Theta Functions and Eta Function

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Tryingg to understand better modular properties of $\theta = \Sigma\, q^{n^2}$ and $\eta = q^{1/24}\,\Pi(1-q^n)$

1 Theta Functions and $\Gamma_0(4)$

Don Zagier's lecture notes on modular forms starts off with a couple of examples. Number theory is insane, and the motivation for the very difficult theorems can sometimes follow from very simple situations and looking at them closely.

Let $n = a^2 + b^2 + c^2$ have a solution in the integers $a, b, c \in \mathbb{Z}$. Then we could ask how the set

$$V_n = \{\tfrac{1}{\sqrt{n}}(a,b,c) : a^2+b^2+c^2 = n\} \in S^2$$

is distributed around the sphere. It is remarkably dfficult to prove this set becomes equidistributed in the sphere. If we collect the averages over all possibl n:

$$\theta(u,z) := \sum_{n\geq 0} \left[\frac{1}{r_3(n)} \sum_{\xi \in V_n} u(\xi) \right] e^{2\pi i\, nz}$$

is a $\left(\frac{1}{2}+\ell\right)$-weight modular form over $\Gamma_0(4)$ for any **spherical harmonic** $u \in L^2[SO(3)]$.

The trouble is I don't know what a modular form is. Or it's weight. Or remember spherical harmonics very well. Nor what $\Gamma_0(4)$ is. So I can't demonstrate that θ transforms how it should over this group.

The theory of half-integer weight modular forms tells us in this case:

$$\frac{1}{r_3(n)} \sum_{\xi \in V_n} u(\xi) \ll n^{-1/28}$$

This is a very slow decay. In the case of a circle $L^2[SO(2)]$, the Riemann-Lebesgue lemma states:

$$\lim_{|n|\to\infty} \hat{\theta}(n) = 0$$

but we do not know how fast[1].

If our function were absolutely continuous, we know $\hat{\theta}(n) = o(1/n)$. Our function decays at $n^{-1/28}$ in momentum space and is definitely **not** absolutely continuous.

1.1 Zagier's review and Poisson Summation

Jacobi's theta function is the sum over all perfect squares. Let $q = e^{2\pi i n z}$ and

$$\theta(z) = \Sigma\, q^{n^2} = 1 + q^4 + q^9 + q^{16} \ldots$$

Zagier says for any function which is smooth and small at infinity:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

[1]Conversely $\delta_0(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z}$ so that $\hat{\delta_0}(n) \equiv 1 \not\to 0$. The way out is δ_0 is not a function.

If we pretend to be quantum physicists we say both sides are traces over different basis. If $A = f$ were an operator:

$$\mathrm{Tr}(A) = \sum_{p\in\mathbb{Z}} \langle p|A|p\rangle = \sum_{x\in\mathbb{Z}} \langle x|A|x\rangle$$

Now I am confused. Perhaps I can stick to Zagier's own derivation. By the **method of images**

$$\sum_{n\in\mathbb{Z}} f(n+x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x} \in L^2(\circ)$$

Therefore it can be expanded in Fourier series. In fact, set $x = 0$

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

In our case $f(x) = e^{-\pi t x^2}$ and we have identity of Gaussians[2]:

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}$$

The two symmetries of the theta function $z \to -\frac{1}{4z}$ and $z \to z+1$ generate some of $SL(2,\mathbb{Z})$ – a subgroup of index 6, meaning there are

[2]All kinds of strange and magical formulas like:

$$\sum_{k=1}^{\infty} e^{-\pi k^2}\left(\pi k^2 - \frac{1}{4}\right) = \frac{1}{8}$$

See: http://math.stackexchange.com/q/297900/4997

6 copies of this group inside.

$$\theta(-\frac{1}{4z}) = \sqrt{\frac{2z}{i}}\theta(z)$$

Except what happens under $\frac{1}{z}$? There must be some other theta functions floating around, "conjugate" to this one.

2 Nilsequences, Complexity and Theta Functions

In the process of showing that $\theta(-\frac{1}{4z}) = \theta(z)$, we needed to use the Poisson summation formula. Certainly the Gaussian is its own Fourier transform, but we can use the distributional and Fourier series:

$$\sum_{m\in\mathbb{Z}} \delta_m(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i\, nz}$$

The left side is obviously 0 away from integers and countably infitite for $z \in \mathbb{Z}$. That the RHS "is" 0 for $z \notin \mathbb{Q}$ is known related to the fact that $\{nz\}$ is equidistributed on $[0,1]$ for $z \notin \mathbb{Q}$. All fractions - not just integers, deserve special attention, since the convergence of either side as $|n| \to \infty$ has to do with the continued fraction expanion of $z = [z_1, z_2, \ldots]$

