Curve Fitting

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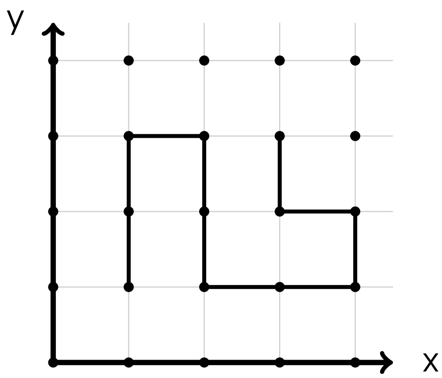
I feel there are ways to do curve-fitting that look up and down the ladder of abstractions. Let's start with something pretty vanilla:



Can we draw a curve that passes through all these points? There are two strategies to try to hook around this many constraints:

- Lagrange Interpolation
- Bezier Curves

And maybe it depends on the type of constraint problem you are trying to solve:

- passing through points
- weaving arround points

I think one way to motivate a theory is to start with a problem you want to solve.[1] We like to brag about how good we are at weaving around constraints. How bad can we be? Let $f : [0,1] \to \mathbb{R}$ be a real-valued function:

$$f(x) = \left\{ \begin{array}{cc} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{array} \right.$$

Then we can ask what the integral is between 0 and 1. Overwhelmingly, the answer should be:

$$\int_0^1 f(x)\,dx = 1 \times \Big| \{x \notin \mathbb{Q}\} \cap [0,1] \Big| + 0 \times \Big| \{x \in \mathbb{Q}\} \cap [0,1] \Big| = 1$$

Relatively innocent-functions like these are sufficient Riemann integration. I reasoned there are vastly more irrational numbers than rational numbers.

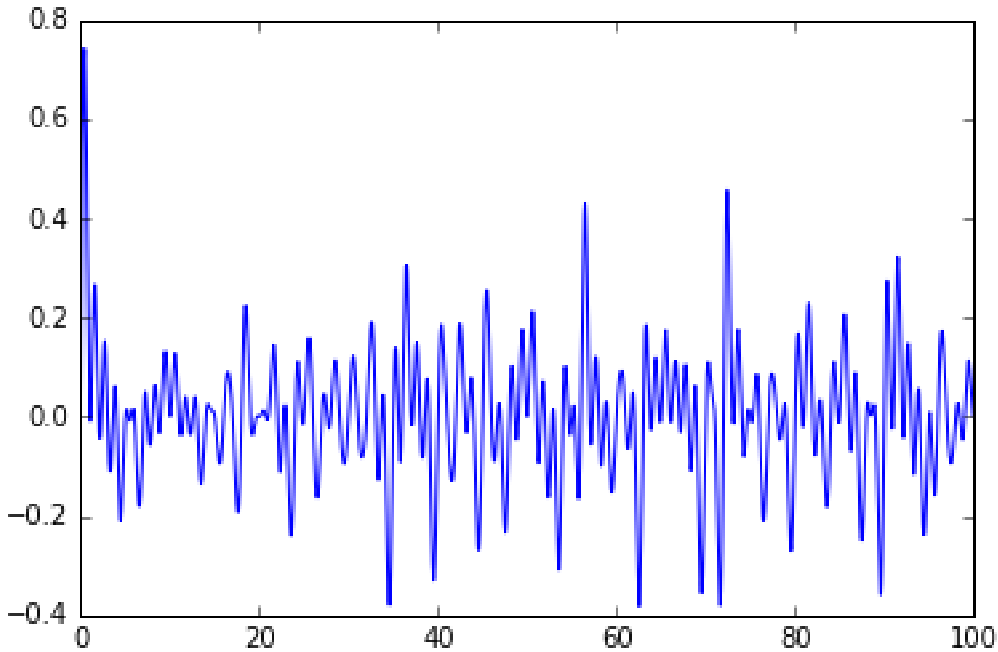
[1]Other times, theories comes out of the box, complete. I think such theories are unusuable, or they leave from for input from you and me. A comeback from that camp could be, "John, why are you so obsessed with this particular problem?" And I don't have any good reason. Just because. Sometimes about me, and the time and place makes me interested. And I could be wrong!

This turns out to be one of the worse functions around, because it looks like $\mathbf{1}$ but isn't:

$$f(x) \approx \mathbf{1} \quad \text{therefore} \quad \int f \approx \int \mathbf{1}$$

If I recall, we placed intervals around every single fraction $\frac{a}{b} \in \mathbb{Q}$ we get an upper estimate for how large this integral could be, and that upper estimate $\to 0$.

How to deal with a function that often looks like $f(x) \equiv 1$ but isn't?[2]



Here is a plot if we add all of the pure tones at all frequencies $\frac{a}{b}$ with $0 < a < b < 10$. Theoretically we have found:

$$\frac{1}{28} \sum_{0<a<b<10} e^{2\pi i \frac{a}{b} t} \approx \frac{1}{c\,N^2} \sum_{0<a<b<N} e^{2\pi i \frac{a}{b} t} = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

The constant in front is an oversimplification. The odds of two numbers being relatively primes is a famous one:

$$\phi(1) + \phi(2) + \cdots + \phi(N) = \#\{(a,b) : a < b \text{ and } \gcd(a,b) = 1\} \approx \frac{2}{\zeta(2)^2}$$

Analysis is the branch of math where we account for all the uses of the $\approx$ symbol. What if I told you the function we just charted is $\approx \mathbf{0}$?

