

## Examples: WKB Approximation

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We hit the ground running a bit here<sup>1</sup> - why are we drawing all these curves in the first place. I'll draw a few in a moment.

The **WKB approximation** says if the gauge where:

$$\phi = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

there are two independent  $\mathcal{A}$ -flat sections of the form:

$$\psi^1 \sim \begin{pmatrix} e^{-\frac{R}{\xi} \int^z \lambda} \\ 0 \end{pmatrix}, \psi^2 \sim \begin{pmatrix} 0 \\ e^{-\frac{R}{\xi} \int^z \lambda} \end{pmatrix}$$

therefore in the limit as  $\xi \rightarrow 0$  we obtain **essential singularities**, which behave like  $e^{1/z}$  near  $z = 0$ .

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<sup>1</sup>I have been encouraged to look at this topic by various sources and have had the privilege to meet Davide Gaiotto during my one-time visit to Perimeter and Andrew Neitzke at various conferences. Since the papers were written about 7 years ago. These read somewhat like Harry Potter, and they are quite lengthy. The reward, potentially is a new look at holomorphic functions and complex analysis (at least for me).

Locally the essential singularity looks like a spiral. Let's solve:

$$e^{1/z} \in \mathbb{R}$$

then  $z = Re^{i\theta}$ ,

$$z(t) = z_0 e^{t e^{i\theta}}$$

so the generic WKB curve is a **logarithmic spiral**.

$$x^K + \sum_{k=2}^K u_k(z) x^{K-k} = 0$$

the coefficients polynomials of suitable degree<sup>2</sup>

This is a polynomial in  $x$  of degree  $K$ , but let's use some earth-shattering language and say that  $\Sigma$  it is  $K$ -fold cover of  $C$ , and that coefficients are sections of the sheaf  $u_k \in H^0(C, K^{\otimes k})$ . Which in turn which define a curve  $\Sigma \subset T^*C$ .

And we will set  $K = 2$ .

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<sup>2</sup>There are lecture notes of Nigel Hitchin or Simon Donaldson – one of them – for undergraduates and he gives you the Riemann existence theorem. I didn't think too much of this but it's **rare** for any good explanation of this result.

## References

- (1) Davide Gaiotto, Gregory W. Moore, Andrew Neitzke. **Wall-Crossing in Coupled 2d-4d Systems.** arXiv:1103.2598v1
- (2) Davide Gaiotto, Gregory W. Moore, Andrew Neitzke. **Wall-crossing, Hitchin Systems, and the WKB Approximation.** arXiv:0907.3987