Tune-Up: Pythagoras Triples

I seem to confuse two slightly different problems in Number Theory: pythagoras triples and primes as the sum of two squares. Without these, there's not much hope for anything else. The more we solve it, the more we can ask questions about the various methods, which seem a bit arbitrary. These advanced methods, seem to lump together entire classes of problems into a single bucket, without looking at any individual problem too carefully.

Ferma's Theorem says $p = a^2 + b^2$ if $p = 4k + 1$. This is true for $p \in \mathbb{Z}$. In order to do such a thing, we have actually called on *complex numbers* since we could say:

$$p = a^2 + b^2 = (a + bi)(a - bi) \in \mathbb{Z}[i]$$

In that case, we could define this as a statement of **ideals** in $\mathbb{Z}[i]$. We are trying to find the prime ideals $\mathfrak{p} \subset \mathbb{Z}[i]$.

$$[p\mathbb{Z}[i] : \mathbb{Z}[i]] = [p\mathbb{Z}[i] : (a + bi)\mathbb{Z}[i]][(a + bi)\mathbb{Z}[i] : \mathbb{Z}[i]]$$

So that $p = \mathfrak{p}\overline{\mathfrak{p}}$, with $\mathfrak{p} = a + bi \in \mathbb{Z}[i]$. We are counting the sizes of the ideals

A numerial example would to find a large prime number.

$$271 \times 271 + 476 \times 476 = 300017$$

and therefore we get a factorization of prime ideals.

$$\big[300017\,\mathbb{Z}[i] : \mathbb{Z}[i]\big] = \big[300017\,\mathbb{Z}[i] : (271 + 476i)\mathbb{Z}[i]\big]\ \big[(271 + 476i)\mathbb{Z}[i] : \mathbb{Z}[i]\big]$$

How does this neat and tidy world translate to the messy world of empirical data and statistics where nothing is certain? Where $\mathbb{R}$ is no longer adequate, since these are just limits of sequences of other measurements.

Computationally, how would we try to find numbers $a, b \in \mathbb{Z}$ such that $a^2 + b^2 = p = 300017$ and how did we know it was a prime number?

- $\sqrt{300017 - 178^2} \approx 518$ in that it's off by a small amount $\sqrt{p - a_1^2} = [\dots] + 0.0087$

This is an *approximation* type problem which gives us one more close solution in addition to the exact answer.

Examples

- $66 + 100100 = 10036$
- $\sqrt{10037 - 6 \times 6} \approx [...] + 0.005$
- $3636 + 9494 = 10132$
- $\sqrt{10133 - 36 \times 36} \approx [...] + 0.00532$
- $2424 + 9898 = 10180$
- $\sqrt{10181 - 24 \times 24} \approx [...] + 0.0051$
- $5656 + 8484 = 10192$
- $\sqrt{10193 - 56 \times 56} \approx [...] + 0.00595$
- $5454 + 8686 = 10312$
- $\sqrt{10313 - 54 \times 54} \approx [...] + 0.00581$
- $7272 + 7272 = 10368$
- $\sqrt{10369 - 72 \times 72} \approx [...] + 0.00694$
- $3636 + 9696 = 10512$
- $\sqrt{10513 - 36 \times 36} \approx [...] + 0.00521$
- $1414 + 102102 = 10600$
- $\sqrt{10601 - 14 \times 14} \approx [...] + 0.0049$
- $6060 + 8484 = 10656$
- $\sqrt{10657 - 60 \times 60} \approx [...] + 0.00595$
- $6868 + 7878 = 10708$
- $\sqrt{10709 - 68 \times 68} \approx [...] + 0.00641$
- $1818 + 102102 = 10728$
- $\sqrt{10729 - 18 \times 18} \approx [...] + 0.0049$
- $66 + 104104 = 10852$
- $\sqrt{10853 - 6 \times 6} \approx [...] + 0.00481$
- $2222 + 102102 = 10888$
- $\sqrt{10889 - 22 \times 22} \approx [...] + 0.0049$
- $66 + 106106 = 11272$
- $\sqrt{11273 - 6 \times 6} \approx [...] + 0.00472$
- $4242 + 9898 = 11368$
- $\sqrt{11369 - 42 \times 42} \approx [...] + 0.0051$
- $2424 + 104104 = 11392$
- $\sqrt{11393 - 24 \times 24} \approx [...] + 0.00481$
- $44 + 108108 = 11680$
- $\sqrt{11681 - 4 \times 4} \approx [...] + 0.00463$
- $66 + 108108 = 11700$
- $\sqrt{11701 - 6 \times 6} \approx [...] + 0.00463$
- $3030 + 104104 = 11716$
- $\sqrt{11717 - 30 \times 30} \approx [...] + 0.00481$
- $2424 + 106106 = 11812$
- $\sqrt{11813 - 24 \times 24} \approx [...] + 0.00472$
- $4848 + 9898 = 11908$
- $\sqrt{11909 - 48 \times 48} \approx [...] + 0.0051$
- $00 + 110110 = 12100$
- $\sqrt{12101 - 0 \times 0} \approx [...] + 0.00455$
- $3636 + 104104 = 12112$
- $\sqrt{12113 - 36 \times 36} \approx [...] + 0.00481$
- $2222 + 108108 = 12148$
- $\sqrt{12149 - 22 \times 22} \approx [...] + 0.00463$
- $6464 + 9090 = 12196$
- $\sqrt{12197 - 64 \times 64} \approx [...] + 0.00556$
- $2424 + 108108 = 12240$
- $\sqrt{12241 - 24 \times 24} \approx [...] + 0.00463$
- $6060 + 9494 = 12436$
- $\sqrt{12437 - 60 \times 60} \approx [...] + 0.00532$
- $6666 + 9090 = 12456$
- $\sqrt{12457 - 66 \times 66} \approx [...] + 0.00556$
- $1212 + 112112 = 12688$
- $\sqrt{12689 - 12 \times 12} \approx [...] + 0.00446$
- $7878 + 8282 = 12808$

