Theta Functions

John D Mangual

$$\theta(x;p) = (x;p)_\infty (px^{-1};p)_\infty = \exp\left(- \sum_{m\neq 0} \frac{x^m}{m(1-p^m)} \right)$$

another one

$$\theta(z;q) := (z;q)_\infty (q/z;q)_\infty = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} z^k q^{\binom{k}{2}}$$

the shifted factorials are defined by:

$$(z;q)_\infty = \prod_{i\geq 0} (1 - zq^i)$$

Let's see if

$$\binom{k}{2} = \frac{k(k-1)}{2} = \frac{k^2}{2} - \frac{k}{2}$$

Then it could be:

$$\theta(q^2;q) = \frac{1}{(q;q^2)} \sum_{k\in\mathbb{Z}} q^k q^{2\binom{k}{2}} = \frac{1}{(q;q^2)} \sum_{n\in\mathbb{Z}} q^{n^2}$$

Wikipedia has

$$\sum_{n\in\mathbb{Z}} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

and we can set $a = b = q$:

$$\sum_{n\in\mathbb{Z}} q^{n^2} = (-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty$$

This also seems odd we can try

$$\theta(q;q^2) = (q;q^2)_\infty(q;q)_\infty(q^2;q^2)_\infty = \sum_{n\in\mathbb{Z}} q^{n^2}$$

It might parameterized in terms of two angles:

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$$

which has another triple product

$$\prod_{m=1}^{\infty}(1-e^{2\imath m i\tau})\left[1 + e^{(2m-1)\pi i\tau + 2\pi i z}\right]\left[1 + e^{(2m-1)\pi i\tau - 2\pi i z}\right]$$

Then $q = e^{2\pi i\tau}$ and $x = e^{2\pi i z}$:

$$\theta(0;q) = \prod (1-q^2)(1+q^{2m-1})(1-q^{2m+1})$$

This is a beautiful triple product but we have to write in terms of rising and falling factorials.

$$\sum_{n\in\mathbb{Z}} q^{n^2} = (-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty$$

The exponent formula looks like

$$\log(1-x) = \sum \frac{x^m}{m}$$

and the geometric series formula:

$$\sum p^{km} = \frac{1}{1-p^m}$$

If we put two of them together it says:

$$\sum_m \sum_k \frac{1}{m} x^m p^{km} = \sum_m \frac{1}{m} \frac{x^m}{1 - p^m}$$

This is very much the logarithm in the beginning of this article.

Part II

So one big problem I will have with a lot of elliptic index paper with θ functions eveywhere is the normalization. And their endless obsession with modular invariance[1]

Uh... so before I get into that we rewind to 2003 before a lot of this paper and read through Appendix A of Nekrasov-Okounkov:

$$\gamma_\hbar(x;\Lambda) = \frac{d}{ds}\bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(e^{\hbar t}-1)(e^{-\hbar t}-1)}$$

This is a mouth-ful but notice right away this is a **Mellin transform** and also this is **zeta regularization**. The Nekrasov partition function is *badly* divergent (as most physics formulas are) and here is one way to fix it.

However, these gentlement have used a very common idea in number theory. Here is a baby example:

$$\sqrt{1}-\sqrt{2}+\sqrt{3}-\sqrt{4}+\cdots = (\sqrt{1}-2\sqrt{2}+\sqrt{3})+(\sqrt{2}-2\sqrt{3}+\sqrt{}$$

Uh... hopefully I remember later[2]

[1] If the object is invariant under $\mathrm{SL}_2(\mathbb{Z})$ or a congruence subgroup $[\Gamma_0(N):\mathrm{SL}_2(\mathbb{Z})] = N$ or a *non-congruenc* group. There are many possibilities that Nekrasov and Shatashvili do not account for ('cuz they're not interested).

[2] but **you** and read `http://math.stackexchange.com/q/1896464/4997`

Nekrasov and Okounkov state this really is zeta-function regularization so we have

$$\gamma_\hbar(0;\Lambda) = -\frac{1}{12}$$

and even some instance of the volumes of the unitary groups:

$$\log\big(\text{Vol } U(N)\big) = \gamma_1(N;1)$$

and the other functions $\gamma_{\epsilon_1,\epsilon_2}$ are embellishments[3].