2.1 A "Curious" Example of Hardy

GH Hardy was already quantifying equidistribution when he wrote an estimate over adding the fractional parts over intgers:

$$\sum_{\nu \leq n} \{\nu x\} = \frac{n}{12} + O(1)$$

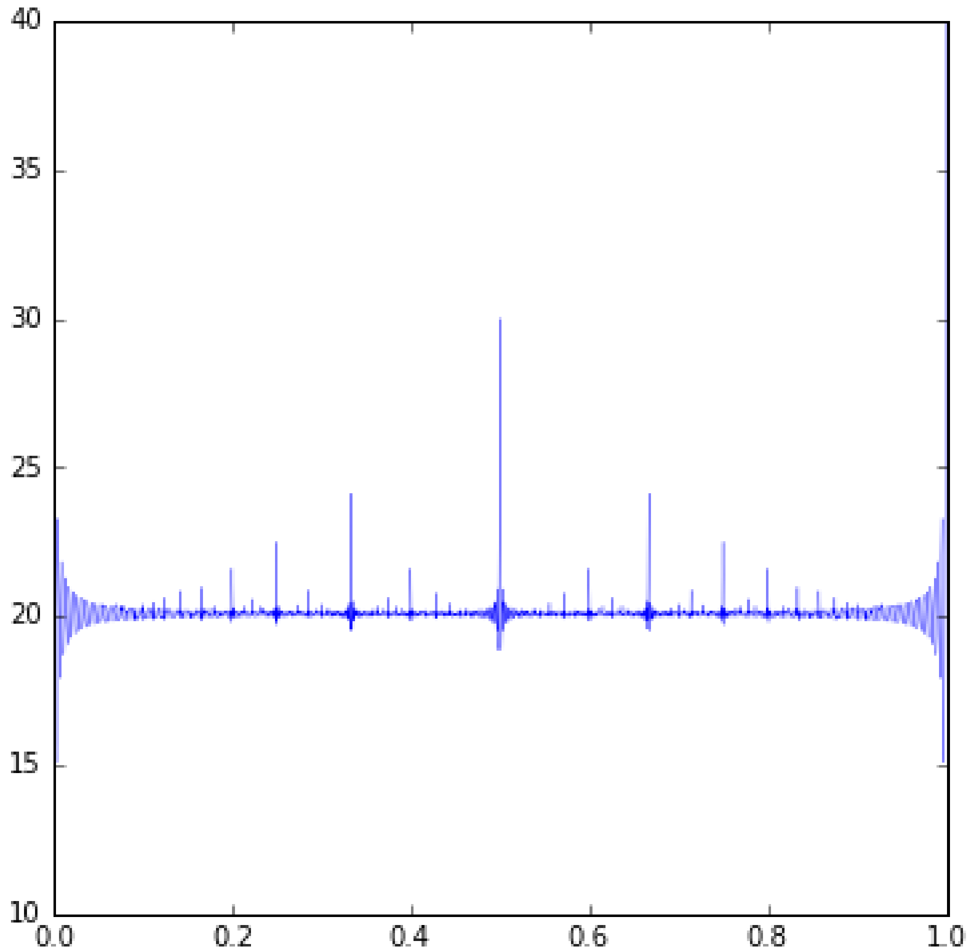
This is not true when $x \in \mathbb{Q}$ - unless we bargain about about the definition of "=" for a bit. Even if $x \notin \mathbb{Q}$ there is an issue if $x = [x_1, x_2, \ldots]$ and $|x_i| \neq O(1)$. A very familiar number in might be:

$$e = [2; 1, 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$$

Setting $x = e$ whould lead to a divergent series. An even faster might be:

$$\frac{e-1}{e+1} = [0; 2, 6, 10, 14, \ldots]$$

Other examples like $x = \sqrt{2}+\sqrt{3}$ or $x = \sqrt[3]{4}$ are unknown[3]. Maybe $x = \sqrt{2}$, while not generic, could have some divergent behavior as well[4].



One huge problem with Hardy's result is that it no longer falls out of Weyl

[3] http://mathoverflow.net/a/177491/1358
[4] http://front.math.ucdavis.edu/1004.3790

equidistribution. We would like to argue that:

$$\frac{1}{n}\sum_{\nu<n}\{\nu x\}^2 \approx \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dx = \frac{1}{12} + O(\dots)$$

I would like to just use the equal sign, but that's not really true. There is "fuzz" along a set slightly thicker than the rationals, $\mathbb{Q}$. I have not read much more of Hardy's paer to see if he pursues this thought further.

2.2 Quantitative Equidistribution

If we search on google for "quantitative equidistribution" the result of Green and Tao[5] show that something like Hardy's conjucture is true for all **nilsequences** but the argument is arbitrarily complex and not necessarily optimal. However, we are told there is some polynomial bound on the error which we might like to pursue. Host and Kra demonstrate how to extract 2-step nilsquences - such as $n\sqrt{2}\{n\sqrt{3}\}$ - from other dynamical systems[6] and from the theta function itself.

[5] http://front.math.ucdavis.edu/0709.3562
[6] http://front.math.ucdavis.edu/0709.3241

“Effective Equidistribution” seems like another helpful Google phrase.

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References

(1) arXiv:1503.05884 Effective equidistribution and property tau. Manfred Einsiedler, Grigory Margulis, Amir Mohammadi, Akshay Venkatesh.

(2) S. Lang and H. Trotter, **Continued fractions for some algebraic numbers**, J. Reine Angew. Math. 255 (1972), 112-134. https://eudml.org/doc/151239

(3) Philipp Fleig, Henrik P. A. Gustafsson, Axel Kleinschmidt, Daniel Persson Eisenstein series and automorphic representations arXiv:1511.04265

(4) Thomas C. Hull (April 2011). "Solving Cubics With Creases: The Work of Beloch and Lill" (PDF). American Mathematical Monthly: 307âĂŞ315. doi:10.4169/amer.math.monthly.118.04.307.

(5) S. Kharchev, A. Zabrodin
arXiv:1502.04603 Theta vocabulary I.
arXiv:1510.02699 Theta vocabulary II.