Attempt at: the Arctic Circle Theorem

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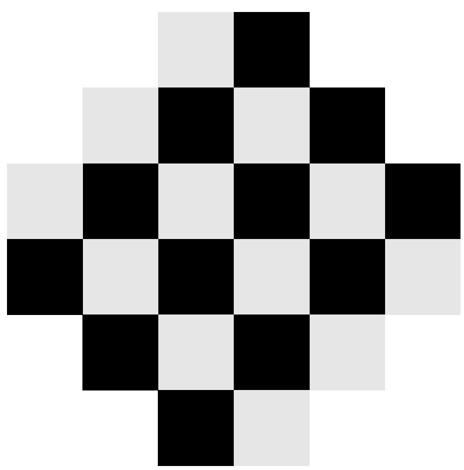
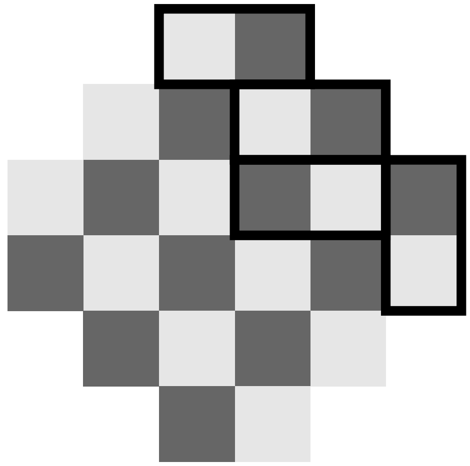
Domino Tilings is notoriously hard to code and to theorize about at the same time. All my scrawled pages are lost.

Exercise 1: Draw an Aztec Diamond

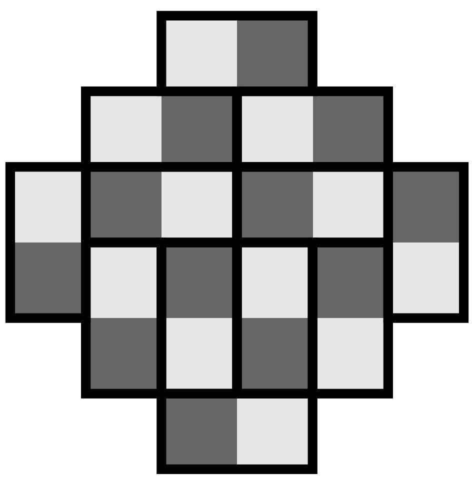


Exercise 2: Draw some dominoes



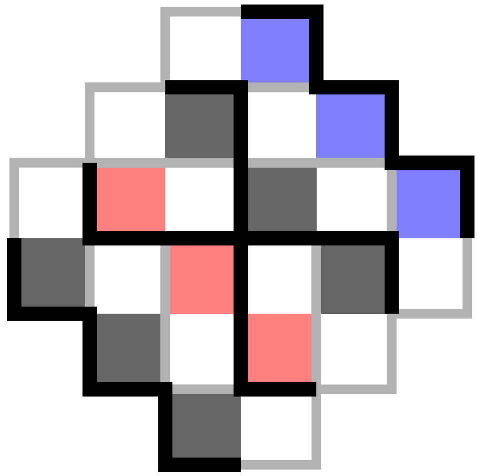
How to encode one domino? The middle line is at (1,0) and (1,1) so we will encode it as `1-0-1`, and we specify that it is horizontal[1].

Exercise 3: finish the domino tiling



[1]We'll always get stuck since there's no language for having one item in **two** places. Maybe it's not so bad. This would be a redundant way to store this information on a computer, but hey we're drawing. This is a separate problem!

Exercise 4: draw a zig-zag path[2]



Exercise 5: prove limiting behavior of zig-zag

[2] I am stressing the colors a bit.

