Examples: Gamma Functions

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I found this neat little formula on the internet:

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3} \cdot \sqrt{2 + \sqrt{3}}$$

My question was answered by Noam Elkies[1] using various cheap multiplication tricks, he derives th formula in question. He explains to me a bit what I am looking at, and why some of these equations might be happening.[2]

The core equation: **mirror formula** is really kind of the only formula there is for the Gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

[1]Fellow Stuyesant alumnus

[2]Equations like these are divorced from applications. I go to an engineer's desk and read one equation on page of his notes - completely irrelevant to the application he has in mind - and run with it.

Given the connection between the Gamma function and the factorial: $\Gamma(n+1) = n!$ we get a relation between the factorial and the sine.[3]

Here is one more:

$$F(\tfrac{1}{4}, \tfrac{1}{4}; 1; \tfrac{1}{64}) = \sqrt{\frac{2}{7\pi}} \times \left[\frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{\Gamma(\frac{3}{7})\Gamma(\frac{5}{7})\Gamma(\frac{6}{7})} \right]^{1/2}$$

expressed in terms of the hypergeometric function. I could not find an infinte product for general hypergeometric funtions, but there could be for special values.

$$F(\tfrac{1}{4}, \tfrac{1}{4}; 1; \tfrac{1}{64}) = \frac{\Gamma(1)}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})} \int_0^1 \frac{dz}{\sqrt[4]{z^3(1-z)(1-zx)}}$$

and $\Gamma(1) = 0! = 1!$ Just trying to make it look like binomial coefficients.

In general there is something called **Chowla-Selberg** formula. Legendre knew

$$\int_0^{\frac{\pi}{2}} \frac{dt}{1 - k^2 \sin^2 t} = \frac{2^{2/3} 3^{1/4}}{8\pi})\Gamma(\frac{1}{3})^3$$

[3] De Moivre's formula $e^{ix} = \cos\theta + i \sin\theta$ is already quite exotic since it claims that exponentials and trigonometry are related. More fudamentally:

$$\textbf{size} \asymp \textbf{angle}$$

The relationship between factorial $n!$ and trig function $\sin\theta$ is a bit more exotic.

where $k = \sin \frac{\pi}{12}$ and there is an Elliptic curve related to $\mathbb{Q}(\sqrt{-3})$.

And Elkies knew these special integrals are artifacts of possibly

- Colmez conjecture
- Abelian varieties or Shimura Varieties
- Chowla-Selberg or Gross-Zagier formulas
- Complex multiplication
- Motives, Homology, etc
- Andre-Oort conjecture

Unfortunately these are written in very complicated abstract language. It is very likely that classical computations (with an $\int$-sign) could exhibit they phenomenon they are talking about.

One shorthand they use is to say:

$$\phi \in H^1 \longleftrightarrow \int_a^b \in H^1$$

I have written the correspondence in schematic and incorrect fashion.

With zero knowledge of this field a few surprises already:

- Why is there no special Gamma function for numberfields $\Gamma_{\mathbb{Q}(i)}$, $\Gamma_{\mathbb{Q}}(\sqrt{3})$? etc.

At the heart is the first contour integral we always know:

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}} = \tan^{-1} x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi$$

There is lots of questions here.[4] In between the lines this is a question about the curve:

$$y^2 = x^2 + 1$$

This is a hyperbola over real numbers $\mathbb{R}$, and is a **sphere** (genus 0) over $\mathbb{C}$.

Here is the example from Colmez own paper. Let $\epsilon = e^{i\pi/8}$ (this is an octogon)

$$\int_{\epsilon}^{\epsilon^3} \frac{x^3 - x}{\sqrt{x^8+1}} \frac{dx}{x} = \frac{2\pi i}{8}(\epsilon^6 - \epsilon^2) \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} \right)$$

Here it is instructive to draw the octogon where the poles should lie, and the line between two corners.

[4] I just made up this formula I don't even remember – I assume Cauchy residue formula is correct, without verifying the approximations made in the proof work.

So what is this new-fangled language Math professors are talking about? Here is a formula for a Faltings height:

$$h_{\mathsf{Fal}}(X_{y^2=x^5+1}) = \log 2\pi - \frac{1}{2}\log\left(\Gamma(\frac{1}{5})^5\Gamma(\frac{2}{5})^3\Gamma(\frac{3}{5})\Gamma(\frac{1}{5})^{-1}\right)$$

Obviously this is an **entropy**. Except I don't know what a Jacobian variety or a Faltings height.

At least here are some of my own thoughts. I might start by using Euler's formula for the factorial:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)(x+2)\ldots(x+n)}$$

so what could we mean by re-ordering half an object?

$$\Gamma(\tfrac{1}{2}) = \lim_{n\to\infty} \frac{\sqrt{n}\; n!}{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\cdots\times\left(\frac{1}{2}+n\right)}$$

This number is related to the middle binomial coefficient. We have that:

$$\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\cdots\times\left(\frac{1}{2}+(n-1)\right) = \frac{(2n)!}{2^n\, n!}$$

The DeMoivre-Laplace limit formula - for the middle binomial coefficient

$$\frac{(2n)!}{n!\times n!} = \binom{2n}{n} \asymp\; 2^{2n}\times\frac{1}{\sqrt{\pi n}}$$

Somehow these stupid objects have to yield statements about Galois theory, L-functions, etc.

References

(1) Xinyi Yuan, Shou-Wu Zhang **On the Averaged Colmez Conjecture** `arXiv:1507.06903`

(2) Pierre Colmez **Periodes des Varietes Abeliennes a Multiplication Complexe** Annals of Mathematics Vol. 138, No. 3 (Nov., 1993), pp. 625-683

(3) David Mumford **Abelian Varieties** American Mathematical Society, 2012.

(4) Bruno Klingler, Emmanuel Ullmo, Andrei Yafaev **Bi-algebraic geometry and the AndrÃl'-Oort conjecture** Preprint.