Scratchwork: $\zeta_K(2)$ with $K = \mathbb{Q}(\sqrt{2})$

For today's warmup problem we show that $x = \sqrt{2} + \sqrt{3}$ is an algebric number. Certainly we have that $\sqrt{2}$ and $\sqrt{3}$ are algebraic numbers, since:

$$(\sqrt{2})^2 - 2 = 0 \text{ and } (\sqrt{3})^2 - 3 = 0$$

and it seems plausible that the sum of two algebraic numbers is an algebraic number, but I can't think of the polynomial. Let's do some trial an error:

- $1 = 1$
- $\sqrt{2} + \sqrt{3} = \sqrt{2} + \sqrt{3}$
- $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$
- $(\sqrt{2} + \sqrt{3})^3 = (2 + 3 \times 3)\sqrt{2} + (3 + 3 \times 2)\sqrt{3} = 11\sqrt{2} + 9\sqrt{3}$
- $(\sqrt{2} + \sqrt{3})^4 = 2 \times 2 + 3 \times 3 + 6 \times 6 + 4 \times (2 + 3) \times \sqrt{6} = 49 + 20\sqrt{6}$

I am not sure if I made an arthmetic error. Let me check these with a computer:

```
>>> ( 5 + 2*6**0.5 ) - ( 2**0.5 + 3**0.5 )**2

-1.7763568394002505e-15

>>> (49 + 20*6**0.5) - ( 2**0.5 + 3**0.5 )**4

-2.842170943040401e-14
```

These decimal errors are just somewhat scary, but at least we obtain statements like:

$$\left|(\sqrt{2} + \sqrt{3})^4 - (49 + 20\sqrt{6})\right| < 10^{-12}$$

in fact the difference is zero. However, that thing we called "$\sqrt{2}$" or "$\sqrt{3}$" was likely something else anyway, with even a small error.

What if we used integers? Let's try representing the $\sqrt{2}$ and $\sqrt{3}$ as 2×2 matrices:

$$\sqrt{2} = \left[\begin{array}{cc} 0 & 2 \\ 1 & 0 \end{array}\right] \text{ and } \sqrt{3} = \left[\begin{array}{cc} 0 & 3 \\ 1 & 0 \end{array}\right]$$

and these aren't the only representations either. Let's merge these into a single matrix:

$$\sqrt{2} \otimes 1 = \left[\begin{array}{cc} 0 & 2 \\ 1 & 0 \end{array}\right] \otimes \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$1 \otimes \sqrt{3} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \otimes \left[\begin{array}{cc} 0 & 3 \\ 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

These matrices come out of nowhere:

```
>>> import numpy as np
>>> a = np.array([[0,2,0,0],[1,0,0,0],[0,0,0,2],[0,0,1,0]])
>>> b = np.array([[0,0,3,0],[0,0,0,3],[1,0,0,0],[0,1,0,0]])
```

but they come out with answers related to what we obtained before:

```
>>> np.dot(a+b,a+b)
array([[ 5,  0,  0, 12],
       [ 0,  5,  6,  0],
       [ 0,  4,  5,  0],
       [ 2,  0,  0,  5]])

>>> f = lambda x : np.dot(x,a+b)
>>> f(f(a+b))
array([[ 0, 22, 27,  0],
       [11,  0,  0, 27],
       [ 9,  0,  0, 22],
       [ 0,  9, 11,  0]])

>>> f(f(f(a+b)))
array([[ 49,   0,   0, 120],
       [  0,  49,  60,   0],
       [  0,  40,  49,   0],
       [ 20,   0,   0,  49]])
```

Running these tensor product comutations backwards, are we sure this map is true?

$$\left[\begin{array}{cc|cc} 49 & 0 & 0 & 120 \\ 0 & 49 & 60 & 0 \\ \hline 0 & 40 & 49 & 0 \\ 20 & 0 & 0 & 49 \end{array}\right] \mapsto \left(49\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} + 20\begin{bmatrix}0&2\\1&0\end{bmatrix}\otimes\begin{bmatrix}0&3\\1&0\end{bmatrix}\right)^{\otimes 2}$$

I was able to find this answer because I know the answer already and I'm making up the rules as I go along. In fact, we have re-construted the **tensor product** and the $\mathbb{Z}$-**module**.

