Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

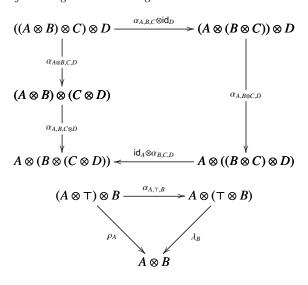
1.1 Symmetric Monoidal Categories

Definition 1 A monoidal category is a category, M, with the following data:

- An object \top of \mathcal{M} ,
- A bi-functor \otimes : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• Subject to the following coherence diagrams:



Definition 2 A symmetric monoidal category (SMC) is a category, M, with the following data:

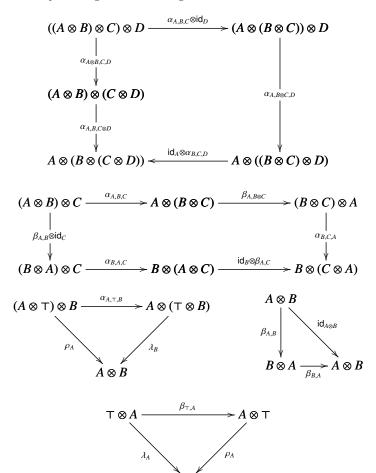
- An object \top of \mathcal{M} ,
- A bi-functor \otimes : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• A symmetry natural transformation:

$$\beta_{A,B}: A \otimes B \longrightarrow B \otimes A$$

• Subject to the following coherence diagrams:

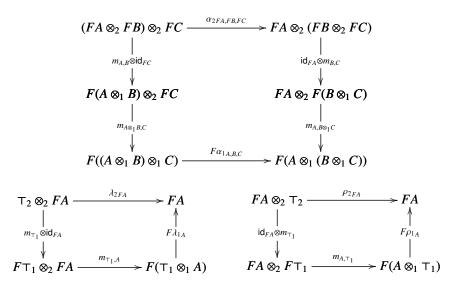


Definition 3 A symmetric monoidal closed category (SMCC) is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of M, the functor $-\otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of M there is an object $B \multimap C$ of M and a natural bijection:

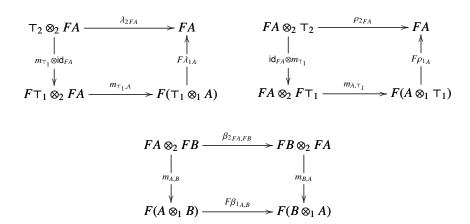
$$\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A \otimes B, C) \cong \operatorname{\mathsf{Hom}}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap: \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of \mathcal{M} .

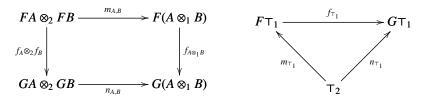
Definition 4 Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F \top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:



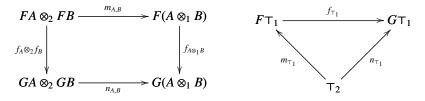
Definition 5 Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal** functor is a functor $F: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1}: \top_2 \longrightarrow F \top_1$ and a natural transformation $m_{A,B}: FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:



Definition 6 Suppose (M_1, \top_1, \otimes_1) and (M_2, \top_2, \otimes_2) are monoidal categories, and (F, m) and (G, n) are monoidal functors between M_1 and M_2 . Then a **monoidal natural transformation** is a natural transformation, $f: F \longrightarrow G$, subject to the following coherence diagrams:

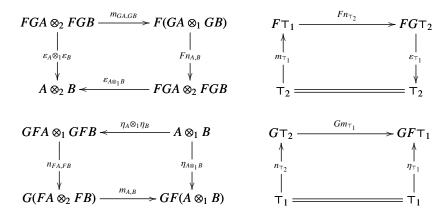


Definition 7 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a symmetric monoidal natural transformation is a natural transformation, $f: F \longrightarrow G$, subject to the following coherence diagrams:

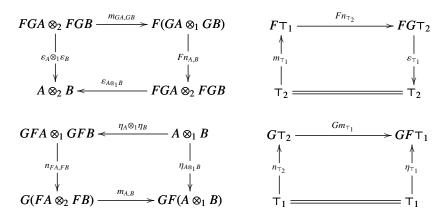


Definition 8 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1: F \dashv G: \mathcal{M}_2$ such that the unit, $\eta_A: A \to GFA$, and the counit, $\varepsilon_A: FGA \to A$,

