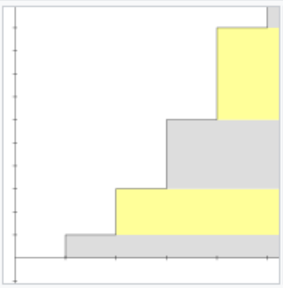
Scratchwork: Divergences

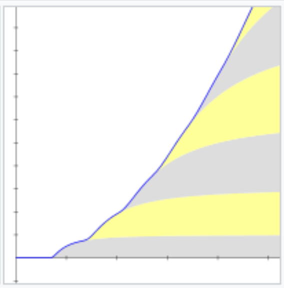
John D Mangual

Somewhat paranoid overview of PNT. Very difficult to type of. Scrapbook-style
Here is from the Wikipedia article on divergent series:

g



The series 1 + 2 + 3 + 4 + ⋯

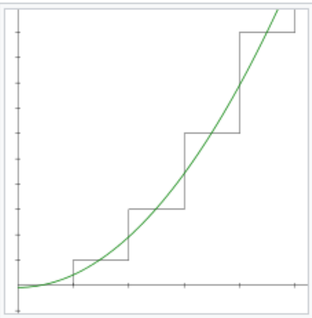


After smoothing

cutoff
is

ote asking me to study carefully
ory. If I tell you this you will at once
a single letter. …"[13]

s $\sum_{k=1}^{\infty} f(k)$ is defined as



Asymptotic behavior of the smoothing. The y-intercept of the parabola is $-\frac{1}{12}$.[1]

Here's a discussion of the prime number theorem that has almost the same figure

2 SIMON RUBINSTEIN-SALZEDO

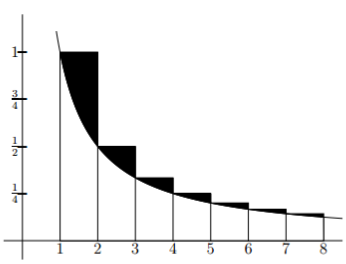


FIGURE 1. A "proof by picture" showing that $0 < \gamma < 1$.

instance [Apo97]. Despite the fact that neither side of (1) makes sense when $s = 1$, Euler bravely set $s = 1$ to obtain

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \left(1 - \frac{1}{2}\right)^{-1}\left(1 - \frac{1}{3}\right)^{-1}\left(1 - \frac{1}{5}\right)^{-1}\left(1 - \frac{1}{7}\right)^{-1}\cdots. \qquad (2)$$

Here is a formal discussion of divergences by Tao. A few basic ones:

for $s = 1, 2, \ldots$, where B_n are the Bernoulli numbers. If one *formally* applies (1) at these values of s, one obtains the somewhat bizarre formulae

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \ldots = -1/2 \qquad (4)$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots = -1/12 \qquad (5)$$

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + \ldots = 0 \qquad (6)$$

and

$$\sum_{n=1}^{\infty} n^s = 1 + 2^s + 3^s + \ldots = -\frac{B_{s+1}}{s+1}. \qquad (7)$$

and his explanation of what is going on

The problem here is the discrete nature of the partial sum

$$\sum_{n=1}^{N} n^s = \sum_{n \le N} n^s,$$

which (if N is viewed as a real number) has jump discontinuities at each positive integer value of N. These discontinuities yield various artefacts when trying to approximate this sum by a polynomial in N. (These artefacts also occur in (2), but happen in that case to be obscured in the error term $O(1/N)$; but for the divergent sums (4), (5), (6), (7), they are large enough to cause real trouble.)

However, these issues can be resolved by replacing the abruptly truncated partial sums $\sum_{n=1}^{N} n^s$ with smoothed sums $\sum_{n=1}^{\infty} \eta(n/N) n^s$, where $\eta : \mathbf{R}^+ \to \mathbf{R}$ is a *cutoff function*, or more precisely a compactly supported bounded function that equals 1 at 0. The case when η is the indicator function $1_{[0,1]}$ then corresponds to the traditional partial sums, with all the attendant discretisation artefacts; but if one chooses a smoother cutoff, then these artefacts begin to disappear (or at least become lower order), and the true asymptotic expansion becomes more manifest.

Note that smoothing does not affect the asymptotic value of sums that were already absolutely convergent, thanks to the dominated convergence theorem. For instance, we have

$$\sum_{n=1}^{\infty} \eta(n/N) \frac{1}{n^2} = \frac{\pi^2}{6} + o(1)$$

Here how it works for the zeta function $\zeta(s)$:

for N large enough. Thus we have

$$\zeta(s) = \frac{1}{s-1} + \lim_{N \to \infty} \sum_{n=1}^{\infty} \frac{1}{n^s} \eta(n/N) - \int_1^N x^{-s} \eta(x/N)\ dx$$

for $\mathrm{Re} s < 1$. The point of doing this is that this definition also makes sense in the region $\mathrm{Re}(s) > 1$ (due to the absolute convergence of the sum $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and integral $\int_1^{\infty} x^{-s} dx$. By using the trapezoidal rule, one also sees that this definition makes sense in the region $\mathrm{Re}(s) > 0$, with locally uniform convergence there also. So we in fact have a globally complex analytic definition of $\zeta(s) - \frac{1}{s-1}$, and thus a meromorphic definition of $\zeta(s)$ on the complex plane. Note also that this definition gives the asymptotic

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|) \qquad (25)$$

near $s = 1$, where γ is Euler's constant.

and he generalizes it to the van Mangoldt function

where ρ ranges over the poles of D, and r_ρ are the residues at those poles. For instance, one has the famous *explicit formula*

$$\sum_{n=1}^{\infty} \Lambda(n)\eta(n/N) = c_{\eta,0} N - \sum_{\rho} c_{\eta,\rho-1} N^{\rho} + \dots$$

where Λ is the von Mangoldt function, ρ are the non-trivial zeroes of the Riemann zeta function (counting multiplicity, if any), and $\dots$ is an error term (basically arising from the trivial zeroes of zeta); this ultimately reflects the fact that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$$

has a simple pole at $s = 1$ (with residue $+1$) and simple poles at every zero of the zeta function with residue -1 (weighted again by multiplicity, though it is not believed that multiple zeroes actually exist).

That pole is not sufficient to prove the prime number theorem, however we get many many intermediate result. If we include $\zeta(1+it) \neq 0$ it will be, this is already outside the scope of Tao's blog (he will cover it in other places, though).

What happens if we tried it with the divisor function?

$$\sum_{n=1}^{\infty} \tau(n)\eta(n/N) = \int_1^{\infty} \log x\eta(x/N)\ dx + 2\gamma c_{\eta,0}N + O(\sqrt{N})$$

where $\tau(n) := \sum_{d|n} 1$ is the divisor function (and in fact one can improve the $O(\sqrt{N})$ bound substantially by being more careful); this corresponds to the fact that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \zeta(s)^2$$

has a double pole at $s = 1$ with expansion

$$\zeta(s)^2 = \frac{1}{(s-1)^2} + 2\gamma\frac{1}{s-1} + O(1)$$

and no other poles, which of course follows by multiplying (25) with itself.

I left out the biggest generalizations. Can you see it?

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References

(1) Terence Tao **The Euler-Maclaurin formula, Bernoulli numbers, the zeta function, and real-variable analytic continuation** `https://terrytao.wordpress.com/2010/04/10/the-euler-maclaurin-formula-bernoulli-numbers-the-`

(2) GH Hardy **Divergent Series** Oxford University Press, 1949/1973.

(3) Simon Rubinstein-Salzedo **Could Euler have conjectured the prime number theorem?** `arXiv:1701.04718`