Later, this perfect algebric description should also have a defect, since we'll observe that $\overline{\mathbb{Q}(\sqrt{2}+\sqrt{3})} = \mathbb{R}$ and wonder how that is happening.

Here is more code:

```
>>> f1 = lambda a,b,c,d : a + b*2**0.5 + c*3**0.5 + d*6**0.5
>>> f2 = lambda a,b,c,d : a - b*2**0.5 + c*3**0.5 - d*6**0.5
>>> f3 = lambda a,b,c,d : a + b*2**0.5 - c*3**0.5 - d*6**0.5
>>> f4 = lambda a,b,c,d : a - b*2**0.5 - c*3**0.5 + d*6**0.5
>>> g = lambda a,b,c,d: np.round( f1(a,b,c,d)*f2(a,b,c,d)*f3(a,b,c,d)*f4(a,b,c,d)).astype(int)
>>> [g(a,b,c,d) for a in range(5) for b in range(5) for c in range(5) for d in range(5)]
```

Then we obtain some silly factorizations like this:

$$-23 = (2+\sqrt{2}+\sqrt{3}) \times (2-\sqrt{2}+\sqrt{3}) \times (2+\sqrt{2}-\sqrt{3}) \times (2-\sqrt{2}-\sqrt{3})$$

Let's restrict our attention to subfields $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$. This is also a quadratic extension, with the base field $K = \mathbb{Q}(\sqrt{2})$. By the same token we have another quadratic extension $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] = 2$.

We could ask about the ring of integers, $\mathcal{O}_L$ with $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and solve the quadratic equations:

$$x^2 + (a_0 + a_1\sqrt{2})x + (b_0 + b_1\sqrt{2}) = 0$$

with $x = x_0 + x_1\sqrt{2} + x_2\sqrt{3} + x_3\sqrt{6}$ (or the basis of your choice). The discriminant condition determines a variety:

$$\sqrt{(a_0 + a_1\sqrt{2})^2 - 4(b_0 + b_1\sqrt{2})} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$$

Even for the extension $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ I don't remember how this works, why this would obtain the ring of integers $\mathbb{Z}[i]$ (or even that this is closed under addition, that if $a \in \mathbb{Z}[i]$ and $b \in \mathbb{Z}[i]$ then $a + b \in \mathbb{Z}[i]$.

— —

The goal of today's talk is much more modest, to write down the zeta function $\zeta_K(2)$ with $K = \mathbb{Q}(\sqrt{2})$. This barely requires understanding the primes in $\mathbb{Z}[\sqrt{2}]$, since all we care about are the norms and ideal classes. However, naïvely I don't know what those are, just integers and maybe complex conjugates. So a certain amount of trial and error must occur.

Also in elementary school the multiplication and division tables $2 \times 3 = 6$ were *memorized*. And we spent 5 or 6 years in school developing the appropriate notions of number and applying them. High school learning to apply these ideas in general, in college applying this knowledge to other subjects. It suggests that even in 4 years of university we can't learn very much.

Then after all this we complain that $\mathbb{Z}$ wasn't a very good concept of number at all, but we have to build up to that.

4/27 Back to work. We're going to build a zeta function for all integers in $\mathbb{Z}[\sqrt{2}]$. These rings no longer totally agree. If I say "$\mathbb{Q}(\sqrt{2})$ is the set of all quotients of $\mathbb{Z}[\sqrt{2}]$" then we have an equation like this:

$$\frac{m_1 + n_1\sqrt{2}}{m_2 + n_2\sqrt{2}} = a + b\sqrt{2} \text{ with } (m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 \text{ and } a, b \in \mathbb{Q}$$

In that case we could solve for both a and b and we obtain a map:

$$(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 \mapsto (a, b) = \left(\frac{m_1 m_2 + 2n_1 n_2}{m_2^2 - 2n_2^2}, \frac{m_1 n_2 + m_2 n_1}{m_2^2 - 2n_2^2} \right) \in \mathbb{Q}^2$$

