COM S 531

Theory of Computation

HW #2

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Problem 9: For $A \subseteq \Sigma^*$ and $n \in N$, we define the nth slice of A language to be language

$$A_n = \{y \in \Sigma^* \mid \langle n, y \rangle \in A\}$$

where $\langle n, y \rangle = \langle s_n, y \rangle$ and $s_0, s_1, s_2 \ldots$ is the standard enumeration of Σ^*.

Let C and D be classes of languages.

1. C parametrizes D (or C is universal for D) if there exists $A \in C$ such that $D = \{A_n \mid n \in N\}$.
2. D is C-countable if there exists $A \in C$ such that $D \subseteq \{A_n \mid n \in N\}$

(a) Prove: A class of D languages is countable if and only if D is $P(\Sigma^*)$ countable.

(b) Prove: DEC is not DEC-countable.

(a) Proof: We have to proof that

(I) D is countable

(II) D is $P(\Sigma^*)$ countable

From (I) $\rightarrow$ (II)

Here, given that D is countable. So, we can write

$$h: N \xrightarrow{onto} D \text{ or, } h(n) = D \text{ where } n \in N$$

where N is a set of natural numbers.

Now, let's define a language ~~A ⊆~~ $A \in P(\Sigma^*)$ as

$$A = \{\langle n, y\rangle \mid n \in N\}$$

let us define the second part of string pairing function as $y \in h(n)$

So, $A = \{\langle n, h(n)\rangle \mid n \in N\}$

From the given definition of n-th slice of a language, we can write

$$A_n = \{h(n) \mid \langle n, h(n)\rangle \in A\}$$
$$= h(n) \text{ where } n \in N$$

Now, $D = h(n),\ n \in N$
$$= \{h(n) \mid n \in N\}$$
$$= \{A_n \mid n \in N\}$$

So, we can write $D = \{A_n \mid n \in N\}$, as a result we can definitely write $D \subseteq \{A_n \mid n \in N\}$. We have taken $A \in P(\Sigma^*)$ and $D \subseteq \{A_n \mid n \in N\}$. So, from given condition (1), we can conclude that D is $P(\Sigma^*)$ countable.

(II) $\rightarrow$ (I)

here given D is $P(\Sigma^*)$ countable. So, we can say for any $A \in P(\Sigma^*)$, $D \subseteq \{A_n | n \in N\}$. Now to prove D is countable, we have to design an onto function $h: N \xrightarrow{onto} D$. Let $a \in D$, then we can define $h: N \longrightarrow D$ ~~such~~ by

$$h(n) = \begin{cases} A_n & \text{if } A_n \in D \\ a & \text{else if } A_n \notin D \end{cases}$$

From the given definition of h function, the domain of h is set of natural ~~na~~ numbers. Now for any $n \in N$ when $A_n \notin D$, we are mapping function h to a element of co-domain set D. So, by ~~fx~~ h, we will hit every element of D, thus the range of function h will be D. So, $h: N \longrightarrow D$ is an onto function.

$\square$

(b) <u>Proof:</u> We will prove the statement by using contradiction. Assume DEC is DEC-countable. Then from statement (2), we can write for every $A \in DEC$, we can write $DEC \subseteq \{A_n | n \in N\}$.

So, we can see that for all $D \in DEC$, D can be written as a form $D = A_n$, where $n \in N$

As, A is decidable, there ~~as~~ is a Turing machine M such that it decides every ~~language~~ string A. Let, $B = \{0,1\}^* - A$ a language which will be also decidable. Because TM M will also decide B by just interchanging it's (TM) input and output. So, B is also decidable, then it can also be written as $B = A_m$ where $m \in N$.

Now, we can write $A = \{0,1\}^* - B = \{0,1\}^* - A_m$. However, this is contradicting that a decidable language can be written as A_n form.

So, we can conclude that, DEC can not be DEC-countable.

