Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

2 Categorical Models

2.1 Lambek Categories

TODO: Define Lambek Categories

2.2 Lambek Categories with Weakening and Contraction

Definition 1. *A **Lambek category with weakening**, $(\mathcal{L}, w, \mathsf{weak})$, is a Lambek category equipped with a monoidal comonad (w, ε, δ), and a monoidal natural transformation* $\mathsf{weak}_A : wA \longrightarrow I$*. Furthermore,* weak *must be a coalgebra morphism. That is, the following digram must commute:*

$$\begin{tikzcd}
wA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta_A"'] & I \arrow[d, "\mathsf{q}_I"] \\
w^2A \arrow[r, "w\mathsf{weak}_A"'] & wI
\end{tikzcd}$$

Definition 2. *A **Lambek category with contraction**, $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$, is a Lambek category equipped with a monoidal comonad (c, ε, δ), and two monoidal natural transformations:*

$$\mathsf{contraL}_{A,B} : cA \otimes B \longrightarrow (cA \otimes B) \otimes cA$$
$$\mathsf{contraR}_{A,B} : B \otimes cA \longrightarrow cA \otimes (B \otimes cA)$$

Furthermore, the following diagrams must commute:

$$\begin{tikzcd}
cA \otimes I \arrow[d, "\mathsf{contraL}_{A,I}"'] & cA \arrow[l, "\rho^{-1}_{cA}"'] \arrow[r, "\lambda^{-1}_{cA}"] & I \otimes cA \arrow[d, "\mathsf{contraR}_{A,I}"] \\
(cA \otimes I) \otimes cA \arrow[rr, "\alpha_{cA,I,cA}"'] & & cA \otimes (I \otimes cA)
\end{tikzcd}$$

$$\begin{tikzcd}
cA \otimes cA \arrow[r, "id_{cA} \otimes \rho^{-1}_{cA}"] \arrow[dd, "\lambda^{-1}_{cA} \otimes id_{cA}"'] & cA \otimes (cA \otimes I) \arrow[r, "id_{cA} \otimes \mathsf{contraL}_{A,I}"] & cA \otimes ((cA \otimes I) \otimes cA) \arrow[d, "id_{cA} \otimes (\rho_{cA} \otimes id_{cA})"] \\
 & & cA \otimes (cA \otimes cA) \\
(I \otimes cA) \otimes cA \arrow[r, "\mathsf{contraR}_{A,I} \otimes id_{cA}"'] & (cA \otimes (I \otimes cA)) \otimes cA \arrow[r, "(id_{cA} \otimes \lambda_{cA}) \otimes id_{cA}"'] & (cA \otimes cA) \otimes cA \arrow[u, "\alpha_{cA,cA,cA}"']
\end{tikzcd}$$

2.3 Lambek Categories with Exchange

Definition 3. *A **Lambek category with exchange**, $(\mathcal{L}, e, \mathsf{ex})$, is a Lambek category equipped with a monoidal comonad (e, ε, δ) on $\mathcal{L}$, and a monoidal natural transformation $\mathsf{ex}_{A,B} : eA \otimes eB \longrightarrow eB \otimes eA$. We require ex to be a coalgebra morphism, and the following diagrams must commute:*

$$\begin{tikzcd}
eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] \arrow[dr, equal] & eB \otimes eA \arrow[d, "\mathsf{ex}_{B,A}"] \\
 & eA \otimes eB
\end{tikzcd}
\qquad
\begin{tikzcd}
e^2A \otimes e^2B \arrow[r, "\mathsf{ex}_{eA,eB}"] \arrow[d, "\mathsf{q}_{eA,eB}"] & e^2B \otimes e^2A \arrow[d, "\mathsf{q}_{eB,eA}"] \\
e(eA \otimes eB) \arrow[r, "e\mathsf{ex}_{A,B}"] & e(eB \otimes eA)
\end{tikzcd}$$

Furthermore, for any coalgebra morphisms $f : (eA, \delta) \longrightarrow (eB, \delta)$ and $g : (eC, \delta) \longrightarrow (eD, \delta)$ between free coalgebras the following diagram must commute:

$$\begin{tikzcd}
eA \otimes eC \arrow[r, "f \otimes g"] \arrow[d, "\mathsf{ex}_{A,C}"] & eB \otimes eD \arrow[d, "\mathsf{ex}_{B,D}"] \\
eC \otimes eA \arrow[r, "g \otimes f"] & eD \otimes eB
\end{tikzcd}$$

The morphism $\mathsf{q}_{A,B} : eA \otimes eB \longrightarrow e(A \otimes B)$ makes (e, q) a monoidal functor.

The first diagram in the previous definition makes ex an involution, and the second and third diagrams are required in the proof that the Eilenberg-Moore category is symmetric; see the proofs of Lemma 10 and Lemma 11.

Definition 4. *Suppose* $(\mathcal{L}, e, \mathsf{ex})$ *is a Lambek category with exchange. Then the* ***Eilenberg Moore category****,* $\mathcal{L}^e$*, of the comonad* (e, ε, δ) *has as objects all the e-coalgebras* $(A, h_A : A \longrightarrow eA)$*, and as morphisms all the coalgebra morphisms. We call* h_A *the action of the coalgebra. Furthermore, the following (action) diagrams must commute:*

$$\begin{tikzcd} A \arrow[r, "h_A"] \arrow[d, "h_A"'] & eA \arrow[d, "eh_A"] \\ eA \arrow[r, "\delta_A"'] & e^2A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "h_A"'] \arrow[dr, equal] & \\ eA \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

Lemma 5 (The Eilenberg Moore Category is Monoidal)**.** *Then the category* $\mathcal{L}^e$ *is monoidal.*

Proof. We must first define the unitors, and then the associator. Then we show that they respect the symmetry monoidal coherence diagrams. Throughout this proof we will make use of the coalgebra (A, h_A), (B, h_B), and (C, h_C).

The tensor product of (A, h_A) and (B, h_b) is $(A \otimes B, q_{A,B} \circ (h_A \otimes h_B))$, and the unit of the tensor product is (I, q_I); both actions are easily shown to satisfies the action diagrams of the Eilenberg Moore category. The left and right unitors are $\lambda : I \otimes A \longrightarrow A$ and $\rho : A \otimes I \longrightarrow A$, because they are indeed coalgebra morphisms.

The respective diagram for the right unitor is as follows:

$$\begin{tikzcd} A \otimes I \arrow[r, "h_A \otimes \mathsf{id}"] \arrow[d, "\rho"'] & eA \otimes I \arrow[r, "\mathsf{id} \otimes \mathsf{q}_I"] \arrow[drr, "\rho"] & eA \otimes eI \arrow[r, "\mathsf{q}_{A,I}"] & e(A \otimes I) \arrow[d, "e\rho"] \\ A \arrow[rrr, "h_A"] & & & eA \end{tikzcd}$$

The left diagram commutes by naturality of ρ, the right diagram commutes by the fact that e is a monoidal functor. Showing the left unitor is a coalgebra morphism is similar.

The unitors are natural and isomorphisms, because they are essentially inherited from the underlying Lambek category.

The associator $\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ is also a coalgebra morphism. First, notice that:

$$\mathsf{q}_{A \otimes B,C} \circ ((\mathsf{q}_{A,B} \circ (h_A \otimes h_B)) \otimes h_c) = \mathsf{q}_{A \otimes B,C} \circ (\mathsf{q}_{A,B} \otimes \mathsf{id}) \circ ((h_A \otimes h_B) \otimes h_C)$$

where the left-hand side is the action of the coalgebra $(A \otimes B) \otimes C$. Similarly, the following is the action of the coalgebra $A \otimes (B \otimes C)$:

$$\mathsf{q}_{A,B \otimes C} \circ (h_A \otimes (\mathsf{q}_{B,C} \circ (h_B \otimes h_C))) = \mathsf{q}_{A,B \otimes C} \circ (\mathsf{id} \otimes \mathsf{q}_{B,C}) \circ (h_A \otimes (h_B \otimes h_C))$$

The following diagram must commute:

$$
\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "(h_A\otimes h_B)\otimes h_C"] \arrow[d, "\alpha"] & (eA\otimes eB)\otimes eC \arrow[r, "\mathsf{q}\otimes\mathsf{id}"] \arrow[d, "\alpha"] & e(A\otimes B)\otimes eC \arrow[r, "\mathsf{q}"] & e((A\otimes B)\otimes C) \arrow[d, "e\alpha"] \\
A\otimes(B\otimes C) \arrow[r, "h_A\otimes(h_B\otimes h_C)"] & eA\otimes(eB\otimes eC) \arrow[r, "\mathsf{id}\otimes\mathsf{q}"] & eA\otimes e(B\otimes C) \arrow[r, "\mathsf{q}"] & e(A\otimes(B\otimes C))
\end{tikzcd}
$$

The left diagram commutes by naturality of α, and the right diagram commutes because e is a monoidal functor.