These $\gamma_\hbar$ satisfy a second-difference equation:

$$\gamma_\hbar(x-\hbar,\Lambda) - 2\gamma_\hbar(x,\Lambda) + \gamma_\hbar(x+\hbar,\Lambda) = \log\left(\frac{x}{\Lambda}\right)$$

Theres so many logs floating around but I really want to talk about this Λ:

$$\sum[\Lambda(n)-1]\frac{e^{-ny}}{1-e^{-ny}} \sim -\frac{2\gamma}{y}$$

Then by the Hardy-Littlewood Tauberian theorem (for Lambert series)[4]:

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)-1}{n} = -2\gamma$$

[3] and a nuisiance to read – painful on the eyes

[4] which we will argue is the same kind of regularization as Nikita Nekrasov uses

If this thing converges at all the coefficients must have been small:

$$\sum_{n \le x} [\Lambda(n) - 1] = o(x)$$

and this is very much equivalent to the Prime Number Theorem.

Here the Λ in question is the Van Mangold function:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

and we regularize the Lambert sum to a normal finite sum

$$y \sum_{n=1}^{\infty} \frac{(\Lambda(n) - 1)e^{-ny}}{1 - e^{-ny}} \approx \sum_{n=1}^{\infty} \frac{(\Lambda(n) - 1)}{n} e^{-ny} \to -2\gamma$$

if we let $y \to 0$

Part III – Review with some more details[5]

Let me digress on the values of the Riemann Zeta function. Here is a formula for $\zeta(2)$:

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \int_{1>t_1>t_2>0} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}$$

Then there's the formula by Eugenio Calabi

$$\frac{3}{8}\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \int_0^1 \int_0^1 \frac{dx\,dy}{1-(xy)^2}$$

There is another dissimilar looking formula:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^{\infty} \log(1-e^{-x})$$

Even before trying our hand at the double-shuffle identities, or relating $\zeta(2)$ to famous constants[6] how to transform the first formula into the third?

I dug up the fist formula from a paper of Zagier, the second in a paper of Elkies and a third in a paper by Passare[7]

[5]So as you might guess the theme is ambiguity in the literature. The confusion as an outider to see the same classical formula in 20 different places – each one with their opinion how it should be developed (or ignored completely).

[6]Why stop at π, there's the Glaisher constant and the Euler-Masceroni number and the Twin Prime Constant etc.

[7]and there is more... Papers flying everywhere!

Elkies showed a sum linking L-functions and ζ-functions:

$$S(n) = \begin{cases} (1-2^{-n})\zeta(n) & \text{if } n \text{ is even} \\ L(n,\chi_4) & \text{if } n \text{ is odd} \end{cases}$$

Elkies formula also links the Euler numbers and Bernoulli numbers.

$$\frac{A_n}{n!} = \left(\frac{2}{\pi}\right)^n \left(\frac{4}{\pi}\right) S(n+1)$$

The $A(n)$ numbers count the volume of a certain polytope:

$$t_1 < t_2 > t_3 < t_4 > \cdots < t_n > t_1$$

this high-dimensional shape splits into a certain number of "triangular" parts:

$$0 < t_1 < t_2 < \cdots < t_n < 1$$

The volume integrals are different too:

$$\int_{\frac{\pi}{2}} 1 du_1, \ldots du_n$$

and the integral is:

$$\int_0^1 \cdots \int_0^1 \frac{dx_1 \ldots dx_n}{1 \pm (x_1 \ldots x_n)^2}$$

The middle integral is an odd duck.

$$\frac{3}{8}\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \int_0^1 \int_0^1 \frac{dx\, dy}{1-(xy)^2}$$

There's no magic change of variables turning into the iterated integral over the triangle $0 < s < t < 1$.

I am pushing a square peg into a round hole.

