

# On Linear Modalities for Exchange, Weakening, and Contraction

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## Abstract

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## 1 Introduction

TODO [1]

## 2 Categorical Models

### 2.1 Lambek Categories

► **Definition 1.** A **monoidal category**,  $(\mathcal{L}, \otimes, I, \lambda, \rho)$ , is a category,  $\mathcal{L}$ , equipped with a bifunctor,  $\otimes : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ , called the tensor product, a distinguished object  $I$  of  $\mathcal{L}$  called the unit, and three natural isomorphisms  $\lambda_A : I \otimes A \rightarrow A$ ,  $\rho_A : A \otimes I \rightarrow A$ , and  $\alpha_{A,B,C} : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$  called the left and right unitors and the associator respectively. Finally, these are subject to the following coherence diagrams:

$$\begin{array}{ccccc}
 ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes \text{id}_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\
 \downarrow \alpha_{A \otimes B,C,D} & & & & \downarrow \text{id}_A \otimes \alpha_{B,C,D} \\
 (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} & & & A \otimes (B \otimes (C \otimes D))
 \end{array}$$
  

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\
 \downarrow \rho_A \otimes \text{id}_B & & \downarrow \text{id}_A \otimes \lambda_B \\
 A \otimes B & & A \otimes B
 \end{array}$$

► **Definition 2.** A **Lambek category** is a monoidal category  $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$  equipped with two bifunctors  $\multimap : \mathcal{L}^{\text{op}} \times \mathcal{L} \rightarrow \mathcal{L}$  and  $\multimap : \mathcal{L} \times \mathcal{L}^{\text{op}} \rightarrow \mathcal{L}$  that are both right adjoint to the tensor product. That is, the following natural bijections hold:

$$\text{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \text{Hom}_{\mathcal{L}}(X, A \multimap B)$$

$$\text{Hom}_{\mathcal{L}}(A \otimes X, B) \cong \text{Hom}_{\mathcal{L}}(X, B \multimap A)$$



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One might call Lambek categories biclosed monoidal categories, but we name them in homage to Lambek for his contributions to non-commutative linear logic.

► **Definition 3.** A monoidal category  $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha)$  is **symmetric** if there is a natural transformation  $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$  such that  $\beta_{B,A} \circ \beta_{A,B} = \text{id}_{A \otimes B}$  and the following commute:

$$\begin{array}{ccccc}
 (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\beta_{A,B \otimes C}} & (B \otimes C) \otimes A \\
 \downarrow \beta_{A,B} \otimes \text{id}_C & & & & \downarrow \alpha_{B,C,A} \\
 (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{\text{id}_B \otimes \beta_{A,C}} & B \otimes (C \otimes A)
 \end{array}
 \quad
 \begin{array}{ccc}
 I \otimes A & \xrightarrow{\beta_{I,A}} & A \otimes I \\
 \searrow \lambda_A & & \swarrow \rho_A \\
 & A &
 \end{array}$$

► **Definition 4.** A symmetric monoidal category  $(\mathcal{L}, \otimes, I, \lambda, \rho, \alpha, \beta)$  is **closed** if it comes equipped with a bifunctor  $\multimap : \mathcal{L}^{\text{op}} \times \mathcal{L} \longrightarrow \mathcal{L}$  that is right adjoint to the tensor product. That is, the following natural bijection  $\text{Hom}_{\mathcal{L}}(X \otimes A, B) \cong \text{Hom}_{\mathcal{L}}(X, A \multimap B)$  holds.

## 2.2 Lambek Categories with Weakening and Contraction

► **Definition 5.** A **Lambek category with weakening**,  $(\mathcal{L}, w, \text{weak})$ , is a Lambek category equipped with a monoidal comonad  $(w, \varepsilon, \delta)$ , and a monoidal natural transformation  $\text{weak}_A : wA \longrightarrow I$ . Furthermore,  $\text{weak}$  must be a coalgebra morphism. That is, the following digram must commute:

$$\begin{array}{ccc}
 wA & \xrightarrow{\text{weak}_A} & I \\
 \delta_A \downarrow & & \downarrow q_I \\
 w^2A & \xrightarrow{w\text{weak}_A} & wI
 \end{array}$$

► **Definition 6.** A **Lambek category with contraction**,  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$ , is a Lambek category equipped with a monoidal comonad  $(c, \varepsilon, \delta)$ , and two monoidal natural transformations:

$$\begin{aligned}
 \text{contraL}_{A,B} &: cA \otimes B \longrightarrow (cA \otimes B) \otimes cA \\
 \text{contraR}_{A,B} &: B \otimes cA \longrightarrow cA \otimes (B \otimes cA)
 \end{aligned}$$

Furthermore, the following diagrams must commute:

$$\begin{array}{ccc}
 cA \otimes I & \xleftarrow{\rho_{cA}^{-1}} & cA \xrightarrow{\lambda_{cA}^{-1}} I \otimes cA \\
 \text{contraL}_{A,I} \downarrow & & \downarrow \text{contraR}_{A,I} \\
 (cA \otimes I) \otimes cA & \xrightarrow{\alpha_{cA,I,cA}} & cA \otimes (I \otimes cA)
 \end{array}$$
  

$$\begin{array}{ccccc}
 cA \otimes cA & \xrightarrow{id_{cA} \otimes \rho_{cA}^{-1}} & cA \otimes (cA \otimes I) & \xrightarrow{id_{cA} \otimes \text{contraL}_{A,I}} & cA \otimes ((cA \otimes I) \otimes cA) \\
 \downarrow \lambda_{cA}^{-1} \otimes id_{cA} & & & & \downarrow id_{cA} \otimes (\rho_{cA} \otimes id_{cA}) \\
 (I \otimes cA) \otimes cA & \xrightarrow{\text{contraR}_{A,I} \otimes id_{cA}} & (cA \otimes (I \otimes cA)) \otimes cA & \xrightarrow{(id_{cA} \otimes \lambda_{cA}) \otimes id_{cA}} & (cA \otimes cA) \otimes cA \\
 & & & & \uparrow \alpha_{cA,cA,cA}
 \end{array}$$

### 2.3 Lambek Categories with Exchange

► **Definition 7.** A **Lambek category with exchange**,  $(\mathcal{L}, e, \text{ex})$ , is a Lambek category equipped with a monoidal comonad  $(e, \varepsilon, \delta)$  on  $\mathcal{L}$ , and a monoidal natural transformation  $\text{ex}_{A,B} : eA \otimes eB \longrightarrow eB \otimes eA$ . We require  $\text{ex}$  to be a coalgebra morphism, and the following diagrams must commute:

$$\begin{array}{c}
 \begin{array}{ccc}
 e^2A \otimes e^2B & \xrightarrow{\text{ex}_{eA,eB}} & e^2B \otimes e^2A \\
 \downarrow q_{eA,eB} & & \downarrow q_{eB,eA} \\
 e(eA \otimes eB) & \xrightarrow{e\text{ex}_{A,B}} & e(eB \otimes eA)
 \end{array}
 \quad
 \begin{array}{ccc}
 eA \otimes eB & \xrightarrow{\text{ex}_{A,B}} & eB \otimes eA \\
 \searrow & & \downarrow \text{ex}_{B,A} \\
 & & eA \otimes eB
 \end{array}
 \quad
 \begin{array}{ccc}
 eI \otimes eA & \xrightarrow{\text{ex}_{I,A}} & eA \otimes eI \\
 \downarrow \varepsilon_I \otimes \text{id} & & \downarrow \text{id} \otimes \varepsilon_I \\
 I \otimes eA & & eA \otimes I \\
 \searrow \lambda_{eA} & & \swarrow \rho_{eA} \\
 & eA &
 \end{array}
 \\
 \\
 \begin{array}{ccccc}
 (eA \otimes eB) \otimes eC & \xrightarrow{\alpha_{eA,eB,eC}} & eA \otimes (eB \otimes eC) & \xrightarrow{\text{id}_{eA} \otimes (\delta_B \otimes \delta_C)} & eA \otimes (e^2B \otimes e^2C) & \xrightarrow{\text{id}_{eA} \otimes q_{eB,eC}} & eA \otimes e(eB \otimes eC) \\
 \downarrow \text{ex}_{A,B} \otimes \text{id}_{eC} & & \downarrow \text{ex}_{A,B} \otimes \text{id}_{eC} & & \downarrow \text{ex}_{A,B} \otimes \text{id}_{eC} & & \downarrow \text{ex}_{A,B} \otimes \text{id}_{eC} \\
 (eB \otimes eA) \otimes eC & & & & & & e(eB \otimes eC) \otimes eA \\
 \downarrow \alpha_{eB,eA,eC} & & & & & & \downarrow \varepsilon_{eB \otimes eC} \otimes \text{id}_{eA} \\
 eB \otimes (eA \otimes eC) & \xrightarrow{\text{id}_{eB} \otimes \text{ex}_{A,C}} & eB \otimes (eC \otimes eA) & \xleftarrow{\alpha_{eB,eC,eA}} & (eB \otimes eC) \otimes eA & & 
 \end{array}
 \end{array}$$

Furthermore, for any coalgebra morphisms  $f : (eA, \delta) \longrightarrow (eB, \delta)$  and  $g : (eC, \delta) \longrightarrow (eD, \delta)$  between free coalgebras the following diagram must commute:

$$\begin{array}{ccc}
 eA \otimes eC & \xrightarrow{f \otimes g} & eB \otimes eD \\
 \downarrow \text{ex}_{A,C} & & \downarrow \text{ex}_{B,D} \\
 eC \otimes eA & \xrightarrow{g \otimes f} & eD \otimes eB
 \end{array}$$

The morphism  $q_{A,B} : eA \otimes eB \longrightarrow e(A \otimes B)$  makes  $(e, q)$  a monoidal functor.

The first diagram in the previous definition makes  $e : \mathcal{L} \longrightarrow \mathcal{L}$  a symmetric monoidal functor, and the second, third, and forth diagrams make the category of free coalgebras (the Kleisli category) symmetric monoidal.

