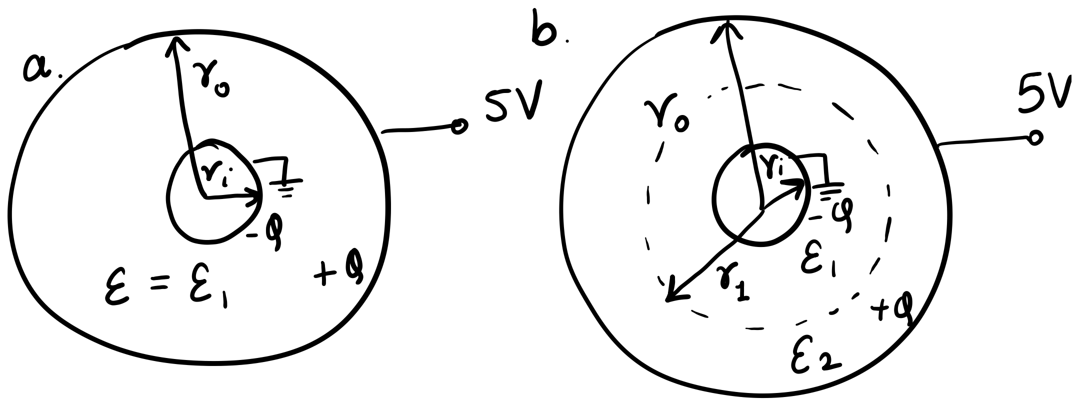
EE 724: HW 2

1. In between two concentric metal spherical shells of radius r_i=5nm and r_o=15nm there are 3 cases



   a. There lies a dielectric with dielectric constant of ε_1 =10*ε_o. A potential difference of 5V is applied to the outer cylinder.
   b. There are 2 concentric dielectrics of dielectric constant ε_1 =10*ε_o (r<10 nm) and ε_2 =30*ε_o (r>10 nm). A potential difference of 5V is applied to the outer cylinder.

   For each case, analytically solve and draw the charge density, Electric Field and Potential profile vs radius. Plot with MATLAB. Solve for capacitance.

   Solution: We will start from the general case and then come to specific cases in (a) and (b). Starting from the Gauss' law for electric displacement:

$$\nabla.\vec{D} = \rho_f$$

   A dielectric has no free charges in the volume and we are not placing any fixed charges either in any case, hence:

$$\nabla.\vec{D} = 0$$

   Since this is a spherically symmetric system, using the divergence expression for spherical radial coordinate, r:

$$\frac{1}{r^2}\frac{\partial r^2 D_r}{\partial r} = 0$$

   Or,

$$D_r(r) = \frac{c}{r^2}$$

   Where c is some constant;

   Assuming the dielectric has linear polarization, we can write the electric field as:

$$E_r(r) = \frac{D_r(r)}{\varepsilon(r)}$$

Or,

$$E_r(r) = \frac{c}{\varepsilon(r)r^2}$$

In order to find the constant, c, we calculate the potential:

$$V(r) = -\int_{r_i}^{r} E_r(r)\, dr$$

i.e.,

$$V(r) = -c\int_{r_i}^{r} \frac{dr}{\varepsilon(r)r^2} = -cf(r, r_i)$$

The total potential difference is known to be ΔV:

$$\Delta V = V(r_o) = -cf(r_o, r_i)$$

The constant, c is hence:

$$c = -\frac{\Delta V}{f(r_o, r_i)}$$

The total charge density (provided by battery as well as polarization):

$$\rho(r) = \varepsilon_0 \nabla.\vec{E} = \varepsilon_0 \frac{1}{r^2}\frac{\partial r^2 E_r}{\partial r}$$

Or,

$$\rho(r) = \varepsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{c}{\varepsilon(r)}\right)$$

The surface charge density is then:

$$\sigma(r) = \rho(r)\, dr$$

And the charge is:

$$Q(r) = \sigma(r)\, 4\pi r^2$$

The free charge on the outer metal is:

$$Q_f(r_o) = -D_r(r_o)4\pi r_o^2$$

The capacitance is hence:

$$C = \frac{Q_f(r_o)}{\Delta V}$$

Now collecting all the expressions for the general case:

$D_r(r) = -\frac{\Delta V}{f(r_o,r_i)}\frac{1}{r^2}$; $E_r(r) = -\frac{\Delta V}{f(r_o,r_i)}\frac{1}{\varepsilon(r)r^2}$; $V(r) = \Delta V\frac{f(r,r_i)}{f(r_o,r_i)}$

$\rho(r) = -\frac{\varepsilon_0}{r^2}\frac{\Delta V}{f(r_o,r_i)}\frac{\partial}{\partial r}\left(\frac{1}{\varepsilon(r)}\right)$; $\sigma(r) = -\frac{\varepsilon_0}{r^2}\frac{\Delta V}{f(r_o,r_i)}\partial\left(\frac{1}{\varepsilon(r)}\right)$; $Q(r) = -4\pi\varepsilon_0\frac{\Delta V}{f(r_o,r_i)}\partial\left(\frac{1}{\varepsilon(r)}\right)$

$C = \frac{4\pi}{f(r_o,r_i)}$

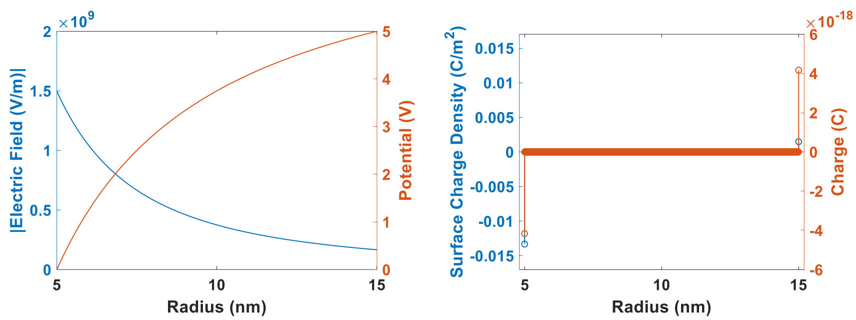
Now for part (a), $\varepsilon(r) = \varepsilon_1 = 10\varepsilon_0$,

$$f(r_o,r_i) = \frac{1}{\varepsilon_1}\int_{r_i}^{r_o}\frac{dr}{r^2} = \frac{1}{\varepsilon_1}\left(\frac{1}{r_i}-\frac{1}{r_o}\right)$$

$$f(r,r_i) = \frac{1}{\varepsilon_1}\left(\frac{1}{r_i}-\frac{1}{r}\right)$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right) = \frac{1}{\varepsilon_1}\ @\ r = r_i$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right) = -\frac{1}{\varepsilon_1}\ @\ r = r_o$$



And for part (b), $\varepsilon(r < r_1) = \varepsilon_1 = 10\varepsilon_0$ and $\varepsilon(r > r_1) = \varepsilon_2 = 30\varepsilon_0$,

$$f(r_o,r_i) = \frac{1}{\varepsilon_1}\int_{r_i}^{r_1}\frac{dr}{r^2} + \frac{1}{\varepsilon_2}\int_{r_1}^{r_o}\frac{dr}{r^2} = \frac{1}{\varepsilon_1}\left(\frac{1}{r_i}-\frac{1}{r_1}\right) + \frac{1}{\varepsilon_2}\left(\frac{1}{r_1}-\frac{1}{r_o}\right)$$

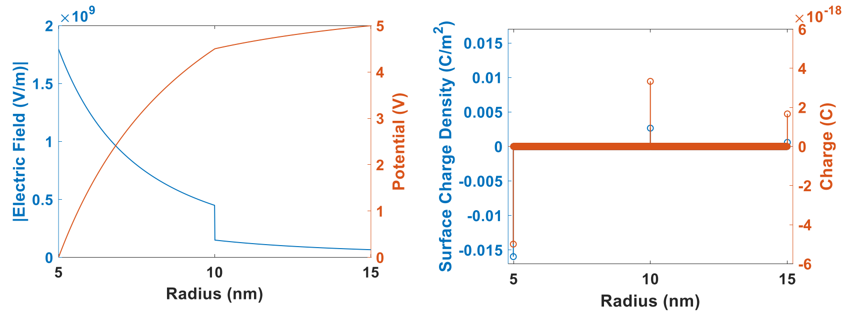
$$f(r,r_i) = \frac{1}{\varepsilon_1}\left(\frac{1}{r_i}-\frac{1}{r}\right)\ \forall\, r < r_1$$

$$f(r,r_i) = \frac{1}{\varepsilon_1}\left(\frac{1}{r_i}-\frac{1}{r_1}\right) + \frac{1}{\varepsilon_2}\left(\frac{1}{r_1}-\frac{1}{r}\right)\ \forall\, r > r_1$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right) = \frac{1}{\varepsilon_1}\ @\ r = r_i$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right)=\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\ @\, r=r_1$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right)=-\frac{1}{\varepsilon_2}\ @\, r=r_o$$



Show that b part can be easily solved alternatively by a capacitive network.

