Scratchwork: Basic Theorems of Lebesgue Integration

9/20 Why do we need Riemann integration or even Lebesgue integration. Here's an integral formula:

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

In calculus class we just memorize a rule: $\int : x^n \to \frac{1}{n+1}x^{n+1}$ and apply the formula. Here's one more:

$$\int_0^x \cos x\,dx = \sin x$$

So there is a second rule $\int : \cos x \mapsto \sin x$. Once we buy into these one or two or a dozen rules, we can corner ourselves very quickly. **Ex**: Show that

$$\int_0^{2\pi} f(x)\,\big(\sin Nx\big)^2\,dx \to \pi f(x)$$

The formula looks right, we have that $0 < \sin^2 x < 1$ and it oscillates fairly evenly so the average should be $\frac{1}{2}$.

Riemann integration was already a formality, Lebesgue integration was an even bigger formality. Here's a common-sense looking theorem that requires Lebesgue integration.

Problem: Exchange integration and summation. Let f_k be a sequence of L^1 integrable functions such that $\sum_1^\infty |f_j| < \infty$. Then $\sum f_k$ converges (almost everywhere) to a function in L^1 and

$$\int \sum_{i=1}^{\infty} f_k(x)\,dx = \sum_{i=1}^{\infty} \int f_k(x)\,dx$$

If the function sequences look forbidding, let $f_n(x) = a_n\,\sin nx$ with $0 \le x \le 1$. We want to know if

$$\int \sum_{i=1}^{\infty} a_n\,\sin(2\pi n x)\,dx = \sum_{i=1}^{\infty} \frac{a_n}{n}\,\cos(2\pi n x)$$

There's even an easy choice $a_n = \frac{1}{\pi k}$. We obtain the sawtooth wave:

$$\sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{\pi k} = \frac{1}{2} - \{x\}$$

It's the only one we ever study. Despite it's usefulness in music, I'm mostly concered about it's arithmetic and geometric properties. This "$=$" sign is really shaky too. At $x = \pi$

$$\sum_{m=1}^{\infty} \frac{0 \pm \epsilon}{\pi k} \approx 0 \pm \left(\frac{1}{2} - \epsilon\right)$$

Fortunately for us, nature (and certainly Mathemtical Physics) will offer us examples of natural processes that require Lebesgue theory to understand.

Problem When do limits and integrals converge?

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$$

If we look on the previous page of the Lebesgue theory textbook, we get conditions when this common-sense formula might work:

- f_n is a sequence of L^1 functins (Lebesgue integrable)[1] with $f_n \to f$ almost everywhere.
- There's a (non-negative) function $g \geq 0$ such that $|f_n| \leq g$ (almost everywhere).

Certainly the authors of these textbooks have lost their minds, and they are writing such a textbook for their health.

Thm with these conditions, the limit $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

If we ever want this common-sense result to whole we need to more results from Lebesgue integration theory:

- Dominated Convergence Theorem
- Fatou's Lemma
- Monotone Convergence Theorem

Lebesgue measure is impossible to construct. Half of probability (e.g. the Law of Large Numbers) is showing that a measure either converges to Lebesgue measure or to a point (or to a Gaussian centered at that point).

Example Here's a set that requires measure theory to even try to estimate the size of. Let $\alpha = \sqrt[4]{2}$ and consider the set $\{(m,n) : 0 < m^2 + \sqrt[4]{2}\, n^2 < X\}$. Here is a measure we could study:

$$\mu(x) = \sum 1_{m^2 + \sqrt[4]{2}\, n^2}(x)$$

There were two possible ways to write this set. Any major differences?

- $\{m^2 + \sqrt[4]{2}\, n^2 : 0 \leq m, n \leq N\}$
- $\{m^2 + \sqrt[4]{2}\, n^2 : 0 \leq m \leq M \ 0 \leq n \leq N\}$
- $\{(m,n) : 0 < m^2 + \sqrt[4]{2}\, n^2 < X\}$

The last set is a collection of pairs of integers (coordinates) and the first two sets are collections of real numbers.

Here's a function we might study with Lebesgue theory:

$$f(x) = \left|\{(m,n) : 0 < m^2 + \sqrt[4]{2}\, n^2 < X\}\right| - \frac{\pi}{4\sqrt[4]{2}} X \approx 0$$

we've used nothing but familiar household objects and the equations describe a natural thing.

References

[1] Gerald B. Folland **Real Analysis: Modern Techniques and their Applications**. Wiley, 1999.

[2] Valentin Blomer, Jean Bourgain, Maksym Radziwiłł, Zeev Rudnick **Small gaps in the spectrum of the rectangular billiard** `arXiv:1604.02413`

[1] Did we ever compute a Lebesgue integral in our lives? So why we bother talking about Lebesgue integrable