Primes in Arithmetic Sequences, Entropy, Hyperbolic 3-Manifolds

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One way to show there are infinitely many primes is to show certain series diverges[1].

$$\sum_{p \in \mathcal{P}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty$$

By **fundamental theorem of arithmetic** $n = p_1^{a_1} \dots p_n^{a_n}$ and then

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p \in \mathcal{P}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) \asymp \exp \left[\sum_{p \in \mathcal{P}} \frac{1}{p} \right] = \infty$$

Both of these sequences diverge. I left out a tiny bit of work.

[1] I get bored just looking at such a series. "Music of the primes" my foot!

In representation theory we learn about character theory over finite groups. In our case $G = (\mathbb{Z}/q\mathbb{Z})^\times$ the multiplicative group of mod q arithmetic. And q is prime (so there are no zero-divisors[2]).

The group ring we are dealing with in this representation theory is: $\mathbb{C}[(\mathbb{Z}/q\mathbb{Z})^\times](s)$ and instead of doing Pigeonhole Principle[3] over $\mathbb{Z}/q\mathbb{Z}$ or even $(\mathbb{Z}/q\mathbb{Z})^\times$ we are doing it over the dual space of functions $f : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$.

One of the character $\chi \equiv 1$ has infinitely many primes and the rest $\chi \not\equiv 1$ have finitely many at best[4].

I don't know if Schur's Lemma exists on a space like $(\mathbb{Z}/q\mathbb{Z})^\times$ but Dirichlet didn't worry about that.

[2] In Haskell these could be **Maybe** types

[3] this is **Dirichlet's** Pigeonhole Principle

[4] In quantum mechanics one might have $\langle p|x\rangle = e^{ipx}$ - with the position-space and momentum space. Then $(x, p) \in \mathbb{C}^2$ is "symplectic".

Let $L(1,\chi) = \sum \chi(n) n^{-s}$, then using finite fourier analysis:

$$\frac{1}{q-1}\sum_{\chi} \log L(1,\chi) = \sum_{p\equiv a(q)} \sum_{m=1}^{\infty} m^{-1} p^{-ms} \approx \sum_{p\equiv a(q)} p^{-s} \geq 0$$

I left out $p^m \equiv 1 \mod q$ since I am just copying the formula[5][6].

The pigeonhole principle reads:

$$\prod_{\chi} L(1,\chi) \geq 1$$

and the problem is not all the characters can "fit" inside a certain space[7]. The volume probably **is** an L-function.

$$L_1(s) < (1-q^{-2})\zeta(s) < \frac{A}{s-1}$$

Plausible.

[5] Furstenberg might be pleased the solution set to $a^m \equiv 1 \mod q$ is the union of arithmetic sequences.

[6] OK. The only terms that matter are $m = 1$. This is also pigeonhole since we have subtracted away only finitely many.

[7] I am willing to bet this is a volume of some kind in the space of adeles $\mathbb{A} = \prod' \mathbb{Q}_p$.

If $\omega \neq 1$, by the **Mean Value Theorem**

$$L_\omega(s) - L_\omega(1) = (s-1)L'_\omega(s_1)$$

I would be challenged to find the value of s_1... this is a far cry from moving towards the tangent line of the parabola.

$$L'_\omega(s) = \sum_{n \not\equiv 0(q)} \chi(n) \log n \, n^{-s} \ll 1$$

We have approximated $L(1,\chi)$ with a line. The contradiction is something like:

$$L(s,\mathbf{1}) \times \cdots \times L(s,\chi)\overline{L(s,\chi)} < A(s-1) \times \left[\frac{B}{s-1}\right]^2$$

and if we let $s \to 1$ the product is zero, but we know this product is ≥ 1.

I can't decipher how this isabout prime number in arithmetic sequence. We have yet to show $L(1,\chi) \neq 0$ for $\chi \in \{1,-1\}$

A tidier way to look at this... at maybe to motivate some geometry is a matrix $\mathcal{A}(s)$ which changes between two bases of L-functions:

$$\mathcal{A}(s)L(s, q\mathbb{Z} + \vec{a}) = L(s, \vec{\chi})$$

Then by the mean Value theorem we can approximate: $\mathcal{A}(s) = \mathcal{A}'(1)(s-1) + \mathcal{A}(1)$ near $s = 1$. We are constantly adding and discarding parts which do not diverge near $s = 1$...

Or we can multiply by $s = 1$ (taking the "residue") and study the behavior of $L(s)$ at $s \approx 1$ by studying the operator $\mathcal{A}(s)$ at $s \approx 1$.

This could be called **Hadamard first-variation formula** or **first-order perturbation theory** or **Newton's definition of derivative** or **Feynman-Hellman formula**.

This page is speculative.

References

(1) JP Serre **Course on Arithmetic** Springer-Verlag

(2) Davenport **Multiplicative Number Theory** Springer-Verlag