Solution:

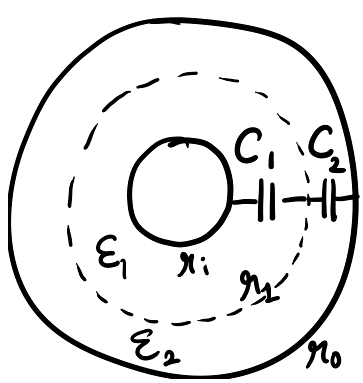
The total capacitance of part (b) was calculated earlier as:

$$C=\frac{4\pi}{f(r_o,r_i)}=\frac{4\pi}{\frac{1}{\varepsilon_1}\left(\frac{1}{r_i}-\frac{1}{r_1}\right)+\frac{1}{\varepsilon_2}\left(\frac{1}{r_1}-\frac{1}{r_o}\right)}$$

The inverse capacitance is hence:

$$\frac{1}{C}=\frac{1}{4\pi\varepsilon_1}\left(\frac{1}{r_i}-\frac{1}{r_1}\right)+\frac{1}{4\pi\varepsilon_2}\left(\frac{1}{r_1}-\frac{1}{r_o}\right)=\frac{1}{C_1}+\frac{1}{C_2}$$

Where C_1 and C_2 are the capacitances of different but constant dielectric constant capacitors with different inner and outer radii as calculated in part(a). Hence this is clearly a series connection of two capacitors.



Note: Similar electrostatics maybe seen in metal nanoparticles surrounded by dielectrics. Electrodes are used in single electron transistors (SETs).

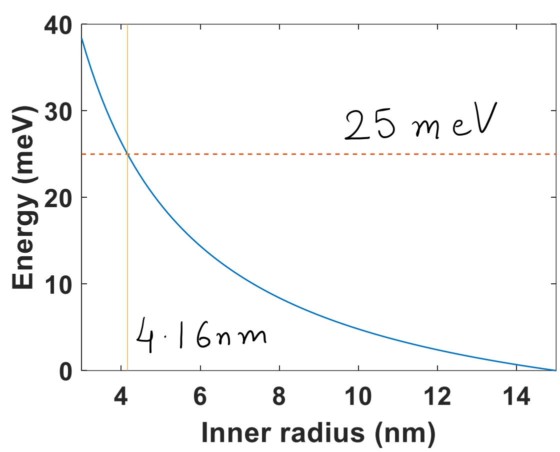
c. Given a case (a), if 1 electron is transferred from from outer to inner shell, how much energy does the next electron transfer need? This increase in energy for each additional electron is known as coulomb blockade to injection. How does it depend on radius of the inner and outer shells? Plot using MATLAB, energy vs. inner radius. Annotate the radius at which energy exceeds thermal energy of 25mV at 300K. At these radii, the coulomb blockade can be observed at room temperature in nano-scale SETs.

Solution: If 1 electron is transferred from outer to inner shell, an additional voltage is developed:

$$\Delta V_{1e} = \frac{e}{C}$$

Where e is the magnitude of electronic charge. Now, the next electron transfer needs some additional energy corresponding to this change in potential:

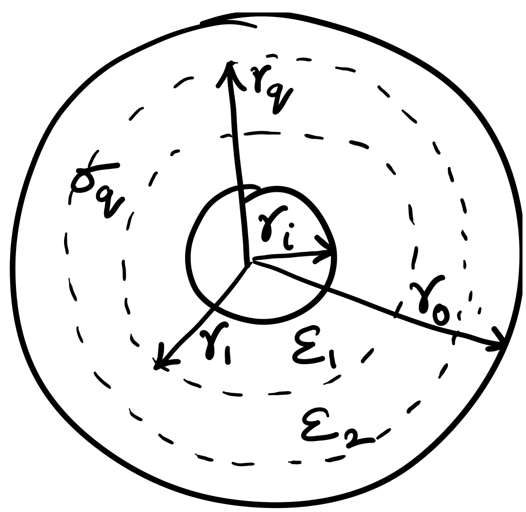
$$\Delta W_{next} = e\Delta V_{1e} = \frac{e^2}{C} = \frac{e^2}{4\pi\varepsilon_1}\left(\frac{1}{r_i} - \frac{1}{r_o}\right)$$



2. In between two concentric metal cylinders of radius r_i=5nm and r_o=15nm there are 3 cases
   a. There lies a dielectric with dielectric constant of $\varepsilon_1 = 10*\varepsilon_o$. A potential difference of 5V is applied to the outer cylinder.
   b. There are 2 concentric dielectrics of dielectric constant $\varepsilon_1 = 10*\varepsilon_o$ (r<10 nm) and $\varepsilon_2 = 30*\varepsilon_o$ (r>10 nm). A potential difference of 5V is applied to the outer cylinder.
   c. There is a fixed charge of density $\sigma = 10^{12}/cm^2$ at between r_q=11nm. A potential difference of 0V is applied to the outer cylinder. Also solve for 5V potential difference.

For each case, analytically solve and draw the charge density, Electric Field and Potential profile vs radius. Plot with MATLAB. Solve for capacitance.

Show that b & c parts can be easily solved alternatively by a capacitive network.



Solution: Considering the general case first, starting from the Gauss' law for electric displacement:

$$\nabla.\vec{D} = \rho_f$$

ρ_f has all the charges from either the battery or the charges that we place (like in (c)),

Since this is a cylindrically symmetric system, using the divergence expression for cylindrical radial coordinate, r:

$$\frac{1}{r}\frac{\partial\, rD_r}{\partial r} = \rho_f(r)$$

Or,

For $r < r_q$,

$$rD_r(r) - r_q D_r(r_q^-) = \int_{r_q^-}^{r} r\,\rho_f(r)\,dr = 0$$

Or,

$$D_r(r) = \frac{r_q D_r(r_q^-)}{r}$$

For $r > r_q$,

$$rD_r(r) - r_q D_r(r_q^+) = \int_{r_q^+}^{r} r\, \rho_f(r)\, dr = 0$$

$$D_r(r) = \frac{r_q D_r(r_q^+)}{r}$$

Assuming the dielectric has linear polarization, we can write the electric field as:

$$E_r(r) = \frac{D_r(r)}{\varepsilon(r)}$$

The potential is calculated as:

$$V(r) = -\int_{r_i}^{r} E_r(r)\, dr$$

For $r < r_q$,

$$V(r) = -\int_{r_i}^{r} \frac{r_q D_r(r_q^-)}{\varepsilon(r) r} dr = -r_q D_r(r_q^-) f(r, r_i)$$

For $r > r_q$,

$$V(r) = -\left(\int_{r_i}^{r_q} \frac{r_q D_r(r_q^-)}{\varepsilon(r) r} dr + \int_{r_q}^{r} \frac{r_q D_r(r_q^+)}{\varepsilon(r) r} dr\right) = -r_q \left(D_r(r_q^-) f(r_q, r_i) + D_r(r_q^+) f(r, r_q)\right)$$

The total potential difference is known to be ΔV:

$$\Delta V = V(r_o)$$

$$\Delta V = -r_q \left(D_r(r_q^-) f(r_q, r_i) + D_r(r_q^+) f(r_o, r_q)\right)$$

Also, the boundary condition at r_q:

$$D_r(r_q^+) - D_r(r_q^-) = \sigma_q$$

These equations allow us to solve for $D_r(r_q^+)$ and $D_r(r_q^-)$:

$$D_r(r_q^-) = \frac{\frac{-\Delta V}{r_q} - f(r_o, r_q)\sigma_q}{f(r_o, r_i)}$$

$$D_r(r_q^+) = \frac{\frac{-\Delta V}{r_q} + f(r_q, r_i)\sigma_q}{f(r_o, r_i)}$$

The total charge density:

$$\rho(r) = \varepsilon_0 \nabla . \vec{E} = \varepsilon_0 \frac{1}{r} \frac{\partial \, rE_r}{\partial r}$$

The surface charge density is then:

$$\sigma(r) = \rho(r) \, dr$$

And the charge per unit length is:

$$Q(r) = \sigma(r) \, 2\pi r$$

The free charge per unit length on the outer plate:

$$Q_f(r_o) = -D_r(r_o) 2\pi r_o$$

The capacitance per unit length is hence:

$$C = \frac{Q_f(r_o)}{\Delta V}$$

Collecting together all the general expressions:

For $r < r_q$,

$D_r(r) = \frac{-\Delta V - f(r_o, r_q)\sigma_q r_q}{f(r_o, r_i)} \frac{1}{r}$; $\quad E_r(r) = \frac{-\Delta V - f(r_o, r_q)\sigma_q r_q}{f(r_o, r_i)} \frac{1}{\varepsilon(r) r}$; $\quad V(r) = \frac{\Delta V + f(r_o, r_q)\sigma_q r_q}{f(r_o, r_i)} \, f(r, r_i)$

$\rho(r) = \frac{-\Delta V - f(r_o, r_q)\sigma_q r_q}{f(r_o, r_i)} \frac{\varepsilon_0}{r} \frac{\partial}{\partial r}\left(\frac{1}{\varepsilon(r)}\right)$; $\quad \sigma(r) = \frac{-\Delta V - f(r_o, r_q)\sigma_q r_q}{f(r_o, r_i)} \frac{\varepsilon_0}{r} \partial \left(\frac{1}{\varepsilon(r)}\right)$;

$Q(r) = 2\pi\varepsilon_0 \frac{-\Delta V - f(r_o, r_q)\sigma_q r_q}{f(r_o, r_i)} \partial \left(\frac{1}{\varepsilon(r)}\right)$

For $r > r_q$,

$D_r(r) = \frac{-\Delta V + f(r_q, r_i)\sigma_q r_q}{f(r_o, r_i)} \frac{1}{r}$; $\quad E_r(r) = \frac{-\Delta V + f(r_q, r_i)\sigma_q r_q}{f(r_o, r_i)} \frac{1}{\varepsilon(r) r}$;

