Squarefree Numbers

John Mangual

1 Orbits of Group Actions

As [?] indicates, number theory problems can often be turned into statements about group actions.

Ex: The odds of two numbers being relatively prime is $\frac{6}{\pi^2}$. How to express this as a problem in group theory?

It is related to the action of $GL(2,\mathbb{Z})$ on the integer lattice $\mathbb{Z}^2$. The orbit of $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ is itself. The orbit of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is all integer vectors with relatively prime coordinates[1].

$$GL(2,\mathbb{Z})\begin{pmatrix}1\\0\end{pmatrix} = \left\{ \begin{pmatrix}a\\b\end{pmatrix} : (a,b) = 1 \right\}$$

Visual inspection of the set $\{(a,b)=1\} \subset \mathbb{Z}^2$ does not "look" symmetric. In fact[?]:

- $\mathbb{Z}^2 = \bigcup_{m\in\mathbb{N}_0} \{(a,b)=m\}$ disjoint union of $GL(2,\mathbb{Z})$ invariant sets.
- $\{(a,b)=1\}$ contains holes of arbitrary size $\rho > 0$ which repeat as some copy of $\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\mathbb{Z}^2 + \left(\begin{smallmatrix}c\\d\end{smallmatrix}\right)0$.
- The different set $\{(a,b)=1\} - \{(a,b)=1\} = \mathbb{Z}^2$ is the entire lattice plane.
- The natural density is $\frac{6}{\pi^2}$ despite no obvious rotation symmetry.

That paper will develop the dynamical system related to the relatively prime integers.

Ex Solutions to $x^2+y^2+z^2=n$ do have group theory interpretation, but only with much difficulty. [?] The trouble is, we can multiply complex numbers sure $(a+bi)(c+di) = (ac-bd)+i(ad+bc)$, there is no multiplily triples of numbers (x,y,z).

Ex In a separate note we tackle the diophantine equation $x^2+y^2+z^2 = 3xyz$. For example, one solution is $(x,y,z) = (1,1,1)$. We can generate more solutions by replacing $z \leftrightarrow 3xy - z$, leading to the solution $(1,1,2)$.

Ex Pythagorean triples $x^2+y^2=z^2$ have an $\mathrm{SL}(2,\mathbb{Z})$ group structure. [?]

[1] Brian Conrad **Group Actions** http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/gpaction.pdf

References

[1] Manfred Einsiedler, Grigory Margulis, Amir Mohammadi, Akshay Venkatesh. *Effective Equidistribution and Property τ*. arXiv:1503.05884

[2] Harry Furstenberg. *On the Infinitude of Primes* American Mathematical Monthly, 62, (1955), 353.

[3] Michael Baake, Christian Huck. *Dynamical properties of k-free lattice points* arXiv:1402.2202

[4] Michael Baake, Christian Huck. *Ergodic properties of visible lattice points* arXiv:1501.01198

[5] Jordan S. Ellenberg, Philippe Michel, Akshay Venkatesh. *Linnik's ergodic method and the distribution of integer points on spheres* arXiv:1001.0897

[6] Jean Bourgain, Alex Gamburd, Peter Sarnak *Markoff Triples and Strong Approximation*

[7] Alex Kontorovich *Levels of Distribution and the Affine Sieve* arXiv:1406.1375