Composition in $\mathcal{L}^e$ is the same as $\mathcal{L}$, and thus, the monoidal coherence diagrams hold in $\mathcal{L}^e$ as well. Thus, $\mathcal{L}^e$ is monoidal. We now show that it is symmetric. $\square$

Lemma 6. *In $\mathcal{L}^e$ there is a natural transformation $\beta_{A,B} : A\otimes B \longrightarrow B\otimes A$.*

Proof. We define β as follows:

$$\beta_{A,B} := A\otimes B \xrightarrow{h_A\otimes h_B} eA\otimes eB \xrightarrow{ex_{A,B}} eB\otimes eA \xrightarrow{\varepsilon_B\otimes\varepsilon_A} B\otimes A$$

Suppose $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ are two coalgebra morphisms. Then the following diagram shows that $\beta_{A,B}$ is a natural transformation:

$$\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[d, "f \otimes g"] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] \arrow[d, "ef \otimes eg"] & eB \otimes eA \arrow[r, "\varepsilon_B \otimes \varepsilon_A"] \arrow[d, "eg \otimes ef"] & B \otimes A \arrow[d, "g \otimes f"] \\
A' \otimes B' \arrow[r, "h_{A'} \otimes h_{B'}"] & eA' \otimes eB' \arrow[r, "\mathsf{ex}_{A',B'}"] & eB' \otimes eA' \arrow[r, "\varepsilon_{B'} \otimes \varepsilon_{A'}"] & B' \otimes A'
\end{tikzcd}$$

The left diagram commutes because f and g are both coalgebra morphisms, the middle diagram commutes because $\mathsf{ex}_{A,B}$ is a natural transformation, and the right diagram commutes by naturality of ε. □

Corollary 7. *For any coalgebras (A, h_A) and (B, h_B) the following commutes:*

$$\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[d, equal] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] & eB \otimes eA \arrow[r, "\varepsilon_B \otimes \varepsilon_A"] & B \otimes A \arrow[d, "h_B \otimes h_A"] \\
A \otimes B \arrow[r, "h_A \otimes h_B"] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] & eB \otimes eA \arrow[r, equal] & eB \otimes eA
\end{tikzcd}$$

Proof. This proof follows by the fact that the following diagram commutes:

$$\begin{tikzcd}
A \otimes B \arrow[r, equal] \arrow[d, "h_A \otimes h_B"] & A \otimes B \arrow[r, equal] \arrow[d, "h_A \otimes h_B"] & A \otimes B \arrow[d, "h_A \otimes h_B"] \\
eA \otimes eB \arrow[r, equal] \arrow[dd, "\mathsf{ex}_{A,B}"'] & eA \otimes eB \arrow[d, "h_{eA} \otimes h_{eB}"] & eA \otimes eB \arrow[ddd, "\mathsf{ex}_{A,B}"] \\
 & e^2A \otimes e^2B \arrow[ul, "\varepsilon_{eA} \otimes \varepsilon_{eB}"'] \arrow[d, "\mathsf{ex}_{eA,eB}"] & \\
eB \otimes eA & e^2B \otimes e^2A \arrow[l, "\varepsilon_{eB} \otimes \varepsilon_{eA}"'] & \\
 & B \otimes A \arrow[ul, "h_B \otimes h_A"] & eB \otimes eA \arrow[l, "\varepsilon_B \otimes \varepsilon_A"']
\end{tikzcd}$$

The diagram on the right commutes because $\beta_{A,B}$ is a natural transformation, and the other diagrams commute either because $\mathcal{L}$ is a Lambek category with exchange, or by the action diagrams. □

Definition 8. *Given two parallel arrows $f, g : B \longrightarrow C$ in a category $\mathcal{C}$, a **cofork** is a morphism $c : A \longrightarrow B$ such that the following diagram commutes:*

$$
\begin{tikzcd}
A \arrow[r, "c"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C
\end{tikzcd}
$$

That is, $f \circ c = g \circ c$.

Lemma 9. *The morphism $\mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B)$ is a cofork of the morphisms $(\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\varepsilon_B \otimes \varepsilon_A)$ and $(e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A)$.*

Proof. We prove this by equational reasoning as follows:

$$
\begin{array}{ll}
(\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) & \\
\quad = (\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\varepsilon_B \otimes \varepsilon_A) \circ (\mathsf{h}_B \otimes \mathsf{h}_A) \circ \beta_{A,B} & \text{(Corollary 7)} \\
\quad = (\mathsf{h}_B \otimes \mathsf{h}_A) \circ ((\varepsilon_B \circ \mathsf{h}_B) \otimes (\varepsilon_A \circ \mathsf{h}_A)) \circ \beta_{A,B} & \\
\quad = (\mathsf{h}_B \otimes \mathsf{h}_A) \circ (\mathsf{id}_B \otimes \mathsf{id}_A) \circ \beta_{A,B} & \text{(Action diagrams)} \\
\quad = (\mathsf{h}_B \otimes \mathsf{h}_A) \circ \beta_{A,B} & \\
\quad = \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) & \text{(Corollary 7)} \\
\quad = (\mathsf{id}_B \otimes \mathsf{id}_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) & \\
\quad = ((e\varepsilon_B \circ \delta_B) \otimes (e\varepsilon_A \circ \delta_A)) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) & \text{(Monoidal Comonad)} \\
\quad = (e\varepsilon_B \otimes e\varepsilon_A) \circ (\delta_B \otimes \delta_A) \circ \mathsf{ex}_{A,B} \circ (\mathsf{h}_A \otimes \mathsf{h}_B) &
\end{array}
$$

□

Lemma 10. *In $\mathcal{L}^e$, β is a coalgebra morphism.*

Proof. The proof follows from the commutativity of the following diagram:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[d, "h_A \otimes h_B"'] \arrow[dr, phantom, "(1)"] & eA \otimes eB \arrow[r, "\mathsf{ex}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"] \arrow[dr, phantom, "(2)"] & eB \otimes eA \arrow[r, "\varepsilon_B \otimes \varepsilon_A"] \arrow[d, "\delta_B \otimes \delta_A"] \arrow[dr, phantom, "(3)"] & B \otimes A \arrow[d, "h_B \otimes h_A"] \\
eA \otimes eB \arrow[r, "eh_A \otimes eh_B"] \arrow[d, "\mathsf{q}_{A,B}"'] \arrow[dr, phantom, "(4)"] & e^2A \otimes e^2B \arrow[r, "\mathsf{ex}_{eA,eB}"] \arrow[d, "\mathsf{q}_{eA,eB}"] \arrow[dr, phantom, "(5)"] & e^2B \otimes e^2A \arrow[r, "e\varepsilon_B \otimes e\varepsilon_A"] \arrow[d, "\mathsf{q}_{eB,eA}"] \arrow[dr, phantom, "(6)"] & eA \otimes eB \arrow[d, "\mathsf{q}_{B,A}"] \\
e(A \otimes B) \arrow[r, "e(h_A \otimes h_B)"] & e(eA \otimes eB) \arrow[r, "e\mathsf{ex}_{A,B}"] & e(eB \otimes eA) \arrow[r, "e(\varepsilon_B \otimes \varepsilon_A)"] & e(B \otimes A)
\end{tikzcd}
$$

Diagram one commutes by the action diagrams for the coalgebras (A, h_A) and (B, h_B), diagram two commutes because $\mathcal{L}$ is a Lambek category with exchange, diagram three does not commute, but holds by Lemma 9, diagram four and six commute by naturality of q, and diagram five commutes because $\mathcal{L}$ is a Lambek category with exchange. □

Lemma 11 (The Eilenberg-Moore Category is Symmetric Monoidal)**.** *The category $\mathcal{L}^e$ is symmetric monoidal.*

Proof. The following diagram shows that $\beta_{B,A} \circ \beta_{A,B} = \mathsf{id}_{A\otimes B}$:

$$\begin{tikzcd}
A\otimes B \arrow[r, "h_A\otimes h_B"] \arrow[d, "h_A\otimes h_B"'] \arrow[dr, phantom, "(1)"] \arrow[rrrddd, equal, bend left] & eA\otimes eB \arrow[d, "\delta_A\otimes\delta_B"] \arrow[rd, phantom, "(6)"] & & \\
eA\otimes eB \arrow[r, "\delta_A\otimes\delta_B"] \arrow[d, "\mathsf{ex}_{A,B}"'] \arrow[dr, phantom, "(2)"] & e^2A\otimes e^2B \arrow[d, "\mathsf{ex}_{eA,eB}"] \arrow[rd, equal] \arrow[rd, phantom, "(4)"'] & & \\
eB\otimes eA \arrow[r, "\delta_B\otimes\delta_A"] \arrow[d, "\varepsilon_B\otimes\varepsilon_A"'] \arrow[dr, phantom, "(3)"] & e^2B\otimes e^2A \arrow[r, "\mathsf{ex}_{eB,eA}"] \arrow[d, "e\varepsilon_B\otimes e\varepsilon_A"] \arrow[dr, phantom, "(5)"] & e^2A\otimes e^2B \arrow[d, "e\varepsilon_A\otimes e\varepsilon_B"] & \\
B\otimes A \arrow[r, "h_B\otimes h_A"] & eB\otimes eA \arrow[r, "\mathsf{ex}_{B,A}"] & eA\otimes eB \arrow[r, "\varepsilon_A\otimes\varepsilon_B"] & A\otimes B
\end{tikzcd}$$

Diagram one commutes trivially commutes, diagram two commutes because $\mathcal{L}$ is a Lambek category with exchange, diagram three does not commute, but holds by Lemma 9, diagram four and five commute by the coherence diagrams of ex, diagram six clearly commutes, diagram seven commutes because (e, ε, δ) is a comonad, and diagram eight commutes by both the action diagrams of the Eilenberg Moore category and the fact that (e, ε, δ) is a comonad.