Problem 10: (a) Assume that C and D are set of languages and $g: C \xrightarrow{onto} D$. Prove: if C is countable, then D is countable.

(b) Prove: if C is a countable set of languages, then $\exists C$ and $\forall C$ are countable.

(a) Proof: Here, we are given,

$$g: C \xrightarrow{onto} D.$$

if C is countable, there will be a function h such that

$$h: N \xrightarrow{onto} C$$

we have to prove that D is countable. So, we have to design a function $f: N \xrightarrow{onto} D$.

Here N is the set of natural numbers.

For, h, every $x \in C$ will have at least one pre-image in domain N. So, for every $n \in N$, $h(n) = C$, where range of h is C.

Now, in function g, every element $y \in D$ will have at least

one pre-image in domain set C. For every ~~$x \in N$~~ $x \in C$

$g(x) = D$, where range is D.

So, we can design a composite function (f) which is $(g \circ h)(n) = g(h(n))$, the domain of this function will be $n \in N$ and the range of the function will be D (Range of ~~h~~ g).

So, we can write ~~$f = g(h$~~ $f = g(h(n))$ as

~~$h: N \xrightarrow{onto} D$~~ $f: N \xrightarrow{onto} D$

This function f is onto because every element in D will have at least one pre-image in N. So, we can conclude that D is also countable.

(b) <u>Proof</u>: Here, given that, C is a countable set of languages. So, there is a function h

$$h: N \xrightarrow{onto} C$$

Now, from lecture we know that

$$\exists C = \{\exists B \mid B \in C\} \quad \text{———} \quad (1)$$

Now, let, $D \in C$, for any D, we can define a function f.

$$f(D) = \exists D \text{ where, } f: C \to \exists C$$

When $D \in C$, from (1) we can see that $\exists D \in \exists C$. So, for every $\exists C$, there will be at least one pre-image in domain of f, which is C. Then, it can be said that $f: C \xrightarrow{onto} \exists C$. Now, from previous proof 10(a) a function g can be defined, $g: N \xrightarrow{onto} \exists C$. So, $\exists C$ is countable.

Similarly we can proof $\forall C$ is countable. For, $\forall C$, we know from the lecture,

$$\forall C = \{\forall B \mid B \in C\} \quad \text{———} \quad (2)$$

Then for every $F \in C$, we can define function f'

$$f'(F) = \forall F \text{ where } f': C \to \forall C$$

When, $F \in C$, from (2) we can see $\forall F \in \forall C$. So, every element of domain C will map every element of co-domain $\forall C$, which satisfies the condition for onto function. So, we can say, $f': C \xrightarrow{onto} \forall C$. In conclusion, by using 10(a), g' can be designed as $g': N \xrightarrow{onto} \forall C$. It proves that $\forall C$ is countable.

Problem 11: Prove that the class of countable classes of languages (defined as CTBL in class) is a σ-ideal on $P(\Sigma^*)$.

Proof: From the lecture, we know that, An ideal on a set X is a collection of subsets of X with the following two properties

1. For all $A, B \subseteq X$

$$A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$$

[Closed under subsets]

2. For all $A, B \subseteq X$

$$[A \in \mathcal{I} \text{ and } B \in \mathcal{I}] \rightarrow A \cup B \in \mathcal{I}$$

[Closed under finite union]

Let,

$$CTBL = \{A_0, A_1, A_2, \ldots\ldots A_\alpha\}$$

where all elements of set CTBL are countable classes and subset of $P(\Sigma^*)$.

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Now, for CTBL set, the mentioned two conditions are suffice to proof.

1. <u>Closed under subset</u>

let, $A, B \subseteq P(\Sigma^*)$

and $A \subseteq B \in CTBL$

here, B is countable as it belongs to CTBL. From Lemma 7, as A is a subset of B, A will also be countable. So, They are closed under subset.