$V(r) = \frac{\Delta V + f(r_o, r_q)\sigma_q r_q}{f(r_o, r_i)} \, f(r_q, r_i) + \frac{\Delta V - f(r_q, r_i)\sigma_q r_q}{f(r_o, r_i)} \, f(r, r_q)$;

$\rho(r) = \frac{-\Delta V + f(r_q, r_i)\sigma_q r_q}{f(r_o, r_i)} \frac{\varepsilon_0}{r} \frac{\partial}{\partial r}\left(\frac{1}{\varepsilon(r)}\right)$; $\quad \sigma(r) = \frac{-\Delta V + f(r_q, r_i)\sigma_q r_q}{f(r_o, r_i)} \frac{\varepsilon_0}{r} \partial \left(\frac{1}{\varepsilon(r)}\right)$;

$Q(r) = 2\pi\varepsilon_0 \frac{-\Delta V + f(r_q, r_i)\sigma_q r_q}{f(r_o, r_i)} \partial \left(\frac{1}{\varepsilon(r)}\right)$

For part (a), $\sigma_q = 0$ and $\varepsilon(r) = \varepsilon_1 = 10\varepsilon_0$,

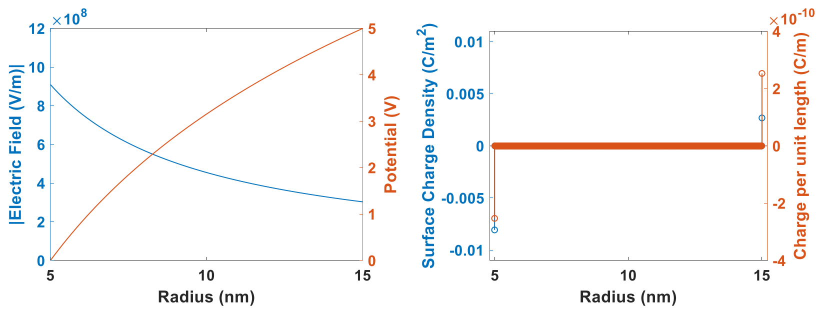
$$f(r_o, r_i) = \frac{1}{\varepsilon_1}\int_{r_i}^{r_o}\frac{dr}{r} = \frac{1}{\varepsilon_1}\ln\frac{r_o}{r_i}$$

$$f(r, r_i) = \frac{1}{\varepsilon_1}\int_{r_i}^{r}\frac{dr}{r} = \frac{1}{\varepsilon_1}\ln\frac{r}{r_i}$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right) = \frac{1}{\varepsilon_1} \ @\ r = r_i$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right) = -\frac{1}{\varepsilon_1} \ @\ r = r_o$$

$$C = \frac{Q_f(r_o)}{\Delta V} = \frac{2\pi\varepsilon_1}{\ln\frac{r_o}{r_i}}$$



For part (b), $\sigma_q = 0$ and $\varepsilon(r < r_1) = \varepsilon_1 = 10\varepsilon_0$ and $\varepsilon(r > r_1) = \varepsilon_2 = 30\varepsilon_0$,

$$f(r_o, r_i) = \frac{1}{\varepsilon_1}\int_{r_i}^{r_1}\frac{dr}{r} + \frac{1}{\varepsilon_2}\int_{r_1}^{r_o}\frac{dr}{r} = \frac{1}{\varepsilon_1}\ln\frac{r_1}{r_i} + \frac{1}{\varepsilon_2}\ln\frac{r_o}{r_1}$$

$$f(r, r_i) = \frac{1}{\varepsilon_1}\ln\frac{r}{r_i} \ \forall\, r < r_1$$

$$f(r, r_i) = \frac{1}{\varepsilon_1}\ln\frac{r_1}{r_i} + \frac{1}{\varepsilon_2}\ln\frac{r}{r_1} \ \forall\, r > r_1$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right) = \frac{1}{\varepsilon_1} \ @\ r = r_i$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right) = \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \ @\ r = r_1$$

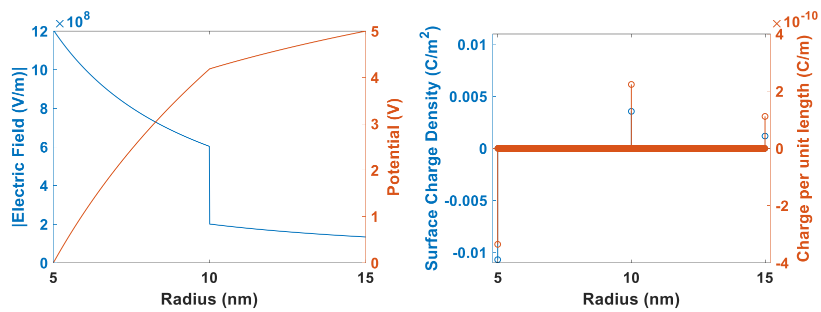
$$\partial\left(\frac{1}{\varepsilon(r)}\right) = -\frac{1}{\varepsilon_2} \ @\ r = r_o$$

$$C = \frac{Q_f(r_o)}{\Delta V} = \frac{2\pi}{\frac{1}{\varepsilon_1}\ln\frac{r_1}{r_i} + \frac{1}{\varepsilon_2}\ln\frac{r_o}{r_1}}$$

The inverse of capacitance is:

$$\frac{1}{C} = \frac{1}{2\pi}\frac{1}{\varepsilon_1}\ln\frac{r_1}{r_i} + \frac{1}{2\pi}\frac{1}{\varepsilon_2}\ln\frac{r_o}{r_1} = \frac{1}{C_1} + \frac{1}{C_2}$$

Where C_1 and C_2 are the capacitances of different but constant dielectric constant capacitors with different inner and outer radii as calculated in part(a). Hence this is clearly a series connection of two capacitors.



For part (c), $\sigma_q = 10^{12}cm^{-2}$ and $\varepsilon(r) = \varepsilon_1 = 10\varepsilon_0$,

$$f(r_o, r_i) = \frac{1}{\varepsilon_1}\int_{r_i}^{r_o}\frac{dr}{r} = \frac{1}{\varepsilon_1}\ln\frac{r_o}{r_i}$$

$$f(r_o, r_q) = \frac{1}{\varepsilon_1}\int_{r_q}^{r_o}\frac{dr}{r} = \frac{1}{\varepsilon_1}\ln\frac{r_o}{r_q}$$

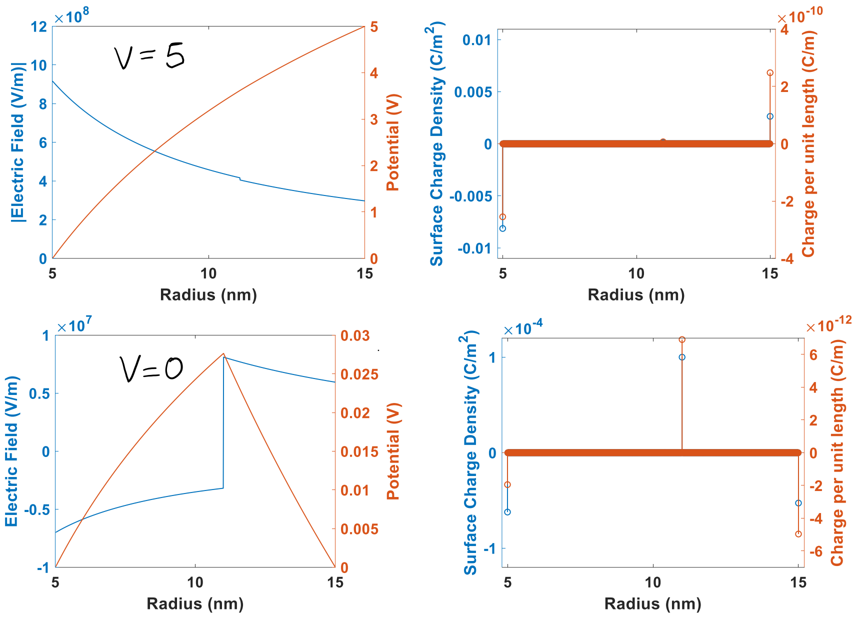
$$f(r_q, r_i) = \frac{1}{\varepsilon_1}\int_{r_i}^{r_q}\frac{dr}{r} = \frac{1}{\varepsilon_1}\ln\frac{r_q}{r_i}$$

$$f(r, r_i) = \frac{1}{\varepsilon_1}\int_{r_i}^{r}\frac{dr}{r} = \frac{1}{\varepsilon_1}\ln\frac{r}{r_i}$$

$$f(r, r_q) = \frac{1}{\varepsilon_1}\int_{r_q}^{r}\frac{dr}{r} = \frac{1}{\varepsilon_1}\ln\frac{r}{r_q}$$