At this point we must verify that β respects the coherence diagrams of a symmetric monoidal category; see Definition 24. Thus, we must show that each of the following diagrams hold:

Case

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B}\otimes\mathsf{id}_C"'] & A\otimes(B\otimes C) \arrow[r, "\beta_{A,B\otimes C}"] & (B\otimes C)\otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B\otimes A)\otimes C \arrow[r, "\alpha_{B,A,C}"] & B\otimes(A\otimes C) \arrow[r, "\mathsf{id}_B\otimes\beta_{A,C}"] & B\otimes(C\otimes A)
\end{tikzcd}$$

Case

$$\begin{tikzcd}
A\otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A\otimes B}"] & \\
B\otimes A \arrow[r, "\beta_{B,A}"'] & A\otimes B
\end{tikzcd}$$

Case

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

□

Definition 12. *Let* $(\mathcal{L}, e, \mathsf{ex})$ *be a Lambek category with exchange. The* ***coKleisli Category of*** e*,* $\mathcal{L}_e$*, is a category with the same objects as* $\mathcal{L}$*. There is an arrow* $\hat{f} : A \longrightarrow B$ *in* $\mathcal{L}_e$ *if there is an arrow* $f : eA \longrightarrow B$ *in* $\mathcal{L}$*. The identity arrow* $\hat{id_A} : A \longrightarrow A$ *is the arrow* $\varepsilon_A : eA \longrightarrow A$ *in* $\mathcal{L}$*. Given* $\hat{f} : A \longrightarrow B$ *and* $\hat{g} : B \longrightarrow C$ *in* $\mathcal{L}_e$*, which are arrows* $f : eA \longrightarrow B$ *and* $g : eB \longrightarrow C$ *in* $\mathcal{L}$*, the composition* $\hat{g} \circ \hat{f} : A \longrightarrow C$ *is defined as* $g \circ ef \circ \delta_A$*.*

2.4 Linear Categories

Definition 13. *A* ***linear category****,* $(\mathcal{L}, !, \mathsf{weak}, \mathsf{contra})$*, is a symmetric monoidal closed category* $(\mathcal{L}, I, \otimes, \multimap)$ *equipped with a symmetric monoidal comonad* $(!, \varepsilon, \delta)$ *with* $\mathsf{q}_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ *and* $\mathsf{q}_I : I \longrightarrow !I$*, and two monoidal natural transformations with components* $\mathsf{weak}_A : !A \longrightarrow I$ *and* $\mathsf{contra}_A : !A \longrightarrow !A \otimes !A$*, satisfying the following conditions:*

- *each* $(!A, \mathsf{weak}_A, \mathsf{contra}_A)$ *is a commutative comonoid, i.e. the following diagrams commute and* $\beta \circ \mathsf{contra}_A = \mathsf{contra}_A$ *where* $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ *is the symmetry natural transformation of* $\mathcal{L}$*;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & !A \otimes !A \arrow[r, "id_{!A} \otimes \mathsf{contra}_A"] & !A \otimes (!A \otimes !A) \\
!A \otimes !A \arrow[rr, "\mathsf{contra}_A \otimes id_{!A}"'] & & (!A \otimes !A) \otimes !A \arrow[u, "\alpha_{!A,!A,!A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
& !A \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes !A & !A \otimes !A \arrow[l, "\mathsf{weak}_A \otimes id_{!A}"] \arrow[r, "id_{!A} \otimes \mathsf{weak}_A"'] & !A \otimes I
\end{tikzcd}$$

- weak_A *and* contra_A *are coalgebra morphisms, i.e. the following diagrams commute;*

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
!!A \arrow[r, "!\mathsf{weak}_A"'] & !I
\end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[r, "\delta_A \otimes \delta_A"] & !!A \otimes !!A \arrow[d, "\mathsf{q}_{!A,!A}"] \\
!!A \arrow[rr, "!\mathsf{contra}_A"'] & & !(!A \otimes !A)
\end{tikzcd}$$

- *any coalgebra morphism $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ between free coalgebras preserve the comonoid structure given by* weak *and* contra*, i.e. the following diagrams commute.*

$$\begin{tikzcd}
!A \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & !B \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & !A \otimes !A \arrow[d, "f \otimes f"] \\
!B \arrow[r, "\mathsf{contra}_B"'] & !B \otimes !B
\end{tikzcd}$$

Definition 14. *Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction and $(\mathcal{L}, w, \mathsf{weak})$ is a Lambek category with weakening, we define a **distributive law** of c over w to be a natural transformation with components $dist_A : cwA \longrightarrow wcA$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
wA & & cwA \arrow[ll, "\varepsilon^c_{wA}"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "w\varepsilon^c_A"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cA & & cwA \arrow[ll, "c\varepsilon^w_A"'] \arrow[dl, "dist_A"] \\
& wcA \arrow[ul, "\varepsilon^w_{cA}"] &
\end{tikzcd}$$

Lemma 15. *Given two comonads $(c, \varepsilon^c, \delta^c)$ and $(w, \varepsilon^w, \delta^w)$ on a category $\mathcal{L}$ such that $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction and $(\mathcal{L}, w, \mathsf{weak})$ is a Lambek category with weakening, the following two diagrams commute:*

$$\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, "\delta^c_{wcA}"'] & & cwc^2A \\
& c^2wcA \arrow[ur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wdist_{wA}"] \\
& w^2cwA &
\end{tikzcd}$$

Proof. The two diagrams above commute because the following ones commute by the distributive law and the comonad laws for c and w.

$$\begin{tikzcd}
cwcA \arrow[rr, "cw\delta^c_A"] \arrow[dr, equal] \arrow[ddr, "\delta^c_{wA}"'] & & cwc^2A \arrow[dl, "cw\varepsilon^c_{cA}"'] \\
& cwcA & \\
& c^2wcA \arrow[u, "c\varepsilon^c_{wcA}"] \arrow[uur, "cdist_{cA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
wcwA \arrow[rr, "wc\delta^w_A"] \arrow[dr, equal] \arrow[ddr, "\delta^w_{cwA}"'] & & wcw^2A \arrow[dl, "wc\varepsilon^w_{wA}"'] \arrow[ddl, "wdist_{wA}"] \\
& wcwA & \\
& w^2cwA \arrow[u, "w\varepsilon^w_{cwA}"] &
\end{tikzcd}$$

□

Lemma 16 (Composition of Weakening and Contraction)**.** *Suppose* $(\mathcal{L}, I, \otimes, w, \mathsf{weak}^w, c, \mathsf{contraL}, \mathsf{contraR})$ *is a Lambek category with weakening and contraction, where* $(w, \varepsilon^w, \delta^w)$ *and* $(c, \varepsilon^c, \delta^c)$ *are the respective monoidal comonads. Then the composition of c and w using the distributive law* $dist_A : cwA \longrightarrow wcA$ *is a monoidal comonad on* $\mathcal{L}$*.*

Proof. For the complete proof see Appendix B.1. $\square$

Definition 17. *A **Lambek category with** cw,* $(\mathcal{L}, cw, \mathsf{weak}^w, \mathsf{contraL}, \mathsf{contraR}, dist)$*, is a Lambek category with weakening and contraction, and a distributive law. Furthermore, the following coherence diagrams commute:*

$$\begin{tikzcd}
I \otimes cwA \arrow[r, "\lambda^{-1}_{I\otimes cwA}"] \arrow[d, "\mathsf{contraR}_{wA,I}"'] & I \otimes (I \otimes cwA) \\
cwA \otimes (I \otimes cwA) \arrow[r, "\varepsilon^c_{wA}\otimes id_{I\otimes cwA}"] & wA \otimes (I \otimes cwA) \arrow[u, "\mathsf{weak}^w_A\otimes id_{I\otimes cwA}"']
\end{tikzcd}$$

$$\begin{tikzcd}
cwA \otimes I \arrow[r, "\rho^{-1}_{cwA\otimes I}"] \arrow[d, "\mathsf{contraL}_{wA,I}"'] & (cwA \otimes I) \otimes I \\
(cwA \otimes I) \otimes cwA \arrow[r, "id_{cwA\otimes I}\otimes\varepsilon^c_{wA}"] & (cwA \otimes I) \otimes wA \arrow[u, "id_{cwA\otimes I}\otimes\mathsf{weak}^w_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[d, "\varepsilon^c_{wA}"'] & & cwB \arrow[d, "\varepsilon^c_{wB}"] \\
wA \arrow[r, "\mathsf{weak}^w_A"] & I & wB \arrow[l, "\mathsf{weak}^w_B"']
\end{tikzcd}$$

where $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ *is any coalgebra morphism between free coalgebras.*

Lemma 18. *Let* $(\mathcal{L}, cw, \mathsf{weak}^w, \mathsf{contraL}, \mathsf{contraR})$ *be a Lambek category with cw. Then the following conditions are satisfied:*

1. *There exist two natural transformations* $\mathsf{weak}_A : cwA \longrightarrow I$ *and* $\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$*.*

2. *Each* $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ *is a comonoid.*

3. weak_A *and* contra_A *are coalgebra morphisms.*

4. *Any coalgebra morphism* $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ *between free coalgebras preserves the comonoid structure given by* weak *and* contra.

