Scratchwork: Prime Number Theorem

Let's go straight for the Analytic Theorem.

> Let $f(t)$ with $t \geq 0$ be a bounded and locally integrable function and suppose that the function
>
> $$g(z) = \int_0^\infty f(t)\, e^{-zt}\, dt \text{ with } \mathrm{Re}(z) > 0$$
>
> extends holomorphically to $\mathbb{R}(z) \geq 0$. Then $\int_0^\infty f(t)\, dt$ and it equals $g(0)$.

On the one hand the language is standard, but the wording might be of some concern.

- $f : \mathbb{R} \to \mathbb{C}$ is defined only on the real axis $\mathbb{R}$ and it is bounded $|f(t)| < M$ always.
- $g(z) = \int_0^\infty f(t) e^{-zt} dt$ is the **Laplace transform** extended to complex arguments $z \in \mathbb{C}$.
- "Locally integrable" is what we mean by "integrable" when we forget the domain of definition is $\mathbb{R}$ or $\mathbb{R}_{\geq 0}$; Our example of a **locally integrable** function that is not **integrable** is:

  $$f(x) = 1 \text{ for all } x \in \mathbb{R}$$

  In fact, $\int_a^b f(x) = b - a$ however $\int_{\mathbb{R}} f(x) = \infty$. Another excellent example is:

  $$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ \\ 0 & x = 0 \end{cases}$$

  This funciton is not locally integrable at $x = 0$, no neighborhood has of $x = 0$ has a integral $\int_{[-\epsilon,\epsilon]} f(x) \notin \mathbb{R}$ does not even converge to a number.

  Could $\int_{[-\epsilon,\epsilon]} f(x)$ be something more exotic than a number? It is a **distribution** and in fact there is the **Cauchy Principal Value**.

- The theorem is designed with a specific choice of $f(t)$ in mind, it is:

  $$f(t) = \theta(e^t)e^{-t} - 1 = \frac{1}{e^t}\left[\sum_{p \leq e^t} \log p\right] - 1$$

  and then the Laplace transform is the zeta function (rather Dirichlet series) that we know stuff about:

  $$g(z) = \frac{1}{z+1}\,\Phi(z+1) - \frac{1}{z} = \frac{1}{z+1}\sum_p \frac{\log p}{p^s} - \frac{1}{z}$$

  so the question is whether we can extract information about $f(t)$ from $g(z)$:

  $$\left[\sum_p \frac{\log p}{p^s}\right] \to \left[\sum_{p \leq x} \log p\right]$$

Hopefully that explains a bit what Zagier Theorem (or Newman's Theorem) was designed to do and why where was a need in the first place. There are these two measures of "size" and we'd like to explain why one is related to the other.

What could go wrong with the limit:

$$\left[g(z) = \int_0^\infty f(t)\, e^{-zt}\, dt \right] \overset{?}{\to} \left[g(0) = \int_0^\infty f(t)\, dt \right]$$

What goes wrong with plugging in $z = 0$? There was a great textbook called "Counterexamples in Analysis" and every sentence in this discussion requires about a dozen counterexamples. At least I need them, this counterfactual reasoning is very very difficult to me.[1] When I did analysis proofs, I would write a "wishful thinking" proof or a "bad" argument, and then gradually turn all the wrong steps into correct steps by accounting for all the exceptions that could occur.

Bad Proof #1 Laplace transforms are always *very* well-behaved.

$$g(z) = \int_0^\infty f(t)\, e^{-zt}\, dt \longrightarrow g(0) = \int_0^\infty f(t)\, dt$$

This is what we just said. By continuity , this limit should hold, for all $f : \mathbb{R} \to \mathbb{C}$.

References

[1] Don Zagier "On Newman's Short Proof of the Prime Number Theorem" American Mathematical Monthly, Vol 104 No 8, Oct 1997.

[2] Gelbaum, Olmstead. **Counterexamples in Analysis** Holden Day, 1965 / Dover , 2003.

[1] Analysis would become my best subject because I knew I had to allocate the most time.