Roth's Theorem on Sequences

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0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Square-Free Numbers No prime divisor is repeated twice. These have density about $2/3$ in the number line, $\mathbb{Z}$

$$90 = 2 \times 3 \times 3 \times 5$$

is not a square-free number. However the red number:

$$85 = 5 \times 13$$

The classic exercise is to show the probability of a number being square-free is $\frac{\pi^2}{6}$

$$\mathbb{P}\big[\min_{x,y\in\mathbb{Z}} |ax+by| = 1\big] = \prod_p \left(1 - \frac{1}{p^2}\right) = \sum_{n>0} \frac{1}{n^2} = \frac{\pi^2}{6}$$

And we will need a proof of the last step to pipe into further work - when we actually try to solve Roth's Theorem.

Proof # 15 Every natural number can be written as the sum of four squares (if you look hard enough).

$$n = x^2 + y^2 + z^2 + t^2$$

This can be proven using geometry of numbers and the fact that $\pi^2 > 8$.

Let $r(n) = \#\{(x,y,z,t) : x^2+y^2+z^2+t^2 = n\}$ be the number of solutions, which are scattered around the 3-sphere S^3.

How many integer points are there **inside** the 3-sphere?

$$R(n) = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 \leq n\} \sim \frac{\pi}{2} \cdot N^2$$

using the formula for $r(n)$ we can find a formula for $R(n)$:

$$r(n) = 8 \sum_{m|n, 4 \nmid m} m$$

This is really clumsy to omit 4's like this and I don't explain why:

$$R(N) = 1 + \sum_{n=1}^{n} \sum_{m|n, 4 \not{d}ivm} m = 1 + 8 \left(\sum_{m \leq N} m \left[\frac{N}{m} \right] - \sum_{m \leq N} m \left[\frac{N}{4m} \right] \right)$$

One can show that $\zeta(2)$ mysterious pops out of lattice point counting under a hyperbola:

$$\theta(x) = \sum_{mr \leq x} x = \zeta(2) \cdot \frac{1}{2} x^2 + O(x \ \log x)$$

Finally:

$$R(N) \sim \frac{\pi^2}{2} N^2 \sim 1 + 8\big(\theta(N) - \theta(N/4)\big) \approx 4\zeta(2) \left(N^2 - \frac{N^2}{4}\right)$$

This is from a textbook “Introduction to Number Theory” by Hua Look Keng, but I found it in Robin Chaptman’s list of proofs of $\zeta(2)$.

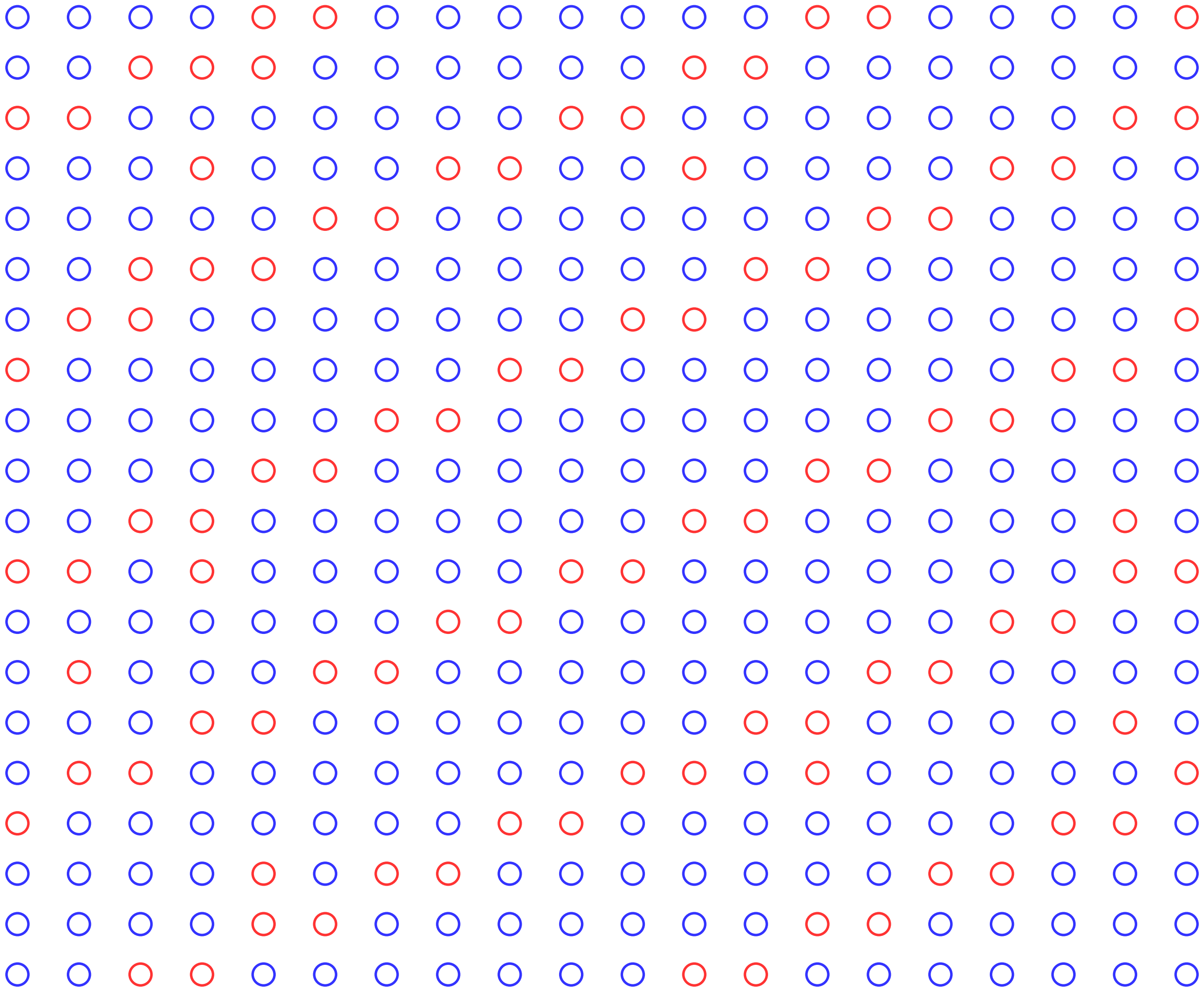
The big questions to me here are why theta functions appear at all... since:

