1/

$$A = \begin{bmatrix} -1 & 0 & -3 \\ -2 & 2 & -2 \\ 5 & -4 & 7 \end{bmatrix} \qquad \det(A - \lambda I_3) = 0$$

$$= \begin{bmatrix} -1 & 0 & -3 \\ -2 & 2 & -2 \\ 5 & -4 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -3 \\ -2 & 2 & -2 \\ 5 & -4 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -1-\lambda & 0 & -3 \\ -2 & 2-\lambda & -2 \\ 5 & -4 & 7-\lambda \end{bmatrix}$$

$$= -1-\lambda \{ 14 - 2\lambda - 7\lambda + \lambda^2 - 8 \} + (-3) \{ 8 - (10 - 5\lambda) \}$$

$$= (-1-\lambda) \{ 6 - 9\lambda + \lambda^2 \} + (-3) \{ 8 - 10 + 5\lambda \}$$

$$= -6 + 9\lambda - \lambda^2 - 6\lambda + 9\lambda^2 - \lambda^3 - 24 + 30 - 15\lambda = 0$$

$$= -\lambda^3 + 8\lambda^2 - 12\lambda = 0$$

$$= \lambda(-\lambda^2 + 8\lambda - 12) = 0$$

$$\therefore -\lambda^2 + 8\lambda - 12 = 0$$

$$= \lambda^2 - 8\lambda + 12 = 0$$

$$= \lambda^2 - 6\lambda - 2\lambda + 12 = 0$$

$$= \lambda(\lambda-6) - 2(\lambda-6) = 0$$

$$= (\lambda-6)(\lambda-2) = 0$$

$\therefore \lambda - 6 = 0$

$\Rightarrow \lambda = 6$

or

$\lambda - 2 = 0$

$\lambda = 2$

1) $$[A|b] = \left[\begin{array}{cccc|c} 1 & -7 & 2 & -5 & 8 \\ 0 & 1 & -3 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$x_1 - 7x_2 + 2x_3 - 5x_4 = 8 \quad \text{--- (i)}$$

$$x_2 - 3x_3 + 3x_4 = 1 \quad \text{--- (ii)}$$

$$x_4 = -1 \quad \text{--- (iii)}$$

(ii) $\Rightarrow x_2 - 3x_3 + 3(-1) = 1$

$\Rightarrow x_2 - 3x_3 - 3 - 1 = 0$

$\Rightarrow x_2 - 3x_3 - 4 = 0$

$\Rightarrow x_2 = 3x_3 + 4$

$$(III) \Rightarrow x_1 - 7(3x_3 + 4) + 2x_3 - 5(-1) = 8$$

$$\Rightarrow x_1 - 21x_3 - 28 + 2x_3 + 5 = 8$$

$$\Rightarrow x_1 - 19x_3 - 23 - 8 = 0$$

$$\Rightarrow x_1 - 19x_3 - 31 = 0$$

$$\Rightarrow x_1 = 19x_3 + 31$$

let $x_3 = t$

now, we can write the solution in the vector form.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 31 + 19t \\ 4 + 3t \\ t \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 31 \\ 4 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 19 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

here t is any real number.

2) (i)

For, Mr x the rate will be

$$x = 20 + 15h$$

For, Mr y the rate will be

$$y = 50 + 10h$$

(ii)

The equations from the model are.

$$x = 20 + 15h$$
$$y = 50 + 10h$$

OR, $x - 15h = 20$

$y - 15h = 50$

Augmented matrix from the equations.

$$\left[\begin{array}{cc|c} 1 & -15 & 20 \\ 1 & -10 & 50 \end{array}\right] \quad [R_2' = R_2 - R_1]$$

$$\left[\begin{array}{cc|c} 1 & -15 & 20 \\ 0 & 5 & 30 \end{array}\right]$$

now,

the equations are,

$$k \quad 15h = 20$$

$$5h = 30$$

$$\therefore h = \frac{30}{5} = 6 \qquad \therefore h = 6$$

so,

$$k = 20 + (15 \times 6) = 110$$

From the above calculations we can say that Mr X and Mr Y will change the same amount which is \$ 110. Therefore the intersection point for Mr X and Mr Y is at six hours mark.