The Krawtchouk ensemble is the correct ensemble for the Aztec Diamond problem. We only focus on $p = \frac{1}{2}$:

$$\mathbb{P}_{\mathsf{Kr}}[(h_1, \ldots h_n)] = \frac{1}{Z} \times \frac{1}{2^{\binom{N}{2}}} \times \Delta_N(h) \times \prod_{j=1}^{N} \binom{K}{h_j}$$

where the normalization Z is a number:

$$\frac{N!}{0! \times 1! \times 2! \times \cdots \times (N-1)!} \times \frac{1}{2^{\binom{N}{2}}} \times \prod_{j=1}^{N} \binom{K}{j}$$

the meaning of these number would be clear to a professor Combinatorics[3] these are related to the Krawtchouk polynomials.

Johansson attributes the circular shape o Timo Sepalainen. We settle for this summary by Persi Diaconis and Robert Griffiths

> Orthogonal polynomials for the multinomial distribution of N balls dropped into d boxes are called multivariate Krawtchouk polynomials.

These ball-and-urn models are commonplace.

[3] or various branches of Computer Science

The limit shape seems to be known from the theory of **first-passage percolation**. It would not hurt to read the book by Grimmett.

$$\mu(x,y) = \begin{cases} y & \text{if } x < y \\ y + (\sqrt{x/2} + \sqrt{y/2})^2 & \text{if } x \geq y \end{cases}$$

The earliest proof by Jockusch-Shor-Propp uses the TASEP model[4]

The most important is that we have a formula for all possible $\vec{h}$ on a given row (and later, the joint probability density over all rows).

Johansson says it is an easy analysis from here.

[4]which can be converted to this type of "first-passage percoltion". More on this once I figure it out.

2 - Determinantal Processes

The surprising fact that[5]:

$$\mathbb{P}(h) = \frac{1}{Z_M\, 2^{\binom{M}{2}}} \prod_{1\le i<j\le M} (h_i - h_j)^2 \prod_{k=1}^{M} \binom{M}{k}$$

I could not figure out the ball-and-urn model this is associated to this combination formula.

The theory of determinantal processes says once we know the kernel we know everything else about that process[6]. The Krawtchouk polynomials are orthogonal with respect to:

$$\frac{1}{2^N}\binom{N}{x} = \frac{1}{2^N}\binom{N}{\frac{N}{2}+t\sqrt{N}} \to \frac{1}{2\pi}e^{-t^2/2}$$

if $x = \frac{N}{2} + t\sqrt{N}$ where we magically know the correct scaling :-)

The probabilities are **Slater determinant** of the different wave functions:

$$\mathbb{P}[h] = \det\Big[\sum \psi_n(h_i)\psi_n(h_j)\Big]_{i,j}$$

[5]The surprising fact that there is any formula at all ...

[6]The theory of determinantal processes - especially the **geometry** - will be important once I assemble the basic discussion.

A kind of smart-alec way of finding the Kernel is to say: $K(x,y) = \sum_{n=1}^{N} \psi_n(x)\psi_n(y)$ equal to

$$\oint dz\ (1+qz)^x (1-pz)^{N-x} (1+q\overline{z})^y (1-p\overline{z})^{N-y}$$

Then the particle density would be if we set $x=y$:

$$\rho(x) = \oint dz\ \left|(1+z/2)^x (1-z/2)^{N-x}\right|^2$$

and I had better hope the middle coefficient convergest to the semicircle:

$$\rho(x) = \frac{1}{\pi}\sqrt{1-x^2}$$

Oh it seems we have bypassed the Gaussians.

Oops if we define Krawtchouk polynomials by:

$$\sum_{n=0}^{N} \psi_n(x) \frac{z^n}{n!} = (1+z/2)^x (1-z/2)^{N-x}$$

then the formula we have written is different from $\sum \psi_n(x)\psi_n(y)$ instead it's $\frac{1}{n!^2}\sum \psi_n(x)\psi_n(y)$. And it's length^2 will be:

$$\frac{1}{2^{2n}}\binom{N}{n}$$

These definitions are so confusing. I may have to start from scratch.

