Nilsequences

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Terence Tao sees a lot of things, but he writes in an obfuscated way, and I think he misses a lot of things. About 10 years ago, I was introduced to the topic of **nilsequences** in a course of Dynamics and Number Theory. I did nothing with it. Let's read Terry's latest blog on this topic[1].

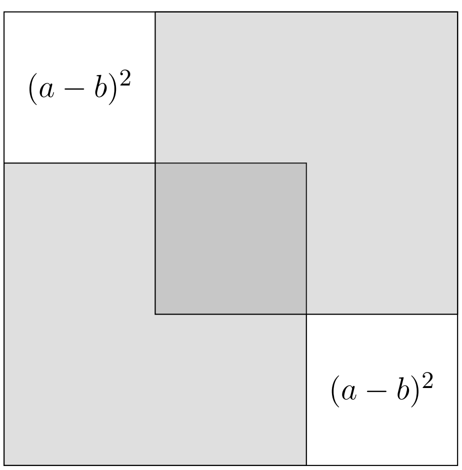
Part of it is like. . . where do fractions come from?

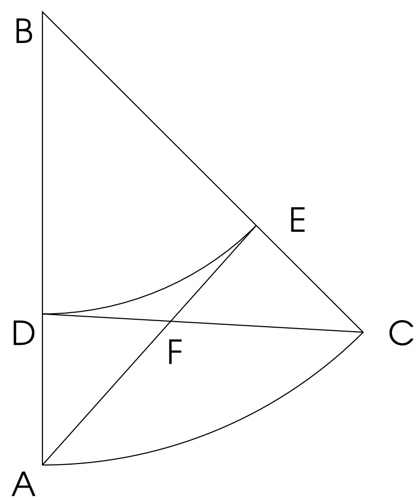
If we take scientific measurement, there's quite a bit error that obstructs us from observing the most delicate patterns. In fact, shielding us completely from finding them (or protecting us).

$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885 03$

If you examine the digits carefully[2] we can prove the decimals do not exhibit any pattern in the decimal expension. However, if we use continued fractions:

$\sqrt{2} = [1; 2, 2, 2, 2, 2, \ldots]$ and this is a nicer system since we have exponential convergence of the the number. Here the error is 10^{-6} (microscopic).