Proof. We will only prove the first condition by defining weak and contra. For the complete proof see Appendix B.2. Each of weak and contracan be given two equivalent definitions. $\mathsf{weak}_A : cwA \longrightarrow I$ is defined as in the diagram below. The left triangle commutes by the definition of *dist* and the right triangle commutes by the definition of weak^w.

$$\begin{tikzcd}
 & wcA \arrow[d, "w\varepsilon^c_A"] \arrow[dr, "\mathsf{weak}^w_{cA}"] & \\
cwA \arrow[ur, "dist_A"] \arrow[r, "\varepsilon^c_{wA}"'] & wA \arrow[r, "\mathsf{weak}^w_A"'] & I
\end{tikzcd}$$

$\mathsf{contra}_A : cwA \longrightarrow cwA \otimes cwA$ is defined as below. The left part of the diagram commutes by the definitions of contraL and of contraR, and the right part commutes because $\mathcal{L}$ is monoidal.

$$\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "\lambda^{-1}_{cwA}"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[dl, "\alpha_{cwA,I,cwA}"'] \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
I \otimes cwA \arrow[r, "\mathsf{contraR}_{wA,I}"'] & cwA \otimes (I \otimes cwA) \arrow[r, "id_{cwA} \otimes \lambda_{cwA}"'] & cwA \otimes cwA
\end{tikzcd}$$

□

Definition 19. *Given two comonads $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ on a category $\mathcal{L}$ such that $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ is a Lambek category with cw and $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange, we define a* ***distributive law for exchange*** *of cw over e to be a natural isomorphism with components $distEx_A : cweA \longrightarrow ecwA$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
eA & & cweA \arrow[ll, "\varepsilon^{cw}_{eA}"'] \arrow[dl, "distEx_A"] \\
& ecwA \arrow[ul, "e\varepsilon^{cw}_{A}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
cwA & & cweA \arrow[ll, "cw\varepsilon^{e}_{A}"'] \arrow[dl, "distEx_A"] \\
& ecwA \arrow[ul, "\varepsilon^{e}_{cwA}"] &
\end{tikzcd}$$

Lemma 20. *Given two comonads $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ on a category $\mathcal{L}$ such that $(\mathcal{L}, cw, \mathsf{weak}, \mathsf{contra})$ is a Lambek category with cw and $(\mathcal{L}, e, \mathsf{ex})$ is a Lambek category with exchange, the following two digrams also commute:*

$$\begin{tikzcd}
cwecwA \arrow[rr, "cwe\delta^{cw}_{A}"] \arrow[dr, "\delta^{cw}_{cweA}"'] & & cwe(cw)^2A \\
& (cw)^2ecwA \arrow[ur, "cwdistEx_{cwA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
ecweA \arrow[rr, "ecw\delta^{e}_{A}"] \arrow[dr, "\delta^{e}_{cweA}"'] & & ecwe^2A \arrow[dl, "edistEx_{eA}"] \\
& e^2cweA &
\end{tikzcd}$$

The proof is similar with the proof of Lemma 15 and we will not elaborate it here. Also, notice the difference between *dist* of *c* over *w* and *distEx* of *cw* over *e*. While *dist* is a natural transformation, *distEx* is a natural isomorphism.

Lemma 21. *let $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ be two monoidal comonads on a Lambek category with cw and exchange $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$. Then the composition of cw and e using the distributive law for exchange $distEx_A : cweA \longrightarrow ecwA$ is a monoidal comonad $(cwe, \varepsilon, \delta)$ on $\mathcal{L}$.*

Proof. Suppose $(cw, \varepsilon^{cw}, \delta^{cw})$ and $(e, \varepsilon^{e}, \delta^{e})$ are monoidal comonads, and $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with cw and exchange. Since by definition $cw, e : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors, we know that their composition

$cwe : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$
\begin{aligned}
\mathsf{q}_{A,B} &: cweA \otimes cweB \longrightarrow cwe(A \otimes B) \\
\mathsf{q}_{A,B} &= cw\mathsf{q}^{e}_{A,B} \circ \mathsf{q}^{cw}_{eA,eB} \\
\mathsf{q}_{I} &: I \longrightarrow cweI \\
\mathsf{q}_{I} &= cw\mathsf{q}^{e}_{I} \circ \mathsf{q}^{cw}_{I}
\end{aligned}
$$

Analogous to the proof of Lemma 16, each of ε and δ can be given two equivalent definitions:

$$
\begin{tikzcd}
cweA \arrow[r, "\varepsilon^{cw}_{eA}"] \arrow[d, "cw\varepsilon^{e}_{A}"'] & eA \arrow[d, "\varepsilon^{e}_{A}"] \\
cwA \arrow[r, "\varepsilon^{cw}_{A}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
cweA \arrow[r, "cw\delta^{e}_{A}"] \arrow[d, "\delta^{cw}_{eA}"'] & cwe^2A \arrow[r, "\delta^{cw}_{e^2A}"] \arrow[d, "\delta^{cw}_{e^2A}"] & (cw)^2e^2A \arrow[d, "cwdist_{eA}"] \\
(cw)^2eA \arrow[r, "(cw)^2\delta^{e}_{A}"'] & (cw)^2e^2A \arrow[r, "cwdist_{eA}"'] & cwecweA
\end{tikzcd}
$$

And the comonad laws can be proved similarly, which we will not elaborate for simplicity. □

Lemma 22. *Let $(cwe, \varepsilon, \delta)$ be a monoidal comonad over a monoidal category $(\mathcal{L}, I, \otimes)$ such that $(\mathcal{L}, I, \otimes, cw, \mathsf{weak}, \mathsf{contra}, e, \mathsf{ex})$ is a Lambek category with cw and exchange. Then the co-Kleisli category of $\mathcal{L}$, $\mathcal{L}_{cwe}$, is a linear category.*

changed to liner category. Finish the proof when lemma 5 is proved.

Proof. The identity object of $\mathcal{L}_{cwe}$ is still I.

The left and right unitors, $\hat{\lambda}_A : I \otimes A \longrightarrow A$ and $\hat{\rho}_A : A \otimes I \longrightarrow A$, in $\mathcal{L}_{cwe}$ are morphisms $cwe(I \otimes A) \longrightarrow A$ and $cwe(A \otimes I) \longrightarrow A$ in $\mathcal{L}$, respectively. Then we define $\hat{\lambda}$ and $\hat{\rho}$ as:

$$
\begin{aligned}
\hat{\lambda}_A &= \varepsilon_A \circ cwe\lambda_A \\
\hat{\rho}_A &= \varepsilon_A \circ cwe\rho_A,
\end{aligned}
$$

where λ and ρ are the left and right unitors in $\mathcal{L}$, respectively. And we define their inverses as:

$$
\begin{aligned}
\hat{\lambda}^{-1}_A &= \varepsilon_{I\otimes A} \circ cwe\lambda^{-1}_A \\
\hat{\rho}^{-1}_A &= \varepsilon_{A\otimes I} \circ cwe\rho^{-1}_A
\end{aligned}
$$

$\hat{\lambda}$ is a nautral isomorphism with inverse $\hat{\lambda}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
cwe(I \otimes A) \arrow[r, "\delta_{I\otimes A}"] \arrow[dr, "cwe\lambda_A"] \arrow[dd, "\varepsilon_{I\otimes A}"'] \arrow[ddr, equal] \arrow[ddr, phantom, "(2)"'] & (cwe)^2(I \otimes A) \arrow[r, "(cwe)^2\lambda_A"] \arrow[d, phantom, "(1)"] & (cwe)^2A \arrow[dd, "cwe\varepsilon_A"] \\
& (3)\ cweA \arrow[ur, "\delta_A"'] \arrow[d, "cwe\lambda^{-1}_A"] \arrow[dr, equal] \arrow[r, phantom, "(5)"] \arrow[dr, phantom, "(4)"'] & {} \\
I \otimes A & cwe(I \otimes A) \arrow[l, "\varepsilon_{I\otimes A}"] & cweA \arrow[l, "cwe\lambda^{-1}_A"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ. (2), (3) and (4) commute trivially. And (5) commutes because cwe is a comonad.

Similarly, $\hat{\rho}$ is a natural isomorphism with inverse $\hat{\rho}^{-1}$ by the following diagram chasing:

$$
\begin{tikzcd}
cwe(A \otimes I) \arrow[r, "\delta_{A\otimes I}"] \arrow[dd, "\varepsilon_{A\otimes I}"'] \arrow[dr, "cwe\rho_A"] \arrow[ddr, equal] & (cwe)^2(A \otimes I) \arrow[r, "(cwe)^2\rho_A"] & (cwe)^2A \arrow[dd, "cwe\varepsilon_A"] \\
& cweA \arrow[ur, "\delta_A"] \arrow[d, "cwe\rho_A^{-1}"] \arrow[dr, equal] & \\
A \otimes I & cwe(A \otimes I) \arrow[l, "\varepsilon_{A\otimes I}"] & cweA \arrow[l, "cwe\rho_A^{-1}"]
\end{tikzcd}
$$

The associator $\hat{\alpha}_A : (A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}_{cwe}$ is the morphism $cwe((A \otimes B) \otimes C) \longrightarrow A \otimes (B \otimes C)$ in $\mathcal{L}$. We define $\hat{\alpha}$ as:

$$\hat{\alpha}_{A,B,C} = \varepsilon_{A\otimes(B\otimes C)} \circ cwe\alpha_{A,B,C},$$

where α is the associator of $\mathcal{L}$. And its inverse is

$$\hat{\alpha}^{-1}_{A,B,C} = \varepsilon_{(A\otimes B)\otimes C} \circ cwe\alpha^{-1}_{A,B,C}$$

$\hat{\alpha}$ is a natural isomorphism with inverse $\hat{\alpha}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
cwe((A \otimes B) \otimes C) \arrow[r, "\delta_{(A\otimes B)\otimes C}"] \arrow[dd, "\varepsilon_{(A\otimes B)\otimes C}"'] \arrow[dr, "cwe\alpha_{A,B,C}"] \arrow[ddr, equal] & (cwe)^2((A \otimes B) \otimes C) \arrow[r, "(cwe)^2\alpha_{A,B,C}"] & (cwe)^2(A \otimes (B \otimes C)) \arrow[dd, "cwe\varepsilon_{A\otimes(B\otimes C)}"] \\
& cwe(A \otimes (B \otimes C)) \arrow[ur, "\delta_{A\otimes(B\otimes C)}"] \arrow[d, "cwe\alpha^{-1}_{A,B,C}"] \arrow[dr, equal] & \\
(A \otimes B) \otimes C & cwe((A \otimes B) \otimes C) \arrow[l, "\varepsilon_{(A\otimes B)\otimes C}"] & cwe(A \otimes (B \otimes C)) \arrow[l, "cwe\alpha^{-1}_{A,B,C}"]
\end{tikzcd}
$$

Therefore, $\mathcal{L}_{cwe}$ is a monoidal category.

