Arithmetic Co-Transformations in the Real and Complex Logarithmic Number Systems

Mark G. Arnold, Thomas A. Bailey, John R. Cowles, and Mark D. Winkel

Abstract—The real logarithmic number system, which represents a value with a sign bit and a quantized logarithm, can be generalized to create the complex logarithmic number system, which replaces the sign bit with a quantized angle in a log/polar coordinate system. Although multiplication and related operations are easy in both real and complex systems, addition and subtraction are hard, especially when interpolation is used to implement the system. Both real and complex logarithmic arithmetic benefit from the use of co-transformation, which converts an addition or subtraction from a region where interpolation is expensive to a region where it is easier. Two co-transformations that accomplish this goal are introduced. The first is an approximation based on real analysis of the subtraction logarithm. The second is based on simple algebra that applies for both real and complex values and that works for both addition and subtraction.

Index Terms—Arithmetic co-transforamtions, logarithmic number systems, complex logarithms.

1 Introduction

The real logarithmic number system (also known as LNS [32], exponential floating point [26], CRD [7], sign/logarithm number system [29], and Gaussian logarithmic arithmetic [11]) uses a finite approximation for the logarithm of the absolute value of a real and a bit for the sign of that real to represent that value. From a theoretical standpoint, the numerical properties of this number system are, for the most part, an idealization [17] of floating point. It has been shown that floating point and real logarithmic arithmetic are extreme cases in a spectrum of "semi-logarithmic number systems" [20], and logarithmic representation has been proposed as part of a new standard for "composite arithmetic" [12]. From a practical standpoint, the real logarithmic number system has a tremendous advantage over floating point: Multiplication and division can be implemented using low cost fixed point addition and subtraction.

The difficulty faced by an implementor deals with addition and, especially, subtraction [2] of values represented as logarithms. These two operations require computing functions that represent incrementation and decrementation of the corresponding real values. For example, given a fixed point logarithm, Z_L, of a positive real value, Z, the system must compute the *addition logarithm*,

$$S_b(Z_L) = \log_b\left(1 + b^{Z_L}\right), \tag{1}$$

in order to obtain the representation of $Z + 1$. For low precision systems, these functions can be precomputed and placed in a read only memory (ROM). For systems requiring precision approaching that of single precision IEEE 754 floating point (23 bits), interpolation [15], [3] may be used to approximate these functions. For real (i.e., fixed point) Z_L, the $S_b(Z_L)$ function is well suited for linear or higher order interpolation because the extrema of its second and higher order derivatives are reasonably small. On the other hand, the function used for subtraction, known as the *subtraction logarithm*,

$$D_b(Z_L) = \log_b\left|1 - b^{Z_L}\right|, \tag{2}$$

has a singularity at zero and, so, it is costly to interpolate.

Several approaches for high precision real valued logarithmic subtraction have been proposed. Stouraitis [27] proposed partitioning the ROM at powers of two into increasingly denser intervals as Z_L approaches zero. Lewis [14] fabricated a logarithmic arithmetic unit (with 20 bits of precision) that combines partitioning with interpolation to achieve a constant fixed point accuracy for $D_b(Z_L)$ as Z_L approaches zero. A similar combination of partitioning and interpolation was disclosed independently in [5]. Arnold et al. [1] proposed that, due to catastrophic cancellation that occurs during floating point subtraction, the required accuracy for $D_b(Z_L)$ interpolation diminishes as Z_L approaches zero. Lewis [16] fabricated a logarithmic arithmetic unit (with 23 bits of precision) using second order interpolation that capitalizes on this diminished required accuracy. Paliouras and Stouraitis [24] suggested computing $\log_b((1 - b^{Z_L}) / Z_L) + \log_b(Z_L)$ for Z_L near zero instead of directly interpolating $D_b(Z_L)$, but the drawback of this approach is that it requires interpolating for two functions instead of one. Orginos et al. [23] avoided the problem by converting to fixed point all but the maximum operand in a multi–operand subtraction; however, this also requires interpolating more than one function.

An alternative approach is to rearrange a series of computations that involves a reasonable proportion of subtraction in

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order to reduce the number of subtractions that must be performed. For example, if $X_1, X_2, \ldots, X_n$ are positive real numbers, the computation $(((\cdots(((X_1 - X_2) + X_3) - X_4)\cdots) + X_{n-1}) - X_n)$ can be rearranged as $(X_1 + X_3 + \cdots + X_{n-1}) - (X_2 + X_4 + \cdots + X_n)$. Arnold et al. [1] proposed a redundant logarithmic number system which represents both a positive and negative component of each real value. This redundant representation often becomes ill conditioned and loses considerable accuracy compared to the conventional real valued logarithmic number system. It also doubles the storage requirements. For these reasons, the redundant logarithmic number system is only suitable in specialized applications where the designer has detailed knowledge of the nature of the computations to be performed.

Of course, real logarithmic arithmetic was widely used in manual computation for over three centuries until the widespread adoption of digital electronics. Despite the ubiquity of fixed and floating point hardware available today, real logarithmic arithmetic has found recent practical application primarily in two areas: low precision applications running on specialized hardware (image and digital signal processing [30], graphics [13], aircraft controls [25], hearing aids [21], and neural nets [6]) and higher precision applications running on microprocessors that lack floating point hardware [10], such as the Fujitsu SparcLite 930 [28]. Sections 4 and 6 describe two new *co-transformation* techniques for real valued logarithmic subtraction that improve its cost effectiveness while maintaining reasonable accuracy.

Many of the practical applications of low precision logarithmic arithmetic involve complex numbers, for instance, the Fast Fourier Transform (FFT). The conventional way of implementing complex arithmetic is to work with pairs of real numbers that represent points in a rectangular coordinate system. Multiplication of two complex numbers, X and Y, in the rectangular coordinate system requires separate computation of $\Re[X] \cdot \Re[Y] - \Im[X] \cdot \Im[Y]$ and $\Re[X] \cdot \Im[Y] + \Im[X] \cdot \Re[Y]$, where $\Re[X]$ is the real part of X and $\Im[X]$ is the imaginary part. This approach can be used regardless of the underlying implementation of the real addition, subtraction, and multiplications. Swartzlander et al. [31] compared floating point, fixed point, and real logarithmic number systems for an FFT implemented with rectangular coordinate complex arithmetic. Section 9 describes a new number system, known as the complex logarithmic number system, which represents each complex point in log/polar coordinates. Co-transformation techniques similar to those for the real logarithmic number system are useful for practical implementation of the complex logarithmic number system.

