

# Reading: Arctic Circle Theorem

Let's read the statement of the **Arctic Circle Theorems** how did the authors state it. What were they concerned about?

**Thm #1** Fix  $\epsilon > 0$ . Then, for all sufficiently large  $n$ , all but an  $\epsilon$  fraction of domino tilings of the Aztec diamond of order  $n$  will have a temperate zone whose boundary stays uniformly within distance  $\epsilon n$  of the inscribed circle.

The proof is somewhat difficult to read.

For starts we have a collection of random shapes, and we need to discern the "temperature" in different parts of the picture based on a single observation. The temperate zone exists over the entire collection of shapes. What does it mean to say that each shape (in this case, a tiling) is "equally likely"? How does one identify the "frozen" region and the "temperate" region?

The Arctic Circle Theorem can be reduced to a statement about the totally asymmetric exclusion process (TASEP).

$$x^*(n) = \begin{cases} 1 & n \leq 0 \\ 0 & n > 0 \end{cases}$$

**Thm #2** Fix  $\alpha \leq \beta$ . If one runs the exclusion process starting at state  $x^*$ , the number of particles in  $[\alpha n, \beta n]$  at time  $n$ , normalized by dividing by  $n$ , converges in probability to  $h(\alpha) - h(\beta)$  where

$$h(x) = \begin{cases} -x & x < -\frac{1}{2} \\ \frac{1-x}{2} - \frac{1}{2}\sqrt{\frac{1}{2} - x^2} & |x| < \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$$

There's even a question about which totally asymmetric exclusion process this could be. This is a discrete-time process, . The "stochastic function"  $x \xrightarrow{?} x'$  is now called a "random process", a sequence of random variables where each state is connected to the previous one in some way. This problem is solved *before* we know what stochastic processes are, so there's lot of content.

This is a mix of geometry and computer science (and not a lot of Arithmetic). One possibility to look at are **bin-packing** problems. Also, these kind of interesting particle processes and phase transitions appear in Statistical Mechanics.

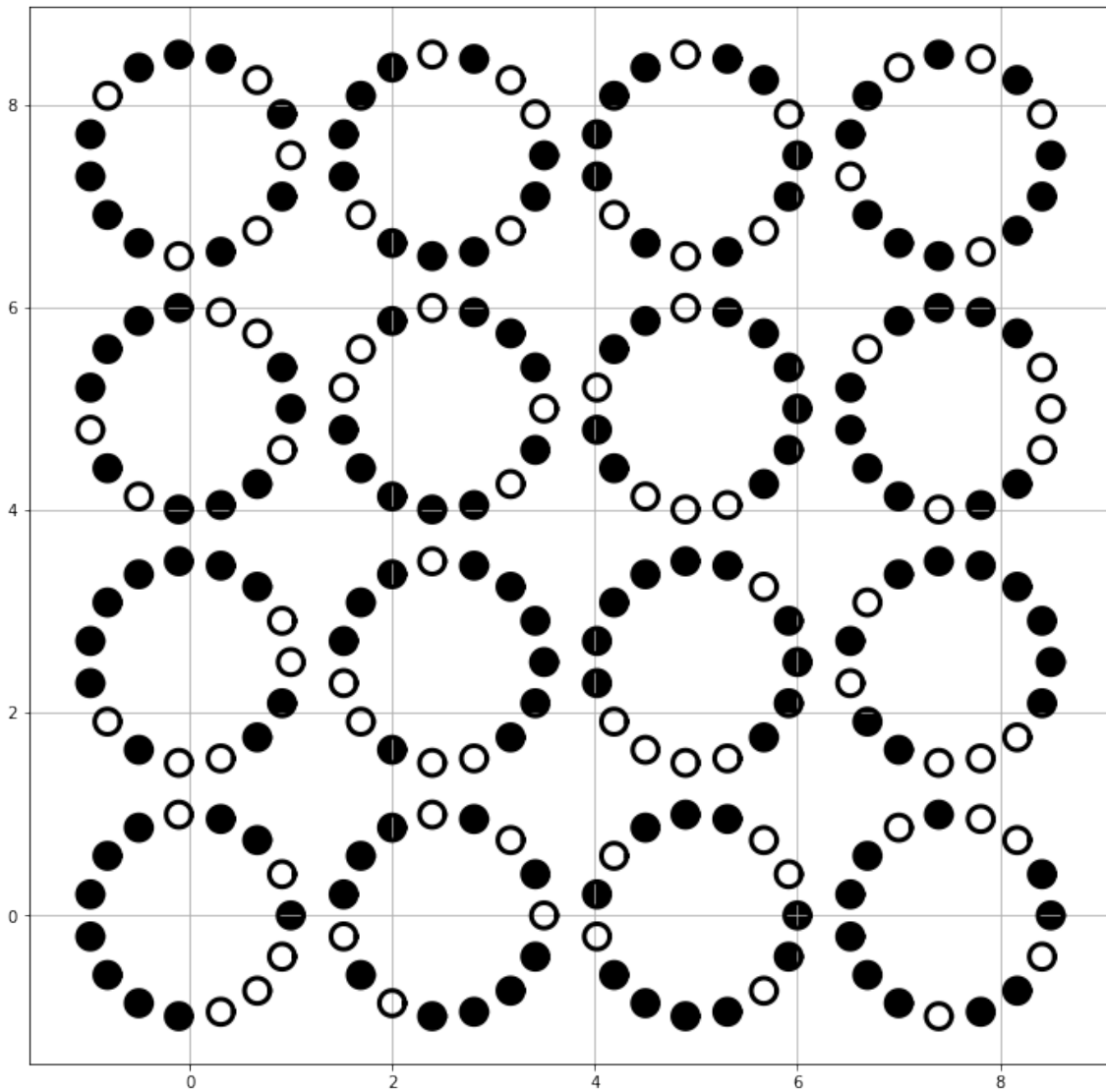
Another challenge is to state this random process problem selecting only the parts that we need or are helpful. Too much details can be a problem, too. Here all the details carry a lot of weight, and maybe there is some negative space as well.

