

# Tune-Up: Clock Arithmetic

**Thm** (2010) “Non-Conventional” ergodic theorem:

- $T : \mathbb{Z}^r \curvearrowright (X, \Sigma, \mu)$  commuting probability-preserving  $\mathbb{Z}^r$  actions
- $(I_N)_{N \geq 1}$  Følner sequence of subsets of  $\mathbb{Z}^r$ .
- $(a_N)_{N \geq 1}$  sequence of points in  $\mathbb{Z}^r$
- $f_1, f_2, \dots, f_d \in L^\infty(\mu)$

The sequence of “non-conventional” ergodic averages converges in  $L^2(\mu)$ .

$$\frac{1}{|I_N|} \sum_{n \in I_N + a_N} \prod_{i=1}^d f_i \circ T_i^n$$

Let’s try a re-phrasing of the theorem.

**Thm** (2010) The sequence  $x_n$  converges.  $x_n \rightarrow x$ .

Let’s try with a couple of more details. We have the sequence of averages of other functions.

**Thm** (2010) The sequence of functions  $\text{Avg}_1(f), \dots, \text{Avg}_n(f)$  converges in the function space  $L^2(\mu)$ . Here  $\text{Avg}_n = \sum_{i=0}^{n-1} T^i$  or “shuffle  $n$  times”, we have separated the procedure from the thing it’s acting on.

**Ex**  $\mathbb{Q}^\times$  is a multiplicative group. I am being sloppy we could write  $\mathbb{Q}^\times \simeq \mathbb{Z}^\infty$ . This is a contradiction because we could write an equally true statement for any number field  $F^\times \simeq \mathbb{Z}^\infty$ . The term seems to be “locally compact abelian group”.

**Ex**  $\times 2 \times 3$  these are commuting operations. The arithmetic makes it instantly clear even though

- “divide by 2 parts and shuffle”
- “divide by 3 parts and shuffle”

could be very different if we switch the operations.

**d=1** This is the **Von Neumann ergodic theorem** which is already in the textbook<sup>1</sup>.

**Thm** (Mean Ergodic Theorem) Let  $(X, \mathcal{B}, \mu, T)$  be a measure-preserving system.

$$\frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n$$

converges to a limit for any function  $f \in L^2(\mu)$ .

Here  $U_T : L^2(\mu) \rightarrow L^2(\mu)$  is the operator associated to the measure-preserving transformation.

$$U_T(f) = f \circ T.$$

How do we use the ergodic theorem?

- mostly likely we will observe the dynamical system *first*
- Then we can guess the limiting average behavior(s).
  - chaotic
  - various patterns
- we can guess or prove that the behavior is ergodic or possibly mixing

**Example** A radio channel has an average sound that's just slightly over *silent*.

**Example** A highway is 50% occupied and half the cars are moving “left” and half the cars are moving “right”.

**Example** The average color of a TV screen is gray.

In colloquial speech we are comparing the two ways of averaging a dynamical system.

$$[\text{space average}] = [\text{time average}]$$

so if we observe a dynamical system over time we compute the average we could have also computed the average over space. Ergodicity is a spectral property so this really depends on the “color” of the “noise” that we are observing.<sup>2</sup>

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<sup>1</sup>it's new textbook as of 2011

<sup>2</sup>Hence they are called “observables”.

## References

- [1] Tim Austin **On the Norm Convergence of non-conventional ergodic averages** arXiv:0805.0320
- [2] Manfred Einsiedler, Thomas Ward **Ergodic Theorem with a View Towards Number Theory** GTM#259. Springer, 2011.
- [3] solenoids ...*Ergodic Theory and Dynamical Systems*