Primes in Arithmetic Sequences, Entropy, Hyperbolic 3-Manifolds

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One way to show there are infinitely many primes is to show certain series diverges[1].

$$\sum_{p \in \mathcal{P}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots = \infty$$

By **fundamental theorem of arithmetic** $n = p_1^{a_1} \dots p_n^{a_n}$ and then

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p \in \mathcal{P}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) \asymp \exp\left[\sum_{p \in \mathcal{P}} \frac{1}{p}\right] = \infty$$

Both of these sequences diverge. I left out a tiny bit of work.

[1] I get bored just looking at such a series. "Music of the primes" my foot!

Another way is to consider the set of numbers not divisible by any prime[2], call it A.

$$\frac{1}{N}|A \cap \{1, \dots, N\}| = \frac{1}{N}|A_k \cap \{1, \dots, N\}| - O\left(\sum_{p>p_k} \frac{1}{p}\right)$$

The density[3] of A is 0. Here A_k is the integers after crossing out multiples of the prime number p_k.

$$0 = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)$$

As we run over all primes, we cross out all natural numbers[4].

There's not a whole lot more to say, but I thought this was a good test case. Remember:

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots = \infty$$

[2] Always watch this words "consider" and "let" in mth papers.

[3] By **fundamental theorem of arithmetic**, $A = \{1\}$, which has density 0.

[4] I do not like arguing by contradiction. In the future, it could matter which one I pick. Here it seems to be a matter of style. It is best we continue...

I wonder if there are other starting points for the infinitude of primes:

$$0 < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < 1$$

No obvious contradiction here. Maybe there are enough prime powers to fill up the number line[5].

In order to prove Legendre's 3-squares theorem, we need to show every arithmetic sequence has infinitely many primes

$$\sum_{p \in \mathcal{P}_{a\mathbb{Z}+b}} \frac{1}{p}$$

Even though the proof uses L-series, we can get away with just $L(1)$. In fact we have written it.

[5] This is how I waste my day... with dead-end questions like these.

Even before we do that let's meditate a little bit

- $n \neq 4^a(8k+7)$
- Can be solved with "geometry of numbers" but still quite a mess[6]

I thought I found a counterexample: $4 = 2^2 + 0^2 + 0^2$

$3 = 1^2 + 1^2 + 1^2$ and $7 \neq$ but $21 = 3 \cdot 7 = 4^2 + 2^2 + 1^2$

Some degenerate examples: $49 = 7^2$ and $25 = 5^2$

$2 \times 7 = 3^2 + 2^2 + 1^2$ so we can get $8k+7$ sometimes.

$3 \times 5 = 2 \cdot 8 + 7$ and we can't have $15 = \square + \square + \square$

$5 \times 7 = 35 = 5^2 + 3^2 + 1^2$

[6] spoiler: we need Adelic geometry of numbers in order to unify all the places

References

(1) JP Serre **Course on Arithmetic** Springer-Verlag

(2) Davenport **Multiplicative Number Theory** Springer-Verlag