Scratchwork: Quaternionic Shimura Varieties

We are going to try to parse whatever we can in one of the latest editions of Advances in Mathematics. My theory is that a lot of these papers at least start off about things we are familiar with. Too much symbolic manipulation makes me a bit critical that we've lost track.

> The aim of this paper is to compare arithmetic intersection numbers on Shimura varieties attached to inner forms of GL_2 over a real quadratic field F.
>
> To illustrate our approach, first consider the arithmetic volume of a Shimura variety attached to an inner form of GL_2 over $\mathbb{Q}$.
>
> ...
>
> We begin with a division quaternion algebra B over F whose discriminant $D_B := (p_1, \dots, p_r) \cdot \mathcal{O}_F$ is a non-empty product of split rational primes.

While I may be grossly underestimating the difficulty of such a task, I feel there should be instances where the problem is easy to state and describe. Even if solving is a ton of work.[1] Even when they solve it, they understate the meaning so much it's just totally lost.

5/16 Because we waited, we get for free, a 1000 pages book on Quaternions by John Voight[2]. Between sections 1 and 2 we obtain the definitions we are looking for:

- B is a totally indefinite division quaternion algebra
- $\mathbf{G}(\mathbb{R}) \simeq \{(A_1, A_2) \in \mathsf{GL}_2(\mathbb{R}) \times \mathsf{GL}_2(\mathbb{R}) : \det A_1 = \det A_2\}$
- $X = \{(z_1, z_2) \in \mathbb{C}^2 : \mathsf{Im}(z_1) \cdot \mathsf{Im}(z_2) > 0\}$
- There's a group action of $\mathbf{G}(\mathbb{R})$ by fractional linear transformations:

$$(A_1, A_2) \cdot (z_1, z_2) = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2} \right)$$

Now there's two objects. The **complex twisted Hilbert modular surface** and the **quaternionic Shimura curve**.

$$M_K := \mathbf{G}(\mathbb{Q}) \bigg\backslash \Big[X \times \big(\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})/K \big) \Big] \tag{$*$}$$

and we're not even ready to discuss the other space. Many thing here the author has stated as defintions can be turned back into *exercises* or even *theorems*. This jargon "arithmetic intersections" should mean two equations we have in the arithmetic object we are describing... arrangements of the fractions over various number systems.

[1] Then I have a job.

[2] *Quaternion algebras*, current version (v.0.9.12, March 29, 2018). https://math.dartmouth.edu/ jvoight/quat.html

What do you think he means by $\mathbb{R}/\log|\mathbb{Q}^\times|$ could this object exist outside of number theory?

Continuing, let $\mathbb{H}^\pm = \{z \in \mathbb{C} : \mathsf{im}(z) \neq 0\}$ be the union of the upper and lower half planes.

$$\Big[\mathbf{G}_1 = \mathsf{Res}_{F/\mathbb{Q}} B_1^\times\Big] \to \Big[\mathbf{G}_1(\mathbb{R}) \simeq \mathbb{H} \times \mathsf{GL}_2(\mathbb{R})\Big]$$

And for the time being we have to parrot these identities because the authors are moving so quickly. There is a group action:

$$(h, \left(\begin{array}{cc} a & b \\ c & d \end{array}\right))z = \frac{az+b}{cz+d}$$

and then we can define the quaternionic Shimura variety:

$$S = \mathbf{G}_1(\mathbb{Q}) \Big\backslash \big[\mathbb{H}^\pm \times (\mathbf{G}_1(\mathbb{A}_{\mathbb{Q},f})/K_1)\big] \qquad (*)$$

I'll concede the author sort-of knows what these definitions are. When the authors talk about a "canonical model" not only am I not sure, I'm not even interested. However I do know what is **fractional linear transformation** and a **quaternion**. I even know slightly what is a **Todd class**. And so we keep reading.

References

[1] Gerard Freixas i Montplet, Siddarth Sankaran "Twisted Hilbert modular surfaces, arithmetic intersections and the JacquetLanglands correspondence" Advances in Mathematics Volume 329, 30 April 2018, Pages 1-84