw\gamma_{B,A}"] & & & cwecw(B \otimes A) \arrow[lll, "cwe\varepsilon^{cw}_{B\otimes A}"] \arrow[uull, "cwdistEx^{-1}_{B\otimes A}"']
\end{tikzcd}
$$

(1), (7) and (9) commute trivially. (2) is the comonad law for cw. (3) commutes by the naturality of δ^{cw}. (4) commutes by the naturality of ε^{cw}. (5) commutes because γ is a natural isomorphism (Lemma **??**). (6) is the definition of $distEx$. (8) is the naturality of $distEx$.

In conclusion, $\mathcal{L}_{cwe}$ is a symmetric monoidal category. □

3 Related Work

TODO

4 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.

A Appendix

A.1 Symmetric Monoidal Categories

Definition 16. *A **monoidal category** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$
\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A \\
\rho_A &: A \otimes \top \longrightarrow A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}
$$

- *Subject to the following coherence diagrams:*

$$
\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
$$

Definition 17. *A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$
\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A \\
\rho_A &: A \otimes \top \longrightarrow A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}
$$

- *A symmetry natural isomorphism:*

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A\otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B\otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C\otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A\otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 18. *A* ***monoidal biclosed category*** *is a monoidal category $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, each of the functors $-\otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ and $B \otimes - : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any object A and C of $\mathcal{M}$, there are two objects $C \leftharpoondown B$ and $B \rightharpoonup C$ of $\mathcal{M}$ and two natural bijections:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, C \leftharpoondown B)$$
$$\mathsf{Hom}_{\mathcal{M}}(B \otimes A, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \rightharpoonup C)$$

Definition 19. *A* ***symmetric monoidal closed category (SMCC)*** *is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of $\mathcal{M}$, the functor $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$*

has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Definition 20. *Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1, A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A, \top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"]
\end{tikzcd}$$

Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

Definition 21. *Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"'] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"'] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"'] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"'] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"'] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"'] & F(B \otimes_1 A)
\end{tikzcd}$$

Definition 22. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 23. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***symmetric monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 24. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are* *monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd} FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\ A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"] \end{tikzcd} \qquad \begin{tikzcd} F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\ \top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2 \end{tikzcd}$$

$$\begin{tikzcd} GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\ G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B) \end{tikzcd} \qquad \begin{tikzcd} G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\ \top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"'] \end{tikzcd}$$

Definition 25. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***symmetric monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd} FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\ A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"] \end{tikzcd} \qquad \begin{tikzcd} F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\ \top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2 \end{tikzcd}$$

$$\begin{tikzcd} GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\ G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B) \end{tikzcd} \qquad \begin{tikzcd} G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\ \top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"'] \end{tikzcd}$$

Definition 26. *A* ***monoidal comonad*** *on a monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd} TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\ T^2A \arrow[r, "\delta_{TA}"'] & T^3A \end{tikzcd} \qquad \begin{tikzcd} & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\ TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA \end{tikzcd}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$
\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\
& \top &
\end{tikzcd}
$$

$$
\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}
$$

Definition 27. *A **symmetric monoidal comonad** on a symmetric monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$
\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}
$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$
\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\
& \top &
\end{tikzcd}
$$

$$
\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}
$$