Examples: Gamma Functions

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I found this neat little formula on the internet:

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3} \cdot \sqrt{2 + \sqrt{3}}$$

My question was answered by Noam Elkies[1] using various cheap multiplication tricks, he derives th formula in question. He explains to me a bit what I am looking at, and why some of these equations might be happening.[2]

The core equation: **mirror formula** is really kind of the only formula there is for the Gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

[1]Fellow Stuyesant alumnus

[2]Equations like these are divorced from applications. I go to an engineer's desk and read one equation on page of his notes - completely irrelevant to the application he has in mind - and run with it.

Given the connection between the Gamma function and the factorial: $\Gamma(n+1) = n!$ we get a relation between the factorial and the sine.[3]

Here is one more:

$$F(\tfrac{1}{4}, \tfrac{1}{4}; 1; \tfrac{1}{64}) = \sqrt{\frac{2}{7\pi}} \times \left[\frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{\Gamma(\frac{3}{7})\Gamma(\frac{5}{7})\Gamma(\frac{6}{7})}\right]^{1/2}$$

expressed in terms of the hypergeometric function. I could not find an infinte product for general hypergeometric funtions, but there could be for special values.

$$F(\tfrac{1}{4}, \tfrac{1}{4}; 1; \tfrac{1}{64}) = \frac{\Gamma(1)}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})} \int_0^1 \frac{dz}{\sqrt[4]{z^3(1-z)(1-zx)}}$$

and $\Gamma(1) = 0! = 1!$ Just trying to make it look like binomial coefficients.

In general there is something called **Chowla-Selberg** formula. Legendre knew

$$\int_0^{\frac{\pi}{2}} \frac{dt}{1-k^2\sin^2 t} = \frac{2^{2/3}3^{1/4}}{8\pi})\Gamma(\frac{1}{3})^3$$

[3]De Moivre's formula $e^{ix} = \cos\theta + i\sin\theta$ is already quite exotic since it claims that exponentials and trigonometry are related. More fudamentally:

size $\asymp$ angle

The relationship between factorial $n!$ and trig function $\sin\theta$ is a bit more exotic.

where $k = \sin\frac{\pi}{12}$ and there is an Elliptic curve related to $\mathbb{Q}(\sqrt{-3})$.

And Elkies knew these special integrals are artifacts of possibly

- Colmez conjecture
- Abelian varieties or Shimura Varieties
- Chowla-Selberg or Gross-Zagier formulas
- Complex multiplication
- Motives, Homology, etc
- Andre-Oort conjecture

Unfortunately these are written in very complicated abstract language. It is very likely that classical computations (with an $\int$-sign) could exhibit they phenomenon they are talking about.

One shorthand they use is to say:

$$\phi \in H^1 \longleftrightarrow \int_a^b \in H^1$$

I have written the correspondence in schematic and incorrect fashion.

With zero knowledge of this field a few surprises already:

- Why is there no special Gamma function for numberfields $\Gamma_{\mathbb{Q}(i)}$, $\Gamma_{\mathbb{Q}}(\sqrt{3})$? etc.

At the heart is the first contour integral we always know:

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}} = \tan^{-1} x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi$$

There is lots of questions here.[4] In between the lines this is a question about the curve:

$$y^2 = x^2 + 1$$

This is a hyperbola over real numbers $\mathbb{R}$, and is a **sphere** (genus 0) over $\mathbb{C}$.

Here is the example from Colmez own paper. Let $\epsilon = e^{i\pi/8}$ (this is an octogon)

$$\int_{\epsilon}^{\epsilon^3} \frac{x^3 - x}{\sqrt{x^8+1}} \frac{dx}{x} = \frac{2\pi i}{8}(\epsilon^6 - \epsilon^2) \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} \right)$$

Here it is instructive to draw the octogon where the poles should lie, and the line between two corners.

[4] I just made up this formula I don't even remember – I assume Cauchy residue formula is correct, without verifying the approximations made in the proof work.

So what is this new-fangled language Math professors are talking about? Here is a formula for a Faltings height:

$$h_{\mathsf{Fal}}(X_{y^2=x^5+1}) = \log 2\pi - \frac{1}{2}\log\left(\Gamma(\frac{1}{5})^5\Gamma(\frac{2}{5})^3\Gamma(\frac{3}{5})\Gamma(\frac{1}{5})^{-1}\right)$$

Obviously this is an **entropy**. Except I don't know what a Jacobian variety or a Faltings height.

