Fibonacci Numbers

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Every issue of Mathematics Magazine is flooded with proofs of Fibonacci identity. I myself have solved a few. So it is surprising to see a discussion by leading dynamicist and Fields Medallalist Curtis McMullen.

Let $\epsilon \in \mathbb{R}$ be an algebraic unit of degree two over $\mathbb{Q}$. Then $x = \epsilon$ solves a quadratic equation:

$$x^2 - ax + b = 0$$

with $a, b \in \mathbb{Q}$. McMullen writes instead:

$$\epsilon^2 = t\epsilon - n$$

with $t = \mathrm{tr}_{\mathbb{Q}}^K(\epsilon)$ and $n = \mathrm{N}_{\mathbb{Q}}^K(\epsilon) = \pm 1$.

In the number theory jargon:

- $\mathbb{Z}[\epsilon]$ is called a **order** in the field $K = \mathbb{Q}(\epsilon)$.
- The discriminant is $D = t^2 - 4n > 0$.
- $(1, \epsilon)$ is a basis for $\mathbb{Z}[\epsilon] \subset \mathbb{R}$.

We represent algebraic numbers by 2×2 matrices[1]

$$\epsilon = \begin{pmatrix} 0 & -n \\ 1 & t \end{pmatrix}, 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sqrt{D} = \begin{pmatrix} -t & -2n \\ 1 & t \end{pmatrix}$$

Here's one place where Curtis gets tricky. He says:

$$\mathrm{tr}^K_{\mathbb{Q}} : \mathrm{M}_2(K) \to \mathrm{M}_2(Q)$$

a 2×2 matrix in $K = \mathbb{Q}(\epsilon)$ is like a 4×4 matrix in $\mathbb{Q}$ (with some rules)[2].

$$\left[\begin{array}{cc|cc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right]$$

[1] this should be extremely bothersome... and we haven't even done cubic fields...

[2] Schemes could be defined as matrices satisfying certain equations. If Alexander Grothendieck hadn't been around, we could say these equations define a "variety" but with additional problems. And spend hours hunting through our commutative algebra textbooks for the properties of these rings. So there you have it a **scheme** is a set of **equations** with certain **problems**.

2 - some scratchwork

How to turn 2×2 matrix into 3×3 matrix?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

There is "isomorphism" $\mathrm{SL}_2(\mathbb{R}) \simeq \mathrm{SO}_{2,1}(\mathbb{R})$:

$$\frac{1}{ad-bc}\begin{pmatrix} \frac{1}{2}(a^2-b^2-c^2+d^2) & ac-bd & \frac{1}{2}(a^2-b^2+c^2-d^2) \\ ab-cd & bc+ad & ab+cd \\ \frac{1}{2}(a^2+b^2-c^2-d^2) & ac+bd & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix}$$

If I set $a = d = 1$, $c = 0$ and $b = t$:

$$\begin{pmatrix} 1-\frac{1}{2}(a+b)^2 & -(a+b) & -\frac{1}{2}(a+b)^2 \\ (a+b) & 1 & (a+b) \\ \frac{1}{2}(a+b)^2 & (a+b) & 1+\frac{1}{2}(a+b)^2 \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{2}a^2 & -a & -\frac{1}{2}a^2 \\ a & 1 & a \\ \frac{1}{2}a^2 & a & 1+\frac{1}{2}a^2 \end{pmatrix}\begin{pmatrix} 1-\frac{1}{2}b^2 & -b & -\frac{1}{2}b^2 \\ b & 1 & b \\ \frac{1}{2}b^2 & b & 1+\frac{1}{2}b^2 \end{pmatrix}$$

These help us solve the Pythagorean equation:

$$x^2 + y^2 = z^2$$

since we can reverse the equation[3] one has:

$$x^2 + y^2 - z^2 = 0$$

This is the quadratic form being preserved by $\mathrm{SO}_{2,1}(\mathbb{R})$ also known as a "spinor".

This is the metric used in **Special Relativity** and it is also used in the **Pythagorean Theorem**. Nobody talks about this!

[3] why don't we do this in real life? $A + B = C$ so we deduce that $B = C - A$ and other deductions of this type?