2 The Real Logarithmic Number System

In Sections 2 through 8, upper case variables and functions are used for real values and lower case variables are used for integer data. If $X \neq 0$ and $Y \neq 0$ are real numbers, the following identities hold when the transcendental S_b, D_b, and $\log_b$ functions are computed *exactly*:

$$\log_b(|X| \cdot |Y|) = \log_b|X| + \log_b|Y| \tag{3}$$

$$\log_b(|X| / |Y|) = \log_b|X| - \log_b|Y| \tag{4}$$

$$\log_b(|X| + |Y|) = \log_b(|Y| \cdot (|X| / |Y| + 1)) = \log_b|Y| + S_b(\log_b|X| - \log_b|Y|) \tag{5}$$

$$\log_b(||X| - |Y||) = \log_b(|Y| \cdot (||X| / |Y| - 1|)) = \log_b|Y| + D_b(\log_b|X| - \log_b|Y|), \tag{6}$$

where $b > 0$ is the base of the logarithms and S_b and D_b are defined in (1) and (2). In typical modern implementations, $b = 2$ and, in the nineteenth-century literature [11], $b = 10$. In an actual implementation, the transcendental functions cannot be computed exactly since $\log_b|X|$ will be stored in a fixed size word. The computation of $\log_b|X|$ can be thought of as an input conversion, analogous to the conversion of a decimal input to IEEE 754 floating point and, so, the error is related to the number of bits allowed in the representation. To understand this error, one must describe the effects of input quantization that occur when a real value, X, is converted to a logarithmic representation, x, composed of an integer, x_L and sign bit, x_θ:

$$x_\theta = \begin{cases} 0 & \text{if } X \geq 0 \\ 1 & \text{if } X < 0 \end{cases} \tag{7}$$

and

$$x_L = o_L\left(\left\lfloor \frac{\log_b|X|}{\Delta_L} \right\rfloor\right) = o_L\left(\left\lfloor \frac{X_L}{\Delta_L} \right\rfloor\right) = o_L\left(\left\lfloor \log_b|X| \right\rfloor\right), \tag{8}$$

where overflow and underflow are dealt with by

$$o_L(x) = \begin{cases} m_L & \text{if } x > m_L \\ x & \text{if } -m_L \leq x \leq m_L \\ -m_L & \text{if } x < -m_L \end{cases} \tag{9}$$

and where Δ_L and m_L are chosen so that $B = b^{\Delta_L}$ is the closest real value larger than 1.0 that can be represented exactly ($x_L = 1$) and B^{m_L} is the largest real value that can be represented exactly ($x_L = m_L$). Therefore, $-\log_2(\Delta_L)$ is roughly the number of bits of precision. We introduce B as a notational simplification which may be omitted in an actual implementation. Use of base B with the integer x_L is equivalent to the use of base b with the fixed point, $x_L \cdot \Delta_L$. The latter has $-\log_2(\Delta_L)$ bits after the binary point, as is often described for implementations in the literature. For instance, $B = 1.0000000826$ and $\Delta_L = 2^{-23}$ are roughly equivalent to IEEE 754 single precision.

Given the real logarithmic representations, x and y, the result, z, of division is:

$$z_\theta = (x_\theta - y_\theta) \bmod m_\theta \tag{10}$$
$$z_L = o_L(x_L - y_L)$$

where $m_\theta = 2$ for the real logarithmic number system and, so, the modulo two subtraction can be implemented simply with an exclusive OR. When x and y are exact representations, (10) is exact. If x and y are not exact, (10) propagates the error analogously to floating point. A similar situation applies to multiplication, except the "–"s are replaced with "+"s. Negation is implemented as multiplication by the representation of minus one ($y_\theta = 1$, $y_L = 0$).

The algorithm for real logarithmic addition and subtraction requires first computing z, the representation of the

ratio of the numbers, as shown in (10). The representation, r, of the sum is:

$$r_\theta = f_\theta(y, z) \bmod m_\theta \qquad r_L = o_L(f_L(y, z)) \tag{11}$$

where

$$f_\theta(y,z) = \begin{cases} (y_\theta + z_\theta) \bmod 2 & \text{if } z_L > 0 \\ y_\theta & \text{if } z_L \le 0 \end{cases} \tag{12}$$

and

$$f_L(y,z) = \begin{cases} y_L + d_B(z_L) & \text{if } z_\theta = 1 \\ y_L + s_B(z_L) & \text{if } z_\theta = 0 \end{cases} \tag{13}$$

We will use the notation $f(y, z)$ to describe computing the quantized logarithmic representation, $(f_L(y, z), f_\theta(y, z))$, of the sum, $X + Y$. Note $f_\theta(y, z)$ is either x_θ or y_θ, depending on whether $|X| > |Y|$ or $|X| \le |Y|$. Subtraction is implemented as negation followed by addition. Equation (13) uses the quantized addition logarithm,

$$s_B(z_L) = \lfloor S_B(z_L) + E_s(z_L) \rfloor = \lfloor S_B(z_L) \rfloor + e_s(z_L), \tag{14}$$

and the quantized subtraction logarithm,

$$d_B(z_L) = \lfloor D_B(z_L) + E_d(z_L) \rfloor = \lfloor D_B(z_L) \rfloor + e_d(z_L), \tag{15}$$

where $e_s(z_L)$ and $e_d(z_L)$ are quantization errors determined by the approximation method(s) used. The problem addressed in Sections 4 and 6 is that given equal resources for interpolation, $\max|e_d| \gg \max|e_s|$.

3 Co-Transformations in Computer Arithmetic

Similarly to Chen [9], we define a co-transformation of two values Y_k and Z_k as:

$$Y_{k+1} = U_k(Y_k, Z_k) \tag{16}$$

and, simultaneously,

$$Z_{k+1} = V_k(Y_k, Z_k), \tag{17}$$

where U_k and V_k are functions chosen to preserve some approximate relationship:

$$F(Y_k, Z_k) \approx F(Y_{k+1}, Z_{k+1}). \tag{18}$$

The function F is chosen so that one or more applications of the co-transformation will yield a meaningful result. For a co-transformation to have practical utility in computer arithmetic, the functions U_k and V_k should be relatively economical to implement, and the result of n applications of the co-transformation should make $F(X_{k+n}, Y_{k+n})$ "closer" to a desired goal than the original $F(X_k, Y_k)$. In Sections 4, 6, 8, and 10, we introduce novel co-transformations that are specifically designed to make computation of S_b and D_b easier.

Some simple co-transformations that reduce the cost of implementing logarithmic arithmetic are well known, such as that for commutativity. As we will do for the novel co-transformations described later, we begin the discussion of this trivial commutativity co-transformation by giving analytical and/or algebraic background. Then, we use this background to derive the actual co-transformation found in the implementation.

In this trivial example, we start with the commutativity of X and Y for real addition, $X + Y = Y + X$. With (5) in mind, commutativity implies $X \cdot (1 + Y/X) = Y \cdot (1 + X/Y)$ for $X \neq 0$ and $Y \neq 0$. In a quantized implementation, this is equivalent to $f(x, \tilde{z}) = f(y, z)$, where $\tilde{z}$ represents the reciprocal of the value represented by z, in other words, $\tilde{z}_L = -z_L$ and $\tilde{z}_\theta = (-z_\theta) \bmod 2$. Because of the properties of modulo two arithmetic, $\tilde{z}_\theta = z_\theta$.