The symmetry, $\hat{\beta}_{A,B} : A \otimes B \longrightarrow B \otimes A$, in $\mathcal{L}_{cwe}$ is the morphism $cwe(A \otimes B) \longrightarrow B \otimes A$ in $\mathcal{L}$, which is defined as:

$$\hat{\beta}_{A,B} = \varepsilon^{cw}_{B\otimes A} \circ cw\gamma_{A,B},$$

where $\varepsilon^{cw}_A : cwA \longrightarrow A$ is a natural transformation associated with the comonad cw, and γ is the natural isomorphism defined in Lemma **??**. Then its inverse is

$$\hat{\beta}^{-1}_{A,B} = \varepsilon^{cw}_{A\otimes B} \circ cw\gamma_{B,A}$$

$\hat{\beta}$ is a natural isomorphism with inverse $\hat{\beta}^{-1}$ because the following diagram chasing commutes:

$$
\begin{tikzcd}
A \otimes B \arrow[dr, phantom, "(1)"] & cw(A \otimes B) \arrow[l, "\varepsilon^{cw}_{A\otimes B}"'] \arrow[d, "\delta^{cw}_{A\otimes B}"] \arrow[dl, equal] \arrow[rr, phantom, "(3)"', shift right=6] & & cwe(A \otimes B) \arrow[ll, "cw\varepsilon^{e}_{A\otimes B}"'] \arrow[d, "\delta^{cw}_{e(A\otimes B)}"] \\
cw(A \otimes B) \arrow[u, "\varepsilon^{cw}_{A\otimes B}"] \arrow[dr, phantom, "(4)"] & (cw)^2(A \otimes B) \arrow[l, "cw\varepsilon^{cw}_{A\otimes B}"] \arrow[ul, phantom, "(2)"] \arrow[rr, phantom, "(5)"', shift right=6] & & (cw)^2e(A \otimes B) \arrow[ll, "(cw)^2\varepsilon^{e}_{A\otimes B}"] \arrow[d, "(cw)^2\delta^{e}_{A\otimes B}"] \\
 & (cw)^2e(B \otimes A) \arrow[u, "(cw)^2\gamma_{B,A}"] \arrow[d, equal] \arrow[ddl, "cw\varepsilon^{cw}_{e(B\otimes A)}"'] \arrow[dr, phantom, "(7)"] & (cw)^2e^2(A \otimes B) \arrow[l, "(cw)^2e\gamma_{A,B}"'] \arrow[r, equal] \arrow[d, phantom, "(8)"] & (cw)^2e^2(A \otimes B) \arrow[d, "cwdistEx_{e(A\otimes B)}"] \arrow[d, phantom, "(9)"', shift right=4] \\
 & (cw)^2e(B \otimes A) \arrow[drr, "cwdistEx_{B\otimes A}"] \arrow[d, phantom, "(6)"] & & (cwe)^2(A \otimes B) \arrow[ul, "cwdistEx^{-1}_{e(A\otimes B)}"] \arrow[d, "cwecw\gamma_{A,B}"] \\
cwe(B \otimes A) \arrow[uuu, "cw\gamma_{B,A}"] & & & cwecw(B \otimes A) \arrow[lll, "cwe\varepsilon^{cw}_{B\otimes A}"] \arrow[uull, "cwdistEx^{-1}_{B\otimes A}"']
\end{tikzcd}
$$

(1), (7) and (9) commute trivially. (2) is the comonad law for cw. (3) commutes by the naturality of δ^{cw}. (4) commutes by the naturality of ε^{cw}. (5) commutes because γ is a natural isomorphism (Lemma **??**). (6) is the definition of $distEx$. (8) is the naturality of $distEx$.

In conclusion, $\mathcal{L}_{cwe}$ is a symmetric monoidal category. □

3 Related Work

TODO

4 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at `http://research.microsoft.com/en-us/um/people/nick/mixed3.ps`.

A Appendix

A.1 Symmetric Monoidal Categories

Definition 23. *A **monoidal category** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A \\
\rho_A &: A \otimes \top \longrightarrow A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Definition 24. *A **symmetric monoidal category (SMC)** is a category, $\mathcal{M}$, with the following data:*

- *An object $\top$ of $\mathcal{M}$,*
- *A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,*
- *The following natural isomorphisms:*

$$\begin{aligned}
\lambda_A &: \top \otimes A \longrightarrow A \\
\rho_A &: A \otimes \top \longrightarrow A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

- *A symmetry natural isomorphism:*

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- *Subject to the following coherence diagrams:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes \mathsf{id}_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dd, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & \\
A \otimes (B \otimes (C \otimes D)) & A \otimes ((B \otimes C) \otimes D) \arrow[l, "\mathsf{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathsf{id}_C"] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathsf{id}_B \otimes \beta_{A,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes \top) \otimes B \arrow[rr, "\alpha_{A,\top,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (\top \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[d, "\beta_{A,B}"'] \arrow[dr, "\mathsf{id}_{A \otimes B}"] & \\
B \otimes A \arrow[r, "\beta_{B,A}"'] & A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\top \otimes A \arrow[rr, "\beta_{\top,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes \top \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

Definition 25. *A* ***monoidal biclosed category*** *is a monoidal category* $(\mathcal{M}, \top, \otimes)$*, such that, for any object B of* $\mathcal{M}$*, each of the functors* $-\otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ *and* $B \otimes - : \mathcal{M} \longrightarrow \mathcal{M}$ *has a specified right adjoint. Hence, for any object A and C of* $\mathcal{M}$*, there are two objects* $C \leftharpoonup B$ *and* $B \rightharpoonup C$ *of* $\mathcal{M}$ *and two natural bijections:*

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, C \leftharpoonup B)$$
$$\mathsf{Hom}_{\mathcal{M}}(B \otimes A, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \rightharpoonup C)$$

Definition 26. *A* ***symmetric monoidal closed category (SMCC)*** *is a symmetric monoidal category,* $(\mathcal{M}, \top, \otimes)$*, such that, for any object B of* $\mathcal{M}$*, the functor* $- \otimes B : \mathcal{M} \longrightarrow \mathcal{M}$

has a specified right adjoint. Hence, for any objects A and C of $\mathcal{M}$ there is an object $B \multimap C$ of $\mathcal{M}$ and a natural bijection:

$$\mathsf{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of $\mathcal{M}$.

Definition 27. *Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a* ***monoidal functor*** *is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1, A}"] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A, \top_1}"] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"]
\end{tikzcd}$$

Need to notice that the composition of monoidal functors is also monoidal, subject to the above coherence conditions.

Definition 28. *Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a* ***symmetric monoidal functor*** *is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:*

$$\begin{tikzcd}
(FA \otimes_2 FB) \otimes_2 FC \arrow[r, "\alpha_{2FA,FB,FC}"] \arrow[d, "m_{A,B} \otimes \mathsf{id}_{FC}"] & FA \otimes_2 (FB \otimes_2 FC) \arrow[d, "\mathsf{id}_{FA} \otimes m_{B,C}"] \\
F(A \otimes_1 B) \otimes_2 FC \arrow[d, "m_{A \otimes_1 B, C}"] & FA \otimes_2 F(B \otimes_1 C) \arrow[d, "m_{A, B \otimes_1 C}"] \\
F((A \otimes_1 B) \otimes_1 C) \arrow[r, "F\alpha_{1A,B,C}"] & F(A \otimes_1 (B \otimes_1 C))
\end{tikzcd}$$

$$\begin{tikzcd}
\top_2 \otimes_2 FA \arrow[r, "\lambda_{2FA}"] \arrow[d, "m_{\top_1} \otimes \mathsf{id}_{FA}"'] & FA \\
F\top_1 \otimes_2 FA \arrow[r, "m_{\top_1,A}"'] & F(\top_1 \otimes_1 A) \arrow[u, "F\lambda_{1A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_2 \top_2 \arrow[r, "\rho_{2FA}"] \arrow[d, "\mathsf{id}_{FA} \otimes m_{\top_1}"'] & FA \\
FA \otimes_2 F\top_1 \arrow[r, "m_{A,\top_1}"'] & F(A \otimes_1 \top_1) \arrow[u, "F\rho_{1A}"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "\beta_{2FA,FB}"] \arrow[d, "m_{A,B}"'] & FB \otimes_2 FA \arrow[d, "m_{B,A}"] \\
F(A \otimes_1 B) \arrow[r, "F\beta_{1A,B}"'] & F(B \otimes_1 A)
\end{tikzcd}$$

