the Schrödinger Equation

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When I opened up Quantum Mechanics by Enrico Fermi, I just expected just some kind of review. These course notes are facisimile of a class at University of Chicago in 1954 written by one of the "greats".

There are some things that are unusual. He opens with a **optics-mechanics** analogy:

mass point	wave packet
trajectory	ray
velocity	group velocity
???	phase velocity
potential function	refractive index
energy	frequency

Hopefully, if you've taken a quantum mechanics course, you may have heard of "particle-wave duality" even if you can't qualify what it meant.

What is. . . **particle-wave duality**?

$$\begin{array}{lcl} \text{Trajectory} & = & \text{Ray} \\ \downarrow & & \downarrow \\ \text{from Maupertuis} & & \text{from Fermat} \\ \downarrow & & \downarrow \\ \int \sqrt{E-U}\, ds = min & & \int \frac{ds}{v} = min \end{array}$$

He then proceeds to prove the Maupertuis and Fermat principles (of optics).
A beam of light "searches for" the optimal path in one of two different ways:

- principle of least action
- principle of least time

For clarification: E is the total energy. Maupertuis principle is **not** the principle of least action.

is that the path followed by a physical system is the one of least "length"

OK. Maupertuis $\neq$ Huygens, which is the one I really like. I will give a fake derivation

First of all $v = \frac{ds}{dt}$. Velocity is the derivative of time. Therefore $\frac{ds}{v} = dt$, and in fact we should get time:

$$T = \int \frac{ds}{v}$$

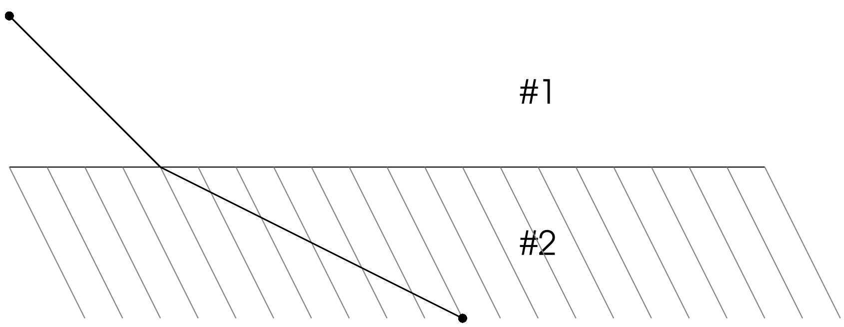
That was too easy. Let's do the other one now.[1] Let $U = 0$ and $E = \frac{1}{2}mv^2$, and if we write:

$$\int \sqrt{E}\, ds = \int \sqrt{\tfrac{1}{2}mv^2}\, ds = \int v\, ds = \int \frac{ds}{dt} ds = \int \left(\frac{ds}{dt}\right)^2 dt = \int v^2\, dt$$

and this should be zero for all possible $\delta v = 0$. The first variation should be:

$$\delta \int \sqrt{E}\, ds = \delta \int v^2\, dt = \int (v + \delta v)^2\, dt - \int v^2\, dt = 2\delta v \int v\, dt = \delta v \int ds$$

And we have shown both Fermat and Mauperthuis lead to length minmization.



This is the worst derivation ever. Yet it reflects the equality of derivations we have in a typical Lagrangian mechanics textbook. And likely, as must as Mr. Fermi had in mind.

We'll see his derivation wasn't much better, but at least he can get us to Schrödinger.

[1] Optics is not a field I now very well (in fact it's a special case of **electromagnetism**, yet Optics has been studied since Newton and Electromagnetism those equations are due to Maxwell. I don't know a good analogue for *refractive index*)

It's hard to know how to continue this because Fermi's development is just so far-fetched. I'm a bit rusty, however:

$$\mathbf{p} = m\mathbf{v} \quad \text{and} \quad E = \frac{1}{2}mv^2 \qquad \text{therefore} \qquad E = \frac{p^2}{2m}$$

In physics class, quantization is just a procedure for generating the Schrödinger equations:

$$\mathbf{p} = i\hbar\nabla \text{ and we just said } E = \frac{p^2}{2m} \text{ so that } -\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

and that is the free particle Schrödinger equation. And it's solutions are the standing waves:

$$\psi(r,t) = A\, e^{-\frac{1}{i\hbar}(\mathbf{p}\cdot\mathbf{r} - E\,t)}$$

That's really well. The reason we solve th wave equation in class is because it's the only differential equation we know how to solve. All other cases, have boundedness issues and well-posedness issues that have never been resolved.

