Worksheet: Lagrange Interpolation

John D Mangual

There are functions which satisfy a bewildering number of constraints. Or we might find in a paper, a the existence of a function - satisfying very reasonable constraints - which is nearly impossible to construct.

Let's try to find an $f(x)$ that passes through a few points.

- $f(0) = 1$
- $f(1) = 2$
- $f(2) = 2$
- $f(3) = 3$
- $f(4) = 0$

How will we find such a function? Let's guess a structure for it:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Next plug in the values $x = 0$, $x = 1$, etc to solve a system of simultaneous equations:

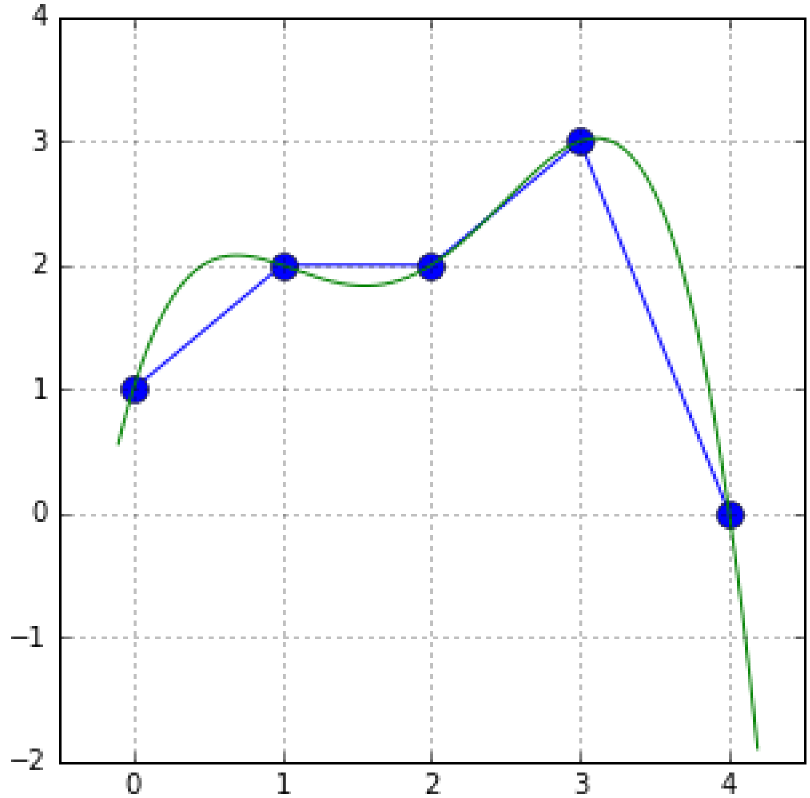
$$\begin{aligned}
f(0) = 1 &= a_0 + a_1 \times 0 + a_2 \times 0^2 + a_3 \times 0^3 + a_4 \times 0^4 + a_5 \times 0^5 \\
f(1) = 2 &= a_0 + a_1 \times 1 + a_2 \times 1^2 + a_3 \times 1^3 + a_4 \times 1^4 + a_5 \times 1^5 \\
f(2) = 2 &= a_0 + a_1 \times 2 + a_2 \times 2^2 + a_3 \times 2^3 + a_4 \times 2^4 + a_5 \times 2^5 \\
f(3) = 3 &= a_0 + a_1 \times 3 + a_2 \times 3^2 + a_3 \times 3^3 + a_4 \times 3^4 + a_5 \times 3^5 \\
f(4) = 0 &= a_0 + a_1 \times 4 + a_2 \times 4^2 + a_3 \times 4^3 + a_4 \times 4^4 + a_5 \times 4^5
\end{aligned}$$

The equation is solved in a standard way called **Lagrange Interpolation**

$$\begin{aligned}
f(x) &= f(0)\,\frac{(x-1)(x-2)(x-3)(x-4)}{(0-1)(0-2)(0-3)(0-4)} \\
&+ f(1)\,\frac{(x-0)(x-2)(x-3)(x-4)}{(1-0)(1-2)(1-3)(1-4)} \\
&+ f(2)\,\frac{(x-1)(x-2)(x-3)(x-4)}{(2-0)(2-1)(2-3)(2-4)} \\
&+ f(3)\,\frac{(x-1)(x-2)(x-3)(x-4)}{(3-0)(3-1)(3-2)(3-4)} \\
&+ f(4)\,\frac{(x-1)(x-2)(x-3)(x-4)}{(4-0)(4-1)(4-2)(4-3)}
\end{aligned}$$

A little bit complicated to write down, we can set $x = 0, 1, 2, 3, 4$ and check it works.

