Unipotent Flows

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The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ also known as $(x, y) \mapsto (x + y, y)$

Ratner's Theorem says[1]

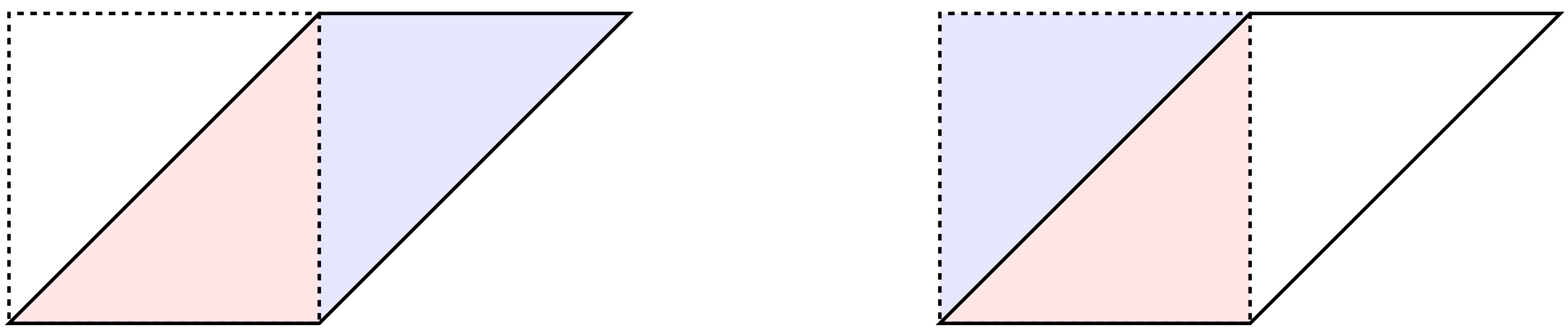
> Let G be a Lie group, $\Gamma < G$ be a discrete subgroup, and $H < G$ be a subgroup isomorphic to $\mathrm{SL}(2, \mathbb{R})$.
>
> Then any H-invariant and ergodic probability measure μ on $X = \Gamma \backslash G$ is homogeneous.
>
> i.e. there is
>
> - a closed connected subgroup $L < G$ containing H such that μ is L-invariant and
> - some $x_0 \in X$ such the L-orbit $x_0 L$ is closed and supports μ
>
> That means μ is an L-invariant volume measure on $x_0 L$

[1] Somtimes in a very difficult theorem we start of great with a clear discussion with lots of illustration. Then after some point we decide everything is "technical" and go for dozens of pages which basically can be omitted. This statement is due to Manfred Einsiedler, whose exposition is a simplification of a simplification of a simplification. . . The work is endless.

Why Learn Ratner's Theorem?

There are integers (a, b, c) such that $|a^2 + b^2 - \sqrt{2}c^2| < \epsilon$
Why can this have arbitrarily small numbers? This is proven by Dani and Margulis[2]

Ratner's Theorem on the equidistribution of the horocycle flow seems to do with the shear of a rhombus.



This in turn has to do with Euclid's Elements. The shear preserves area, these two rectangles have equal area. Just take a pair of scissors and cut.

[2] Although I find a closed proof of Bourgain of certain cases in 2016

Our goal is to try to express some of these "entropy" considerations in simple language. And I need to re-write all of these statements since I am not an expert on Ratner Theorem.

References

(1) Davide Gaiotto, Peter Koroteev **On Three Dimensional Quiver Gauge Theories and Integrability** `arXiv:1304.0779`

(2) Titchmarsh **Theory of Functions** `https://archive.org/details/TheTheoryOfFunctions`