At least here are some of my own thoughts. I might start by using Euler's formula for the factorial:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)(x+2)\dots(x+n)}$$

so what could we mean by re-ordering half an object?

$$\Gamma(\tfrac{1}{2}) = \lim_{n\to\infty} \frac{\sqrt{n}\; n!}{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\cdots\times\left(\frac{1}{2}+n\right)}$$

This number is related to the middle binomial coefficient. We have that:

$$\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\cdots\times\left(\frac{1}{2}+(n-1)\right) = \frac{(2n)!}{2^n\, n!}$$

The DeMoivre-Laplace limit formula - for the middle binomial coefficient

$$\frac{(2n)!}{n!\times n!} = \binom{2n}{n} \asymp\; 2^{2n}\times\frac{1}{\sqrt{\pi n}}$$

Somehow these stupid objects have to yield statements about Galois theory, L-functions, etc.

$$\sin\pi x = \pi x\prod_{n\neq 0}\left(1-\frac{x}{n}\right)$$

Let's do an attempt at the modern language:

$$
\begin{array}{ccccc}
\frac{\zeta_E'(0)}{\zeta_E(0)} & = & \frac{1}{12h}\sum_{i=1}^{h}\log\big(\Delta(\mathfrak{a}_i)\big)\Delta(\mathfrak{a}_i^{-1})\big) & = & -2h_{\mathsf{Fal}}(X)+\frac{1}{2}\log D \\
\uparrow\downarrow & & & & || \\
\frac{\zeta'(0)}{\zeta(0)}+\frac{L'(\chi,0)}{L(\chi,0)} & = & \log 2\pi-\log D+\frac{w}{2h}\sum_{x=1}^{D-1}\chi(x)\log\Gamma(\frac{x}{D}) & = & ht(1+\chi_{\mathsf{Art}})
\end{array}
$$

well I drew "||" as $\uparrow\downarrow$ for a sideways equal sign. The equal signs turn into arrows anyway. This is Colmez diagram of all the different terms.[5]

- ζ_E is the zeta function for elliptic curve
- $L(\chi),0)$ is the L-function at 0 for basically $1+1+1\ldots$ (forever)

  $$1\cdot\chi(0)+1\cdot\chi(1)+\cdots=\sum_{n\geq 0}1\cdot\chi(n)$$

  Remember that $\zeta(0)=\frac{1}{2}$ and $\zeta'(0)=-(1/2)\log|2\pi|_\infty$ where $|\cdot|_\infty$ is the infinite place corresponding to $\mathbb{R}$.
- Does this normalization look right with the $\log 2\pi$?

  $$\log|2\pi|_\infty-\sum_{p<\infty}\frac{\log p}{p-1}=0$$

- X is an elliptic curve exhibiting **complex multiplication** and specifically the ring is $\mathcal{O}_E$.
- I don't know what an "Artin character" is
- Δ is the modular form of weight 12: $\Delta(a)=q\prod(1-q^n)^{-24}$
- $E=\mathbb{Q}(\sqrt{-D})\subset\overline{\mathbb{Q}}$ is a quadratic extention of the fractions. and $\mathfrak{a}$ is a "representative of the group of clsses of fraction ideals of E that on considers as a «net» of $\mathbb{C}$"

And then one looks at the Faltings height conjecture - named after Colmez - and they look quite similar. **Not** doing that today.

[5]Notice the **logarithmic derivative** $d\log f(x)=\frac{f'(x)}{f(x)}\,dx$ For example: $p(x)=\prod(x-c)$ then $d\log p(x)=\frac{p'(x)}{p(x)}=\sum\frac{1}{x-c}$

What seems to be called into question is our use of the very integers $\mathbb{Z}$ and the number π or circle $\bigcirc$. Our entire education is build on learning to use these two symbols as a model of the real world.

Colmez offers two proof (strategies) for the Chowla-Selberg formula. The two formulas in the middle are equivalent either:

- moving to the left (**analysis**)
- moving to the right (**geometry**)

So here are all the ingredients we need: the middle binomial coefficients, the eta function, some number fiels, some crazy height, some tori.

And we'll be almost not quite modern - Colmez conjecture (this is almost purely geometric) involve:

- Logarithmic derivative of Artin L-functions at $s = 0$
- Faltings heights of Abelian varieties with complex multiplication
- It generalizes $\zeta'(0)/\zeta(0) = \log 2\pi$.

Pierre Colmez takes some effort to make it accessible[6] writing:

$$\prod_{p\in\mathcal{P}} \prod_{\sigma\in H_k} |\langle \omega_\tau^\sigma, \omega_{c\tau}^\sigma, u_\sigma \rangle_p|_p = 1$$

Or there is something about the Faltings height and abelian varieties

$$ht(a) = Z(a^*, 0) \text{ for all } a \in \mathcal{CM}^0$$

perhaps CM stands for "complex multiplication". For all abelian varieties this height should equal this partition function defined over a very exotic number system.

[6]absolute complete gibberish

Where is the factorial? I started asking about $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and got this mess.

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} = \lim_{n\to\infty} \frac{\sqrt{n}\; n!}{\frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \times \cdots \times \left(\frac{1}{2} + n\right)}$$

References

(1) Xinyi Yuan, Shou-Wu Zhang **On the Averaged Colmez Conjecture** `arXiv:1507.06903`

(2) Pierre Colmez **Periodes des Varietes Abeliennes a Multiplication Complexe** Annals of Mathematics Vol. 138, No. 3 (Nov., 1993), pp. 625-683

(3) David Mumford **Abelian Varieties** American Mathematical Society, 2012.

(4) Bruno Klingler, Emmanuel Ullmo, Andrei Yafaev **Bi-algebraic geometry and the AndrÃľ-Oort conjecture** Preprint.

(5) Stephen Hawking **Zeta function regularization of path integrals in curved spacetime** Comm. Math. Phys. Volume 55, Number 2 (1977), 133-148.