Theta Functions and Ford Circles

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Learn about modular forms $\theta(z) = \sum q^{n^2}$ and $\eta(z) = q^{1/24} \prod (1 - q^n)$

1 Theta Functions and $\Gamma_0(4)$

Lagrange's theorem states an integer can be represented as the sum of three squares, $n = a^2 + b^2 + c^2$ if and only if $n \neq 4^j(8k+7)$. This borderline case between easier statements in 2 and 4 dimensions

- $p = a^2 + b^2$ iff $p = 4k + 1$ (Ex. $p = 7481$ or $p = 3641332172344000 3717$)
- Every integer is the sum of 4 squares $n = a^2 + b^2 + c^2 + d^2$.

While studying 3 squares, Duke uses a property of spherical harmonics, that this series is a $\Gamma_0(4)$ modular form for all $u \in L^2[SO(3)]$.

$$\theta(u; z) = \sum_{n \in \mathbb{Z}} \left[\frac{1}{r_3(n)} \sum_{\xi \in V_n} u(\xi) \right] e^{2\pi i\, nz}$$

Here $V_n = \{\vec{m} : m_1^2 + m_2^2 + m_3^2 = n\} = nS^2 \cap \mathbb{Z}^3$ are the lattice points distance n from the origin. As a half-integer weight modular form, Duke knew the coefficiets decayed fairly slowly:

$$\frac{1}{r_3(n)} \sum_{\xi \in V_n} u(\xi) \ll n^{-1/28}$$

but slowly enough to show these points (eventually) equidistribute around the sphere.

Unfortunately, I had no idea what a half-integer modular form was, or why the coefficients decays so slowly, or why that was relevant .

Modular Forms textbooks start with a few examples: Eisenstein series, Poincare series, eta functions and finally one that kind of resembles our theta function:

$$\theta(z) = \sum e^{2\pi i\, n^2 z}$$

This function has an obvious symmetry, $z \mapsto z+1$ and a less obvious one:

$$\theta\left(-\frac{1}{4z}\right) = \sqrt{-2\pi i z}\,\theta(z)$$

These two symmetries in conjunction, are our copy of $\Gamma_0(4)$. Viz:

$$\Gamma_0(4) \equiv \left\langle z \mapsto z+1, z \mapsto -\frac{1}{4z}\right\rangle$$

The second symmetry is known as Poisson summation:

$$\sum_{n\in\mathbb{Z}} e^{\pi x^2 t} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{\pi x^2/t}$$

It's tempting to say there is $\mathrm{SL}_2(\mathbb{Z})$ symmetry. After repeating the same mistake over and over, observe $\langle z \mapsto -\frac{1}{z}, z \mapsto z + {\color{red}2}\rangle$ is the same as $\Gamma_0(4)$.

$SL(2, \mathbb{Z})$ is known to be related to continued fractions. We content ourselves with one example:

$$\pi = 3.14 = 3 + 0.14 \approx 3 + \frac{1}{7} = [3; 7] = \frac{22}{7}$$

The maps $z \to z + 1$ as well as $z \to -\frac{1}{z}$ behave nicely with this new format:

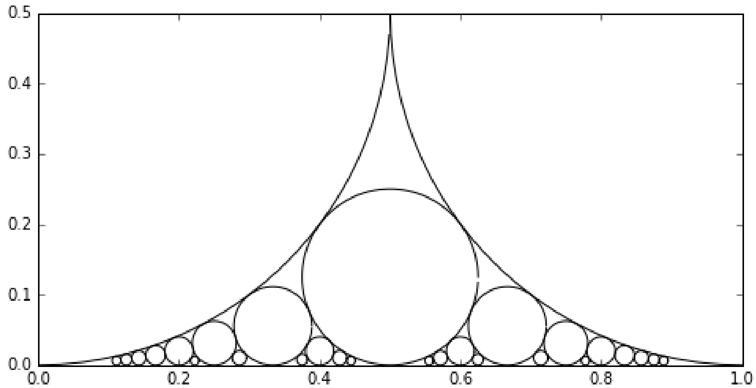
$$[a; b] + 1 = [a + b; b] \quad \text{and} \quad c + \frac{1}{[a; b]} = [c; a, b]$$

In fact we do not have $z \to z + 1$ instead there is $z \to z + 2$, so we cannot change the parity of our continued fraction digits. We have 6 classes:

