

Examples: Theta Functions

John D Mangual

Conformal field theory is a central topic in mathematical physics¹. What is Conformal Field Theory?

The rational gaussian model has a single scalar free field ϕ which is compactified on a circle with $R^2 \in \mathbb{Q}$.

This theory has a large group of symmetries – an extension of the the $U(1)$ current algebra.

The N primary fields $[\phi_p]$ are vertex-operators with momentum $\frac{p}{\sqrt{N}}$ and $p \in \mathbb{Z}_N$.

The fusion rules are $\phi_p \times \phi_q = \phi_{p+q}$ with $p, q \in \mathbb{Z}$.

¹What does that even mean? Here it means there are other math problems in fields like Number Theory and Topology and Chern Simons Theory – and specific sources – which point to the paper we will review today.

The discussion the last page is already quite problematic. Why is $R^2 \in \mathbb{Q}$, e.g. $R = \sqrt{3}$?

It seems I have confused ϕ and φ . These are related by the exponential function:

$$\phi_p(\mathbf{c}) = \exp \left(\frac{p}{\sqrt{N}} \int_{\mathbf{c}} \partial \varphi \right)$$

this is such a nice looking integrals with nice properties:

$$\phi_p(\mathbf{a})\phi_q(\mathbf{b}) = e^{2\pi i pq/N} \phi_q(\mathbf{b})\phi_p(\mathbf{a})$$

Now I want to know why these feel like the basic commutation operators from Quantum Mechanics:

$$[x, p] = -i\hbar$$

and you can even prove these yourself, right?

Let $p = -i\hbar \frac{d}{dx}$ then:

$$\left[x, i\hbar \frac{d}{dx} \right] f(x) = i\hbar \left(x \frac{df}{dx} - x \frac{df}{dx} + \frac{dx}{dx} f \right) = -i\hbar$$

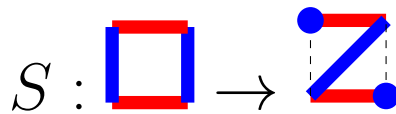
Expect we are getting the exponentiated form. That's it²

²I am not going to review anything more from Verlinde's paper – I will be too busy making sense of these objects to discuss anything else.

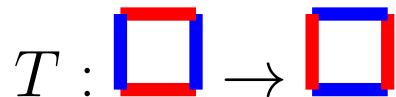
Excuse me, Verline talks about the S and T operators. The first one is clear:

$$S : \chi_p \rightarrow \frac{1}{\sqrt{N}} \sum_{q \in \mathbb{Z}_N} e^{2\pi i pq/N} \chi_q$$

The only S and T operators that I know very well act on a Torus:



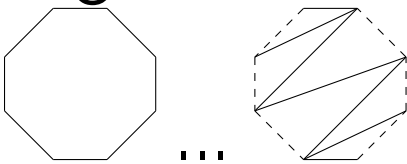
The other operator flips the torus (a technical term):



This action lifts to **observable** that happens on that torus.

$$T : \chi_p \rightarrow e^{2\pi i p/N} \chi_p$$

With some effort these can happen on an octagon³



Verlinde's big result is: **the modular transformation S diagonalizes the fusion rules**⁴

$$\phi_p \times \phi_q = \phi_{p+q}$$

³Perhaps I should draw these by hand first!

⁴See? We can recite these formulas over and over many times with no idea what they mean :-)

How do we compactify the free scalar field on a circle?

Verlinde could have reminded us this scalar field was a section⁵ $\varphi : \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \rightarrow \mathbb{R}$. I think it's over \mathbb{R} . It could be a complex valued field.

Our torus is $S^1 \times S^1$ but I wish to remember the complex structure, so we remember⁶ the number: $\tau \in \mathbb{C}/\text{SL}(2, \mathbb{Z})$ that section is generated by two transformations:

$$z \mapsto z + 1 \qquad z \mapsto -\frac{1}{z}$$

For that matter we could pick other very simple transformations and it makes a huge world of different. For the record:

$$z \mapsto z + 1 \qquad z \mapsto -\frac{1}{4z}$$

That is because holomorphic function's don't just take partial derivatives.

⁵or function

⁶This space makes more sense after you've done a lot of cut-and-paste as on the previous page.

$$\begin{array}{ccc}
 f(z + dy) & \xrightarrow{+dx f'(z)} & f(z + dz) \\
 \vdots & & \vdots \\
 +dy f'(z) & & +dy f'(z) \\
 \\
 f(z) & \xrightarrow{+dx f'(z)} & f(z + dx)
 \end{array}$$

It's hard to draw the picture of what holomorphic means... that the real and imaginary parts change in a compatible way.

A **free field** or more specifically a **free boson** or **gaussian model** as an action:

$$S = \int \mathcal{L} = \frac{1}{2\pi} \int \partial X \bar{\partial} X$$

and Ginsparg tells us the propagator should be:

$$\langle X(z, \bar{z}) X(w, \bar{w}) \rangle = -\frac{1}{2} \log |z - w|$$

What is a conformal field anyway? Length behaves nicely under conformal mappings:

$$ds^2 \mapsto \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial \bar{f}}{\partial \bar{z}} \right) ds^2$$

This field splits into a holomorphic and anti-holomorphic part⁷

$$X(z, \bar{z}) = \frac{1}{2}(x(z) + \bar{x}(\bar{z}))$$

and I can't seem to find how X behaves under conformal mappings $z \mapsto f(z)$.