- $\sqrt{12809 - 78 \times 78} \approx [...] + 0.0061$
- $3434 + 108108 = 12820$
- $\sqrt{12821 - 34 \times 34} \approx [...] + 0.00463$
- $5454 + 100100 = 12916$
- $\sqrt{12917 - 54 \times 54} \approx [...] + 0.005$
- $22 + 114114 = 13000$
- $\sqrt{13001 - 2 \times 2} \approx [...] + 0.00439$
- $66 + 114114 = 13032$
- $\sqrt{13033 - 6 \times 6} \approx [...] + 0.00439$
- $3838 + 108108 = 13108$
- $\sqrt{13109 - 38 \times 38} \approx [...] + 0.00463$
- $2424 + 112112 = 13120$
- $\sqrt{13121 - 24 \times 24} \approx [...] + 0.00446$
- $6464 + 9696 = 13312$
- $\sqrt{13313 - 64 \times 64} \approx [...] + 0.00521$
- $2020 + 114114 = 13396$
- $\sqrt{13397 - 20 \times 20} \approx [...] + 0.00439$
- $00 + 116116 = 13456$
- $\sqrt{13457 - 0 \times 0} \approx [...] + 0.00431$
- $7474 + 9090 = 13576$
- $\sqrt{13577 - 74 \times 74} \approx [...] + 0.00556$
- $7272 + 9292 = 13648$
- $\sqrt{13649 - 72 \times 72} \approx [...] + 0.00543$
- $1818 + 116116 = 13780$
- $\sqrt{13781 - 18 \times 18} \approx [...] + 0.00431$
- $7878 + 8888 = 13828$
- $\sqrt{13829 - 78 \times 78} \approx [...] + 0.00568$
- $3636 + 112112 = 13840$
- $\sqrt{13841 - 36 \times 36} \approx [...] + 0.00446$
- $7676 + 9090 = 13876$
- $\sqrt{13877 - 76 \times 76} \approx [...] + 0.00556$
- $2424 + 116116 = 14032$
- $\sqrt{14033 - 24 \times 24} \approx [...] + 0.00431$
- $3434 + 114114 = 14152$
- $\sqrt{14153 - 34 \times 34} \approx [...] + 0.00439$
- $1818 + 118118 = 14248$
- $\sqrt{14249 - 18 \times 18} \approx [...] + 0.00424$
- $3636 + 114114 = 14292$
- $\sqrt{14293 - 36 \times 36} \approx [...] + 0.00439$
- $5252 + 108108 = 14368$
- $\sqrt{14369 - 52 \times 52} \approx [...] + 0.00463$
- $00 + 120120 = 14400$
- $\sqrt{14401 - 0 \times 0} \approx [...] + 0.00417$
- $66 + 120120 = 14436$
- $\sqrt{14437 - 6 \times 6} \approx [...] + 0.00417$
- $7878 + 9292 = 14548$
- $\sqrt{14549 - 78 \times 78} \approx [...] + 0.00543$
- $1616 + 120120 = 14656$
- $\sqrt{14657 - 16 \times 16} \approx [...] + 0.00417$
- $3636 + 116116 = 14752$
- $\sqrt{14753 - 36 \times 36} \approx [...] + 0.00431$
- $5454 + 110110 = 15016$
- $\sqrt{15017 - 54 \times 54} \approx [...] + 0.00455$
- $2626 + 120120 = 15076$
- $\sqrt{15077 - 26 \times 26} \approx [...] + 0.00417$
- $6666 + 104104 = 15172$
- $\sqrt{15173 - 66 \times 66} \approx [...] + 0.00481$
- $00 + 124124 = 15376$
- $\sqrt{15377 - 0 \times 0} \approx [...] + 0.00403$
- $66 + 124124 = 15412$
- $\sqrt{15413 - 6 \times 6} \approx [...] + 0.00403$

