Examples: Theta Functions

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Conformal field theory is a central topic in mathematical physics[1]. What is Conformal Field Theory?

The rational gaussian model has a single scalar free field ϕ which is compactified on a circle with $R^2 \in \mathbb{Q}$.

This theory has a large group of symmetries – an extension of the the $U(1)$ current algebra.

The N primary fields $[\phi_p]$ are vertex-operators with momentum $\frac{p}{\sqrt{N}}$ and $p \in \mathbb{Z}_N$.

The fusion rules are $\phi_p \times \phi_q = \phi_{p+q}$ with $p, q \in \mathbb{Z}$.

[1]What does that even mean? Here it means there are other math problems in fields like Number Theory and Topology and Chern Simons Theory – and specific sources – which point to the paper we will review today.

The discussion the last page is already quite problematic. Why is $R^2 \in \mathbb{Q}$, e.g. $R = \sqrt{3}$?

It seems I have confused ϕ and φ. These are related by the exponential function:

$$\phi_p(\mathbf{c}) = \exp\left(\frac{p}{\sqrt{N}} \int_{\mathbf{c}} \partial\varphi\right)$$

this is such a nice looking integrals with nice properties:

$$\phi_p(\mathbf{a})\phi_q(\mathbf{b}) = e^{2\pi i\, pq/N}\phi_q(\mathbf{b})\phi_p(\mathbf{a})$$

Now I want to know why these feel like the basic commutation operators from Quantum Mechanics:

$$[x, p] = -i\hbar$$

and you can even prove these yourself, right? Let $p = -i\hbar\frac{d}{dx}$ then:

$$\left[x, i\hbar\frac{d}{dx}\right] f(x) = i\hbar\left(x\frac{df}{dx} - x\frac{df}{dx} + \frac{dx}{dx}f\right) = -i\hbar$$

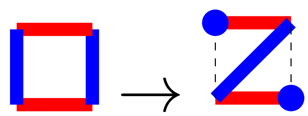
Expect we are getting the exponentiated form. That's it[2]

[2] I am not going to review anything more from Verlinde's paper – I will be too busy making sense of these objects to discuss anything else.

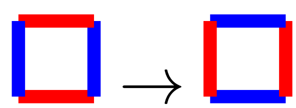
Excuse me, Verline talks about the S and T operators. The first one is clear:

$$S : \chi_p \to \frac{1}{\sqrt{N}} \sum_{q \in \mathbb{Z}_N} e^{2\pi i\, pq/N} \chi_q$$

The only *S* and *T* operators that I know very well act on a Torus:

$S :$ 

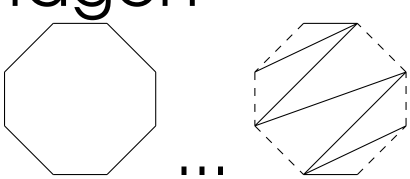
The other operator flips the torus (a technical term):

$T :$ 

This action lifts to **observable** that happens on that torus.

$$T : \chi_p \to e^{2\pi i\, p/N} \chi_p$$

With some effort these can happen on an octagon[3]



Verlinde's big result is: **the modular transformation S diagonalizes the fusion rules**[4]

$$\phi_p \times \phi_q = \phi_{p+q}$$

[3]Perhaps I should draw these by hand first!

[4]See? We can recite these formulas over and over many times with no idea what they mean :-)

How do we compactify the free scalar field on a circle?

Verlinde could have reminded us this scalar field was a section[5] $\varphi : \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \to \mathbb{R}$. I think it's over $\mathbb{R}$. It could be a complex valued field.

Our torus is $S^1 \times S^1$ but I wish to remember the complex structure, so we remember[6] the number: $\tau \in \mathbb{C}/\mathrm{SL}(2,\mathbb{Z})$ that section is generated by two transformations:

$$z \mapsto z + 1 \qquad z \mapsto -\frac{1}{z}$$

For that matter we could pick other very simple transformations and it makes a huge world of different. For the record:

$$z \mapsto z + 1 \qquad z \mapsto -\frac{1}{4z}$$

That is because holomorphic function's don't just take partial derivatives.

[5]or function

[6]This space makes more sense after you've done a lot of cut-and-paste as on the previous page.

$$\begin{tikzcd}
f(z+dy) \arrow[r, dash, dashed, "+dx\, f'(z)"] & f(z+dz) \\
f(z) \arrow[u, dash, dashed, "+dy\, f'(z)"] \arrow[r, dash, dashed, "+dx\, f'(z)"] & f(z+dx) \arrow[u, dash, dashed, "+dy\, f'(z)"']
\end{tikzcd}$$

It's hard to draw the picture of what holomorphic means... that the real and imaginary parts change in a compatible way.

A **free field** or more specifically a **free boson** or **gaussian model** as an action:

$$S = \int \mathcal{L} = \frac{1}{2\pi} \int \partial X \overline{\partial X}$$

and Ginsparg tells us the propagator should be:

$$\langle X(z, \overline{z})\, X(w, \overline{w}) \rangle = -\frac{1}{2} \log |z - w|$$

What is a conformal field anyway? Length behaves nicely under conformal mappings:

$$ds^2 \mapsto \left(\frac{\partial f}{\partial z}\right) \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right) ds^2$$

This field splits into a holomorphic and anti-holomorphic part[7]

$$X(z, \overline{z}) = \frac{1}{2}(x(z) + \overline{x}(\overline{z}))$$

and I can't seem to find how X behaves under conformal mappings $z \mapsto f(z)$.

[7] they are not complex conjugates... x depends on z and $\overline{x}$ depends holomorphically on $\overline{z}$. Nature somehow requires the real and imaginary parts to behave separate but equal – in reality creating two different worlds.

Conformal fields seem to be **representations of the Virasoro algebra** describing things that transform nicely under conformal maps:

$$\ell_n : z \mapsto z + \epsilon\, z^n$$

I am looking for a formula for $x(z)$. Reading many many times I try a formula I saw elsewhere:

$$x(z) = \sum_{n=0}^{\infty} a_n z^n$$

where a_n is either an[8]

- Gaussian random variable with mean 0 and norm 1
- a simple harmonic oscillator $[a_m, a_n] = \delta_{m,n}$

There is a way to pass between a Gaussian and harmonic oscillator that I reviwed before[9]

Oh, here is a free fermion:

$$S = \frac{1}{8\pi} \int \psi\, \overline{\partial} \psi + \overline{\psi}\, \partial \overline{\psi}$$

[8]The first version – which makes the most sense to me – I learned from Yuval Peres at PCMI 2007. However it comes from work of Mikhail Sodin.

[9]There was a trick in symplectic geometry that takes one random walk and turns it into a harmonic oscillator. It could be the Distermaat-Heckman theorem. When I wrote one thing it magically turned into the other. This is a linear space, so it's reasonable to assume this will work in infinite dimensions.

There is a lot to be unhappy with here. And yet we continue.

Section 8 of Ginsparg shows us how to put free bosons on a torus. . .

References

(1) Erik Verlinde. **Fusion rules and modular transformations in 2D conformal field theory.** Nuclear Physics B Volume 300, 1988, Pages 360-376

(2) Luis Alvarez-Gaume **Topics in Conformal Field Theory** `https://cds.cern.ch/record/204721/files/CERN-TH-5540-89.pdf`

(3) Gregory Moore, Nathan Seiberg **Lectures on Rational Conformal Field Theory** . . .

(4) Paul Ginsparg. **Applied Conformal Field Theory** `arXiv:hep-th/9108028`

(5) Mikhail Sodin. **Zeroes of Gaussian analytic functions** `arXiv:math/0410343`