Scratchwork: Class Field Theory

One common mistake is to say the ring of integers of $K = \mathbb{Q}(\sqrt{-5})$ is $\mathbb{Z}[\sqrt{-5}]$. In fact it's $\mathbb{Z}[\frac{1+\sqrt{-5}}{2}]$. This example is important because it's the first time we observe the failure of unique factorization in "integers":

$$2 \times 3 = (1+\sqrt{-5}) \times (1-\sqrt{-5})$$

Despite being quite well-known, I feel this is the kind of result that needs to be checked very carefully. Number Theory in particular, is known to re-arrange obvious facts in shocking ways:

$$\left(\frac{1+\sqrt{-5}}{2}\right)^2 = \frac{1}{4} + \sqrt{-5} - \frac{5}{4} = 2 \times \left(\frac{1+\sqrt{-5}}{2}\right) - 2 \times 1$$

What's so special about the $\sqrt{-5}$ that we obtain a number field with class number $h(K) = 2$?

Ex Factor the numbers $1 \leq n \leq 100$ in each of the two orders, $\mathcal{O}_1 = \mathbb{Z}\left[\frac{1+\sqrt{-5}}{2}\right]$ and $\mathcal{O}_2 = \mathbb{Z}[\sqrt{-5}]$.

Ex Show that the ring of integers of $\mathbb{Q}(\sqrt{-5})$ is $\mathbb{Z}\left[\frac{1+\sqrt{-5}}{2}\right]$.

References

[1] Henri Cohen **Computational Number Theory in Relation with L-Functions** `arXiv:1809.10904`