Lookup: Ring Theory

Our prototypical example of a ring is the number line, abstractly written as $\mathbb{Z}$. This didn't happen until the 20th century with Emmy Noether (or something like that).

We're pretty sure we need the tensor product. A *tensor* is just a **box**. We needed tensors to build the theory of General Relativity and described "curved" 3- and 4-dimensional spaces. They are used to describe the **curvature** of different types of "shapes" or "spaces". By the 1920's academics realized tensors would play a role in topology (the most *qualitative* study of shapes – invariant under high levels of distortion).

The naïve way of constructing the tensor product would be merely to write $x \otimes y$ with $x, y \in R$ two elements of a ring. We should have:

$$x \otimes y \neq y \otimes x$$

Then we could have the basic properties of tensor produts:

- $x \otimes (y_1 + y_2) = (x \otimes y_1) + (x \otimes y_2)$
- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$
- $(r\, x) \otimes y = x \otimes (r\, y)$

so we could have $x \in M$ and $y \in N$ elements of two R-modules (a generalization of matrices). Then we continue the inspection of the properties of the ring. Instead of building the tensor product element by element, there is a **bilinear map**:

$$\otimes : M \times N \to M \otimes N$$

In our case, $x = \vec{x} \in M$ and $y = \vec{y} \in N$ are vector spaces.

Ex if $M = \mathbb{R}$ and $N = \mathbb{R}$ then $M \times N = R \otimes R$.

Ex $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^2$.

Ex $\mathbb{R} \otimes \mathbb{Z}[i] = \mathbb{R}[i] = \mathbb{C}$.

Ex $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$. (This an example of **torsion**.)

Ex $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z} = 0$.

Ex $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$.

So there are well-behaved rules for generating entire number systems. In a graduate-level textbook or reference book, the category is called R-Mod.

Category theory could let us organize the many different number systems and "geometric" objects that arise in our computations. The inner product which is just $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$. It turns out to be more correctly written as:

$$x \cdot y = x_1 y^1 + x_2 y^2 + x_3 y^3 \in \mathbb{R}$$

could be thought us as a map from $\mathbb{R}^3 \times \mathbb{R}_3 \to \mathbb{R}$.

$$(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}) \otimes (\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}) = 9(\mathbb{R} \otimes \mathbb{R}) = \mathbb{R}^9$$

$M \otimes_R -$ and $- \otimes_R N$ are **right-exact functors** so they have well-behaved properties.

Let's do the following typing exercise, a **triangle**:

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "\tilde{f}"] \\ & G \end{tikzcd}$$

Here $\tilde{f} \circ \otimes = f$.

The danger of "universality" is that we have to at some point recover the original object. Yet it's a succinct way of dealing with **everything** at one.

It would look funny to write the matrix object in terms of matrix objects:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (e_1 \otimes e^1) + (e_2 \otimes e^2) + (e_3 \otimes e^3)$$

There's confusion about the algebraic objects that we are dealing with. Here it's called a "balanced product" or a "tensor product".

- $M \times N$ is called a balance product even though ther are only two factors.
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The category theory way is nice and clean, yet it might not be obvious to interpret. Maybe we can be suspicious about the role of $\mathbb{Z}$ everywhere. [1]

[1] Example if $n \in \mathbb{Z}$ then $n+1 \in \mathbb{Z}$. Our number line rarely runs to more than $n = 100$ or $n = 1000$ and what if $n = 10^{100}$ or something like that. We can't use a pocket calculator which is limited to 8 decimal places. So there's always a notion of **next**. Worse than that the operations of $+$ and $\times$ might behave oddly with the spillover with the decimal places. And "decimal" here is just the map $T : n \mapsto 10 \times x$. Example $2048 = 2 \times 10^3 + 4 \times 10 + 8 = (2 \times T^3 + 4 \times T + 8 \times I) \times 1$ this is also $2^1 0 = (T_2)^{10} \times 1$.

$$2 \times T^3 + 4 \times T + 8 \times I = (T_2)^{10}$$

Here we try to make decimals look like another operation.

The balanced products are written in terms of the Hom functor.

$$\mathsf{Hom}_{\mathbb{Z}}(M \otimes_R N, G) \simeq \mathsf{Hom}_R(M, \mathsf{Hom}_{\mathbb{Z}}(N, G))$$

So that our simple notations of $\sum v_i v^i = v^2$ had basic categorical content where our notions of “number” had to be revised.

05/13 Let's just parrot some of the definitions from Wikipedia. Once we have these giant boxes of numbers we can product others. Our calculation is reduced to a single arrow and we went up with something categorical $A \to B$.

Here is the way **engineers** look at tensor product:

$$v \otimes w = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 w_1 & v_1 w_2 \\ v_2 w_1 & v_2 w_2 \end{bmatrix}$$

Then we get the abstract representation where these are two copies of any ring (e.g. $R = \mathbb{Z}$) and we get:

$$(\mathbb{R} \oplus \mathbb{R}) \oplus (\mathbb{R} \oplus \mathbb{R}) \simeq (\mathbb{R} \otimes \mathbb{R})^4 \simeq \mathbb{R}^4$$

The symbol $\simeq$ is misleading because the space of possible maps from one vector space (or module, since "scalars" or "rescaling" doesns't necessarily have a reverse) is a space of matrices.

Such a tensor product construction lifts to mappings. If we have $S : V \to X$ and $T : W \to Y$ then we can have tensor of the two mappings:

$$(S : V \to X) \otimes (T : W \to Y) = (S \otimes T) : V \otimes W) \to (X \otimes Y)$$

In math, describing the geometry of one "shape" often leads to another geometry or another "shape".

Ex The "wedge product" of two vector spaces:

$$V \wedge V = (V \otimes V)/\{v \otimes v : v \in V\} \simeq V \otimes V/\{v_1 \otimes v_2 + v_2 \otimes v_1$$

These are getting non-constructive proofs. Here the instructions just say:

- Set $v \otimes v = 0$ for all vectors in the space $v \in V$.
- Set $v_1 \otimes v_2 + v_2 \otimes v_1$ for all $v_1, v_2 \in V$.

Q does this equivalent work for $\mathbb{Z}$-modules (instead of $\mathbb{R}$-vector space). This wedge product could be how we represent **Area** (from Geometry) or **Determinant** (from Linear Algebra).

References

[1]