(III)

From, the number (II) question we get at the 6 hour mark both mr x and mr y's price are same.

when the time $h<6$ then.

$x = 20 + 15h$

let, $h = 5$

$x = 20 + 15 \times 5$

$= 95$

and,

$y = 50 + 10 \times 5$

$= 100$

let

$h = 4$

$x = 20 + 15 \times 4$

$= 80$

and,

$y = 50 + 10 \times 4$

$= 90$

let

$h = 3$

$x = 20 + 15 \times 3$

$= 65$

and,

$y = 50 + 10 \times 3$

$= 80$

From the above calculation we can see if the time rented is $h < 6$ then the survices of mr x will be benefitial and when it is $h > 6$ then.

let,

$h = 7$

$x = 20 + 15 \times 7$

$= 125$

when,

$h = 8$

$x = 20 + 15 \times 8$

$= 140$

when,

$h = 9$

$x = 20 + 15 \times 9$

$= 155$

and,

$h = 7$

$y = 50 + 10 \times 7$

$= 120$

and,

$h = 8$

$y = 50 + 10 \times 9$

$= 130$

and,

$h = 9$

$y = 50 + 10 \times 9$

$= 140$

so, it can be said when the rented time is $h < 6$ the survices of mr x are benefitial and when $h > 6$ ~~mr~~ the survices of mr. y are benefitial and when $h = 6$ mr x and mr y will offer the same prices.

3.

$$5x_3 + 15x_5 = 5$$
$$2x_2 + 4x_3 + 7x_4 + x_5 = 3$$
$$x_2 + 2x_3 + 3x_4 = 1$$
$$x_2 + 2x_3 + 4x_4 + x_5 = 2$$

$$\left[\begin{array}{cccc|c} 0 & 5 & 0 & 15 & 5 \\ 2 & 4 & 7 & 1 & 3 \\ 1 & 2 & 3 & 0 & 1 \\ 1 & 2 & 4 & 1 & 2 \end{array}\right] R_2 = R_1$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 7 & 1 & 3 \\ 0 & 5 & 0 & 15 & 5 \\ 1 & 2 & 3 & 0 & 1 \\ 1 & 2 & 4 & 1 & 2 \end{array}\right] R_1 = R_3$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 5 & 0 & 15 & 5 \\ 2 & 4 & 7 & 1 & 3 \\ 1 & 2 & 4 & 1 & 2 \end{array}\right] R_3' = R_3 - 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 5 & 0 & 15 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 & 2 \end{array}\right] R_4' = R_4 - R_1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 5 & 0 & 15 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \quad R_2' = \frac{R_2}{5}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \quad R_4' = R_3 - R_4$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = R_1 - 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -6 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = 3R_3 - R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -9 & -4 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equation

$$x_2 - 9x_5 = -4$$
$$x_3 + 3x_5 = 1$$
$$x_4 + x_5 = 1$$

let,

$$x_5 = t$$

so, $x_2 - 9t = -4$

$x_3 + 3t = 1$

$x_4 + t = 1$

so, the general solution is,

$$x_2 = -4 + 9t$$
$$x_3 = 1 - 3t$$
$$x_4 = 1 - t$$

4) (i)

Here, $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$$\therefore A^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+1 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+1+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 1+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

Because this is an identity matrix the inverse of this matrix will also be an identity matrix

$$\therefore (A.A^T)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$2x+3y = -1$$
$$3x+2y = 1$$

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A^{-1} = [A \mid I] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] R_1' = \frac{R_1}{2}$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 1/2 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] r_2' = 3r_1 - r_2$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 1/2 & 0 \\ 0 & \frac{5}{2} & \frac{3}{2} & -1 \end{array}\right] \quad r_2' = \frac{2r_2}{5}$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 1/2 & 0 \\ 0 & 1 & \frac{3}{5} & -\frac{2}{5} \end{array}\right] \quad r_1' = r_1 - \frac{3}{2}r_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & -\frac{2}{5} & \frac{3}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{2}{5} \end{array}\right]$$

so, inverse matrix is

$$\begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix}$$

$\therefore$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\cancel{\Rightarrow \begin{bmatrix} \frac{2}{5} + \frac{3}{5} \\ \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + \frac{3}{5} \\ -\frac{3}{5} - \frac{2}{5} \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(Ans)