These rational numbers could be slopes of line lines. We could arrange all these possible sloes in a circle:

$$\cos^2\theta + \sin^2\theta = (a + b\sqrt{2})^2 + (c + d\sqrt{2})^2 = \big(a^2 + c^2 + 2b^2 + 2d^2\big) + \sqrt{2}\big(2ab + 2cd\big) = 1 + 0\sqrt{2}$$

I needed to consult the definition. The **Dedekind Zeta function** of a number field (such as $K = \mathbb{Q}(\sqrt{2})$) is the sum over integral ideals of K. And I don't quite know enough ring theory to decide what those are. There's a bit of Euclidean geometry there...

$$\zeta_K(s) = \sum_a \frac{1}{N(a)^s}$$

as $(a) = (a_1 + a_2\sqrt{2}) \subseteq K$ varies over the integral ideals, with $a_1, a_2 \in \mathbb{Z}$. An alternative definition is as the product over all primes:

$$\zeta_K(s) = \prod_p \frac{1}{1 - N(p)^s}$$

Detailed examinations of these zeta functions exist in the literature, but what matters is our own understanding. Do we know what prime numbers $\mathfrak{p} \in \mathbb{Z}$ look like in this ring? Books do, I don't:

- $(\sqrt{2}+1)(\sqrt{2}-1) = 1$ and this means the same prime can apper infinitely many times.
- $(2\sqrt{2}+1)(2\sqrt{2}-1) = 7 \in \mathbb{Z}$ this means that prime numbers can split in two.

By quadratic reciprocity we know how primes split over this field. However, these two theorems: quadratic reciprocity and sum of two squares theorem, I don't know if I take them for granted:

$$p = x^2 + 2y^2 \text{ iff } \{x : x^2 \equiv 2 \pmod{p}\} \neq \varnothing \text{ iff } p \equiv \pm 1 \pmod{8}$$

and this is a fairly well-studied but somewhat random sequence. Every time I learn a new number, it's quite informative:

```
  7,  17,  23,  31,  41,  47,  71,  73,  79,  89,  97,
103, 113, 127, 137, 151, 167, 191, 193, 199, 223, 233,
239, 241, 257, 263, 271, 281, 311
```

These are cute, but every time I get a fact like this, I need to check the logic, and also every time we have even a basic question over $\mathbb{Z}[\sqrt{2}]$ it factors through one like this. What if we try to compute the GCD of 31 and 41:

$$\frac{31}{41} = \frac{7+3\sqrt{2}}{9+5\sqrt{2}} \times \frac{7-3\sqrt{2}}{9-5\sqrt{2}}$$

To me, these multiplication facts are much more interesting than the actual answer. What if we run the Euclidean algorithm on the first two factors:

$$9 + 5\sqrt{2} = a(7 + 3\sqrt{2}) + b$$

with some "small" a, b. Do we know the answer? We take as a given that the answer can be found, but we don't really know. Do we know an isomorphism:

$$\mathbb{Z}[\sqrt{2}]/(7+3\sqrt{2})\mathbb{Z}[\sqrt{2}] \simeq \mathbb{F}_{31} = \mathbb{Z}/31\mathbb{Z}$$

It's not instantly obvious to me. This is clearly a statement about lattices and requires Euclidean geometry, or Fermat's little theorem.

Here's quotation from a paper by Adler I found that talks about decimals:

> The decimal expansion of real numbers, familiar to us all, has a dramatic generalization to representation of dynamical system orbits by symbolic sequences. The natural way to associate a symbolic sequence with an orbit is to track its history through a partition. But in order to get a useful symbolism, one needs to construct a partition with special properties. In this work we develop a general theory of representing dynamical systems by symbolic systems by means of so-called Markov partitions
>
> **Symbolic Dynamics and Markov Parititions**
> Bulletin of the American Mathematical Society, Vol 35, No 1. Jan 1998, 1-56

In section three, two examples are worked out:

- $x \mapsto 2x$
- $x \mapsto \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)$

They find partitions of $\mathbb{R}^2$ (or $S^1 \times S^1$) that behave nicely with respect to these multiplications. For our discussion of $\mathbb{Q}(\sqrt{2})$ the map is $\times(1+\sqrt{2})$

$$(1+\sqrt{2})(a+b\sqrt{2}) = (a+2b)+\sqrt{2}(a+b) \text{ or } \left(\begin{array}{cc} 1 & 2 \\ 1 & 1 \end{array}\right)$$

This is another multiplication we might use in the context of number theory and dynamical systems. A "**number**" will be encoded by a trajectory in this dynamical system that we'll call the "**digits**". Our decimal system is expressed in terms of the dynamical system $\times 10$, using a slightly imperfect map $\{1,2,\dots,10\}^{\mathbb{Z}} \to \mathbb{R}$

$$\sqrt{2} = 1.4142135623730951$$

When I think of "Additive Combinatorics" I wonder how carries of addition behave under this system. That's not quite what they mean, but it's fair fame. It's just the books will lead you in a different direction.

