

Scratchwork: Jacobi Triple-Product Formula

Comparing apples and oranges. There is lots of discussion of theta functions in the literature. How do we normalize our definitions? Here's an example from Integrable Systems:

$$\theta_1(u; p) = \theta_1(u) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos 2u + p^{4n})(1 - p^{2n})$$

These are called **elliptic theta functions**, and the branch of mathematics is called *integrable systems*. Here is another theta function:

$$\theta_4(u; p) = \theta_4(u) = \prod_{n=1}^{\infty} (1 - 2p^{2n-1} \cos 2u + p^{4n})(1 - p^{2n})$$

Why is this going to be so confusing. Here's another definition of theta function.

$$\bar{\theta}_1(x|\tau) = 2e^{i\pi\frac{\tau}{4}} \sin(x) \prod_{n=1}^{\infty} (1 - e^{2ix} e^{\pi i\tau(2n)})(1 - e^{-\pi ix} e^{\pi i\tau(2n)})$$

$$\bar{\theta}_4(x|\tau) = \prod_{n=1}^{\infty} (1 - e^{2ix} e^{\pi i\tau(2n-1)})(1 - e^{-\pi ix} e^{\pi i\tau(2n-1)})$$

and the Jacobi theta function, which is missing a factor of an infinite product.

$$\theta_i(x|\tau) = \prod_{n=1}^{\infty} (1 - e^{2\pi i\tau n}) \theta_i(x|\tau)$$

The same discussion tells is there are some modularity transformations available to us:

$$\begin{aligned} \theta_1(z|\tau+1) &\stackrel{S}{=} \exp\left(\frac{\pi i}{4}\right) \theta_1(z|\tau) \\ \theta_1\left(z\left|-\frac{1}{\tau}\right.\right) &\stackrel{T}{=} -i\sqrt{-\tau i} \exp\left(-\frac{1}{\pi i}\tau z^2\right) \theta_1(z\tau|\tau) \\ \theta_1\left(z\left|\frac{\tau}{1-\tau}\right.\right) &\stackrel{STS}{=} \exp\left(\frac{\pi i}{4}\right) \sqrt{1-\tau} \exp\left(-\frac{1}{i\pi}(\tau-1)\right) \theta_1(z(\tau-1)|\tau) \end{aligned}$$

These formulas are unfortunately, very complicated have an unclear meaning as stated and we don't know where they come from. Here's another source of examples:

$$\begin{aligned} \Theta(z|\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z} \\ \theta(\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \tau \in \mathbb{H} \\ \vartheta(t) &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, t > 0 \end{aligned}$$

Between these three definitions, we get there is a symmetry within the set of perfect squares of integers $\square = \{n^2 : n \in \mathbb{Z}\}$.

Here's a fourth, more varied definition of theta function which follows the same idea:

$$\theta(z; u) = \sum_{m \in \mathbb{Z}^3} u(m) e(|m|z)$$

with u is a spherical harmonic of degree ℓ . The Fourier coefficients are given by:

$$a(n) = n^{\ell/2} r_3(n) W_u(n) \quad \text{with} \quad W_u(n) = \frac{1}{r_3(n)} \sum_{\xi \in V_n} u(\xi)$$

and we have that $\theta(z; u)$ is a holomorphic cusp form on $\Gamma_0(4)$.

Let's see a statement of the Jacobi Triple Product formula. Here's the exercise in Apostol's **Modular Function and Dirichlet Series in Number Theory**.

$$\theta(\tau) = 1 + 2 \sum_{n=1}^{\infty} e^{\pi i n^2 \tau} = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}$$

The statement of the Jacobi triple product formula is the sum is equal to the product.

$$\prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n-1}z^2)(1 + x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m}$$

Here's the exercise in **Conformal Field Theory**.

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Here's the exercise in Elias Stein's textbook on **Complex Analysis**.

$$\begin{aligned} \Theta(z|\tau) &= \sum_{n=-\infty}^{\infty} e^{i n^2 \tau} e^{2 \pi i n z} \\ \Pi(z|\tau) &= \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1} e^{2 \pi i z})(1 + q^{2n-1} e^{-2 \pi i z}) \\ \Theta(z|\tau) &= \Pi(z|\tau) \end{aligned}$$

Θ is entire in $z \in \mathbb{C}$ and holomorphic in $\tau \in \mathbb{H}$.

- $\Theta(z+1|\tau) = \Theta(z|\tau)$
- $\Theta(z+\tau|\tau) = \Theta(z|\tau) e^{-\pi i \tau} e^{-2 \pi i z}$
- $\Theta(z|\tau) = 0$ whenever $z \in \frac{1}{2}(1+\tau) + \mathbb{Z} + \tau \mathbb{Z}$.

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Let's see this in action. If $\text{Im}(\tau) = t \geq t_0 > 0$ and $z = x + iy$, then $|q| \leq e^{-\pi t_0} < 1$ and

$$(1 - q^{2n})(1 + q^{2n-1} e^{2 \pi i z})(1 + q^{2n-1} e^{-2 \pi i z}) = 1 + O(|q|^{2n-1} e^{2 \pi |z|})$$

In off-handed way, we could conclude that $\Theta(z|\tau) \approx 1 + O(1)$.

$$\Pi(z|\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1} e^{2 \pi i z})(1 + q^{2n-1} e^{-2 \pi i z}) = 1 + O\left(\sum_{n=1}^{\infty} |q|^{2n-1} e^{2 \pi |z|}\right) < \infty$$

Example The addition formula for Elliptic Theta function

$$\theta_1(u+x)\theta_1(u-x)\theta_1(v+y)\theta_1(v-y) - \theta_1(u+y)\theta_1(u-y)\theta_1(v+x)\theta_1(v-x) = \theta_1(u+v)\theta_1(u-v)\theta_1(x+y)\theta_1(x-y)$$

Example The Elliptic Gamma Function

$$\Phi_1(z; \mathbf{p}, \mathbf{q}) = \prod_{j,k=0}^{\infty} \frac{1 - e^{+2iz} \mathbf{p}^{2j+1} \mathbf{q}^{2k+1}}{1 - e^{-2iz} \mathbf{p}^{2j+1} \mathbf{q}^{2k+1}} = \prod_{j=0}^{\infty} \frac{(e^{+2iz} \mathbf{p}^{2j-1} \mathbf{q}; \mathbf{q}^2)_{\infty}}{(e^{-2iz} \mathbf{p}^{2j-1} \mathbf{q}; \mathbf{q}^2)_{\infty}}$$

References

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- [4] Elias Stein **Complex Analysis** (Princeton Lectures in Analysis, Book II) Princeton University Press, 2006.
- [5] Rodney Baxter **Exactly Solved Models in Statistical Mechanics** 197?.