► **Definition 8.** Suppose  $(\mathcal{L}, e, \text{ex})$  is a Lambek category with exchange. Then the **Eilenberg Moore category**,  $\mathcal{L}^e$ , of the comonad  $(e, \varepsilon, \delta)$  has as objects all the  $e$ -coalgebras  $(A, h_A : A \longrightarrow eA)$ , and as morphisms all the coalgebra morphisms. We call  $h_A$  the action of the coalgebra. Furthermore, the following (action) diagrams must commute:

$$\begin{array}{ccc}
 A & \xrightarrow{h_A} & eA \\
 h_A \downarrow & & \downarrow eh_A \\
 eA & \xrightarrow{\delta_A} & e^2A
 \end{array}
 \quad
 \begin{array}{ccc}
 A & & \\
 h_A \downarrow & \searrow & \\
 eA & \xrightarrow{\varepsilon_A} & A
 \end{array}$$

► **Lemma 9** (The Eilenberg Moore Category is Monoidal). *The category  $\mathcal{L}^e$  is monoidal.*

**Proof.** For the complete proof see Appendix B.1.1. ◀

► **Lemma 10.** *In  $\mathcal{L}^e$  there is a natural transformation  $\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$ .*

**Proof.** We define  $\beta$  as follows:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{ex_{A,B}} eB \otimes eA \xrightarrow{\varepsilon_B \otimes \varepsilon_A} B \otimes A$$

For the proof that it is natural please see Appendix B.1.2.  $\blacktriangleleft$

► **Corollary 11.** For any coalgebras  $(A, h_A)$  and  $(B, h_B)$  the following commutes:

$$\begin{array}{ccccccc} A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{ex_{A,B}} & eB \otimes eA & \xrightarrow{\varepsilon_B \otimes \varepsilon_A} & B \otimes A \\ \parallel & & & & & & \downarrow h_B \otimes h_A \\ A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{ex_{A,B}} & eB \otimes eA & \xlongequal{\quad} & eB \otimes eA \end{array}$$

**Proof.** For the complete proof please see Appendix B.1.3.  $\blacktriangleleft$

► **Definition 12.** Given two parallel arrows  $f, g : B \rightarrow C$  in a category  $\mathcal{C}$ , a **cofork** is a morphism  $c : A \rightarrow B$  such that the diagram  $A \xrightarrow{c} B \xrightarrow[f]{g} C$  commutes. That is,  $f \circ c = g \circ c$ .

► **Lemma 13.** The morphism  $ex_{A,B} \circ (h_A \otimes h_B)$  is a cofork of the morphisms  $(h_B \otimes h_A) \circ (\varepsilon_B \otimes \varepsilon_A)$  and  $(e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A)$ .

**Proof.** This proof holds by straightforward equational reasoning. For the complete proof please see Appendix B.1.4.  $\blacktriangleleft$

► **Lemma 14.** In  $\mathcal{L}^e$ ,  $\beta$  is a coalgebra morphism.

**Proof.** For the complete proof see Appendix B.1.5.  $\blacktriangleleft$

► **Lemma 15** (The Eilenberg-Moore Category is Symmetric Monoidal). The category  $\mathcal{L}^e$  is symmetric monoidal.

**Proof.** For the complete proof please see Appendix B.1.6.  $\blacktriangleleft$

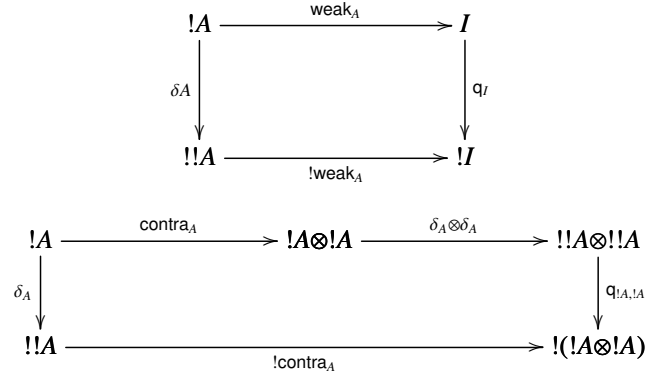
## 2.4 Linear Categories

► **Definition 16.** A **linear category**,  $(\mathcal{L}, !, \text{weak}, \text{contra})$ , is a symmetric monoidal closed category  $(\mathcal{L}, I, \otimes, \multimap)$  equipped with a symmetric monoidal comonad  $(!, \varepsilon, \delta)$  with  $q_{A,B} : !A \otimes !B \rightarrow !(A \otimes B)$  and  $q_I : I \rightarrow !I$ , and two monoidal natural transformations with components  $\text{weak}_A : !A \rightarrow I$  and  $\text{contra}_A : !A \rightarrow !A \otimes !A$ , satisfying the following conditions:

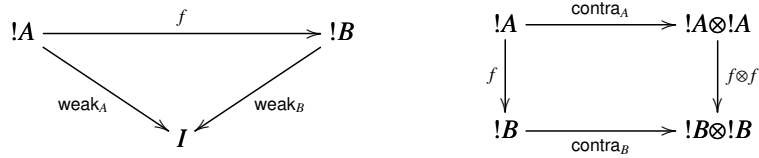
- each  $(!A, \text{weak}_A, \text{contra}_A)$  is a commutative comonoid, i.e. the following diagrams commute and  $\beta \circ \text{contra}_A = \text{contra}_A$  where  $\beta_{B,C} : B \otimes C \rightarrow C \otimes B$  is the symmetry natural transformation of  $\mathcal{L}$ ;

$$\begin{array}{ccccc} !A & \xrightarrow{\text{contra}_A} & !A \otimes !A & \xrightarrow{id_{!A} \otimes \text{contra}_A} & !A \otimes (!A \otimes !A) \\ \text{contra}_A \downarrow & & & & \uparrow \alpha_{!A, !A, !A} \\ !A \otimes !A & \xrightarrow{\text{contra}_A \otimes id_{!A}} & & & (!A \otimes !A) \otimes !A \\ & & & & \\ & & !A & & \\ & \swarrow \lambda^{-1} \quad \downarrow \text{contra}_A \quad \searrow \rho^{-1} & & & \\ I \otimes !A & \xleftarrow{\text{weak}_A \otimes id_{!A}} & !A \otimes !A & \xrightarrow{id_{!A} \otimes \text{weak}_A} & !A \otimes I \end{array}$$

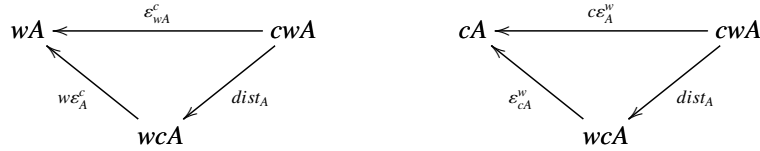
- $\text{weak}_A$  and  $\text{contra}_A$  are coalgebra morphisms, i.e. the following diagrams commute;



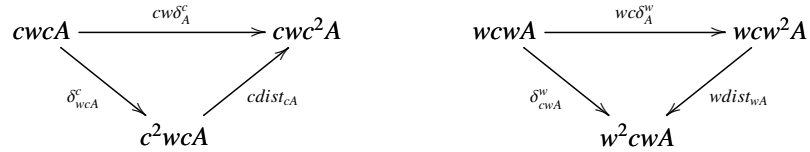
- any coalgebra morphism  $f : (!A, \delta_A) \rightarrow (!B, \delta_B)$  between free coalgebras preserve the comonoid structure given by  $\text{weak}$  and  $\text{contra}$ , i.e. the following diagrams commute.



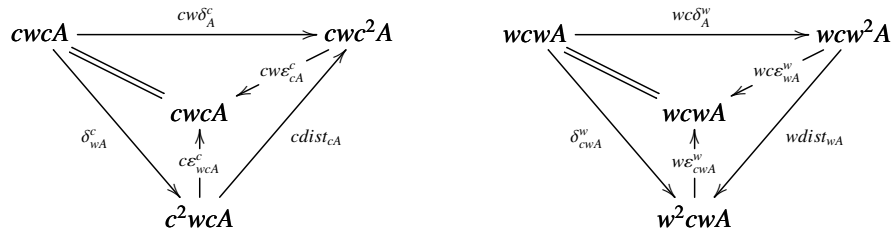
► **Definition 17.** Given two comonads  $(c, \varepsilon^c, \delta^c)$  and  $(w, \varepsilon^w, \delta^w)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction and  $(\mathcal{L}, w, \text{weak})$  is a Lambek category with weakening, we define a **distributive law** of  $c$  over  $w$  to be a natural transformation with components  $\text{dist}_A : cwA \rightarrow wcA$ , subject to the following coherence diagrams:



► **Lemma 18.** Given two comonads  $(c, \varepsilon^c, \delta^c)$  and  $(w, \varepsilon^w, \delta^w)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction and  $(\mathcal{L}, w, \text{weak})$  is a Lambek category with weakening, the following two diagrams commute:



**Proof.** The two diagrams above commute because the following ones commute by the distributive law and the comonad laws for  $c$  and  $w$ .



► **Lemma 19** (Composition of Weakening and Contraction). *Suppose  $(\mathcal{L}, I, \otimes, w, \text{weak}^w, c, \text{contraL}, \text{contraR})$  is a Lambek category with weakening and contraction, where  $(w, \varepsilon^w, \delta^w)$  and  $(c, \varepsilon^c, \delta^c)$  are the respective monoidal comonads. Then the composition of  $c$  and  $w$  using the distributive law  $\text{dist}_A : cwA \longrightarrow wcA$  is a monoidal comonad on  $\mathcal{L}$ .*

**Proof.** For the complete proof see Appendix B.3. ◀

► **Definition 20.** A **Lambek category with  $cw$** ,  $(\mathcal{L}, cw, \text{weak}^w, \text{contraL}, \text{contraR}, \text{dist})$ , is a Lambek category with weakening and contraction, and a distributive law. Furthermore, the following coherence diagrams commute:

$$\begin{array}{ccc}
 I \otimes cwA & \xrightarrow{\lambda_{I \otimes cwA}^{-1}} & I \otimes (I \otimes cwA) \\
 \downarrow \text{contraR}_{wA, I} & & \uparrow \text{weak}_A^w \otimes id_{I \otimes cwA} \\
 cwA \otimes (I \otimes cwA) & \xrightarrow{\varepsilon_{wA}^c \otimes id_{I \otimes cwA}} & wA \otimes (I \otimes cwA)
 \end{array}$$
  

$$\begin{array}{ccc}
 cwA \otimes I & \xrightarrow{\rho_{cwA \otimes I}^{-1}} & (cwA \otimes I) \otimes I \\
 \downarrow \text{contraL}_{wA, I} & & \uparrow id_{cwA \otimes I} \otimes \text{weak}_A^w \\
 (cwA \otimes I) \otimes cwA & \xrightarrow{id_{cwA \otimes I} \otimes \varepsilon_{wA}^c} & (cwA \otimes I) \otimes wA
 \end{array}$$
  

$$\begin{array}{ccc}
 cwA & \xrightarrow{f} & cwB \\
 \downarrow \varepsilon_{wA}^c & & \downarrow \varepsilon_{wB}^c \\
 wA & \xrightarrow{\text{weak}_A^w} I & \xleftarrow{\text{weak}_B^w} wB
 \end{array}$$

where  $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$  is any coalgebra morphism between free coalgebras.

