Scratchwork: Eisenstein Series

There's this paper which gives an estimate in terms of theta functions, and then the same concept in terms of automorphic representations. It took so much energy to learn the first one, that I could barely get into the second. Uh…he says it's the Burgess bound, he says it's the theta correspondence. It's missing all sorts of constants, one papers written entirely in French, nobody has a clear explanation.

The modular forms textbook does not have that many examples, so I chose theta functions. There are only a few theta functions, I found *even more* theta functions. He uses eisenstein series. I found *even more* eisenstein series, and Hilbert modular forms adn the works. The research paper immediately uses automorphic forms and L-functions.

What's missing is (yet another person) slowly connecting all the pieces, as if they were LEGO's. If you look back, to mid 19th century the pre-cursors to modular forms were always there. I found in a library today, works of Weber, Kronecker, Fricke, Klein, Tannery, Ferrers, Darboux, and etc. The language of varieties and divisors was just getting started if they could think of it at all. There was classical functions, without understanding the representation theory – the symmetries motivating them.

With that pep talk, let's do some math. $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is invariant under two maps $z \mapsto z+1$ and $z \mapsto -\frac{1}{4z}$ (where $q = e^{2\pi i\, z}$); this group is called $\Gamma_0(4)$ and we have $[SL(2,\mathbb{Z}) : \Gamma_0(4)] = 6$. The recipe for Eisenstein series looks like this:

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma\, z)^s \quad \text{w/} \quad \mathrm{Im}(\gamma s) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

Yes, I need the formula. Our choice is $\Gamma = \Gamma_0(4)$. It remains to find this quotient $\Gamma_\infty\backslash\Gamma_0(4)$. By definition:

$$\begin{aligned} \Gamma_0(4) &= \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) : c \equiv 0 \pmod 4 \right\} \\ \Gamma_\infty &= \left\{ \left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) : x \in \mathbb{Z} \right\} \end{aligned}$$

If instead we used $\Gamma_0(1) \simeq SL(2,\mathbb{Z})$, the cusps are indexed by relatively prime pairs of integers (a.k.a. "reduced fractions").

$$\Gamma_{0,\infty}\backslash\Gamma_0 = \left\{ \pm(x,y) \in \mathbb{Z}^2/\{\pm 1\} : \mathsf{gcd}(x,y) = 1 \right\}$$

$$\Gamma_{0,\infty} \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto \pm(c,d)$$

As usual, the argument is really basic but I haven't done it yet, and if we set $\Gamma = \Gamma_0(4)$, I can't find it in any textbook. We'd have $ad - bc = 1$ and

$$\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = \left(\begin{array}{cc} a+cx & b+dx \\ c & d \end{array} \right)$$

It's been a long time since I've done abstract algebra computations, but I'd guess that (as **cosets**):

$$\big[\Gamma_\infty \backslash \Gamma_0(4) : \Gamma_\infty \backslash \mathsf{SL}(2,\mathbb{Z})\big] = \big[\Gamma_0(4) : \mathsf{SL}(2,\mathbb{Z})\big] = 6$$

I also remember finding that $\Gamma_0(4) \simeq \Gamma(2)$, which looks crazy but I did a change of variables:

$$\left[z \mapsto z+1, z \mapsto -\frac{1}{4z} \right] \overset{z \mapsto 2z}{\to} \left[z/2 \mapsto z/2+1, z/2 \mapsto -\frac{1}{4(z/2)} \right] = \left[z \mapsto z+2, z \mapsto -\frac{1}{z} \right]$$

This could be done using a similarity transform (I'm not even checking the other one):

$$\left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & 2 \end{array} \right)^{-1} \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & 2 \end{array} \right) = \left(\begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right)$$

Mostly I'm just scrared it'll be wrong. But look... um...

$$\left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & 2 \end{array} \right)^{-1} \left(\begin{array}{cc} 0 & 1 \\ -4 & 0 \end{array} \right) \left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & 2 \end{array} \right) = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$

That's a funny way of writing the identity matrix. That's because this is $\mathsf{PSL}_2(\cdot)$. Can I find 6 cosets? A long time ago I learned Bezout theorem:

$$\mathsf{gcd}\,(c,d) = \mathsf{min}\,\big\{ cx + dy > 0 : x, y \in \mathbb{Z} \big\}$$

Our's is a little different. Under the restriction that $c \equiv 0 \pmod 4$ how many classes can we get for this "line"?

$$(a,b) + x\,(c,d) \in \mathbb{Z}^2 \text{ for } x \in \mathbb{Z}$$

You get six chances to write things in $SL(2,\mathbb{Z})$ that are not in $\Gamma_0(4)$ what could they be?

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right), \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \left(\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right), \left(\begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \in \Gamma(2) \simeq \Gamma_0(4)$$

Not quite right.