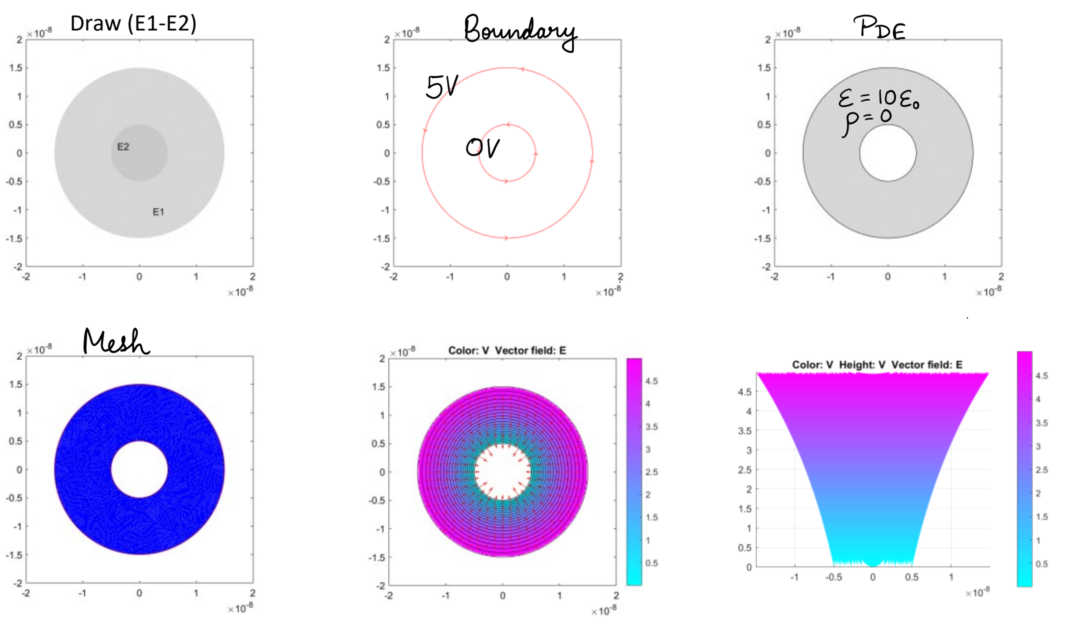
$$\partial\left(\frac{1}{\varepsilon(r)}\right) = \frac{1}{\varepsilon_1}\ @\, r = r_i$$

$$\partial\left(\frac{1}{\varepsilon(r)}\right) = -\frac{1}{\varepsilon_1}\ @\, r = r_o$$

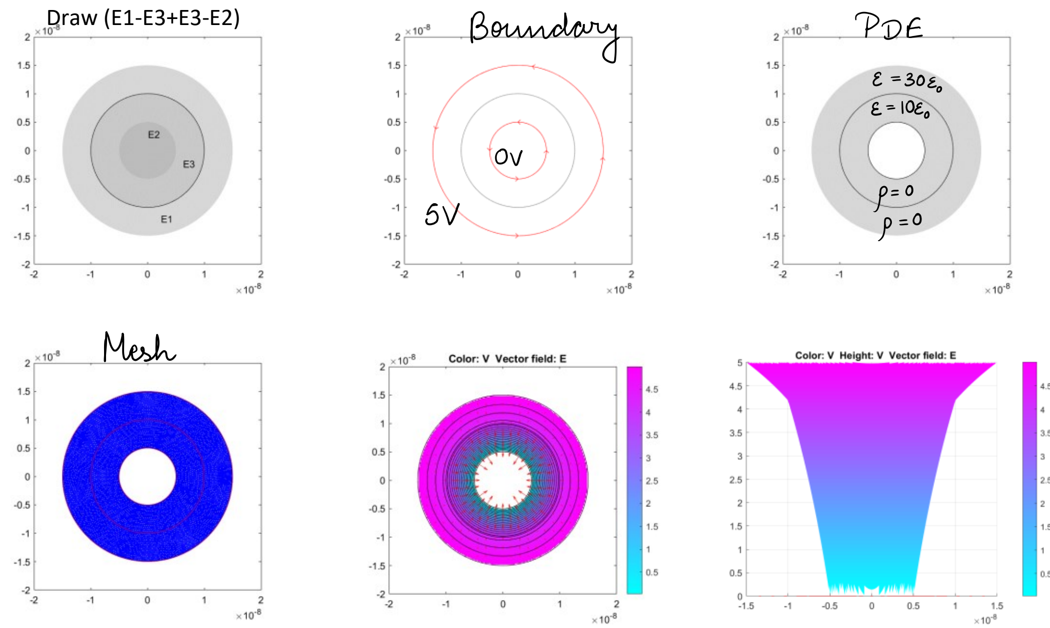


Qualitatively verify using Matlab PDE solver.

For part (a),

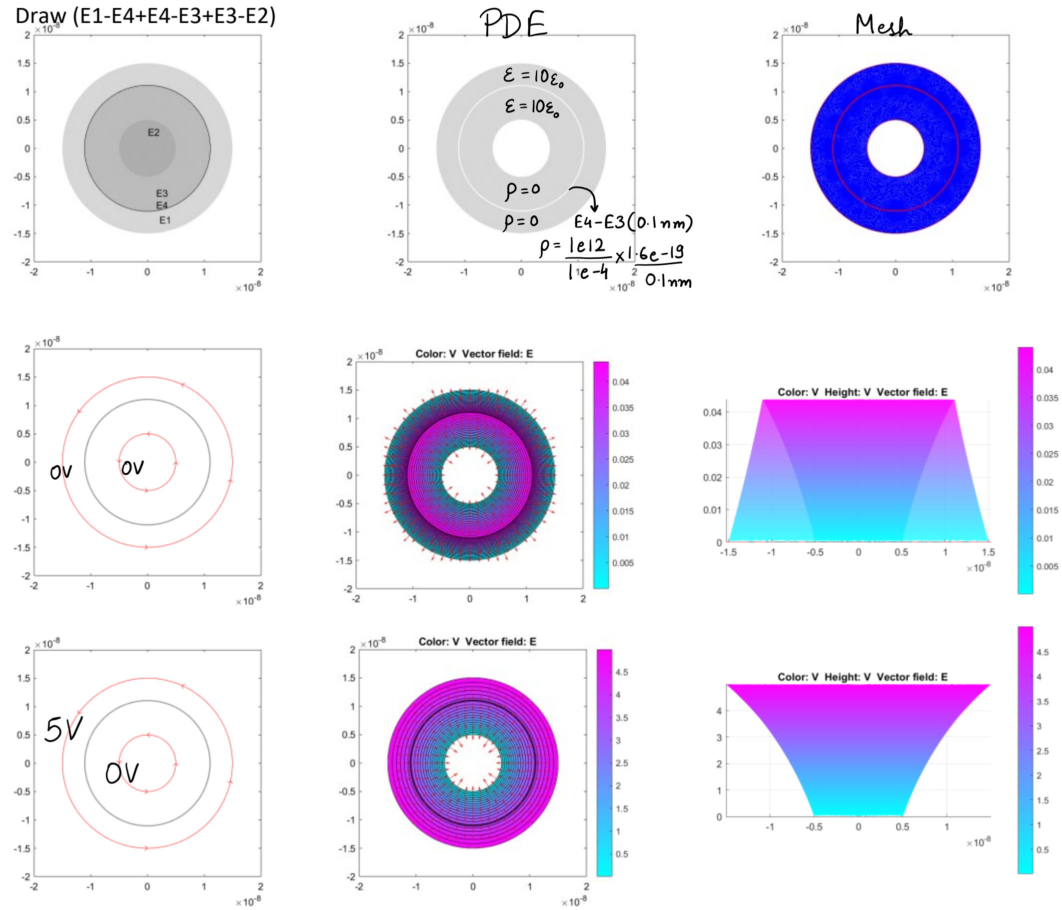


For part (b)



Draw (E1-E3+E3-E2)
E2
E3
E1
Boundary
0V
5V
PDE
ε = 30ε0
ε = 10ε0
ρ = 0
ρ = 0
Mesh
Color: V Vector field: E
Color: V Height: V Vector field: E

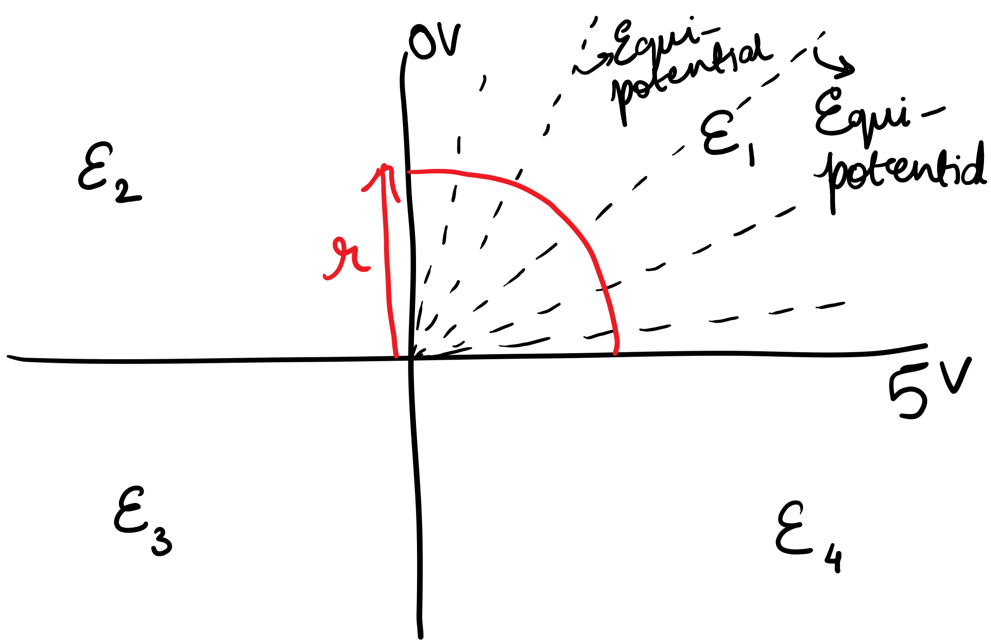
For part (c),



Note: Such electrostatics is seen in nanowire FETs where the nanowire channel is surrounded by a concentric gate dielectric and electrode. When the device is on, the channel has many conductors and can be approximated as a metal.