► **Lemma 21.** *Let  $(\mathcal{L}, cw, \text{weak}^w, \text{contraL}, \text{contraR})$  be a Lambek category with  $cw$ . Then the following conditions are satisfied:*

1. *There exist two natural transformations  $\text{weak}_A : cwA \longrightarrow I$  and  $\text{contra}_A : cwA \longrightarrow cwA \otimes cwA$ .*
2. *Each  $(cwA, \text{weak}_A, \text{contra}_A)$  is a comonoid.*
3.  *$\text{weak}_A$  and  $\text{contra}_A$  are coalgebra morphisms.*
4. *Any coalgebra morphism  $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$  between free coalgebras preserves the comonoid structure given by  $\text{weak}$  and  $\text{contra}$ .*

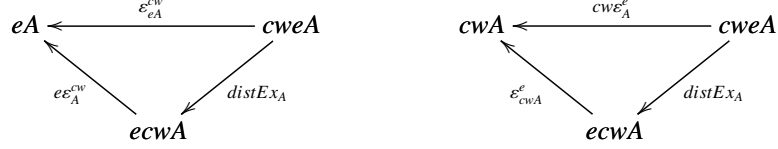
**Proof.** We will only prove the first condition by defining  $\text{weak}$  and  $\text{contra}$ . For the complete proof see Appendix B.4. Each of  $\text{weak}$  and  $\text{contra}$  can be given two equivalent definitions.  $\text{weak}_A : cwA \longrightarrow I$  is defined as in the diagram below. The left triangle commutes by the definition of  $\text{dist}$  and the right triangle commutes by the definition of  $\text{weak}^w$ .

$$\begin{array}{ccccc}
 & & wcA & & \\
 & \nearrow \text{dist}_A & \downarrow w\varepsilon_A^c & \searrow \text{weak}_{cA}^w & \\
 cwA & \xrightarrow{\varepsilon_{wA}^c} & wA & \xrightarrow{\text{weak}_A^w} & I
 \end{array}$$

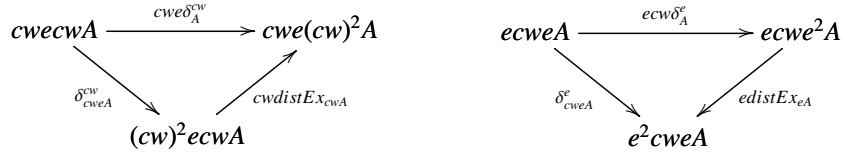
$\text{contra}_A : cwA \longrightarrow cwA \otimes cwA$  is defined as below. The left part of the diagram commutes by the definitions of  $\text{contraL}$  and of  $\text{contraR}$ , and the right part commutes because  $\mathcal{L}$  is monoidal.

$$\begin{array}{ccccc}
 cwA & \xrightarrow{\rho_{cwA}^{-1}} & cwA \otimes I & \xrightarrow{\text{contraL}_{wA, I}} & (cwA \otimes I) \otimes cwA \\
 \downarrow \lambda_{cwA}^{-1} & & & \swarrow \alpha_{cwA, I, cwA} & \downarrow \rho_{cwA} \otimes id_{cwA} \\
 I \otimes cwA & \xrightarrow{\text{contraR}_{wA, I}} & cwA \otimes (I \otimes cwA) & \xrightarrow{id_{cwA} \otimes \lambda_{cwA}} & cwA \otimes cwA
 \end{array}$$

► **Definition 22.** Given two comonads  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, cw, \text{weak}, \text{contra})$  is a Lambek category with  $cw$  and  $(\mathcal{L}, e, \text{ex})$  is a Lambek category with exchange, we define a **distributive law for exchange** of  $cw$  over  $e$  to be a natural isomorphism with components  $\text{distEx}_A : cweA \longrightarrow ecwA$ , subject to the following coherence diagrams:



► **Lemma 23.** Given two comonads  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  on a category  $\mathcal{L}$  such that  $(\mathcal{L}, cw, \text{weak}, \text{contra})$  is a Lambek category with  $cw$  and  $(\mathcal{L}, e, \text{ex})$  is a Lambek category with exchange, the following two digrams also commute:



The proof is similar with the proof of Lemma 18 and we will not elaborate it here. Also, notice the difference between  $\text{dist}$  of  $c$  over  $w$  and  $\text{distEx}$  of  $cw$  over  $e$ . While  $\text{dist}$  is a natural transformation,  $\text{distEx}$  is a natural isomorphism.

► **Lemma 24.** let  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  be two monoidal comonads on a Lambek category with  $cw$  and exchange  $(\mathcal{L}, I, \otimes, cw, \text{weak}, \text{contra}, e, \text{ex})$ . Then the composition of  $cw$  and  $e$  using the distributive law for exchange  $\text{distEx}_A : cweA \longrightarrow ecwA$  is a monoidal comonad  $(cwe, \varepsilon, \delta)$  on  $\mathcal{L}$ .

**Proof.** Suppose  $(cw, \varepsilon^{cw}, \delta^{cw})$  and  $(e, \varepsilon^e, \delta^e)$  are monoidal comonads, and  $(\mathcal{L}, I, \otimes, cw, \text{weak}, \text{contra}, e, \text{ex})$  is a Lambek category with  $cw$  and exchange. Since by definition  $cw, e : \mathcal{L} \longrightarrow \mathcal{L}$  are monoidal functors, we know that their composition  $cwe : \mathcal{L} \longrightarrow \mathcal{L}$  is a monoidal functor:

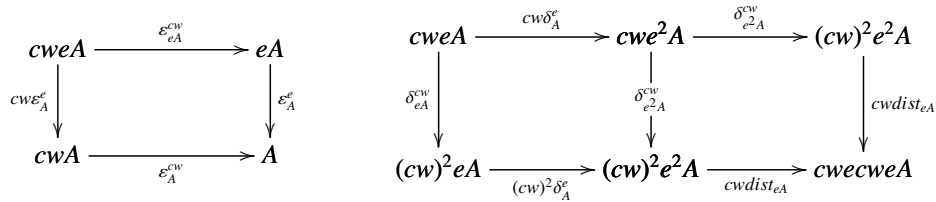
$$q_{A,B} : cweA \otimes cweB \longrightarrow cwe(A \otimes B)$$

$$q_{A,B} = cwq_{A,B}^e \circ q_{eA,eB}^{cw}$$

$$q_I : I \longrightarrow cweI$$

$$q_I = cwq_I^e \circ q_I^{cw}$$

Analogous to the proof of Lemma 19, each of  $\varepsilon$  and  $\delta$  can be given two equivalent definitions:



And the comonad laws can be proved similarly, which we will not elaborate for simplicity. ◀

► **Lemma 25.** Let  $(cwe, \varepsilon, \delta)$  be a monoidal comonad over a monoidal category  $(\mathcal{L}, I, \otimes)$  such that  $(\mathcal{L}, I, \otimes, cw, \text{weak}, \text{contra}, e, \text{ex})$  is a Lambek category with  $cw$  and exchange. Then the co-Kleisli category of  $\mathcal{L}$ ,  $\mathcal{L}_{cwe}$ , is a linear category.

changed to  
linar category.  
Finish the  
proof when  
proved.

**Proof.** The identity object of  $\mathcal{L}_{cwe}$  is still  $I$ .

The left and right unitors,  $\hat{\lambda}_A : I \otimes A \rightarrow A$  and  $\hat{\rho}_A : A \otimes I \rightarrow A$ , in  $\mathcal{L}_{cwe}$  are morphisms  $cwe(I \otimes A) \rightarrow A$  and  $cwe(A \otimes I) \rightarrow A$  in  $\mathcal{L}$ , respectively. Then we define  $\hat{\lambda}$  and  $\hat{\rho}$  as:

$$\hat{\lambda}_A = \varepsilon_A \circ cwe\lambda_A$$

$$\hat{\rho}_A = \varepsilon_A \circ cwe\rho_A,$$

where  $\lambda$  and  $\rho$  are the left and right unitors in  $\mathcal{L}$ , respectively. And we define their inverses as:

$$\hat{\lambda}_A^{-1} = \varepsilon_{I \otimes A} \circ cwe\lambda_A^{-1}$$

$$\hat{\rho}_A^{-1} = \varepsilon_{A \otimes I} \circ cwe\rho_A^{-1}$$

$\hat{\lambda}$  is a natural isomorphism with inverse  $\hat{\lambda}^{-1}$  because the following diagram chasing commutes:

$$\begin{array}{ccccc}
 cwe(I \otimes A) & \xrightarrow{\delta_{I \otimes A}} & (cwe)^2(I \otimes A) & \xrightarrow{(cwe)^2\lambda_A} & (cwe)^2A \\
 \downarrow \varepsilon_{I \otimes A} & \searrow cwe\lambda_A & \downarrow & \nearrow \delta_A & \downarrow cwe\varepsilon_A \\
 & (1) & & & \\
 & (3) \ cweA & & (5) & \\
 & \downarrow cwe\lambda_A^{-1} & & & \\
 I \otimes A & \xleftarrow{\varepsilon_{I \otimes A}} & cwe(I \otimes A) & \xleftarrow{cwe\lambda_A^{-1}} & cweA
 \end{array}$$

(1) commutes by the naturality of  $\delta$ . (2), (3) and (4) commute trivially. And (5) commutes because  $cwe$  is a comonad.

