Theta Functions and Ford Circles

John D Mangual

Learn about modular forms $\theta(z) = \sum q^{n^2}$ and $\eta(z) = q^{1/24} \prod (1 - q^n)$

1 Theta Functions and $\Gamma_0(4)$

Lagrange's theorem states an integer can be represented as the sum of three squares, $n = a^2 + b^2 + c^2$ if and only if $n \neq 4^j(8k+7)$. This borderline case between easier statements in 2 and 4 dimensions

- $p = a^2 + b^2$ iff $p = 4k + 1$ (Ex. $p = 7481$ or $p = 364133217234400037l7$)
- Every integer is the sum of 4 squares $n = a^2 + b^2 + c^2 + d^2$.

While studying 3 squares, Duke uses a property of spherical harmonics, that this series is a $\Gamma_0(4)$ modular form for all $u \in L^2[SO(3)]$.

$$\theta(u; z) = \sum_{n \in \mathbb{Z}} \left[\frac{1}{r_3(n)} \sum_{\xi \in V_n} u(\xi) \right] e^{2\pi i\, nz}$$

Here $V_n = \{\vec{m} : m_1^2 + m_2^2 + m_3^2 = n\} = nS^2 \cap \mathbb{Z}^3$ are the lattice points distance n from the origin. As a half-integer weight modular form, Duke knew the coefficiets decayed fairly slowly:

$$\frac{1}{r_3(n)} \sum_{\xi \in V_n} u(\xi) \ll n^{-1/28}$$

but slowly enough to show these points (eventually) equidistribute around the sphere.

Unfortunately, I had no idea what a half-integer modular form was, or why the coefficients decays so slowly, or why that was relevant .

Modular Forms textbooks start with a few examples: Eisenstein series, Poincare series, eta functions and finally one that kind of resembles our theta function:

$$\theta(z) = \sum e^{2\pi i\, n^2 z}$$

This function has an obvious symmetry, $z \mapsto z + 1$ and a less obvious one:

$$\theta\left(-\frac{1}{4z}\right) = \sqrt{-2\pi i z}\,\theta(z)$$

These two symmetries in conjunction, are our copy of $\Gamma_0(4)$. Viz:

$$\Gamma_0(4) \equiv \left\langle z \mapsto z+1, z \mapsto -\frac{1}{4z} \right\rangle$$

The second symmetry is known as Poisson summation:

$$\sum_{n \in \mathbb{Z}} e^{\pi x^2\, t} = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{\pi x^2 / t}$$

It's tempting to say there is $\mathrm{SL}_2(\mathbb{Z})$ symmetry. After repeating the same mistake over and over, observe $\langle z \mapsto -\frac{1}{z}, z \mapsto z + 2 \rangle$ is the same as $\Gamma_0(4)$.

References

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