[2] One set of problems that plagued me was if we had two competing definition of limit, maybe one returns a number and the other does not. If we have two limiting procedures 1 and 2, maybe:

$$\big[\lim\big]_1 a_n = A \quad \text{implies} \quad \big[\lim\big]_2 a_n = A$$

Most of the time we don't really care how the limit procedure is defined. Who cares really?

$$N \gg 1 \quad \text{implies} \quad a_N \approx A$$

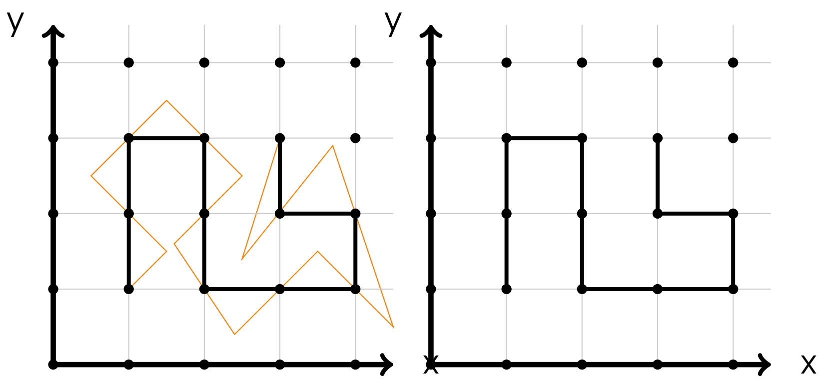
and the impliciation looks self-evident and most people won't question it. The only reason we remember the exception is because that particular conversation went on record.

When I read someone else's paper, a lot of my surprise stems from the author suggesting there is enough from in function space for something (very unlikely) to occur. Let's make a little problem set:

1 Show constant function on the rational numbers integrates to zero:

$$\int_{[0,1]} \mathbf{1}_{\mathbb{Q}} = 0$$

2 Find a curve that weaves around the obstacle course on page one:



With a pencil it's always possible to draw a curve that passes through all the dots. Our challenge is to find an equation that passes through all the loops. This is not so obvious it can always be done.

3 While I am remembering that my other example was going to be, we are going to solve the Prime Number Theorem.

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p \\ 0 & \text{not prime} \end{cases}$$

The prime number says that the density of primes is roughly $\frac{1}{\text{\# digits}}$ it can also be phrased as:

$$\sum_{n \leq x} \Lambda(n) = x + o(x)$$

where $o(x)$ is a really small, unpredictible number[3] This won't be a review of number theory. Just ironing out one or two parts in a previous discussion. It's pretty hard.

4 Look at a real paper. There are one or two short papers of Bourgain that I try to get through from time to time.[4]

$$f(t) = \sum_{|x|,|y|,|z|<N} e^{2\pi i t\,(x^2+y^2-\sqrt{2}z^2)}$$

This is my favorite way to cause trouble and I get the sense this is the recommended way to start. A sense that comes from nowhere. .

[3] if you looked at my other projects, a great question to ask would be "how small", and "how unpredictible". We could ask even though $o(x)$ is a small number, maybe it is bigger than 1 or 5 or 100.

[4] He does not write in a very forgiving way, and occasionally it's so bad, we may as well try to write the step ourselves.

All good project start from raw ingredients and tell a story.[5] For me pictures form an engine that start projects. Or it could be a memory or something that happened.

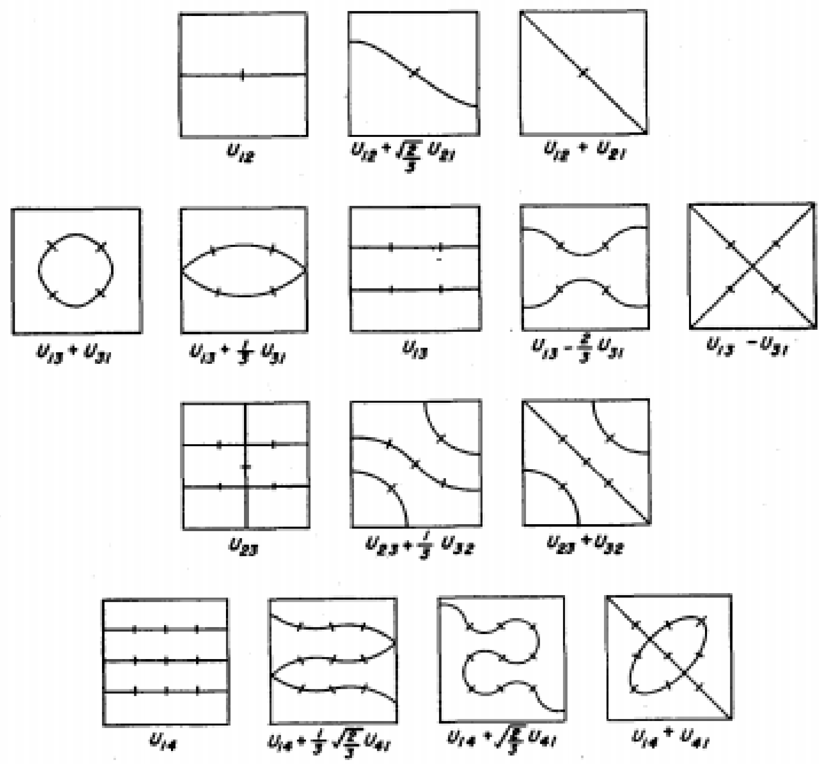


Figure 3. Nodal lines for a square membrane.