With the algebraic background above, we are ready to define the well known co-transformation that reduces the table size in half for logarithmic addition and subtraction:

$$\begin{aligned} u_L(y,z) &= \begin{cases} y_L & \text{if } z_L \ge 0 \\ y_L + z_L & \text{if } z_L < 0 \end{cases} \\ u_\theta(y,z) &= \begin{cases} y_\theta & \text{if } z_L \ge 0 \\ (y_\theta + z_\theta) \bmod m_\theta & \text{if } z_L < 0 \end{cases} \\ v_L(y,z) &= |z_L| \\ v_\theta(y,z) &= \begin{cases} z_\theta & \text{if } z_L \ge 0 \\ (-z_\theta) \bmod m_\theta & \text{if } z_L < 0 \end{cases} \end{aligned} \tag{19}$$

There are no preconditions for applying the commutativity co-transformation. The postcondition guaranteed as a result of applying it is $v_L(y, z) \ge 0 \wedge f(y, z) = f(u(y, z), v(y, z))$. Note that $v_\theta(y, z) = z_\theta$ because of the properties of modulo two arithmetic. From (19), it is easy to show that $f(y, z) = f(u(y, z), v(y, z))$ because

$$u(y,z) = \begin{cases} y & \text{if } z_L \ge 0 \\ x & \text{if } z_L < 0 \end{cases} \qquad v(y,z) = \begin{cases} z & \text{if } z_L \ge 0 \\ \tilde{z} & \text{if } z_L < 0 \end{cases}$$

This co-transformation is only applied once. The effect is to insure that s_B and d_B are only computed for positive arguments, effectively reducing the implementation cost by one half. To those familiar with previous real logarithmic number system implementations, this derivation may seem a bit tortured, but it will be useful in Section 10 when these ideas are generalized to complex values. In a real valued implementation, the determination of the sign, $f_\theta(u(y, z), v(y, z))$ can be simplified considerably.

4 Co-Transformation Derived from a Truncated Series

As the bibliography of [2] shows, the following series has been given in the literature several times for real Z_L:

$$D_b(Z_L) = \log_b|Z_L| - \log_b(\log_b e) + \frac{Z_L}{2} + \frac{Z_L^2 \cdot \ln b}{24} - \frac{Z_L^4 \cdot (\ln b)^3}{2880} + \frac{Z_L^6 \cdot (\ln b)^5}{181440} - \cdots.$$

However, we will use (4) in a novel way to derive an approximate co-transformation that converts certain logarithmic subtraction (d_B, which is costly for interpolation near the singularity) to logarithmic addition (s_B, which is much cheaper for interpolation). For Z_L near zero [11], [22],

$$D_B(Z_L) \approx \log_b|Z_L| - \log_b(\log_b e) + \frac{Z_L}{2}, \qquad (20)$$

and the error is bounded by $Z_L^2 \cdot \ln b/24$. As described in [1], the required accuracy (due to unavoidable catastrophic cancellation) diminishes as Z_L approaches zero:

$$E_d(Z_L) \approx C/Z_L, \qquad (21)$$

where C is a constant which most naturally would be $\Delta_L \cdot \log_b e$. By solving for $(Z_L^2 \cdot \ln b)/24 < (\Delta_L \cdot \log_b e)/Z_L$, we find that, when $Z_L < \sqrt[3]{24 \cdot (\log_b e)^2 \cdot \Delta_L}$, the truncated series produces less error than is inherent in the subtraction being implemented by this approximation. For example, for 23 bits of precision, $Z_L < 0.0181255$.

The constant $-\log_b(\log_b e)$ and the linear term, $Z_L/2$, are trivial to implement. The only difficulty is $\log_b(Z_L)$. Let $g = \lfloor -\log_2(24 \cdot (\log_b e)^2 \cdot \Delta_L)/3 \rfloor$ be the number of bits of Z_L after the binary point that are zero when this truncated series is used. Suppose there are $2n$ additional bits in the fixed point Z_L. It is then possible to break Z_L apart into two n bit integers, z_1 and z_2, such that $Z_L = 2^{-g-n} \cdot z_1 + 2^{-g-2n} \cdot z_2$ and $\Delta_L = 2^{-g-2n}$. For example, with 23 bit precision, $g = 5$ and $n = 9$. Applying (5), we have:

$$\begin{aligned}
\log_b(Z) &= \log_b\left(2^{-g-n} \cdot z_1 + 2^{-g-2n} \cdot z_2\right) \\
&= \log_b\left(2^{-g-2n} \cdot z_2\right) + S_b\left(\log_b\left(2^{-g-n} \cdot z_1\right)\right. \\
&\quad \left. - \log_b\left(2^{-g-2n} \cdot z_2\right)\right) \\
&= -\log_b\left(2^{g+2n}\right) + \log_b(z_2) + S_b\left(\log_b(z_1)\right. \\
&\quad \left. - \log_b(z_2) + \log_b\left(2^n\right)\right). \qquad (22)
\end{aligned}$$

Since n is a small integer, the size of the lookup table for $\log_b(z_1)$ and $\log_b(z_2)$ is affordable.

The real analytical background above justifies the co-transformation described below which assumes the quantized addition logarithm, s_B, is used rather than the exact S_b. Furthermore, it assumes that the commutativity co-transformation described in (19) has already been applied so that $z_L > 0$. When $z_L \cdot \Delta_L < 2^{-g}$ (equivalently, $z_L < 2^{2n}$), $z_L = 2^n \cdot z_1 + z_2$. From this, we can define the co-transformation:

$$\begin{aligned}
u_L(y,z) &= \begin{cases} y_L & \text{if } z_\theta = 0 \vee z_L \geq 2^{2n} \\ c_2 + o_L\left(\left\lceil \log_B(z_2) \right\rceil + \left\lceil \frac{z}{2} \right\rceil + y_L\right) & \text{if } z_\theta = 1 \wedge z_L < 2^{2n} \end{cases} \\
u_\theta(y,z) &= y_\theta \\
v_L(y,z) &= \begin{cases} z_L & \text{if } z_\theta = 0 \vee z_L \geq 2^{2n} \\ o_L\left(\left\lceil \log_B(z_1) \right\rceil\right) - o_L\left(\left\lceil \log_B(z_2) \right\rceil\right) & \text{if } z_\theta = 1 \wedge z_L < 2^{2n} \\ + \; c_1 - c_2 & \end{cases} \\
v_\theta(y,z) &= \begin{cases} z_\theta & \text{if } z_\theta = 0 \vee z_L \geq 2^{2n} \\ 0 & \text{if } z_\theta = 1 \wedge z_L < 2^{2n} \end{cases} \qquad (23)
\end{aligned}$$

where

$$c_1 = -\lfloor \log_B(2^{g+n}) - \log_B(\log_B e) \rfloor,$$

$$c_2 = -\lfloor \log_B(2^{g+2n}) - \log_B(\log_B e) \rfloor,$$

and, so, $c_1 - c_2 = \lfloor \log_B(2^n) \rfloor = n \cdot 2^{g+2n}$ and $c_2 = (g + 2^n) \cdot 2^{g+2n} - \lfloor \log_B(\log_B e) \rfloor$ for $b = 2$. Note that (23) insures that $v_L(y, z) > 0$ for $z_L > 0$. The precondition for applying this series co-transformation is that the commutativity co-transformation has already been applied, i.e., $z_L \geq 0$. The postcondition that is guaranteed as a result of applying it under these circumstances is $(v_L(y, z) \geq 2^{2n} \vee (v_\theta(y, z) = 0 \wedge v_L(y, z) \geq 0)) \wedge f(y, z) \approx f(u(y, z), v(y, z))$.