**Thm #3** The translation-invariant, stationary probability measures on  $\mathcal{X}$  are precisely those that can be expressed as convex combinations of the measures  $\mu_d$  with  $0 < d < 1$ .

Here  $\mathcal{X} = \{0, 1\}^{\mathbb{Z}}$ .

- **translation-invariant** means  $T^*\mu = \mu$ .
- $M(\mathcal{X})$  is the set of probability measures on  $\mathcal{X}$ .
- a convex combination of the measure is defined by:  $(\alpha\mu_1 + (1 - \alpha)\mu_2)(S) = \alpha\mu_1(S) + (1 - \alpha)\mu_2(S)$
- **stationary** means for  $\mu \in M(\mathcal{X})$ ,  $P_\mu(x' \in A) = P_\mu(x \in A) = \mu(A)$ .

The situation where we have to juggle the time-averages and space-averages of the random process is called “Ergodic theory”. We observe an area over time, or we take a single observation and compare different regions in the area. There’s no reason *a priori* two answers are nearly the same. These are 2D random processes.



There’s been quite a few discussions since [JSP], we can certainly look for ourselves. These pictures are states of a circular TASEP, they are just necklaces - in this case with 15 beads, 5 of which are colored white and the rest are colored black. The beads randomly move in a clockwise direction when there is nothing ahead of them.

This would be a good time to argue about what is meant by “random” as in Probability or to decide how to attach a probability distribution to what we are observing, as in Statistics. There’s also the “Probabilistic Method” where we try to use probability theory to solve another problem that we actually care about such as Geometry or Number Theory.

These problems are fairly carefully studied already (e.g. KPZ universality class). We’re going to study the proof for ourselves and the result should be another random process or a map between a random process. Or we’ll say it’s not quite random and then it will be Ergodic Theory. Or it is random after all and then it’s Probability again.

There’s a good discussion about what our instincts should predict vs. what actually happens, so let’s try to see a bunch of that here.

## References

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- [4] William Jockusch, James Propp, Peter Shor.  
**Random Domino Tilings and the Arctic Circle Theorem** arXiv:math/9801068