Prime Number Theorem

John D Mangual

The Wiener-Ikehera Tauberian Theorem implies the Prime Number Theorem, as can be shown in various places.

None of those discussions really explain to me:

- connection to divergent series
- how we can proceed on our own

Norber Weiner's original argument is very easy to follow: Möbius inversion is what gives the series to start:

$$\sum_{m=1}^{\infty} x^m \log m = \sum_{n=1}^{\infty} \Lambda(n) \frac{x^n}{1-x^n}$$

and then set $x = e^{-\xi}$ and $\xi \to 0$ (or $x \to 1$).

The two limiting behaviors are rather diffrent

$$\frac{x^n}{1-x^n} = \begin{cases} 1/n\epsilon & \text{as } x \to 1 \\ \epsilon^n & \text{as } x \to 0 \end{cases}$$

and the cuttoff point is when $(1-\epsilon)^n$ is getting small (near $n = 1/\epsilon$)

$$\sum_{n=1}^{\infty} \Lambda(n) \frac{\xi e^{-n\xi}}{1-e^{-n\xi}} \approx \sum_{n=1}^{1/\xi} \frac{\Lambda(n)}{n}$$

Maybe if we take the derivative of both sides:

$$\sum_{n=1}^{\infty} \Lambda(n) \, \frac{d}{d(n\xi)} \left[\frac{n\xi}{1-e^{n\xi}} \right] \approx \sum_{n=1}^{1/\xi} \Lambda(n)$$

And clearly the two sides are approximate so we are done.

$$\frac{d}{du}\left(\frac{u}{1-e^u}\right) = \begin{cases} 1 & \text{as } u \to 0 \\ ue^{-u} & \text{as } u \to \infty \end{cases}$$

The left side can be shown to be 1 through "elementary" arguments.

$$\sum_{n=1}^{\infty} \Lambda(n) \, \frac{d}{d(n\xi)} \left[\frac{n\xi}{1-e^{n\xi}} \right] \approx \frac{1}{\xi} + O(\log \xi)$$

References

(1) David Vernon Widder **The Laplace Transform** Princeton University Press, 1948.

(2) Norbert Wiener **Tauberian Theorems** Annals of Mathematics Vol. 33, No. 1 , pp. 1-100

(3) G. H. Hardy **Divergent Series** Oxford University Press 1973