Definition 29. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 30. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then a* ***symmetric monoidal natural transformation*** *is a natural transformation, $f : F \longrightarrow G$, subject to the following coherence diagrams:*

$$\begin{tikzcd}
FA \otimes_2 FB \arrow[r, "m_{A,B}"] \arrow[d, "f_A \otimes_2 f_B"'] & F(A \otimes_1 B) \arrow[d, "f_{A \otimes_1 B}"] \\
GA \otimes_2 GB \arrow[r, "n_{A,B}"'] & G(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[rr, "f_{\top_1}"] & & G\top_1 \\
& \top_2 \arrow[ul, "m_{\top_1}"] \arrow[ur, "n_{\top_1}"'] &
\end{tikzcd}$$

Definition 31. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are* *monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 32. *Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between $\mathcal{M}_1$ and $\mathcal{M}_2$ and (G, n) is a symmetric monoidal functor between $\mathcal{M}_2$ and $\mathcal{M}_1$. Then a* ***symmetric monoidal adjunction*** *is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \to GFA$, and the counit, $\varepsilon_A : FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:*

$$\begin{tikzcd}
FGA \otimes_2 FGB \arrow[r, "m_{GA,GB}"] \arrow[d, "\varepsilon_A \otimes_1 \varepsilon_B"'] & F(GA \otimes_1 GB) \arrow[d, "Fn_{A,B}"] \\
A \otimes_2 B & FGA \otimes_2 FGB \arrow[l, "\varepsilon_{A \otimes_1 B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\top_1 \arrow[r, "Fn_{\top_2}"] & FG\top_2 \arrow[d, "\varepsilon_{\top_1}"] \\
\top_2 \arrow[u, "m_{\top_1}"] \arrow[r, equal] & \top_2
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes_1 GFB \arrow[d, "n_{FA,FB}"'] & A \otimes_1 B \arrow[l, "\eta_A \otimes_1 \eta_B"'] \arrow[d, "\eta_{A \otimes_1 B}"] \\
G(FA \otimes_2 FB) \arrow[r, "m_{A,B}"'] & GF(A \otimes_1 B)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\top_2 \arrow[r, "Gm_{\top_1}"] & GF\top_1 \\
\top_1 \arrow[u, "n_{\top_2}"] \arrow[r, equal] & \top_1 \arrow[u, "\eta_{\top_1}"']
\end{tikzcd}$$

Definition 33. *A* ***monoidal comonad*** *on a monoidal category $\mathcal{C}$ is a triple (T, ε, δ), where (T, m) is a monoidal endofunctor on $\mathcal{C}$, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:*

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$
\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\
& \top &
\end{tikzcd}
$$

$$
\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}
$$

Definition 34. *A* ***symmetric monoidal comonad*** *on a symmetric monoidal category C is a triple (T, ε, δ), where (T, m) is a symmetric monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \rightarrow T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:*

$$
\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & T^2A \arrow[d, "T\delta_A"] \\
T^2A \arrow[r, "\delta_{TA}"'] & T^3A
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[dl, equal] \arrow[d, "\delta_A"] \arrow[dr, equal] & \\
TA & T^2A \arrow[l, "\varepsilon_{TA}"] \arrow[r, "T\varepsilon_A"'] & TA
\end{tikzcd}
$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$
\begin{tikzcd}
TA \otimes TB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[dr, "\varepsilon_A \otimes \varepsilon_B"'] & T(A \otimes B) \arrow[d, "\varepsilon_{A \otimes B}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
T\top \arrow[dr, "\varepsilon_\top"'] & & \top \arrow[ll, "\mathsf{m}_\top"'] \arrow[dl, equal] \\
& \top &
\end{tikzcd}
$$

$$\begin{tikzcd}
TA \otimes TB \arrow[rr, "\mathsf{m}_{A,B}"] \arrow[d, "\delta_A \otimes \delta_B"'] & & T(A \otimes B) \arrow[d, "\delta_{A \otimes B}"] \\
T^2A \otimes T^2B \arrow[r, "\mathsf{m}_{TA,TB}"'] & T(TA \otimes TB) \arrow[r, "T\mathsf{m}_{A,B}"'] & T^2(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
\top \arrow[r, "\mathsf{m}_\top"] \arrow[d, "\mathsf{m}_\top"'] & T\top \arrow[d, "\delta_\top"] \\
T\top \arrow[r, "T\mathsf{m}_\top"'] & T^2\top
\end{tikzcd}$$

B Proofs

B.1 Proof of Composition of Weakening and Contraction (Lemma 16)

Since by definition $w : \mathcal{L} \longrightarrow \mathcal{L}$ and $c : \mathcal{L} \longrightarrow \mathcal{L}$ are monoidal functors we know that their composition $cw : \mathcal{L} \longrightarrow \mathcal{L}$ is a monoidal functor:

$$\begin{aligned}
&\mathsf{q}_{A,B} : cwA \otimes cwB \longrightarrow cw(A \otimes B)\\
&\mathsf{q}_{A,B} = c\mathsf{q}^w_{A,B} \circ \mathsf{q}^c_{wA,wB}\\
&\mathsf{q}_I : I \longrightarrow cwI\\
&\mathsf{q}_I = c\mathsf{q}^w_I \circ \mathsf{q}^c_I
\end{aligned}$$

We must now define both $\varepsilon_A : cwA \longrightarrow A$ and $\delta_A : cwA \longrightarrow cwcwA$, and then show that they are monoidal natural transformations subject to the comonad laws. Since we are composing two comonads each of ε and δ can be given two definitions, but they are equivalent:

- $\varepsilon_A : cwA \longrightarrow A$ is defined as in the diagram below, which commutes by the naturality of ε^c.

$$\begin{tikzcd}
cwA \arrow[r, "\varepsilon^c_{wA}"] \arrow[d, "c\varepsilon^w_A"'] & wA \arrow[d, "\varepsilon^w_A"] \\
cA \arrow[r, "\varepsilon^c_A"'] & A
\end{tikzcd}$$

- $\delta_A : cwA \longrightarrow cwcwA$ is defined as in the diagram:

$$
\begin{tikzcd}
cwA \arrow[r, "c\delta^w_A"] \arrow[d, "\delta^c_{wA}"'] & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, "\delta^c_{w^2A}"] & c^2w^2A \arrow[d, "cdist_{wA}"] \\
c^2wA \arrow[r, "c^2\delta^w_A"'] & c^2w^2A \arrow[r, "cdist_{wA}"'] & cwcwA
\end{tikzcd}
$$

The left part of the diagram commutes by the naturality of δ^c and the right part commutes trivially.

The remainder of the proof shows that the comonad laws hold.

Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & cwcwA \arrow[d, "cw\delta_A"] \\
cwcwA \arrow[r, "\delta_{cwA}"'] & cwcwcwA
\end{tikzcd}
$$

The previous diagram commutes because the following one does.

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, phantom, "(1)"] & cwcwA \arrow[r, "cwc\delta^w_A"] \arrow[d, "\delta^c_{wcwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwcw^2A \arrow[r, "cw\delta^c_{w^2A}"] \arrow[d, "\delta^c_{wcw^2A}"] \arrow[dr, phantom, "(5)"] & cwc^2w^2A \arrow[dd, "cwcdist_{wA}"] \\
cwcwA \arrow[d, "c\delta^w_{cwA}"'] \arrow[dr, phantom, "(4)"] & c^2wcwA \arrow[r, "c^2wc\delta^w_A"] \arrow[dr, "c^2\delta^w_{cwA}"'] \arrow[r, phantom, shift right=6, "(3)"] & c^2wcw^2A \arrow[ur, "cdist_{cw^2A}"] \arrow[d, "c^2wdist_{wA}"] \arrow[r, phantom, shift right=6, "(6)"] & \\
cw^2cwA \arrow[rr, "\delta^c_{w^2cwA}"'] & & c^2w^2cwA \arrow[r, "cdist_{wcwA}"'] & cwcwcwA
\end{tikzcd}
$$

(1) commutes by equality and we will not expand δ_A for simplicity. (2) and (4) commutes by the naturality of δ^c. (3), (5) commutes by the conditions of $dist$. (6) commutes by the naturality of $dist$.

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\delta_A"] \arrow[dr, equal] & cwcwA \arrow[d, "cw\varepsilon_A"] \\
& cwA
\end{tikzcd}
$$

The triangle commutes because of the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\delta^w_A"] \arrow[ddd, equal] \arrow[ddr, "\varepsilon^c_{wA}"] & & cw^2A \arrow[r, "\delta^c_{w^2A}"] \arrow[d, equal] & c^2w^2A \arrow[dl, "c\varepsilon^c_{w^2A}"'] \arrow[ddd, "cdist_{wA}"] \\
& & cw^2A \arrow[ull, "c\varepsilon^w_{wA}"'] \arrow[d, "\varepsilon^c_{w^2A}"] \arrow[ul, phantom, "(1)"] \arrow[ur, phantom, "(2)"] & \\
& wA \arrow[u, phantom, "(3)"] \arrow[l, phantom, "(5)"] \arrow[d, phantom, "(6)"] & w^2A \arrow[l, "\varepsilon^w_{wA}"'] \arrow[ur, phantom, "(4)"] \arrow[dr, phantom, "(7)"] & \\
cwA \arrow[ur, "\varepsilon^c_{wA}"] & & wcwA \arrow[ll, "\varepsilon^w_{cwA}"] \arrow[u, "w\varepsilon^c_{wA}"'] & cwcwA \arrow[l, "\varepsilon^c_{wcwA}"] \arrow[uul, "cw\varepsilon^c_{wA}"']
\end{tikzcd}
$$

(1) commutes by the comonad law for w with components δ^w_A and ε^w_{wA}. (2) commutes by the comonad law for c with components $\delta^c_{w^2A}$ and $\varepsilon^c_{w^2A}$. (3) and (7) commute by the naturality of ε^c. (4) commutes by the condition of $dist$. (5) commutes trivially. And (6) commutes by the naturality of ε^w.

