Unipotent Flows

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I audited a course on homogeneous systems

- Roth's Theorem
- Szemeredi Theorem
- Ratner Theorem

It was much too fast to absorb. And, year later, I still have questions. The teacher was brilliant and that may have been $\frac{1}{2}$ a design feature and $\frac{1}{2}$ my own curiosity.

Now, when I read the actual papers I notice the material is much more complicated than he had shown us; that he had (rather calmly) manoeuvered around various technicalities in order to present the material to us. Also, there are now textbooks covering many features.

In particular, I tried to understand the Oppenheim conjecture, as usual thinking it would be very simple to resolve.

$$\overline{\left\{ x^2 + y^2 - \sqrt{2}\, z^2 : (x,y,z) \in \mathbb{Z}^3 \right\}} = \mathbb{R}$$

My strategoy would have been to say here is $a \in \mathbb{R}$ and $\epsilon \ll 1$ and we'd like to find (x,y,z) such that $q(x,y,z) \approx a$.

$$\left| (x^2 + y^2 - \sqrt{2}\, z^2) - a \right| < \epsilon$$

This is ϵ-δ definition the way we might see in calculus class. I have drawn the picture a few times. Instead, the main step is to deform the equation itself:

The saying goes: **Algebra is generous: she often gives more than is asked for**; it is also hiding things from us as well. Things that are quite complicated.[1] This is also part of a common strategy in mathematics, to make the problem more severe and more terrifying, but we also see the big picture.

In this case, we know a few things that are going on:

- Lie groups over $\mathbb{Q}$ are more complicated than Lie groups over $\mathbb{R}$ (e.g. it's possible to take limits in $SO(3,\mathbb{Q})$ that go outside the group).
- ?? (I forgot)

[1] This is due to Jean d'Alembert (1717-1783) and I thought it was David Hilbert (1862-1943).

Cognitively, I see a blind spot *among professors* about Ratner's theorem. We treat it as a black box. **By Ratner's Orbit-Closure Theorem the values of Q are dense.** We have no idea inside, I have learned nothing. It makes asking for help very difficult.

When procedure comes along that covers *all* unipotent flows. First of all, we are going to Oppenheim conjucture into a single giant procedure that will absorb all equidistribution problems, including ones we haven't though of. We need two pieces of information:

- $G = \mathrm{SL}(3, \mathbb{R})$
- $H = \mathrm{SO}(Q) \simeq \mathrm{SO}(1,2) \simeq \mathrm{PSL}(2, \mathbb{R})$

and the algebra says the closure of a certain orbit is the entire group: $\overline{G_{\mathbb{Z}} H} = G$. That's excellent. What does that even mean?

Another similar statement $Q(H\mathbb{Z}^3) = Q(\mathbb{R}^3)$. (Yes, that's a 4th root of 2) :

$$[Q] = \left[x^2 + y^2 - \sqrt{2}\, z^2\right] \leftrightarrow \left[\begin{array}{ccc} 1 & & \\ & 1 & \\ & & -\sqrt[4]{2} \end{array}\right]$$

The H could literally be thought of a group of substitutions.[2] If $a_+^2 + b_+^2 = 1$ then:

$$\left[(a_+\, x + b_+\, y)^2 + (-b_+\, x + a_+\, y)^2 - \sqrt{2}\, z^2\right] \leftrightarrow \left[\begin{array}{ccc} a_+ & b_+ & \\ -b_+ & a_+ & \\ & & 1 \end{array}\right] \left[\begin{array}{ccc} 1 & & \\ & 1 & \\ & & -\sqrt[4]{2} \end{array}\right]$$

and we need the other substitution: If $a_-^2 - \sqrt{2}\, b_-^2 = 1$ then:

$$\left[x^2 + \left(a_-\, y + \sqrt{2}\, b_-\, z\right)^2 - \sqrt{2}\left(b_-\, y + a_-\, z\right)^2\right] \leftrightarrow \left[\begin{array}{ccc} 1 & & \\ & a_- & \sqrt{2}\, b_- \\ & b_- & a_- \end{array}\right] \left[\begin{array}{ccc} 1 & & \\ & 1 & \\ & & -\sqrt[4]{2} \end{array}\right]$$

and hopefully the $\sqrt{2}$'s are correct. This is as far as I got on my previous attempts.

I think I'll go try something else.

Later... how do we solve Lagrange's 3 squares problem as a unipotent flow?

$$x^2 + y^2 + z^2 = n$$

This equation is **positive-definite**.

References

(1) Dave Witte Morris **Introduction to Arithmetic Groups** `arXiv:math/0106063`

[2] There might even be a professional-soundeding name for it, like "representation"