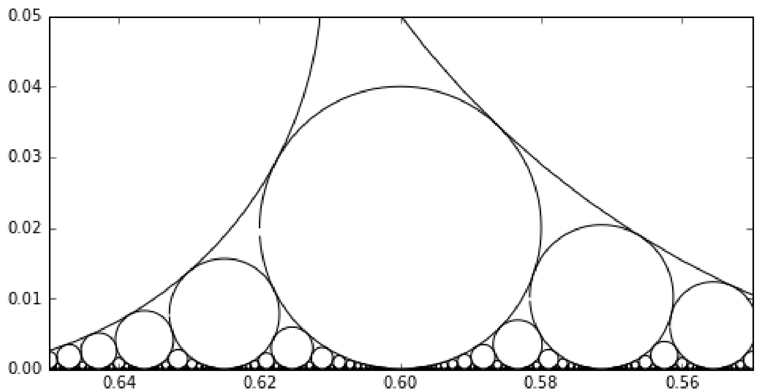
- $[\mathtt{EVEN}; \mathtt{EVEN}]$
- $[\mathtt{EVEN}; \mathtt{ODD}]$, $[\mathtt{ODD}; \mathtt{ODD}]$, $[\mathtt{ODD}; \mathtt{EVEN}]$

The first class always maps to itself and the second group of three turn into each other observing the rules: $\mathtt{ODD} + \mathtt{ODD} = \mathtt{EVEN}$ and $\mathtt{ODD} + \mathtt{EVEN} = \mathtt{ODD}$

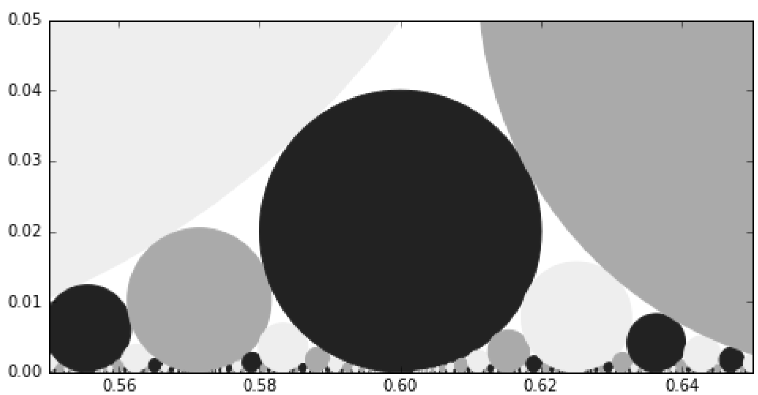
These magical rules I just made up are also found in Mumford's *Lectures on Theta* - which has everything one could ask.



The Ford Circles - crated by Lester Ford



It makes a different if we zoom near 0.6



In $\Gamma_0(4)$ we use 3 or 6 colors for circles

Here's a quick way to get a $\frac{3}{2}$-weight $\Gamma_0(4)$ modular form as I read in a paper by Yves Meyer. Just cube the theta function:

$$\theta(z)^3 = \left[\sum_{n\in\mathbb{Z}} q^{n^2}\right]^3 = \sum_{n\in\mathbb{Z}} r_3(n)\, q^{n^2}$$

And it's exactly the one we wanted. In fact, we want a slightly different one, but if we take the cube of the S-duality equation:

$$\left[\sum_{x\in\mathbb{Z}} e^{\pi x^2 t}\right]^3 = \left[\frac{1}{\sqrt{t}}\sum_{x\in\mathbb{Z}} e^{\pi x^2/t}\right]^3$$

The sum of 3-squares function returns. We add over $n = a^2 + b^2 + c^2$

$$\sum_{n\in\mathbb{Z}} r_3(n) e^{\pi n t} = \frac{1}{t^{3/2}} \sum_{n\in\mathbb{Z}} r_3(n) e^{\pi n/t}$$

This is Poisson summation even though there are no squares.

If we try to recover the f and $\hat{f}$ we get Guinand's formula:

$$\frac{df_x}{dt}(0) + \sum_{n\in\mathbb{N}} \frac{r_3(n)}{\sqrt{n}} f_x(\sqrt{n}) = \frac{d\hat{f}_x}{dt}(0) + \sum_{n\in\mathbb{N}} \frac{r_3(n)}{\sqrt{n}} \hat{f}_x(\sqrt{n})$$

References

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(3) Thomas C. Hull (April 2011). "Solving Cubics With Creases: The Work of Beloch and Lill" (PDF). American Mathematical Monthly: 307âĂŞ315. doi:10.4169/amer.math.monthly.118.04.307.

(4) S. Kharchev, A. Zabrodin
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arXiv:1510.02699 Theta vocabulary II.