Proposal: The Factorial Function

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It would not be a stretch to categorize my work under the topics in my 3 previous blog posts and this one. I talk about the Stirling formula:

$$n! \approx \sqrt{2\pi e}\left(\frac{n}{e}\right)^n$$

Think about it. If you don't know much math, there are only so many starting points. So no matter how complicated our analysis gets, there shold only be a few principles at play.

Here we can get away with just the trapezoid rule:

$$\log n! \; = \; \log 1 + \log 2 + \cdots + \log n$$

$$\approx \; \int_1^n \log x \, dx = n \log n - n$$

The goal of this project is to state and make towards specific conjectures, built from these.

This concludes my proposals for the Spring.

A My frustration with physics literature or math literature, I am unlikely to come up with observation of interest to others.

For one things, since my interest is about $n!$ I am maybe interested in counting things. Within combinatorics there is:

- counting things
- finding rare and exotic objects
- studying how things are connected
- words

and many other things. I am mainly focusing on the first one here. It is called **enumrative combinatorics**. There are two kinds of applications:

- counting things predicted by mathematics
- counting things related to the outside world

The permutation group can be used to prove quadratic reciprocity.

There's a nice example of card shuffling, where measures on the permutation group converges to the uniform distribution.

The domino shuffling of the Aztec diamond can be expressed in terms of measures on the permutation group.

Binomial coefficients can be completed to geometric objects in the field of math called "integral geometry"

Galois theory, literally permutes the zeros of a polynomial and I always liked those inequalities from contests (which I could never solve)

The Taylor series uses factorials. That's by far the strangest one. What is being permuted there?

I would say combinatorics is a very healthy field and not much I could contribute there. Nothing profound or exciting.

B

The modern theory of factorials I can't even read the formulas.

$$\mathbf{D}_{-n} \prod_{i=1}^{\infty} f(x_i) = \frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n} \prod_{i=1}^{\infty} f(x_i) \operatorname{Res} \left[\frac{\sum_{i=1}^{n} \frac{z_n t^{n-i}}{z_i q^{n-i}}}{\left(1-\frac{tz_2}{qz_1}\right)\cdots\left(1-\frac{tz_n}{qz_{n-1}}\right)} \prod_{i<j} \frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{qz_i}{tz_j}\right)}{\left(1-\frac{z_i}{tz_j}\right)\left(1-\frac{qz_i}{z_j}\right)} \right.$$
$$\left. \times \prod_{i=1}^{n} \left(\prod_{i'=1}^{\infty} \frac{1-t^{-1}q^{\frac{x_{i'}}{z_i}}}{1-q^{\frac{x_{i'}}{z_i}}} \cdot \frac{f(z_i)}{f(q^{-1}z_i)} z_i^{-1} \right) \right].$$

Here is a very important random process

$$\frac{\Gamma(N\theta)}{\Gamma(\theta)^M \Gamma(N\theta - M\theta)} \prod_{1 \le i < j \le M} (x_i^{N-1} - x_j^{N-1})(x_i^N - x_j^N)^{1-2\theta}$$
$$\times \prod_{j=1}^{M} (x_i^N)^{(N-1)\theta} (1 - x_i^N)^{\theta(M-N-1)+1} \prod_{i,j=1}^{M} |x_j^N - x_i^{N-1}|^{\theta-1} \prod_{i=1}^{M} \frac{(1-x_i^{N-1})^{\theta(N-M)-1}}{(x_i^{N-1})^{N\theta}}.$$

Here is another discussion of that random process

$$(29) \quad \sum_{a_1,\ldots,a_J:|a|=j} \frac{q^{\sum_{m=1}^{J}(m-1)a_m}}{Z_j(J)} w_{\{u,qu,\ldots,q^{J-1}u\}}\Big(i_0, \{a_1,\ldots,a_J\}; k_0, \{c_1,\ldots,c_J\}\Big)$$
$$= q^{\sum_{m=1}^{J}(m-1)c_m} \sum_{a_1,\ldots,a_J:|a|=j} \frac{q^{\sum_{m=1}^{J}(m-1)a_m}}{Z_j(J)} w_{\{u,qu,\ldots,q^{J-1}u\}}\Big(i_0, \{a_1,\ldots,a_J\}; k_0, \{\underbrace{1,\ldots,1}_{n},\underbrace{0,\ldots,0}_{J-n}\}\Big)$$
$$= \frac{q^{\sum_{m=1}^{J}(m-1)c_m}}{Z_n(J)} \sum_{\substack{a_1,\ldots,a_J:|a|=j \\ d_1,\ldots,d_J:|d|=n}} \frac{q^{\sum_{m=1}^{J}(m-1)a_m}}{Z_j(J)} w_{\{u,qu,\ldots,q^{J-1}u\}}\Big(i_0, \{a_1,\ldots,a_J\}; k_0, \{d_1,\ldots,d_J\}\Big),$$

