Fibonacci Numbers

John D Mangual

Every issue of Mathematics Magazine is flooded with proofs of Fibonacci identity. I myself have solved a few. So it is surprising to see a discussion by leading dynamicist and Fields Medallalist Curtis McMullen.

Let $\epsilon \in \mathbb{R}$ be an algebraic unit of degree two over $\mathbb{Q}$. Then $x = \epsilon$ solves a quadratic equation:

$$x^2 - ax + b = 0$$

with $a, b \in \mathbb{Q}$. McMullen writes instead:

$$\epsilon^2 = t\epsilon - n$$

with $t = \mathrm{tr}^K_{\mathbb{Q}}(\epsilon)$ and $n = \mathrm{N}^K_{\mathbb{Q}}(\epsilon) = \pm 1$.

In the number theory jargon:

- $\mathbb{Z}[\epsilon]$ is called a **order** in the field $K = \mathbb{Q}(\epsilon)$.
- The discriminant is $D = t^2 - 4n > 0$.
- $(1, \epsilon)$ is a basis for $\mathbb{Z}[\epsilon] \subset \mathbb{R}$.

We represent algebraic numbers by 2×2 matrices[1]

$$\epsilon = \begin{pmatrix} 0 & -n \\ 1 & t \end{pmatrix}, 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sqrt{D} = \begin{pmatrix} -t & -2n \\ 1 & t \end{pmatrix}$$

Here's one place where Curtis gets tricky. He says:

$$\mathrm{tr}_{\mathbb{Q}}^{K} : \mathrm{M}_2(K) \to \mathrm{M}_2(Q)$$

a 2×2 matrix in $K = \mathbb{Q}(\epsilon)$ is like a 4×4 matrix in $\mathbb{Q}$ (with some rules)[2].

$$\left[\begin{array}{cc|cc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right]$$

[1] this should be extremely bothersome... and we haven't even done cubic fields...

[2] Schemes could be defined as matrices satisfying certain equations. If Alexander Grothendieck hadn't been around, we could say these equations define a "variety" but with additional problems. And spend hours hunting through our commutative algebra textbooks for the properties of these rings. So there you have it a **scheme** is a set of **equations** with certain **problems**.

References

(1) Curtis McMullen. **Uniformly Diophantine Fixed Numbers in a Real Quadratic Field**

(2) Jean Bourgain, Alex Kontorovich. **Beyond Expansion II: Traces of Thin Semigroups** `arXiv:1310.7190v1`