Erdos-Szekeres Problem

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One place where my nilsequences project is getting stuck is that I can't verify that certain averages form nilsequences.

$$n \in \mathbb{N} \mapsto \int f(x) \, \big[f \circ T^n \big](x) \, \big[f \circ T^{2n} \big](x) \, d\mu(x) \in \mathbb{C}$$

Logically speaking, there's a good abstact discussion showing this is true (and much more). This result would mean a lot more to me, if I could define a specific dynamical system $T : X \to X$ and an observable[1] $f : X \to \mathbb{R}$.

None of the examples I can think of are terribly exciting, and dynamical systems textbooks only study a few template cases. These experts, feel that any interesting dynamical system is "conjugate" to a few basic cases.

Textbooks provide the example of the circle: $T : x \mapsto x + \sqrt{2}$ or any number $T : x \mapsto x + \alpha$ with $\alpha \notin \mathbb{Q}$. We have this dichotomy, either:

- $\alpha \notin \mathbb{Q}$ and T is ergodic
- $\alpha \in \mathbb{Q}$ and T is **not** ergodic

So that $X = S^1 \to S^1$ and $T : X \to X$. I observe none of these rotations will be mixing, the dynamical system T is simply not violent enough. Textbooks may feel the circle example is fundamental:

- . . .
- Any time we are looking prove a dynamical system is ergodic, we're looking for eigenfunctions of a unitary operators:

  $$U_T : f(x) \in L^2(\mu) \mapsto \big[f \circ T \big](x) \in L^2(\mu)$$

  After much practice[2] on simply knows, the eigenfunctions of unitary opeartors are the roots of unity; we are solving:

  $$U_T \, f = \lambda \, f \text{ and we know } \lambda = e^{2\pi i \, \alpha} \text{ with } \alpha \notin \mathbb{Q}$$

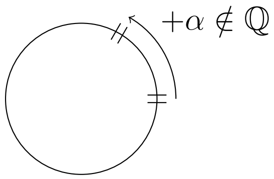
  There is a copy of the rotation dynamical system $T' : x \mapsto x + \alpha \in S^1$ Ergodic Theory says that $\lambda = \mathbf{1}$ is an eigenvalue.

[1] This word "observable" is rather loaded. Maybe I am pushing an analogy with Quantum Mechanics? Studying the dynamical system $T : X \to X$ is (I think) the same as studying the action of $T : L^2(X, \mu) \to L^2(X, \mu)$.

In quantum mechanics, observables are operators (matrices) yet in our setting observables are functions. And. . . if I really step back I think about how Ergodic Theory is meant to justify taking certain averages in Statistical Mechanics.

[2] E.g. a semseter of Quantum Mechanics

We've shown that whenever we try to prove a dynamical system, the rotation $T : x \mapsto x + \alpha$ is always there. Perhaps there will be a nil-rotation as well? That is Terry's point.



So all dynamical systems have rotations embedded in them (either in the physical space, or the hilbert space of "observables") by looking for eigenfunctions of

$$U_T : f \mapsto \big(f \circ T\big)(x) \equiv f\big[T(x)\big]$$

If we find an eigenfunction f, the dynamical system restricts to a rotation:

$$T\Big|_f : S^1 \subseteq X \to S^1 \subseteq X \text{ possibly with } \overline{S^1} = X$$

possibly this circle is dense in the entire circle.

We could ask to quantify ergodicity: (how much) ergodic is it. I could find irrational numbers[3] α that are just barely outside of $\mathbb{Q}$:

$$\left| x - \frac{p}{q} \right| > \frac{1}{\sqrt{8}\, q^2}$$

There are infinitely many numbers $x \in \mathbb{Q}``+"\frac{1}{\sqrt{8}\,q^2}$ it's hard to write this notation because the correct notation is not "$+$". And these kinds of rotations must also be embedding in the spectal theory of the unitary operators.

The real input to many of these problems will not be a dynamical systems, just a sequence of numbers $a_n \in \mathbb{R}$ (or $\mathbb{C}$) and it will our job to find a realistic T (or just be OK that one might exist).

If we found an eigenfunction ϕ that was invariant, we could take the Fourier coefficiets:

$$\widehat{\phi}(n) = \int_0^1 e^{2\pi i\, kx} \phi(x) \; d\mu(x)$$

These Fourier coefficients should also be invariant under the shift map. We could find $\widehat{\phi \circ T}(n)$:

$$\widehat{(\phi \circ T)}(n) = \int_0^1 e^{-2\pi i\, kx} (\phi \circ T)(x) \; d\mu(x) = e^{-2\pi i\, k\alpha} \int_0^1 e^{-2\pi i\, kx} \phi(x) \; d\mu(T^{-1}x) = e^{-2\pi i\, k\alpha}\, \widehat{\phi}(n)$$

If ϕ is T-invariant, then α had better be 0. So there we have it $\alpha = 0$.
It could be the Fourier coefficients always agree and yet the functions are slightly different at a few points, so we use the special term T-**invariant almost everywhere**.

$$\widehat{\phi}(n) = \widehat{(\phi \circ T)}(n)$$

In the case of a *rotation* this means that $\hat{\phi}(n) = 0$ for $n \neq 0$. $\phi(x) = 0 + \epsilon(x)$ where $\epsilon \neq 0$ finitely many points (or a set of measure zero).

[3] `https://en.wikipedia.org/wiki/Diophantine_approximation`

Thinking about it a long time, Szmeredi's theorem most helpful when a sequence of number has no good separation between structure and chance.

$$\text{sequence} \stackrel{?}{=} \text{structure} + \text{chance}$$

and we have no idea how to separate them. This could be anything. It defies all language.

If I choose an example from a textbook, the risk is separation is too good, and another – must simpler – theory could apply.[4] Off the top of my head:

$$T : x \mapsto 2x \pmod 1$$

This map is ergodic, it's even mixing. Still if I am dealing with the number line, it is a little too predictible. Maybe go a step further, I found this one after digging around:

$$\left\{ 2^k \, 3^l \, \frac{m}{N} : 0 < k,\, l < 3 \, \log N \right\}$$

I don't know which dyamical system generates these numbers, I think you need both $T : x \mapsto 2x$ and $T' : x \mapsto 3x$ ...These numbers are $\kappa_1 (\log \log \log N)^{-\kappa_2/100}$-dense on sequenes of numbes relatively prime to 6: gcd$(6, N) = 1$.

Multiplication by $\times 2$ has an entropy of $H = \log 2$ since $T^{-1}(x) = \{2x, 2x + 0.5\}$, so that T^n should have 2^n pre-images.

We could try it. Let $f(x+1) = f(x)$:

$$n \mapsto \int_0^1 f(x)\, f(2^n x)\, f(2^{2n} x)\, dx = \mu \left\{ x : \left\{ x \in [0, \tfrac{1}{2}] \right\} \bigcap \left\{ 2^n \, x \in [0, \tfrac{1}{2}] \right\} \bigcap \left\{ 4^n \, x \in [0, \tfrac{1}{2}] \right\} \right\}$$

There must be all sorts of examples, very difficult to name just one.

[4] Empirical data is full of great examples of "noise" or "stuff" or "garbage". You subtract away, the best-fit-line or whatever a good Statistics textbook is telling you to do. Try to find an explanation for what's left. You can't.

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References

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