Similarly,  $\hat{\rho}$  is a natural isomorphism with inverse  $\hat{\rho}^{-1}$  by the following diagram chasing:

$$\begin{array}{ccccc}
 cwe(A \otimes I) & \xrightarrow{\delta_{A \otimes I}} & (cwe)^2(A \otimes I) & \xrightarrow{(cwe)^2\rho_A} & (cwe)^2A \\
 \downarrow \varepsilon_{A \otimes I} & \searrow cwe\rho_A & \downarrow & \nearrow \delta_A & \downarrow cwe\varepsilon_A \\
 & & & & \\
 & & cweA & & \\
 & \downarrow cwe\rho_A^{-1} & & & \\
 A \otimes I & \xleftarrow{\varepsilon_{A \otimes I}} & cwe(A \otimes I) & \xleftarrow{cwe\rho_A^{-1}} & cweA
 \end{array}$$

The associator  $\hat{\alpha}_A : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$  in  $\mathcal{L}_{cwe}$  is the morphism  $cwe((A \otimes B) \otimes C) \rightarrow A \otimes (B \otimes C)$  in  $\mathcal{L}$ . We define  $\hat{\alpha}$  as:

$$\hat{\alpha}_{A,B,C} = \varepsilon_{A \otimes (B \otimes C)} \circ cwe\alpha_{A,B,C},$$

where  $\alpha$  is the associator of  $\mathcal{L}$ . And its inverse is

$$\hat{\alpha}_{A,B,C}^{-1} = \varepsilon_{(A \otimes B) \otimes C} \circ cwe\alpha_{A,B,C}^{-1}$$

$\hat{\alpha}$  is a natural isomorphism with inverse  $\hat{\alpha}^{-1}$  because the following diagram chasing commutes:

$$\begin{array}{ccccc}
 cwe((A \otimes B) \otimes C) & \xrightarrow{\delta_{(A \otimes B) \otimes C}} & (cwe)^2((A \otimes B) \otimes C) & \xrightarrow{(cwe)^2\alpha_{A,B,C}} & (cwe)^2(A \otimes (B \otimes C)) \\
 \downarrow \varepsilon_{(A \otimes B) \otimes C} & \searrow cwe\alpha_{A,B,C} & \downarrow & \nearrow \delta_{A \otimes (B \otimes C)} & \downarrow cwe\varepsilon_{A \otimes (B \otimes C)} \\
 & & & & \\
 & & cwe(A \otimes (B \otimes C)) & & \\
 & \downarrow cwe\alpha_{A,B,C}^{-1} & & & \\
 (A \otimes B) \otimes C & \xleftarrow{\varepsilon_{(A \otimes B) \otimes C}} & cwe((A \otimes B) \otimes C) & \xleftarrow{cwe\alpha_{A,B,C}^{-1}} & cwe(A \otimes (B \otimes C))
 \end{array}$$





**XX:10** On Linear Modalities for Exchange, Weakening, and Contraction

$FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$  subject to the following coherence conditions:

$$\begin{array}{ccc}
 (FA \otimes_2 FB) \otimes_2 FC & \xrightarrow{\alpha_{2FA,FB,FC}} & FA \otimes_2 (FB \otimes_2 FC) \\
 \downarrow m_{A,B} \otimes \text{id}_{FC} & & \downarrow \text{id}_{FA} \otimes m_{B,C} \\
 F(A \otimes_1 B) \otimes_2 FC & & FA \otimes_2 F(B \otimes_1 C) \\
 \downarrow m_{A \otimes_1 B, C} & & \downarrow m_{A, B \otimes_1 C} \\
 F((A \otimes_1 B) \otimes_1 C) & \xrightarrow{F\alpha_{1A,B,C}} & F(A \otimes_1 (B \otimes_1 C))
 \end{array}$$
  

$$\begin{array}{ccc}
 I_2 \otimes_2 FA & \xrightarrow{\lambda_{2FA}} & FA \\
 \downarrow m_{I_1} \otimes \text{id}_{FA} & & \uparrow F\lambda_{1A} \\
 FI_1 \otimes_2 FA & \xrightarrow{m_{I_1, A}} & F(I_1 \otimes_1 A)
 \end{array}
 \qquad
 \begin{array}{ccc}
 FA \otimes_2 I_2 & \xrightarrow{\rho_{2FA}} & FA \\
 \downarrow \text{id}_{FA} \otimes m_{I_1} & & \uparrow F\rho_{1A} \\
 FA \otimes_2 FI_1 & \xrightarrow{m_{A, I_1}} & F(A \otimes_1 I_1)
 \end{array}$$

Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

► **Definition 27.** Suppose we are given two symmetric monoidal closed categories  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Then a **symmetric monoidal functor** is a monoidal functor  $(F, m) : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$  subject to the following additional coherence condition:

$$\begin{array}{ccc}
 FA \otimes_2 FB & \xrightarrow{\beta_{2FA,FB}} & FB \otimes_2 FA \\
 \downarrow m_{A,B} & & \downarrow m_{B,A} \\
 F(A \otimes_1 B) & \xrightarrow{F\beta_{1A,B}} & F(B \otimes_1 A)
 \end{array}$$

► **Definition 28.** Suppose  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are monoidal categories, and  $(F, m)$  and  $(G, n)$  are monoidal functors between  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Then a **monoidal natural transformation** is a natural transformation,  $f : F \longrightarrow G$ , subject to the following coherence diagrams:

$$\begin{array}{ccc}
 FA \otimes_2 FB & \xrightarrow{m_{A,B}} & F(A \otimes_1 B) \\
 \downarrow f_A \otimes_2 f_B & & \downarrow f_{A \otimes_1 B} \\
 GA \otimes_2 GB & \xrightarrow{n_{A,B}} & G(A \otimes_1 B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 FI_1 & \xrightarrow{f_{I_1}} & GI_1 \\
 \swarrow m_{I_1} & & \searrow n_{I_1} \\
 & I_2 &
 \end{array}$$

► **Definition 29.** Suppose  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are monoidal categories, and  $(F, m)$  is a monoidal functor between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  and  $(G, n)$  is a monoidal functor between  $\mathcal{M}_2$  and  $\mathcal{M}_1$ . Then a **monoidal adjunction** is an ordinary adjunction  $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$  such that the unit,  $\eta_A : A \rightarrow GFA$ , and the counit,  $\varepsilon_A : FGA \rightarrow A$ , are monoidal natural transformations. Thus, the following diagrams must

commute:

$$\begin{array}{ccc}
 FGA \otimes_2 FGB & \xrightarrow{m_{GA,GB}} & F(GA \otimes_1 GB) \\
 \downarrow \varepsilon_A \otimes_1 \varepsilon_B & & \downarrow F\eta_{A,B} \\
 A \otimes_2 B & \xleftarrow{\varepsilon_{A \otimes_1 B}} & FGA \otimes_2 FGB
 \end{array}
 \qquad
 \begin{array}{ccc}
 FI_1 & \xrightarrow{Fn_{I_2}} & FGI_2 \\
 \uparrow m_{I_1} & & \downarrow \varepsilon_{I_1} \\
 I_2 & \xlongequal{\quad} & I_2
 \end{array}$$

$$\begin{array}{ccc}
 GFA \otimes_1 GFB & \xleftarrow{\eta_A \otimes_1 \eta_B} & A \otimes_1 B \\
 \downarrow n_{FA,FB} & & \downarrow \eta_{A \otimes_1 B} \\
 G(FA \otimes_2 FB) & \xrightarrow{m_{A,B}} & GF(A \otimes_1 B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 GI_2 & \xrightarrow{Gm_{I_1}} & GFI_1 \\
 \uparrow n_{I_2} & & \uparrow \eta_{I_1} \\
 I_1 & \xlongequal{\quad} & I_1
 \end{array}$$

► **Definition 30.** A **monoidal comonad** on a monoidal category  $\mathcal{C}$  is a triple  $(T, \varepsilon, \delta)$ , where  $(T, m)$  is a monoidal endofunctor on  $\mathcal{C}$ ,  $\varepsilon_A : TA \rightarrow A$  and  $\delta_A : TA \rightarrow T^2A$  are monoidal natural transformations, which make the following diagrams commute:

$$\begin{array}{ccc}
 TA & \xrightarrow{\delta_A} & T^2A \\
 \downarrow \delta_A & & \downarrow T\delta_A \\
 T^2A & \xrightarrow{\delta_{TA}} & T^3A
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & TA & & \\
 & \swarrow & \downarrow \delta_A & \searrow & \\
 TA & \xleftarrow{\varepsilon_{TA}} & T^2A & \xrightarrow{T\varepsilon_A} & TA
 \end{array}$$

The assumption that  $\varepsilon$  and  $\delta$  are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{array}{ccc}
 TA \otimes TB & \xrightarrow{m_{A,B}} & T(A \otimes B) \\
 \searrow \varepsilon_A \otimes \varepsilon_B & & \downarrow \varepsilon_{A \otimes B} \\
 & & A \otimes B
 \end{array}
 \qquad
 \begin{array}{ccc}
 TI & \xleftarrow{m_I} & I \\
 \searrow \varepsilon_I & & \downarrow \\
 & & I
 \end{array}$$

$$\begin{array}{ccccc}
 TA \otimes TB & \xrightarrow{m_{A,B}} & T(A \otimes B) & & \\
 \downarrow \delta_A \otimes \delta_B & & \downarrow \delta_{A \otimes B} & & \\
 T^2A \otimes T^2B & \xrightarrow{m_{TA,TB}} & T(TA \otimes TB) & \xrightarrow{Tm_{A,B}} & T^2(A \otimes B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 I & \xrightarrow{m_I} & TI \\
 \downarrow m_I & & \downarrow \delta_I \\
 TI & \xrightarrow{Tm_I} & T^2I
 \end{array}$$

## B Proofs

### B.1 Lambek Categories with Exchange

#### B.1.1 Proof of The Eilenberg-Moore Category is Monoidal (Lemma 9)

We must first define the unitors, and then the associator. Then we show that they respect the symmetry monoidal coherence diagrams. Throughout this proof we will make use of the coalgebra  $(A, h_A)$ ,  $(B, h_B)$ , and  $(C, h_C)$ .