What about $(3, 4, 5)$ triangle?

$$3^2 + 4^2 = 5^2$$

That works. This matrix equation has:

$$\begin{aligned} x &= 3(1 - \frac{1}{2}a^2) + 4(-a) + 5(-\frac{1}{2}a) && (1) \\ y &= 3a + 4 + 5a && (2) \\ z &= 3(\frac{1}{2}a^2) + 4a + 5(1 + \frac{1}{2}a^2) && (3) \end{aligned}$$

and then we always have a Pythagorean triple:

$$x^2 + y^2 = z^2$$

I might even finish off the algebra just a bit:

$$\begin{aligned} x &= 3 - 4a - 4a^2 && (4) \\ y &= 8a + 4 && (5) \\ z &= 5 + 4a + 4a^2 && (6) \end{aligned}$$

Doesn't it make sense? This is true for all a:

$$(4 - 1 - 4a - 4a^2)^2 + (8a + 4)^2 = (4 + 1 + 4a + 4a^2)^2$$

All Pythagorean triples can be written this way for $m, n \in \mathbb{Z}$ – here $m = 2$ and $n = 1 + 2a$

$$\begin{aligned} x &= m^2 - n^2 && (7) \\ y &= 2mn && (8) \\ z &= m^2 + n^2 && (9) \end{aligned}$$

2 - What Fibonacci sequence does McMullen have in mind?

$$f_{m+1} = tf_m - nf_{m-1}$$

with $f_0 = 0$ and $f_1 = 1$.

With are 2×2 matrices in $K = \mathbb{Q}(\epsilon)$ or 4×4 matrices in $\mathbb{Q}$.

$$f_m = \mathrm{tr}^K_{\mathbb{Q}}(\epsilon^m/\sqrt{D})$$

and asymptotic $f_m \asymp \epsilon^m$ when m is very large.[4]

As numbers we get Fibonacci identity:

$$\epsilon^m = f_m \epsilon - nf_{m-1}$$

as $U \leftrightarrow \epsilon$ as 2×2 matrices:

$$U^m = f_m U - nf_{m-1} I$$

and he even puts for us:

$$U^m = \begin{pmatrix} -n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_{m-1} & f_m \\ f_m & f_{m+1} \end{pmatrix} \equiv f_{m+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod f_m$$

These 2×2 congruence formulas are delighful. Stanley's Enumerative Combinatorics should be full of them[5]

[4] So in the usual Fibonacci sequence:

$$f_m \asymp \left(\frac{1+\sqrt{5}}{2}\right)^m$$

[5] And a discussion of "trace" as defined by numbertheorists discussed in short article by Keith Conrad http://bit.ly/2bXUjNL or in textbooks on Algebraic Number Theory

The lattice Curtis McMullen winds up picking is:

$$\left(\begin{array}{c|c} f_{m+1} - f_{m-1} & f_{m+2} - f_{m+1} - f_m \\ \hline 0 & f_m \end{array}\right)$$

and if you want we can pick the simplest one:

$$f_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m$$

the numbers in this case are $\mathbb{Q}(\sqrt{5})$ and our 2×2 matrix supplies infinitely many geodesics in half-plane $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.

If I read correctly the numbers:

$$\frac{f_{m+1} - f_{m-1}}{f_m} \in \mathbb{Q}$$

all have distinct infinitely repeating continued fractions with bounded coefficients.

someting is not quite right I am afraid

3 In Section 4 - Curtis McMullen recommends:

$$\frac{f_{m+1} - f_{m-1}}{f_m} \cdot \frac{1+\sqrt{5}}{2}$$

or that $\mathbb{Q}(\sqrt{5})$ has infinitely elements:

$$\overline{[(1,1)^m, 1, 2, 0, (1,1)^m, 1, 2, 4]} \in \mathbb{Q}(\sqrt{5})$$

setting $s = 1$ into his formula. He leaves generating other examples as an exercise[6]

We also have Fibonacci identities for Bianchi groups $\mathrm{SL}_2(\mathcal{O}_d)$. We can writing down a definition of the Bianchi groups, but saying much about the structure of these groups of substitutions.[7]

[6] We can replace the $(1,1)^m$ with $(1,s)^m$ but his theoretical examples are even more varied.

[7] You'd think it would be as simple as $\mathrm{SL}_2(\mathbb{Z}[i])$ is just $\mathrm{SL}_2(\mathbb{Z})$ with an extra letter. That's certainly what I thought until I worked it out. There's a nice book by Luigi Bianchi in Italian:
Bianchi, Luigi (1892). "Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginar". Mathematische Annalen. `http://gdz.sub.uni-goettingen.de/dms/load/img/?PID=GDZPPN00225364X`

References

(1) Curtis McMullen. **Uniformly Diophantine Fixed Numbers in a Real Quadratic Field**

(2) Jean Bourgain, Alex Kontorovich. **Beyond Expansion II: Traces of Thin Semigroups** arXiv:1310.7190v1