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^{\infty} \log(1-e^{-x})$$

Passare also shows this is the volume of a polygon just a triangle:

$$\int_{x<y<\pi} 1\, dx\, dy$$

and the reason the triangle and the exponential integral are the same is they are the real and comlpex parts of the Amoeba.

$$x + y + 1 = e^u + e^v + 1 = 0$$

I've checked already. This special coincidence doesn't always work. Just for Harnak curves[8]

[8] and the variables on this page are screwed up

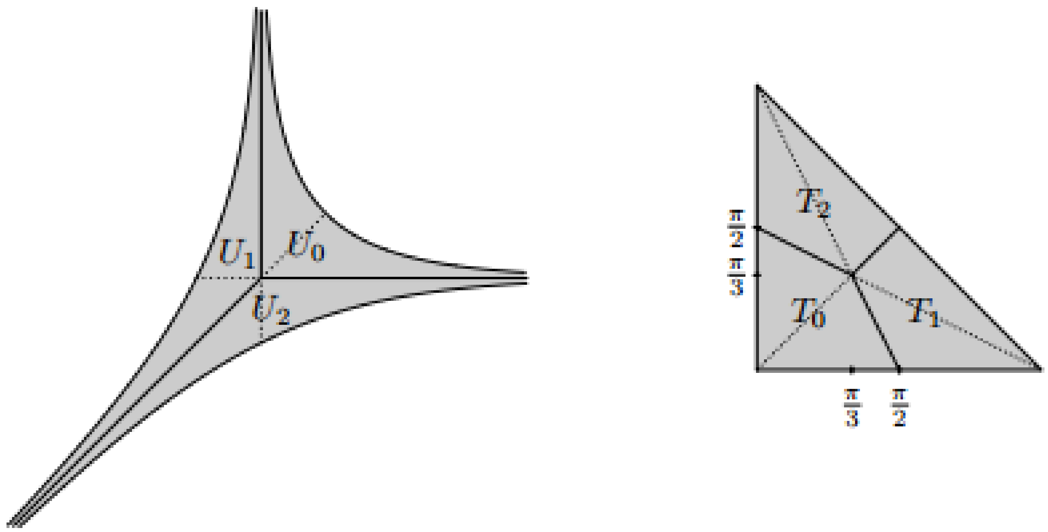

FIGURE 4. *All six subsets have equal area.*

THEOREM: *The area-preserving bijective map $(x, y) \mapsto (-y, x - y)$ permutes the amoeba subsets cyclically: $U_0 \mapsto U_1 \mapsto U_2 \mapsto U_0$.*

Does Elkies polygon decomposition match up with Passare. We're stuck in a really lame situation[9] with these two polygon decomposition's don't match up in a fundamental way.

- Do we have one triangle?
- or many triangles?

There is a paper by Alexander Goncharov[10] **Multiple zeta-values, Galois groups, and geometry of modular varieties** `arXiv:math/0005069`

[9]There's an OK paper by Zurab Silagadze that says yes, but his explanation is a bit disorderly. It's almost better to try again and I have some stuff he won't think of.

[10]it's still "geometry" but it is quite varied.

At will we can find source that find $\zeta(2n)$ by induction. First

$$\zeta(2)=\frac{\pi^2}{6}, \zeta(4)=\frac{\pi^4}{90}, \zeta(6)=\frac{\pi^6}{945}, \ldots$$

skipping $n \mapsto n+2$. The link to even Bernoulli numbers was not that direct anyway:

$$\zeta(2n)=(-1)^{n+1}B_{2n}\frac{(2\pi)^{2n}}{2(2n)!}$$

What about $\zeta(3)$? We know it is irrational but not transcendental. Elkies' chain looks more like:

$$L(1,\chi_4)\to\zeta(2)\to L(3,\chi_4)\to\zeta(4)\to\ldots$$

switching between the ζ and L-functions.

References

(1) Taro Kimura, Vasily Pestun **Quiver elliptic W-algebras** `arXiv:1608.04651`

(2) Wikipedia "Jacobi Triple Product", "Ramanujan Theta Function"

(3) Eric M. Rains, S. Ole Warnaar **Bounded Littlewood identities** `arXiv:1506.02755`

(4) GH Hardy **Divergent Series** texttthttps://archive.org/details/DivergentSeries

(5) David Vernon Widder **The Laplace Transform** `https://archive.org/details/laplacetransform03181`