Examples: Quadratic Reciprocity

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Here is a theorem from the modern number theory literature (about 10 years old):[1]

> Let π be a unitary cuspical representation of $\mathrm{GL}_2(\mathbb{A}_F)$ and χ is a unitary character of $\mathbb{A}_F^\times/F^\times$ with finite conductor $\mathfrak{f}$.
>
> There is an $N > 0$ such that:
> $L(\frac{1}{2}, \pi \times \chi) \ll \mathrm{Cond}(\pi)^N \mathrm{Cond}_\infty(\chi)^N N(\mathfrak{f})^{1/2-\frac{1}{24}}$ and also a bound for thse other L-functions
> $L(\frac{1}{2}, \chi) \ll \mathrm{Cond}_\infty(\chi)^N N(\mathfrak{f})^{1/4-\frac{1}{200}}$

This is unfortunately written at such a level of abstraction that we have no idea what is going on.

I barely know what a modular form or an L-function is (though I am seeing them constantly)

[1] Akshay Venkatesh – Sparese Equdistribution Problems, Period Bounds and Subconvexity – *Annals of Mathematics* (2005)

Then the author says if we set $F = \mathbb{Q}$ this is a *subconvexity* result due to Burgess.

I know what "convex" means - a circle is convex. A square is convex. I don't know what **subconvex** means.

Burgess proved $L(\frac{1}{2}, \chi) \ll k^{\frac{7}{32}+\epsilon}$ and the "principal difficulty" (Burgess' words) is to show an estimate like this:

$$\sum_{x=1}^{k} \left| \sum_{y=1}^{h} \chi(x+y) \right|^{2r}$$

here χ is a Dirichlet charcter (such the Legendre symbol $(\frac{\cdot}{p})$.

There is nothing convex about this. And in a way it doesn't matter since we can write down the formula:

$$L(\frac{1}{2}, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}}$$

This is divergent if $\chi \equiv 1$ – the **trivial character** but what about other sequences of ± 1 ?

References

(1) Jared Weinstein. **Reciprocity laws and Galois representations: recent breakthroughs** Bull. Amer. Math. Soc. 53 (2016), 1-39

(2) David A Cox. **Primes of the Form $x^2 + ny^2$: Fermat, Class Field Theory, and Complex Multiplication** Wiley, 2013.

(3) **A prime ideal $\mathfrak{p}$ decomposes in $\mathbb{Q}(\zeta_{24})/\mathbb{Q}(\sqrt{-6})$ iff it is generated by $\alpha \in 1 + 2\mathbb{Z}[\sqrt{-6}]$** http://mathoverflow.net/q/234570/1358

(4) Roy L. Adler **Symbolic dynamics and Markov partitions** Bull. Amer. Math. Soc. 35 (1998), 1-56 http://www.ams.org/journals/bull/1998-35-01/S0273-0979-98-00737-X/