

# Tune-Up: Poisson Summation

I wanted to understand the Four squares theorem a little better.

**Theorem** (Lagrange) Every positive integer is the sum of four squares.  $n = a^2 + b^2 + c^2 + d^2 \in \mathbb{Z}$ .

The number of representations of  $n$  as the sum of four squares, call it  $r_4(n)$ . This is also the coefficients of the power of the theta function  $\theta(z)^4$ . This is a modular form of weight 2 on  $\Gamma_0(4)$ .

- $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^2 + \dots$  and  $\theta(z)^4 = \sum_{n \geq 0} r_4(n) q^n = 1 + 8q + 24q^2 + \dots$  OEIS <sup>1</sup>
- The dimension of the space of Modular forms of weight 2 on  $\Gamma_0(4)$  is at most 2. This requires some residue calculus and the shape of  $\mathbb{H}/\Gamma_0(4)$ . There are no cusp forms at this level. It is spanned by two Eisenstein series:
  - $\mathbb{G}_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n = -\frac{1}{24} + q + 3q^2 + 4q^3 + \dots$  and  $G_2(z) = -4\pi^2 \mathbb{G}_2(z)$ .
  - $\mathbb{G}_2^*(z) = \mathbb{G}_2(z) + \frac{\pi}{8\pi y}$  (this is a “regularization”). These two functions  $\mathbb{G}_2$  and  $\mathbb{G}_2^*$  form a basis.
  - The three functions  $\mathbb{G}_2^*(z)$ ,  $\mathbb{G}_2^*(2z)$  and  $\mathbb{G}_2^*(4z)$  span  $M_2(\Gamma_0(4))$  so there must be a relation.
  - $\theta(z)^4 = 8(\mathbb{G}_2(z) - 2\mathbb{G}_2(2z)) + 16(\mathbb{G}_2(2z) - 2\mathbb{G}_2(4z))$ .

**Theorem**  $\theta(z)$  satisfies two functional equations.  $\theta(z+1) = \theta(z)$  and  $\theta(-\frac{1}{4z}) = \sqrt{-2iz} \theta(z)$  for  $z \in \mathbb{H}$ .

Zagier's article is unfortunately quite dense, so we'll supplement it with other sources. Here's is proof of the Four-Squares Theorem.

Stein. Let's define the theta function for real numbers  $s > 0$ . Then:

$$\theta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$$

**Theorem**  $s^{-1/2} \theta(1/s) = \theta(s)$  whenever  $s > 0$ . And this follows from Poisson summation formula and:

$$\mathcal{F} : [f : x \mapsto e^{-\pi s x^2}] \mapsto [\hat{f} : \xi \mapsto s^{-1/2} e^{-\pi \xi^2 / s}]$$

As a separate question we have to ask how  $\theta(s)$  extends to  $s \in \mathbb{C}$ . So there are two items to check:

- Poisson summation  $\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$ .
- Fourier transform is it's own Gaussian.  $f(x) = e^{-\pi s x^2}$  and  $\hat{f}(\xi) = s^{1/2} e^{-\pi \xi^2 / s}$ .

---

<sup>1</sup><https://oeis.org/A000118>

Step 1A: If  $f \in \mathcal{S}(\mathbb{R})$ . Then  $\sum_{n \in \mathbb{Z}} f(x+n)$  is continuous and  $\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$  is continuous.

Wouldn't we like to check the definition of continuity? Let's see what happens if we set  $f(x) = e^{\pi s x^2}$ .

Certainly  $f(x) = e^{-\pi s x^2} \in \mathcal{S}(\mathbb{R})$ . Show that  $\sum_{n \in \mathbb{Z}} e^{\pi n^2 s}$  is continuous and  $\frac{1}{\sqrt{s}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/s} e^{2\pi i n x}$  is continuous.

Step 1B: We'd like to say that since the Fourier coefficients of two functions are the same, then the two functions are the same.

$$\int_0^1 dx e^{-2\pi i n x} \left[ \sum_{n \in \mathbb{Z}} e^{\pi n^2 s} \right] \stackrel{?}{=} \int_0^1 dx e^{-2\pi i n x} \left[ \frac{1}{\sqrt{s}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/s} e^{2\pi i n x} \right]$$

Is it obvious this is true? This still be checked in Step 2. Even if we decide these two things are the same, do we have any other possible functions this could be?

**Thm** Suppose that  $g$  is an integrable function on the circle with  $\hat{g}(n) = 0$  for all  $n \in \mathbb{Z}$ . Then  $g(\theta_0) = 0$  whenever  $g$  is continuous "at" the point  $\theta_0$ .

*Sketch* Let  $p(\theta) = \epsilon + \cos \theta$  and  $p_k(\theta) = [p(\theta)]^k$ . And choose  $B$  such that  $|g(\theta)| \leq B$  for all  $\theta$ . Here  $g(\theta) = f_1(\theta) - f_2(\theta)$  where we think  $f_1 \equiv f_2$ .  $g(\theta)$  is integrable on  $S^1$  and so it is bounded.

$$p_k(\theta) = [p(\theta)]^k = [1 + \epsilon \cos \theta]^k \approx [1 + \epsilon(1 - \frac{1}{2}\theta^2)]^k \approx 1 + k\theta$$

These Heuristic approximation are not in Stein's book. In fact what we're really going for is  $p_k(\theta) = [p(\theta)]^k \approx 1$ .

