Tauberian Theorems

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Abstract

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0.1 Abel's Theorem

Section 7.5 of Titchmarsh says if $s_n = a_1 + \cdots + a_n \sim n$ then $f(x) \propto \frac{1}{1-x}$ i.e. a **pole**

What exactly does "$\sim$" mean? $A \sim B$ means $\lim_{n\to\infty} \frac{A}{B} = 1$. For example: $n^2 \sim n^2 + n + 1$.

For our theorem we have that if the average value of the coefficients is 1 there is a pole:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \left[\lim_{n\to\infty} \frac{a_1 + \ldots a_n}{n}\right] \cdot \frac{1}{1-x} + O(1)$$

The residue of this pole could be 0 and that $f(x)$ is **holomorphic** at $x = 1$. Usually is the case.

What does "$x = 1$" mean anyway? On a computer it means $x = 0.9999995991$ or even $x = 1.02$ Floating point arithmetic on computers means $x = 1$ is actually $x = 1 \pm \epsilon$ where ϵ is our **tolerance**, a very small error since we can never be 100% sure.

Our theorem says that evaluating at $x = 1$ we either get a number, which Hardy bluntly puts as $O(1)$ or a simple divergent function proportional to the average of **all** coefficients, $\frac{\mathbf{A}}{x-1}$.

If we use telescoping series we can extract information about the coefficients from information about the totals $s_n = a_1 + \cdots + a_n$:

$$\sum a_n x^n = (1-x) \sum s_n\, x^n \sim (1-x) \sum A \cdot n x^{n-1} = (x-1) \cdot \frac{A}{(x-1)^2} = \frac{A}{x-1}$$

If we know that $s_n \sim nA$ then we conclude $\boxed{\sum a_n x^n \sim \frac{A}{x-1}}$ tending to an average as $x \to 1$

The proof uses the **triangle inequality** $|a + b| < |a| + |b|$ the length of any side of a triangle is smaller than the sum of the other two sides.

$$\left| \sum s_n\, x^n - \frac{A}{(x-1)^2} \right| \le \left| \sum_{n=0}^{N} \big[s_n - A\,(n+1)\big] x^n \right| + \left| \sum_{n=N+1}^{\infty} \big[s_n - A\,(n+1)\big] x^n \right|$$

Don't be alrarmed by $n-1$ it is just n, but I am trying to keep the formulas exact[1]. We need to explain in a convincing way that both terms are "small" or "close". N and x have constraints:

$$\begin{aligned}
\left| \left[\frac{1}{n+1} \sum_{k=0}^{n} a_k \right] - A \right| &= \epsilon \text{ for } n > N \\
\left| \left[(x-1) \sum_{n=0}^{N} a_n\, x^n \right] - A \right| &= \epsilon \text{ for } x > 1-\delta \text{ once we choose } N
\end{aligned}$$

Tolerances play a big role in our ability to verify this always works. And yet **Why should we bother to check?** Think about if this really matters to you.

The typical case of this theorem is that $f(z) = \sum a_n z^n$ is some hypergeometric series or some kind of power series that solves a differential equation. Even though we can crank out the power series solution term by term, we have no freakin' clue what it should mean. We don't even know basic information such as **radius of convergence**.

$$R = \liminf_{n\to\infty} |a_n|^{1/n}$$

Now, at least if we can average the coefficients we can assert there is a pole. There will be more to say on this topic if we move on and come back later. We need to wait until there is an interesting power series we wish to run through the machinery of **Complex Analysis**.

Example If $a_n \sim \log n$ then as $x \to 1$ or let $N = \frac{1}{1-x} \to \infty$ then convergence to the **entropy**

$$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x} \log \frac{1}{1-x} = N \log N$$

The largest term in the expansion is where $x \approx 1/a_n$ and a few of the neighboring terms.

Example In Hardy and Wright's book *An Introduction to the Theory of Numbers* we find a formula in the back of chapter 17:

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = x$$

It would be nice to make the conclusion that $\sum \frac{\mu(n)}{n} \approx x(1-x) \to 0$ but sadly I am not sure.

0.2 Tauber's Theorem

What if we are in the opposite situation. We know our function $f(x)$ but can't really say much about the coefficients in the expansion: $a_n x^n + \dots$ maybe they are hard to compute?

[1] If you really must know... we can start from the **geometric series formula** and take it's square. First of all:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum x^n$$

True as long as $|x| < 1$, but we can make sense of it even if $|x| = 1$ and even if $|x| > 1$ we have ways... then:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum (n+1)\, x^n$$

This involves all the possible ways of splitting n into **two** numbers: $5 = 5+0, 4+1, 3+2, 2+3, 1+4, 0+5$

References

[1] Titchmarsh. *Theory of Functions* Clarendon Press. Oxford, 1932.