3 - Determintanal Processes

In this iteration, we try to become accountable for the formulas we use, and making sure they represent what we want.

What ever the Krawtchuk polynomials are there is a formula for them here:

$$\sum_{n=0}^{N} K_n(x; N, p)\frac{z^n}{n!} = (1+qz)^x(1-pz)^{N-x}$$

For simplicity I am going to set $p = q = \frac{1}{2}$. What does $N = 2$ look like?

$$\begin{aligned} K_0(x) + K_1(x)z + K_2(x)\frac{z^2}{2} &= (1+\tfrac{1}{2}z)^x(1-\tfrac{1}{2}z)^{2-x} \\ &= ? \end{aligned}$$

If we expand to x places that doesn't simplify very much now, does it?

Here is another version:

$$\sum_{n=0}^{N} Q_n(x) z^n = (1-z)^x (1+z)^{N-x}$$

with duality $Q_n(x) = Q_x(n)$. Does it matter if we use K or Q?

Here is the formula again:

$$\mathbb{P}(h) = \frac{1}{Z_M \, 2^{\binom{M}{2}}} \prod_{1 \le i < j \le M} (h_i - h_j)^2 \prod_{k=1}^{M} \binom{M}{k}$$

and I could set $M = 3$:

$$\mathbb{P}(h) = \begin{vmatrix} 1\binom{3}{1} & \frac{1}{2}a\binom{3}{2} & \frac{1}{2^2}a^2\binom{3}{3} \\ 1\binom{3}{1} & \frac{1}{2}b\binom{3}{2} & \frac{1}{2^2}b^2\binom{3}{3} \\ 1\binom{3}{1} & \frac{1}{2}c\binom{3}{2} & \frac{1}{2^2}c^2\binom{3}{3} \end{vmatrix}$$

Then using row operations (such as Gram-Schmid we can pain-stakingly make these three columns orthogonal with respect to $w(x) = \binom{N}{x}\frac{1}{2^N}$

If we look at the formula at let n be random

$$\mathbb{E}[K_n(x,N)] = \sum_{n=0}^{N} K_n(x;N)\frac{z^n}{n!} = (1+z/2)^x(1-z/2)^{N-x}$$

in the limit $n \to \infty$ $\mathbb{P}(n) = \frac{z^n}{n!}$ is a Poisson distribution, with average z.

$*$ We are still learning, let's use the simpliest formula: let $X = \xi_1 + \cdots + \xi_n$

$$G(z;X) = \prod_{j=1}^{N} \big(1 + u(\xi_j)z\big)$$

and the Krawtchuk polynomials simply fall out:

$$K_n(X) = n! \sum_{\sigma \in S_n} u(\xi_{\sigma(1)}) \dots u(\xi_{\sigma(1)})$$

This notation is getting hard to read – **defeating the purpose of notation**. The $\xi \in \{0,1\}$ so that $u(\xi) = \xi - \frac{1}{2} \in \{\frac{1}{2}, -\frac{1}{2}\}$.

$$K_n(X) = n! \sum_{\sigma \in S_N} (-1)^{f(\sigma,n,N)} \frac{1}{2^n}$$

for some indeterminate function f. I think[7]

[7] when do we have a formula in real life? never!

Since I have no idea what I am doing let's try:

$$Q_n(x) = \sum_{k=0}^{N} \frac{\binom{x}{k}\binom{N-x}{n-k}}{\binom{N}{n}} = \sum_{k=0}^{N} \frac{\binom{n}{k}\binom{N-n}{x-k}}{\binom{N}{x}}$$

then we are trying to evaluate a very complicated determinant:

$$\det\left[\sum_{n=0}^{N}\left(\sum_{k=0}^{N}\frac{(-1)^k\binom{n}{k}\binom{N-n}{x_i-k}}{\binom{N}{x_i}}\times\sum_{k=0}^{N}\frac{(-1)^k\binom{n}{k}\binom{N-n}{x_j-k}}{\binom{N}{x_j}}\right)\right]_{i,j}$$

again this is pretty dreadful to read and we don't know what these objects are counting.