**Lemma** Suppose  $f$  is integrable on the circle and bounded by  $B$ . Then there exists a sequence  $\{f_k\}_{k=1}^\infty$  of continuous functions on the circle so that

$$\sum_{x \in [-\pi, \pi]} |f_k(x)| \leq B \quad \text{and} \quad \int_{-\pi}^{\pi} (f(x) - f_k(x)) dx \rightarrow 0$$

another question will be how we even know that the Fourier transform approaches  $f$ . It is not even true in many reasonable circumstances.

When does  $\sum_{|n| < N} \hat{f}(n) e^{2\pi i n x} \rightarrow f(x)$  in the limit? There could be a pointwise convergence or a sup convergence or an  $L^2$  convergence (e.g. using Pythagoras theorem).

Back to the proof. It seems we have:

$$\int_{[-\pi, \pi]} g(\theta) [p(\theta)]^k d\theta = \int_{[-\pi, \pi]} g(\theta) (1 + \epsilon \cos \theta)^k d\theta \approx g(0)$$

This looks about right  $p(\theta) = 1 + \epsilon \cos \theta \approx 1$  and  $p_k(\theta) = [p(\theta)]^k = [1 + \epsilon \cos \theta]^k \approx 1$ . A closer look:

$$(1 + \epsilon \cos \theta)^k \approx \begin{cases} (1 + \epsilon/2)^k & |\theta| < \eta \\ (1 - \epsilon/2)^k & |\theta| > \delta \end{cases}$$

where  $p(\theta) = 1 + \epsilon \cos \theta \geq 1 + \epsilon/2$  for  $|\theta| < \eta$  and  $|p(\theta)| < 1 - \epsilon/2$  for  $|\theta| \geq \delta$ . There's still a little bit of room left over, at least this is more specific than saying overall that  $[p(\theta)]^k \approx 1$ . These angles are more-or-less

$\theta = 60^\circ$  and  $\theta = 120^\circ$ . There were two parameters to play with  $k$  and  $\theta$ .

**Definition** A family of Kernels  $\{K_n(x)\}_{n=1}^\infty$  on the circle is said to be a family of **good kernels** if

- For all  $n \geq 1$   $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) dx = 1$
- There exists  $M > 0$  such that for all  $n \geq 1$ ,  $\int_{-\pi}^{\pi} |K_n(x)| dx \leq M$
- For every  $\delta > 0$  then  $\int_{\delta \leq |x| \leq \pi} |K_n(x)| dx \rightarrow 0$  as  $n \rightarrow \infty$ .

**Theorem 4.1** Let  $\{K_n\}_{n=1}^\infty$  be a family of good kernels, and  $f$  is an integrable function on the circle. Then

$$\lim_{n \rightarrow \infty} (f * K_n)(x) = f(x)$$

whenever  $f$  is continuous at  $x$ . If  $f$  is continuous, then the above limit is uniform.  $\{K_n\}$  is called an **approximation to the identity**. Basically:

$$(f * K_n)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) K_n(y) dy \approx f(x)$$

Let's see if we can verify the properties of a good Kernel with our very crude example:  $[p(\theta)]^k = (1 + \epsilon \cos \theta)^k$ .

- $\frac{1}{2\pi} \int [1 + \epsilon \cos \theta]^n dx \asymp \frac{1}{2\pi} \int 1 + (\sum \epsilon^k \cos k\theta) dx = 1$  (so far so good)
- $\int_{-\pi}^{\pi} |[1 + \epsilon \cos \theta]^n| dx \leq \int_{-\pi}^{\pi} |1 + \epsilon|^n dx = 2\pi(1 + \epsilon)^n$ . We need  $\epsilon \ll \frac{1}{n}$ .
- For every  $\delta > 0$ ,  $\int_{\delta \leq |x| \leq \pi} |(1 + \epsilon \cos \theta)^n| dx \xrightarrow{?} 0$ . This looks false since  $1 + \epsilon \cos \theta > 0$  for  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ .

This is advantageous, since it says we can manoeuvre Poisson summation and prove it using only very blunt tools, without an approximation to the identity, pointing at each value with our index finger and smudging a little bit. Say,  $g(x) \approx \frac{1}{1+2\epsilon} [g(x-\epsilon) + g(x) + g(x+\epsilon)]$  in a very common sense way.

In Electrical engineering this would be called “filter design”. In the long run, what can we say about the difference between the space average and the frequency average since we only have a finite amount of computing power.

$$\sum_{|n| < N} f(n+x) - \sum_{|m| < M} \hat{f}(m) e^{2\pi i m x} \approx 0$$

What about that?

## References

- [1] Don Zagier, et. al. **The 1-2-3 of Modular Forms** Springer, 2008.
- [2] Elias Stein **Fourier Analysis** (Princeton Lectures in Analysis, Book I) Princeton University Press, 2006.