A Family of Adjoint Models as the Foundation of Process Tree Based Threat Analysis

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Abstract

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1 Introduction

TODO [1]

2 Categorical Models

We develop a categorical framework in which many different intuitionistic substructural logics may be modeled. The locus of this framework is an adjunction. We initially take a monoidal category, $\mathcal{L}$, as a base, and then extend it with one or more structural morphisms – a morphism corresponding to a structural rule in logic – to obtain a second category $\hat{\mathcal{L}}$. Then we form a monoidal adjunction $\hat{\mathcal{L}} : F \dashv G : \mathcal{L}$ just as Benton [1] did for intuitionistic linear logic. Depending on which structural morphisms we add to $\hat{\mathcal{L}}$ we will obtain different models. In particular, each model will come endowed with a comonad on $\mathcal{L}$ which equips $\mathcal{L}$ with the ability to track the corresponding structural rule(s).

We will show that by adding the morphisms for either weakening, contraction, or exchange, to $\mathcal{L}$ will yield an adjoint model of non-commutative relevance logic/linear logic, non-commutative contraction logic/linear logic, and commutative/non-commutative linear logic. The latter model will come with a monoidal comonad $e : \mathcal{L} \longrightarrow \mathcal{L}$ such that there is a symmetry $\mathsf{ex}_{A,B} : eA \triangleright eB \longrightarrow eB \triangleright eA$, where $- \triangleright -$ denotes a non-commutative tensor product. In fact, this is the first adjoint model of the Lambek calculus with the exchange comonad.

At this point we will have adjoint models for each individual structural rule. What if we want more than one structural rule? There are a few different choices that one can choose from depending on the scenario. First, if $\hat{\mathcal{L}}$ contains more than one structural rule, then $\mathcal{L}$ will have a single comonad that adds all of those structural rules to $\mathcal{L}$. For example, if $\hat{\mathcal{L}}$ contains weakening, contraction, and exchange, then $\hat{\mathcal{L}}$ is cartesian closed and $\mathcal{L}$ will have the usual $! : \mathcal{L} \longrightarrow \mathcal{L}$ comonad. The second scenario is when $\mathcal{L}$ also contains some structural rules. For example, if $\hat{\mathcal{L}}$ contains exchange and weakening and $\mathcal{L}$ contains exchange, then $\mathcal{L}$ will have a comonad, $r : \mathcal{L} \longrightarrow \mathcal{L}$, which combines linear logic with relevance logic. Thus, how we instantiate the two categories in the adjunction influences which logic one may model.

What if we want multiple comonads tracking different logics? In this scenario the different comonads would allow us to mix the different logics in interesting ways. Suppose $\mathcal{L}$ has no structural

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rules and $\mathcal{E}$ is $\mathcal{L}$ with exchange and $\mathcal{EW}$ is $\mathcal{E}$ with weakening. Then we can form two adjunctions $\mathcal{E} : F \dashv G : \mathcal{L}$ and $\mathcal{EW} : H \dashv J : \mathcal{L}$, but the categories $\mathcal{E}$ and $\mathcal{EW}$ have a structural rule in common. So instead, we form the adjunction $\mathcal{EW} : H \dashv J : \mathcal{E} : F \dashv G : \mathcal{L}$. Thus, $\mathcal{L}$ has the exchange comonad $e = FG : \mathcal{L} \longrightarrow \mathcal{L}$ as well as the relevance logic comonad $r = FHJG : \mathcal{L} \longrightarrow \mathcal{L}$. Additionally, there is a comonad $w = JH : \mathcal{E} \longrightarrow \mathcal{E}$ adding weakening to $\mathcal{E}$.

