

# Proposal: The Factorial Function

John D Mangual

It would not be a stretch to categorize my work under the topics in my 3 previous blog posts and this one. I talk about the Stirling formula:

$$n! \approx \sqrt{2\pi e} \left(\frac{n}{e}\right)^n$$

Think about it. If you don't know much math, there are only so many starting points. So no matter how complicated our analysis gets, there should only be a few principles at play.

Here we can get away with just the trapezoid rule:

$$\log n! = \log 1 + \log 2 + \cdots + \log n$$

$$\approx \int_1^n \log x \, dx = n \log n - n$$

The goal of this project is to state and make towards specific conjectures, built from these.

This concludes my proposals for the Spring.

**A** My frustration with physics literature or math literature, I am unlikely to come up with observation of interest to others.

For one things, since my interest is about  $n!$  I am maybe interested in counting things. Within combinatorics there is:

- counting things
- finding rare and exotic objects
- studying how things are connected
- words

and many other things. I am mainly focusing on the first one here. It is called **enumerative combinatorics**. There are two kinds of applications:

- counting things predicted by mathematics
- counting things related to the outside world

The permutation group can be used to prove quadratic reciprocity.

There's a nice example of card shuffling, where measures on the permutation group converges to the uniform distribution.

The domino shuffling of the Aztec diamond can be expressed in terms of measures on the permutation group.

Binomial coefficients can be completed to geometric objects in the field of math called "integral geometry"

Galois theory, literally permutes the zeros of a polynomial and I always liked those inequalities from contests (which I could never solve)

The Taylor series uses factorials. That's by far the strangest one. What is being permuted there?

I would say combinatorics is a very healthy field and not much I could contribute there. Nothing profound or exciting.

## B

The modern theory of factorials I can't even read the formulas.

$$D_{-n} \prod_{i=1}^{\infty} f(x_i) = \frac{(-1)^{n-1}}{(2\pi i)^n} \prod_{i=1}^{\infty} f(x_i) \operatorname{Res} \left[ \frac{\sum_{i=1}^n \frac{z_n t^{n-i}}{z_i q^{n-i}}}{\left(1 - \frac{tz_2}{qz_1}\right) \cdots \left(1 - \frac{tz_n}{qz_{n-1}}\right)} \prod_{i < j} \frac{\left(1 - \frac{z_i}{z_j}\right) \left(1 - \frac{qz_i}{tz_j}\right)}{\left(1 - \frac{z_i}{tz_j}\right) \left(1 - \frac{qz_i}{z_j}\right)} \right. \\ \left. \times \prod_{i=1}^n \left( \prod_{i'=1}^{\infty} \frac{1 - t^{-1} q \frac{x_{i'}}{z_i}}{1 - q \frac{x_{i'}}{z_i}} \cdot \frac{f(z_i)}{f(q^{-1} z_i)} z_i^{-1} \right) \right].$$

Here is a very important random process

$$\frac{\Gamma(N\theta)}{\Gamma(\theta)^M \Gamma(N\theta - M\theta)} \prod_{1 \leq i < j \leq M} (x_i^{N-1} - x_j^{N-1})(x_i^N - x_j^N)^{1-2\theta} \\ \times \prod_{j=1}^M (x_i^N)^{(N-1)\theta} (1 - x_i^N)^{\theta(M-N-1)+1} \prod_{i,j=1}^M |x_j^N - x_i^{N-1}|^{\theta-1} \prod_{i=1}^M \frac{(1 - x_i^{N-1})^{\theta(N-M)-1}}{(x_i^{N-1})^{N\theta}}.$$

Here is another discussion of that random process

$$(29) \quad \sum_{a_1, \dots, a_J: |a|=j} \frac{q^{\sum_{m=1}^J (m-1)a_m}}{Z_j(J)} w_{\{u, qu, \dots, q^{J-1}u\}} \left( i_0, \{a_1, \dots, a_J\}; k_0, \{c_1, \dots, c_J\} \right) \\ = q^{\sum_{m=1}^J (m-1)c_m} \sum_{a_1, \dots, a_J: |a|=j} \frac{q^{\sum_{m=1}^J (m-1)a_m}}{Z_j(J)} w_{\{u, qu, \dots, q^{J-1}u\}} \left( i_0, \{a_1, \dots, a_J\}; k_0, \underbrace{\{1, \dots, 1\}}_n, \underbrace{\{0, \dots, 0\}}_{J-n} \right) \\ = \frac{q^{\sum_{m=1}^J (m-1)c_m}}{Z_n(J)} \sum_{\substack{a_1, \dots, a_J: |a|=j \\ d_1, \dots, d_J: |d|=n}} \frac{q^{\sum_{m=1}^J (m-1)a_m}}{Z_j(J)} w_{\{u, qu, \dots, q^{J-1}u\}} \left( i_0, \{a_1, \dots, a_J\}; k_0, \{d_1, \dots, d_J\} \right),$$

here is a physics partition function

$$Z_T(t) = \sum_{\mathcal{X} \in \mathfrak{M}^T} \exp \left( - \sum_{(x_L > x_R)} \left( \left( c_{i(x_L), i(x_R)}^+ \right)^{[0]} \beta \log \frac{x_R}{x_L} + \sum_{m \neq 0} \frac{1 - q_1^m}{m(1 - p^m)(1 - q_2^{-m})} \left( c_{i(x_L), i(x_R)} \right)^{[m]} \frac{x_R^m}{x_L^m} \right) \right) \\ \times \exp \left( \sum_{x \in \mathcal{X}} \left( \log q_{i(x)} \log_{q_2} \frac{x}{\tilde{x}} + \sum_{m=1}^{\infty} \left( \frac{1 - q_1^m}{1 - p^m} t_{i(x), m}^{(+)} x^m + \frac{1 - q_1^{-m}}{1 - p^{-m}} t_{i(x), m}^{(-)} x^{-m} \right) \right) \right) \quad (2.15)$$

Reality is complex, physical systems ultimately describe a very complicated reality.<sup>1</sup> I am betting these formulas are very important and useful. If I could read them!

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<sup>1</sup>An iPhone is a very complicated thing and we don't know how to use it, except for basic features.

## References

(1) **Wikipedia** "Factorial Function" <https://en.wikipedia.org/wiki/Factorial>