

# Unipotent Flows

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I audited a course on homogeneous systems

- Roth's Theorem
- Szemerédi Theorem
- Ratner Theorem

It was much too fast to absorb. And, year later, I still have questions. The teacher was brilliant and that may have been  $\frac{1}{2}$  a design feature and  $\frac{1}{2}$  my own curiosity.

Now, when I read the actual papers I notice the material is much more complicated than he had shown us; that he had (rather calmly) manoeuvred around various technicalities in order to present the material to us. Also, there are now textbooks covering many features.

In particular, I tried to understand the Oppenheim conjecture, as usual thinking it would be very simple to resolve.

$$\overline{\{x^2 + y^2 - \sqrt{2} z^2 : (x, y, z) \in \mathbb{Z}^3\}} = \mathbb{R}$$

My strategy would have been to say here is  $a \in \mathbb{R}$  and  $\epsilon \ll 1$  and we'd like to find  $(x, y, z)$  such that  $q(x, y, z) \approx a$ .

$$\left| (x^2 + y^2 - \sqrt{2} z^2) - a \right| < \epsilon$$

This is  $\epsilon$ - $\delta$  definition the way we might see in calculus class. I have drawn the picture a few times. Instead, the main step is to deform the equation itself:

The saying goes: **Algebra is generous: she often gives more than is asked for**; it is also hiding things from us as well. Things that are quite complicated.<sup>1</sup> This is also part of a common strategy in mathematics, to make the problem more severe and more terrifying, but we also see the big picture.

In this case, we know a few things that are going on:

- Lie groups over  $\mathbb{Q}$  are more complicated than Lie groups over  $\mathbb{R}$  (e.g. it's possible to take limits in  $SO(3, \mathbb{Q})$  that go outside the group).
- ?? (I forgot)

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<sup>1</sup>This is due to Jean d'Alembert (1717-1783) and I thought it was David Hilbert (1862-1943).

Cognitively, I see a blind spot *among professors* about Ratner's theorem. We treat it as a black box. **By Ratner's Orbit-Closure Theorem the values of  $Q$  are dense.** We have no idea inside, I have learned nothing. It makes asking for help very difficult.

When procedure comes along that covers *all* unipotent flows. First of all, we are going to Oppenheim conjecture into a single giant procedure that will absorb all equidistribution problems, including ones we haven't thought of. We need two pieces of information:

- $G = \mathrm{SL}(3, \mathbb{R})$
- $H = \mathrm{SO}(Q) \simeq \mathrm{SO}(1, 2) \simeq \mathrm{PSL}(2, \mathbb{R})$

and the algebra says the closure of a certain orbit is the entire group:  $\overline{G_{\mathbb{Z}}H} = G$ . That's excellent. What does that even mean?

Another similar statement  $Q(H\mathbb{Z}^3) = Q(\mathbb{R}^3)$ . (Yes, that's a 4th root of 2) :

$$[Q] = [x^2 + y^2 - \sqrt{2}z^2] \leftrightarrow \begin{bmatrix} 1 & & \\ & 1 & \\ & & -\sqrt{2} \end{bmatrix}$$

The  $H$  could literally be thought of a groups of substitutions.<sup>2</sup>

$$[(x + tz)^2 + y^2 - \sqrt{2}z^2] \leftrightarrow \begin{bmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -\sqrt{2} \end{bmatrix}$$

and we need the other substitution:

$$[x^2 + (y + sz)^2 - \sqrt{2}z^2] \leftrightarrow \begin{bmatrix} 1 & & \\ & 1 & s \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -\sqrt{2} \end{bmatrix}$$

and this is as far as I got on my previous attempts. I think I'll go try something else.

Later... how do we solve Lagrange's 3 squares problem as a unipotent flow?

$$x^2 + y^2 + z^2 = n$$

This equation is **positive-definite**.

## References

(1) Dave Witte Morris **Introduction to Arithmetic Groups** [arXiv:math/0106063](https://arxiv.org/abs/math/0106063)

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<sup>2</sup>There might even be a professional-sounding name for it, like "representation"