2.1 Lambek Categories

▶ **Definition 1.** A **monoidal category**, $(\mathcal{L}, \otimes, I, \lambda, \rho)$, is a category, $\mathcal{L}$, equipped with a bifunctor, $\otimes : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$, called the tensor product, a distinguished object I of $\mathcal{L}$ called the unit, and three natural isomorphisms $\lambda_A : I \otimes A \longrightarrow A$, $\rho_A : A \otimes I \longrightarrow A$, and $\alpha_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$ called the left and right unitors and the associator respectively. Finally, these are subject to the following coherence diagrams:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B \otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "\mathsf{id}_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

▶ **Definition 2.** A **Lambek category** is a monoidal category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ equipped with two bifunctors $\rightharpoonup : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ and $\leftharpoonup : \mathcal{L} \times \mathcal{L}^{\mathsf{op}} \longrightarrow \mathcal{L}$ that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \rightharpoonup B) \qquad \mathsf{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, B \leftharpoonup A)$$

One might call Lambek categories biclosed monoidal categories, but we name them in homage to Lambek for his contributions to non-commutative linear logic.

▶ **Definition 3.** A monoidal category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ is **symmetric** if there is a natural transformation $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$ such that $\beta_{B,A} \circ \beta_{A,B} = \mathsf{id}_{A \otimes B}$ and the following commute:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"'] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes A \arrow[rr, "\beta_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes I \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

▶ **Definition 4.** A symmetric monoidal category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \beta)$ is **closed** if it comes equipped with a bifunctor $\multimap : \mathcal{L}^{\mathsf{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$ that is right adjoint to the tensor product. That is, the following natural bijection $\mathsf{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \mathsf{Hom}_{\mathcal{L}}(X, A \multimap B)$ holds.

▶ **Definition 5.** A **Lambek category with weakening**, $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \mathsf{weak})$, is a Lambek category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ equipped with a natural transformation $\mathsf{weak}_A : A \longrightarrow I$.

▶ **Definition 6.** A **Lambek category with contraction**, $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \mathsf{contraL}, \mathsf{contraR})$, is a Lambek category $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$ equipped with natural transformations:

$$\mathsf{contraL}_{A,B} : (A \otimes B) \longrightarrow (A \otimes B) \otimes A \qquad \mathsf{contraR}_{A,B} : (B \otimes A) \longrightarrow A \otimes (B \otimes A)$$

Furthermore, the following diagrams must commute:

$$\begin{tikzcd}
A \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & & A \arrow[ll, "\rho_A^{-1}"'] \arrow[rr, "\lambda_A^{-1}"] & & I \otimes A \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(A \otimes I) \otimes A \arrow[rrrr, "\alpha_{A,I,A}"'] & & & & A \otimes (I \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[r, "id_A \otimes \rho_A^{-1}"] \arrow[dd, "\lambda_A^{-1} \otimes id_A"'] & A \otimes (A \otimes I) \arrow[r, "id_A \otimes \mathsf{contraL}_{A,I}"] & A \otimes ((A \otimes I) \otimes A) \arrow[d, "id_A \otimes (\rho_A \otimes id_A)"] \\
& & A \otimes (A \otimes A) \\
(I \otimes A) \otimes A \arrow[r, "\mathsf{contraR}_{A,I} \otimes id_A"'] & (A \otimes (I \otimes A)) \otimes A \arrow[r, "(id_A \otimes \lambda_A) \otimes id_A"'] & (A \otimes A) \otimes A \arrow[u, "\alpha_{A,A,A}"']
\end{tikzcd}$$

▶ **Definition 7.** A **Lambek category with exchange**, $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \mathsf{ex})$, is a Lambek category, $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$, such that $\mathcal{L}$ is symmetric monoidal, where $\mathsf{ex}_{A,B} : A \otimes B \longrightarrow B \otimes A$ is the symmetry.

▶ **Lemma 8.** *Let A and B be two objects in a Lambek category with exchange. Then $(A \rightharpoonup B) \cong (B \leftharpoonup A)$.*

Proof. First, notice that for any object C we have

$$\begin{array}{rcll}
Hom[C, A \rightharpoonup B] & \cong & Hom[C \otimes A, B] & \mathcal{L} \text{ is a Lambek category} \\
& \cong & Hom[A \otimes C, B] & \text{By the symmetry } \mathsf{ex}_{C,A} \\
& \cong & Hom[C, B \leftharpoonup A] & \mathcal{L} \text{ is a Lambek category}
\end{array}$$

Thus, $A \rightharpoonup B \cong B \leftharpoonup A$ by the Yoneda lemma. ◀

3 Related Work

TODO

4 Conclusion

TODO

References

1 P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.

A Appendix