Approximating $\sum$ with $\int$ in 2 Dimensions

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1 Euler-Maclaurin

What is the 2-dimensional version of the Euler-Maclaurin formula?

$$\tfrac{1}{4}[f(0,0)+f(1,0)+f(1,1)+f(1,0)] \approx \int_0^1 f(x,y)\,dx\,dy$$

This integral is sort of correct, but what is more accurate? After lots of scratchwork I came up with a complicated formula:

$$\begin{aligned}
\sum_a^b \sum_c^d f(m,n) &= \int_a^b \int_c^d f(x,y)\,dx\,dy \\
&- \frac{1}{2}\left[\int_a^b \Big[f(x,d) - f(x,c) \Big] dx + \int_c^d \Big[f(b,y) - f(a,y) \Big] dy \right] \\
&+ \frac{1}{4}\left[f(a,c) - f(a,d) - f(b,c) + f(b,d) \right]
\end{aligned}$$

It is so complicated I don't even know if it works. I would like to estimate the factorial function over complex numbers:

$$(M+iN)!_{\mathbb{Z}[i]} = M!N! \prod_{m\geq 1, n \geq 1}^{M,N} (m+in)$$

Not even sure this is such a great definition of factorial over Gaussian integers. When in doubt, we can consult Manjul Bhargava's definition of S-factorial (originally over $\mathbb{Z}$) [**?**].

There are two ways to validate the above formula. One is to check that separation of variables works. Let $f(m,n) = f_0(m) + f_1(n)$.

$$\sum_a^b \sum_c^d f(m,n) \approx \sum_a^b \sum_c^d \big[f_0(m) + f_1(n) \big] = (d-c) \sum_a^b f_0(m) + (b-a) \sum_c^d f_1(n)$$

The integral formula should have identical check. The last line is worth a quick check:

$$f(a,c) - f(a,d) - f(b,c) + f(b,d) = \big(f_0(a)+f_1(c)\big) - \big(f_0(a)+f_1(d)\big) - \big(f_0(b)+f_1(c)\big) + \big(f_0(b)+f_1(d)\big) = 0$$

These corner terms play no input if there is separation of variables. The middle term we can validate:

$$\int_a^b \Big[f(x,d) - f(x,c)\Big] dx = \int_a^b \Big[\big(f_0(x) + f_1(d)\big) - \big(f_0(x) + f_1(c)\big)\Big] dx = (b-a)\Big[f_1(d) - f_1(c)\Big]$$

So our summation formula behaves correctly under separation of variables.

1.1 Another Derivation

Sadly I am still not 100% sure so we are going to derive the 2-variable formula another way, simply by iterating the EM formula twice:

$$\begin{aligned} \sum_a^b \sum_c^d f(m,n) &\approx \sum_a^b \left[\int_c^d f(m,y)\, dx + \frac{1}{2}\big[f(m,d) - f(m,c)\big] \right] \\ &= \int_c^d \sum_a^b f(m,y)\, dx + \sum_a^b \frac{1}{2}\big[f(m,d) - f(m,c)\big] \end{aligned}$$

Looks like we are recovering the previous steps as before:

$$\sum_a^b \frac{1}{2}\big[f(m,d) - f(m,c)\big] = \frac{1}{2}\int_a^b \Big[f(x,d) - f(x,c)\Big]\, dx + \frac{1}{4}\Big[\big(f(b,d) - f(b,c)\big) - \big(f(a,d) - f(a,c)\big)\Big]$$

Then summation in the integral can also be approximated using single-variable Euler-Maclaurin

$$\int_c^d \sum_a^b f(m,y)\, dy = \int_c^d \left(\int_a^b f(x,y)\, dx + \frac{1}{2}\Big[f(b,y) - f(a,y)\Big] \right) dy$$

Sorry for this sucky explanation. After some introspection it should be clear the two formulas are the same. These types of formulas were very usefulf or engineers or actuaries who deal with large tables of numbers. These days even though a computer might do it for you in certain circumstances, it won't be the case if your information has not been digitized.

Stirling's formula involves estimating the integral of $\log(x+iy)$ over a square. Stokes theorem or Green's formula could play a role in such approximations.

References

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[3] Manjul Bhargava. *The Factorial Function and Generalizations* American Mathematical Monthly, Vol. 107, No. 9 (Nov., 2000), pp. 783-799

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[5] Michael Spivak *Calculus on Manifolds* Westview Press, 1971.