2. <u>Closed under finite unions</u>

let, $A, B \subseteq P(\Sigma^*)$

and $A, B \in CTBL$

As, Both A, B belong to CTBL, they are both countable set. So, from Lemma 8, we can conclude that $A \cup B$ will also be countable. So, the condition holds.

So, from the given argument, we can conclude that CTBL is a σ-ideal on $P(\Sigma^*)$.

$\square$

Problem 12 Prove all the inclusions in the infinite diagram.

$$\begin{tikzcd}
 & \vdots & \\
\Sigma_3^0 \arrow[ur, phantom, "\subseteq"] & & \Pi_3^0 \arrow[ul, phantom, "\supseteq"] \\
 & \Delta_3^0 \arrow[ul, phantom, "\subseteq"] \arrow[ur, phantom, "\supseteq"] & \\
\Sigma_2^0 \arrow[ur, phantom, "\subseteq"] & & \Pi_2^0 \arrow[ul, phantom, "\supseteq"] \\
 & \Delta_2^0 \arrow[ul, phantom, "\subseteq"] \arrow[ur, phantom, "\supseteq"] & \\
\Sigma_1^0 \arrow[ur, phantom, "\subseteq"] & & \Pi_1^0 \arrow[ul, phantom, "\supseteq"] \\
 & \Delta_1^0 \arrow[ul, phantom, "\subseteq"] \arrow[ur, phantom, "\subseteq"] &
\end{tikzcd}$$

Proof: From the class lecture, we have the following recursions,

(1) $\Sigma_1^0 = CE$

(2) $\Pi_n^0 = co\Sigma_n^0$

(3) $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

(4) $\Sigma_{n+1}^0 = \exists \Pi_n^0$

Thus for example, we have to following

(5) $\Pi_1^0 \overset{(2)}{=} co\Sigma_1^0 \overset{(1)}{=} coCE \overset{cor\ 30}{=} \forall DEC$

(6) $\Delta_1^0 \overset{(3)}{=} \Sigma_1^0 \cap \Pi_0^1 \overset{(1,5)}{=} CE \cap coCE \overset{Lemma5}{=} DEC$

$$(7)\ \Sigma_2^\circ \overset{(4)}{=} \exists \Pi_1^\circ \overset{(5)}{=} \exists \mathrm{co}CE \overset{\text{Cor } 30}{=} \exists\forall DEC$$

$$(8)\ \Pi_2^\circ \overset{(2)}{=} \mathrm{co}\Sigma_2^\circ \overset{(7)}{=} \mathrm{co}\exists\forall DEC = \forall\exists \mathrm{co}DEC = \forall\exists DEC$$

From the given diagram, we have to prove that

$$\Delta_n^\circ \subseteq \Sigma_n^\circ$$

$$\Delta_n^\circ \subseteq \Pi_n^\circ$$

$$\Pi_n^\circ \subseteq \Delta_{n+1}^\circ$$

$$\Sigma_n^\circ \subseteq \Delta_{n+1}^\circ$$

We will prove these using the method of induction.

So, let, $n=1$

Then we have to prove

$$\Delta_1^\circ \subseteq \Sigma_1^\circ$$

$$\Delta_1^\circ \subseteq \Pi_1^\circ$$

$$\Pi_1^\circ \subseteq \Delta_2^\circ$$

$$\Sigma_1^\circ \subseteq \Delta_2^\circ$$

~~Here, Δ_1°~~ Now, for $\Delta_1^\circ \subseteq \Sigma_1^\circ$ and $\Delta_1^\circ \subseteq \Pi_1^\circ$

$\Delta_1^\circ = DEC$ (from 6)

$\Sigma_1^\circ = CE$ (from 1)

$= \exists DEC$ (Theorem 29)

$\Pi_1^0 = \forall DEC$ (from 5)

From Lemma 22, we know that DEC is closed under $\leq_m$ reduction. Then $\exists DEC$ and $\forall DEC$ will also be closed under $\leq_m$ reduction (Lemma 24).