$$\det\left[\sum_{n=0}^{N} Q_n(x_i)Q_n(x_j)\right]_{i,j}$$

This is simpler but less informative. The kernel looks like this:

$$\sum_{n=0}^{N}\left[\sum_{k=0}^{N}\frac{(-1)^k\binom{n}{k}\binom{N-n}{x-k}}{\binom{N}{x}}\times\sum_{l=0}^{N}\frac{(-1)^l\binom{n}{l}\binom{N-n}{y-l}}{\binom{N}{y}}\right]$$

4 - The Ehrenfest Urn model.

In a way the Krawtchouk polynomials are pretty stupid. They are orthogonal to the binomial coefficients:

$$\sum_{x=0}^{N} \psi_m(x)\psi_n(x) \times \frac{1}{2^N}\binom{N}{x} = \begin{cases} 0 & \text{if } m \neq n \\ ? & \text{otherwise} \end{cases}$$

but actually the steady state of a random walk on $\{0, \ldots, N\}$ with odds of jumping half to the left and to the right, is the uniform distribution.

This is known was the **gambler's ruin**[8]. Instead we need the Ehrenfest urn model.

> An urn has N balls coloured red or blue. Transitions in a Markov chain are made by selecting a ball at random and changing its colour. X_t is the number of red balls after t transitions.
>
> $\{X_t\}_{t\in\mathbb{N}}$ is a reversible **Markov Chain** with a Binomial $(N, \frac{1}{2})$ stationary distribution.

[8] Gambler's ruin says if you random walk left and right on $0, \ldots, n$ you eventually hit zero.

So this Ehrenfest model can also be considered a random walk on $(\mathbb{Z}_2)^N$. 

Q: How many circles are red? How many circles are blue?

References

(1) Kurt Johansson **Discrete orthogonal polynomial ensembles and the Plancherel measure.** `math/9906120` Annals of Mathematics (2) 153 (2001), No. 2, 259–296.

(2) Benjamin J. Fleming, Peter J. Forrester. **Interlaced particle systems and tilings of the Aztec diamond.** `arXiv:1004.0474`

(3) Manuel Fendler, Daniel Grieser. **A new simple proof of the Aztec diamond theorem.** `arXiv:1410.5590`

(4) Frédéric Bosio, Marc A. A. Van Leeuwen. **A bijection proving the Aztec diamond theorem by combing lattice paths.** `arXiv:1209.5373`

(5) David E. Speyer **Variations on a theme of Kasteleyn, with application to the totally nonnegative Grassmannian** `arXiv:1510.03501`

(6) Persi Diaconis, Robert Griffiths. **An introduction to multivariate Krawtchouk polynomials and their applications** `arXiv:1309.0112`

(7) Tom Koornwinder **Krawtchouk Polynomials, a Unification of Two Different Group Theoretic Interpretations** `SIAM J. Math. Anal., 13(6), 10111023.`

(8) Richard Feynman **Statistical Mechanics**

"The historically first example of such a system goes back to **De Moivre** (1738) and **Laplace** (1812) who considered the problem of finding the asymptotic distribution of a sum of i.i.d. random variables for Bernoulli trials, when the pre-limit distribution is explicit, and took the limit of the resulting expression.

While this computation may look like a simple exercise when viewed from the heights of modern probability, in its time it likely served the role of a key stepping stone - first rigorous proofs of central limit theorems appeared only in the beginning of the XXth century.

At the moment we are arguably in a "De Moivre-Laplace stage" for a certain class of stochastic systems which is often referred to as the **KPZ universality class**, after an influential work of Kardar-Parisi-Zhang in mid-80's. . . "

taken from:

Alexei Borodin, Vadim Gorin. **Lectures on Integrable probability** `1212.3351`