5 Implementation of Series Co-Transformation

The terms $o_L(\lfloor \log_B(z_1) \rfloor)$ and $o_L(\lfloor \log_B(z_2) \rfloor)$ in (23) can be implemented with one small (2^n word) ROM containing $-m_L$ as an approximation for $-\infty$ in its first word. Such an approach assumes that two or more clock cycles may be spent computing $v_L(y, z)$, as would be acceptable in a software implementation. An advantage that (23) has for software implementation is that, unlike hardware partitioning schemes [16], [5], the point of separation between z_1 and z_2 is fixed and, therefore, easier to implement on a processor that lacks normalization hardware. The address calculation for a fully partitioned d_B table involves normalization of z_L, as does Paliouras' approach [24]. Although some high performance CISC processors have instructions that find the position of the leading bit in a word, on many processors that lack floating point this can only be accomplished by a loop that shifts z_L while maintaining a count. There is often a penalty associated with loops on such processors. For these processors, co-transformation software may be faster than full partitioning of d_B since the table addresses can be computed without a loop.

A real logarithmic arithmetic software package [10] that utilizes this co-transformation was implemented in 80 × 86 assembly language for 23 bit precision ($g = 6$, $n = 9$).

A slight variation of (23) allows for a hardware implementation [5] that computes $v_L(y, z)$ in one clock cycle by storing $o_L(\lfloor \log_B(z_1) \rfloor) + c_1$ and $o_L(\lfloor \log_B(z_2) \rfloor) + c_2$ in two separate ROMs.

6 Algebraic Co-Transformation

There is an alternative co-transformation that achieves a similar effect to the one described in the previous section, but which has many additional desirable properties. Rather than being based on analysis of a series expansion, as in the last section, this alternative co-transformation is derived from simple algebra.

If $Z_L > 0$ is the fixed point argument to the real S_b or D_b functions, there are many ways we can select two positive fixed point numbers, Z_1 and Z_2 such that $Z_L = Z_1 + Z_2$. Equivalently, there are corresponding ways to select the integers, z_1 and z_2, such that $z_L = z_1 + z_2$. Unlike the previous co-transformation (23), the co-transformation derived below does not depend on any particular choice of Z_1 and Z_2.

$$
\begin{aligned}
D_b(Z_L) &= \log_b\left|1-b^{Z_L}\right| \\
&= \log_b\left|1-b^{Z_1+Z_2}\right| \\
&= \log_b\left|1-b^{Z_2}+\left(b^{Z_2}-b^{Z_1}\cdot b^{Z_2}\right)\right| \\
&= \log_b\left|1-b^{Z_2}+b^{Z_2}\cdot\left(1-b^{Z_1}\right)\right| \\
&= \log_b\left|1-b^{Z_2}+b^{Z_2}\cdot b^{\log_b\left(1-b^{Z_1}\right)}\right| \\
&= \log_b\left|1-b^{Z_2}+b^{Z_2}\cdot b^{D_b(Z_1)}\right| \\
&= \log_b\left|1-b^{Z_2}+b^{Z_2+D_b(Z_1)}\right| \\
&= \log_b\left|1-b^{Z_2}+\frac{1-b^{Z_2}}{1-b^{Z_2}}\cdot b^{Z_2+D_b(Z_1)}\right| \\
&= \log_b\left|\left(1-b^{Z_2}\right)\cdot\left(1+\frac{b^{Z_2+D_b(Z_1)}}{1-b^{Z_2}}\right)\right| \\
&= \log_b\left|b^{\log_b\left|1-b^{Z_2}\right|}\cdot\left(1+\frac{b^{Z_2+D_b(Z_1)}}{b^{\log_b\left|1-b^{Z_2}\right|}}\right)\right| \\
&= \log_b\left|b^{D_b(Z_2)}\cdot\left(1+\frac{b^{Z_2+D_b(Z_1)}}{b^{D_b(Z_2)}}\right)\right| \\
&= \log_b\left|b^{D_b(Z_2)}\cdot\left(1+b^{Z_2+D_b(Z_1)-D_b(Z_2)}\right)\right| \\
&= \log_b\left|b^{D_b(Z_2)}\right| \\
&\quad+\log_b\left(1+b^{Z_2+D_b(Z_1)-D_b(Z_2)}\right) \\
&= D_b(Z_2)+S_b\left(Z_2+D_b(Z_1)-D_b(Z_2)\right).
\end{aligned}
$$

As a notational convenience, we will define an auxiliary function,

$$H(P, Q, R) = Q + S_b(R + P - Q), \tag{24}$$

and, so,

$$D_b(Z_L) = H(D_b(Z_1), D_b(Z_2), Z_2) \tag{25}$$

is an identity that holds when H is computed *exactly*. Similarly, we define a quantized auxiliary function,

$$h(p, q, r) = q + s_B(r + p - q), \tag{26}$$

and, so,

$$d_B(z) \approx h(d_B(z_1), d_B(z_2), z_2). \tag{27}$$

Analogously to (15), we can define the error in (27) as $e_h(z_L)$. Since $d_B(z_1)$ and $d_B(z_2)$ are implemented by table lookup, $0 \le e_d(z_1) \le 1$ and $0 \le e_d(z_2) \le 1$. Since $0 \le S_B' \le 1$,

$$
\begin{aligned}
\left|e_h(z_L)\right| &= h\left(d_B(z_1), d_B(z_2), z_2\right) - \left\lfloor D_b\left(z_L \cdot \Delta_L\right)\right\rfloor \\
&\le e_d(z_2) + S_B' \cdot \left(e_d(z_1) - e_d(z_2)\right) \\
&\quad + e_s\left(z_2 + d_B(z_1) - d_B(z_2)\right) \\
&\le 1 + e_s\left(z_2 + d_B(z_1) - d_B(z_2)\right).
\end{aligned}
$$

Using (27) to approximate d_B introduces no more than one additional machine unit of relative error (perceived by the end user as a relative error of $B - 1$) beyond whatever error the s_B approximation method introduced. An exhaustive simulation for 23 bits of precision was implemented whose results agree with the above analysis. In this simulation, the error in the s_B approximation was modeled as a random variable, e_s, uniformly distributed between 0 and a chosen maximum. For all quantized values, z_L, in the range of interest ($1 \le z_L < 2^{2n}$), the result of (27) was compared with the result of (2), computed using double precision floating point arithmetic for 10 randomly chosen values of e_s. With over two million values tested, in no case was the error of the approximation (27) ever observed to be greater than $1 + e_s$. Therefore, (27) is a much more accurate approximation of the subtraction logarithm than (20) in most cases.