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The tensor product of  $(A, h_A)$  and  $(B, h_B)$  is  $(A \otimes B, q_{A,B} \circ (h_A \otimes h_B))$ , and the unit of the tensor product is  $(I, q_I)$ ; both actions are easily shown to satisfy the action diagrams of the Eilenberg Moore category. The left and right unitors are  $\lambda : I \otimes A \rightarrow A$  and  $\rho : A \otimes I \rightarrow A$ , because they are indeed coalgebra morphisms.

The respective diagram for the right unitor is as follows:

$$\begin{array}{ccccc}
 A \otimes I & \xrightarrow{h_A \otimes \text{id}} & eA \otimes I & \xrightarrow{\text{id} \otimes q_I} & eA \otimes eI & \xrightarrow{q_{A,I}} & e(A \otimes I) \\
 \downarrow \rho & & & \searrow \rho & & & \downarrow e\rho \\
 A & \xrightarrow{h_A} & & & & & eA
 \end{array}$$

The left diagram commutes by naturality of  $\rho$ , the right diagram commutes by the fact that  $e$  is a monoidal functor. Showing the left unitor is a coalgebra morphism is similar.

The unitors are natural isomorphisms, because they are essentially inherited from the underlying Lambek category.

The associator  $\alpha : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$  is also a coalgebra morphism. First, notice that:

$$q_{A \otimes B, C} \circ ((q_{A,B} \circ (h_A \otimes h_B)) \otimes h_C) = q_{A \otimes B, C} \circ (q_{A,B} \otimes \text{id}) \circ ((h_A \otimes h_B) \otimes h_C)$$

where the left-hand side is the action of the coalgebra  $(A \otimes B) \otimes C$ . Similarly, the following is the action of the coalgebra  $A \otimes (B \otimes C)$ :

$$q_{A, B \otimes C} \circ (h_A \otimes (q_{B,C} \circ (h_B \otimes h_C))) = q_{A, B \otimes C} \circ (\text{id} \otimes q_{B,C}) \circ (h_A \otimes (h_B \otimes h_C))$$

The following diagram must commute:

$$\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{(h_A \otimes h_B) \otimes h_C} & (eA \otimes eB) \otimes eC \\
\downarrow \alpha & & \downarrow \alpha \\
A \otimes (B \otimes C) & \xrightarrow{h_A \otimes (h_B \otimes h_C)} & eA \otimes (eB \otimes eC) \\
& \text{id} \otimes q \searrow & \\
& e(A \otimes B) \otimes eC & \xrightarrow{q} e((A \otimes B) \otimes C) \\
& \downarrow e\alpha & \\
& eA \otimes e(B \otimes C) & \xrightarrow{q} e(A \otimes (B \otimes C))
\end{array}$$

The left diagram commutes by naturality of  $\alpha$ , and the right diagram commutes because  $e$  is a monoidal functor.

Composition in  $\mathcal{L}^e$  is the same as  $\mathcal{L}$ , and thus, the monoidal coherence diagrams hold in  $\mathcal{L}^e$  as well. Thus,  $\mathcal{L}^e$  is monoidal. We now show that it is symmetric.

### B.1.2 Proof of Lemma 10

We define  $\beta$  as follows:

$$\beta_{A,B} := A \otimes B \xrightarrow{h_A \otimes h_B} eA \otimes eB \xrightarrow{e\chi_{A,B}} eB \otimes eA \xrightarrow{e\chi_{B,A}} B \otimes A$$

## XX:14 On Linear Modalities for Exchange, Weakening, and Contraction

Suppose  $f : A \longrightarrow A'$  and  $g : B \longrightarrow B'$  are two coalgebra morphisms. Then the following diagram shows that  $\beta_{A,B}$  is a natural transformation:

$$\begin{array}{ccccccc}
 A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{\text{ex}_{A,B}} & eB \otimes eA & \xrightarrow{\varepsilon_B \otimes \varepsilon_A} & B \otimes A \\
 \downarrow f \otimes g & & \downarrow ef \otimes eg & & \downarrow eg \otimes ef & & \downarrow g \otimes f \\
 A' \otimes B' & \xrightarrow{h_{A'} \otimes h_{B'}} & eA' \otimes eB' & \xrightarrow{\text{ex}_{A',B'}} & eB' \otimes eA' & \xrightarrow{\varepsilon_{B'} \otimes \varepsilon_{A'}} & B' \otimes A'
 \end{array}$$

The left diagram commutes because  $f$  and  $g$  are both coalgebra morphisms, the middle diagram commutes because  $\text{ex}_{A,B}$  is a natural transformation, and the right diagram commutes by naturality of  $\varepsilon$ .

### B.1.3 Proof of Corollary 11

This proof follows by the fact that the following diagram commutes:

$$\begin{array}{ccccc}
 A \otimes B & \xlongequal{\quad} & A \otimes B & \xlongequal{\quad} & A \otimes B \\
 \downarrow h_A \otimes h_B & & \downarrow h_A \otimes h_B & & \downarrow h_A \otimes h_B \\
 eA \otimes eB & \xlongequal{\quad} & eA \otimes eB & & eA \otimes eB \\
 \downarrow \text{ex}_{A,B} & \swarrow \varepsilon_{eA} \otimes \varepsilon_{eB} & \downarrow h_{eA} \otimes h_{eB} & & \downarrow \text{ex}_{A,B} \\
 & & e^2A \otimes e^2B & & \\
 & \swarrow \varepsilon_{eB} \otimes \varepsilon_{eA} & \downarrow \text{ex}_{eA,eB} & & \\
 eB \otimes eA & & e^2B \otimes e^2A & & \\
 & \swarrow h_B \otimes h_A & & & \\
 & & B \otimes A & \xleftarrow{\varepsilon_B \otimes \varepsilon_A} & eB \otimes eA
 \end{array}$$

The diagram on the right commutes because  $\beta_{A,B}$  is a natural transformation, and the other diagrams commute either because  $\mathcal{L}$  is a Lambek category with exchange, or by the action diagrams.

### B.1.4 Proof of Lemma 13

We prove this by equational reasoning as follows:

$$\begin{aligned}
 & (h_B \otimes h_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) \\
 &= (h_B \otimes h_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ (h_B \otimes h_A) \circ \beta_{A,B} && \text{(Corollary 11)} \\
 &= (h_B \otimes h_A) \circ ((\varepsilon_B \circ h_B) \otimes (\varepsilon_A \circ h_A)) \circ \beta_{A,B} \\
 &= (h_B \otimes h_A) \circ (\text{id}_B \otimes \text{id}_A) \circ \beta_{A,B} && \text{(Action diagrams)} \\
 &= (h_B \otimes h_A) \circ \beta_{A,B} \\
 &= \text{ex}_{A,B} \circ (h_A \otimes h_B) && \text{(Corollary 11)} \\
 &= (\text{id}_B \otimes \text{id}_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) \\
 &= ((e\varepsilon_B \circ \delta_B) \otimes (e\varepsilon_A \circ \delta_A)) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B) && \text{(Monoidal Comonad)} \\
 &= (e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A) \circ \text{ex}_{A,B} \circ (h_A \otimes h_B)
 \end{aligned}$$

### B.1.5 Proof of Lemma 14

The proof follows from the commutativity of the following diagram:

$$\begin{array}{ccccccc}
 A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & \xrightarrow{\Theta_{A,B}} & eB \otimes eA & \xrightarrow{\varepsilon_B \otimes \varepsilon_A} & B \otimes A \\
 \downarrow h_A \otimes h_B & (1) & \downarrow \delta_A \otimes \delta_B & (2) & \downarrow \delta_B \otimes \delta_A & (3) & \downarrow h_B \otimes h_A \\
 eA \otimes eB & \xrightarrow{eh_A \otimes eh_B} & e^2 A \otimes e^2 B & \xrightarrow{\Theta_{eA,eB}} & e^2 B \otimes e^2 A & \xrightarrow{e\varepsilon_B \otimes e\varepsilon_A} & eA \otimes eB \\
 \downarrow q_{A,B} & (4) & \downarrow q_{eA,eB} & (5) & \downarrow q_{eB,eA} & (6) & \downarrow q_{B,A} \\
 e(A \otimes B) & \xrightarrow{e(h_A \otimes h_B)} & e(eA \otimes eB) & \xrightarrow{e\Theta_{A,B}} & e(eB \otimes eA) & \xrightarrow{e(\varepsilon_B \otimes \varepsilon_A)} & e(B \otimes A)
 \end{array}$$

Diagram one commutes by the action diagrams for the coalgebras  $(A, h_A)$  and  $(B, h_B)$ , diagram two commutes because  $\mathcal{L}$  is a Lambek category with exchange, diagram three does not commute, but holds by Lemma 13, diagram four and six commute by naturality of  $q$ , and diagram five commutes because  $\mathcal{L}$  is a Lambek category with exchange.

### B.1.6 Proof of The Eilenberg-Moore Category is Symmetric Monoidal (Lemma 15)

The following diagram shows that  $\beta_{B,A} \circ \beta_{A,B} = \text{id}_{A \otimes B}$ :

$$\begin{array}{ccccccc}
 A \otimes B & \xrightarrow{h_A \otimes h_B} & eA \otimes eB & & & & \\
 \downarrow h_A \otimes h_B & (1) & \downarrow \delta_A \otimes \delta_B & (6) & & & \\
 eA \otimes eB & \xrightarrow{\delta_A \otimes \delta_B} & e^2 A \otimes e^2 B & & & & \\
 \downarrow \Theta_{A,B} & (2) & \downarrow \Theta_{eA,eB} & (4) & & & \\
 eB \otimes eA & \xrightarrow{\delta_B \otimes \delta_A} & e^2 B \otimes e^2 A & \xrightarrow{\Theta_{eB,eA}} & e^2 A \otimes e^2 B & & \\
 \downarrow \varepsilon_B \otimes \varepsilon_A & (3) & \downarrow e\varepsilon_B \otimes e\varepsilon_A & (5) & \downarrow e\varepsilon_A \otimes e\varepsilon_B & & \\
 B \otimes A & \xrightarrow{h_B \otimes h_A} & eB \otimes eA & \xrightarrow{\Theta_{B,A}} & eA \otimes eB & \xrightarrow{\varepsilon_A \otimes \varepsilon_B} & A \otimes B
 \end{array}$$

Diagram one trivially commutes, diagram two, four, and five commute because  $\mathcal{L}$  is a Lambek category with exchange, diagram three does not commute, but holds by Lemma 13, diagrams six, seven, and eight commute by the fact that  $(e, \varepsilon, \delta)$  is a comonad and the action diagrams of the Eilenberg Moore category.