I have found two problems on Math.StackExchange that cover my questions:

- `https://math.stackexchange.com/q/1288478/4997`
  Modular Forms: Find a set of representatives for the cusps of $\Gamma_0(4)$
- `https://math.stackexchange.com/q/1328680/4997`
  $\Gamma(2)$ **and** $\Gamma_0(4)$ **are conjugate**
- `https://math.stackexchange.com/q/2008204/4997`
  Inequivalent cusps of $\Gamma_0(4)$

Online Solution Let me begin with some generalities:

Let R be a ring with identity and $\mathbf{P}^1(R)$ the projective line over R. To define this gadget, consider the equivalence relation $\sim$ on $R \times R$ defined by $(a, b) \sim (c, d)$ if there is a unit u such that $ua = c$ and $ub = d$. Note now that the ideal $aR + bR$ generated by (a, b) depends only on the equivalence class of (a, b). Put $\mathbf{P}^1(R)$ for the set of all equivalence classes for which this ideal generated is the full ring R.

In the case of the ring $\mathbf{Z}/N\mathbf{Z}$, there is the following fact:

> **Fact** If c, d are positive integers with $(c, d, N) = 1$, then, there is $c' \equiv c \bmod N$ and $d' \equiv d \bmod N$ so that $(c', d') = 1$.

This fact allows us to choose representatives for our class $(c : d)$ so that $(c, d) = 1$.

Now, we have the following general proposition:

> **Proposition** The map
>
> $$(c : d) \mapsto \begin{pmatrix} * & * \\ c & d \end{pmatrix} : \mathbf{P}^1(\mathbf{Z}/N\mathbf{Z}) \to [SL_2(\mathbf{Z}) : \Gamma_0(N)]$$
>
> between $\mathbf{P}^1(\mathbf{Z}/N\mathbf{Z})$ and coset representatives $[SL_2(\mathbf{Z}) : \Gamma_0(N)]$ for $\Gamma_0(N)$ in $SL_2(\mathbf{Z})$ is a bijection.

Proof The map is firstly well-defined: as remarked, we work with "points" $(c : d)$ where $(c, d) = 1$; thus, there are integers a, b so that $ad - bc = 1$ and these integers are what the $*$ represents. Check that the class that $(c : d)$ maps to is independent of the choice of (a, b) chosen above!

The map is clearly surjective. Showing injectivity is a small computation which also I am happy to leave for you. $\square$

In this case, we have

$$\mathbf{P}^1(\mathbf{Z}/4\mathbf{Z}) = \{(0 : 1), (1 : 0), (1 : 1), (1 : 2), (1 : 3), (2 : 1)\}$$

and a set of coset representatives for $\Gamma_0(4)$ in $SL_2(\mathbf{Z})$ is:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}.$$

Edit Indeed, there are theorems that give coset representatives for other important congruence subgroups (viz. $\Gamma_1(N)$ and $\Gamma(N)$). See L. Kilford's book starting from Proposition 2.11. He discusses this at length. (Let me add that you'll find the following group theory fact useful: if $K \subseteq H \subseteq G$ are groups, then, there is an explicit bijection between $[G : K]$ and $[G : H] \times [H : K]$.)

Remarks: All the most important computations were skipped. We should be nervious that this instance of "modular arithmetic" on 2×2 matrices has escaped us!

Here are random formulas taken verbatim from a research paper:

$$\lambda(m) := [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(m)] = \frac{m^3}{2} \prod_{\substack{p \mid m \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right), \quad m > 2,$$

$$\lambda_1(m) := [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma_1(m)] = \frac{m^2}{2} \prod_{\substack{p \mid m \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right), \quad m > 2,$$

$$\lambda_0(m) := [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma_0(m)] = m \prod_{\substack{p \mid m \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right), \quad m \geq 1.$$

These group theory problems never go away. They just get more complex.

THEOREM 6.3

Up to modular conjugacy, there are exactly 33 congruence subgroups of the modular group which are torsion-free and of genus zero, all of which are given in Table 1.