here is a physics partition function

$$Z_{\mathsf{T}}(t) = \sum_{\mathcal{X}\in\mathfrak{M}^{\mathsf{T}}} \exp\left(- \sum_{(x_L \succ x_R)} \left(\left(c^{+}_{\mathsf{i}(x_L),\mathsf{i}(x_R)} \right)^{[0]} \beta \log\frac{x_R}{x_L} + \sum_{m\neq 0} \frac{1-q_1^m}{m(1-p^m)(1-q_2^{-m})} \left(c_{\mathsf{i}(x_L),\mathsf{i}(x_R)} \right)^{[m]} \frac{x_R^m}{x_L^m} \right) \right)$$
$$\times \exp\left(\sum_{x\in\mathcal{X}} \left(\log \mathfrak{q}_{\mathsf{i}(x)} \log_{q_2} \frac{x}{\mathring{x}} + \sum_{m=1}^{\infty} \left(\frac{1-q_1^m}{1-p^m} t^{(+)}_{\mathsf{i}(x),m} x^m + \frac{1-q_1^{-m}}{1-p^{-m}} t^{(-)}_{\mathsf{i}(x),m} x^{-m} \right) \right) \right) \quad (2.15)$$

Reality is complex, physical systems ultimately describe a very complicated reality.[1] I am betting these formulas are very important and useful. If I could read them!

Here is one more just in as of yesterday:

$$\mathcal{I}_{\{z_1=\frac{\beta}{\gamma},z_2=\frac{pq}{t}\},\mathbf{v},a,b} = (p;p)^2(q;q)^2 \oint \frac{dw_1}{4\pi i w_1} \oint \frac{dw_2}{4\pi i w_2} \frac{\Gamma_e(\frac{pq}{t}(\beta\gamma)^{\pm1}w_1^{\pm1}w_2^{\pm1})}{\Gamma_e(w_1^{\pm2})\Gamma_e(w_2^{\pm2})} \tag{B.4}$$
$$\Gamma_e(t^{\frac{1}{2}}\frac{\beta^2}{\gamma}b^{-1}w_1^{\pm1})\Gamma_e(\frac{t^{\frac{3}{2}}}{pq}\gamma^{-1}bw_1^{\pm1})\Gamma_e(t^{\frac{1}{2}}\frac{\gamma^2}{\beta}bw_2^{\pm1})\Gamma_e(\frac{t^{\frac{3}{2}}}{pq}\beta^{-1}b^{-1}w_2^{\pm1})$$
$$\Gamma_e(t^{\frac{1}{2}}\gamma a w_1^{\pm1}v_1^{\pm1})\Gamma_e(t^{\frac{1}{2}}\beta^{-1}a^{-1}w_1^{\pm1}v_2^{\pm1})\Gamma_e(t^{\frac{1}{2}}\beta a^{-1}w_2^{\pm1}v_1^{\pm1})\Gamma_e(t^{\frac{1}{2}}\gamma^{-1}aw_2^{\pm1}v_2^{\pm1})\,.$$

It's really great there is all these wonderful formulas. arXiv really is a LIFO queue. Every day there is something new and you forget about what happened just before.

Now to find some common ground:

- why are they doing all of this?
- there are lots of generalized factorials (and we have found some) these formulas are very specific and what had motivated the authors to build them
- what's in it for the rest of us? what are they counting?

Just like an elementary school reading comprehension exercise. What is the main idea?

[1] An iPhone is a very complicated thing and we don't know how to use it, except for basic features.

C These people are intrested in the **superconformal index** or **partition function** of certain supersymmetric gauge theories.

Here's a claim:

- superconformal indices generalize the factorial function

There's immediate defects about this idea. Uh...

- (supersymmetric) gauge theories are geometric objects
- and the answer we are getting is a "cardinality" of (unknown object).
- So we have discarded most of the information available to us I believe we are counting fixed points of torus orbits, or various combinations of $SU(2)$.

We have no idea what they are talking about anyway... so why not start with that?

One mantra floating around is that **combinatorics objects are shadows of geometric objects** and it's corollary **geometric objects are shadows of integrable systems, K-theory, etc**

And there are new-age one's like "entropy is cardinality" or even "entropy is area". And we can cross our legs and meditate on that.

C-1 To summarize part C, some physicists are generating sequences of numbers, which behave like the factorial, to arbitrary accuracy but with unclear meaning or patterns.

One thing I notice is the factorial and the Vandermonde matrix can go under the same style

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

and the Wikipedia page on Schur polynomials has other formulas of this type:

$$\frac{1}{(a-b)(b-c)(c-a)} \begin{vmatrix} a & a^2 & a^4 \\ b & b^2 & b^4 \\ c & c^2 & c^4 \end{vmatrix} = abc(a+b+c)$$

To me this is the oldest style of problem. Take a sequence of numbers and say something about them.

References

(1) **Wikipedia** "Factorial Function" https://en.wikipedia.org/wiki/Factorial