Lookup: Ring Theory

We're pretty sure we need the tensor product. A *tensor* is just a **box**. We needed tensors to build the theory of General Relativity and described "curved" 3- and 4-dimensional spaces. They are used to describe the **curvature** of different types of "shapes" or "spaces". By the 1920's academics realized tensors would play a role in topology (the most *qualitative* study of shapes – invariant under high levels of distortion).

The naïve way of constructing the tensor product would be merely to write $x \otimes y$ with $x, y \in R$ two elements of a ring. We should have:

$$x \otimes y \neq y \otimes x$$

Then we could have the basic properties of tensor produts:

- $x \otimes (y_1 + y_2) = (x \otimes y_1) + (x \otimes y_2)$
- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$
- $(r\, x) \otimes y = x \otimes (r\, y)$

so we could have $x \in M$ and $y \in N$ elements of two R-modules (a generalization of matrices). Then we continue the inspection of the properties of the ring. Instead of building the tensor product element by element, there is a **bilinear map**:

$$\otimes : M \times N \to M \otimes N$$

In our case, $x = \vec{x} \in M$ and $y = \vec{y} \in N$ are vector spaces.

Ex if $M = \mathbb{R}$ and $N = \mathbb{R}$ then $M \times N = R \otimes R$.

Ex $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^2$.

Ex $\mathbb{R} \otimes \mathbb{Z}[i] = \mathbb{R}[i] = \mathbb{C}$.

Ex $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$. (This an example of **torsion**.)

Ex $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z} = 0$.

Ex $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$.

So there are well-behaved rules for generating entire number systems. In a graduate-level textbook or reference book, the category is called R-Mod.

Category theory could let us organize the many different number systems and "geometric" objects that arise in our computations. The inner product which is just $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$. It turns out to be more correctly written as:

$$x \cdot y = x_1 y^1 + x_2 y^2 + x_3 y^3 \in \mathbb{R}$$

could be thought us as a map from $\mathbb{R}^3 \times \mathbb{R}_3 \to \mathbb{R}$.

$$(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}) \otimes (\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}) = 9(\mathbb{R} \otimes \mathbb{R}) = \mathbb{R}^9$$

$M \otimes_R -$ and $- \otimes_R N$ are **right-exact functors** so they have well-behaved properties.

References

[1]