

Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

1.1 Symmetric Monoidal Categories

Definition 1 A *monoidal category* is a category, \mathcal{M} , with the following data:

- An object \top of \mathcal{M} ,
- A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\lambda_A : \top \otimes A \longrightarrow A$$

$$\rho_A : A \otimes \top \longrightarrow A$$

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

- Subject to the following coherence diagrams:

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes \text{id}_D} & (A \otimes (B \otimes C)) \otimes D \\ \downarrow \alpha_{A \otimes B, C, D} & & \downarrow \alpha_{A, B \otimes C, D} \\ (A \otimes B) \otimes (C \otimes D) & & A \otimes ((B \otimes C) \otimes D) \\ \downarrow \alpha_{A, B, C \otimes D} & \xleftarrow{\text{id}_A \otimes \alpha_{B,C,D}} & \\ A \otimes (B \otimes (C \otimes D)) & & A \otimes ((B \otimes C) \otimes D) \end{array}$$

$$\begin{array}{ccc} (A \otimes \top) \otimes B & \xrightarrow{\alpha_{A, \top, B}} & A \otimes (\top \otimes B) \\ \searrow \rho_A & & \swarrow \lambda_B \\ & A \otimes B & \end{array}$$

Definition 2 A *symmetric monoidal category (SMC)* is a category, \mathcal{M} , with the following data:

- An object \top of \mathcal{M} ,
- A bi-functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{aligned}\lambda_A &: \top \otimes A \longrightarrow A \\ \rho_A &: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)\end{aligned}$$

- A symmetry natural transformation:

$$\beta_{A,B} : A \otimes B \longrightarrow B \otimes A$$

- Subject to the following coherence diagrams:

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes \text{id}_D} & (A \otimes (B \otimes C)) \otimes D \\ \downarrow \alpha_{A \otimes B, C, D} & & \downarrow \alpha_{A, B \otimes C, D} \\ (A \otimes B) \otimes (C \otimes D) & & A \otimes ((B \otimes C) \otimes D) \\ \downarrow \alpha_{A, B, C \otimes D} & \xleftarrow{\text{id}_A \otimes \alpha_{B,C,D}} & \\ A \otimes (B \otimes (C \otimes D)) & & \end{array}$$

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\beta_{A, B \otimes C}} & (B \otimes C) \otimes A \\ \downarrow \beta_{A,B} \otimes \text{id}_C & & & & \downarrow \alpha_{B,C,A} \\ (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{\text{id}_B \otimes \beta_{A,C}} & B \otimes (C \otimes A) \end{array}$$

$$\begin{array}{ccc} (A \otimes \top) \otimes B & \xrightarrow{\alpha_{A,\top,B}} & A \otimes (\top \otimes B) \\ \downarrow \rho_A & & \downarrow \lambda_B \\ & A \otimes B & \end{array}$$

$$\begin{array}{ccc} A \otimes B & & \\ \downarrow \beta_{A,B} & \searrow \text{id}_{A \otimes B} & \\ B \otimes A & \xrightarrow{\beta_{B,A}} & A \otimes B \end{array}$$

$$\begin{array}{ccc} \top \otimes A & \xrightarrow{\beta_{\top,A}} & A \otimes \top \\ \downarrow \lambda_A & & \downarrow \rho_A \\ & A & \end{array}$$

Definition 3 A *symmetric monoidal closed category (SMCC)* is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of \mathcal{M} , the functor $- \otimes B : \mathcal{M} \rightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of \mathcal{M} there is an object $B \multimap C$ of \mathcal{M} and a natural bijection:

$$\text{Hom}_{\mathcal{M}}(A \otimes B, C) \cong \text{Hom}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ the internal hom of \mathcal{M} .