Not without artifacts. . . However, looks very good in some places!

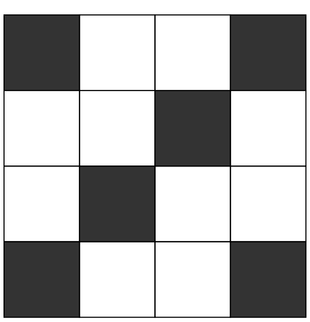


We don't really know what is happending between the values of $x = 0$ and $x = 1$.

NEXT more constraint problems. Maybe in two dimensions.

There are lots of good textbook discussions of Lagrange interpolation, the example I had in mind was the Prime Number Theorem, where the common-sense interolation seems to fail.[1]

I wonder if it's possible to write a function which is positive in some parts and negative in the other. Here's a picture. It's actually really hard to write a good pattern without going up to 4×4.



Can we define a function which is positive on the black square and negative on the white squares? **Yes** we have to be more specific:

- monomials $\{x^m y^n : m, n \in \mathbb{N}\}$

[1]We could be in the other situation, where everybody is in the field and three's no theory.

- trig functions $\{\sin mx \, \sin ny : m, n \in \mathbb{Z}\}$

The second one is more likely, but we can find algebraic curves whose level sets have exactly the contraints we specify (at a cost).

My knowledge of 2D interpolation is slim to none and I have to ask around. So even though PNT[2] is much more difficult, we might proceed first there.

Lastly of my outline of examples: **find a circle that passes through three points**. Thanks to Decartes, we know a circle has equation like this:

$$(x-h)^2 + (y-k)^2 = r^2$$

However we may never exactly know what the triples (h, k, r) could be. In my work, I have to do transformations on the circles, and for each transformation I make, I keep losing accuracy. Settle for:

$$(x^2 + y^2) + a\,x + b\,y + c = 0$$

There are three equations (and three unknowns) so if I add a 4th point, this system will be overdetermined:

$$\left| \begin{array}{cccc} x^2+y^2 & x & y & 1 \\ \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{array} \right| = 0$$

It is possible to determine the circle this way. Let's pick three points: $(0,0), (1,0), (1,1)$ and find the center and radius:

$$0 = \left| \begin{array}{cccc} x^2+y^2 & x & y & 1 \\ 0^2+0^2 & 0 & 0 & 1 \\ 1^2+0^2 & 1 & 0 & 1 \\ 1^2+1^2 & 1 & 1 & 1 \end{array} \right| = \left| \begin{array}{ccc} x^2+y^2 & x & y \\ 1^2+0^2 & 1 & 0 \\ 1^2+1^2 & 1 & 1 \end{array} \right| = \left| \begin{array}{ccc} x^2+y^2 & x & y \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right| == (x^2+y^2)\times 1 - x \times 1 - y \times 1$$

The equation is $x^2 - x + y^2 - y = (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 - \frac{1}{2} = 0$ so the center is $(x, y,) = (\frac{1}{2}, \frac{1}{2})$ and the radius is $r = \frac{1}{\sqrt{2}}$.

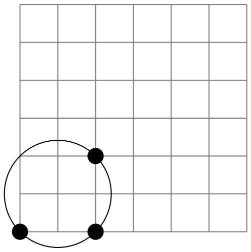
More difficult is to observe if we map this circle under an **inversion** the image is still a circle, whose center and radius is less clear:

$$z \mapsto -\frac{1}{z} \text{ or } (x, y) \mapsto \left(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

some proofs are more convincing than others. Our goal would be to find the radius and center more efficiently.

In the modern day, since we have computers, we might try to **learn** the center and radius by trial and error. And we can see how to do that.

[2] Prime Number Theorem



References

[1] ...