Worksheet: Lagrange Interpolation

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There are functions which satisfy a bewildering number of constraints. Or we might find in a paper, a the existence of a function - satisfying very reasonable constraints - which is nearly impossible to construct.

Let's try to find an $f(x)$ that passes through a few points.

- $f(0) = 1$
- $f(1) = 2$
- $f(2) = 2$
- $f(3) = 3$
- $f(4) = 0$

How will we find such a function? Let's guess a structure for it:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Next plug in the values $x = 0$, $x = 1$, etc to solve a system of simultaneous equations:

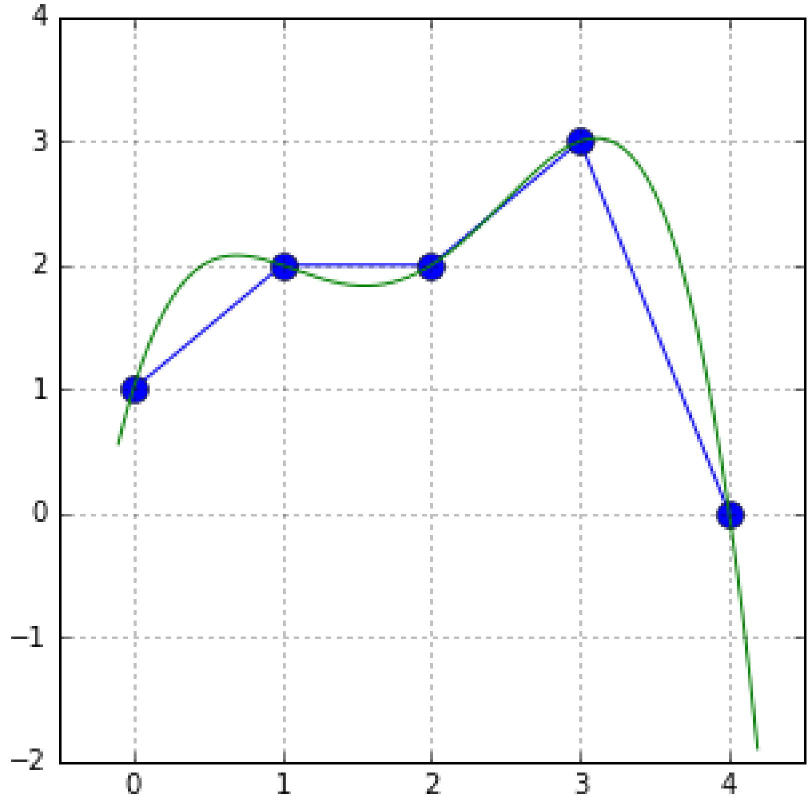
$$\begin{aligned}
f(0) = 1 &= a_0 + a_1 \times 0 + a_2 \times 0^2 + a_3 \times 0^3 + a_4 \times 0^4 + a_5 \times 0^5 \\
f(1) = 2 &= a_0 + a_1 \times 1 + a_2 \times 1^2 + a_3 \times 1^3 + a_4 \times 1^4 + a_5 \times 1^5 \\
f(2) = 2 &= a_0 + a_1 \times 2 + a_2 \times 2^2 + a_3 \times 2^3 + a_4 \times 2^4 + a_5 \times 2^5 \\
f(3) = 3 &= a_0 + a_1 \times 3 + a_2 \times 3^2 + a_3 \times 3^3 + a_4 \times 3^4 + a_5 \times 3^5 \\
f(4) = 0 &= a_0 + a_1 \times 4 + a_2 \times 4^2 + a_3 \times 4^3 + a_4 \times 4^4 + a_5 \times 4^5
\end{aligned}$$

The equation is solved in a standard way called **Lagrange Interpolation**

$$\begin{aligned}
f(x) &= f(0)\,\frac{(x-1)(x-2)(x-3)(x-4)}{(0-1)(0-2)(0-3)(0-4)} \\
&+ f(1)\,\frac{(x-0)(x-2)(x-3)(x-4)}{(1-0)(1-2)(1-3)(1-4)} \\
&+ f(2)\,\frac{(x-1)(x-2)(x-3)(x-4)}{(2-0)(2-1)(2-3)(2-4)} \\
&+ f(3)\,\frac{(x-1)(x-2)(x-3)(x-4)}{(3-0)(3-1)(3-2)(3-4)} \\
&+ f(4)\,\frac{(x-1)(x-2)(x-3)(x-4)}{(4-0)(4-1)(4-2)(4-3)}
\end{aligned}$$

A little bit complicated to write down, we can set $x = 0, 1, 2, 3, 4$ and check it works.

Not without artifacts...However, looks very good in some places!



We don't really know what is happending between the values of $x = 0$ and $x = 1$.

NEXT more constraint problems. Maybe in two dimensions.

References

(1) ...