Spin Chain Dualities

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Two physicists at Perimeter Institute - Davide Gaiotto and Peter Koroteev - discuss promising “dualities” between a spin chain and an integrable system.

- Twisted anisotropic XXZ spin chain
- Trigonometric Ruijsenaars-Schneider model

Here and there, I have read of a dynamical system that is “integrable” or a spin-chain.

In fact, for a long time I thought these two terms were interchangeable.

Here is some more information about this duality:

- $SU(L+1)$ XXZ spin chain
- $GL(Q)$ tRS model

What aspects of XXZ map to what aspects of tRS?

- impurities $\longleftrightarrow$ Eigenvalues of M
- twist $\longleftrightarrow$ Eigenvalues of T
- anisotropy $\longleftrightarrow$ Eigenvalue of E

I gleaned that tRS involves matrices which satisfy an eq:

$$MTM^{-1}T^{-1} = E$$

Does $Q = L + 1$?

I left out some important details about a quiver, the Bethe-Ansatz equations look to complicated to read[1].

[1] This is common in integrable systems... Unfortunately, does not mean "easy to read" or "intuitive".

What is Twisted Anisotropic XXZ Spin Chain over $SU(L+1)$?

Gaiotto-Koroteev may have borrowed their interpretaton of spin chains from Nikita Nekrasov and Samson Shatashvili. These authors write about a **gauge-Bethe correspondence** just like the one we are trying to figure out now.

The $SU(2)$ XXX spin chain[2] describes a set of spins on a lattice of length N. The Hilbert space is a tensor product:

$$\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$$

and $SU(2)$ acts as a representation on this space.

$$H = J \sum_{a=1}^{L} (S_a^x S_{a+1}^x + S_a^y S_{a+1}^y + S_a^z S_{a+1}^z)$$

where $\vec{S}_a = \frac{i}{2}\vec{\sigma}_a$ are te Pauli spin matrices.

[2] so I guess $L = 1$ here...

The state of this system can only have one of 2^N outcomes, such as $|\uparrow\ldots\uparrow\rangle, |\downarrow\ldots\uparrow\rangle, |\downarrow\ldots\downarrow\rangle \in \mathcal{H}$.

The boundary conditions are "twisted" around the circle...

$$\vec{S}_{N+1} = e^{\frac{i}{2}\theta\sigma_3}\vec{S}_1 e^{-\frac{i}{2}\theta\sigma_3}$$

The **total amount of spin** $\vec{S} = \sum \vec{S}_a$ commutes with the Hamiltonian... this is a very fance way of saying **spin is conserved**.

This narrows down the dynamics a tiny bit since, even though there are 2^N possibilities for spin, we can put them into groups of $\binom{N}{M}$ spins for $M = 0, 1, \ldots, N$.

A **wavefunction** is a map $\psi : \mathbb{C}^{\otimes n} \to \mathbb{C}$ and we would like to find wavefunctions that are eigenvectors of H.

Skipping very important work for now... the eigenspaces are related to the Bethe Ansatz equations:

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^L = e^{i\theta} \prod_{k \neq j} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}$$

Since I can't do these equations justice in a few sentences I am not going to motivate them at all[3] - the *inhomogeneous* XXX spin chain looks almst the same:

$$\prod_{a=1}^{L} \frac{\lambda_j - \nu_a + i s_a}{\lambda_j - \nu_a - i s_a} = e^{i\theta} \prod_{k \neq j} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}$$

There is the "analytic" Bethe Ansatz and the "algebraic" Bethe Ansatz. . . all of them are way too complicated.

[3]We should not the cameo appearances of the **Cauchy transform** and **algebraic invariant theory** since we are doing all these index manipulations!!!

We are looking for XXZ spin chain over $SU(L+1)$ not just $SU(2)$.

$$H = \sum_{a=1}^{L} (J\, S^x_a S^x_{a+1} + J\, S^y_a S^y_{a+1} + J_z\, S^z_a S^z_{a+1})$$

One possiblity is to **embed** $\rho : SU(2) \to SU(L+1)$ and get a representation that way. The representations of $SU(2)$ are indexed by the half-integers[4]

Domenico Orlando and Susanne Reffert, elegantly show us how xxx$_{1/2}$ spin chain is related to the equivariant cohomology of the tangent bundle of the grassmanian.

$$H_*\big[T^* Gr(N, L)\big]$$

[4]a fact that I must review over and over

References

(1) Davide Gaiotto, Peter Koroteev **On Three Dimensional Quiver Gauge Theories and Integrability** `arXiv:1304.0779`

(2) Domenico Orlando, Susanne Reffert **The Gauge-Bethe Correspondence and Geometric Representation Theory** `arXiv:1011.6120`

(3) Nikita A.Nekrasov, Samson L.Shatashvili **Supersymmetric vacua and Bethe ansatz** `arXiv:0901.4744`