Definition 4 Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \rightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \rightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \rightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:

$$\begin{array}{ccc} (FA \otimes_2 FB) \otimes_2 FC & \xrightarrow{\alpha_{2FA,FB,FC}} & FA \otimes_2 (FB \otimes_2 FC) \\ \downarrow m_{A,B} \otimes \text{id}_{FC} & & \downarrow \text{id}_{FA} \otimes m_{B,C} \\ F(A \otimes_1 B) \otimes_2 FC & & FA \otimes_2 F(B \otimes_1 C) \\ \downarrow m_{A \otimes_1 B, C} & & \downarrow m_{A, B \otimes_1 C} \\ F((A \otimes_1 B) \otimes_1 C) & \xrightarrow{F\alpha_{1A,B,C}} & F(A \otimes_1 (B \otimes_1 C)) \end{array}$$

$$\begin{array}{ccc} \top_2 \otimes_2 FA & \xrightarrow{\lambda_{2FA}} & FA \\ \downarrow m_{\top_1} \otimes \text{id}_{FA} & & \uparrow F\lambda_{1A} \\ F\top_1 \otimes_2 FA & \xrightarrow{m_{\top_1, A}} & F(\top_1 \otimes_1 A) \end{array} \quad \begin{array}{ccc} FA \otimes_2 \top_2 & \xrightarrow{\rho_{2FA}} & FA \\ \downarrow \text{id}_{FA} \otimes m_{\top_1} & & \uparrow F\rho_{1A} \\ FA \otimes_2 F\top_1 & \xrightarrow{m_{A, \top_1}} & F(A \otimes_1 \top_1) \end{array}$$

Definition 5 Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal functor** is a functor $F : \mathcal{M}_1 \rightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \rightarrow F\top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \rightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:

$$\begin{array}{ccc} (FA \otimes_2 FB) \otimes_2 FC & \xrightarrow{\alpha_{2FA,FB,FC}} & FA \otimes_2 (FB \otimes_2 FC) \\ \downarrow m_{A,B} \otimes \text{id}_{FC} & & \downarrow \text{id}_{FA} \otimes m_{B,C} \\ F(A \otimes_1 B) \otimes_2 FC & & FA \otimes_2 F(B \otimes_1 C) \\ \downarrow m_{A \otimes_1 B, C} & & \downarrow m_{A, B \otimes_1 C} \\ F((A \otimes_1 B) \otimes_1 C) & \xrightarrow{F\alpha_{1A,B,C}} & F(A \otimes_1 (B \otimes_1 C)) \end{array}$$

$$\begin{array}{ccc}
\top_2 \otimes_2 FA & \xrightarrow{\lambda_{2FA}} & FA \\
\downarrow m_{\top_1} \otimes \text{id}_{FA} & & \uparrow F\lambda_{1A} \\
F\top_1 \otimes_2 FA & \xrightarrow{m_{\top_1, A}} & F(\top_1 \otimes_1 A)
\end{array}
\quad
\begin{array}{ccc}
FA \otimes_2 \top_2 & \xrightarrow{\rho_{2FA}} & FA \\
\downarrow \text{id}_{FA} \otimes m_{\top_1} & & \uparrow F\rho_{1A} \\
FA \otimes_2 F\top_1 & \xrightarrow{m_{A, \top_1}} & F(A \otimes_1 \top_1)
\end{array}$$

$$\begin{array}{ccc}
FA \otimes_2 FB & \xrightarrow{\beta_{2FA, FB}} & FB \otimes_2 FA \\
\downarrow m_{A, B} & & \downarrow m_{B, A} \\
F(A \otimes_1 B) & \xrightarrow{F\beta_{1A, B}} & F(B \otimes_1 A)
\end{array}$$

Definition 6 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **monoidal natural transformation** is a natural transformation, $f : F \rightarrow G$, subject to the following coherence diagrams:

$$\begin{array}{ccc}
FA \otimes_2 FB & \xrightarrow{m_{A, B}} & F(A \otimes_1 B) \\
\downarrow f_A \otimes_2 f_B & & \downarrow f_{A \otimes_1 B} \\
GA \otimes_2 GB & \xrightarrow{n_{A, B}} & G(A \otimes_1 B)
\end{array}
\quad
\begin{array}{ccc}
F\top_1 & \xrightarrow{f_{\top_1}} & G\top_1 \\
\swarrow m_{\top_1} & & \searrow n_{\top_1} \\
& \top_2 &
\end{array}$$