These pictures of the vibrating nodes ("notes" ♩♫♩) of a square drum were made are before computer graphics (the textbook was written in the 1930s). Drafting and technical drawing (without computers) seems like stupid and arcane technical skill, but our ability to make things of good quality is constrained by this. Maybe the textbooks fail to make the case.

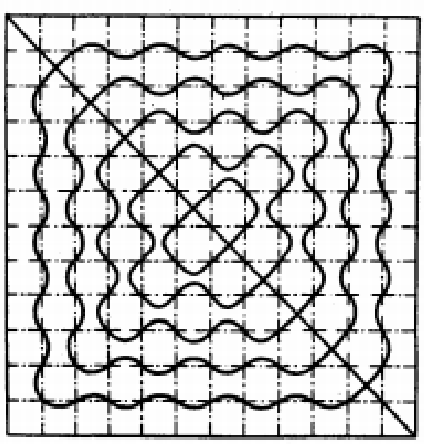


Figure 7

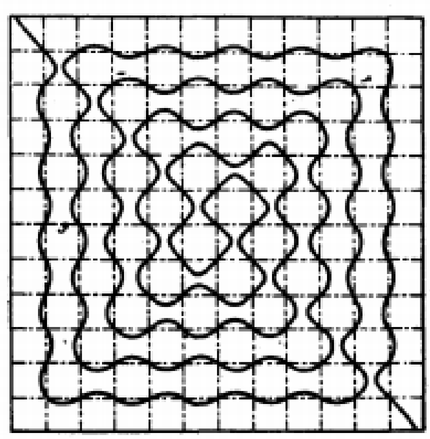


Figure 8

Generating these curves with computer is not so easy. The equation for both of these is:

$$f(x, y) = 0$$

[5] On day I got a textbook "Methods in Mathematical Physics (Vol 1)" The book is in German so maybe that's why it was so cheap; it is available in English. These math-for-phyisicists textbooks are pretty dull. This is the middle of the 20th century and Math and Physics, which you'd assume were related had taken very different paths. Mathematicians were tired of calculations and instead proposed to build a world entirely of abstrct principles. Physicists needed to build things and used math whenever they needed it, or fabricated their own... Courant-Hilbert managed to write a textbook that assembles hundreds of unrelated techniques into a pretty-good story. And it's still the story in the modern literature, more or less.

for some judiciously chosen function f. In the case of the square we can try sines and cosines:

$$f(x,y) = \sum_{m,n \in \mathbb{Z}} a_{m,n} \sin mx \sin ny$$

Maybe I need to include cosines as well. Different variations. Why stop at two? Maybe I can get *every possible pattern* using a level-set of bunch of sine and cosine.

Additionally, even when we have the equation, solving $f(x,y) = 0$ is not very realistic, maybe solving $f(x,y) < \epsilon$ where $\epsilon = 10^{-3}$ or 10^{-6} or however small you are intereted in. And this becomes an algorithms problem.

In the process I may have mixed up two kinds of problems:

- Drawing the curve (maybe with Bezier curves)
- Finding a function with reasonable level set (maybe with Lagrange Interpolation)

I think they're almost the same but after reading Wikipedia I might be wrong. When I started off in number theory, there are all these cute averaging statements, such as:

$$\frac{1}{r_3(n)} \sum_{x^2+y^2+z^2=n} \phi(x,y,z) \ll n^{-1/28}$$

All of these theorems rest on the interpolation results I'm trying to explore here...and perhaps all of these number theory problems are sources of ill-behaved functions I am outlining.

All the papers I have seen resort to horrible complicated rearrangements that deserve a better explanation.

 # 1 Starting small, let's show that $\int \mathbf{1}_{\mathbb{Q}} = 0$. The rational numbers are dense but form a set of measure zero.

- $\overline{\mathbb{Q}} = \mathbb{R}$
- $\mu(\mathbb{Q}) = 0$

You could take for granted that you can always find a fraction close enough to your number

$$47\% = \frac{47}{100} \approx \frac{12}{25}$$

12 in 25 sounds pretty good instead of **47 in 100**. Maybe one is more persuasive than the other.[6] Even though we say $\mathbb{Q}$ is dense in $\mathbb{R}$, finding any fraction is easy, finding a really good fraction can be some work. If $\alpha \notin \mathbb{Q}$:

$$\exists\, p, q \text{ such that } \left| \frac{p}{q} - \alpha \right| < \frac{1}{q^2}$$

Therefore density misses other more refined statements we could make about $\overline{\mathbb{Q}}$.

