Some Interesting Formulas Involving the GCD

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Sometimes, when I read a String Theory paper, I try to find a verifiable statement. Here is one I found in a paper:

$$I^{\mathcal{N}=1^*}(N,1,0) = N \sum_{d|N} 1 = N\sigma_0(N)$$

This number is called a **superconformal index** and it also equals:

$$I^{\mathcal{N}=1^*}(N,N,n) = \sum_{d|N} \sum_{l=1}^{N} \gcd(d,l)$$

I was heckled on MathOverflow for posting such an elementary formula. It's not mine, it's his.

Perhaps the general formula can show us the pattern:

$$I^{\mathcal{N}=1^*}(N,m,n) = \frac{N}{m}\sum_{d|N}\sum_{l=1}^{\gcd(d,m)} \gcd\left(\gcd(d,m), n+\frac{ld}{\gcd(d,m)}\right)$$

This formula is later shown to be equal to:

$$I^{\mathcal{N}=1^*}(N,m,n) = \sum_{d|N}\sum_{t=0}^{d-1}\gcd\left(N\frac{d}{m}, N\frac{m}{d}, N\left(\frac{t}{m}+\frac{m}{d}\right)\right)$$

In order for this equation to make sense, I eventually found $m|N$ and $d|N$ - I hope I guessed correctly.

By **Möbius inversion** we should have:

$$\frac{N}{m}\sum_{l=1}^{\gcd(d,m)} \gcd\left(\gcd(d,m), n+\frac{ld}{\gcd(d,m)}\right) = \sum_{t=0}^{d-1}\gcd\left(N\frac{d}{m}, N\frac{m}{d}, N\left(\frac{t}{m}+\right.\right.$$

These seem rather tedious to verify and their meaning unclear.

A starting point could be the Bezout theorem that:

$$\gcd(a, b) = \min_{x,y \in \mathbb{Z}} |ax + by|$$

Buried in the paper is his original statements about lattices when explain the appearance of **GCD** everywhere.

References

(1) arXiv:1606.01022 The Arithmetic of Supersymmetric Vacua. Antoine Bourget, Jan Troost. physics.hep-th.

(2) arXiv:1511.03116 On the N=1* Gauge Theory on a Circle and Elliptic Integrable Systems. Antoine Bourget, Jan Troost. physics.hep-th.

(3) arXiv:1506.03222 Counting the Massive Vacua of N=1* Super Yang-Mills Theory. Antoine Bourget, Jan Troost. physics.hep-th.

(4) arXiv:1305.0318 Reading between the lines of four-dimensional gauge theories. Ofer Aharony, Nathan Seiberg, Yuji Tachikawa. WIS/03/13-APR-DPPA, UT-13-15, IPMU13-0081. physics.hep-th.