Attempt at: the Arctic Circle Theorem

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Working backwards, I will try to prove the Artic Circle Theorem, then I will state the Arctic Circle Theorem and finally explain why it is important to me[1].

I wonder why dominos have height functions at all. One set of computer science notes shows perfect matchings as an instance of the max-flow problem.

The goal here is to keep the animals at bay, to keep the complexity at a manageable level.

[1]There are many proofs of the Arctic Circle Theorem using a wide range of techniques. Often the Aztec Diamond cases is considered "known" or "settled" which doesn't help our cases any. I am trying to find a proof that can be read from end to end without too much difficulty.

I recall speaking to David Speyer about the advances and proofs in the Aztec Diamond and yet – an expert in the field – he claimed he wasn't aware. This was very much confusing. He did also say that all the proofs he is seeing are very similar. There must be Kasteleyn formula and maybe the Smith determinant.

The proofs I have seen often involve difficult complex analysis... steepest descent and/or Riemann-Hilbert equations. The equations look a big mess and I am hoping there is away out. The only case of Riemann-Hilbert problem I understand are **Stirling Formula** $n! \approx \sqrt{2\pi n}(n/e)^n$ and the **Szegö** curve, $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = 0$ which is almost like $|xe^{1-x}| = 1$

One big turn-of-the-millenium mathematical idea is determinantal processes[2].

However, the Aztec Diamond domino shuffling can be turned into a determinantal process in various ways.

Kurt Johansson in 2000 writes about a few generalizations of the Plancherel Measure. All of these have to do with the permutation group, S_n.

$$\sum_{|\lambda|=n} (\chi_\lambda)^2 = n!$$

In a way all these permutations occur randomly (like shuffling a deck of cards), so the representations also occur randomly

$$\frac{1}{n!} \sum_{|\lambda|=n} \mathbb{P}[\chi = \lambda] = 1$$

Then, why do domino tilings of the Aztec Diamond constitute a measure on the representations of the permutation group[3].

[2]They were a canidate to help solve the Riemann Hypothesis (e.g. the Keating Snaith conjecture. Certainly they help you talk about the behavior of the Riemann zeta function on the critical line $\zeta(\frac{1}{2} + it)$.

[3]shuffling a deck of cards

And what would happen if I used a really crappy source of randomness. Do I get a convincing shape? LOL

I would like the minimum amount of difficuly that still constitute a proof[4]. Let me not stall any more:

$$\mathbb{P}[Z_r(h)] \asymp \prod_{1\le i<j\le r} \frac{h_j - h_i}{j-i} \prod_{1\le i<j\le n+1-r} \frac{k_j - k_i}{j-i}$$

I should be more careful... there's just a number which I am too lazy to type[5]

This is known as the Krawtchouk Ensemble. A lot is known about these equations (here they are discribing "zig-zag paths" in the tiling).

I would like to show as $n, r \to \infty$ with $r/n = k$, this approaches the GUE. This has been done before.

All I need to show is that if $\frac{h}{n} < \sqrt{1 - (\frac{r}{n})^2}$ the region is frozen... but we know more details.

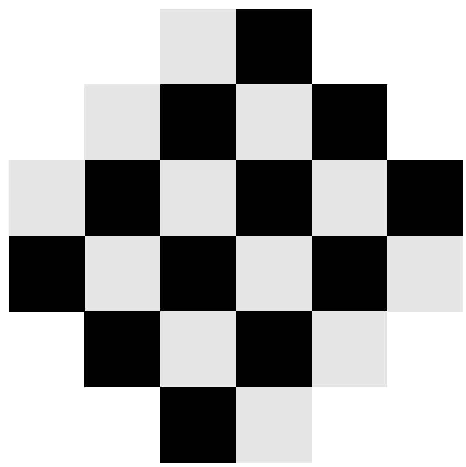
[4] A physicist emphasizes the result and skips all the details, the analyst is showing off their calculation skills. I am a geometer and emphasize shapes :-/

[5] $n \to \infty$ anyway, so why not just start now?