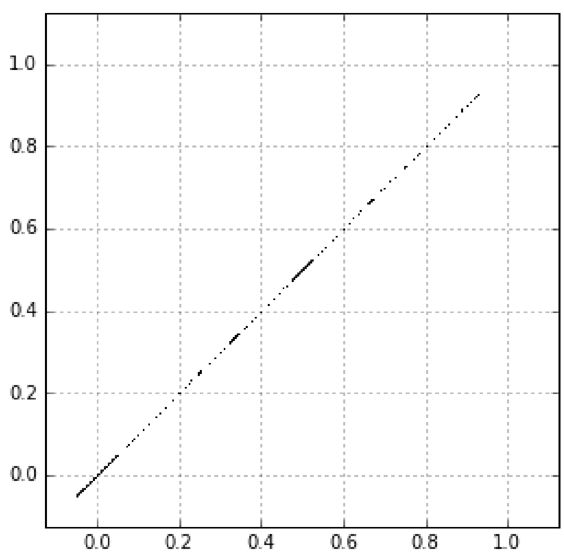
Let's walk through a few sample arguments. How did I first learn that $\mathbb{Q}$ is a set of measure zero? This was self-evident to Stein. A point has measure zero, there are countably many fractions, and therefore $\mathbb{Q}$ has measure zero.

$$\mu(\{pt\}) = 0 \text{ and } \mu(\mathbb{Q}) \leq \bigcup_{\frac{a}{b} \in \mathbb{Q}} \mu(\{|x - \frac{a}{b}| < \frac{\epsilon}{2^k}\}) = \epsilon \sum \frac{1}{2^k} = \epsilon \to 0$$

If we index the fractions one by one we can get some kind of geometric series.

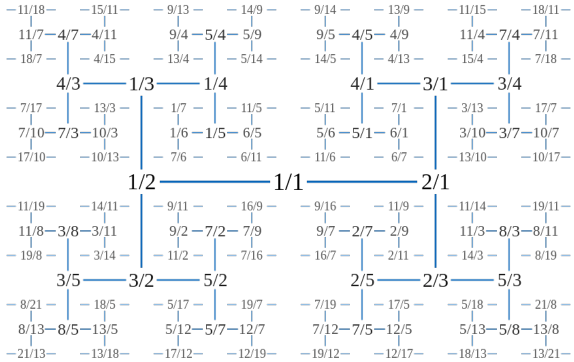
$$(\frac{a_1}{b_1}, \frac{1}{2}) \to (\frac{a_2}{b_2}, \frac{1}{2^2}) \to (\frac{a_3}{b_3}, \frac{1}{2^3}) \to (\frac{a_4}{b_4}, \frac{1}{2^4}) \to \dots (\frac{a_k}{b_k}, \frac{1}{2^k}) \to \dots$$

There are several enuerations of the fractions we can use, such as the Farey Fractions or the Calkin-Wilf tree or the Stern-Brocot Tree. The textbook says **any** enumeration of fractions. Personally, I drew a picture:

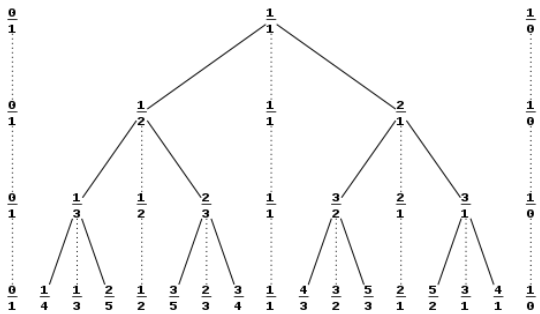


[6]Here we use a tiny bit of projective space $\frac{a}{b} = [a : b] = \mathbb{Q}P^1$ and maybe even a tiny bit of the Galois cohomology of the field $\mathbb{Q}$. That is my theory what all these H^1's are doing there.

The Calkin-Wilf tree (Wikipedia)



The Stern-Brocot tree



I really needed to make the countability of the rationals in a tangible, almost deadpan kind of way. Even there is some research you can do:

- Carlo Carminati, Giulio Tiozzo **A canonical thickening of Q and the dynamics of continued fractions** `arXiv:1004.3790`

I have to look to see if there is more. But now we can appreciate how abstract real analysis can be: we are taking inventory of something infinitely complex and we're just saying it has been done.

Flipping through my old Real Anaysis textbook I found some gems.[7] I can't look at this picture and not see Dehn's theorem, that if the length and the width are rational, then every single rectangle must have rational length and width.