Case 3:

$$
\begin{tikzcd}
cwA \arrow[d, "\delta_A"'] \arrow[dr, equal] & \\
cwcwA \arrow[r, "\varepsilon_{cwA}"'] & cwA
\end{tikzcd}
$$

The previous triangle commutes because the following diagram chasing does.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\delta^w_A"] \arrow[drr, "\delta^c_{wA}"] \arrow[ddd, equal] & & cw^2A \arrow[rr, "\delta^c_{w^2A}"] \arrow[d, phantom, "(1)"] & & c^2w^2A \arrow[ddd, "cdist_{wA}"] \arrow[ddll, "c^2w\varepsilon^w_A"'] \\
& & c^2wA \arrow[urr, "c^2\delta^w_A"'] \arrow[d, equal] \arrow[l, phantom, "(2)"] \arrow[r, phantom, "(3)"] & & \\
& & c^2wA \arrow[uull, "c\varepsilon^c_{wA}"] \arrow[d, "cdist_A"] \arrow[dl, phantom, "(4)"] \arrow[dr, phantom, "(5)"] & & \\
cwA & & cwcA \arrow[ll, "cw\varepsilon^c_A"] & & cwcwA \arrow[ll, "cwc\varepsilon^w_A"]
\end{tikzcd}
$$

(1) commutes by the naturality of δ^c. (2) is the comonad law for c with components δ^c_{wA} and ε^c_{wA}. (3) is the comonad law for w with components δ^w_A and ε^w_A. (4) commutes by the condition of *dist*. And (5) commute by the naturality of *dist*.

¡¡¡¡¡¡¡ HEAD

B.2 Proof of Conditions of Lambek category with cw (Lemma 18)

1. As shown in the paper.
2. Each $(cwA, \mathsf{weak}_A, \mathsf{contra}_A)$ is a comonoid.

Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] & cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\mathsf{contra}_A"] & cwA \otimes (cwA \otimes cwA) \\
cwA \otimes cwA \arrow[rr, "\mathsf{contra}_A\otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}
$$

The previous diagram commutes by the following diagram chasing.

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\mathsf{contra}_A"'] \arrow[dr, phantom, "(1)"] & cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\rho^{-1}_{cwA}"] \arrow[dl, equal] \arrow[dr, phantom, "(2)"] & cwA \otimes (cwA \otimes I) \arrow[d, "id_{cwA}\otimes\mathsf{contraL}_{wA,I}"] \\
cwA \otimes cwA \arrow[r, "id_{cwA}\otimes\lambda^{-1}_{cwA}"] \arrow[d, "\rho^{-1}_{cwA}\otimes id_{cwA}"'] & cwA \otimes (I \otimes cwA) \arrow[d, "id_{cwA}\otimes\mathsf{contraR}_{wA,I}"] \arrow[dr, phantom, "(3)"] & cwA \otimes ((cwA \otimes I) \otimes cwA) \arrow[d, "id_{cwA}\otimes(\rho_{cwA}\otimes id_{cwA})"] \\
(cwA \otimes I) \otimes cwA \arrow[d, "\mathsf{contraL}_{wA,I}\otimes id_{cwA}"'] & cwA \otimes (cwA \otimes (I \otimes cwA)) \arrow[r, "id_{cwA}\otimes(id_{cwA}\otimes\lambda_{cwA})"] \arrow[ur, "id_{cwA}\otimes\alpha_{cwA,I,cwA}"] \arrow[d, phantom, "(4)"] & cwA \otimes (cwA \otimes cwA) \\
((cwA \otimes I) \otimes cwA) \otimes cwA \arrow[rr, "(\rho_{cwA}\otimes id_{cwA})\otimes id_{cwA}"'] & & (cwA \otimes cwA) \otimes cwA \arrow[u, "\alpha_{cwA,cwA,cwA}"']
\end{tikzcd}
$$

(1) commutes trivially and we would not expand contra for simplicity. (2) and (4) commute because $(\mathcal{L}, c, \mathsf{contraL}, \mathsf{contraR})$ is a Lambek category with contraction. (3) commutes because $\mathcal{L}$ is monoidal.

Case 2:

$$
\begin{tikzcd}
 & cwA \arrow[dl, "\lambda^{-1}"'] \arrow[d, "\mathsf{contra}_A"] \arrow[dr, "\rho^{-1}"] & \\
I \otimes cwA & cwA \otimes cwA \arrow[l, "\mathsf{weak}_A \otimes id_{cwA}"] \arrow[r, "id_{cwA} \otimes \mathsf{weak}_A"'] & cwA \otimes I
\end{tikzcd}
$$

The diagram above commutes by the following diagram chasing.

$$
\begin{tikzcd}
I \otimes cwA & & & wA \otimes cwA \arrow[lll, "\mathsf{weak}^w_A \otimes id_{cwA}"'] \\
 & I \otimes (I \otimes cwA) \arrow[ul, "id_I \otimes \lambda_{cwA}"] \arrow[ul, phantom, "(1)"', shift right=6] & wA \otimes (I \otimes cwA) \arrow[l, "\mathsf{weak}^w_A \otimes id_{I \otimes cwA}"'] \arrow[ur, "id_{wA} \otimes \lambda_{cwA}"'] \arrow[ul, phantom, "(2)"'] \arrow[ur, phantom, "(3)", shift left=6] & \\
 & I \otimes cwA \arrow[u, "\lambda^{-1}_{I \otimes cwA}"] \arrow[r, "\mathsf{contraR}_{wA,I}"] \arrow[ur, phantom, "(4)"] & cwA \otimes (I \otimes cwA) \arrow[u, "\varepsilon^c_{wA} \otimes id_{I \otimes cwA}"'] \arrow[dr, "id_{cwA} \otimes \lambda_{cwA}"] & \\
cwA \arrow[uuu, "\lambda^{-1}_{cwA}"] \arrow[ur, "\lambda^{-1}_{cwA}"] \arrow[dr, "\rho^{-1}_{cwA}"'] \arrow[ddd, "\rho^{-1}_{cwA}"'] \arrow[rrr, phantom, "(5)"] & & & cwA \otimes cwA \arrow[uuu, "\varepsilon^c_{wA} \otimes cwA"'] \arrow[ddd, "id_{cwA} \otimes \varepsilon^c_{wA}"] \\
 & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] \arrow[d, "\rho^{-1}_{cwA}"'] \arrow[dr, phantom, "(9)"] & (cwA \otimes I) \otimes cwA \arrow[ur, "\rho_{cwA} \otimes id_{cwA}"] \arrow[d, "id_{cwA \otimes I} \otimes \varepsilon^c_{wA}"] & \\
 & (cwA \otimes I) \otimes I \arrow[dl, "\rho_{cwA} \otimes id_I"'] \arrow[dl, phantom, "(6)", shift left=6] & (cwA \otimes I) \otimes wA \arrow[l, "id_{cwA \otimes I} \otimes \mathsf{weak}^w_A"'] \arrow[dr, "\rho_{cwA} \otimes id_{wA}"] \arrow[dl, phantom, "(7)"] \arrow[dr, phantom, "(8)"', shift right=6] & \\
cwA \otimes I & & & cwA \otimes wA \arrow[lll, "id_{cwA} \otimes \mathsf{weak}_A"]
\end{tikzcd}
$$

(1), (2) and (3) commute by the functionality of λ. (6), (7) and (8) commute by the functionality of ρ. (4) and (9) are conditions of the Lambek category with cw. And (5) is the definition of contra.

3. weak and contra are coalgebra morphisms.

Case 1:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{weak}_A"] \arrow[d, "\delta A"'] & I \arrow[d, "\mathsf{q}_I"] \\
cwcwA \arrow[r, "cw\mathsf{weak}_A"'] & cwI
\end{tikzcd}
$$

The previous diagram commutes by the diagram below. (1) commutes by the naturality of δ^c. (2) commutes by the condition of $dist_{wA}$. (3), (5) and (6) commute because c is a monoidal comonad. (4) commutes because $(\mathcal{L}, w, \mathsf{weak}^w)$ is a Lambek category with weakening. (7) commutes because c and w are monoidal comonads.

$$
\begin{tikzcd}
cwA \arrow[rr, "c\mathsf{weak}^w_A"] \arrow[d, "\delta^c_{wA}"'] \arrow[drr, "c\delta^w_A"] & & cI \arrow[rr, "\epsilon^c_I"] \arrow[ddrr, "c\mathsf{q}^w_I"'] & & I \arrow[dl, equal] \arrow[dd, "\mathsf{q}^c_I"] \\
c^2wA \arrow[d, "c^2\delta^w_A"'] \arrow[r, phantom, "(1)"] & \arrow[r, phantom, "(4)"] & \arrow[r, phantom, "(5)"] & I \arrow[ul, "\mathsf{q}^c_I"'] \arrow[r, phantom, "(6)"] & \\
c^2w^2A \arrow[d, "cdist_{wA}"'] \arrow[drr, "c\varepsilon^c_{wA}"] \arrow[dr, phantom, "(2)"] & & cw^2A \arrow[ll, "\delta^c_{w^2A}"'] \arrow[d, equal] \arrow[dl, phantom, "(3)"] & \arrow[r, phantom, "(7)"] & cI \arrow[ul, "\varepsilon^c_I"'] \arrow[d, "c\mathsf{q}^w_I"] \\
cwcwA \arrow[rr, "cw\varepsilon^c_{wA}"'] & & cw^2A & & cwI \arrow[ll, "cw\mathsf{weak}^w_A"]
\end{tikzcd}
$$

