Tune-Up: Continuity

Here is the Poisson summation formula:

Thm If $f \in \mathcal{S}(\mathbb{R})$ then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$ and in fact $\sum_{n\in\mathbb{Z}} f(n+x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i nx}$ for $x \in \mathbb{R}$.

Here $\mathcal{S}(\mathbb{R})$ is the **Schwartz space** of function on the real line.

Schwartz space on $\mathbb{R}$ is the set of indefinintely differentiable functions, so that f and all it's derivatives $f^{(n)}$ are rapidly decreasing: $\sup_{x\in\mathbb{R}} |x|^k |f^(\ell)(x)| < \infty$ for every $k, \ell > 0$.

Proposition

- $\sum_{n\in\mathbb{Z}} f(x+n)$ is continuous in $x \in \mathbb{R}$.
- $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$ is continuous in $x \in \mathbb{R}$.

How do we decide when an infinite series is continous? Between the two statements we need information both about $f(n)$ and $\hat{f}(n)$ with $n \in \mathbb{Z}$. So there's no free lunch here.

Theorem If $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$.

Theorem If $f(x) = e^{-\pi x^2}$ then $\hat{f}(\xi) = f(\xi)$. Also $\int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$.

This will help us show that $s^{-1/2}\theta(1/s) = \theta(s)$ for $\theta(s) = \sum e^{-(\pi n^2)s}$ and $s > 0$.

Theorem If $f \in \mathcal{S}(\mathbb{R})$ then $f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi ix\xi}\, d\xi$.

Despite all these theorems the only Schwartz functon we have so far is $f(x) = e^{-\pi x^2}$ and $\theta(s)$ is not in $\mathcal{S}(\mathbb{R})$.

Definition A function $f(x)$ is "continuous" at the point x if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$

$$|x-a| < \delta \to |f(x) - f(a)| < \epsilon$$

These nested quantifiers seem rather difficult to understand, and they're implicit in all our other theorems. The math still works.

Example The sequence of functions $f_n(x) = x^n - x^{2n}$ converges (but not uniformly) $f_n(x) \to 0$ on $[0,1]$. Here we use a tiny bit of topology and say that $[0,1]$ is "closed".

M-test If we can show $\sup_{xinE} |a_n(x)| < M_n$ and $\sum M_n$ converges then $\sum a_n(x)$ converges absolutely and uniformly on the set E.

References

[1] Vladimir Zorich **Mathematical Analysis I** (Universitext) Springer, 2015.