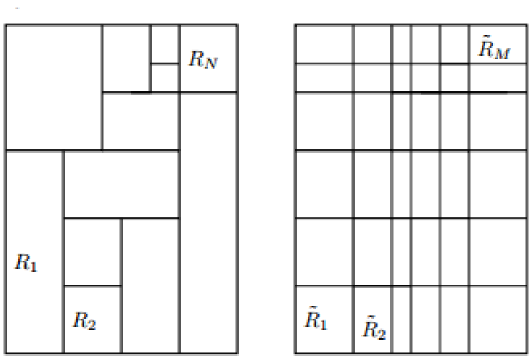


Figure 2. The grid formed by the rectangles R_k

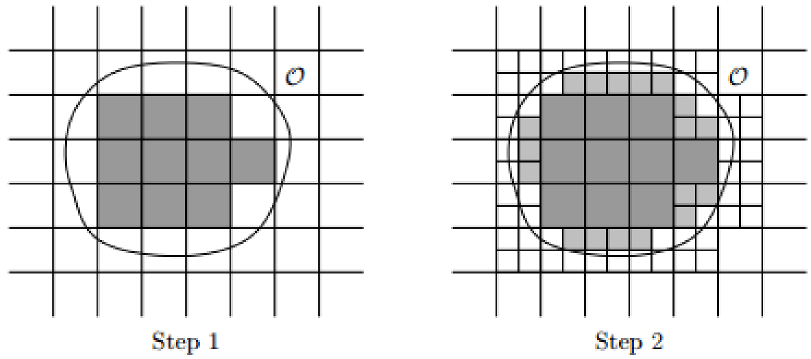


Figure 3. Decomposition of $\mathcal{O}$ into almost disjoint cubes

I could have spent an entire semester drawing ever more complicated shapes and putting rectangles on them.

References

(1) Richard Courant, David Hilbert. **Methoden der mathematischen Physik** (Satz I) Springer, 1937.

(2) Elias Stein, Rami Shakarchi. **Analysis III: Real Analysis - Measure Theory, Integration and Hilbert Space** (Princeton Lectures in Analysis) Princeton University Press, 2004.

[7] At my school, real analysis at the level of Lebesgue Theory was taught to 3rd year undergraduates who were pretty serious about math. I failed it. And passed the second time with a B. These days I almost agree with their assessment because I look at the textbook and still don't really get it.

3 There are two approaches I can think of two the prime number theorem that should be written:

- Real Analysis
- Ergodicity of the Horocycle Flow

and I can put this up here, because if someone beats me to it, I will gladly take their existing argument simplify it and settle for the worse constant.

The starting point is a bit arbitrary, an artifact of how we have developed and do arithmetic over the centuries. . . just the way you and me do:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p \\ 0 & \text{not prime} \end{cases}$$

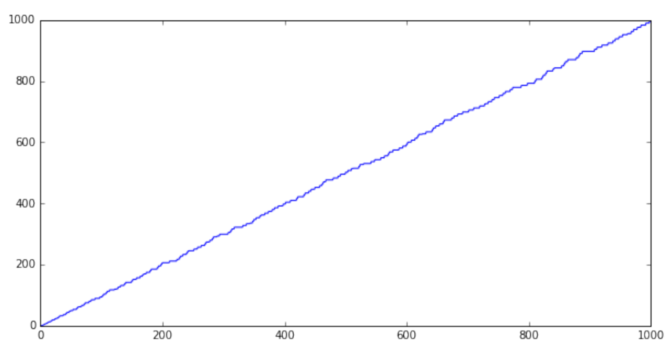
The prime number says that the density of primes is roughly $\frac{1}{\text{\# digits}}$ it can also be phrased as:

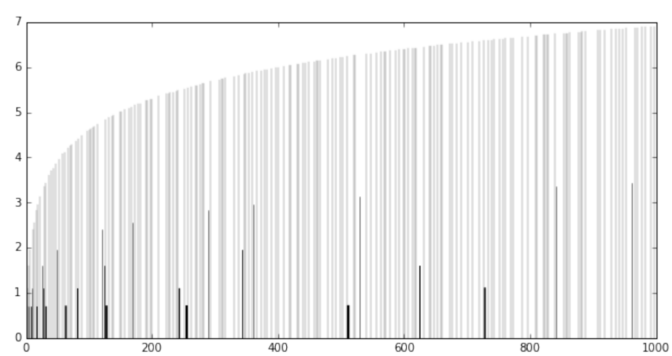
$$\sum_{n \leq x} \Lambda(n) = x + o(x)$$

With a little bit of work, this function can be charted, just like any other. We'd like to be able to say the derivative of the second equation is the first one:

$$\frac{d}{dx} \sum_{n \leq x} \Lambda(n) = \frac{d}{dx}\big(x + o(x)\big) = 1 + o(1) \quad \text{but} \quad \frac{d}{dx} o(x) \stackrel{?}{=} o(1)$$

So our instinct – mine at least – really falls apart on this problem. Unfortunately, I can try to get other instinct and it might look weird, but at least it's more accurate. For one thing: the line is not prefectly straight:



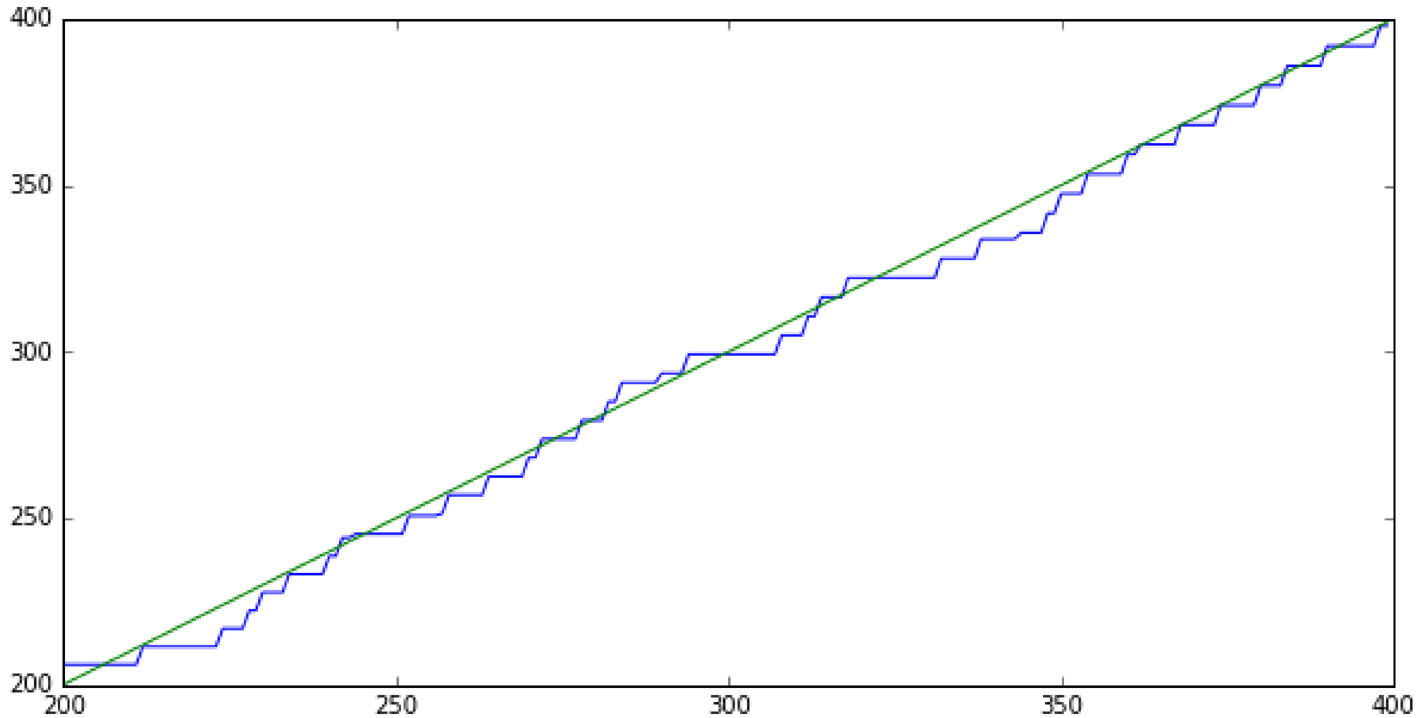


This took several years to state the problem in a visual way that I felt comforatble with. It exists in the literature. Always after you find it, suddenly the paper is right there:

- Don Zagier **The First Million Prime Numbers**[8]

I really want to drive the point home, how decidedly not straigth these lines are.

[8] http://people.mpim-bonn.mpg.de/zagier/files/doi/10.1007/BF03039306/fulltext.pdf



The prime number theorem is important because there are many other statements that I care about that are equivalent to PNT, so that it has become a sort of "gatekeeper" for me. And I can't exactly say I've overcome, but we have a reasonable shot of pinning the last details here: ... ...

For now I'm an a hurry I'll state some variants of Prime Number Theorem.

- the angle of a prime number in $\mathbb{Z}[i]$ is equidistriuted as $p \gg 1$. This one doesn't look like PNT at all, it starts from noticing that $5 = 2^2 + 1^2$ but $7 \neq \square + \square$.
- the Sato-Tate conjecture
- the Chebotarev Density Theorem

Despite their profound modernity, all of these are slightly old fashioned. Here I tend to check the most recent arXiv articles to see what the latest conjecture is.[9]

I think by the 1920's, 30's and 40's I started to find really good, convincing proofs of PNT. Parts of it were still being settled the 80's and 90's and up to now. I heard the version I am about to give, has really bad constants.

If we wanted to we could sequester the Prime Number Theorem to five very basic pages:

- Don Zagier **On Newman's Short Proof of the Prime Number Theorem** American Mahtematical Monthly (1997)

[9] And experts in other fields are not always sure where the frontier is. You and I should feel free to settle here, or if you know a lot, to take a brand new position! Here's one:

- Kaisa Matomäki, Maksym Radziwiłł, Terence Tao **An averaged form of Chowla's conjecture** `arXiv:1503.05121`

The time is certainly approaching!

Like I said won't spend much time on this. A lot has changed since the 1950's. I worked out a few solutions in a Starbucks. Just a summary[10]

Buzzwords like "resurgence" and "regularization" do not exist yet. Instead, the textbook proposes a clever use of the Laplace transform:

$$f(x) = \sum_{n=1}^{\infty} \frac{\big(\Lambda(n) - 1\big)e^{-nx}}{1 - e^{-nx}}$$

Why could this series be solve our problem or natural from number theory? The weight:

$$\frac{e^{-nx}}{1-e^{-nx}} \approx \left\{ \begin{array}{cc} \frac{1}{nx} & x \to 0 \\ \\ e^{-nx} & x \gg 1 \end{array} \right. \text{ with turning point at } n \approx \frac{1}{x}$$