3. Two electrodes start at origin and extend to +x and +y to infinity (i.e. perpendicular to each other) with a bias of 5V across them.
   a) For the 1st quadrant (Note: this is like the Gate/Source or Gate Drain capacitance that is a parasitic capacitance).
   b) For the rest of the quadrants
   c) If each quadrant has dielectrics with different dielectric constant (as opposed to same dielectric constant in a & b)

   For each case, analytically solve and draw the charge density, Electric Field and Potential profile vs radius. Plot with MATLAB.



Solution: Here we will use the symmetry of the setup:

a. In any quadrant, the bisector line is an equipotential line by symmetry. Now we have two sub-quadrants. The bisectors of those sub-quadrants are again equipotential lines**. This process can be repeated to fill the whole quadrant with equipotential lines, all separated from each other by the same angle.**
b. This is a critical statement because this means that the density of equipotential lines is constant at any radius, in other words, the electric field is a function of only radius and has no theta dependence within a quadrant of constant dielectric constant.

For the first quadrant, this means,

$$E_1(r).\frac{\pi}{2}.r = \Delta V$$

Or,

$$E_1(r) = \frac{2\Delta V}{\pi r}$$

The clockwise field is assumed positive.

The free charge density on the positive electrode can also be written as:

$$\sigma_1(r) = D_1(r) = \frac{2\varepsilon_1 \Delta V}{\pi r}$$

For the other quadrants, this means,

$$E_4(r).\frac{\pi}{2}.r + E_3(r).\frac{\pi}{2}.r + E_2(r).\frac{\pi}{2}.r = -\Delta V$$

But,

$$E_i(r) = \frac{D(r)}{\varepsilon_i}$$

$D(r)$ has no quadrant dependence because there are no free or externally placed charges in the bulk.

Which means,

$$D(r) = -\frac{2\Delta V}{\pi r\left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4}\right)}$$

$$E_i(r) = -\frac{2\Delta V}{\pi r \varepsilon_i\left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4}\right)}$$

The field is counter-clockwise in these quadrants (hence negative),

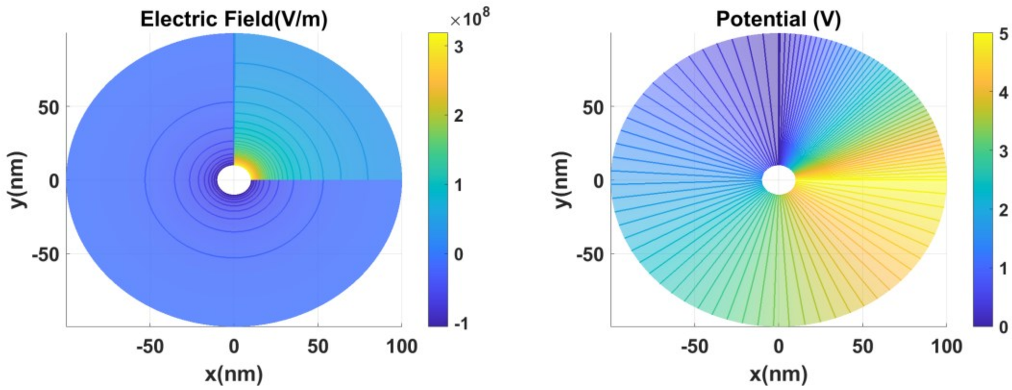
The potential for any quadrant can now be calculated:

$$V(\theta) = \Delta V - \int_0^\theta E(r).r d\theta$$

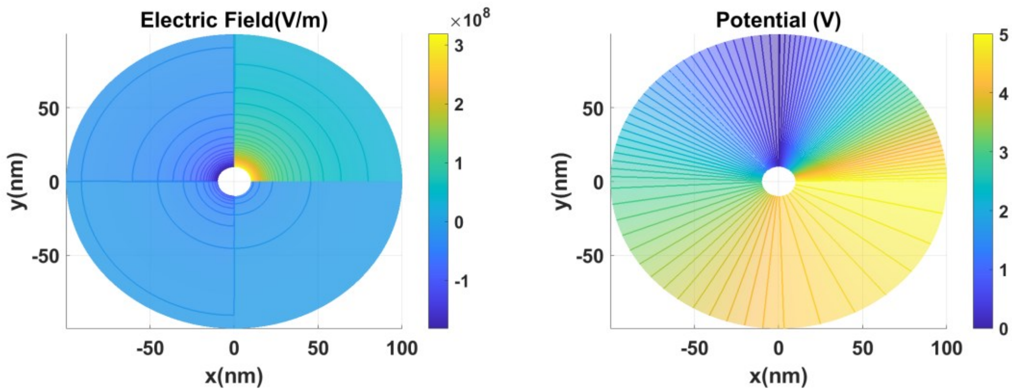
Here θ is measured anti-clockwise from the x-axis and $E(r)$ depends on which quadrant we are in.

For MATLAB plots, we assume $\varepsilon_1 = 10\varepsilon_0$, for part (a),

For part (b) we assume: $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 10\varepsilon_0$,



For part (c) we assume: $\varepsilon_1 = 10\varepsilon_0, \varepsilon_2 = 20\varepsilon_0, \varepsilon_3 = 40\varepsilon_0$ and $\varepsilon_4 = 80\varepsilon_0$,



Solve for the total capacitance.

Solution: The capacitance per unit area can be found as:

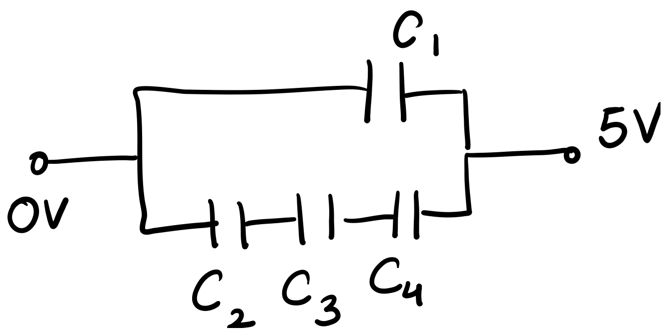
$$C_1(r) = \frac{\sigma_1(r)}{\Delta V} = \frac{2\varepsilon_1}{\pi r}$$

The total capacitance per unit length of first quadrant between r_i and r_o radii is the integral of the capacitance per unit area and is hence:

$$C_1 = \frac{2\varepsilon_1}{\pi} \ln \frac{r_o}{r_i}$$

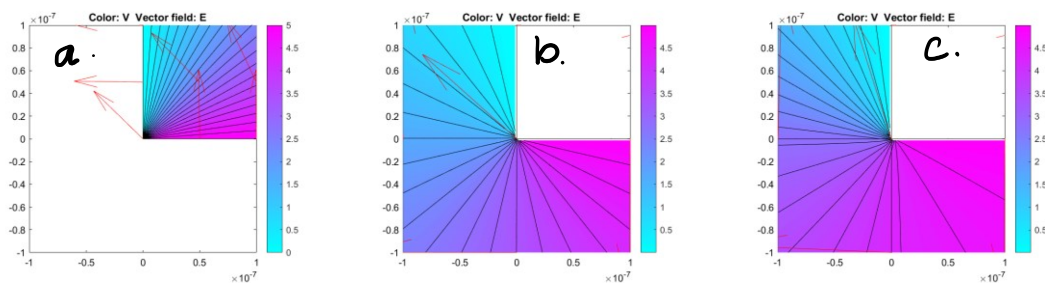
Can (c) be solved by capacitor network easily?

Solution: Since the boundaries of dielectrics are all equipotential surfaces, these capacitors can be considered as series and parallel connections:



Qualitatively verify using Matlab PDE solver.

Solution: The structures have been generated in PDEmodeler to verify the results:



For 3 (a), identify the outer radius by which the charge reduces to 10% of the inner radius.

Solution:

$$\sigma_1(r) = \frac{2\varepsilon_1 \Delta V}{\pi r}$$

At $10 \times r_i$, the charge is reduced to 10% of the charge at the inner radius, r_i.

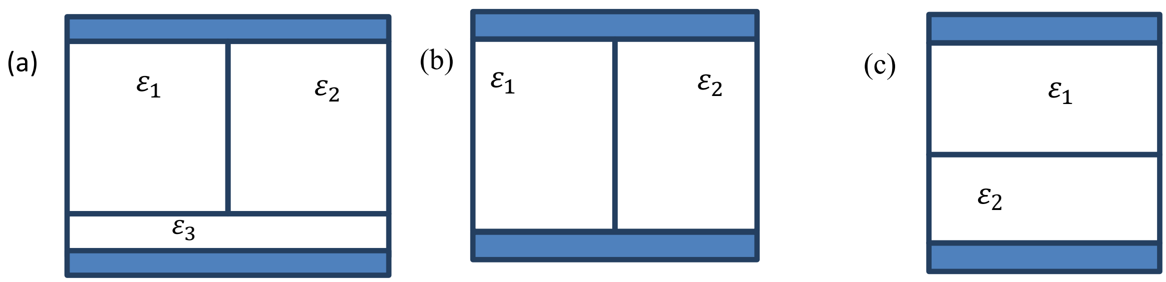
Note that this outer radius captures 90% of the capacitance; So, the plates do not need to be infinite for this approximation to be valid.