There are many possible variations of co-transformations that can be derived from (27). For example, the one which is most analogous to (23) is:

$$
\begin{aligned}
u_L(y, z) &= \begin{cases} y_L & \text{if } z_\theta = 0 \vee z_L \ge 2^{2n} \\ y_L + d_B(z_2) & \text{if } z_\theta = 1 \wedge z_L < 2^{2n} \end{cases} \\
u_\theta(y, z) &= y_\theta \\
v_L(y, z) &= \begin{cases} z_L & \text{if } z_\theta = 0 \vee z_L \ge 2^{2n} \\ z_2 + d_B(z_1) & \text{if } z_\theta = 1 \wedge z_L < 2^{2n} \\ -d_B(z_2) & \end{cases} \\
v_\theta(y, z) &= \begin{cases} z_\theta & \text{if } z_\theta = 0 \vee z_L \ge 2^{2n} \\ 0 & \text{if } z_\theta = 1 \wedge z_L < 2^{2n} \end{cases}
\end{aligned} \tag{28}
$$

The choice of n in (28) is arbitrary. The hardware is similar to that in the last section, except there is no need to add $z/2$. The preconditions and postconditions for this form (28) of the algebraic co-transformation are the same as for the series co-transformation (23).

There is no limitation on the size of Z_L in (25) and, so, there is no limitation that restricts the number of bits in z_1 or z_2, which means z_1 and z_2 could comprise all of z. So, (27) may be used to derive a simpler co-transformation that converts every logarithmic subtraction into a logarithmic addition:

$$
\begin{aligned}
u_L(y, z) &= \begin{cases} y_L & \text{if } z_\theta = 0 \\ y_L + d_B(z_2) & \text{if } z_\theta = 1 \end{cases} \\
u_\theta(y, z) &= y_\theta \\
v_L(y, z) &= \begin{cases} z_L & \text{if } z_\theta = 0 \\ z_2 + d_B(z_1) - d_B(z_2) & \text{if } z_\theta = 1 \end{cases} \\
v_\theta(y, z) &= 0.
\end{aligned} \tag{29}
$$

There are no preconditions for this alternate form (29) of the algebraic co-transformation. The postcondition is $v_\theta(y, z) = 0 \wedge f(y, z) \approx f(u(y, z), v(y, z))$. Equation (29) eliminates the

need to interpolate for d_B at all if the table sizes for $d_B(z_1)$ and $d_B(z_2)$ are considered acceptable.

7 Iterative Co-Transformations

The function H can be applied iteratively to reduce these table sizes. For example, if $Z_L = Z_1 + Z_2 + Z_3$,

$$\begin{aligned} D_b(Z_L) &= H(D_b(Z_1 + Z_2), D_b(Z_3), Z_3) \\ &= H(H(D_b(Z_1), D_b(Z_2), Z_2), D_b(Z_3), Z_3). \end{aligned} \quad (30)$$

Software [10] that approximates $D_b(Z_L)$ using (30) was implemented for 23 bits of precision, where there are 10 bits in Z_1, and nine bits each in Z_2 and Z_3. These 28 bits are sufficient to approximate the entire range of D_b within the 32-bit word because of the essential zero concept [32].

When (25) is carried to its logical conclusion, each bit of Z_L could be processed separately in an iterative series of co-transformations,

$$U_k(Y, Z) = \begin{cases} H\left(Y, D_b\left(2^{k-n}\right), 2^{k-n}\right) & \text{if } Z > 2^{k-n} \\ Y & \text{if } Z \le 2^{k-n} \end{cases} \quad (31)$$

and

$$V_k(Y, Z) = \begin{cases} Z - 2^{k-n} & \text{if } Z > 2^{k-n} \\ Z & \text{if } Z \le 2^{k-n} \end{cases} \quad (32)$$

where, now, n is the number of fractional bits, $F(Y, Z) = Z - S_b(Y)$, $Z_0 = Z_L$, and $Y_0 = -\infty$. When, after j iterations, $Z_j = 0$, we have $Y_j \approx D_b(Z_L)$.

8 Addition Co-Transformations

There are relationships analogous to (25) that describe how to convert a particular logarithmic addition into another, possibly simpler, logarithmic addition:

$$S_b(Z_L) = H(S_b(Z_1), D_b(Z_2), Z_2) \quad (33)$$

and

$$S_b\left(Z_L\right) = H\left(S_b^{-1}\left(Z_1\right), S_b\left(Z_2\right), Z_2\right), \quad (34)$$

where $S_b^{-1}(Z_1) = D_b(Z_1)$ because Z_1 is real.

Assuming the commutativity co-transformation (19) and the algebraic co-transformation (29) have already been applied, a third co-transformation, derived from (33), can, under certain circumstances, eliminate the need for any interpolation. The precondition for applying the following addition co-transformation is that the commutativity and algebraic co-transformations have already been applied, i.e., $z_L \ge 0 \wedge z_\theta = 0$:

$$\begin{aligned} u_L(y, z) &= y_L + d_B\left(z_2\right) \\ u_\theta(y, z) &= y_\theta \\ v_L(y, z) &= z_2 + s_B\left(z_1\right) - d_B\left(z_2\right) \\ v_\theta(y, z) &= 0. \end{aligned} \quad (35)$$

where z_1 contains the high order bits of z_L and z_2 contains the low order bits of z_L. The postcondition that is guaranteed as a result of applying the addition co-transformation given the above preconditions is $v_L(y, z) >> 0 \wedge f(y, z) \approx f(u(y, z), v(y, z))$. For $b = 2$, $v_L(y, z) \cdot \Delta_L$ is at least equal to the number of fractional bits[1] in Z_1, which means (13) can be computed more easily. In particular, for 23 bits of precision, the s_B in (13) can be approximated without interpolation using the same table that gives $s_B(z_1)$ when z_2 contains 12 bits. If z_2 contains more bits,[2] interpolation would be required, but would use a much smaller multiplier than if (35) had not been applied.

9 Complex Logarithmic Number System

There is a natural generalization of the real logarithmic number system that allows representation of complex values. Instead of allocating a single bit for the sign, one can allocate an appropriate number of bits to represent an angle in the complex plane. To convert a complex value, X, to a quantized complex logarithmic representation, $x = x_L + x_\theta \cdot i$, composed of the quantized logarithm of the length of a vector in the complex plane, x_L, and the quantized angle of that vector, x_θ:

$$x_\theta = \left\lfloor \frac{\arctan\left(\Re[X], \Im[X]\right)}{2\pi} \cdot m_\theta \right\rfloor \bmod m_\theta \quad (36)$$

and

$$\begin{aligned} x_L &= o_L\left(\left\lfloor \frac{0.5 \cdot \log_b\left(\Re[X]^2 + \Im[X]^2\right)}{\Delta_L} \right\rfloor\right) \\ &= o_L\left(\left\lfloor \log_B[X] \right\rfloor\right), \end{aligned} \quad (37)$$

where $i = \sqrt{-1}$ and $\arctan(\Re[X], \Im[X])$ returns an angle between $-\pi$ and π. This four quadrant arctan function handles all cases such as $\Re[X] = 0$ or $\Im[X] = 0$. The constant m_θ determines the precision with which the angle is represented. Obviously, when $m_\theta = 2$, this is equivalent to (7) and (8), but to represent imaginary numbers, $m_\theta \bmod 4 = 0$. Multiplication and division, as in (10), generalize without difficulty. The conjugate can be formed by negating $x_\theta \bmod m_\theta$.

