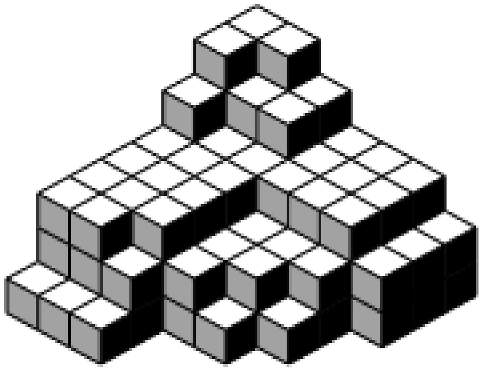
Examples: $n!$ the Factorial

John D Mangual

There are famous McMahon formulas and I could be mixing them up. Here is one, which counts literally *all* plane paritions:

$$\sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^n$$

My apology in advance for not having a good picture. They are take some work to draw. Here we take example from Mirjana Vuletic.



This is good but not really what I am looking for today.

I am looking for those partitions which fit inside an $a \times b \times c$ box. There is an exact number on Wikipedia:

$$\#\{boxes\} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

and you might wonder why so much attention might be drawn to a simple equation like this:

- Inside Mathematics - literally thousands of papers and they keep coming
- Outside Mathematics - maybe a statistician or data analyst might find this structure relates to something in the real world[1]

Today we will do something totally useless and set $a = b = c = \frac{1}{2}$. How many ways to pack $1 \times 1 \times 1$ cubes into a box of side $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$?

The procedure for guessing a value of $(\frac{1}{2})!$ might stem from Euler's definition of factorial:

$$x! = \lim_{n\to\infty} \frac{n^x\, n!}{x \times (x+1) \times (x+2) \times \ldots (x+n)}$$

[1]Baseball, elections, the human genome, the weather, instagram, the radio... and the end of the day these are all tables of numbers and there exist procedures where these can processed in a similar ways.

Let's try to shift the product lattice by (x, y, z):

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{(i+j+k)+(x+y+z)-1}{(i+j+k)+(x+y+z)-2}$$

I don't even know what this number is. OK. Let

$$n!! = 0! \times 1! \times \cdots \times (n-1)!$$

and we can even define the q-factorial

$$[n]_q! := \prod_{i=1}^{n}\frac{1-q^n}{1-q}$$

The hyperfactorial (and the Barnes G-function)

$$\sum_{\pi} q^{|\pi|} = \frac{a!!_q b!!_q c!!_q (a+b+c)!!_q}{(a+b)!!_q (b+c)!!_q (c+a)!!_q}$$

and we can set $q = 1$ to get one McMahon formula. Or let $a, b, c \to \infty$ to get the other.[2]

And now we can set $a = b = c = \frac{1}{2}$, so that $(\frac{1}{2})! = \sqrt{\pi}$ and $(\frac{1}{2})!! = \ldots$ Euler never wrote a formula for the Barnes G-function.

Oh my this got complicated. Have an answer in a bit.

[2]Everything fits together in McMahon world. Everything is beautiful!

One reason I want to move on is because I have a lot of interesting factorials to talk about.

$$n! = \sum_{\lambda \in P_n} (t_\lambda)^2$$

P_n is the set of all partitions of n. Any way you can think of to split a number:

$$10 = 5 + 3 + 2$$

for each such partition you could play a game and draw a **tableau**

10	8	7	4	3
9	6	5		
2	1			

10	9	8	3	1
7	5	4		
6	2			

the Robinson Schensted correspondence is a way of taking permutations and turning them into pairs of shapes like this. And the miracle is there are exactly $n!$ as can be bound by playing **jeu de taquin**.

And now we ask the obvious question, how do we play jeu de taquin with $\frac{1}{2} \times \frac{1}{2}$ square?

Interesting - badly divergent infinite products - can arise in supersymmetric localization computations in theoretical physics. Even if you don't know what that term means you can appreciate the strange number it produces:

$$\prod_{\alpha}\prod_{\ell=1}^{\infty}\left((\ell+1)^2+\alpha(\sigma_0)^2\right)^{2\ell(\ell+2)}$$

Again I am not beginning to ask why or how these occur. These are determinant of Laplacian. Here is a matrix:

$$\det\begin{bmatrix}1&0&0\\0&2&0\\0&0&c\end{bmatrix}=1\times 2\times 3=6$$

and you multiply all the numbers of the diagonal. And somehow we compute

$$\det\nabla=\det\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=\prod_{n=1}^{\infty}\cdots$$

and the question is how we can replace Δ by an appropriate large matrix. In the case of a sphere

$$S^3=\{x^2+y^2+z^2+w^2=1\}$$

we can find a convincing basis using **spherical harmonics**. Even the basic case:

$$\frac{df}{dx} \approx \frac{f(x+1) - f(x-1)}{2}$$

leads to a matrix which can be diagonalized. Or what about the less symmetric definition:

$$\frac{df}{dx} \approx f(x+1) - f(x)$$

Or what if we incorporate scale. Then maybe we can get appropriate answer:

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

and we have many other choices we can make which are more or less the same, but could give wildly different answers in important situations!

$$\prod_{n=1}^{\infty} n^4 \stackrel{?}{=} \big(\prod_{n=1}^{\infty} n\big)^4$$

if you like just multiply all the numbers and take fourth powers. the tradtional rules of arithmetic break down.

$$(1 \times 2 \times 3 \times 4)^4 \approx 1^4 \times 2^4 \times 3^4 \times 4^4$$

and - if you're really cynical - even multiplication is called into question. why are we doing this?

References

(1) Mirjana Vuletic **A generalization of MacMahon's formula** `arXiv:0707.0532`

(2) Paul Zinn-Justin **Six-Vertex, Loop and Tiling models: Integrability and Combinatorics** `arXiv:0901.0665`

(3) Anton Kapustin, Brian Willett, Itamar Yaakov **Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter** `arXiv:0909.4559`