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "\delta_A"'] & cwA \otimes cwA \arrow[r, "\delta_A \otimes \delta_A"] & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}_{cwA,cwA}"] \\
cwcwA \arrow[rr, "cw\mathsf{contra}_A"'] & & cw(cwA \otimes cwA)
\end{tikzcd}
$$

To prove the previous diagram commute, we first expand it, Then we divide it into five parts as shown belovee, and prove each part commutes.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, phantom, "(c)"] & cwA \otimes I \arrow[rrr, "\mathsf{contraL}_{wA,I}"] & & & (cwA \otimes I) \otimes cwA \arrow[d, "\rho_{cwA} \otimes id_{cwA}"] \\
cw^2A \arrow[d, "\delta^c_{w^2A}"'] & w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA \otimes I}"'] & w(cwA \otimes cwA) \arrow[rr, "\varepsilon^w_{cwA \otimes cwA}"] & & cwA \otimes cwA \arrow[d, "c\delta^w_A \otimes c\delta^w_A"] \\
c^2w^2A \arrow[d, "cdist_{wA}"'] & & & cw^2A \otimes cw^2A \arrow[r, equal] \arrow[ur, "c\varepsilon^c_{wA} \otimes c\varepsilon^c_{wA}"] \arrow[ur, phantom, shift right=3, "(a)"] \arrow[dr, phantom, "(b)"] & cw^2A \otimes cw^2A \arrow[d, "\delta^c_{w^2A} \otimes \delta^c_{w^2A}"] \\
cwcwA \arrow[d, "cw\rho_{cwA}"'] \arrow[rr, phantom, "(d)"] & & & & c^2w^2A \otimes c^2w^2A \arrow[ul, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
cw(cwA \otimes I) \arrow[d, "cw\mathsf{contraL}_{wA,I}"'] \arrow[uuur, "\varepsilon^c_{w(cwA \otimes I)}"] & & & & cwcwA \otimes cwcwA \arrow[d, "\mathsf{q}^c_{wcwA \otimes wcwA}"] \arrow[ll, phantom, "(e)"] \\
cw((cwA \otimes I) \otimes cwA) \arrow[rr, "cw(\rho_{cwA} \otimes id_{cwA})"'] & & cw(cwA \otimes cwA) \arrow[uuuu, "c\varepsilon^w_{cwA \otimes cwA}"] & & c(wcwA \otimes wcwA) \arrow[ll, "c\mathsf{q}^w_{cwA \otimes cwA}"]
\end{tikzcd}
$$

Part (a) and (b) are comonad laws.

Part (c) commutes by the following diagram chase. (1) is equality. (2) is the comonad law for w. (3) is the comonad law for c. (4) commutes by the naturality of ε^c. (5) is one of the conditions for $dist_{wA}$. (6) commutes by the naturality of ε^w. And (7) commutes by the naturality of ε^c.

$$
\begin{tikzcd}
cwA \arrow[rrr, "\rho^{-1}_{cwA}"] \arrow[d, "c\delta^w_A"'] \arrow[dr, "c\delta^w_A"] \arrow[drr, equal] & & & cwA \otimes I \\
cw^2A \arrow[r, equal] \arrow[d, "\delta^c_{w^2A}"'] \arrow[u, phantom, shift left=6, "(1)"] & cw^2A \arrow[r, "c\varepsilon^w_{wA}"] \arrow[d, "dist_{wA}"] \arrow[ur, phantom, "(2)"] & cwA \arrow[r, phantom, "(6)"] & \\
c^2w^2A \arrow[ur, "\varepsilon^c_{cw^2A}"] \arrow[d, "cdist_{wA}"'] \arrow[u, phantom, shift left=6, "(3)"] \arrow[r, phantom, "(4)"] & wcwA \arrow[ur, "\varepsilon^w_{cwA}"] \arrow[rr, "w\rho^{-1}_{cwA}"] \arrow[u, phantom, shift right=6, "(5)"] \arrow[drr, phantom, "(7)"] & & w(cwA \otimes I) \arrow[uu, "\varepsilon^w_{cwA \otimes I}"'] \\
cwcwA \arrow[ur, "\varepsilon^c_{wcwA}"] \arrow[rrr, "cw\rho^{-1}_{cwA}"'] & & & cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA \otimes I)}"']
\end{tikzcd}
$$

Part (d) commutes by the following diagram chase. The upper two squares both commute by the naturality of ε^w, and the lower two squares commute by the naturality of ε^c.

$$
\begin{tikzcd}
cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] & cwA \otimes cwA \\
w(cwA \otimes I) \arrow[u, "\varepsilon^w_{cwA \otimes I}"] \arrow[r, "w\mathsf{contraL}_{wA,I}"] & w((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^w_{(cwA \otimes I) \otimes cwA}"] \arrow[r, "w(\rho_{cwA} \otimes id_{cwA})"] & w(cwA \otimes cwA) \arrow[u, "\varepsilon^w_{cwA \otimes cwA}"] \\
cw(cwA \otimes I) \arrow[u, "\varepsilon^c_{w(cwA \otimes I)}"] \arrow[r, "cw\mathsf{contraL}_{wA,I}"'] & cw((cwA \otimes I) \otimes cwA) \arrow[u, "\varepsilon^c_{w((cwA \otimes I) \otimes A)}"] \arrow[r, "cw(\rho_{cwA} \otimes id_{cwA})"'] & cw(cwA \otimes cwA) \arrow[u, "\varepsilon^c_{w(cwA \otimes cwA)}"']
\end{tikzcd}
$$

Part (e) commutes by the following diagram. (1) commutes by the condition of $dist_{wA}$. (2) and (4) commute by the naturality of ε^c. (3) and (5) commute because w and c are monoidal comonads.

$$
\begin{tikzcd}
cwA \otimes cwA & & cw^2A \otimes cw^2A \arrow[ll, "c\varepsilon^w_{wA} \otimes c\varepsilon^w_{wA}"'] \arrow[d, "dist_{wA} \otimes dist_{wA}"] & & c^2w^2A \otimes c^2w^2A \arrow[ll, "\varepsilon^c_{cw^2A} \otimes \varepsilon^c_{cw^2A}"'] \arrow[d, "cdist_{wA} \otimes cdist_{wA}"] \\
& & wcwA \otimes wcwA \arrow[ull, "\varepsilon^w_{cwA} \otimes \varepsilon^w_{cwA}"] \arrow[dll, "\mathsf{q}^w_{cwA,cwA}"] & & cwcwA \otimes cwcwA \arrow[ll, "\varepsilon^c_{wcwA} \otimes \varepsilon^c_{wcwA}"'] \arrow[d, "\mathsf{q}^c_{wcwA \otimes wcwA}"] \\
w(cwA \otimes cwA) \arrow[uu, "\varepsilon^w_{cwA \otimes cwA}"] & & cw(cwA \otimes cwA) \arrow[ll, "\varepsilon^c_{w(cwA \otimes cwA)}"] & & c(wcwA \otimes wcwA) \arrow[ll, "c\mathsf{q}_{cwA \otimes cwA}"] \arrow[ull, "\varepsilon^c_{wcwA \otimes wcwA}"]
\arrow[from=1-1, to=2-3, phantom, "(1)"]
\arrow[from=1-3, to=2-5, phantom, "(2)"]
\arrow[from=1-1, to=3-3, phantom, "(3)"]
\arrow[from=2-3, to=3-3, phantom, "(4)"]
\arrow[from=2-5, to=3-4, phantom, "(5)"]
\end{tikzcd}
$$

4. Any coalgebra morphism $f : (cwA, \delta_A) \longrightarrow (cwB, \delta_B)$ between free coalgebras preserves the comonoid structure given by weak and contra.

Case 1: This coherence diagram is given in the definition of the Lambek category with cw.

$$
\begin{tikzcd}
cwA \arrow[rr, "f"] \arrow[dr, "\mathsf{weak}_A"'] & & cwB \arrow[dl, "\mathsf{weak}_B"] \\
& I &
\end{tikzcd}
$$

Case 2:

$$
\begin{tikzcd}
cwA \arrow[r, "\mathsf{contra}_A"] \arrow[d, "f"'] & cwA \otimes cwA \arrow[d, "f \otimes f"] \\
cwB \arrow[r, "\mathsf{contra}_B"'] & cwB \otimes cwB
\end{tikzcd}
$$

The square commutes by the diagram chasing below, which commutes by the naturality of ρ and contraL.

$$
\begin{tikzcd}
cwA \arrow[r, "\rho^{-1}_{cwA}"] \arrow[d, "cwf"'] & cwA \otimes I \arrow[r, "\mathsf{contraL}_{wA,I}"] \arrow[d, "cwf \otimes id_I"] & (cwA \otimes I) \otimes cwA \arrow[r, "\rho_{cwA} \otimes id_{cwA}"] \arrow[d, "(cwf \otimes id_I) \otimes cwf"] & cwA \otimes cwA \arrow[d, "cwf \otimes cwf"] \\
cwB \arrow[r, "\rho^{-1}_{cwB}"'] & cwB \otimes I \arrow[r, "\mathsf{contraL}_{wB,I}"'] & (cwB \otimes I) \otimes cwB \arrow[r, "\rho_{cwB} \otimes id_{cwB}"'] & cwB \otimes cwB
\end{tikzcd}
$$

======= ¿¿¿¿¿¿¿ origin/master