$\therefore DEC \subseteq \exists DEC \cap \forall DEC$ (Lemma 25)

We can say that $DEC \subseteq \exists DEC \Rightarrow \Delta_1^0 \subseteq \Sigma_1^0$

and $DEC \subseteq \forall DEC \Rightarrow \Delta_1^0 \subseteq \Pi_1^0$

Now, let's prove $\Pi_1^0 \subseteq \Delta_2^0$

here, $\Pi_1^0 = \forall DEC$ (from 5)

and $\Delta_2^0 = \Sigma_2^0 \cap \Pi_2^0$ (from 3)

$= \exists \Pi_1^0 \cap \forall \exists DEC$ (from 4, 8)

$= \exists \forall DEC \cap \forall \exists DEC$ (from 5)

So, we have to prove $\forall DEC \subseteq \exists \forall DEC \cap \forall \exists DEC$

Now, as $\forall DEC$ is closed under $\leq_m$ reduction, we can say, $\forall DEC \subseteq \exists \forall DEC$ [Lemma 24]

Now, from the previous proof, we get $DEC \subseteq \exists DEC$.
Now, adding $\forall$ to both sides, $\forall DEC \subseteq \forall \exists DEC$. (As if $X \subseteq Y$, then $\forall X \subseteq \forall Y$).

$$\therefore \forall DEC \subseteq \exists \forall DEC \cap \forall \exists DEC$$

$$\Rightarrow \Pi_1^0 \subseteq \Sigma_2^0 \cap \Pi_2^0 = \Delta_2^0$$

Similarly we have to prove $\Sigma_1^0 \subseteq \Delta_2^0$

$$\Rightarrow \exists DEC \subseteq \exists \forall DEC \cap \forall \exists DEC \text{ [From 1, 3, 4, 5, 8]}$$

Here, $\exists DEC$ is $\leq_m$ reducible. ~~So, from~~ (Lemma 22 and 23).
So, from lemma 25, $\exists DEC \subseteq \forall \exists DEC$. Again in the previous part we have proved that $DEC \subseteq \forall DEC$. Adding $\exists$ on both the sides, $\exists DEC \subseteq \exists \forall DEC$. So, we can conclude that

$$\exists DEC \subseteq \forall \exists DEC \cap \exists \forall DEC$$

$$\Rightarrow \Sigma_1^0 \subseteq \Delta_2^0$$

So, we have proved that for $n=1$, all the four conditions hold.

Now, let for $n=k-1$, all the four conditions are true.

So, $\Delta^{\circ}_{k-1} \subseteq \Sigma^{\circ}_{k-1}$

$\Delta^{\circ}_{k-1} \subseteq \Pi^{\circ}_{k-1}$

$\Pi^{\circ}_{k-1} \subseteq \Delta^{\circ}_{k}$

$\Sigma^{\circ}_{k-1} \subseteq \Delta^{\circ}_{k}$

These are true. Now, we have to prove all these conditions hold for $n=k-1+1$ or $n=k$

when $n=k$

$\Delta^{\circ}_{k} \subseteq \Sigma^{\circ}_{k}$

$\Delta^{\circ}_{k} \subseteq \Pi^{\circ}_{k}$

$\Pi^{\circ}_{k} \subseteq \Delta^{\circ}_{k+1}$

$\Sigma^{\circ}_{k} \subseteq \Delta^{\circ}_{k+1}$

We know that

$\Delta^{\circ}_{n} = \Sigma^{\circ}_{n} \cap \Pi^{\circ}_{n}$ [from 3]

So, $\Delta^{\circ}_{k} = \Sigma^{\circ}_{k} \cap \Pi^{\circ}_{k}$

From this we can easily conclude $\boxed{\Delta^{\circ}_{k} \subseteq \Sigma^{\circ}_{k}}$ and $\boxed{\Delta^{\circ}_{k} \subseteq \Pi^{\circ}_{k}}$