It is instructive to compare our proposed complex representation to those that are commonly used. Fig. 1 shows the representable points in a small fixed point complex number system, similar to what might be used in Digital Signal Processing (DSP) applications, such as the FFT. The precision and range of Fig. 1 are too limited to be of much practical value, but larger systems would be similar in appearance. Often in DSP problems, it is known that all values computed by the hardware will lie within a circle whose radius is known a priori. This a priori bound is quite different than one typically finds in general purpose computation, where the dangers of overflow cannot be so easily excluded [33]. For example, a 256 point radix two FFT whose inputs lie within the unit circle only needs a number system that can represent $-256 < |Z| < 256$. In the rectangular fixed point system, this translates into circumscribing the desired circle within a square, $-256 < \Re[Z] < 256$, $-256 <$

1. $v_L(y, z) \cdot \Delta_L > Z_2/2 - \log_b(Z_2)$ from (20), and $-\log_2(Z_2)$ is at least the number of fractional bits in Z_1.
2. So, $Z_1 = z_1 : \Delta_L$ would contain fewer fractional bits.

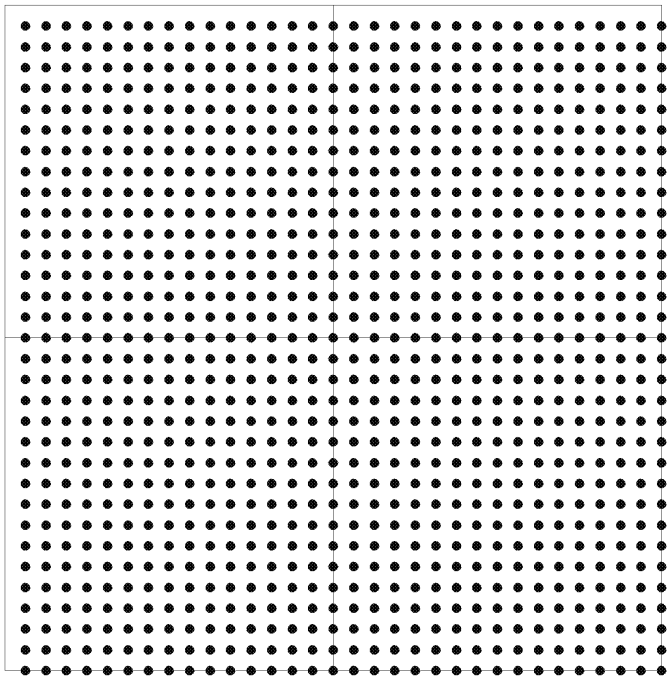

Fig. 1. Rectangular fixed point: 1,024 points circumscribing $|Z| < 16$.

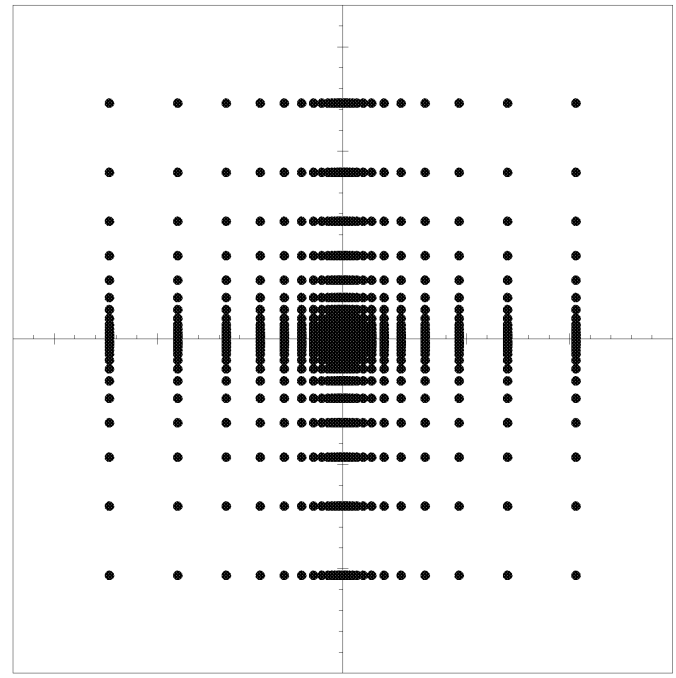

Fig. 2. Rectangular floating point: 1,024 points circumscribing $|Z| < 16$.

$\Im[Z] < 256$. Approximately one quarter $(1 - \pi/4)$ of the fixed point representations are wasted points that will never be used for such an FFT.

Another commonly used representation is a rectangular arrangement of floating point (or real logarithmic) representations. Fig. 2 shows such a complex rectangular floating point system, which has the same number of points and represents the same range of values as Fig. 1. These points are not distributed in the most desirable way. The relative precision of the *complex value* wobbles considerably. The worst precision occurs at angles that are odd multiples of $\pi/2$. The precision near the axis becomes unreasonably high in that dimension while decreasing in the perpendicular dimension. As explained earlier, overflow is not a possibility, but the reciprocal situation, underflow, is a potential problem. Although IEEE 754 diminishes the danger of underflow by including hardware for denormals, Fig. 2 does not include denormals because of the added expense of such hardware. If we take a closer look near zero (Fig. 3), we see there are gaps all along the $\Re[Z]$ and $\Im[Z]$ axes. These gaps indicate underflow may be a serious problem for a floating point representation.

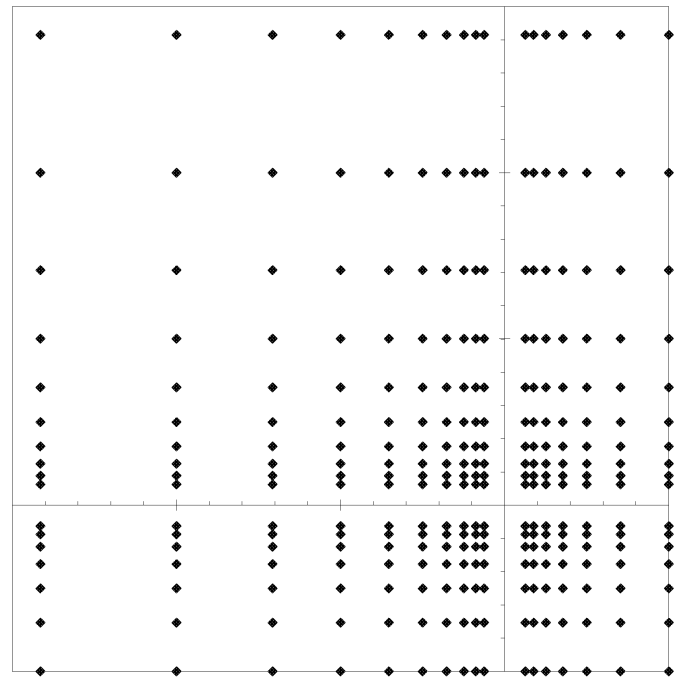

Fig. 3. Close up of Fig. 2 near zero.

