# Scratchwork: Theta Functions

William Duke's proof that the solutions to $n = a^2 + b^2 + c^2$ become equidistributed as $n \to \infty$ takes a quarter of a page:

- Goro Shimura shows there are many theta functions, each invariant under $\Gamma_0(4)$

$$\theta(z;u) = \sum_{m \in \mathbb{Z}^3} u(m)\, e(z|m|^2) = \sum_{n>0} n^{\ell/2}\, r_3(n) \left[ \frac{1}{r_3(n)} \sum_{\xi \in V_3(n)} u(\xi) \right] e(nz)$$

  one for each spherical Harmonic $u \in L^2(S^2)$. Here $|m|^2 = m_1^2 + m_2^2 + m_3^2$ and $V_3(n) = \#\{(a,b,c) : a^2 + b^2 + c^2 = m\}$.

- Henryk Iwaniec offers a bound for the Fourier coffients of cusp forms.

$$a_n \ll_{k,\epsilon} n^{k/2 - 2/7 + \epsilon}$$

  These are tending to zero if we fix a tolerance ($\epsilon$) and a "weight" of modular form ($k$).

- Combining Iwaniec and Shimura's result[1] we obtain an estimate for the sphere averages

$$\frac{1}{r_3(n)} \sum_{\xi \in V_3(n)} u(\xi) \ll_{u,\epsilon} n^{-1/28+\epsilon}$$

  This bound depends on the spherical harmonic ($u$) and the tolerance ($\epsilon$). And we need $n \not\equiv 7 \pmod 8$.

There have been many surprises along the way learning this topic. And I have problems with a lot of this discussion, because these are professors talking to other professors. Slowly turning these objections into contributions of my own!

- $\theta(z)$ is $\Gamma_0(4)$ invariant not $SL(2,\mathbb{Z})$ invariant. As subgroups the index $[SL(2,\mathbb{Z}) : \Gamma_0(4)] = 6$. For the record $\Gamma_0(4) \simeq \langle z \mapsto z+1, z \mapsto -\frac{1}{4z} \rangle$ while $SL(2,\mathbb{Z}) \simeq \langle z \mapsto z+1, z \mapsto -\frac{1}{z} \rangle$. These are like continued fractions $a = [a_0; a_1, \ldots, a_n]$ all of whose digits are multiples of $4$.

- $\theta(z)$ is *not* a cusp form, but if $\deg u > 0$ then $\theta(z;u)$ **is** a cusp form. The Fourier series *is* the $q$-series as $q = e^{2\pi i t}$ is the change of variables.

$$\theta(z;u) = 0 + a_1\, q + a_2\, q^2 + \ldots$$

- Even though we are studying equidistribution of solutions to an equation $f(x) = n$ we don't necessarily study the existence of one solution. After reading Serre's **Course on Arithemtic** (GTM #85) we learn that the existence of one solution is a rather deep problem. Lastly, I don't think it's "done" by the time we discuss equidistribution.

- I didn't know what the symbol "$\ll$" meant. $1 \ll 100$ and also $100 \ll 1$. Here's one more: $10^{100} \ll x$ since *eventually* $x > 10^{100}$. These are things we learn in beginning math classes, but sometimes a brand new symbol is introduced and we have to do it again.

[1] and an estimate of Siegel, which I haven't even looked at, $r_3(n) \gg_\epsilon \sqrt{n^{1-\epsilon}}$.

Then I find out all of this is slightly out of date. This one pricked my last nerve. I believe, in the process of verifying Duke's claim (leaning me through Iwaniec and Shimura and Walspurger and even more) we have made new lemmas. One of them is definitnely new.

OK. Here's a problem statement: let's try to find the constant to go with the $\ll$ sign:

$$\left[a_n \ll n^{k/2-2/7+\epsilon}\right] \to \left[a_n < C\, n^{k/2-2/7+\epsilon}\right]$$

The constant $C$ is not known (probably because nobody cares) if we fix $(\epsilon)$ and $(k)$.

Shimura's constructions are the start of the **theta correspondence**, and I'm choosing to use the out-of-date version that doesn't use any of Waldspurger's technology. Duke makes no use of the adeles, $\mathbb{A}$; any strategy involving them will be new. Iwaniec makes heavy use of Eisenstein series and it's rather mysterious:

$$[\text{theta functions}] \to [\text{Eisenstein series}]$$

Iwaniec says it works, making such a kind of map, risk-free, but do we really understand it? I don't immediately have a problem that is unkown about it. The more we unpack, the scarier it gets.

$$[\text{Kloosterman sums}] \quad || \quad [\text{Bessel functions}]$$

Do we really understand this? In the case of 4-squares, the proof involves the Weil conjectures, yet in 3-squares this just falls out of some very tricky averaging procedures, that I do not wish to replicate.[2]

This discussion has created some deep and lingering doubts: do I understand Weyl's equidistrubiton criterion *for spheres* or requirements for Poisson summation? Do I know what it was so important that $\theta(z;u)$ was a **holomorphic** cusp form, or how to get number-theoretic information out of that? Not really.

[2] at this time