Separating Linear Modalities

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Abstract

TODO

1 Introduction

TODO [1]

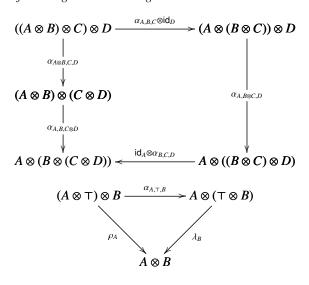
1.1 Symmetric Monoidal Categories

Definition 1 A monoidal category is a category, M, with the following data:

- An object \top of \mathcal{M} ,
- A bi-functor \otimes : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• Subject to the following coherence diagrams:



Definition 2 A symmetric monoidal category (SMC) is a category, M, with the following data:

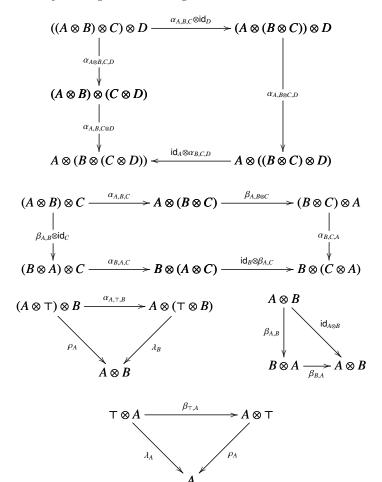
- An object \top of \mathcal{M} ,
- A bi-functor \otimes : $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$,
- The following natural isomorphisms:

$$\begin{array}{l} \lambda_A: \top \otimes A \longrightarrow A \\ \rho_A: A \otimes \top \longrightarrow A \\ \alpha_{A,B,C}: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \end{array}$$

• A symmetry natural transformation:

$$\beta_{A,B}: A \otimes B \longrightarrow B \otimes A$$

• Subject to the following coherence diagrams:

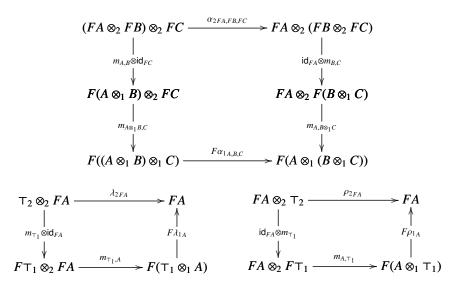


Definition 3 A symmetric monoidal closed category (SMCC) is a symmetric monoidal category, $(\mathcal{M}, \top, \otimes)$, such that, for any object B of M, the functor $-\otimes B : \mathcal{M} \longrightarrow \mathcal{M}$ has a specified right adjoint. Hence, for any objects A and C of M there is an object $B \multimap C$ of M and a natural bijection:

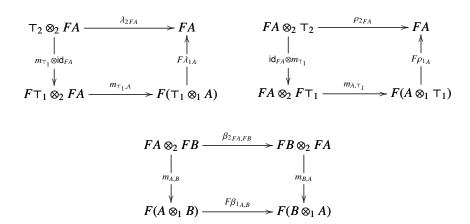
$$\operatorname{\mathsf{Hom}}_{\mathcal{M}}(A \otimes B, C) \cong \operatorname{\mathsf{Hom}}_{\mathcal{M}}(A, B \multimap C)$$

We call the functor $\multimap: \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ the internal hom of \mathcal{M} .

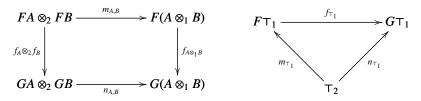
Definition 4 Suppose we are given two monoidal categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **monoidal functor** is a functor $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1} : \top_2 \longrightarrow F \top_1$ and a natural transformation $m_{A,B} : FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:



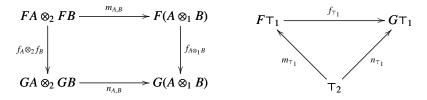
Definition 5 Suppose we are given two symmetric monoidal closed categories $(\mathcal{M}_1, \top_1, \otimes_1, \alpha_1, \lambda_1, \rho_1, \beta_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2, \alpha_2, \lambda_2, \rho_2, \beta_2)$. Then a **symmetric monoidal** functor is a functor $F: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, a map $m_{\top_1}: \top_2 \longrightarrow F \top_1$ and a natural transformation $m_{A,B}: FA \otimes_2 FB \longrightarrow F(A \otimes_1 B)$ subject to the following coherence conditions:



Definition 6 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) and (G, n) are monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a **monoidal natural transformation** is a natural transformation, $f: F \longrightarrow G$, subject to the following coherence diagrams:

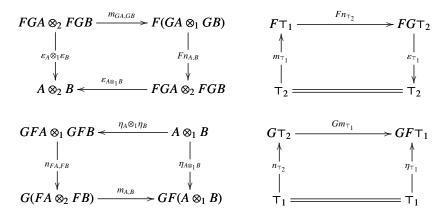


Definition 7 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) and (G, n) are symmetric monoidal functors between \mathcal{M}_1 and \mathcal{M}_2 . Then a symmetric monoidal natural transformation is a natural transformation, $f: F \longrightarrow G$, subject to the following coherence diagrams:

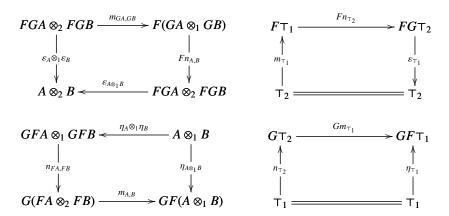


Definition 8 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are monoidal categories, and (F, m) is a monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1: F \dashv G: \mathcal{M}_2$ such that the unit, $\eta_A: A \to GFA$, and the counit, $\varepsilon_A: FGA \to A$,

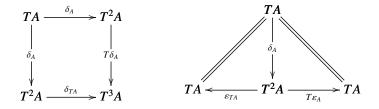
are monoidal natural transformations. Thus, the following diagrams must commute:



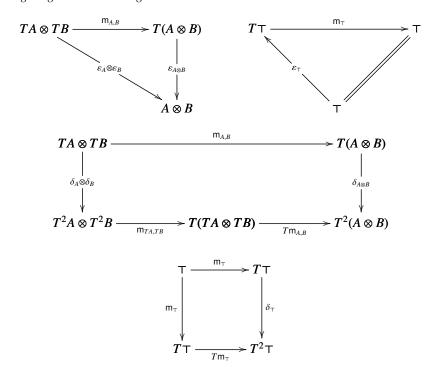
Definition 9 Suppose $(\mathcal{M}_1, \top_1, \otimes_1)$ and $(\mathcal{M}_2, \top_2, \otimes_2)$ are SMCs, and (F, m) is a symmetric monoidal functor between \mathcal{M}_1 and \mathcal{M}_2 and (G, n) is a symmetric monoidal functor between \mathcal{M}_2 and \mathcal{M}_1 . Then a **symmetric monoidal adjunction** is an ordinary adjunction $\mathcal{M}_1: F \dashv G: \mathcal{M}_2$ such that the unit, $\eta_A: A \to GFA$, and the counit, $\varepsilon_A: FGA \to A$, are symmetric monoidal natural transformations. Thus, the following diagrams must commute:



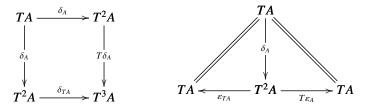
Definition 10 A monoidal comonad on a monoidal category C is a triple (T, ε, δ) , where (T, m) is a monoidal endofunctor on C, $\varepsilon_A : TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are monoidal natural transformations, which make the following diagrams commute:



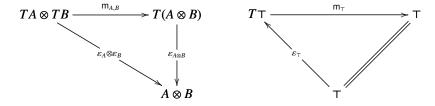
The assumption that ε and δ are monoidal natural transformations amount to the following diagrams commuting:

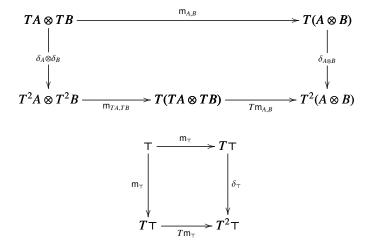


Definition 11 A symmetric monoidal comonad on a symmetric monoidal category C is a triple (T, ε, δ) , where (T, m) is a symmetric monoidal endofunctor on C, ε_A : $TA \longrightarrow A$ and $\delta_A : TA \to T^2A$ are symmetric monoidal natural transformations, which make the following diagrams commute:



The assumption that ε and δ are symmetric monoidal natural transformations amount to the following diagrams commuting:

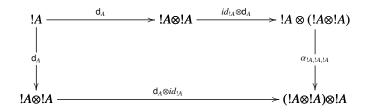


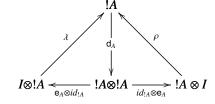


1.2 Linear Category

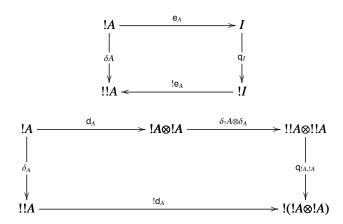
Definition 12 A linear category is specified by

- a symmetric monoidal closed category $(\mathcal{L}, I, \otimes, \multimap)$,
- a symmetric monoidal comonad $(!, \varepsilon, \delta)$ on \mathcal{L} , with $q_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $q_I : I \longrightarrow !I$;
- monoidal natural transformations on \mathcal{L} with components $e_A : !A \longrightarrow I$ and $d_A : !A \longrightarrow !A \otimes !A$, s.t.
 - each (!A, e_A , d_A) is a commutative comonoid, i.e. the following diagrams commute and $\beta \circ d_A = d_A$ where $\beta_{B,C} : B \otimes C \longrightarrow C \otimes B$ is the symmetry natural transformation of \mathcal{L} ;

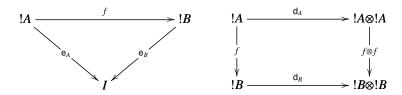




- e_A and d_A are coalgebra morphisms, i.e. the following diagrams commute;

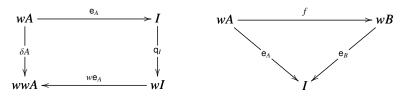


- any coalgebra morphism $f:(!A,\delta_A) \longrightarrow (!B,\delta_B)$ between free coalgebras preserve the comonoid structure given by e and d, i.e. the following diagrams commute.



Definition 13 A (modified) linear category with weakening is specified by

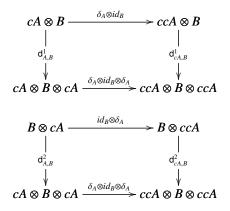
- a monoidal closed category $(\mathcal{L}, I, \otimes)$,
- a monoidal comonad (w, ε, δ) on \mathcal{L} with $q_{A,B} : wA \otimes wB \longrightarrow w(A \otimes B)$ and $q_I : I \longrightarrow wI$, and
- a monoidal natural transformation e on \mathcal{L} with components $e_A: wA \longrightarrow 1$ s.t. the following diagrams commute:



Definition 14 A (modified) linear category with contraction is specified by

- a monoidal closed category $(\mathcal{L}, I, \otimes)$,
- a monoidal comonad (c, ε, δ) on \mathcal{L} with $q_{A,B} : cA \otimes cB \longrightarrow c(A \otimes B)$ and $q_I : I \longrightarrow cI$, and

- monoidal natural transformations d^1 and d^2 on \mathcal{L} with components $d^1_{A,B}: cA \otimes B \longrightarrow cA \otimes B \otimes cA$ and $d^2_{A,B}: B \otimes cA \longrightarrow cA \otimes B \otimes cA$.
- (May need the following diagrams commute:)



2 Related Work

TODO

3 Conclusion

TODO

References

[1] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical Report UCAM-CL-TR-352, University of Cambridge Computer Laboratory, 1994. Accessible online at http://research.microsoft.com/en-us/um/people/nick/mixed3.ps.