

Tune-Up: Decimals

Why are such advanced professors writing about decimals...? Let's see if we can put these results further within reach.

Lemma Let μ be a probability measure on a finite set S , with $|S| = N$ and $H_\mu = \rho \log N$. Let $\frac{\log 2}{\log N} < \delta < \frac{1}{2}\rho$. There is a probability measure $\nu \geq (\rho - \delta)\mu$ and $\|\nu\|_2^2 \leq 4\rho^{-1}N^{-\delta}$.

We are guessing based on the reading the text that “finite subsets” $S \subseteq \mathbb{R}/\mathbb{Z}$ yet the statement at the moment says probability measure on *any* finite set. In fact this section is called “notations and preliminaries”.

Reading comprehension: how do we turn these into statements that actually make sense to us? Also their idea of “finite” is worth remarking, that we could eventually break everything into **individual** or **atomic** or **prime** parts. Also finite here could mean $N = 10^{100}$ that's technically finite.

Let $\mu(\{s\}) = w_s$.

Let $S_1 = \{s \in S : w_s < 2N^{-\delta}\}$ and $S_2 = S \setminus S_1$ and for $i = 1, 2$ let $\nu_i = (1/\mu(S_i))\mu|_{S_i}$.

Write \cdot as the partition of S into singletons.

$$\begin{aligned} \rho \log N &= H_\mu = H_\mu(\{S_1, S_2\}) + H_\mu(\cdot | \{S_1, S_2\}) \\ &= H_\mu(\{S_1, S_2\}) + \mu(S_1)H_{\nu_1} + \mu(S_2)H_{\nu_2} \\ &\leq \mu(S_1) \log N + \delta \mu(S_2) \log N + \mu(S_1) \log 2 \end{aligned}$$

Here ν_1 is supported on at most N elements, so $H_{\nu_1} \leq \log N$.

Here ν_2 is supported on at most $N^\delta/2$ elements, $H_{\nu_2} \leq \delta \log N - \log 2$

Observe that $H_\mu(\{S_1, S_2\}) \leq \log 2$.

Using these three entropy estimates, they measure

$$\nu(S_1) \geq \frac{\rho - \delta}{1 - \delta + \log 2 / \log N} > \rho - \delta$$

The authors claim this is sufficient to prove the theorem by setting $\nu = \nu_1$ and observing

$$\|\nu\|_2^2 \leq \|\nu\|_\infty \leq 2\mu(S_1)^{-1}N^{-\delta} \leq 4\rho^{-1}N^{-\delta}$$

[exercise]

□

H_μ denotes an **entropy** measure of some kind. It's not in our best interest to copy this proof at least without thinking about what all the parts mean. Most of the number theory doesn't happen until section 3.

Effective Rudolph Theorem Let a, b be relatively prime integers. Let μ be an arbitrary probability measure on \mathbb{R}/\mathbb{Z} with entropy condition $H_\mu(\mathcal{P}_N) \geq \rho \log N$. There exist

- $\dots < \delta < \dots$ and $\dots < T < \dots$ and
- $f \in C^1(\mathbb{R}/\mathbb{Z})$ non-negative.
- $0 \leq s \leq \dots$ and $0 \leq t \leq \dots$

solving

$$[a^s b^t \cdot \mu](f) \geq (\rho - 3\delta)\lambda(f) - \kappa T^{-\delta/2} \|f'\|_2$$

Here $\|f'\|_2^2 = \int_0^1 |f'(x)|^2 dx$. How does this build on the Chinese Remainder Theorem? Why is entropy useful there?

Our use of equations is rather interesting here. We are absolutely sure the left thing is the same as the right thing of the equation (that the two sides of the scale balance). Or in “real” calculations we are shuffling around approximations, we just get another example of this kind of theorem. What if we are using computers and the decimal points run out? Then we still get another case of these.

03/27 What does **effective** mean? The other question is on mathematical writing and *statements*. Here is the experimental observation:

$$\overline{\{a^k \times b^\ell \times x : k, \ell \geq 0\}} = \mathbb{R}/\mathbb{Z}$$

These are specialists and have presented this observation as number theory, a statement about the decimal system. Here \overline{X} is the *closure* of X with respect to a certain topology (or notion of “closeness”).

Rudolph-Johnson Theorem

- Let μ be a probability measure on \mathbb{R}/\mathbb{Z} invariant under t_a and t_b
- Let a, b be multiplicatively independent.
- Suppose $h_\mu(\times a) = \mu \log a$

Then $\mu \geq \eta \lambda$ for any $A \subset \mathbb{R}/\mathbb{Z}$ and $\mu(A) \geq \eta \lambda(A)$. Here μ is *any* probability measure, $\lambda = dx$ is Lebesgue measure (which may be an enormous amount of work to compute accurately, η is a *constant* – what’s that?

Effective Rudolph-Johnson Theorem

- Let a, b be multiplicatively independent.
- μ be a probability measure on \mathbb{R}/\mathbb{Z} .
- Entropy condition $H_\mu(\mathcal{P}_N) \geq \rho \log N$ for some $\rho > 0$ $N > N_0(a, b)$
- $100/\log_a N \leq \delta \leq \rho/20$ and $f \in C^1(\mathbb{R}/\mathbb{Z})$.

There is an integer $m = a^s b^t < N$ so that

$$[m.\mu](f) \geq (\rho - 3\delta)\lambda(f) - \kappa_1 \log(N)^{-\kappa_2 \delta} \|f'\|_2$$

where κ_1, κ_2 depend only on a, b . We can’t accept this. We need to see what these constants and measures look like.

Effective Rudolph Theorem

- Let a, b be relatively prime integers.
- Let μ be an arbitrary probability measure on \mathbb{R}/\mathbb{Z}
- Suppose an *entropy condition* $H_\mu(\mathcal{P}_N) \geq \rho \log N$ for some $\rho > 0$ and $N > N_0(a, b)$
- Let $\frac{10}{\log_a N} \leq \delta \leq \frac{\rho}{20}$ and $a^{20/\delta} \leq T \leq \frac{\delta}{4} \log_b(N)$ and $f \in C^1(\mathbb{R}/\mathbb{Z})$.

Then there exist integers s, t with

- $0 \leq s \leq (1 - \delta) \log_a(N)$
- $0 \leq t \leq T$
- $[a^s b^t . \mu](f) \geq (\rho - 3\delta) \lambda(f) - \kappa T^{-\delta/2} \|f'\|_2$

Here κ depends only on a, b and $\|f'\|_2^2 = \int_0^1 |f'|^2 dt$

Out of these difficult statements, we get more elementary statements.

Thm Let a, b be multiplicatively independent with $(ab, N) = 1$ Then for any $m \in (\mathbb{Z}/N\mathbb{Z})^\times$ the set

$$\{a^k \times b^\ell \times \frac{m}{N} : 0 < k, \ell < 3 \log N\}$$

This set is $\kappa(\log \log \log N)^{-\kappa/100}$ with depending only on a, b . So we would spend the rest of the time looking for these scaling constants.

These subsets of the integers seem to be highly chaotic, so complicated that we can barely estimate the entropy to describe them.

There are two different hypotheses:

- Let a, b be **relatively prime**.
- Let a, b be **multiplicatively independent**.

What are the inputs in such a theorem? Why are the constraints so specific? What would we like to see instead?

References

[1] Jean Bourgain, Elon Lindenstrauss, Philippe Michel, Akshay Venkatesh. **Some Effective Results of $\times a \times b$** Volume 29, Issue 6. December 2009 , pp. 1705-1722

[2]

[3]