- $2424 + 122122 = 15460$
- $\sqrt{15461 - 24 \times 24} \approx [...] + 0.0041$
- $5050 + 114114 = 15496$
- $\sqrt{15497 - 50 \times 50} \approx [...] + 0.00439$
- $4848 + 116116 = 15760$
- $\sqrt{15761 - 48 \times 48} \approx [...] + 0.00431$
- $00 + 126126 = 15876$
- $\sqrt{15877 - 0 \times 0} \approx [...] + 0.00397$
- $22 + 126126 = 15880$
- $\sqrt{15881 - 2 \times 2} \approx [...] + 0.00397$
- $66 + 126126 = 15912$
- $\sqrt{15913 - 6 \times 6} \approx [...] + 0.00397$
- $4040 + 120120 = 16000$
- $\sqrt{16001 - 40 \times 40} \approx [...] + 0.00417$
- $1414 + 126126 = 16072$
- $\sqrt{16073 - 14 \times 14} \approx [...] + 0.00397$
- $4848 + 118118 = 16228$
- $\sqrt{16229 - 48 \times 48} \approx [...] + 0.00424$
- $8484 + 9696 = 16272$
- $\sqrt{16273 - 84 \times 84} \approx [...] + 0.00521$
- $2222 + 126126 = 16360$
- $\sqrt{16361 - 22 \times 22} \approx [...] + 0.00397$
- $66 + 128128 = 16420$
- $\sqrt{16421 - 6 \times 6} \approx [...] + 0.00391$
- $2424 + 126126 = 16452$
- $\sqrt{16453 - 24 \times 24} \approx [...] + 0.00397$
- $1212 + 128128 = 16528$
- $\sqrt{16529 - 12 \times 12} \approx [...] + 0.00391$
- $2626 + 126126 = 16552$
- $\sqrt{16553 - 26 \times 26} \approx [...] + 0.00397$
- $4242 + 122122 = 16648$
- $\sqrt{16649 - 42 \times 42} \approx [...] + 0.0041$
- $2828 + 126126 = 16660$
- $\sqrt{16661 - 28 \times 28} \approx [...] + 0.00397$
- $3636 + 124124 = 16672$
- $\sqrt{16673 - 36 \times 36} \approx [...] + 0.00403$
- $00 + 130130 = 16900$
- $\sqrt{16901 - 0 \times 0} \approx [...] + 0.00385$
- $66 + 130130 = 16936$
- $\sqrt{16937 - 6 \times 6} \approx [...] + 0.00385$
- $3434 + 126126 = 17032$
- $\sqrt{17033 - 34 \times 34} \approx [...] + 0.00397$
- $6464 + 114114 = 17092$
- $\sqrt{17093 - 64 \times 64} \approx [...] + 0.00439$
- $4848 + 122122 = 17188$
- $\sqrt{17189 - 48 \times 48} \approx [...] + 0.0041$
- $5454 + 120120 = 17316$
- $\sqrt{17317 - 54 \times 54} \approx [...] + 0.00417$
- $3838 + 126126 = 17320$
- $\sqrt{17321 - 38 \times 38} \approx [...] + 0.00397$
- $2424 + 130130 = 17476$
- $\sqrt{17477 - 24 \times 24} \approx [...] + 0.00385$
- $88 + 132132 = 17488$
- $\sqrt{17489 - 8 \times 8} \approx [...] + 0.00379$
- $1212 + 132132 = 17568$
- $\sqrt{17569 - 12 \times 12} \approx [...] + 0.00379$
- $1616 + 132132 = 17680$
- $\sqrt{17681 - 16 \times 16} \approx [...] + 0.00379$
- $7272 + 112112 = 17728$
- $\sqrt{17729 - 72 \times 72} \approx [...] + 0.00446$
- $1818 + 132132 = 17748$