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Now, for $n=k-1$, ~~$\Delta^0_{k-1} \subseteq \Delta^0_k$~~ $\Pi^0_{k-1} \subseteq \Delta^0_k$ is true.

$$\Rightarrow \Pi^0_{k-1} \subseteq \Sigma^0_k \cap \Pi^0_k \quad [\text{From } 3]$$

From this, we can write,

$$\Pi^0_{k-1} \subseteq \Pi^0_k$$

$$\Rightarrow \exists \Pi^0_{k-1} \subseteq \exists \Pi^0_k$$

$$\Rightarrow \Sigma^0_k \subseteq \Sigma^0_{k+1} \quad [\text{From } 4] \quad \text{——} (9)$$

As, All of these are closed under $\leq_m$ reduction

$$\Sigma^0_k \subseteq \exists \Sigma^0_k \cap \forall \Sigma^0_k \quad [\text{Lemma } 25]$$

Now, we know that

$$\Pi^0_2 = \forall \exists DEC \quad [\text{From } 8]$$

$$= \forall CE \quad [\text{Theorem } 29]$$

$$= \forall \Sigma^0_1$$

So, $\Pi^0_{n+1} = \forall \Sigma^0_n$ —— (10)

$$\therefore \Sigma^0_k \subseteq \exists \Sigma^0_k \cap \forall \Sigma^0_k$$

$$\Rightarrow \Sigma^0_k \subseteq \forall \Sigma^0_k$$

$$\Rightarrow \Sigma^0_k \subseteq \Pi^0_{k+1} \quad [\text{From } 10] \quad \text{——} (11)$$

From (9) and (11), $\Sigma^0_k \subseteq \Sigma^0_{k+1} \cap \Pi^0_{k+1}$

$$\Rightarrow \boxed{\Sigma^0_k \subseteq \Delta^0_{k+1}} \quad [\text{From } 3]$$

Now again, $\Sigma^{\circ}_{k-1} \subseteq \Delta^{\circ}_{k}$ is true. (For $n=k-1$) we have assumed)

$$\therefore \Sigma^{\circ}_{k-1} \subseteq \Delta^{\circ}_{k}$$

$$\Rightarrow \Sigma^{\circ}_{k-1} \subseteq \Sigma^{\circ}_{k} \cap \Pi^{\circ}_{k} \quad \text{[From 3]}$$

$$\Rightarrow \Sigma^{\circ}_{k-1} \subseteq \Sigma^{\circ}_{k}$$

$$\Rightarrow \forall \Sigma^{\circ}_{k-1} \subseteq \forall \Sigma^{\circ}_{k}$$

$$\Rightarrow \Pi^{\circ}_{k} \subseteq \Pi^{\circ}_{k+1} \quad \text{[From 10]} \quad \text{——— (12)}$$

Again

$$\Pi^{\circ}_{k} \subseteq \exists \Pi^{\circ}_{k} \cap \forall \Pi^{\circ}_{k} \quad \text{[Lemma 25]}$$

$$\Rightarrow \Pi^{\circ}_{k} \subseteq \exists \Pi^{\circ}_{k}$$

$$\Rightarrow \Pi^{\circ}_{k} \subseteq \Sigma^{\circ}_{k+1} \quad \text{[From 4]} \quad \text{——— (13)}$$

From (12) and (13) $\Rightarrow$

$$\Pi_{k} \subseteq \Pi^{\circ}_{k+1} \cap \Sigma^{\circ}_{k+1}$$

$$\therefore \boxed{\Delta_{k} \subseteq \Delta^{\circ}_{k+1}} \quad \text{[From 3]}$$

So, we have proved that for, $n=k-1+1$, all the four conditions hold. We can then conclude that all inclusions are true in the given infinite diagram.

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Problem 13: Prove that there is a function $g: N \longrightarrow N$ with the following properties

(i) g is nondecreasing, i.e., $g(n) \leq g(n+1)$ holds for all $n \in N$

(ii) g is unbounded, i.e., for every $m \in N$ there exists $n \in N$ such that $g(n) > m$.