$$\theta(q)^4 = \left[\sum q^{n^2}\right]^4 = \sum r(n) q^n$$

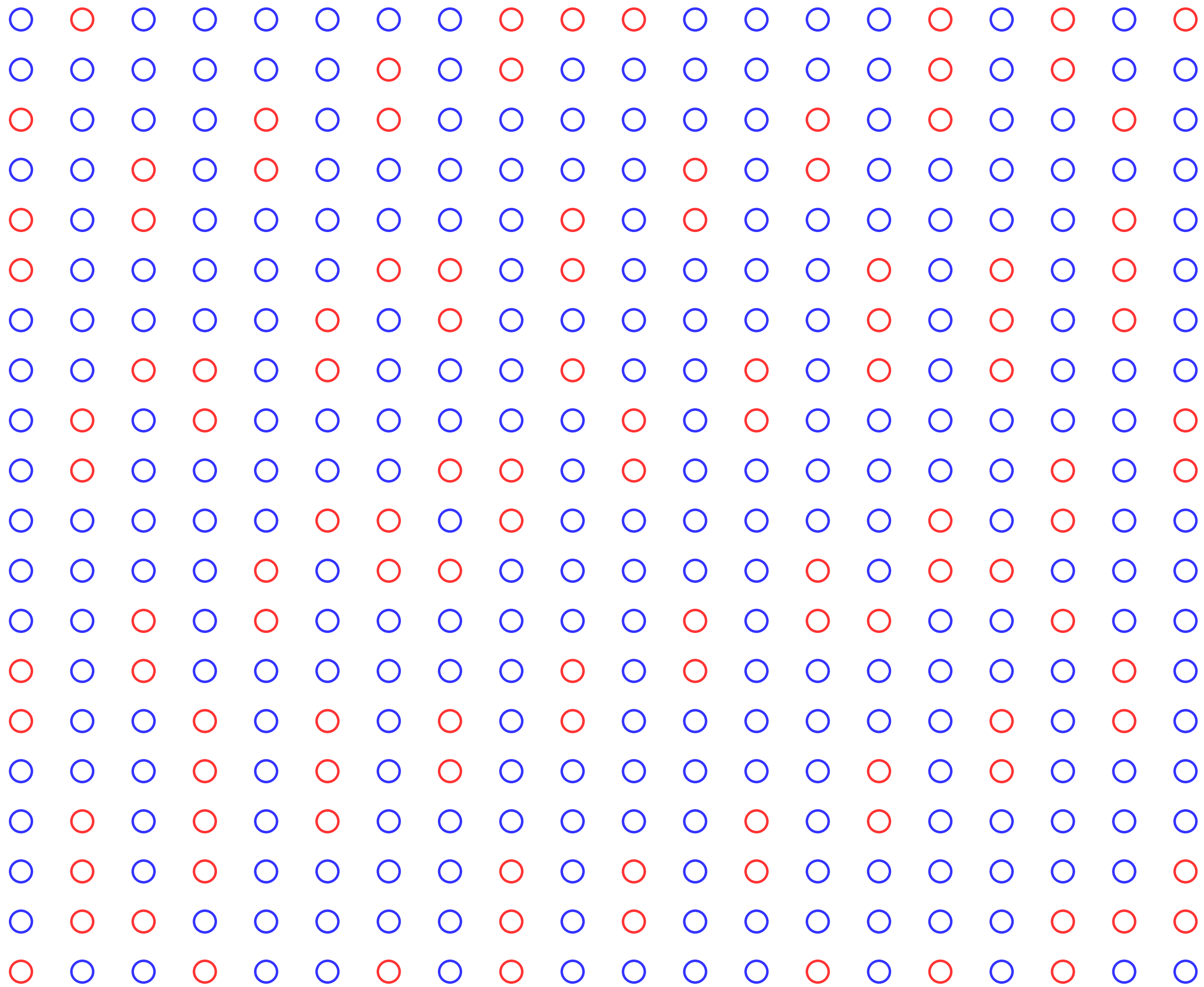
By ommitting his derivation of the sum of 4 squares formula, we lose a big chance to get at the geometry of lattices and number theory.