- $\sqrt{17749 - 18 \times 18} \approx [...] + 0.00379$
- $2222 + 132132 = 17908$
- $\sqrt{17909 - 22 \times 22} \approx [...] + 0.00379$
- $00 + 134134 = 17956$
- $\sqrt{17957 - 0 \times 0} \approx [...] + 0.00373$
- $4242 + 128128 = 18148$
- $\sqrt{18149 - 42 \times 42} \approx [...] + 0.00391$
- $4848 + 126126 = 18180$
- $\sqrt{18181 - 48 \times 48} \approx [...] + 0.00397$
- $9696 + 9696 = 18432$
- $\sqrt{18433 - 96 \times 96} \approx [...] + 0.00521$
- $6666 + 120120 = 18756$
- $\sqrt{18757 - 66 \times 66} \approx [...] + 0.00417$
- $7676 + 114114 = 18772$
- $\sqrt{18773 - 76 \times 76} \approx [...] + 0.00439$
- $5454 + 126126 = 18792$
- $\sqrt{18793 - 54 \times 54} \approx [...] + 0.00397$
- $3838 + 132132 = 18868$
- $\sqrt{18869 - 38 \times 38} \approx [...] + 0.00379$
- $9090 + 104104 = 18916$
- $\sqrt{18917 - 90 \times 90} \approx [...] + 0.00481$
- $5656 + 126126 = 19012$
- $\sqrt{19013 - 56 \times 56} \approx [...] + 0.00397$
- $2424 + 136136 = 19072$
- $\sqrt{19073 - 24 \times 24} \approx [...] + 0.00368$
- $66 + 138138 = 19080$
- $\sqrt{19081 - 6 \times 6} \approx [...] + 0.00362$
- $8484 + 110110 = 19156$
- $\sqrt{19157 - 84 \times 84} \approx [...] + 0.00455$
- $1616 + 138138 = 19300$
- $\sqrt{19301 - 16 \times 16} \approx [...] + 0.00362$
- $6060 + 126126 = 19476$
- $\sqrt{19477 - 60 \times 60} \approx [...] + 0.00397$
- $4646 + 132132 = 19540$
- $\sqrt{19541 - 46 \times 46} \approx [...] + 0.00379$
- $3636 + 136136 = 19792$
- $\sqrt{19793 - 36 \times 36} \approx [...] + 0.00368$
- $6464 + 126126 = 19972$
- $\sqrt{19973 - 64 \times 64} \approx [...] + 0.00397$

These examples are *abundant* yet they cost us time and resources. Should we look for more patterns? How do these compare to what we already have? What do we do about them? Just a quick-look one of them should go to a runoff:

$$19013 - 56^2 = 19973 - 64^2 \approx 0.0039682$$

This one was equality. We are looking for $\approx$ or $\asymp$ and not really $=$. Maybe $\equiv$. And there is some typos.

References

[1] Yann Bugeaud **Effective simultaneous rational approximation to pairs of real quadratic numbers** `1907.10253`

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[3] Zeev Rudnick, Ezra Waxman **Angles of Gaussian Primes** `arXiv:1705.07498`

[4] Alexander Logunov, Eugenia Malinnikova **Nodal Sets of Laplace Eigenfunctions: Estimates of the Hausdorff Measure in Dimension Two and Three** `arXiv:1605.02595`