(iii) For every computable, nondecreasing, unbounded function $f: N \longrightarrow N$, $f(n) > g(n)$ holds for all but finitely many $n \in N$.

Proof: From lecture,

From the enumeration

$$M_0, M_1, M_2 - - -$$

of all TMs, we can extract an enumeration

$$\phi_1, \phi_2, \phi_3 - - -$$

of all computable partial function $\phi_k : \subseteq N \longrightarrow N$

Let's define the slowly growing function

$$g : N \longrightarrow N$$

by

$$g(n) = \min\{\varphi_k(l) : 0 \le k \le n, 0 \le l \le n, \underset{\text{defined}}{\varphi_k(l)\downarrow}\} - 1$$

The value of $g(n)$ can be either positive or negative, but it will work as a slowly growing computable function.

(1) $g(n)$ must be non-decreasing

We have designed function using $\varphi_1, \varphi_2, \ldots$ computable partial functions. Here these computable partial functions are nondecreasing. So, for any value of k and l, $\varphi_k(l) \le \varphi_k(l+1)$. Similarly $\min\{\varphi_k(l)\} \le \min\{\varphi_k(l+1)\}$.

~~For example min{φk~~ For example $\min\{\varphi_0(l), \varphi_1(l), \varphi_2(l), \varphi_3(l) \ldots\} \le \min\{\varphi_0(l+1), \varphi_1(l+1), \varphi_2(l+1), \ldots\}$

So, $\min\{\varphi_k(l) : 0 \le k \le n, 0 \le l \le n, \varphi_k(l) \downarrow\} \le \min\{\varphi_k(l+1), 0 \le k \le n+1, 0 \le l+1 \le n+1, \varphi_k(l+1) \downarrow\}$.

From the discussion, we can conclude that $g(n) \le g(n+1)$.

So, function g is nondecreasing.

(ii) <u>g is unbounded, i.e., for every $m \in \mathbb{N}$, there exist $n \in \mathbb{N}$, such that $g(n) > m$</u>

We know that $\varphi_k(l)$ functions are unbounded. For every $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$, such that $\varphi_k(n) > m$. This is also true for $g(n)$. As we can find value $m \in \mathbb{N}$, such that $\min\{\varphi_k(l) : 0 \le k, l \le n\}$ will be greater than m. From this we can also conclude that for every $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such ~~ta~~ that $g(n) > m$.

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(iii) For every computable, non decreasing, unbounded function $f: \mathbb{N} \longrightarrow \mathbb{N}$, $f(n) > g(n)$ holds for all but finitely many $n \in \mathbb{N}$.

Let $f: \mathbb{N} \longrightarrow \mathbb{N}$ is computable, then there exists $m \in \mathbb{N}$ such that $\varphi_m = f$. ~~Then there exists~~ Then for all $n \geq m$

$$g(n) = -1 + \min \left\{ \varphi_k(n) \;\middle|\; \begin{array}{l} 0 \leq k \leq n \\ 0 \leq n \leq n \\ \varphi_k(n)\downarrow \end{array} \right.$$

$$\leq -1 + \varphi_m(n)$$

$$< \varphi_m(n)$$

$$< f(n)$$

So, we can say that $f(n) > g(n)$ for infinitely many $n \in \mathbb{N}$.

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Problem 14: Prove that a partial function $f: \subseteq \Sigma^* \to \Sigma^*$ is computable if and only if it's graph

$$G_f = \{\langle x, f(x)\rangle \mid x \in \text{dom} f\}$$

is c.e.

Proof:

(I) A partial function $f: \subseteq \Sigma^* \to \Sigma^*$ is computable

(II) $G_f = \{\langle x, f(x)\rangle \mid x \in \text{dom} f\}$ is c.e.