At this point we must verify that  $\beta$  respects the coherence diagrams of a symmetric monoidal category; see Definition 3. Thus, we must show that each of the following diagrams hold:

$$\begin{array}{ccccc}
 (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\beta_{A,B \otimes C}} & (B \otimes C) \otimes A \\
 \downarrow \beta_{A,B} \otimes \text{id}_C & & & & \downarrow \alpha_{B,C,A} \\
 (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{\text{id}_B \otimes \beta_{A,C}} & B \otimes (C \otimes A)
 \end{array}$$

Case

We can show that this diagram commutes, by reducing it to the corresponding diagram on free coalgebras which we know holds by the assumption that  $\mathcal{L}$  is a Lambek category with exchange. This reduction is as follows (due to the size of the diagram it is broken up into three diagrams that can be straightforwardly composed):

Diagram 1:

$$\begin{array}{c}
 (A \otimes B) \otimes C \xlongequal{\quad\quad\quad} (A \otimes B) \otimes C \\
 \downarrow (h_A \otimes h_B) \otimes \text{id} \qquad \qquad \qquad \downarrow (h_A \otimes h_B) \otimes h_C \\
 (eA \otimes eB) \otimes C \xrightarrow{\text{id} \otimes h_C} (eA \otimes eB) \otimes eC \\
 \downarrow \text{ex} \otimes \text{id} \qquad \qquad \qquad \downarrow \text{ex} \otimes \text{id} \\
 (eB \otimes eA) \otimes C \xrightarrow{(\delta \otimes \delta) \otimes \text{id}} (e^2 B \otimes e^2 A) \otimes C \xrightarrow{\text{id} \otimes h_C} (eB \otimes eA) \otimes eC \\
 \downarrow (e \otimes e) \otimes \text{id} \qquad \qquad \qquad \downarrow (e \otimes e \otimes e) \otimes \text{id} \qquad \qquad \qquad \downarrow \alpha \\
 (B \otimes A) \otimes C \xrightarrow{(h_B \otimes h_A) \otimes \text{id}} (eB \otimes eA) \otimes C \xrightarrow{\text{id} \otimes h_C} eB \otimes (eA \otimes eC) \\
 \parallel \qquad \qquad \qquad \downarrow h_B \otimes (h_A \otimes h_C) \\
 (B \otimes A) \otimes C \xrightarrow{\alpha} B \otimes (A \otimes C) \xrightarrow{\text{id} \otimes (h_A \otimes h_C)} B \otimes (eA \otimes eC) \\
 \parallel \\
 (B \otimes A) \otimes C \xrightarrow{\alpha} B \otimes (A \otimes C) \xrightarrow{\text{id} \otimes (h_A \otimes h_C)} B \otimes (eA \otimes eC)
 \end{array}$$

(1) (2) (3)

Diagram one commutes because  $(e, \varepsilon, \delta)$  is a comonad, diagram two does not commute, but holds by Lemma 13, and diagram 3 commutes by naturality of  $\alpha$ . All other diagrams commute trivially.

Diagram 2:



$$\begin{array}{c}
\begin{array}{c}
(A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C) \xrightarrow{h_A \otimes (h_B \otimes h_C)} eA \otimes (eB \otimes eC) \xrightarrow{\text{id} \otimes q} eA \otimes e(B \otimes C) \xrightarrow{\text{ex}} e(B \otimes C) \otimes eA \\
\downarrow (h_A \otimes h_B) \otimes h_C \quad (1) \quad \downarrow h_A \otimes (h_B \otimes h_C) \quad (2) \quad \downarrow \text{id} \otimes (e h_B \otimes e h_C) \quad (3) \quad \downarrow \text{id} \otimes (h_B \otimes h_C) \quad (4) \quad \downarrow e(h_B \otimes h_C) \otimes \text{id} \\
(eA \otimes eB) \otimes eC \xrightarrow{\alpha} eA \otimes (eB \otimes eC) \xrightarrow{\text{id} \otimes (\text{id} \otimes \text{id})} eA \otimes (e^2 B \otimes e^2 C) \xrightarrow{\text{id} \otimes q} eA \otimes e(eB \otimes eC) \xrightarrow{\text{ex}} e(eB \otimes eC) \otimes eA \\
\downarrow \text{ex} \otimes \text{id} \quad (5) \quad \downarrow \alpha \quad \downarrow \alpha \\
(eB \otimes eA) \otimes eC \xrightarrow{\alpha} eB \otimes (eA \otimes eC) \xrightarrow{\text{id} \otimes \text{ex}} eB \otimes (eC \otimes eA) \\
\downarrow h_B \otimes (h_A \otimes h_C) \quad \downarrow \text{id} \otimes (h_A \otimes h_C) \quad \downarrow \text{id} \otimes (h_A \otimes h_C) \\
B \otimes (A \otimes C) \xrightarrow{\text{id} \otimes (h_A \otimes h_C)} B \otimes (eA \otimes eC) \xrightarrow{\text{id} \otimes \text{ex}} B \otimes (eC \otimes eA) \xrightarrow{h_B \otimes \text{id}} eB \otimes (eC \otimes eA) \\
\downarrow \text{id} \otimes (h_A \otimes h_C) \quad \downarrow \text{id} \otimes \text{ex} \quad \downarrow h_B \otimes \text{id} \\
B \otimes (eA \otimes eC) \xrightarrow{\text{id} \otimes \text{ex}} B \otimes (eC \otimes eA) \xrightarrow{h_B \otimes \text{id}} eB \otimes (eC \otimes eA)
\end{array}
\end{array}$$

Diagram one commutes by naturality of  $\alpha$ , diagram two commutes by the action diagrams, diagram three commutes by naturality of  $q$ , diagram four commutes by naturality of  $\text{ex}$ , and diagram five commutes because  $\mathcal{L}$  is a Lambek category with exchange. All other diagrams trivially commute.

Diagram 3:

$$\begin{array}{ccc}
 e(B \otimes C) \otimes eA & \xrightarrow{\varepsilon \otimes \varepsilon} & (B \otimes C) \otimes A \\
 \downarrow e(h_B \otimes h_C) \otimes \text{id} & & \downarrow (h_B \otimes h_C) \otimes \text{id} \\
 e(eB \otimes eC) \otimes eA & \xrightarrow{\varepsilon \otimes \varepsilon} & (eB \otimes eC) \otimes A \\
 \downarrow \varepsilon \otimes \text{id} & & \downarrow (\varepsilon \otimes \varepsilon) \otimes \text{id} \\
 (eB \otimes eC) \otimes eA & \xrightarrow{(\varepsilon \otimes \varepsilon) \otimes \varepsilon} & (B \otimes C) \otimes A \\
 \downarrow \alpha & & \downarrow \alpha \\
 eB \otimes (eC \otimes eA) & \xrightarrow{\varepsilon \otimes (\varepsilon \otimes \varepsilon)} & B \otimes (C \otimes A) \\
 \parallel & & \parallel \\
 eB \otimes (eC \otimes eA) & & \\
 \parallel & & \parallel \\
 eB \otimes (eC \otimes eA) & \xrightarrow{\varepsilon \otimes (\varepsilon \otimes \varepsilon)} & B \otimes (C \otimes A)
 \end{array}$$

The top most diagram commutes by naturality of  $\varepsilon$  and the middle diagram commutes by naturality of  $\alpha$ . All other diagrams trivially commute.

**Case**

$$\begin{array}{ccc}
 I \otimes A & \xrightarrow{\beta_{I,A}} & A \otimes I \\
 \searrow \lambda_A & & \swarrow \rho_A \\
 & A &
 \end{array}$$

Just as we did for the previous case we reduce this diagram down to the corresponding one on free coalebras that we know holds by the assumption that  $\mathcal{L}$  is a Lambek category with exchange. This case follows from the following commutative diagram:

$$\begin{array}{ccccccc}
 I \otimes A & \xrightarrow{h_I \otimes h_A} & eI \otimes eA & \xrightarrow{\text{ex}} & eA \otimes eI & \xrightarrow{\varepsilon \otimes \varepsilon} & A \otimes I \\
 \downarrow \lambda & \searrow \text{id} \otimes h_A & \downarrow \varepsilon \otimes \text{id} & & \downarrow \text{id} \otimes \varepsilon & \swarrow \varepsilon \otimes \text{id} & \downarrow \rho \\
 & & I \otimes eA & & eA \otimes I & & \\
 & & \downarrow \lambda & & \downarrow \rho & & \\
 A & \xrightarrow{h_A} & eA & \xlongequal{\quad} & eA & \xrightarrow{\varepsilon} & A
 \end{array}$$

The left most triangle commutes by the action diagrams and the lower diagram commutes by naturality of  $\lambda$ . Similarly, the right most lower diagram commutes by naturality of  $\rho$ . The middle diagram commutes because  $\mathcal{L}$  is a Lambek category with exchange. All other diagrams trivially commute.