The new number system that we propose is ideally suited for problems such as the FFT. Fig. 4 shows a complex logarithmic number system which has the same number of points and which encloses the same circle as Figs. 1 and 2. Because of the polar arrangement of points, there are no regions of wasted points (in contrast to both fixed and floating point, which have points that will never be used). The complex logarithmic number system maintains the same relative precision, regardless of the angle, and, so, the precision does not wobble as it does with floating point. If we take a closer look near zero (Fig. 5), we see underflow is a much less significant problem, because there is only a small circle where underflow would occur. In all respects, the complex logarithmic number system is a much more efficient arrangement of points in the complex plane than either the fixed point or floating point approaches.

The problem, of course, is addition. In 1895, Mehmke [18] described the idea of complex addition logarithms,

$$\Re\left[S_B(Z_C)\right] = 0.5 \cdot \log_b\left(1 + b^{Z_L}\cos(Z_\theta) + b^{2 \cdot Z_L}\right)$$
$$\Im\left[S_B(Z_C)\right] = \arctan\left(1 + b^{Z_L}\cos(Z_\theta), b^{Z_L}\sin(Z_\theta)\right), \quad (38)$$

where $Z_C = Z_L + i \cdot Z_\theta$, $Z_L = z_L \cdot \Delta_L$, and $Z_\theta = 2\pi \cdot z_\theta / m_\theta$, but, to the authors' knowledge, this complex logarithm number

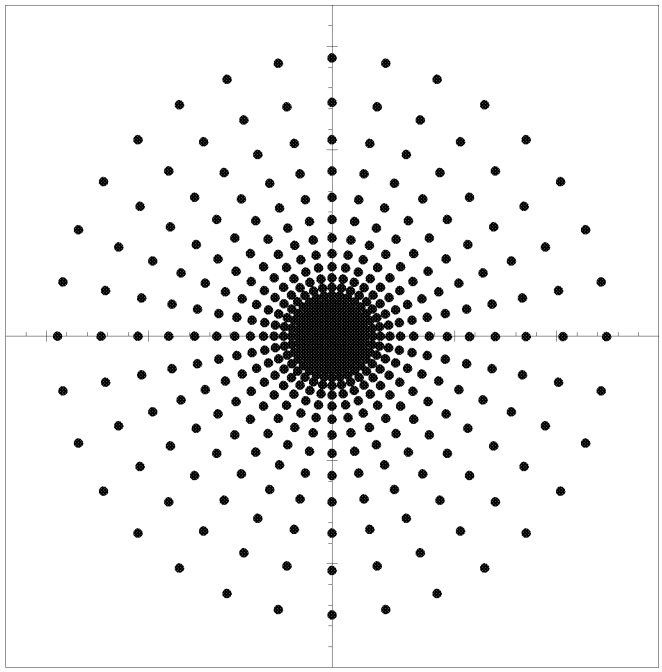

Fig. 4. Complex logarithmic number system: 1,024 points circumscribing $|Z| < 16$.

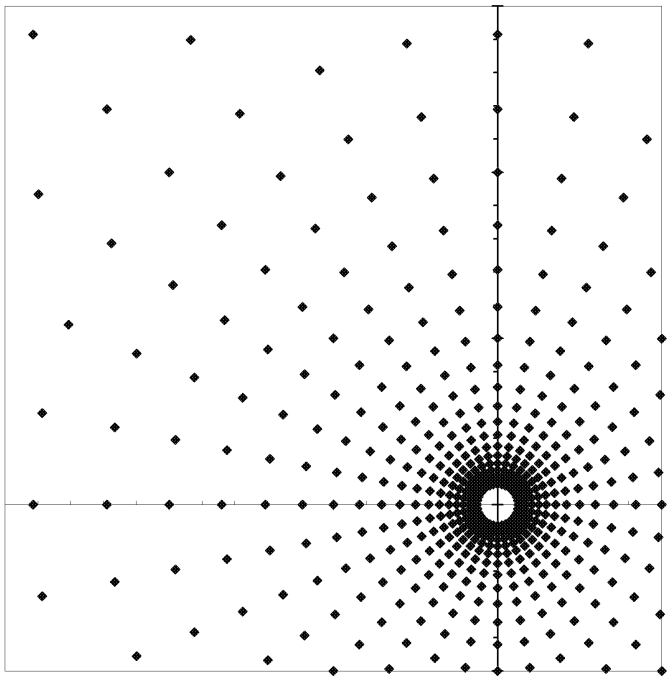

Fig. 5. Close up of Fig. 4 near zero.

system was never analyzed further and has never been used in a modern implementation until now.[3]

Complex relationships, such as

$$S_b(Z_C) = D_b(Z_C + k \cdot \pi i), \tag{39}$$

where k is any odd integer, simplify the description of the addition algorithm, because there is no need to mention the D_b function. Given the complex quantized representations, x and y, of two complex values, X and Y, the addition algorithm is the same as (11), except the definition of (13) and (12) generalize to:

$$f_\theta(y, z) = (y_\theta + \Im[s_B(z)]) \bmod m_\theta \tag{40}$$

$$f_L(y, z) = y_L + \Re[s_B(z)], \tag{41}$$

or, more simply, (40) and (41) can be combined:

$$f(y, z) = y + s_B(z), \tag{42}$$

where $z = z_L + z_\theta i$, and s_B is the quantized complex addition logarithm.

10 Complex Co-Transformation

Prior low precision implementations of the real logarithmic number system have typically used ROM to implement S_b without interpolation. Having both real and imaginary parts of Z_C means the number of ROM address bits required for a noninterpolated complex S_b is almost double what would be required for a noninterpolated real S_b. Such large ROM sizes are unreasonable, even for moderate precision systems and, so, interpolation is important. Furthermore, for values that approach $\pm\pi i$, S_b has the same singularity that D_b has for values that approach zero. Therefore, interpolation of S_b in one half of the complex plane is much more difficult than interpolation of the real S_b function, a fact [18] failed to note. A novel co-transformation, similar to the real ones described earlier, is required to make complex interpolation affordable.

