Newton's Inequalities

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With some difficulty I have reconstructed my train of thought. There are two nice, very approachable results, I wish to cover. I had been looking three gentlement who were studying the carries we do in arithmetic. Towards the end of their discussion they prove:

$$\mathbb{P}\big(\text{ random } n\text{-permutation has } j \text{ descents}\big) = \mathbb{P}\big(j < \text{ sum of } n \text{ uniform in } [0,1] < j+1\big)$$

Diaconis credits Jim Pitman with his discussion. That makes for a funny topic tree it goes:

shuffling cards $\leftrightarrow$ carries in arithmetic $\to$ polynomials with real zeros

Why would three very serious men, study the carries in arithmetic. Or how about one not-very-serious man? I went to see if I could justify it and I found out a few things. Basically, we add numbers and carries are a foundation part of our arithmetic.

Looking at Pitman, there's a whole combinatorial world and it's all just right there. Here's just one:

$$S_k = \frac{\sigma_k}{\binom{n}{k}} \text{ therefore } S_{k-1}S_{k+1} \leq S_k^2$$

where σ_k is the elementary symmetric polynomial. Eg. $\sigma_1 = a+b+c$ and $\sigma_2 = ab+bc+ca$ and $\sigma_3 = abc$. These inequalities are nice and versatile, but your time is limited and mine is too. Is there more going on?

Richard Stanley offers a whole collecton of these **unimodal** sequences. He phrased the result as:

$$P(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k \text{ therefore } a_j^2 \geq a_{j-1}a_{j+1}$$

This is almost the same as what we had before. The proof really blows me away. He uses Rolle's theorem, which is pathetic.

$$f(a) = f(b) = 0 \text{ therefore } f'(c) = 0 \text{ for some } a < c < b$$

Using Rolle's Theorem many many times, he find a quadratic polynomial with only real roots:

$$\frac{n!}{2}(a_{j-1}x^2 + 2a_j x + a_{j+1}) \text{ has only real roots, therfore } (2a_j^2)^2 - 4a_{j-1}a_{j+1} \geq 0$$

using the quadratic formula. Really deep stuff here.

It's hard for me to look at a wrench or a hammer and gain inspiration. Likewise, I've collected these simple but very productive results here but I haven't been sure what to do with them.

Until . . .

References

(1) Richard Stanley. **Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry** Annals of the New York Academy of Sciences. 576: 500âĂŞ535.

(2) Jim Pitman. **Probabilistic Bounds on the Coefficients of Polynomials with Only Real Zeros** Journal of Combinatorial Theory. Series A 77, 279-303 (1997)

(3) Alexei Borodin, Persi Diaconis, Jason Fulman **On adding a list of numbers (and other one-dependent determinantal processes)** `arXiv:0904.3740`

(4) Matthew Baker **Hodge theory in combinatorics** `arXiv:1705.07960`