

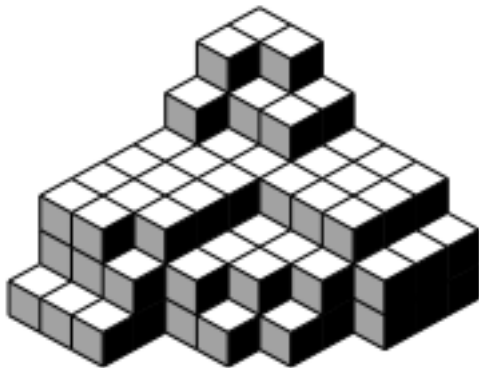
Examples: $n!$ the Factorial

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There are famous McMahon formulas and I could be mixing them up. Here is one, which counts literally *all* plane partitions:

$$\sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^n$$

My apology in advance for not having a good picture. They are take some work to draw. Here we take example from Mirjana Vuletic.



This is good but not really what I am looking for today.

I am looking for those partitions which fit inside an $a \times b \times c$ box. There is an exact number on Wikipedia:

$$\#\{boxes\} = \prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{i+j+k-1}{i+j+k-2}$$

and you might wonder why so much attention might be drawn to a simple equation like this:

- Inside Mathematics - literally thousands of papers and they keep coming
- Outside Mathematics - maybe a statistician or data analyst might find this structure relates to something in the real world¹

Today we will do something totally useless and set $a = b = c = \frac{1}{2}$. How many ways to pack $1 \times 1 \times 1$ cubes into a box of side $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$?

The procedure for guessing a value of $(\frac{1}{2})!$ might stem from Euler's definition of factorial:

$$x! = \lim_{n \rightarrow \infty} \frac{n^x n!}{x \times (x+1) \times (x+2) \times \dots \times (x+n)}$$

¹Baseball, elections, the human genome, the weather, instagram, the radio... and the end of the day these are all tables of numbers and there exist procedures where these can be processed in a similar way.

Let's try to shift the product lattice by (x, y, z) :

$$\prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{(i+j+k) + (x+y+z) - 1}{(i+j+k) + (x+y+z) - 2}$$

I don't even know what this number is. OK. Let

$$n!! = 0! \times 1! \times \dots \times (n-1)!$$

and we can even define the q -factorial

$$[n]_q! := \prod_{i=1}^n \frac{1 - q^n}{1 - q}$$

The hyperfactorial (and the Barnes G-function)

$$\sum_{\pi} q^{|\pi|} = \frac{a!!_q b!!_q c!!_q (a+b+c)!!_q}{(a+b)!!_q (b+c)!!_q (c+a)!!_q}$$

and we can set $q = 1$ to get one McMahon formula. Or let $a, b, c \rightarrow \infty$ to get the other.²

And now we can set $a = b = c = \frac{1}{2}$, so that $(\frac{1}{2})! = \sqrt{\pi}$ and $(\frac{1}{2})!! = \dots$. Euler never wrote a formula for the Barnes G-function.

Oh my this got complicated. Have an answer in a bit.

²Everything fits together in McMahon world. Everything is beautiful!

One reason I want to move on is because I have a lot of interesting factorials to talk about.

$$n! = \sum_{\lambda \in P_n} (t_\lambda)^2$$

P_n is the set of all partitions of n . Any way you can think of to split a number:

$$10 = 5 + 3 + 2$$

for each such partition you could play a game and draw a **tableau**

10	8	7	4	3
9	6	5		
2	1			

10	9	8	3	1
7	5	4		
6	2			

the Robinson Schen-

sted correspondence is a way of taking permutations and turning them into pairs of shapes like this. And the miracle is there are exactly $n!$ as can be bound by playing **jeu de taquin**.

And now we ask the obvious question, how do we play jeu de taquin with $\frac{1}{2} \times \frac{1}{2}$ square?

References

- (1) Mirjana Vuletic **A generalization of MacMahon's formula** [arXiv:0707.0532](#)
- (2) Paul Zinn-Justin **Six-Vertex, Loop and Tiling models: Integrability and Combinatorics**
[arXiv:0901.0665](#)