Reminder: Group Theory

Let's get some help with computers. What are the groups of order $|G| = 400$? Here is the computer program.[1]

```
G := AllSmallGroups(400);;
List(G, g -> StructureDescription(g));
[ "C25 : C16", "C400", "C25 : C16", "C25 : Q16", "C8 x D50",
  "C25 : (C8 : C2)", "C25 : QD16", "D400", "C2 x (C25 : C8)",
  "C25 : (C8 : C2)", "C4 x (C25 : C4)", "C25 : (C4 : C4)",
  "C25 : (C4 : C4)", "C25 : ((C4 x C2) : C2)", "C25 : QD16", "C25 : D16",
  "C25 : Q16", "C25 : QD16", "C25 : ((C4 x C2) : C2)", "C100 x C4",
  "C25 x ((C4 x C2) : C2)", "C25 x (C4 : C4)", "C200 x C2",
  "C25 x (C8 : C2)", "C25 x D16", "C25 x QD16", "C25 x Q16",
  "C25 : (C8 x C2)", "C25 : (C8 : C2)", "C4 x (C25 : C4)",
  "C25 : (C4 : C4)", "C2 x (C25 : C8)", "C25 : (C8 : C2)",
  "C25 : ((C4 x C2) : C2)", "C2 x (C25 : Q8)", "C2 x C4 x D50",
  "C2 x D200", "C25 : ((C4 x C2) : C2)", "D8 x D50",
  "C25 : ((C4 x C2) : C2)", "Q8 x D50", "C25 : ((C4 x C2) : C2)",
  ...
]
```

Notice there is is both $C_{25} \times QD_{16}$ and $C_{25} \ltimes QD_{16}$.

In more commom notation we can write with the symbols $\times$ (direct product) and $\ltimes$ (indirect product):

- $G = (C_5 \ltimes Q_8) \times D_{10}$
- $G = (C_5 \ltimes C_5) \ltimes (C_4 \times C_4)$
- $G = C_2 \times ((C_5 \times C_5) \ltimes C_8)$
- $G = D_8 \times ((C_5 \times C_5) \ltimes C_2)$

[1] https://math.stackexchange.com/questions/4108993/the-221-groups-of-order-g-400
https://www.gap-system.org/

Induced Representations:

$\mathsf{Ind}_H^G \pi = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \otimes V$

- G finite group
- H subgroup
- (π, V) representation of H

Ex Right regular representation.

$$\textbf{reg} = \mathsf{Ind}_1^G 1 = \mathbb{C}[G] \times_{\mathbb{C}} 1$$

Frobenius Reciprocity:

- $\langle \mathsf{Ind}_H^G \psi, \phi \rangle_G = \langle \psi, \mathsf{Res}_H^G \phi \rangle_H$ (category theory)
- $\mathsf{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} M, N) \simeq \mathsf{Hom}_{\mathbb{C}[H]}(M, {}_{\mathbb{C}[H]}N)$ (modules)
- $\mathsf{Ind}_H^G \dashv \mathsf{Res}_H^G$ (adjunction)
  - o $\mathsf{Res}_H^G : \mathsf{Rep}_G \to \mathsf{Rep}_H$
  - o $\mathsf{Ind}_H^G : \mathsf{Rep}_H \to \mathsf{Rep}_G$

Ex Adjoint functors: which categories and functors were used here?

$$\mathsf{hom}_{\mathcal{C}}(FY, X) \simeq \mathsf{hom}_{\mathcal{D}}(Y, GX)$$

Here $F = \mathsf{Ind}_H^G$ (induction) and $G = \mathsf{Res}_H^G$ (restriction). $\mathcal{C} = \mathsf{Rep}_G$ and $\mathcal{D} = \mathsf{Rep}_H$.

References

[1] Wikipedia "induced representation", ""

[2] ...