The other problem is that I learned quantum mechanics about 7 years ago in 2009 back in Califronia. I left there on very bad terms and havn't been since 2012.

$$K_t(x,y) = K_t(x-y) = \frac{1}{\sqrt{2\pi i t}} e^{-\frac{i(x-y)^2}{2t}} \xrightarrow{t\to 0} \delta(x-y)$$

The propagator is away of "generating" the evolution – how the wave function changes oer time. My approach is neither

- rigorous enough for functional analysts
- physically relevant enough for engineers
- visual enough for geometers

There are people who make entire careers out of studying Laplacian operators on surfaces:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and here we specify we are doing the Laplacian on $\mathbb{R}^2$ (flat Euclidean space).

Feynman uses the Lagrangian (instead of the Hamiltonian) for the Path integral:

Lagrangian = Potential − Kinetic $\qquad$ Hamiltonian = Potential + Kinetic

In the case of free particle $L = \frac{1}{2}mv^2$ so that the propagator is:

$$K_0(b,a) = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi i h\epsilon}\right)^{N/2} \times \int\cdots\int \exp\left\{\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i - x_{i-1})\right\} dx_1 \dots dx_{N-1}$$

and if you do the infinitely many Gaussian integrals:

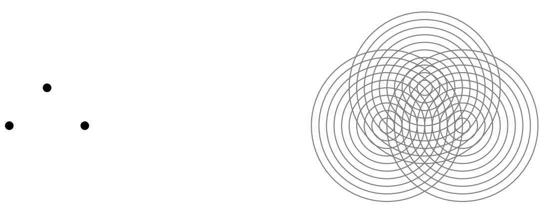
$$K_0(b,a) = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}} \exp\left\{\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right\}$$

This answer makes sense. The quantum free particle starts off concentrated at a point, and diffuses spreading out evenly in a circle.

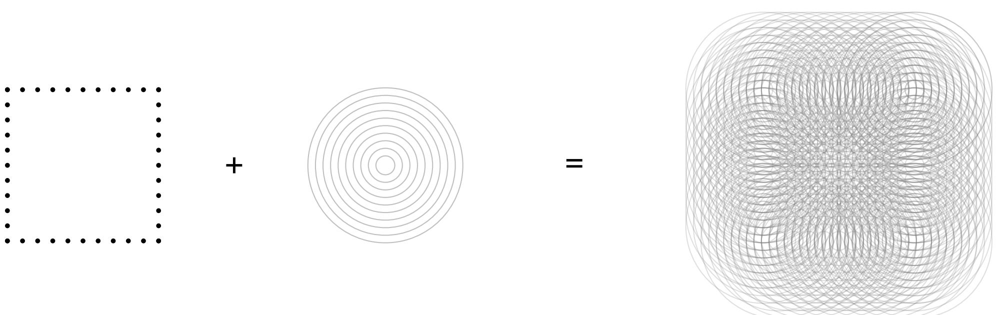
We have now solved the free particle equation twice.

- some kind of Gaussian diffusion
- a wave

Maybe I'll conclude with my picture of the propagator.



Maybe can solve the Schrödinger equation by taking minkowski sums of shapes:



and the wave function equation could read $\psi(t) = \psi(0) + K(t-0)$ as shapes?

5/11 Vladimir Arnol'd was a world unto himself. Let's pick three books:

- Mathematical Aspects of Classical Mechanics
- Catastrophe Theory
- Contact Geometry and Wave Propagation

It's this last one that sets me off: really taking to heart that particle-wave duality is about particles and waves. Who do I believe? I would rather believe Enrico Fermi that Fermat, Huygens and the study of Newtonian optics could be related to the wave equation of Schrödinger and Heisenberg.

It's exhausting to think this hard. It's difficult to have a point of view when many clearly authoritative or expert sources point otherwise.

For starters: there is **sympletic geometry** (in even dimensions $\mathbb{R}^{2n}$) and **contact geometry** (in odd dimensions $\mathbb{R}^{2n+1}$). None of my physics textbooks (I not a physics major) discuss sympletic manifolds. The most basic idea is position and momentum become interchangeable:

$$\text{position} \leftrightarrow \text{momentum} \qquad \text{so that} \qquad (\text{position}, \text{momentum}) \in \mathbb{R}^{2n}$$

and therefore physical degrees of freedom in mechanical systems come in pairs. Yet, wave propagation seems to be about contact geometry.

Whichever one you pick, modern symplectic geometry or contact geometry is unintelligible. We will step back a couple of decades, as the most modern language is very much beyond my reach.

References

(1) HansjÃűrg Geiges **Christiaan Huygens and Contact Geometry**

(2) Richard Feynman. **Quantum Mechanics and Path Integrals** Dover, 2010.

(3) Enrico Fermi. **Quantum Mechanics** University of Chicago Press, 1961.

(4) David Tong **Quantum Field Theory** (Math Tripos, Part III)
`http://www.damtp.cam.ac.uk/user/tong/qft/qft.pdf`

(5) Matthew Schwartz **Quantum Field Theory** (class notes, Fall 2008)
`http://isites.harvard.edu/fs/docs/icb.topic521209.files/QFT-Schwartz.pdf`