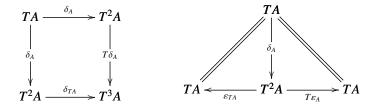
are monoidal natural transformations. Thus, the following diagrams must commute:



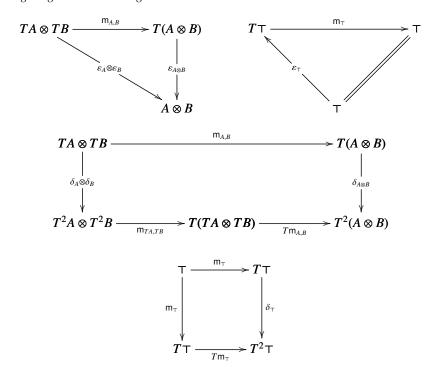
Definition 9 Suppose (M_1, \top_1, \otimes_1) and (M_2, \top_2, \otimes_2) are SMCs, and (F, m) is a symmetric monoidal functor between M_1 and M_2 and (G, n) is a symmetric monoidal functor between M_2 and M_1 . Then a **symmetric monoidal adjunction** is an ordinary adjunction $M_1: F \dashv G: M_2$ such that the unit, $\eta_A: A \to GFA$, and the counit, $\varepsilon_A: FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:



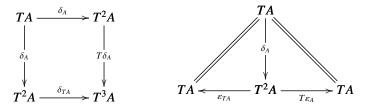
Definition 10 A monoidal comonad on a monoidal category C is a triple (T, ε, δ) , where (T, m) is a monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:



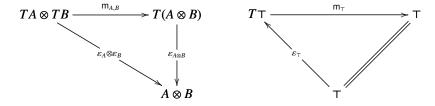
The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

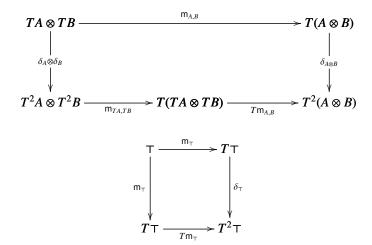


Definition 11 A symmetric monoidal comonad on a symmetric monoidal category C is a triple (T, ε, δ) , where (T, m) is a symmetric monoidal endofunctor on C, ε_A : $TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:



The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

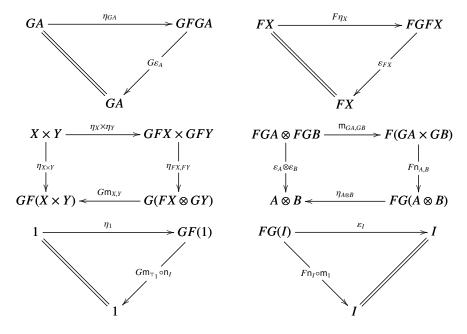




1.2 LNL Model

Definition 12 A linear/non-linear (LNL) model (C, \mathcal{L}, F, G) consists of

- a cartesian closed category $(C, 1, \times, \longrightarrow)$
- a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,
- a pair of symmetric monoidal functors $(G, n): \mathcal{L} \longrightarrow \mathcal{C}$ and $(F, m): \mathcal{C} \longrightarrow \mathcal{L}$ that form a symmetric monoidal adjunction $\mathcal{C}: F \dashv G: \mathcal{L}$, subject to the following coherence conditions, where η and ε are the unit and counit of the adjunction and $A, B \in Ob(\mathcal{L}), X, Y \in Ob(\mathcal{C})$.



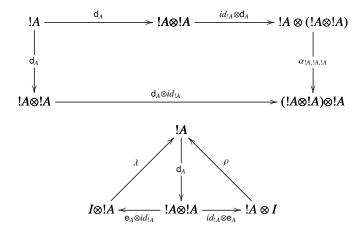
Lemma 13 Given a LNL model (C, \mathcal{L}, F, G) , define $p_{X,Y} : F(X \times Y) \longrightarrow FX \otimes FY$ as the composition $\varepsilon_{FX \otimes FY} \circ F\eta_{FX,FY} \circ F(\eta_X \times \eta_Y)$, and $p_I : F1 \longrightarrow I$ as the composition $\varepsilon_I \circ Fn_I$. Then for $F \dashv G$, $m_{X,Y}$ are components of a natural isomorphism with inverses $p_{X,Y}$, and m_1 is an isomorphism with inverse p_I , i.e. $F(X) \otimes F(Y) \cong F(X \times Y)$, and $I \cong F(1)$.