## B.2 Weakening and Contraction

### B.3 Proof of Composition of Weakening and Contraction (Lemma 19)

Since by definition  $w : \mathcal{L} \rightarrow \mathcal{L}$  and  $c : \mathcal{L} \rightarrow \mathcal{L}$  are monoidal functors we know that their composition  $cw : \mathcal{L} \rightarrow \mathcal{L}$  is a monoidal functor:

$$\begin{aligned} q_{A,B} &: cwA \otimes cwB \rightarrow cw(A \otimes B) \\ q_{A,B} &= cq_{A,B}^w \circ q_{wA,wB}^c \\ q_I &: I \rightarrow cwI \\ q_I &= cq_I^w \circ q_I^c \end{aligned}$$

We must now define both  $\varepsilon_A : cwA \rightarrow A$  and  $\delta_A : cwA \rightarrow cwcwA$ , and then show that they are monoidal natural transformations subject to the comonad laws. Since we are composing two comonads each of  $\varepsilon$  and  $\delta$  can be given two definitions, but they are equivalent:

- $\varepsilon_A : cwA \rightarrow A$  is defined as in the diagram below, which commutes by the naturality of  $\varepsilon^c$ .

$$\begin{array}{ccc} cwA & \xrightarrow{\varepsilon_{wA}^c} & wA \\ \downarrow c\varepsilon_A^w & & \downarrow \varepsilon_A^w \\ cA & \xrightarrow{\varepsilon_A^c} & A \end{array}$$

- $\delta_A : cwA \rightarrow cwcwA$  is defined as in the diagram:

$$\begin{array}{ccccc} cwA & \xrightarrow{c\delta_A^w} & cw^2A & \xrightarrow{\delta_{w^2A}^c} & c^2w^2A \\ \downarrow \delta_{wA}^c & & \downarrow \delta_{w^2A}^c & & \downarrow cdist_{wA} \\ c^2wA & \xrightarrow{c^2\delta_A^w} & c^2w^2A & \xrightarrow{cdist_{wA}} & cwcwA \end{array}$$

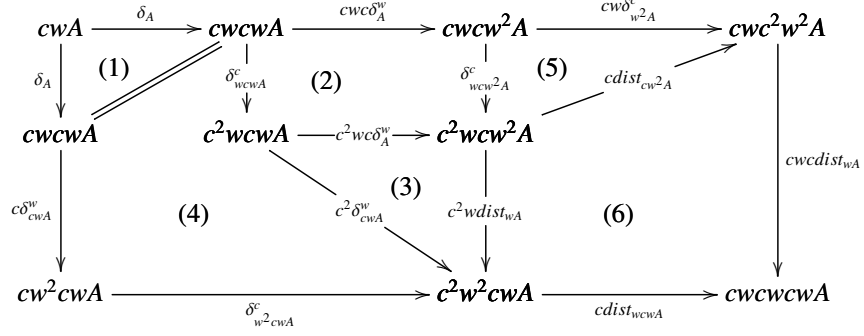
The left part of the diagram commutes by the naturality of  $\delta^c$  and the right part commutes trivially.

The remainder of the proof shows that the comonad laws hold.

**Case 1:**

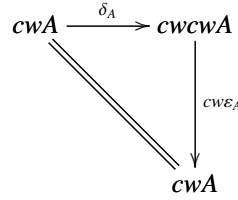
$$\begin{array}{ccc} cwA & \xrightarrow{\delta_A} & cwcwA \\ \downarrow \delta_A & & \downarrow cw\delta_A \\ cwcwA & \xrightarrow{\delta_{cwA}} & cwcwcwA \end{array}$$

The previous diagram commutes because the following one does.

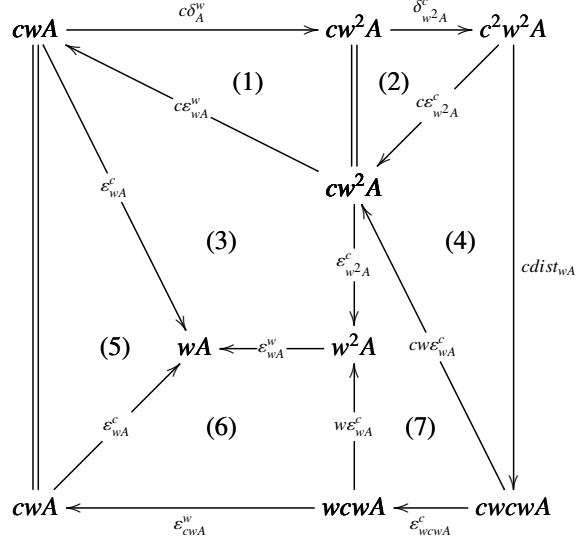


(1) commutes by equality and we will not expand  $\delta_A$  for simplicity. (2) and (4) commutes by the naturality of  $\delta^c$ . (3), (5) commutes by the conditions of *dist*. (6) commutes by the naturality of *dist*.

**Case 2:**



The triangle commutes because of the following diagram chasing.



(1) commutes by the comonad law for  $w$  with components  $\delta_A^w$  and  $\epsilon_{wA}^w$ . (2) commutes by the comonad law for  $c$  with components  $\delta_{w^2A}^c$  and  $\epsilon_{w^2A}^c$ . (3) and (7) commute by the naturality of  $\epsilon^c$ . (4) commutes by the condition of *dist*. (5) commutes trivially. And (6) commutes by the naturality of  $\epsilon^w$ .

Case 3:

$$\begin{array}{ccc}
 & cwA & \\
 \delta_A \downarrow & \searrow & \\
 cwcwA & \xrightarrow{\varepsilon_{cwA}} & cwA
 \end{array}$$

The previous triangle commutes because the following diagram chasing does.

$$\begin{array}{ccccc}
 cwA & \xrightarrow{c\delta_A^w} & cw^2A & \xrightarrow{\delta_{w^2A}^c} & c^2w^2A \\
 & \searrow \delta_{wA}^c & \downarrow c\varepsilon_{wA}^c & \nearrow c^2\delta_A^w & \downarrow cdist_{wA} \\
 & (2) & c^2wA & (3) & \\
 & \nearrow c\varepsilon_{wA}^c & \downarrow cdist_A & \nwarrow c^2w\varepsilon_A^w & \\
 & (4) & c^2wA & (5) & \\
 cwA & \xleftarrow{cw\varepsilon_A^c} & cwcA & \xleftarrow{cwc\varepsilon_A^w} & cwcwA
 \end{array}$$

(1) commutes by the naturality of  $\delta^c$ . (2) is the comonad law for  $c$  with components  $\delta_{wA}^c$  and  $\varepsilon_{wA}^c$ . (3) is the comonad law for  $w$  with components  $\delta_A^w$  and  $\varepsilon_A^w$ . (4) commutes by the condition of  $dist$ . And (5) commute by the naturality of  $dist$ .

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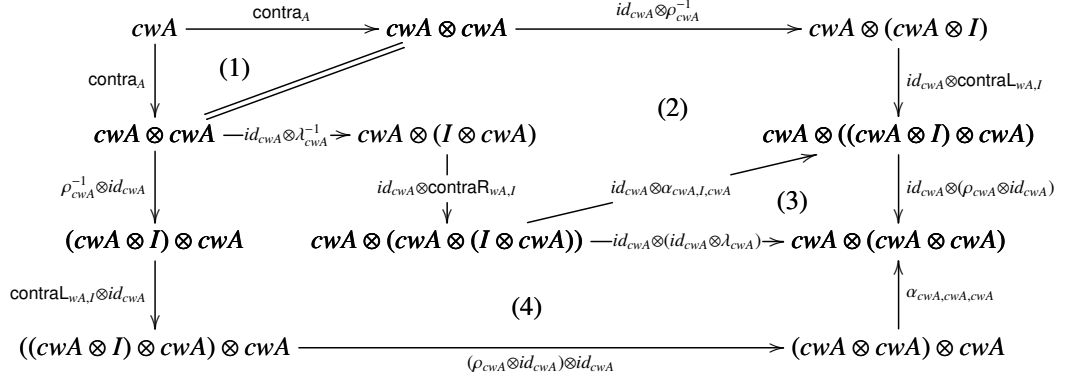
## B.4 Proof of Conditions of Lambek category with $cw$ (Lemma 21)

1. As shown in the paper.
2. Each  $(cwA, \text{weak}_A, \text{contra}_A)$  is a comonoid.

Case 1:

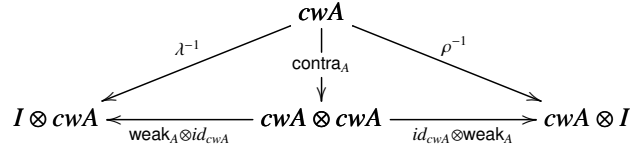
$$\begin{array}{ccccc}
 cwA & \xrightarrow{\text{contra}_A} & cwA \otimes cwA & \xrightarrow{id_{cwA} \otimes \text{contra}_A} & cwA \otimes (cwA \otimes cwA) \\
 \text{contra}_A \downarrow & & & & \uparrow \alpha_{cwA, cwA, cwA} \\
 cwA \otimes cwA & \xrightarrow{\text{contra}_A \otimes id_{cwA}} & & & (cwA \otimes cwA) \otimes cwA
 \end{array}$$

The previous diagram commutes by the following diagram chasing.

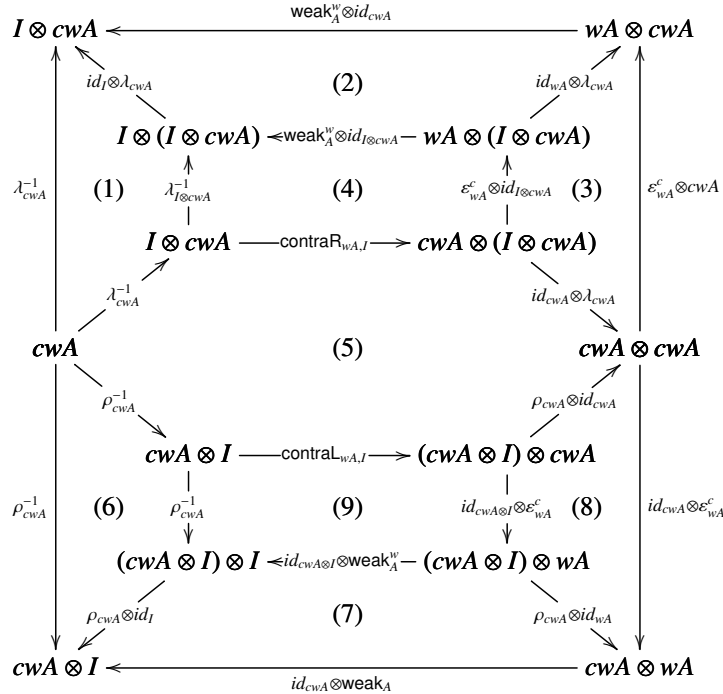


(1) commutes trivially and we would not expand  $\text{contra}$  for simplicity. (2) and (4) commute because  $(\mathcal{L}, c, \text{contraL}, \text{contraR})$  is a Lambek category with contraction. (3) commutes because  $\mathcal{L}$  is monoidal.

**Case 2:**



The diagram above commutes by the following diagram chasing.