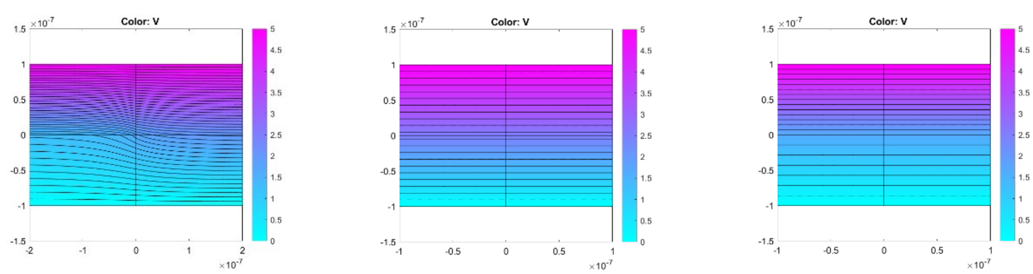
4. In which parallel plate capacitor shown below, capacitor network may not be used. Confirm with MATLAB PDETOOL simulations.

   Solution: (1) Different dielectric regions can be treated as individual capacitors in a network if the field lines in each dielectric are unchanged compared to situations when those dielectrics were standalone capacitors with the same potential difference across them. (2) This also means that for any dielectric to be treated as a series capacitor, its boundaries need to be equipotential surfaces (to be similar to its standalone capacitive state).

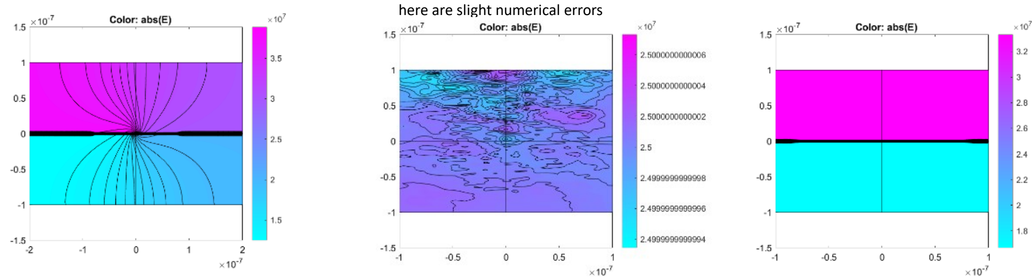
   (b) and (c) can be treated as capacitor networks (parallel and series respectively).
   For (b), the field is same everywhere, however the charge on the metal plates is not uniform due to different dielectric constants along the boundary of metal plate. This is also captured by a parallel combination of capacitors with equal applied potential.
   For (c), since the field lines are perpendicular to the metal everywhere, the equipotential surfaces are parallel to metal, the boundary of the dielectrics can hence be assumed to be a metal or equipotential surface, thus neatly dividing region in to two independent capacitors.

   In each of the examples, plot (a) equipotential surfaces (b) iso-electric field lines. Comment on how we can derive or draw one if we know the other.





E is constant (colorbar), what we see here are slight numerical errors



For PDEtools, we assumed $\varepsilon_1 = 10\varepsilon_0, \varepsilon_2 = 20\varepsilon_0, \varepsilon_3 = 30\varepsilon_0$

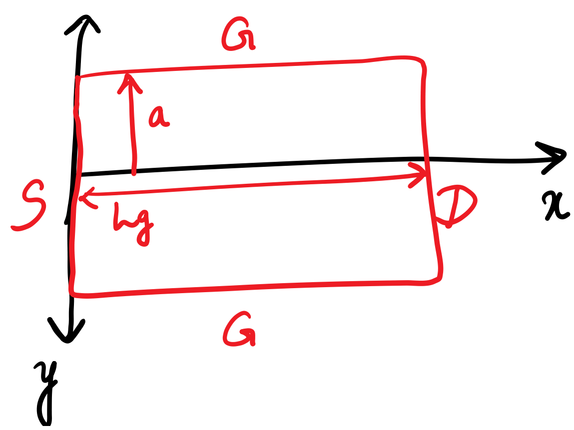
As seen in the above plots, the density of equipotential surfaces gives the electric field magnitude, hence iso-electric field lines go through regions of constant equipotential surface density.

Similarly, if the isoelectric field lines are close-by, that means the field is changing fast and so should the density of corresponding equipotential surfaces.

5. For a finFET of gate length 15nm node and approximate it as infinitely tall fin-height.

a) Derive the expression for V(x,y) at V_G=5V and (i) V_D=V_S=0V; (ii) V_D=-5V, everything else is same;

Plot V(x,y) is matlab using "meshgrid comment and 3D plots". Compare the shape with the PDEtool box solution.



Solution: Starting from the Laplace's equation:

$$\nabla^2 V = 0$$

In the x-y coordinate system this becomes:

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

Assuming the solution is variable separable:

$$V(x,y) = f(x).g(y)$$

$$g(y)\frac{\partial^2 f(x)}{\partial x^2} + f(x)\frac{\partial^2 g(y)}{\partial y^2} = 0$$

Dividing both sides by $V(x,y)$:

$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2}+\frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2}=0$$

The sum of two functions of two different variables sum upto a constant for every value of those two variables implies that the two functions have to be separately constant:

$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2}=-k^2$$

$$\frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2}=k^2$$

$$f(x)=A_k\cos kx+B_k\sin kx$$

$$g(y)=C_k e^{ky}+D_k e^{-ky}$$

The boundary conditions:

(i) $V(0,y)=0$
(ii) $V(L_G,y)=0$
(iii) $V(x,a)=V_G$
(iv) $V(x,-a)=V_G$

(i) and (ii) give,

$$A_k=0$$

$$k_n L_G=n\pi$$

Symmetry about x-axis gives:

$$g(y)=E_k\cosh ky$$

The total solution is hence:

$$V(x,y)=\Sigma_{n=1,2,\ldots}F_n\sin\frac{n\pi}{L_G}x\,.\cosh\frac{n\pi}{L_G}y$$

Using the gate voltage condition, $V(x,a)=V_G$:

$$V_G=\Sigma_n\left(F_n\cosh\frac{n\pi}{L_G}a\right)\sin\frac{n\pi}{L_G}x=\Sigma_n G_n\sin\frac{n\pi}{L_G}x$$

Multiplying both sides by $\sin\frac{m\pi}{L_G}x$ and integrating from $x=0$ to L_G (only $m=n$ terms survive):

$$\frac{V_G L_G}{m\pi}(1-\cos m\pi)=\frac{G_m L_G}{2}$$

$$G_m = \frac{2V_G}{m\pi}(1 - \cos m\pi)$$

This gives the gate potential only solution as:

$$V_1(x,y) = \Sigma_{n=1,2,3\ldots} \frac{2V_G(1-\cos n\pi)}{n\pi \cosh\frac{n\pi}{L_G}a} \sin\frac{n\pi}{L_G}x \, . \cosh\frac{n\pi}{L_G}y$$

$$V_1(x,y) = \Sigma_{n=1,3,5\ldots} \frac{4V_G}{n\pi \cosh\frac{n\pi}{L_G}a} \sin\frac{n\pi}{L_G}x \, . \cosh\frac{n\pi}{L_G}y$$

On the other hand, let:

$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} = k^2$$

$$\frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2} = -k^2$$

$$f(x) = C_k e^{kx} + D_k e^{-kx}$$

$$g(y) = A_k \cos ky + B_k \sin ky$$

The boundary conditions:

(i) $V(0,y) = 0$
(ii) $V(L_G,y) = V_D$
(iii) $V(x,a) = 0$
(iv) $V(x,-a) = 0$

(i) gives:

$$f(x) = E_k \sinh kx$$

(iii) and (iv) and symmetry about x-axis give:

$$g(y) = A_k \cos ky$$

$$k_n a = (2n+1)\frac{\pi}{2}$$

The total solution becomes:

$$V(x,y) = \Sigma_{n=1,3,5\ldots} F_n \left(\cos\frac{n\pi}{2a}y\right)\left(\sinh\frac{n\pi}{2a}x\right)$$

(ii) gives:

$$V_D = \Sigma_{n=1,3,5\ldots} F_n\left(\cos\frac{n\pi}{2a}y\right)\left(\sinh\frac{n\pi}{2a}L_G\right) = \Sigma_{n=1,3,5\ldots} G_n\left(\cos\frac{n\pi}{2a}y\right)$$

Multiplying both sides by $\cos\frac{m\pi}{2a}y$ and integrating from $y = -a$ to a (only $m = n$ terms survive):

$$\frac{V_D 4a}{m\pi}\sin\frac{m\pi}{2} = G_m a$$

This gives the drain potential only solution as:

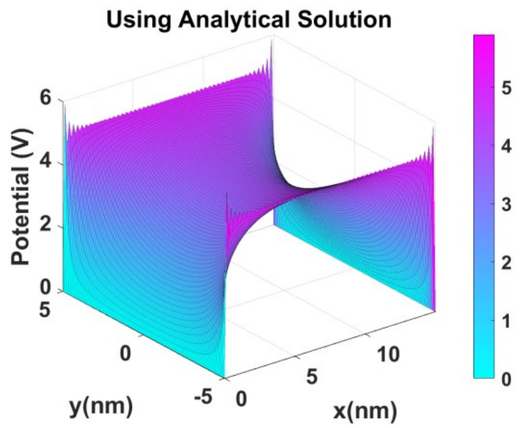
$$V_2(x, y) = \Sigma_{n=1,3,5\ldots}\frac{4V_D \sin\frac{n\pi}{2}}{n\pi \sinh\frac{n\pi}{2a}L_G}\left(\sinh\frac{n\pi}{2a}x\right)\left(\cos\frac{n\pi}{2a}y\right)$$