Using an intelligent choice of filters we'd like to make an approximation, an engineer migth not question it. A mathematicians could be scared of (rather faint) signals that might change the result:

$$f(x) = \sum_{n=1}^{\infty} \frac{\big(\Lambda(n) - 1\big)e^{-nx}}{1 - e^{-nx}} \approx \frac{1}{x} \sum_{n<\frac{1}{x}} \frac{\Lambda(n) - 1}{n}$$

So if we reason this function divergece a certain way, maybe we can collect our result about the primes. Here are other ways to look at f:

$$f(x) = \int_0^{\infty} \frac{te^{-xt}}{1-e^{-xt}}\, d\left[\sum_{n \leq x} \frac{\Lambda(n) - 1}{n} \right]$$

This is a **Riemann-Stieltjes** integral, which is why I drew the **spike train** on the previous page. The measure is no longer dx instead it is

$$d\left[\sum_{n \leq x} \frac{\Lambda(n) - 1}{n} \right] = \left\{ \begin{array}{cc} 0 & x \notin \mathbb{Z} \\ \\ \frac{\Lambda(n)-1}{n} & x = n \in \mathbb{Z} \end{array} \right.$$

measuring all the places where this function jumps. And we are convolving this spike train against the generator of the Bernoulli polynomials:

$$g(x,t) = \frac{te^{-xt}}{1-e^{-xt}} \text{ vs } \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

At least, they resemble each other. Here's another awy to look at f

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \{\Lambda(n) - 1\} e^{-mnx} = \sum_{n=1}^{\infty} \left[\sum_{d|n} \big(\Lambda(d) - 1\big) \right] e^{-nx}$$

He calls it Lambert summation but we can also call it Möbius transformation. So there's a way to pass between the van Mangoldt function Λ and the just plain $\log$ function.

[10] I was surprised to learn from the barista that cold brew coffee was not the same as I coffee. I had one of each.

Just have that $\log = \Lambda * 1$ or $\Lambda = \log * \mu$ where μ is the **Möbius** function. Depending on which source you look, we can prove that:

$$f(x) \sim -\frac{2\gamma}{x}$$

why is that a solution, though? "Abelian" and "Tauberian" theorems are rather heavy tools, that either blow-up elementary calculations into deeper ones, or extract results that are buried in infinite series.

Norbert Weiner solves the theorem in a few pages (plus 20 or 30 pages devoted to proving the Tauberian theorem)

$$\sum_{m=1}^{\infty} (\log m) x^m = \cdots = \sum_{n=1}^{\infty} \Lambda(n) \frac{x^n}{1-x^n}$$

Using a mix of Euler-Maclaurin summation and Moöbius inversion he rearranges everything. Substitute $x = e^{\xi}$ and take derivative with respect to ξ and set $\xi \to 0$. He obtains a result :[11]

$$\sum_{n=1}^{\infty} \Lambda(n) \frac{d}{dn\xi} \left[\frac{n\xi \, e^{-n\xi}}{1-e^{-n\xi}} \right] = \frac{1}{\xi} + O(\log \xi)$$

Why did the grandfather of Stochastic Processes (the kind you see the stock market) take a few minutes of his time to solve the Prime Number Theorem? I can only guess the primes are behaving like random processes.

In modern lingo, **regularization** could refer either to noisy functions or divergent functions.

- $\sum n = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$
- $\prod n = 1 \times 2 \times 3 \times 4 \times \cdots = \sqrt{2\pi}$
- $\prod p = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots = \frac{1}{2\pi}$

My goal had been to understand: What was being regularized? Tauber theorem expand on the old proofs I think:

Why do the Tauberian theorems work?

It seems, I have left a giant mess and unfortunately newer problems and challenges await! Put it to you like this: the more time we spend on this and everything else is going haywire?

References

(1) Norbert Weiner **Tauberian Theorems**
Annals of Mathematics. Vol. 33, No. 1 (Jan., 1932), pp. 1-100

[11] These however, begin to look like more and more desperate attempts to fix a growing problem. As we look at the primes at finer and finer scales (or larger and larger scales) their irregularity is posing various challenges.

4 In this section we attempt to discuss the Oppenheim conjecture in an intelligent way. One way to think of it the example:

$$\left\{x^2 + y^2 - \sqrt{2}z^2 : x, y, z \in \mathbb{Z}\right\} \subseteq \mathbb{R}$$

This subset is in fact dense in $\mathbb{R}$. That's no surprise to me since this thing can never be be zero (except for $x = y = z = 0$). If you want I pick an even more obvious one by making all the numbers different:

$$\left\{\sqrt{2}x^2 + \sqrt{3}y^2 - \sqrt{5}z^2 : x, y, z \in \mathbb{Z}\right\} \subseteq \mathbb{R}$$

Math can be a slightly doubtful field at times. Sometimes we let plausible statements pass. Other times we press the question. Can you find me $x, y, z \in \mathbb{Z}$ such that:

$$|\sqrt{5} - \left(x^2 + y^2 - \sqrt{2}z^2\right)| < 10^{-6}$$

The conjecture was that as long as the three numbers are not multiples of a fraction, the set of integer values is dense.

$$\sqrt{2}\,(x^2 + \tfrac{3}{2}y^2 + \tfrac{1}{5}z^2) \in \sqrt{2}\,\mathbb{Q}$$

I took out a common factor, so this number will always be $\sqrt{2}\times$ a fraction.

