

# **Roth's Theorem on Sequences**

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0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

**Square-Free Numbers** No prime divisor is repeated twice.  
 These have density about  $2/3$  in the number line,  $\mathbb{Z}$

$$90 = 2 \times 3 \times 3 \times 5$$

is not a square-free number. However the red number:

$$85 = 5 \times 17$$

The classic exercise is to show the probability of a number being square-free is  $\frac{\pi^2}{6}$

$$\mathbb{P}\left[\min_{x,y \in \mathbb{Z}} |ax + by| = 1\right] = \prod_p \left(1 - \frac{1}{p^2}\right) = \sum_{n>0} \frac{1}{n^2} = \frac{\pi^2}{6}$$

And we will need a proof of the last step to pipe into further work - when we actually try to solve Roth's Theorem.

**Proof # 15** Every natural number can be written as the sum of four squares (if you look hard enough).

$$n = x^2 + y^2 + z^2 + t^2$$

This can be proven using geometry of numbers and the fact that  $\pi^2 > 8$ .

Let  $r(n) = \#\{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 = n\}$  be the number of solutions, which are scattered around the 3-sphere  $S^3$ .

How many integer points are there **inside** the 3-sphere?

$$R(n) = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 \leq n\} \sim \frac{\pi}{2} \cdot N^2$$

using the formula for  $r(n)$  we can find a formula for  $R(n)$ :

$$r(n) = 8 \sum_{m|n, 4 \nmid m} m$$

This is really clumsy to omit 4's like this and I don't explain why:

$$R(N) = 1 + \sum_{n=1}^N \sum_{m|n, 4 \nmid m} m = 1 + 8 \left( \sum_{m \leq N} m \left[ \frac{N}{m} \right] - \sum_{m \leq N} m \left[ \frac{N}{4m} \right] \right)$$

One can show that  $\zeta(2)$  mysterious pops out of lattice point counting under a hyperbola:

$$\theta(x) = \sum_{mr \leq x} x = \zeta(2) \cdot \frac{1}{2} x^2 + O(x \log x)$$

Finally:

$$R(N) \sim \frac{\pi^2}{2} N^2 \sim 1 + 8(\theta(N) - \theta(N/4)) \approx 4\zeta(2) \left( N^2 - \frac{N^2}{4} \right)$$

This is from a textbook “Introduction to Number Theory” by Hua Look Keng, but I found it in Robin Chaptman’s list of proofs of  $\zeta(2)$ .

The big questions to me here are why theta functions appear at all... since:

$$\theta(q)^4 = \left[ \sum q^{n^2} \right]^4 = \sum r(n) q^n$$

By ommitting his derivation of the sum of 4 squares formula, we lose a big chance to get at the geometry of lattices and number theory.

Moving on... let’s find other sequences of positive upper density!

## References

- (1) JP Serre **Course on Arithmetic** Springer-Verlag