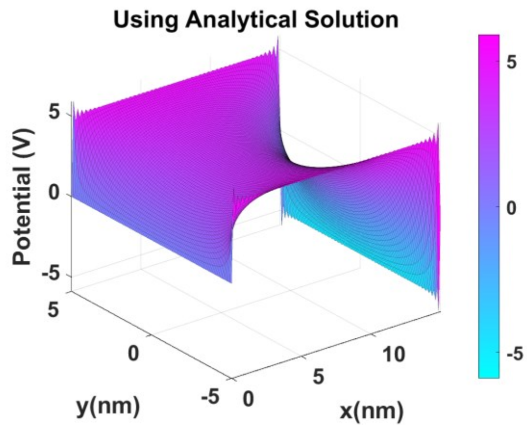
Now superposition reveals the solution in the presence of both drain and gate biases as:

$$V(x, y) = V_1(x, y) + V_2(x, y)$$

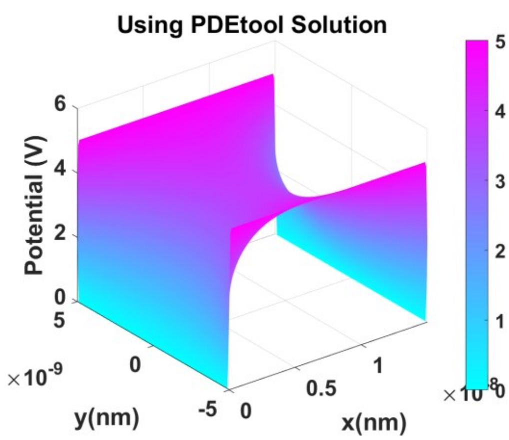
$$V(x, y) = \Sigma_{n=1,3,5\ldots}\frac{4V_G}{n\pi \cosh\frac{n\pi}{L_G}a}\ \sin\frac{n\pi}{L_G}x\,.\cosh\frac{n\pi}{L_G}y + \frac{4V_D \sin\frac{n\pi}{2}}{n\pi \sinh\frac{n\pi}{2a}L_G}\left(\sinh\frac{n\pi}{2a}x\right)\left(\cos\frac{n\pi}{2a}y\right)$$

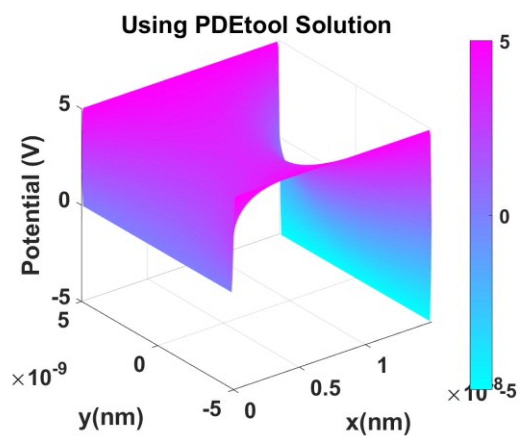
Plots using analytical solution for first 50 terms:





Plots using PDEtools:



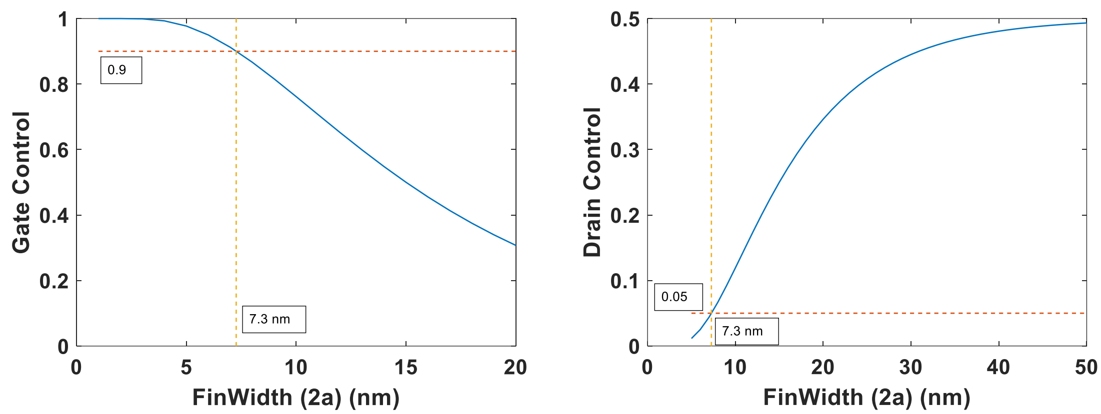


b) Find out analytically $\frac{dV_b}{dV_G}$ at V_G=5V and low V_{DS} which is a measure of gate control of effective barrier position (i.e. V_b). Plot $\frac{dV_b}{dV_G}$ vs. finwidth and find the finwidth limit for 90% effectiveness in barrier control by gate.

Solution: For low V_{DS}, it is safe to assume that $V_b = V\left(\frac{L_G}{2}, 0\right)$, i.e the mid-point of the channel and the fin (barrier for a positive charge).

$$V_b = V\left(\frac{L_G}{2}, 0\right) = \Sigma_{n=1,3,5\ldots} \frac{4V_G}{n\pi \cosh \frac{n\pi}{L_G} a} \sin\frac{n\pi}{2} + \frac{4V_D \sin\frac{n\pi}{2}}{n\pi \sinh\frac{n\pi}{2a} L_G}\left(\sinh\frac{n\pi}{4a} L_G\right)$$

$$\frac{dV_b}{dV_G} = \Sigma_{n=1,3,5\ldots} \frac{4}{n\pi \cosh\frac{n\pi}{L_G} a} \sin\frac{n\pi}{2}$$



c) Find out analytically $\frac{dV_b}{dV_D}$ at V_G=5V and high V_{DS} which is a measure of drain control. Assume that barrier is minimum at the center and its position is voltage independent (an approximation whose validity you can check based on part d). Plot $\frac{dV_b}{dV_D}$ vs. finwidth and find the finwidth limit for 90% effectiveness in barrier control by gate.

Solution: For high V_{DS}, and assuming that $V_b = V\left(\frac{L_G}{2}, 0\right)$, i.e the mid-point of the channel and the fin (barrier for a positive charge).

$$V_b = V\left(\frac{L_G}{2}, 0\right) = \Sigma_{n=1,3,5\ldots} \frac{4V_G}{n\pi \cosh\frac{n\pi}{L_G} a} \sin\frac{n\pi}{2} + \frac{4V_D}{n\pi \sin\frac{n\pi}{2} \sinh\frac{n\pi}{2a} L_G}\left(\sinh\frac{n\pi}{4a} L_G\right)$$

$$\frac{dV_b}{dV_D} = \Sigma_{n=1,3,5\ldots} \frac{4 \sin\frac{n\pi}{2}}{n\pi \sinh\frac{n\pi}{2a} L_G}\left(\sinh\frac{n\pi}{4a} L_G\right)$$

For 90% gate control, we can assume 5% drain control (assuming source control is same as drain control).

d) How does barrier position change as a function as VDS becomes more negative? Write a algorithm and implement it in Matlab to plot V_b vs V_{DS} and x(barrier), y(barrier) vs V_{DS}. Check experimentally based on PDEtool box quantitatively (i.e. plot expt vs calculated).

Solution: The total potential expression is:

$$V(x,y) = \Sigma_{n=1,3,5\ldots} \frac{4V_G}{n\pi \cosh\frac{n\pi}{L_G}a} \sin\frac{n\pi}{L_G}x \,.\cosh\frac{n\pi}{L_G}y + \frac{4V_D \sin\frac{n\pi}{2}}{n\pi \sinh\frac{n\pi}{2a}L_G}\left(\sinh\frac{n\pi}{2a}x\right)\left(\cos\frac{n\pi}{2a}y\right)$$

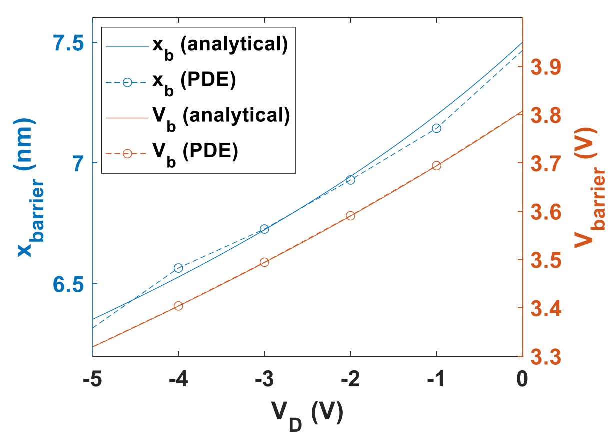
The y_b is clearly 0 by symmetry of the applied gate potentials. Putting in y_b:

$$V(x,0) = \Sigma_{n=1,3,5\ldots} \frac{4V_G}{n\pi \cosh\frac{n\pi}{L_G}a} \sin\frac{n\pi}{L_G}x + \frac{4V_D \sin\frac{n\pi}{2}}{n\pi \sinh\frac{n\pi}{2a}L_G}\left(\sinh\frac{n\pi}{2a}x\right)$$

At the barrier, $\frac{dV(x,0)}{dx} = 0$:

$$\frac{dV(x,0)}{dx} = \Sigma_{n=1,3,5\ldots} \frac{4V_G}{L_G \cosh\frac{n\pi}{L_G}a} \cos\frac{n\pi}{L_G}x + \frac{4V_D \sin\frac{n\pi}{2}}{2a \sinh\frac{n\pi}{2a}L_G}\left(\cosh\frac{n\pi}{2a}x\right) = 0$$

We can find numerically the location of x_b using graphical interpolation methods as given in HW2.m and compare with the PDE solutions at different V_D.



e) If there is fixed charge (e.g. ionic cores during depletion), how does it affect the above analysis?