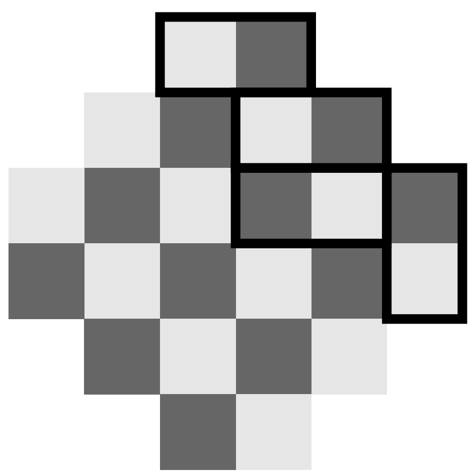
2 Plancherel Measure, Zig-Zag Paths

Domino Tilings is notoriously hard to code and to theorize about at the same time. All my scrawled pages are lost.

Exercise 1: Draw an Aztec Diamond



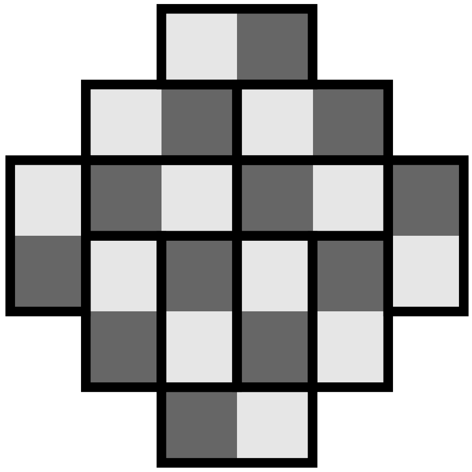
Exercise 2: Draw some dominoes



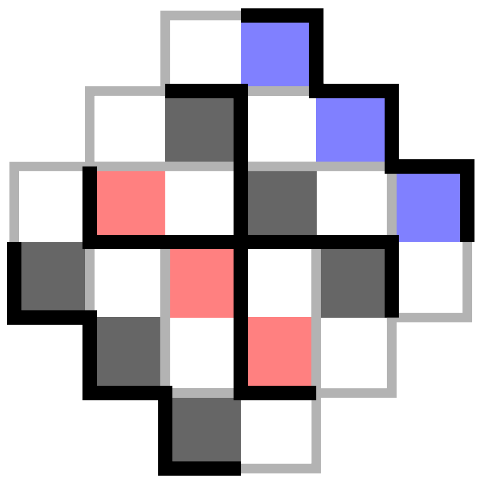
How to encode one domino? The middle line is at (1,0) and (1,1) so we will encode it as `1-0-1`, and we specify that it is horizontal[6].

[6] We'll always get stuck since there's no language for having one item in **two** places. Maybe it's not so bad. This would be a redundant way to store this information on a computer, but hey we're drawing. This is a separate problem!

Exercise 3: finish the domino tiling



Exercise 4: draw a zig-zag path[7]



Exercise 5: prove limiting behavior of zig-zag

References

(1) Kurt Johanssn **Discrete orthogonal polynomial ensembles and the Plancherel measure.** `math/9906120` Annals of Mathematics (2) 153 (2001), No. 2, 259–296.

(2) Benjamin J. Fleming, Peter J. Forrester. **Interlaced particle systems and tilings of the Aztec diamond.** `arXiv:1004.0474`

(3) Manuel Fendler, Daniel Grieser. **A new simple proof of the Aztec diamond theorem.** `arXiv:1410.5590`

(4) Frédéric Bosio, Marc A. A. Van Leeuwen. **A bijection proving the Aztec diamond theorem by combing lattice paths.** `arXiv:1209.5373`

(5) David E. Speyer **Variations on a theme of Kasteleyn, with application to the totally nonnegative Grassmannian** `arXiv:1510.03501`

[7] I am stressing the colors a bit.