Tune-Up: Four Squares

Let g be elements of $SU(2)$ let's define an averaging operator $z : L^2(SU(2)) \to L^2(SU(2))$ with

$$zf(x) = \sum \left(f(gx) + f(g^{-1}x) \right)$$

These operators might have a **spectral gap** $\lambda(z_g) < 2k$. Somewhat fancy, question asked in 1921 is whether Lebesgue measure is the unique finitely additive measure on S^2 (we could construct all sorts of measures – notions of area or volume or mass – on what is geometrically, a sphere).

noncommutative diophantine property we can find a universal constant D such taht for any word $W \in \langle g \rangle$ of length m, the norms are greater then a certain size,

$$||W_m \pm e|| \geq D^{-m}$$

Using the a norm on 2×2 matrices:

$$\left| \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| \right| = a^2 + b^2 + c^2 + d^2$$

Exercise for the case in question we have an easy

$$||g \pm e||^2 = 2|\text{trace}(g) \mp 2|$$

What were the main results:

- if $g \in SU(2) \cap M_{2\times 2}(\overline{\mathbb{Q}})$ (2×2 matrices with algebraic entries) then z_g satisfies non-commutative diophantine property
- Let $\{g_1, \ldots, g_m\}$ be a set of elements in $SU(2)$ generating a free griph and satisfying [noncommuutative diophantine property] then z has a spectral gap.

Just a reminder, irreducible representations $G = SU(2)$ are given by $\pi_n = \text{sym}^N(V)$ it's a linear map on the space of homogeneious polynomials:

$$(x, y) \mapsto (ax + bycx + dy), \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(2)$$

abc

$$\pi_N(z) = \pi_N(g) + \pi_N(g^{-1}) + \cdots + \pi_N(g_m) + \pi_N(g_m^{-1})$$

In all of these theorems, something complex and chaotic occurs. The first moves look pretty systematic, we are going to define point-measures on the compact lie group:

$$\nu = \frac{1}{2k} \sum \delta_g + \delta_{g^{-1}}$$

in order to show that averages tend to a limit, we can use an **approximate identity**

$$P_\delta = \frac{\chi_{B_{1,\delta}}}{|B(1,\delta)|}$$

where $\delta \ll 1$ and $l \sim \log \frac{1}{\delta}$ and $\mu = \nu^{(l)} * P_\delta$ (density is bounded by $||P_\delta||_\infty \sim \delta^{-3}$).

Lemma "L^2-flattening Lemma"

- $\delta < ||\mu||_2 < \delta$
- $||\mu * \mu|| < \delta^\epsilon ||\mu||_2$

These number-theoretic averages – this should be concrete and super-tangible – is compared to **random walk** so we have to measure cost of comparing these number patterns to random. We now have an infinite number of examples of free groups with $\overline{\langle g \rangle} \simeq SU(2)$ the sphere. Before we take the limit, these objects are very close to perfectly symmetric so we an call them **approximate group**.

References

[1] Terence Tao, Van Vu. **Additive Combinatorics** (Cambridge Advanced Studies in Mathematics #105) Cambridge University Press, 2006.