⁷they are not complex conjugates... x depends on z and \bar{x} depends holomorphically on \bar{z} . Nature somehow requires the real and imaginary parts to behave separate but equal – in reality creating two different worlds.

Conformal fields seem to be **representations of the Virasoro algebra** describing things that transform nicely under conformal maps:

$$\ell_n : z \mapsto z + \epsilon z^n$$

I am looking for a formula for $x(z)$. Reading many many times I try a formula I saw elsewhere:

$$x(z) = \sum_{n=0}^{\infty} a_n z^n$$

where a_n is either an⁸

- Gaussian random variable with mean 0 and norm 1
- a simple harmonic oscillator $[a_m, a_n] = \delta_{m,n}$

There is a way to pass between a Gaussian and harmonic oscillator that I reviewed before⁹

Oh, here is a free fermion:

$$S = \frac{1}{8\pi} \int \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}$$

⁸The first version – which makes the most sense to me – I learned from Yuval Peres at PCMI 2007. However it comes from work of Mikhail Sodin.

⁹There was a trick in symplectic geometry that takes one random walk and turns it into a harmonic oscillator. It could be the Dierker-Hellman theorem. When I wrote one thing it magically turned into the other. This is a linear space, so it's reasonable to assume this will work in infinite dimensions.

There is a lot to be unhappy with here. And yet we continue.

Section 8 of Ginsparg shows us how to put free bosons on a torus¹⁰...

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My logic: all scalars are real numbers, so they all commute. CFT has three similar-looking objects:

- fields
- states
- operators

A conformal field theory **is** a quantum field theory:

$$\text{CFT} \subset \text{QFT}$$

A field could be a function  $\phi : \mathbb{R}^2 \rightarrow V$  where  $V$  is a vector space of some kind describing the feelings of the particle.

The group of conformal transformations in the Euclidean plane should act on these fields in a

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<sup>10</sup>and the compactification means the image is a cylinder  $S^1 \times \mathbb{R}$  and  $X \equiv X + 2\pi r$ . All that seems to matter is the ring (order?)  $\frac{1}{2r}\mathbb{Z} + \mathbb{Z}r$  and different  $r$ 's give the same range of numbers. I'm getting there



nice way. These nice actions are called **representations** – and occasionally by other names.

I am not the first person to lose sleep over a mathematical definition of CFT - there is also **Graeme Segal**, who has done much of that for me.

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The representations of the conformal group are known as Verma modules.

Even easier (in flat space...) there is a recipe for quantizing the Hamiltonians<sup>11</sup> that one learns in the QFT class. Start from Classical mechanics:

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p} = \{q, H\} \\ \dot{p} &= -\frac{\partial H}{\partial q} = \{p, H\}\end{aligned}$$

We simply decree that the Poisson bracket<sup>12</sup>

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<sup>11</sup>and now we complain that we don't really know or understand the Hamiltonian formalism. My advice is to find an interesting motion that you like and then compute the Hamiltonian. Advice #2 – they are always rotations.

<sup>12</sup>Newton comes up with his awesome mechanics and we start solving free-body diagrams for two centuries. Eventually we have trains and textiles and so forth. If you solve these things all day, they kind of start to look the same. Hamilton's equations (but first Lagrange) streamlined all of that.

As far as I can tell... the Hamiltonian is just a way to say “total energy” – and the Hamiltonian is *generating* the flow of the position and momentum  $(p, q)$ .

From Hamiltonian's viewpoint, the position and conjugate momentum are interchangeable. So we stop calling it  $x(t)$  and call it  $q(t)$  because it looks just like  $p(t)$ .

are replaced by Commutator brackets.

$$\dot{q} = [q, H]$$

$$\dot{p} = [p, H]$$

and then our QFT class proceeds to write hundreds of pages of algebra without any insight.

In QFT, the bosons still anti-commute. If there is a boson we use a commutator:

$$[a, b] = ab - ba$$

if we have a fermion we use an anti-commutator:

$$\{a, b\} = ab + ba$$

which is just plain addition.

For quantum field theory, the operator valued function<sup>13</sup>

$$[\phi(x), \phi(y)] = 0$$

and an equation for the conjugate momentum

$$[\pi(x), \pi(y)] = 0$$

and the position and momentum should anti-commute

$$[\phi(x), \pi(y)] = i\delta(x - y)$$

where  $x, y \in \mathbb{R}^2$  and should behave nicely under the transformations of the conformal group.

$$\phi : \mathbb{R}^2 \times V \rightarrow V$$

This still not write we basically took the Harmonic oscillator and added a new variable. It is like we had a  $2 \times 2$  matrix with variables  $z \in \mathbb{C}$ :

$$\begin{bmatrix} a(z) & b(z) \\ -\bar{b}(z) & \bar{a}(z) \end{bmatrix}$$

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<sup>13</sup>that's what Tong calls them

I don't think the book ever tells us exactly how  $\phi$  acts on the space of states  $|n(z)\rangle$ , or how the Virasoro algebra acts.

and then I broke out Senechal.

## References

- (1) Erik Verlinde. **Fusion rules and modular transformations in 2D conformal field theory.**  
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- (4) Paul Ginsparg. **Applied Conformal Field Theory** [arXiv:hep-th/9108028](https://arxiv.org/abs/hep-th/9108028)
- (5) Mikhail Sodin. **Zeroes of Gaussian analytic functions** [arXiv:math/0410343](https://arxiv.org/abs/math/0410343)
- (6) Graeme Segal. **The definition of conformal field theory** See:  
<https://ncatlab.org/nlab/show/conformal+field+theory>