Why is the map $\times(1+\sqrt{2})$ going to be such big deal. We've shown that Pell equation solutions are ambiguous – they're infinite:

$$x^2 - 2y^2 = 14 \text{ with } (x,y) = \{(4,1),(8,5),\dots\}$$

I found this number with a computer, but we can use number theory to reason about patterns in the output of our program.

How many orbits of solutions to pell equaion $(x+y\sqrt{2}) \times (1+\sqrt{2})^{\mathbb{Z}}$ sppear when solving $x^2 - 2y^2 = n$? And we can look them up on a table, eventually I found `https://oeis.org/A035185` and we could show that:

$$\left| \Big\{ (x,y) : x^2 - 2y^2 = 98 \Big\} / (1+\sqrt{2})^{\mathbb{Z}} \right| = 3$$

Still haven't check this with a computer. How do you separate the three orbits of infinitely many solutions?

However at least on paper there's an easy answer:

$$\sigma_{\sqrt{2}}(n) = \#\Big\{ r|n : n \equiv 1,7 \pmod 8 \Big\} - \#\Big\{ r|n : n \equiv 3,5 \pmod 8 \Big\}$$

if I even wrote it correctly.

5/8 We'd decided to model the action of $\times(1+\sqrt{2})$ on pairs of numbers $\mathbb{Z}^2$ as a 2×2 matrix:

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

This is not the only one it could be. Also, there could be action on $\mathbb{Q}^2$ or $\mathbb{C}^2$, but beware, these could even be other $\mathbb{Q}$-torus. What is going to be a well-behave set of lines with respect to this matrix? So we have to find the eigenvectors:

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

Then if we subtract both sides, we get the characteristic equation of the matrix:

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \big(1+1\big)\lambda + \big(1\cdot 1 - 1\cdot 2\big) = \lambda^2 - 2\lambda - 1 = 0$$

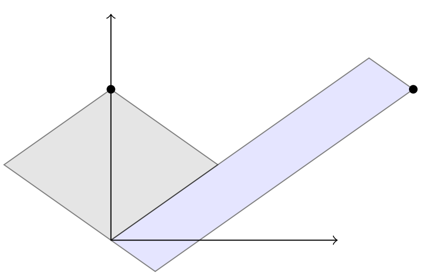
and there are two solution which are called the "Silver numbers"

$$\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2} = 1 \pm \sqrt{2}$$

To find the eigenvectors we could try different things:

$$\begin{aligned} a + 2b &= (1\pm\sqrt{2})a \\ b &= \pm\frac{1}{\sqrt{2}}a \\ \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} \pm\sqrt{2} \\ 1 \end{bmatrix} \end{aligned}$$

This is beginner exercise in linear algebra and the next step is not much more technical. We are going to build Markov partition with respect to this matrix. Notice that $1-\sqrt{2} < 0$.



Next can we write down the rectangles so that the matrix map acts perfectly? Perhaps if we write the equation over:

$$(1+\sqrt{2})^2 = 2\times(1+\sqrt{2}) + 1$$

and maybe it will be easier if I use the diagonal matrix (and we can slant it later):

$$\begin{bmatrix} \sqrt{2}+1 & 0 \\ 0 & \sqrt{2}-1 \end{bmatrix}$$

and perhaps we should act on the torus $[0,1]\times[0,1]$

$$(x,y) \mapsto \Big(\{(\sqrt{2}+1)\times x\}, \{(\sqrt{2}-1)\times y\}\Big)$$

Constructing the actual Markov partition seems to be a bit of an art:

There's arrows everywhere in this thing. Still no good answer,...