

Some Interesting Formulas Involving the GCD

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Sometimes, when I read a String Theory paper, I try to find a verifiable statement. Here is one I found in a paper:

$$I^{\mathcal{N}=1^*}(N, 1, 0) = N \sum_{d|N} 1 = N\sigma_0(N)$$

This number is called a **superconformal index** and it also equals:

$$I^{\mathcal{N}=1^*}(N, N, n) = \sum_{d|N} \sum_{l=1}^N \gcd(d, l)$$

I was heckled on MathOverflow for posting such an elementary formula. It's not mine, it's his.

Perhaps the general formula can show us the pattern:

$$I^{\mathcal{N}=1^*}(N, m, n) = \frac{N}{m} \sum_{d|N} \sum_{l=1}^{\gcd(d, m)} \gcd \left(\gcd(d, m), n + \frac{ld}{\gcd(d, m)} \right)$$

This formula is later shown to be equal to:

$$I^{\mathcal{N}=1^*}(N, m, n) = \sum_{d|N} \sum_{t=0}^{d-1} \gcd \left(N \frac{d}{m}, N \frac{m}{d}, N \left(\frac{t}{m} + \frac{m}{d} \right) \right)$$

In order for this equation to make sense, I eventually found $m|N$ and $d|N$ - I hope I guessed correctly.

By **Möbius inversion** we should have:

$$\frac{N}{m} \sum_{l=1}^{\gcd(d, m)} \gcd \left(\gcd(d, m), n + \frac{ld}{\gcd(d, m)} \right) = \sum_{t=0}^{d-1} \gcd \left(N \frac{d}{m}, N \frac{m}{d}, N \left(\frac{t}{m} + \frac{m}{d} \right) \right)$$

These seem rather tedious to verify and their meaning unclear.

A starting point could be the Bezout theorem that:

$$\gcd(a, b) = \min_{(x,y) \in \mathbb{Z} \setminus (0,0)} |ax + by|$$

Buried in the paper is his original statements about lattices when explain the appearance of **GCD** everywhere.

The inputs seem to be a Lie algebra (such as $\mathfrak{so}(N)$) plus a 4-manifold such as \mathbb{R}^4 or $\mathbb{R}^3 \times S^1$

Algebra	Theory	On \mathbb{R}^4	On $\mathbb{R}^3 \times S^1$
\mathfrak{a}_{N-1}	$(\mathrm{SU}(N)/\mathbb{Z}_m)_n$	N	$I^{N=1}(N, m, n)$
$\mathfrak{b}_{N \geq 2}$	$\mathrm{Spin}(2N+1)$ $\mathrm{SO}(2N+1)_+$ $\mathrm{SO}(2N+1)_-$	$2N-1$	$2N-1$ $2(2N-1)$ $2N-1$
$\mathfrak{c}_{N \geq 2}$	$\mathrm{Sp}(2N)$ $(\mathrm{Sp}(2N)/\mathbb{Z}_2)_+$ $(\mathrm{Sp}(2N)/\mathbb{Z}_2)_-$	$N+1$	$N+1$ $\begin{cases} 2(N+1) & \text{for even } N \\ \frac{3}{2}(N+1) & \text{for odd } N \end{cases}$ $\begin{cases} N+1 & \text{for even } N \\ \frac{3}{2}(N+1) & \text{for odd } N \end{cases}$
$\mathfrak{d}_{N \geq 3}$	$\mathrm{Spin}(2N)$ $\mathrm{SO}(2N)_+$ $\mathrm{SO}(2N)_-$	$2(N-1)$	$2(N-1)$ $4(N-1)$ $2(N-1)$
$\mathfrak{d}_{N \text{ odd}}$	$(\mathrm{Spin}(2N)/\mathbb{Z}_4)_n$	$2(N-1)$	$4(N-1)$
$\mathfrak{d}_{N \equiv 2 \bmod 4}$	$\mathrm{Sc}(2N)_\pm$ and $\mathrm{Ss}(2N)_\pm$ $(\mathrm{SO}(2N)/\mathbb{Z}_2)_{00}^{00}$ $(\mathrm{SO}(2N)/\mathbb{Z}_2)_{11}^{11}$ $(\mathrm{SO}(2N)/\mathbb{Z}_2)_{10}^{00}$ $(\mathrm{SO}(2N)/\mathbb{Z}_2)_{01}^{11}$	$2(N-1)$	$3(N-1)$ $5(N-1)$ $3(N-1)$ $4(N-1)$ $2(N-1)$
$\mathfrak{d}_{N \equiv 0 \bmod 4}$	$\mathrm{Sc}(2N)_+$ and $\mathrm{Ss}(2N)_+$ $\mathrm{Sc}(2N)_-$ and $\mathrm{Ss}(2N)_-$ $(\mathrm{SO}(2N)/\mathbb{Z}_2)_{00}^{00}$ $(\mathrm{SO}(2N)/\mathbb{Z}_2)_{11}^{11}$ $(\mathrm{SO}(2N)/\mathbb{Z}_2)_{01}^{00}$ $(\mathrm{SO}(2N)/\mathbb{Z}_2)_{10}^{00}$	$2(N-1)$	$4(N-1)$ $2(N-1)$ $5(N-1)$ $3(N-1)$ $3(N-1)$ $3(N-1)$
\mathfrak{e}_6	E_6 $(E_6/\mathbb{Z}_3)_n$	12	12 20
\mathfrak{e}_7	E_7 $(E_7/\mathbb{Z}_2)_n$	18	18 27
\mathfrak{e}_8	E_8	30	30
\mathfrak{f}_4	F_4	9	9
\mathfrak{g}_2	G_2	4	4

Table 2. The Pure $\mathcal{N} = 1$ Indices on a Circle. The notations are as in [9].

What is a **Theory**?

There are too many inputs to define $\mathcal{N} = 1^*$ Supersymmetric Yang-Mills Theory with Lie group $SU(N)$ on manifold $\mathbb{R}^3 \times S^1$, and to classify the “Line Operators” ...

What is a reasonable minimum amount of information to describe the Theories discussed here?

What is **S-duality**?

Many of the quantities discussed here are not $SL(2, \mathbb{Z})$ invariant. Instead they transform “nicely” under actions of $z \mapsto z+1$ and $z \mapsto -1/z$. This is called a **group action**.

References

- (1) arXiv:1606.01022 The Arithmetic of Supersymmetric Vacua. Antoine Bourget, Jan Troost. physics.hep-th.
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- (3) arXiv:1506.03222 Counting the Massive Vacua of $N=1^*$ Super Yang-Mills Theory. Antoine Bourget, Jan Troost. physics.hep-th.
- (4) arXiv:1305.0318 Reading between the lines of four-dimensional gauge theories. Ofer Aharony, Nathan Seiberg, Yuji Tachikawa. WIS/03/13-APR-DPPA, UT-13-15, IPMU13-0081. physics.hep-th.