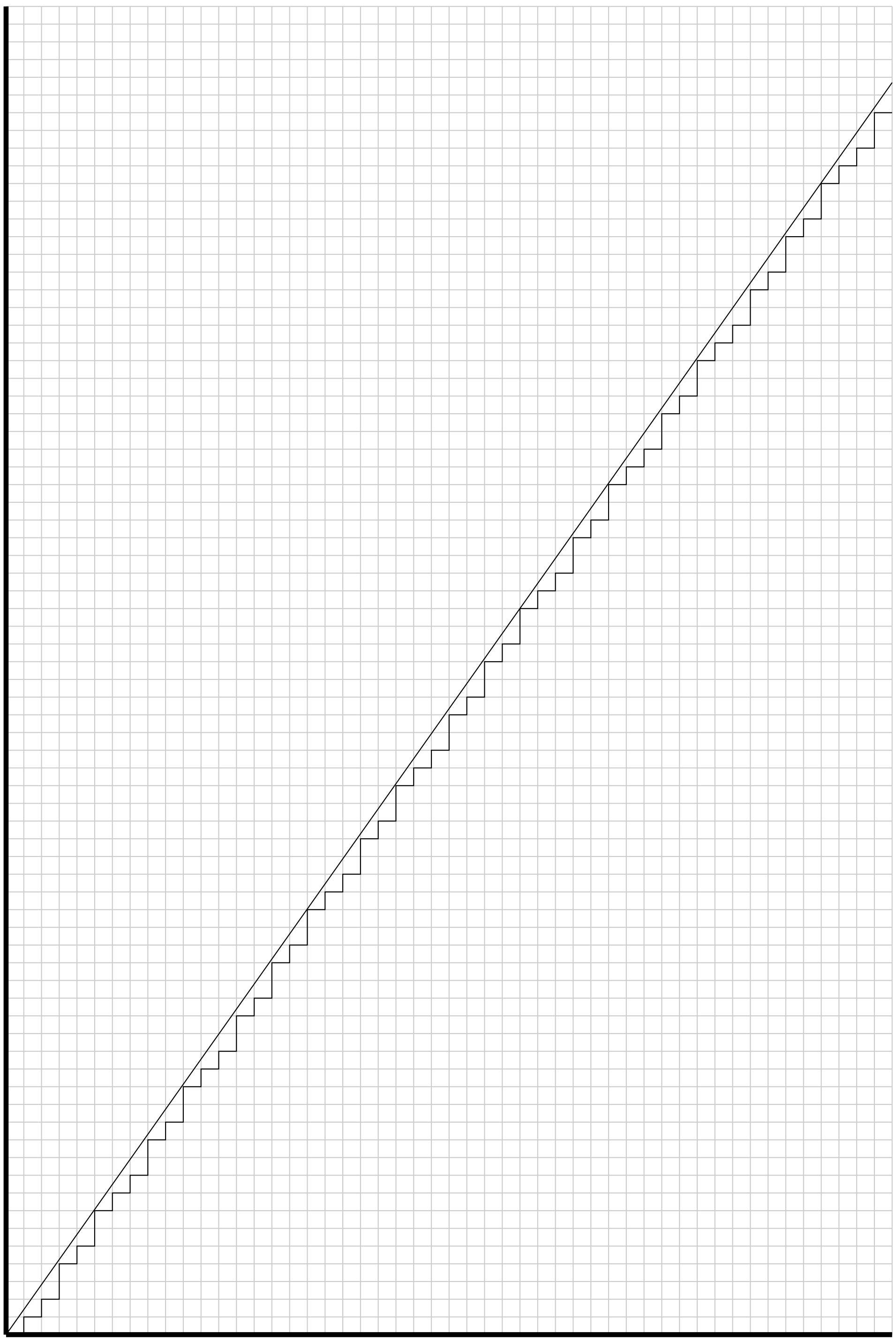


These pictures lead to two different proofs that $\sqrt{2} \notin \mathbb{Q}$. They are both geometric proofs, and argue by infinite descent. Therefore, there must have been an ètale cohomology.

I have no idea what ètale cohomology and the books do not simplify it enough for me. Forget it. One way to get meaningful numbers to arbitrary accuracy is to observe a dynamical system over time and take detailed measurements. And you will see.

[1] `https://terrytao.wordpress.com/2017/04/28/notes-on-nilcharacters-and-their-symbols/`
[2] `http://www.gutenberg.org/files/129/129.txt`

Using a minimimal amount of math we can define most (possibly all) nilsequences. The fractional part of multiples of $\sqrt{2}$ is nilsequence.



We want. . . even more nilsequences. In the Terry Tao blog we get two to get us started:

$$n \mapsto \left(\sqrt{2}n\{\sqrt{3}n\}\right) \text{ or } n \mapsto \left(\sqrt{2}n\{\sqrt{2}n\}\right)$$

Earlier, two professors Vitaly Bergelson and Alexander Leibman studied these kinds of sequences of numbers, to my satisfaction.

Green and Tao are looking for patterns in sequences of numbers. I can never produce a sequence of numbers that would require the techniques they are using. And yet, we kind of see them every day.

In empirical measurements; any time we try to use math to solve a real problem, things get "complicated" and maybe Green and Tao help us reason about that.

Tao's blog today was about **symbols** of nilsequences. I don't understand how you can write an entire theory of numbers and still not write down a single one. We can compute:

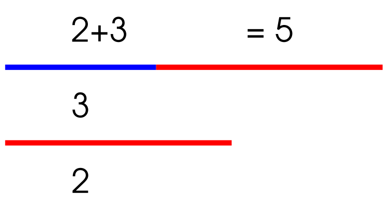
$$n \times \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n\,x & n\,y + \binom{n}{2}\,xz \\ 0 & 1 & n\,z \\ 0 & 0 & 1 \end{pmatrix}$$

The notation $n \times \cdot \equiv (\,\cdot\,)^n$ is short for multiplying a number by itself n times. Here x, y, z are elements of $\mathbb{R}$, but they could also be elements of $\mathbb{Z}$. And it's the number in the upper right corner we are intested. So, if you do the reduction right you get

$$n \mapsto (\{nx\}, \{ny + nx\lfloor nz \rfloor\}, \{nz\})$$

and I'm faking it slightly because I haven't done the arithmetic.

It helps to think like a cave man. What if there were no numbers? We could take two lengths and add them.



Tao - and Green, somewhat - have become suspect of the basic number operations. $+$, $\times$, $-$, $\div$ if you do it too much and then someone tells you that it's wrong. So, let's try new kind of number!

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Here's a reading list. I will leave in the Class Field Theory book since even though don't need it any way (we are solving over $\mathbb{Z}$), in fact we may need it anyway.

References

(1) Ben Green, Terence Tao, Tamar Ziegler. **An inverse theorem for the Gowers $U^{s+1}[N]$-norm** `arXiv:1009.3998`

(2) Ben Green **Approximate algebraic structure** `arXiv:1404.0093`

(3) W. T. Gowers **Generalizations of Fourier analysis, and how to apply them** `arXiv:1608.04127`