Moving on. . . let’s find other sequences of positive upper density!

Square-Free values of $x^2 + 2$



Square-Free values of $x^2 + 17$

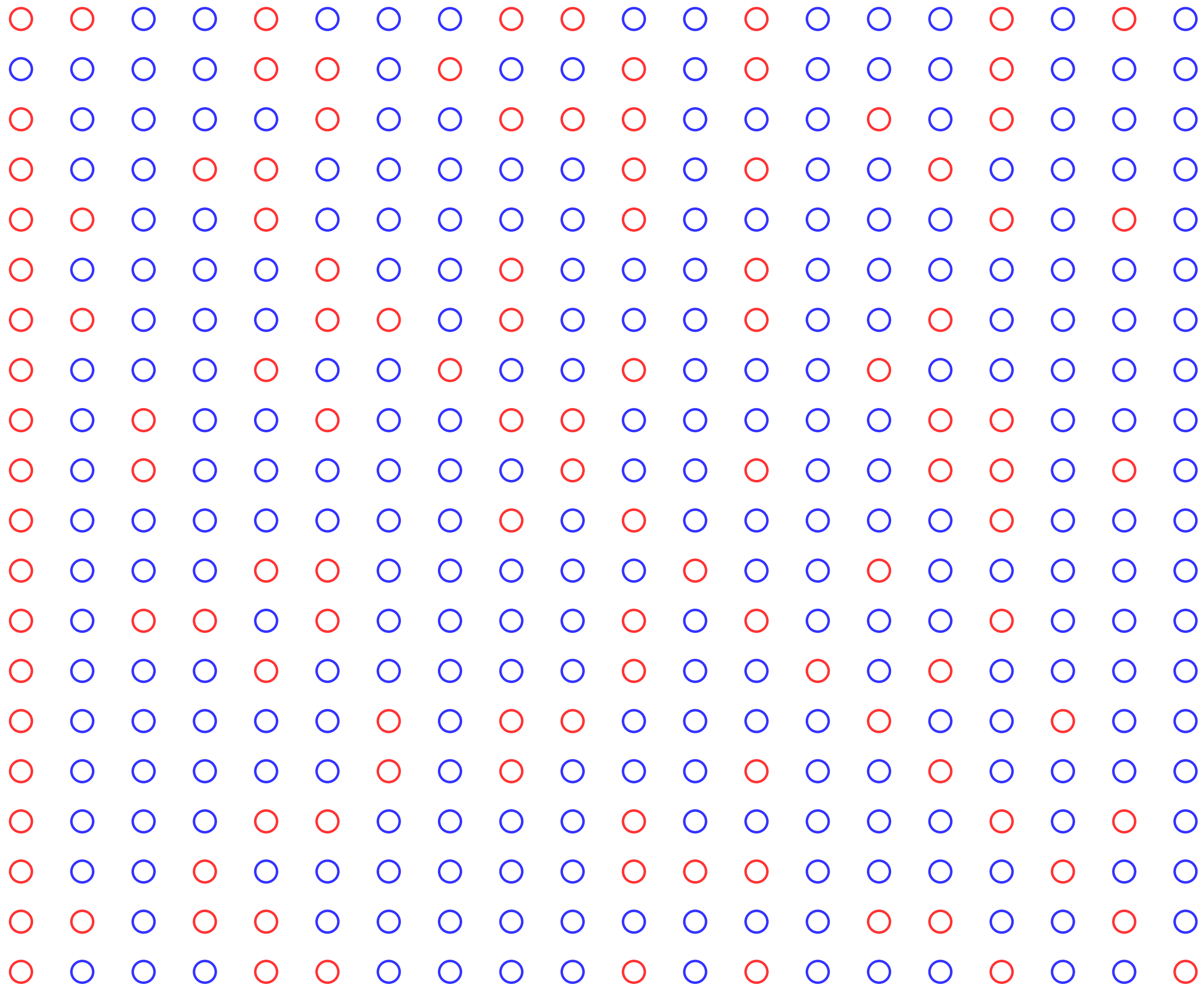


The square-free values $\{n : n^2 + k \not\equiv 0 \mod \Box\}$ have positive density[1]. I think proving existence of arithmetic sequence in these sets (or just the vanilla square-frees) should be much easier than proving for arbitrary sequence.

Square-Free numbers are so well-understood that we are better off not using these sequences to test Roth's Theorem.

[1] I found this on a Sieve Theory course by Zeev Rudnick

Numbers n with prime factors $p > \sqrt{n}$



References

(1) JP Serre **Course on Arithmetic** Springer-Verlag

(2) Francesco Cellarosi, Yakov G. Sinai **Ergodic Properties of Square-Free Numbers** `arXiv:1112.4691`

(3) Michael Baake, Christian Huck **Ergodic properties of visible lattice points** `arXiv:1501.01198`