(1), (2) and (3) commute by the functionality of  $\lambda$ . (6), (7) and (8) commute by the functionality of  $\rho$ . (4) and (9) are conditions of the Lambek category with  $cw$ . And (5) is the definition of  $\text{contra}$ .

3. weak and contra are coalgebra morphisms.

**Case 1:**

$$\begin{array}{ccc}
 cwA & \xrightarrow{\text{weak}_A} & I \\
 \delta_A \downarrow & & \downarrow q_I \\
 cwcwA & \xrightarrow{cw\text{weak}_A} & cwI
 \end{array}$$

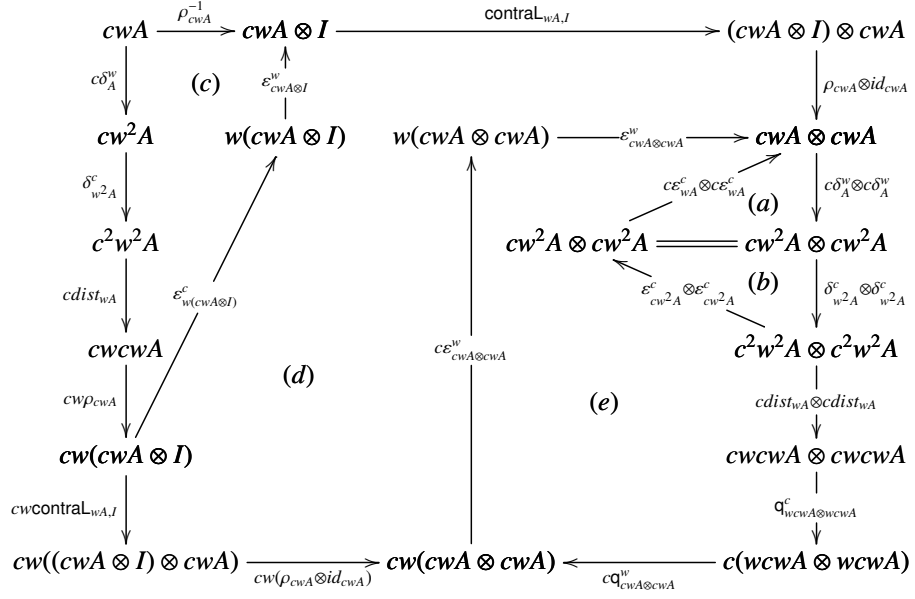
The previous diagram commutes by the diagram below. (1) commutes by the naturality of  $\delta^c$ . (2) commutes by the condition of  $\text{dist}_{wA}$ . (3), (5) and (6) commute because  $c$  is a monoidal comonad. (4) commutes because  $(\mathcal{L}, w, \text{weak}^w)$  is a Lambek category with weakening. (7) commutes because  $c$  and  $w$  are monoidal comonads.

$$\begin{array}{ccccc}
 cwA & \xrightarrow{c\text{weak}_A^w} & cI & \xrightarrow{\epsilon_I^c} & I \\
 \delta_{wA}^c \downarrow & \searrow c\delta_A^w & \swarrow q_I^c & (5) & \parallel \\
 c^2wA & & & & I \\
 c^2\delta_A^w \downarrow & (1) & & & \downarrow q_I^c \\
 c^2w^2A & \xleftarrow{\delta_{w^2A}^c} & cw^2A & & cI \\
 c\text{dist}_{wA} \downarrow & \searrow c\epsilon_{wA}^c & \parallel & (7) & \downarrow c q_I^w \\
 cwcwA & \xrightarrow{cw\epsilon_{wA}^c} & cw^2A & \xleftarrow{cw\text{weak}_A^w} & cwI
 \end{array}$$

**Case 2:**

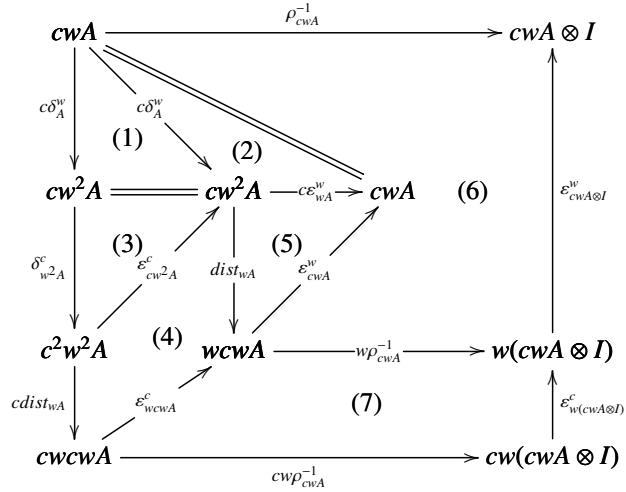
$$\begin{array}{ccccc}
 cwA & \xrightarrow{\text{contra}_A} & cwA \otimes cwA & \xrightarrow{\delta_A \otimes \delta_A} & cwcwA \otimes cwcwA \\
 \delta_A \downarrow & & & & \downarrow q_{cwA, cwA} \\
 cwcwA & \xrightarrow{cw\text{contra}_A} & & & cw(cwA \otimes cwA)
 \end{array}$$

To prove the previous diagram commute, we first expand it, Then we divide it into five parts as shown below, and prove each part commutes.



Part (a) and (b) are comonad laws.

Part (c) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for  $w$ . (3) is the comonad law for  $c$ . (4) commutes by the naturality of  $\varepsilon^c$ . (5) is one of the conditions for  $dist_{wA}$ . (6) commutes by the naturality of  $\varepsilon^w$ . And (7) commutes by the naturality of  $\varepsilon^c$ .



Part (d) commutes by the following diagram chase. The upper two squares both commute by



the naturality of  $\varepsilon^w$ , and the lower two squares commute by the naturality of  $\varepsilon^c$ .

$$\begin{array}{ccccc}
 cwA \otimes I & \xrightarrow{\text{contra}_{wA,I}} & (cwA \otimes I) \otimes cwA & \xrightarrow{\rho_{cwA} \otimes id_{cwA}} & cwA \otimes cwA \\
 \varepsilon_{cwA \otimes I}^w \uparrow & & \varepsilon_{(cwA \otimes I) \otimes cwA}^w \uparrow & & \varepsilon_{cwA \otimes cwA}^w \uparrow \\
 w(cwA \otimes I) & \xrightarrow{w \text{contra}_{wA,I}} & w((cwA \otimes I) \otimes cwA) & \xrightarrow{w(\rho_{cwA} \otimes id_{cwA})} & w(cwA \otimes cwA) \\
 \varepsilon_{w(cwA \otimes I)}^c \uparrow & & \varepsilon_{w((cwA \otimes I) \otimes cwA)}^c \uparrow & & \varepsilon_{w(cwA \otimes cwA)}^c \uparrow \\
 cw(cwA \otimes I) & \xrightarrow{cw \text{contra}_{wA,I}} & cw((cwA \otimes I) \otimes cwA) & \xrightarrow{cw(\rho_{cwA} \otimes id_{cwA})} & cw(cwA \otimes cwA)
 \end{array}$$

Part (e) commutes by the following diagram. (1) commutes by the condition of  $dist_{wA}$ . (2) and (4) commute by the naturality of  $\varepsilon^c$ . (3) and (5) commute because  $w$  and  $c$  are monoidal comonads.

$$\begin{array}{ccccc}
 cwA \otimes cwA & \xleftarrow{c\varepsilon_{wA}^w \otimes c\varepsilon_{wA}^w} & cw^2A \otimes cw^2A & \xleftarrow{\varepsilon_{cw^2A}^c \otimes \varepsilon_{cw^2A}^c} & c^2w^2A \otimes c^2w^2A \\
 \varepsilon_{cwA \otimes cwA}^w \uparrow & & \downarrow & & \downarrow cdist_{wA} \otimes cdist_{wA} \\
 & & (1) \quad dist_{wA} \otimes dist_{wA} & & (2) \\
 & & \downarrow & & \downarrow \\
 & & w cwA \otimes w cwA & \xleftarrow{\varepsilon_{w cwA}^c \otimes \varepsilon_{w cwA}^c} & c w cwA \otimes c w cwA \\
 & & (3) & & (5) \\
 & & \downarrow & & \downarrow q_{w cwA \otimes w cwA}^c \\
 w(cwA \otimes cwA) & \xleftarrow{\varepsilon_{w(cwA \otimes cwA)}^c} & cw(cwA \otimes cwA) & \xleftarrow{c q_{cwA \otimes cwA}^c} & c(w cwA \otimes w cwA)
 \end{array}$$

4. Any coalgebra morphism  $f : (cwA, \delta_A) \rightarrow (cwB, \delta_B)$  between free coalgebras preserves the comonoid structure given by weak and contra.

**Case 1:** This coherence diagram is given in the definition of the Lambek category with  $cw$ .

$$\begin{array}{ccc}
 cwA & \xrightarrow{f} & cwB \\
 & \searrow \text{weak}_A & \swarrow \text{weak}_B \\
 & I &
 \end{array}$$

**Case 2:**

$$\begin{array}{ccc}
 cwA & \xrightarrow{\text{contra}_A} & cwA \otimes cwA \\
 f \downarrow & & \downarrow f \otimes f \\
 cwB & \xrightarrow{\text{contra}_B} & cwB \otimes cwB
 \end{array}$$

The square commutes by the diagram chasing below, which commutes by the naturality of  $\rho$  and  $\text{contra}_L$ .

$$\begin{array}{ccccccc}
 cwA & \xrightarrow{\rho_{cwA}^{-1}} & cwA \otimes I & \xrightarrow{\text{contra}_{wA,I}} & (cwA \otimes I) \otimes cwA & \xrightarrow{\rho_{cwA} \otimes id_{cwA}} & cwA \otimes cwA \\
 \downarrow cw f & & \downarrow cw f \otimes id_I & & \downarrow (cw f \otimes id_I) \otimes cw f & & \downarrow cw f \otimes cw f \\
 cwB & \xrightarrow{\rho_{cwB}^{-1}} & cwB \otimes I & \xrightarrow{\text{contra}_{wB,I}} & (cwB \otimes I) \otimes cwB & \xrightarrow{\rho_{cwB} \otimes id_{cwB}} & cwB \otimes cwB
 \end{array}$$

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