Spin Chain "Dualities"

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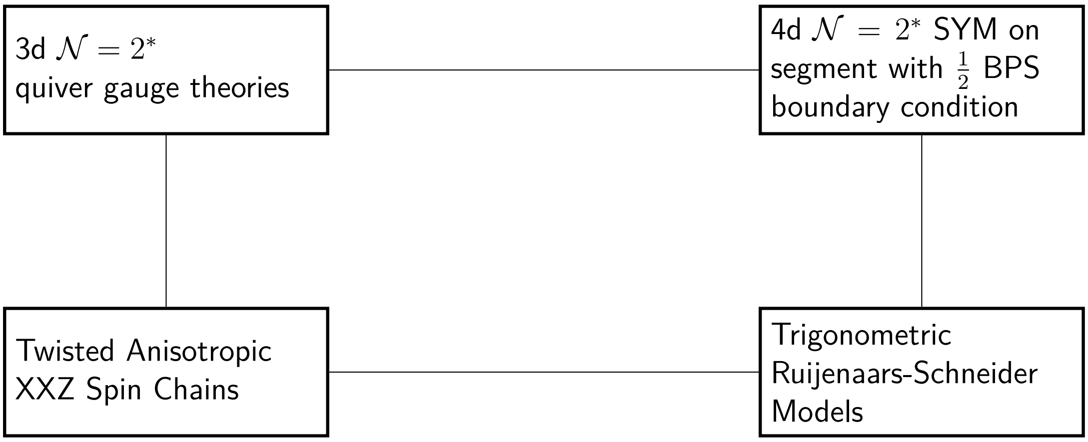
1 Spin Chains and Integrable Systems

Not knowing better, I took for granted that spin chained and integrable systems should be related objects. Yet experts, may not feel as I do. Using insights from Supsersymmetric Gauge Theory, Davide Gaiotto and Peter Koroteev have been able to state (possibly *prove*) relationships between:

- between spin chains
- between integrable systems
- between spin chains and integrable systems

these are deduced as the consequences of the **Gauge-Bethe Correspondence** outlined by Nekrasov and Shatsahvili [**?**].

Our goal is to state dualities more carefully so we can try to match them with classical or modern mathematical ideas.



This correspondence becomes difficult to follow since no one person is familiar with all the terms involved.

- Unclear how to define $\mathcal{N} = 2^*$ gauge theory or do decide which details are important in connection to XXZ spin chains.
- Find a clear basic definition of **twisted** **anisotropic** spin chains. Similar for the trigonomi-etric Ruijsenaars-Scheider model.
- How to decide which aspects of XXZ spin chain map to quiver gauge theory and tRS model and discern a mathematical meaning.
- All these dualities can be stated quite cleanly but the spin chains or integrable systems are messy to calculate with. It's unclear how these symmetries can help us.

The **bispectral duality** of the XXZ spin chain has some interesting consequences.

$$\begin{tikzcd}
\boxed{\mathrm{rGaudin}[m,t]} \arrow[r, dash] \arrow[d, dash] & \boxed{\mathrm{rGaudin}^\vee[t,m]} \arrow[d, dash] \\
\boxed{\mathrm{XXX}[m,\tau;\epsilon]} \arrow[r, dash] \arrow[d, dash] & \boxed{\mathrm{tGaudin}[\tau,m;\epsilon]} \arrow[d, dash] \\
\boxed{\mathrm{XXZ}[m,\tau;\epsilon,R]} \arrow[r, dash, dashed] & \boxed{\mathrm{XXZ}^\vee[\tau,m;-\epsilon,R]}
\end{tikzcd}$$

Next we have to chase around definitions of the XXX models with twists m and anisotropies τ.

Isn't XXZ spin chain anisotropic to begin with? By definition of XXZ Hamiltonian[1]:

$$H = -J\sum_{n=1}^{L}\left[S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta\left(S_n^z S_{n+1}^z - \frac{1}{4}\right)\right]$$

where $\vec{S} = (S^x, S^y, S^z) = \vec{\sigma}_n$ are the Pauli spin matrices for spin $= \frac{1}{2}$. Since half-spin matrices correspond to $\mathfrak{su}(2)$, I thought maybe we just put matrices from $\mathfrak{su}(n)$ into the equation to have higher-spin XXZ. Other sources say this is beyond spin $\frac{1}{2}$ (such as spin 1 and $\frac{3}{2}$) no Hamiltonian has been written - or it is very complicated.