(I) $\longrightarrow$ (II)

Given $f: \subseteq \Sigma^* \to \Sigma^*$ is computable. So, there exist a turing machine M_f that emulates function f. Now, we have to construct an algorithm that will compute G_f as follow

Algorithm for M_g

1. Input $\langle x, y\rangle$ where $x \in \Sigma^*$ and $y \in \Sigma^*$
2. Run x on M_f
3. If M_f halts on x and outputs str. and str == y
4. Output $\langle x, y\rangle$ where $y = f(x)$
5. Else
6. Undefined or loop forever [$x \notin \text{dom} f$]
7. End

This algorithm will output all pairs in G_f. For the values

$x \in$ dom f. TM M_f will output y or $f(x)$ and the algorithm for M_g will output the string pair $\langle x, y \rangle$. when $x \notin$ dom f, the turing machine M_f will be undefined and similarly M_g will also loop forever this making G_f a c.e. language.

Now (II) $\rightarrow$ (I)

Given G_f is c.e. Then we have to proof that the partial function ~~$f: \subseteq \Sigma^*$ is com~~ $f: \subseteq \Sigma^* \rightarrow \Sigma^*$ is computable. As G_f is computably enumerable, then there is TM M_a such that $L(M_a) = G_f$. Now, we can develop TM M_f by following algorithm.

<u>Algorithm for M_f</u>

1. Input $x \in \Sigma^*$
2. For every $y \in \Sigma^*$
3. Run M_a on $\langle x, y \rangle$
4. If M_a accepts $\langle x, y \rangle$
5. output y and Halt
7. Else undefined
8. End

This given algorithm computes $f(x)$ such that $x \in$ dom f. Then there will be a y that makes $\langle x, y \rangle$ to exist in G_f. The algorithm ~~a~~ set f as undefined when y is not in range, for those y, values of x are not defined, thus making f as a partial function.

Problem 15: let $A \subseteq \Sigma^*$ be c.e., and let B be an infinite decidable subset of A. Prove: If A is undecidable, then $A \setminus B$ is undecidable.

Proof: Suppose, $A \setminus B$ is decidable. Here, we are also given that B is an infinite decidable subset. Since, $A \setminus B$ and B are both decidable, then their union set will be also decidable. So, $(A \setminus B) \cup B$ should be a decidable set. But $(A \setminus B) \cup B = A$, which is making A set to be decidable. This contradicts the statement that A is undecidable. So, we can say that $A \setminus B$ can not be ~~un~~decidable.

So, it is proved that if A is undecidable, then $A \setminus B$ is undecidable.

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Problem 16: Let $A = L(U)$ be the universal c.e. language defined in class lecture, and let $B \subseteq \Sigma^*$. Prove if $A \leq_m B$ and $\Sigma^* \setminus A \leq_m B$, then B is either c.e. nor co-c.e.

Proof: From corollary 37, we know that if $A = L(u)$ is the universal c.e. language then Σ^*/A is co-c.e. is not decidable, and A is also not decidable (corollary 36).

Again from the lectures, we know that if A class C of languages is closed under reduction $\leq_m$ for all $B, A \subseteq \Sigma^*$

$$A \leq_m B \in C \Rightarrow A \in C$$

Now, let's try to construct the given proof using ~~contra~~ contradiction. We have to proof that B is not either c.e. or co-c.e. First let $B \in CE$. As, in the given condition, $\Sigma^* \setminus A \leq_m B$ and $B \in CE$, then $\Sigma^* \setminus A \in CE$. But this is a contradiction to corollary 37. So, $\Sigma^* \setminus A \in CE$ can not be a c.e language which means B is also not c.e.

Now, let's assume that $B \in COCE$. Here, we are given that $A \leq_m B$. So, we can say that $A \in COCE$. But A is the universal c.e. language and $A \notin COCE$. That's why $B \notin COCE$.

So, we proved that if $A \leq_m B$ and $\Sigma^* \setminus A \leq_m B$, then B is not either c.e. or co-c.e.

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