Tune-Up: Weak Mixing

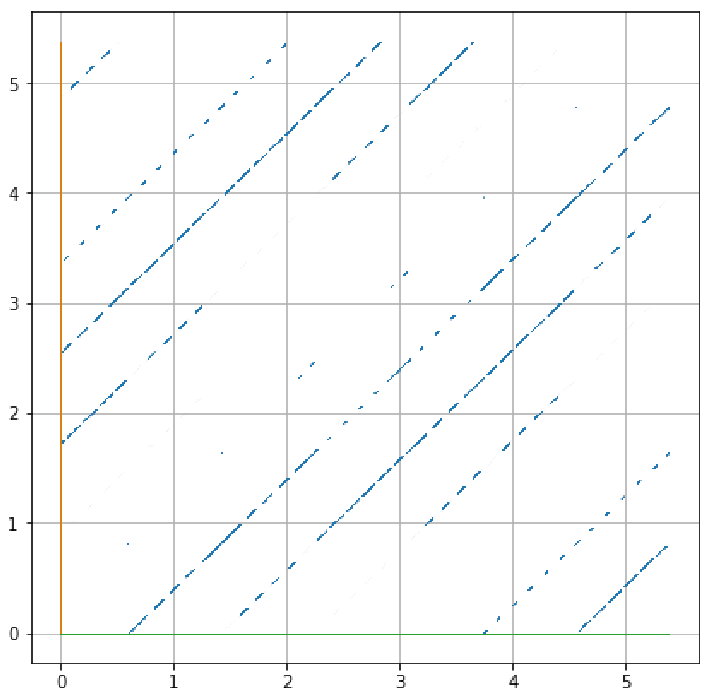
So if you have an average to compute, you might try to find a dynmical system and function (an "observable") decide that it's ergodic, measure-preserving etc and show that the averages line up.

Often our limiting measure could be something very simple, like $(\mathbb{R}, dx)$ or $(S^1, d\theta)$ (the angles of a circle). Or we infer our answer from these simple answer (this is why we do integrals).

This is an example of an **interval exchange transformation**.

$$T(x) = \left\{ \begin{array}{l|c} x + (\sqrt{3} + \sqrt{5}) & 0 < x < \sqrt{2} \\ x + (\sqrt{5} - \sqrt{3}) & \sqrt{2} < x < \sqrt{2} + \sqrt{3} \\ x - (\sqrt{2} + \sqrt{3}) & \sqrt{2} + \sqrt{3} < x < \sqrt{2} + \sqrt{3} + \sqrt{5} \end{array} \right.$$

Here is the graph of the function $x \mapsto T^{1000}(x)$. In fact, we need to be more specific we only sampled on a discrete set of points, $10^{-5}\mathbb{Z} \subseteq \mathbb{R}$ and we . After a million points, we are testing the limits of my laptop computer or at least I have to make an explicit request to my computer, here using `Python 3.6.9`



We can guess from the picture what the limiting measure μ might be.

The statements of the mean ergodic theorem (there's also a point-wise ergodic theorem) could be important or helpful. A measure-preserving system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only iff

$$\frac{1}{N}\sum_{n=0}^{\infty} f \circ T^n \longrightarrow_{L^2_\mu} \int f \, d\mu$$

Here's another systement $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if

$$\frac{1}{N}\sum_{n=0}^{\infty} \langle f \circ T^n, g\rangle \to \int f \, d\mu \int g \, d\mu$$

for any $f, g \in L^2_\mu$. The book says we can optionally use set-theory that $(X, \mathcal{B}, \mu, T)$ is ergodic if

$$\frac{1}{N}\sum_{n=0}^{\infty} \mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$$

these aveages are multiplicative *in the limit*. The definition of **mixing** looks similar.

$$\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$$

The **pre-images** are getting more and more shuffled until they are perfectly mixed ...as mixed as they're ever going to get. In fact, a (measure-preserving) dynamical system

Two very innocent-looking culprits:

- **shape**
- **area** (or "volume" or "measure")

If we try to balance things on a "scale" what did we mean by that. What were we measuring or counting?

References

[1] Manfred Einsiedler, Thomas Ward **Erodic Theory with a View Towards Number Theory** (GTM #259) Springer, 2011.