On Hardy's Curious Rersult $\sum_{n\leq N}\{n\theta\}^2 = \frac{N}{12} + O(1)$

John D Mangual

On archive.org we find the complete papers of mathematician GH Hardy. In the generically titled paper **Some Problems of Diophantine Approximation** we find one curious result:

$$\sum_{\nu\leq n}\{\nu\theta\}^2 = \frac{n}{12} + O(1)$$

Hardy comments: When we consider the great irregularity and obscurity of the behavior of $\sum\{\nu\theta\}$ it is not a little surprising that $\sum\{\nu\theta\}^2$ should behave such marked regularity.

I did not have difficulty finding my own "proof" of the result using the equidistribution of $\{\nu\theta\}$ around the unit circle:

$$\frac{1}{n}\sum_{\nu\le n}\{\nu\theta\}^2 \approx \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dx = \frac{x^3}{3}\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}} = 2\times\frac{1}{24} = \frac{1}{12}$$

Except when I tried to write a more "formal" proof (e.g. the symbol $\approx$ is unacceptable) ... I have two candidates:

- Weyl Equidistribution Theorem:

$$\frac{1}{n}\sum_{\nu\le n} f(\nu\theta) \approx \int_0^1 f(x)\,dx$$

- Birkhoff Ergodic Theorem. Let $T : x \to x+\theta$ be a rotation

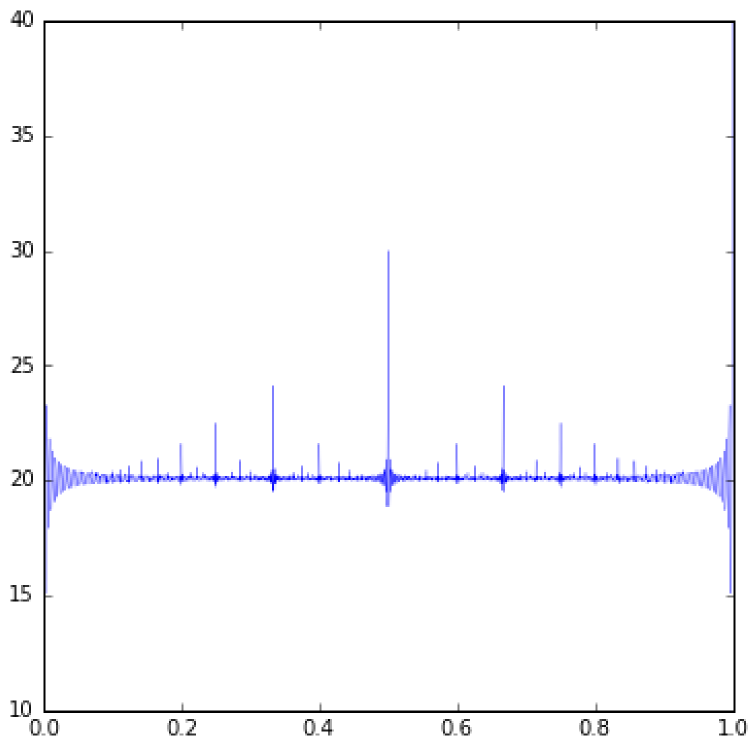
$$\frac{1}{n}\sum_{\nu\le n} f\left(T^{\nu}x\right) = \int_0^1 f(x)\,dx$$

  for almost every $x \in [0,1]$.

Neither theorem tells you how fast the averages converge to the integral:

$$\left|\frac{1}{n}\sum_{\nu\leq n}\{\nu\theta\}^2 - \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dx\right| = O(1)$$

And we don't know the "exceptional set" of measure zero. Here we plot $\sum_{\nu=0}^{20}\{\nu\theta\}^2$ for various θ in th range of $[0,1]$ and we notice spikes at all rational numbers $\theta \in \mathbb{Q}$.



The plot looks more like the spikes of a ruler than a straight line. Hardy in a later paper **The Lattice Points of a Triangle**, he correts himself that not just $\theta \notin \mathbb{Q}$ but $\theta = [a_0; a_1, a_2, a_3, \dots]$ where $a_n < M$ (bounded partial quotients).

A good example is $\theta = \sqrt{2}$ or $\theta = e$ (for a # with unbounded partial quotients). On MathOverflow one user notes the Fourier expansion:

$$\frac{1}{2\pi^2} \sum_{k \neq 0} \frac{e^{2\pi i k x}}{k^2}$$

so that our series converges if $k^2 ||k\theta|| > k^\epsilon$, and problems for well approximated numbers.

These curious formulas (and Hardy has many more of them!) motivate a careful study of the Birkoff ergodic theorem, since every T and every $f \in L^2(X, \mu)$ leads to a "curious" formula of this type.

Mean Ergodic Theorem

Basically $\frac{1}{N}\sum(T^n f)$ converges to $\int f$ – this is what have before $\boxed{f(x) = x^2}$

Let $X = S^1$ be the circle with Lebesgue measure $\mu = dx$ and $T_\theta : S^1 \to S^1$ be $x \to x + \theta$ for $\theta \notin \mathbb{Q}$ (and $\mathcal{X}$ is the Borel sigma-algebra). Then:

$$\frac{1}{N}\sum_{n \leq N}(T_\theta^n f)(x) \to \int f\, d\mu$$

where convergence is in $L^2(S^1, \mu)$. This is not pointwise convergence this is is convergence in L^2.

Why be so careful? The goal is not to be tedious. Our hope is to follow one of the myriad proofs of the von Neumann ergodic theorem, and study the error term.

For T_θ as θ ranges from 0 to 1 perhaps there are some unusual numbers!

References

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