Table 1

Index	*Level*	*Group*
6	2	$\Gamma(2)$
	4	$\Gamma_0(4)$
12	3	$\Gamma(3)$
	4	$\Gamma_0(4) \cap \Gamma(2)$
	5	$\Gamma_1(5)$
	6	$\Gamma_0(6)$
	8	$\Gamma_0(8)$
	9	$\Gamma_0(9)$
24	4	$\Gamma(4)$
	6	$\Gamma_0(3) \cap \Gamma(2)$
	7	$\Gamma_1(7)$
	8	$\Gamma_1(8)$, $\Gamma_0(8) \cap \Gamma(2)$, $\left\{\pm\begin{pmatrix} 1+4a & 2b \\ 4c & 1+4d \end{pmatrix}, a \equiv c \mod 2\right\}$
	12	$\Gamma_0(12)$
	16	$\Gamma_0(16)$, $\left\{\pm\begin{pmatrix} 1+4a & b \\ 8c & 1+4d \end{pmatrix}, a \equiv c \mod 2\right\}$
36	6	$\Gamma_0(2) \cap \Gamma(3)$
	9	$\Gamma_1(9)$, $\left\{\pm\begin{pmatrix} 1+3a & 3b \\ 3c & 1+3d \end{pmatrix}, a \equiv c \mod 3\right\}$
	10	$\Gamma_1(10)$
	18	$\Gamma_0(18)$
	27	$\left\{\pm\begin{pmatrix} 1+3a & b \\ 9c & 1+3d \end{pmatrix}, a \equiv c \mod 3\right\}$
48	8	$\Gamma_1(8) \cap \Gamma(2)$, $\left\{\pm\begin{pmatrix} 1+4a & 4b \\ 4c & 1+4d \end{pmatrix}, a \equiv c \mod 2\right\}$
	12	$\Gamma_1(12)$, $\left\{\pm\begin{pmatrix} 1+6a & 2b \\ 6c & 1+6d \end{pmatrix}, a \equiv c \mod 2\right\}$
	16	$\Gamma_0(16) \cap \Gamma_1(8)$, $\left\{\pm\begin{pmatrix} 1+4a & 2b \\ 8c & 1+4d \end{pmatrix}, a \equiv c \mod 2\right\}$
	24	$\left\{\pm\begin{pmatrix} 1+6a & b \\ 12c & 1+6d \end{pmatrix}, a \equiv c \mod 2\right\}$
	32	$\left\{\pm\begin{pmatrix} 1+4a & b \\ 16c & 1+4d \end{pmatrix}, a \equiv c \mod 2\right\}$
60	5	$\Gamma(5)$
	25	$\Gamma_0(25) \cap \Gamma_1(5)$

These tables are telling us a lot of informtion about these groups:

Table 3

Group	*Hauptmodul*	*Values at the cusps*
$\Gamma(3)$	$\left(\frac{\eta(\tau/3)}{\eta(3\tau)}\right)^3$	$3, z^2+3z+9, \infty$
$\Gamma_0(4)\cap\Gamma(2)$	$\frac{\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8}$	$4, -4, 0, \infty$
$\Gamma_1(5)$	$\frac{1}{q}\prod_{n=1}^{\infty}(1-q^n)^{-5\left(\frac{n}{5}\right)}$	$z^2-11z-1, 0, \infty$
$\Gamma_0(6)$	$\frac{\eta(\tau)^5\eta(3\tau)}{\eta(2\tau)\eta(6\tau)^5}$	$5, -4, -3, \infty$

In the expression of the Hauptmodul for $\Gamma_1(5)$, $\left(\frac{n}{5}\right)$ denotes the Legendre symbol

The abstract:

> We study and classify all the conjugacy classes of the genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$ with no elliptic elements. We show that it suffices to classify those inside the modular group and determine them completely. We also discuss an application to modular curves.

and the paper was called **Torsion-Free Genus Zero Congruence Subgroups of PSL**$_2(\mathbb{R})$ so there are a lot of mysteries in the world of fractions and 2×2 matrices and these elementary computations support the theory of modular forms. Even if it's not stated.

Elsewhere on Math.Stackexchange someone computes the orbit of ∞ under $\Gamma_0(4)$:

$$\Gamma_0(4)\cdot\infty = \left\{ \left(\begin{array}{cc} a & b \\ 4c & d \end{array}\right)\infty : ad-4bc=1 \right\} = \left\{ \frac{a}{4c} : gcd(a,4c)=1 \right\} = \left\{ \frac{p}{q} : gcd(p,q)=1, 4|q \right\}$$

Here maybe we have an orbit of $\Gamma_0(4)\subseteq PSL_2(\mathbb{Q})$ on $\infty\in\widehat{\mathbb{Q}}=\mathbb{Q}\cup\{\infty\}$. The outcome of the discussion is that there are only **three** non-equivalent cusps:

$$z=0=\frac{0}{1}, \quad z=\infty=\frac{1}{0}, \quad z=\frac{1}{2}$$

We also can obtain the orbit of zero in that discussion:

$$\Gamma_0(4)\cdot 0 = \{\frac{p}{q} : 4\nmid q, gcd(p,q)=1\}$$

Eventually we'll learn to do these baby computations for ourselves :-) And there are more like this !
Without futher ado, the rest of modular forms:

References

[1] Anton Deitmar **Automorphic Forms** (Unirsitext) Springer, 2013.

[2] Philipp Fleig, Henrik P. A. Gustafsson, Axel Kleinschmidt, Daniel Persson. **Eisenstein series and automorphic representations** `arXiv:1511.04265`