The rational numbers embedded in the reals $\mathbb{Q} \subseteq \mathbb{R}$ or even the adeles $\mathbb{Q} \subseteq \mathbb{A}$ exhibit all sorts of hidden patterns and faint signals which can just happily ignore. I don't have to entertain them at all – I can just ignore all of them!!

Bourgain only considers quadratic forms of a certain "signature" $(+,+,-)$ and he is going to consider entire "lines" of qudratic forms. I think the geometric term is "pencil".

$$x^2 + y^2 - a\,z^2 \text{ or } x^2 + b\,y^2 - a\,z^2$$

the second equation is the same as the first with $b = 1$. And we let $a \in [0, 1]$ or some other range of values. I might write these as $[1 : 1 : -a]$ and $[1 : b : -a] \notin \mathbb{Q}^3$ or $\mathbb{Q}P^2 = (\mathbb{Q}^\times)^3/\mathbb{Q}$. This shorthand means I consider three numbers the same if I multiply all of them by the same fraction. E.g. $[\sqrt{2} : \sqrt{2} : \sqrt{2}] = [1 : 1 : 1]$.

In 1989 Margulis showed for any quadratic form $[a : b : c] \notin (\mathbb{Q}^\times)^3/\mathbb{Q}$ leads to a quadrtaic equation $a\,x^2 + b\,y^2 + c\,z^2$ whose values are dense in $\mathbb{R}$. Unipotent flows are a way to change quaratic equations into one another. Therefore, Margulis solve all quadratic equations showing their value-sets are dense, but never solves any single equation.

I can solve any single oppenheim type problem using a computer:

$$|Q(x, y, z)| < \epsilon$$

and maybe I can solve it with enough time, but I can never show the values of $\mathbb{Q}(\mathbb{Z}^3)$ are dense. However Ratner's theorem lie very deep, taking over 200 pages to prove.

Unfortunately, I can hardly understand Bourgain's note beyond the first paragraph.

$$\max_{|\xi|<A(N)} \min_{x\in\mathbb{Z}^n\ 0<|x|<N} |Q(x)-\xi| < \delta(N)$$

This is a very stuffy way of phrasing density (that might be easy for a computer to understand). If I want to show the values of $Q(x)$ are dense:

- Choose a test value $\xi \in \mathbb{R}$.
- I am going to try all integers up to size $|x| < N$
- Of all the vaues $|\xi| < A(N)$ the worst error we obtain is $\delta(N)$.

Next, Bourgain proved two sets of results depending on whether or not we believe the Lindelöf hypothesis. I don't know it.

With the Lindelof hypothesis the smallest value of $Q(x)$ is about size $\frac{1}{N}$.

$$\min \left\{ |Q(x)| : x \in \mathbb{Z}^3 \backslash \{0\}, \ \ |x| < N \right\} \ll N^{-1+\epsilon}$$

and there is a trade-off between the range of values ξ we can approximate and how accurately we can estimate them:

$$A(N)\delta(N)^{-2} \ll N^{1-\epsilon}$$

Without the Lindelof Hypothesis we do a bit worse:

$$\min \left\{ |Q(x)| : x \in \mathbb{Z}^3 \backslash \{0\}, \ \ |x| < N \right\} \ll N^{-\frac{2}{5}+\epsilon}$$

and there is the trade-off between also worse (with funny dimensions):

$$A(N)^3\delta(N)^{-\frac{11}{2}} \ll N^{1-\epsilon}$$

So, although he uses some rather expensive theorems, Bourgain whittles down the Oppenheim conjecture from 200 pages down to 8 pages. Except, this is true for **almost all** values.

$$\mu\left\{ a \in [\tfrac{1}{2}, 1] : \min \left\{ x^2 + y^2 - a\, z^2 : |x|, |y|, |z| < N, (x,y,z) \in \mathbb{Z}^3 \backslash \{0\} \right\} \ll N^{-\frac{2}{5}+\epsilon} \right\} = 1$$

This is Jean Bourgain for you. Not only does he tell us the values of $Q(x)$ are dense (or small) he tells us how dense (or small) these values will be – most of the time!

If I point to any one quadratic equation: $x^2 + y^2 - \sqrt{2+\sqrt{2}}\ z^2$ Bourgain's theorem tells us **nothing**. Only how this equation and its neighbors behave collectively.

Personal side note: We can get this full-measure result without doing any Ratner Theory because of Bourgain's arguments. I think the key is not to get lost in the shuffle. Maybe later we study the horocycle flow itself and why it's related to these quaratic equations. I want to know what the big deal is.

– DRAW PICTURES –
– NUMERICALS?? –

References

(1) János Kollár **Quadratic solutions of quadratic forms** `arXiv:1607.01276`

(2) Zeév Rudnick, Ezra Waxman **Angles of Gaussian primes** `arXiv:1705.07498`

(3) Jean Bourgain. **A quantitative Oppenheim Theorem for generic diagonal quadratic forms** `arXiv:1604.02087`

(4) Valentin Blomer, Jean Bourgain, Maksym Radziwiłł, Zeev Rudnick **Small gaps in the spectrum of the rectangular billiard** `arXiv:1604.02413`