Definition 7 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **symmetric monoidal natural transformation** is a natural transformation, $f : F \rightarrow G$, subject to the following coherence diagrams:

$$\begin{array}{ccc}
FA \otimes_2 FB & \xrightarrow{m_{A, B}} & F(A \otimes_1 B) \\
\downarrow f_A \otimes_2 f_B & & \downarrow f_{A \otimes_1 B} \\
GA \otimes_2 GB & \xrightarrow{n_{A, B}} & G(A \otimes_1 B)
\end{array}
\quad
\begin{array}{ccc}
F\top_1 & \xrightarrow{f_{\top_1}} & G\top_1 \\
\swarrow m_{\top_1} & & \searrow n_{\top_1} \\
& \top_2 &
\end{array}$$

Definition 8 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \rightarrow GFA$, and the counit, $\varepsilon_A : FGA \rightarrow A$,

are monoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{array}{ccc}
FGA \otimes_2 FGB & \xrightarrow{m_{GA,GB}} & F(GA \otimes_1 GB) \\
\downarrow \varepsilon_A \otimes_1 \varepsilon_B & & \downarrow Fn_{A,B} \\
A \otimes_2 B & \xleftarrow{\varepsilon_{A \otimes_1 B}} & FGA \otimes_2 FGB
\end{array}
\quad
\begin{array}{ccc}
F\tau_1 & \xrightarrow{Fn_{\tau_2}} & FG\tau_2 \\
\uparrow m_{\tau_1} & & \downarrow \varepsilon_{\tau_1} \\
\tau_2 & \xlongequal{\quad} & \tau_2
\end{array}$$

$$\begin{array}{ccc}
GFA \otimes_1 GFB & \xleftarrow{\eta_A \otimes_1 \eta_B} & A \otimes_1 B \\
\downarrow n_{FA,FB} & & \downarrow \eta_{A \otimes_1 B} \\
G(FA \otimes_2 FB) & \xrightarrow{m_{A,B}} & GF(A \otimes_1 B)
\end{array}
\quad
\begin{array}{ccc}
G\tau_2 & \xrightarrow{Gm_{\tau_1}} & GF\tau_1 \\
\uparrow n_{\tau_2} & & \uparrow \eta_{\tau_1} \\
\tau_1 & \xlongequal{\quad} & \tau_1
\end{array}$$

Definition 9 Suppose $(\mathcal{M}_1, \tau_1, \otimes_1)$ and $(\mathcal{M}_2, \tau_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a symmetric monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1 : F \dashv G : \mathcal{M}_2$ such that the unit, $\eta_A : A \rightarrow GFA$, and the counit, $\varepsilon_A : FGA \rightarrow A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:

$$\begin{array}{ccc}
FGA \otimes_2 FGB & \xrightarrow{m_{GA,GB}} & F(GA \otimes_1 GB) \\
\downarrow \varepsilon_A \otimes_1 \varepsilon_B & & \downarrow Fn_{A,B} \\
A \otimes_2 B & \xleftarrow{\varepsilon_{A \otimes_1 B}} & FGA \otimes_2 FGB
\end{array}
\quad
\begin{array}{ccc}
F\tau_1 & \xrightarrow{Fn_{\tau_2}} & FG\tau_2 \\
\uparrow m_{\tau_1} & & \downarrow \varepsilon_{\tau_1} \\
\tau_2 & \xlongequal{\quad} & \tau_2
\end{array}$$

$$\begin{array}{ccc}
GFA \otimes_1 GFB & \xleftarrow{\eta_A \otimes_1 \eta_B} & A \otimes_1 B \\
\downarrow n_{FA,FB} & & \downarrow \eta_{A \otimes_1 B} \\
G(FA \otimes_2 FB) & \xrightarrow{m_{A,B}} & GF(A \otimes_1 B)
\end{array}
\quad
\begin{array}{ccc}
G\tau_2 & \xrightarrow{Gm_{\tau_1}} & GF\tau_1 \\
\uparrow n_{\tau_2} & & \uparrow \eta_{\tau_1} \\
\tau_1 & \xlongequal{\quad} & \tau_1
\end{array}$$