Lemma 14 Given a LNL model (C, \mathcal{L}, F, G) , the adjunction $F \dashv G$ induces a symmetric monoidal comonad $(!, \varepsilon, \delta)$ on \mathcal{L} , where ! represents FG, $\varepsilon : FG \longrightarrow 1$ is the counit of the adjunction, and $\delta : FG \longrightarrow FGFG$, i.e. ! is a symmetric monoidal functor and ε, δ are monoidal natural transformations.

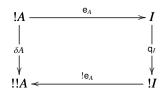
Lemma 15 Given a LNL model (C, \mathcal{L}, F, G) , F is a strong functor, i.e. F preserves the monoidal structure up to an isomorphism. And a strong functor induces a unique monoidal structure.

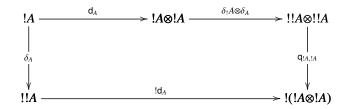
Definition 16 A linear category is specified by

- a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,
- a symmetric monoidal comonad $(!, \varepsilon, \delta)$ on \mathcal{L} , with $q_{A,B} : !A \otimes !A \longrightarrow !(A \otimes B)$ and $q_I : I \longrightarrow !I$;
- monoidal natural transformations on \mathcal{L} with components $e_A : !A \longrightarrow I$ and $d_A : !A \longrightarrow !A \otimes !A$, s.t.
 - each (!A, e_A , d_A) is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ d_A = d_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of \mathcal{L} ;

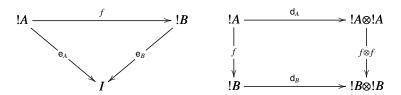


- e_A and d_A are coalgebra morphisms, i.e. the following diagrams commute;





- any coalgebra morphism $f:(!A,\delta_A) \longrightarrow (!B,\delta_B)$ between free coalgebras preserve the comonoid structure given by e and d, i.e. the following diagrams commute.

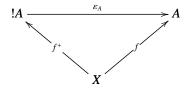


Theorem 17 $(!, \varepsilon, \delta)$ induces a symmetric monoidal adjunction $(\mathcal{L}^!, \otimes, I) : F \dashv G : (\mathcal{L}, \otimes, I)$, where \mathcal{L} is the category of Eilenberg-Moore coalgebras.

Theorem 18 Any LNL model is a linear category.

Theorem 19 Any linear category gives rise to an LNL model, though it is not in general unique.

Definition 20 A Lafont Category is a SMCC $(\mathcal{L}, \otimes, I)$ in which for every object A of \mathcal{L} , there exists a commutative comonoid $(!A, \mathsf{d}_A, \mathsf{e}_A)$ and a morphism $\varepsilon : !A \longrightarrow A$ s.t. for all commutative comonoid (X, d, e) and for all $f : X \longrightarrow A$, there exists a unique comonoid morphism $f^+ : (X, d, e) \longrightarrow (!A, \mathsf{d}_A, \mathsf{e}_A)$ s.t. the following diagram commutes:



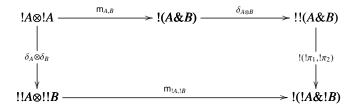
Alternative definition: a SMCC $(\mathcal{L}, \otimes, I)$ in which the forgetful functor $U : Comon(\mathcal{L}, \otimes, I) \longrightarrow \mathcal{L}$ has a right adjoint!, where $Comon(\mathcal{L}, \otimes, I)$ is the category of commutative comonoids in \mathcal{L} .

Theorem 21 $Comon(\mathcal{L}, \otimes, I)$ is cartesion. hence, every Lafont category defines a LNL model.

Definition 22 A *Seely Category* is a *SMCC* $(\mathcal{L}, \otimes, I)$ with products (&) and a terminal object \top , together with

• a comonad $(!, \delta, \varepsilon)$, where $\delta_A : !A \longrightarrow !!A$ and $\varepsilon_A : !A \longrightarrow A$, and

two natural isomorphisms m_{A,B}:!A⊗!B ≅!(A&B) and m_⊤: I ≅!T making (!, m):
(£, &, ⊤) → (£, ⊗, I) a symmetric monoidal functor s.t. the following diagram commutes:



Theorem 23 The comonad $(!, \delta, \varepsilon)$ on \mathcal{L} generates an adjunction $\mathcal{L}_!: L \dashv M: \mathcal{L}$ between \mathcal{L} and the (co-)Kleisli category $\mathcal{L}_!$ associated to the comonad. Thus, a Seely category defines a LNL model.

2 Related Work

TODO

3 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at http://research.microsoft.com/en-us/um/people/nick/mixed3.ps.