Item: **Cohomology of Arithmetic Groups**

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The Eichler-Shimura isomorphism let's you express weight-2 modular forms theory into cohology of $\mathrm{SL}_2(\mathbb{Z})$. This will take a lot of effort to unpack. One version I have found says these two are the same:

- $f(z)\,dz$ is a Γ-invariant form on $\mathbb{H}$
- $[\,f(z)dz\,] \in H^1(\mathbb{H}/\Gamma, \mathbb{C}) = H^1(\Gamma, \mathbb{C})$

but these are two different kinds of cohomology. One of them is a hyperbolic space $\mathbb{H}/\Gamma$ and the other is a group of 2×2 matrices $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. How can we not have a complete understanding of both of these objects?

There's a trade-off between generality and our ability to supply details. I never told you what Γ was, and the entire textbook writes the discussion without naming a specific answer. How can they have the best possible answer? If, I decide to focus on one Γ, let's say $\Gamma_0(4) = \langle z \mapsto z+1, z \mapsto -\frac{1}{4z} \rangle$ maybe I will say things that don't generalize.

In between, would be story where I examine many possible Γ and a statement will be true in cases and not others (in many cases and not others). I might even be able to express this type of meta-logic using a small amount of category theory.

Let's find a modular form of weight 2. The first one I can think of is a theta-function raised to the 4th power:

$$\theta(z) = \left(\sum q^{n^2}\right)^4 = \sum r_4(n)\, q^n$$

and here $\Gamma_0(4) \neq \mathrm{SL}_2(\mathbb{Z})$. How do we know it is modular form of weight 2? This is a great example if we keep in mind the following recipe:

$$\mathsf{M}_2(\Gamma_0(N)) = \mathsf{S}_2(\Gamma_0(N)) \oplus \mathsf{E}_2(\Gamma_0(N))$$

for all congruence groups, not just $N = 4$. This says every weight two modular form splits into to parts:

- Eisenstein series
- Cusp forms (Poincaré series)

The jargon gets worse and worse. Eichler-Shimura theory, Atkin-Lehmer theory. If I have an interestng number theory problem, maybe I can turn it into a modular forms problem:

$$\text{modular forms} \overset{?}{\neq} \text{number theory}$$

I cannot find any modular forms of weight 2 that are not Eisenstein series until $\Gamma_0(11)$

The formula I found on the internet is a **newform**[1] and the coefficient field is $\mathbb{Q}$ (in fact they're in $\mathbb{Z}$) and we get the first fiew terms and Satake parameters (which sound useful).

$$\eta(z)^2\eta(11z)^2 = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + \mathrm{O}(q^{10})$$

It is not obvious to me these coefficients are multiplicative (except $a_{11} = +1$), in that case:

$$a_2 \times a_3 = (-2) \times (-1) = +2 = a_6$$

The coefficients don't seem to match up since perfect powers they are not multiplicative:

$$a_2 \times a_2 \times a_2 = (-2) \times (-2) \times (-2) = -8 \neq 0 = a_8$$

The L-function is found by moving the q-series q^n to Dirichlet-series n^s as done by the **Mellin transform**

$$L(f,s) = 1^s - 2 \times 2^s - 1 \times 3^s + 2 \times 4^s + 1 \times 5^s + 2 \times 6^s + \dots = \left(1 - \frac{a_{11}}{11^s}\right)^{-1} \prod_{p \neq 11}\left(1 - \frac{a_p}{p^s}\right)^{-1}$$

Eichler-Shimura maps the newforms (or Eisenseries) over this group and maps them to elements of group cohomology

$$\int_0^{11} \eta(z)^2\eta(11z)^2 \, dz = \text{???}$$

and now for a homework question: what are the other cycles of $\Gamma_0(11)$? Not so easy.[2]

$$\Gamma_0(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \equiv \left(\begin{array}{cc} * & * \\ 0 & * \end{array}\right) \pmod{N} \right\}$$

Where is also $\Gamma_1(N)$ with a more strict equivalence relation. The equation $ad - bc = 1$ says they are neighbors on the Farey fraction list:

$$\dots < \frac{a}{c} < \frac{b}{d} < \dots$$

so the cosets $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ is indexed by the Farey fractions $\pmod{N}$ (I just took that from a textbook) so there should be **11+1=12** cosets. Neither of these is normal subgroup with exact sequence:

$$1 \to \Gamma(N) \to \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1$$

We will have time to test my intuition for algebra later (as is already happening!).

References

(1) John Cremona **The L-functions and modular forms database project** `arXiv:1511.04289` `http://www.lmfdb.org/`

(2) William Stein **Modular Forms, A Computational Approach** Modular forms of Weight 2 `http://wstein.org/books/modform/modform/index.html`

(3) Christoph Schmitt. **Calculation of L-Functions Associated with Newforms: Implementation, Choice of Parameters and Verification of Zero** (Diplomarbeit, 2010)

[1] `http://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/11/2/1/a/`

[2] `http://wstein.org/books/modform/modform/modform.html`