Solution: We can involve superposition to write the total potential expression as:

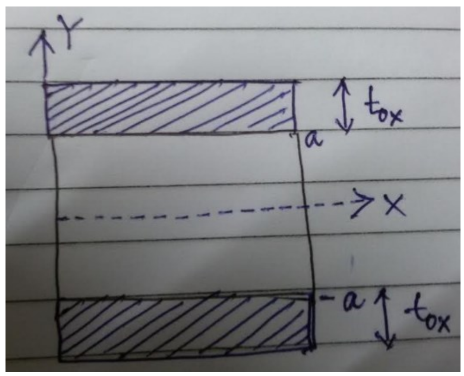
$$V(x,y) = V_{due\ to\ V_G\ and\ V_D} + V_{due\ to\ fixed\ charge}$$

The first term has been calculated in this question, the second term can also be calculated assuming all gate, source and drain potentials to be zero and using the Poisson equation. However, this second term will not have V_G and V_D dependence (since it is a fixed charge). So, the gate control and the drain control analysis will not change, but the barrier location and height will be affected depending on the location and the sign and the amount of the fixed charge in the fin.

f) If there is finite gate oxide thickness (t_{ox}), how does it affect the above analysis for Vbarrier calculations?

Solution: If there is a finite gate oxide thickness, the oxide material (say SiO_2) has a different dielectric constant compared to the rest of the fin (say, Si). This means the Laplace equation cannot be applied as it is (since epsilon has space dependence).

However, consider the scenario when the oxide thickness is very small compared to the fin width and channel lengths. Then the oxide can be assumed as a 1D capacitor:



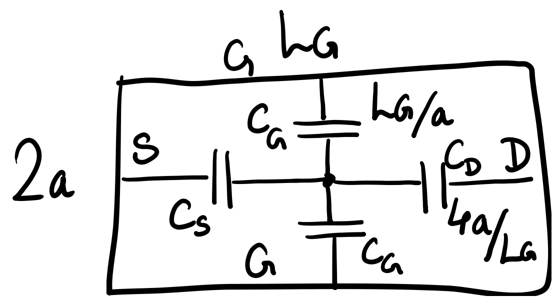
Next, we can replace this oxide thickness with an equivalent amount of Silicon thickness so that the 1D capacitance of the layer remains unchanged.

$$\frac{\varepsilon_{ox}}{t_{ox}} = \frac{\varepsilon_{Si}}{t_{Si}}$$

After replacing t_{ox} with t_{Si}, the whole material is now of constant dielectric constant and the barrier has similar capacitive coupling to gate as in the case with oxide. The Laplace solution on the new “oxide replaced with capacitive equivalent Silicon thickness” can be used for barrier analysis now.

g) For V at center, can a capacitive divider model be created? What is the error compared to 1D models for capacitors compared to capacitances derived from Laplace solutions.

Solution: We can assume a 1D capacitive model to center and compare it with the center voltage from the exact Laplacian solution:



The capacitive effects from boundaries to the center have been drawn using the 1D model.

The potential at center is then:

$$V = \frac{2C_G V_G + C_S V_S + C_D V_D}{2C_G + C_S + C_D}$$

The values for various C are shown in the figure.

For a particular case as used in simulations in this question: $L_G = 15\ nm$, $a = 5\ nm$, $V_G = 5\ V$ and $V_S = V_D = 0$

We get:

$$V_{1D} = 3.46\ V$$

The exact Laplacian solution for this as seen in part (d) is:

$$V_{lap} = 3.81\ V$$

This is a 9% error indicating that extremely simple 1D capacitor model already holds a lot of information about the barrier and the detailed Laplacian solution is a minor quantitative correction over it. However, as we will soon see in the lectures, the carrier densities are very sensitive to barrier heights (exponential dependence), in that regard, this shift is huge. However, the critical part is that the barrier control $\frac{dV_b}{dV_{G/D}}$ (rather than the exact barrier height V_b) by gate, drain and source are still reasonably captured by the 1D model.

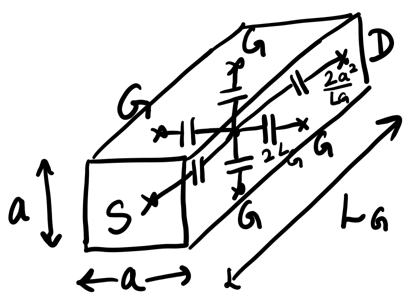
From 1D model:

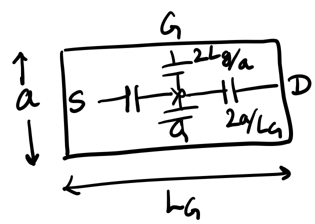
$$\frac{dV_b}{dV_G} = \frac{2C_G}{2C_G + C_S + C_D} = 0.69$$

The Laplacian value for this as seen in part (b) is 0.76, which again is a 9% error.

6. For a nanowire FET of gate length 15nm with a *square* cross-section of finside "a" nm. Repeat part (b) and (c) to show which device has better gate control and poorer drain control? At what dimension (finwidth vs. finlength) do the devices perform similarly i.e. which device requires more scaled features for same performance.

   Solution: We can do the detailed 3D Laplacian solution for nanowire FETs as in question 5 to obtain the barrier control equations. However, as seen in the part (g), the 1D capacitive model is a reasonable approximation to get barrier control:





For the nanowire FET:

$$\frac{dV_b}{dV_G} = \frac{4C_G}{4C_G + C_D + C_S}$$

And,

$$\frac{dV_b}{dV_D} = \frac{C_D}{4C_G + C_D + C_S}$$

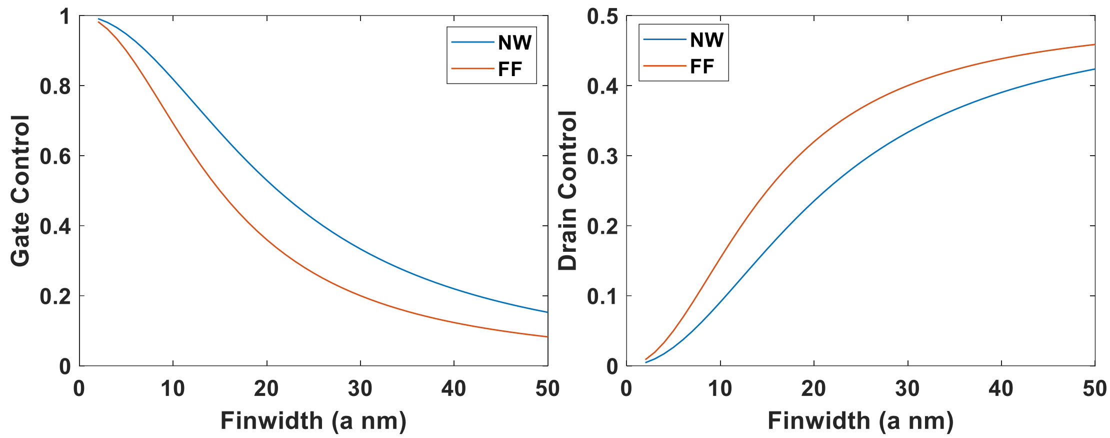
Similarly for the finFET:

$$\frac{dV_b}{dV_G} = \frac{2C_G}{2C_G + C_D + C_S}$$

And,

$$\frac{dV_b}{dV_D} = \frac{C_D}{2C_G + C_D + C_S}$$

Plotting in matlab,



For similar performance, let us equate:

$$\frac{dV_b}{dV_G}|_{NW} = \frac{dV_b}{dV_G}|_{FF}$$

$$\frac{C_D + C_S}{4C_G}|_{NW} = \frac{C_D + C_S}{2C_G}|_{FF}$$

$$\frac{1}{2}\left(\frac{a}{L_G}\right)^2_{NW} = \left(\frac{a}{L_G}\right)^2_{FF}$$

$$\frac{1}{\sqrt{2}}\left(\frac{a}{L_G}\right)_{NW} = \left(\frac{a}{L_G}\right)_{FF}$$

Clearly the finFET requires more scaled a/L_G compared to nanowire FET (about $\sqrt{2} = 1.4 \times$ more scaled).

7. Please explain why even without free charge, potential V(x,y) has a saddle point behavior in 2D or 3D?

Saddle point behaviors are allowed in 2D or 3D even in the absence of free charges because saddle points do not necessarily violate the condition of net electric field divergence to be zero as needed by the Laplace solution. A saddle point in potential can have a minima in the x-direction indicating that field lines are all converging to this point in the x-direction. However, this does not mean there is a negative charge there. The saddle point will have a maxima in the y-direction indicating that field lines are all emanating outward from this point in the y-direction. The saddle point ensures that the net divergence of electric field considering all directions is still zero.