Furstenberg Topology

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1 Infinitude of Primes in Various Domains

In 1955, Harry Furstenberg gave a topological proof of the infinitude of prime numbers in $\mathbb{Z}$. A collection of open sets $\mathcal{B}$ is a **basis**[1] for a topology on $\mathbb{Z}$ if every open set is a union of open sets in $\mathcal{B}$. In our case every open set is the union of arithmetic sequences in $\mathbb{Z}$

$$\mathcal{B} = \{a\mathbb{Z} + b : a \neq 0, b \in \mathbb{Z}\}$$

Ex Show this space is normal. Every arithmetic progression is closed as well as open since:

$$\mathbb{Z} = \big(a\mathbb{Z} + b_0\big) \cup \bigcup_{0 \leq b \neq b_0 < a} \big(a\mathbb{Z} + b\big)$$

So the union of a finite number of arithmetic progressions is closed. Furstenburg asks about the set:

$$\bigcup_p A_p = \mathbb{Z} \backslash \{-1, 1\}$$

The LHS is the finite union of closed sets, but $\{-1, 1\}$ is not open (it only has 2 points). So there must be infinitely many primes.

I guess that's a proof... normality means here compact and Hausdorff. In the case of Furstenberg topology:

- **Compact** that any partition of $\mathbb{Z}$ into arithmetic sequences, can be pared down to a union of finitely many arithmetic sequences.
- **Hausdorff** Any two numbers x, y are elements of disjoint arithmetic sequences. How about $\big(a\mathbb{Z} + x\big) \cup \big(a\mathbb{Z} + y\big) = \varnothing$ not a does not divide $y - x$.

The result says that $\mathbb{Z} \backslash \{-1, 1\}$ cannot be covered by finitely many arithmetic sequences.

How to generalize to $\mathbb{Z}[i]$? Nearly all the steps are the same except we choose a different set.

$$\bigcup_p A_p = \mathbb{Z} \backslash \{1, i, -1, -i\}$$

So now we have shown there are infinitely many primes in $\mathbb{Z}[i]$. For any ring of integers $\mathcal{O}_K$ the story could be the same.

What about primes in arithemetic sequences in $\mathbb{Z}$? It could be that Furstenberg topology still works here. Instead we take the **subspace** topology. For $A \subset \mathbb{Z}$ there is O' is open in $\mathcal{B}_A$ if

[1] Allen Hatcher **Notes on Introductory Point-Set Topology** http://bit.ly/1MxYoHI

$O' = O \cap A$ for some open set in $\mathcal{B}$. The Furstenberg topology relative to an arithmetic sequence $a\mathbb{Z} + b$ is just the Furstenberg topology itself resticted to that arithemtic sequence, since

$$\big(a\mathbb{Z} + b\big) \cap \big(c\mathbb{Z} + d\big) = \varnothing \text{ or } b + \mathrm{lcm}(a, c)\mathbb{Z}$$

What if there are finintely many primes in this arithmetic sequence? There still might be finintely many prime outside of this sequence, so:

$$\mathbb{Z} \backslash \bigcup_{p \notin A} A_p = \prod_{p \in A} p^{\mathbb{N}}$$

Unfortunately, this remainder has *natural density* 0 and cannot be an open set.

References

[1] G. H. Hardy , Edward M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press; 2008.

[2] Harry Furstenberg. *On the Infinitude of Primes* American Mathematical Monthly, 62, (1955), 353.

[3] Idris Mercer. *On Furstenberg's Proof of the Infinitude of Primes* American Mathematical Monthly 116: 355-356