Definition 10 A **monoidal comonad** on a monoidal category \mathcal{C} is a triple (T, ε, δ) , where (T, m) is a monoidal endofunctor on \mathcal{C} , $\varepsilon_A : TA \rightarrow A$ and $\delta_A : TA \rightarrow T^2A$ are monoidal natural transformations, which make the following diagrams commute:

$$\begin{array}{ccc}
TA & \xrightarrow{\delta_A} & T^2A \\
\downarrow \delta_A & & \downarrow T\delta_A \\
T^2A & \xrightarrow{\delta_{TA}} & T^3A
\end{array}
\quad
\begin{array}{ccccc}
& & TA & & \\
& \swarrow & \downarrow \delta_A & \searrow & \\
TA & \xleftarrow{\varepsilon_{TA}} & T^2A & \xrightarrow{T\varepsilon_A} & TA
\end{array}$$

The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

$$\begin{array}{ccc}
TA \otimes TB & \xrightarrow{m_{A,B}} & T(A \otimes B) \\
& \searrow \varepsilon_A \otimes \varepsilon_B & \downarrow \varepsilon_{A \otimes B} \\
& & A \otimes B
\end{array}
\qquad
\begin{array}{ccc}
T\top & \xrightarrow{m_\top} & \top \\
& \searrow \varepsilon_\top & \downarrow \\
& & \top
\end{array}$$

$$\begin{array}{ccc}
TA \otimes TB & \xrightarrow{m_{A,B}} & T(A \otimes B) \\
\downarrow \delta_A \otimes \delta_B & & \downarrow \delta_{A \otimes B} \\
T^2A \otimes T^2B & \xrightarrow{m_{TA,TB}} T(TA \otimes TB) \xrightarrow{Tm_{A,B}} & T^2(A \otimes B)
\end{array}$$

$$\begin{array}{ccc}
\top & \xrightarrow{m_\top} & T\top \\
\downarrow m_\top & & \downarrow \delta_\top \\
T\top & \xrightarrow{Tm_\top} & T^2\top
\end{array}$$

Definition 11 A *symmetric monoidal comonad* on a symmetric monoidal category \mathcal{C} is a triple (T, ε, δ) , where (T, m) is a symmetric monoidal endofunctor on \mathcal{C} , $\varepsilon_A : TA \rightarrow A$ and $\delta_A : TA \rightarrow T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:

$$\begin{array}{ccc}
TA & \xrightarrow{\delta_A} & T^2A \\
\downarrow \delta_A & & \downarrow T\delta_A \\
T^2A & \xrightarrow{\delta_{TA}} & T^3A
\end{array}
\qquad
\begin{array}{ccccc}
& & TA & & \\
& \swarrow & \downarrow \delta_A & \searrow & \\
TA & \xleftarrow{\varepsilon_{TA}} & T^2A & \xrightarrow{T\varepsilon_A} & TA
\end{array}$$

The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

$$\begin{array}{ccc}
TA \otimes TB & \xrightarrow{m_{A,B}} & T(A \otimes B) \\
& \searrow \varepsilon_A \otimes \varepsilon_B & \downarrow \varepsilon_{A \otimes B} \\
& & A \otimes B
\end{array}
\qquad
\begin{array}{ccc}
T\top & \xrightarrow{m_\top} & \top \\
& \searrow \varepsilon_\top & \downarrow \\
& & \top
\end{array}$$

$$\begin{array}{ccccc}
TA \otimes TB & \xrightarrow{m_{A,B}} & T(A \otimes B) \\
\downarrow \delta_A \otimes \delta_B & & \downarrow \delta_{A \otimes B} \\
T^2A \otimes T^2B & \xrightarrow{m_{TA,TB}} T(TA \otimes TB) \xrightarrow{Tm_{A,B}} T^2(A \otimes B)
\end{array}$$

$$\begin{array}{ccc}
\top & \xrightarrow{m_\top} & T\top \\
\downarrow m_\top & & \downarrow \delta_\top \\
T\top & \xrightarrow{Tm_\top} & T^2\top
\end{array}$$

2 Related Work

TODO

3 Conclusion

TODO

References

- [1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at <http://research.microsoft.com/en-us/um/people/nick/mixed3.ps>.