Scratchwork: Mean Value Theorem

Do we understand the Mean Value Theorem at all? Let's try studying an example with sine.

$$f(b) + f(a) = (b-a)f'(c) \text{ with } b > c > a$$

This is used when we want to do a linear approxmation, which we usually write as:

$$f(x+h) \approx f(x) + f'(x) \times h$$

Do we know what these operations are $+$ and $\times$? And can we qualify this symbol $\approx$? And we still haven't set $f(x) = \sin x$.

$$\sin(\theta + h) \approx \sin\theta + h \times \cos\theta$$

and haven't told you that $\sin'\theta = \cos\theta$. It's just something that we religiously beleive. These are, for example, the x and y coordinate charts of a circle

$$x^2 + y^2 = 1 \text{ let's write } x = \cos\theta \text{ and } y = \sin\theta$$

as if we've never seen these objects in our lives.[1]

The mean value theorem let's use quantitative versions:

$$f'(x) \approx \frac{1}{2h}\big(f(x+h) - f(x-h)\big) + \frac{h^2}{2} \times \max_{[x-h,x+h]} |f''(c)|$$

Let's look at a table of exact values of sine:

- $\sin 72^\circ = \sqrt{\frac{1}{8}(5+\sqrt{5})}$
- $\sin 75^\circ = \frac{\sqrt{2}}{4}\big(\sqrt{3}+1\big)$ and $\cos 75^\circ = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$
- $\sin 78^\circ = \frac{1}{8}(\sqrt{30+6\sqrt{5}} + \sqrt{5} - 1)$

These angles are in increments of $3^\circ = \frac{\pi}{120}$ radians. And very innocently put in the values:

$$\begin{aligned}
\cos 75^\circ &= \frac{60}{\pi} \times (\sin 78^\circ - \sin 72^\circ) + (\frac{\pi}{120})^2 \times |\sin(?)| \\
\frac{\pi}{60} \times \frac{\sqrt{2}}{4} \times (\sqrt{3}-1) &= \Big[\frac{1}{8}(\sqrt{30+6\sqrt{5}} + \sqrt{5} - 1)\Big] - \Big[\sqrt{\frac{1}{8}(5+\sqrt{5})}\Big] \pm \frac{1}{4} \times \frac{\pi}{60}
\end{aligned}$$

Not that $|\sin\theta| < 1$. Not quite Ramanujan level, but it's less than obvious.[2]

References

[1] ...

[1] Later, we might try to replace this with another algebraic curve. It could even be birational to a circle. To an algebraic geometer this might be dull. Here's an example $(x, y) = (\cos 2\theta, \sin 3\theta)$ is an algebraic curve.

[2] There will be times when $|\sin\theta| < 1$ less than sufficient. See if we can find ways to bootstrap such inequalities? For the moment...