The relationship defined in (34) holds for complex Z_C and, so, it may be used as the basis for such a complex co-transformation. If we let $Z_1 = Z_L$ and $Z_2 = i \cdot Z_\theta$, we have

$$\begin{aligned} S_b(Z_C) &= H\left(S_b^{-1}(Z_L), S_b(i \cdot Z_\theta), i \cdot Z_\theta\right) \\ &= S_b(i \cdot Z_\theta) + S_b\left(i \cdot Z_\theta + S_b^{-1}(Z_L) - S_b(i \cdot Z_\theta)\right) \\ &= S_b(i \cdot Z_\theta) + S_b(T), \end{aligned} \tag{43}$$

where

$$T = i \cdot Z_\theta + \pi i + D_b(Z_L) - S_b(i \cdot Z_\theta) \tag{44}$$

because $S_b^{-1}(Z_L) = D_b(Z_L) + \pi i$. The commutativity co-transformation (19) allows us to insure that $Z_L \geq 0$ prior to applying (43), therefore, consider how T can be simplified:

$$\begin{aligned} \Re[T] &= \Re\left[D_b(Z_L)\right] - \Re\left[S_b(i \cdot Z_\theta)\right] \\ &= D_b(Z_L) \\ &\quad - \frac{\log_b(2) + \log_b\left(1 + \cos(Z_\theta)\right)}{2}, \\ \Im[T] &= Z_\theta + \pi + \Im\left[D_b(Z_L)\right] - \Im\left[S_b(i \cdot Z_\theta)\right] \\ &= Z_\theta + 2\pi - \Im\left[S_b(i \cdot Z_\theta)\right] \\ &= \frac{Z_\theta}{2}. \end{aligned} \tag{45}$$

3. Except for application of polar representation to complex SLI arithmetic [33], which bears some similarity to the complex logarithmic arithmetic described here.

The relationship $\Im[D_b(Z_L)] = \pi$ holds only for $Z_L \geq 0$ but the other [18] identity used above, $\Im[S_b(i \cdot Z_\theta)] = Z_\theta/2$, holds for all $Z_\theta \neq k \cdot \pi$. Because $|Z_\theta| \leq \pi$ in the complex logarithm number system, $|\Im[T]| \leq \pi/2$, which has the desired effect of making interpolation of $S_b(T)$ much easier than interpolation of $S_b(Z_C)$ when $\pi/2 < |Z_\theta| \leq \pi$. Therefore, the following is equivalent to (43) when $Z_L \geq 0$:

$$\Re\left[S_b\left(Z_C\right)\right] = \frac{\log_b(2) + \log_b\left(1 + \cos\left(Z_\theta\right)\right)}{2} + \Re\left[S_b(T)\right]$$
$$\Im\left[S_b\left(Z_C\right)\right] = \frac{Z_\theta}{2} + \Im\left[S_b(T)\right]. \quad (46)$$

If (43) were applied inappropriately, when $Z_L < 0$, it would have the undesirable effect of making $\pi/2 < |\Im[T]| \leq \pi$.

From (46), we can derive a co-transformation that avoids the interpolation difficulties for complex logarithmic addition. Assuming the commutativity co-transformation (19) has already been applied, the following describes the complex addition co-transformation:

$$u_L(y,z) = y_L + \left\lfloor \frac{\log_B\left(1 + \cos\left(z_\theta \cdot 2\pi/m_\theta\right)\right) + \log_B(2)}{2} \right\rfloor$$
$$u_\theta(y,z) = \left(y_\theta + \left\lfloor \frac{z_\theta}{2} \right\rfloor \right) \bmod m_\theta$$
$$u_L(y,z) = d_B\left(z_L\right) - \left\lfloor \frac{\log_B\left(1 + \cos\left(z_\theta \cdot 2\pi/m_\theta\right)\right) + \log_B(2)}{2} \right\rfloor$$
$$v_\theta(y,z) = \left\lfloor \frac{z_\theta}{2} \right\rfloor. \quad (47)$$

The precondition for applying the complex co-transformation is that the commutativity co-transformation has already been applied, i.e., $z_L \geq 0$. The postcondition that is guaranteed as a result of applying the complex co-transformation under these circumstances is $|v_\theta(y, z)| \leq m_\theta/4 \wedge f(y, z) \approx f(u(y, z), v(y, z))$.

11 Coleman's Co-Transformation

After the original version of this paper was submitted, the authors became aware of the work of Coleman [8], who independently discovered a co-transformation that reduces the cost of interpolating the real valued subtraction logarithm. Coleman's technique, unlike those described here and in [5], transforms the problem of computing $D_b(Z_L)$ for real Z_L near zero into interpolation of D_b for an argument further away from zero. All of the co-transformations given here convert cases near the singularity into interpolation of S_b. Coleman's technique is more limited than those given here because a D_b computation is required in every case. For this reason, Coleman's technique would not be as useful for implementing the complex logarithmic number system as the ones described in this paper are.

12 Conclusions

We have shown that the well-known real logarithmic number system is a special case of the more obscure complex logarithmic number system. Both real and complex systems share the common problem that using interpolation near the singularity (as happens when subtracting nearly equal values) produces more error than is acceptable. We have proposed two co-transformations to eliminate this problem. The first is based on the analysis of the subtraction logarithm and the second is based on simple algebra. For 23 bits of precision, co-transformation with the real logarithmic number system produces acceptable results using small tables, and is more suitable for software than the partitioning previously disclosed for hardware.

The iterative co-transformation described in Section 7 bears some similarity to CORDIC and similar algorithms [9], [19]. Further research into this may be warranted.

Co-transformation with the complex logarithmic number system offers a practical approach to implement this interesting system which has not yet been fully explored. In particular, there is a question on how to represent values near zero, for which there may be analogies to [4]. We hope that future investigation into the complex logarithmic number system will discover how useful it is for DSP applications, such as the FFT, that make extensive use of complex arithmetic.

Acknowledgments

The authors wish to thank Somatogen, Inc. for providing support for the writing of this paper. Also, the authors wish to thank Pharlap and Microway.

From 1988 to 1996, two of the authors (Arnold and Winkel) attempted to market a software package [10] that embodies the real algorithms [5] disclosed here. These authors wish to thank the following people who assisted us in this endeavor: David Bazuki (Knowledge Revolution), Robin Bolz (Microsoft), Walter Bright (Symantec), Tom Brightman (Cyrix), Joel Egberg (Wind River Systems), Nat Goldhaber (Apple), Dan Hammerstrom (ASI), Randy Isaac (IBM-Austin Research Lab), Ray and Sabin Larsen (Altra), David Matula (Texas A&M), Tom Newsom (Hewlett-Packard), Everett Roach (Weitek), Jerry Rogers (Cyrix), Larry Silverman (Intel), Earl Swartzlander (Univ. of Texas-Austin), Duncan Terry (Hewlett-Packard), Richard Tompane (3DO), Susheela Vasan (3DO), Don Watt (Hewlett-Packard), and Gideon Yiguel (Microsoft).

During this time, the authors became aware of the significant contributions [26] made to the field by a competitor, Les Pickett (LogPoint Systems, Inc., www.logpoint.com), who developed one of the most widely used commercial applications of logarithmic arithmetic: the cabin air pressure controls for the Boeing 767 and several other aircraft designed since 1977 [25].

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