XXZ $SU(L)$ spin chain, with N_i Bethe roots at ith level of nesting with spins transforming in various power of asymmetric powers of the fundamental representation:

$$\mathcal{R} = \bigoplus_{i=1}^{M_1}\mathbf{L} \oplus \bigoplus_{i=1}^{M_2}\wedge\mathbf{L} \oplus \cdots \oplus \bigoplus_{i=1}^{M_{L-1}}\wedge^{L-1}\mathbf{L}$$

where there is no Hamiltonian, just Bethe equations. Gaiotto and Koroteev were careful to say "*an* XXZ spin chain" indicating there could be more than one. The eigenstates live somwhere in the representation $\mathcal{R}$ but we don't where.

$\mathcal{N}^* = 2$ SQCD is a $U(N)$ gauge theory with M flavors, with an explicity superpotential and vaccum equations – which coincide with a spin matrix. The twisted anisotropic XXZ $\mathfrak{su}(2)$ spin chain with impurity parmaters m_a, a single twist paramter $e^{2\pi i\theta} \in U(1)$ and explicit Hamiltonian.

[1]See (2.9) in `http://www.reffert.itp.unibe.ch/Lectures.pdf` also recall:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

I have some questions about what it means, a vector of matrices. In Quaternions at least $\vec{\sigma} = (\mathbf{i}, \mathbf{j}, \mathbf{k}) \in \mathbb{H}^3$

$$
\begin{tikzcd}
\boxed{\mathrm{rCM}[m,t]} \arrow[r, dash] \arrow[d, dash] & \boxed{\mathrm{rCM}^{\vee}[t,m]} \arrow[d, dash] \\
\boxed{\mathrm{rRS}[m,\tau;\epsilon]} \arrow[r, dash] \arrow[d, dash] & \boxed{\mathrm{rCM}[\tau,m;\epsilon]} \arrow[d, dash] \\
\boxed{\mathrm{tRS}[m,\tau;\epsilon,R]} \arrow[r, dash] & \boxed{\mathrm{tRS}^{\vee}[\tau,m;-\epsilon,R]}
\end{tikzcd}
$$

These relate variants of **Ruijsenaars-Schneider** models and **Calogero-Moser** theories.

We are not here to challenge the correctness of the arguments in [?] - just need to understand better what Gaiotto and Koroteev said and how they argued[2].

XXX-**rGaudin** duality

Even more classical is duality of Gaudin model with itself.

rGaudin w/ $SL(M)$ group and N sites		rGaudin w/ $SL(N)$ group and M sites

Proof At classical level the spectral curves and action differentials are equivalent. At quantum level, the spectra exhibit symmetry.

The paper [?] talks only about stems from classical systems such as rigid body motion and the Rosochatius system. The proof involves moment maps and loop groups. These get "quantized" into the Gaudin systems appearing in [?] with the same symmetry.

The quantum proof comprares two sets of Bethe equations which we just copy.

$$\frac{M_1 - M_2 - \epsilon}{t_i} + \sum \frac{v_b \epsilon}{t_i - z_b} - \sum \frac{2\epsilon}{t_i - t_j} = 0$$

Here $M = 2, N = 2$. This is for the Gaudin model and next is for the $SL(2)$ XXX spin chain:

$$\prod \frac{\lambda_i + M}{\lambda_i + M + k\epsilon} = \frac{z_2}{z_1} \prod \frac{\lambda_i - \lambda_j - \epsilon}{\lambda_i - \lambda_j + \epsilon}$$

There is a mapping from one set of solutions to the other with twists $\leftrightarrow$ masses interchanging roles $\square$

It's not clear what having the same set of Bethe equations has ensured us. The only integebrable system I vaguely understand is the Harmonic oscillator and the Pentagram map. I have no idea what this proof means or how to use it, but this is what I have to work with. And it's been accepted by (representatives of) the Physics community.

Where are the twists m and impuruties τ? The impurities behave like "poles" in complex analysis while the twists relate $n \to n+1$.

References

[1] Davide Gaiotto, Peter Koroteev. *On Three Dimensional Quiver Gauge Theories and Integrability* `arXiv:1304.0779`

[2] Nikita Nekrasov, Samson Shatashvili.

[3] E. Mukhin, V. Tarasov, A. Varchenko *Bispectral and* $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ *Dualities, Discrete Versus Differential* `arXiv:math/0605172`

[4] M. Adams, J. Harnad and J. Hurtubise, *Dual moment maps into loop algebras*, Letters in Mathematical Physics 20, 299 (1990), `http://dx.doi.org/10.1007/BF00626526`.

[2] `http://media.scgp.stonybrook.edu/presentations/20130618_koroteev.pdf`