Notes 5: Vinogradov Mean Value Theorem

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1 The Deepest and Most Difficult Part

For Terry Tao everything revolves around the Riemann ζ function. $\boxed{\frac{1}{N}\sum_{I} e(f(n)) \ll N^{-c/k^2}}$

It's hard to believe this integral is an integer:

$$J_{\ell,k}(M) = \int_{\mathbb{T}^2} |e(\alpha_1 n + \cdots + \alpha_k n^k)|^{2\ell}\, d\alpha$$

is in $\mathbb{Z}$.

The volume of the box is

$$\begin{aligned} B &= \{\vec{y} \in \mathbb{Z}^k : |y_j| \le \ell M^j\} \\ &= \prod_{j=1}^{k} (2\ell M^j + 1) \end{aligned}$$

Baiscally the ℓ-fold sum of the curve is equidistributed in the box B: $\ell\gamma = \gamma + \cdots + \gamma \approx B$

Let $\gamma_k(a) = (a^1, a^2, \dots, a^k)$ be a "twisted cubic". And let

$$\begin{aligned} J_{\ell,k}(M) &= \#\left\{(\vec{x}, \vec{y}) \in \mathbb{Z}^{2\ell} : \sum \gamma_k(\vec{x}) = \gamma_k(\vec{y})\right\} \\ &\subseteq [1, M]^{2\ell} \end{aligned}$$

The "deepest and most difficult part" of Vinogradov's argument to show:

$$J_{\ell,k}(M) \ll e^{O(\ell^2)} \times M^{O(k^2 e^{-\ell/k^2})} \times \frac{M^{2\ell}}{|B|}$$

The volume of the box is roughly

$$|B| = \prod (2\ell M^j + 1) = e^{O(k \log \ell)} M^{\binom{\ell}{2}}$$

therefore we can rescale $j_{\ell,k}(M) = M^{\binom{k}{2} - 2\ell} J_{\ell,k}(M)$.

The system of equations has a most $e^{O(\ell k) p^{\binom{k}{2}}}$

$$\sum x^j = v_j \mod p$$

Each solution to this system has at most $p^{\binom{k}{2}}$ has representatives. These are like the number of domino tilings of the Aztec diamond. Places to look:

- Noam Elkies, Mokshay Madiman, Bourgain-Demeter-Guth.

Restricting mod p we have $J^{(p)}_{\ell,k}(M) \approx J_{\ell,k}(M)$ for large enough p. The Linnik Lemma states:

$$J_{\ell,k}(M) \ell e^{O(\ell k}(J^p_{\ell,k}(M) + M^{\ell+k-1})$$

for some $kM^{1/k} < p \ll k^4 M^{1/k}$. Lots of symmetric polynomials. We observe

$$y \mapsto \sum \gamma(y)$$

is at most $k!$-to-1 map.

2 Aztec Diamonds

abc def

I have no idea how any of the stuff with the previous section works. Let's try drawing pictures.

References

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