Scratchwork: Gaussian Integral

Matrix identities are infinite.

Ex. In 2003, Fyodorov and Strahov found an identity for $N \times N$ Hermitian Gaussian random matrix, H and let $D(z) = \det(z - H)$ be the characteristic polynomial. Let $u_1, \dots, u_d \in \mathbb{C}\backslash\mathbb{R}$ and $v_1, \dots v_d \in \mathbb{C}$.

$$\left\langle \frac{D(v_1)\dots D(v_d)}{D(u_1)\dots D(u_d)} \right\rangle = \det\left(\frac{1}{u_i - v_j}\right)^{-1} \cdot \det\left\langle \frac{1}{u_i - v_j}\frac{D(v_j)}{D(u_i)} \right\rangle$$

This is a comparison of two $d \times d$ matrices, which are themselves expectation of another random process.

This is my blog, we can work out the numerous proofs that have emerged between the years 2000-2005. This offers a springboard into various things.

Ex. The latest moviation I can think of for studying the Gaussian comes from Mochizuki:

Gaussians $\to \cdots \to$ Degrees of Arithemtic Line Bundles $\to \cdots \to$ Teichmuller Theory $\to \cdots \to$ **IUT**

I always wondered what makes the Gaussian special. Whenenever we want to "solve" something we do a sequences of manipulations (preserving all information) or approximations (losing some info) and deform our old problem into an example we're more familiar with.

Theory All problem solving consists of these types of approximations and deformations. The shape that Mochizuki offers us is just the one:

$$\left(\int_{\mathbb{R}} e^{-x^2/2}\,dx\right)\left(\int_{\mathbb{R}} e^{-y^2/2}\,dy\right) = \int_{\mathbb{R}\times\mathbb{R}} e^{-(x^2+y^2)}\,dxdy = \left(\int_0^\infty r\,dre^{r^2/2}\right)\left(\int_{[0,2\pi]} d\theta\right) = 2\pi$$

and we extracted several ideas from this solutions including but not limited to:

- change of variables
- differential maps
- jacobians
- fibrations
- seperation of variables
- probability

and Mochizuki says there's even more...there's almost all of number theory. As a babe step, he feels the theta function is a kind of "Gaussian":

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i\, n^2 z}$$

There are many many theta functions and these form a mess called the theory of "Abelian varieties". The theory of Gaussians looks to me like the following decomposition:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \left(\mathbb{R}_{\geq 0} \times S^1\right) \cup \{0\}$$

is this something I just made up or is it functorial? It's not a common way to think about the Gaussian. Within the mathematical community (a small group of people) there are a wide range of opinions, and it will still be unconventional.

Mochizuki – who is quite controversial already – is telling us to hit there. I'd take his word for it. Then, we become like him.

Ex. A few more examples from Knot theory.

$$\mathcal{Z}_{\mathsf{CS}}(S^3; q) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi} e^{\frac{1}{2g}x_i^2} \prod_{i<j}^{N} \left(2 \sinh \frac{x_i - x_j}{2}\right)^2$$

This is Gaussian integral. Go!

References

[1] Gerald V. Dunne, Mithat Ünsal **Resurgence and Trans-series in Quantum Field Theory: The $\mathbb{C}P^{N-1}$ Model** `arXiv:1210.2423`

[2] Maxim Kontsevich **Exponential Integral** `https://www.youtube.com/watch?v=tM25X6AI5dY`

References

[1] Alexei Borodin, Grigori Olshanski, Eugene Strahov **Giambelli compatible point processes** `arXiv:0505021`

[2] Bertrand Eynard, Taro Kimura. **Towards U(N—M) knot invariant from ABJM theory** `arXiv:1408.0010`