Tutorial : Quadratic Reciprocity

Reciprocity Theorems: why do they happen? why are they important? Every time I see a proof of quadratic reciprocity it's usually at the end of a textbook just to show off all the "concepts" we have learned. For me, the argument's tell me it's true but if we are trying to solve the equiation:

$$x^2 \equiv p \pmod{q}$$

it doesn't explain why $p \leftrightarrow q$ should be possible, why these numbers should be on equal footing. And if we look at other "reciprocity theorems" can their proofs be told in "elementary" language i.e. with numbers!

These days I find the Legendre symbol misleading. Let's try another way:

$$\Big[x^2 \equiv p \pmod{q}\Big] \leftrightarrow \Big[x^2 \equiv q \pmod{p}\Big]$$

except this is not true when but $p = 4k + 3$ and $q = 4k + 3$ then we have the opposite

$$\Big[x^2 \equiv p \pmod{q}\Big] \leftrightarrow \Big[x^2 \not\equiv q \pmod{p}\Big]$$

How is this applied in the real world? I think that's an open question. All I can say is that the concept of the integers $\mathbb{Z}$ becomes are scaffolding for understanding a chaotic world, that really has no intrinsic concept of number.[1]

Can you come up with a better concept than $\mathbf{0}$ or better than $\mathbb{Z}$? No.

Usually the justification is that Quadratic Reciprocity is connect to other branches of mathematics. Let's try:

$$\big[\text{fermat's little theorem(s)}\big] \to \big[\text{QR}\big] \to \big[\text{three squares}\big] \to \big[\text{Banach-Tarski paradox}\big]$$

[1] Why is there no **dynamical systems** proof of Quadratic Reciprocity? We will not discuss cryptrography in any way, but maywe we could discuss **coding** or **information theory**. Just, number theory is to as "applied" as one would like. Philosophically it's about abstracting away from the real world something extremely pure.

A Wikipedia has a separate page for proofs of Quadratic Reciprocity. Let's try the willy-nilly Eisenstein proof, which I despise.

References

(1) Masanori Morishita **Knots and Primes** (Universitext) Springer, 2012.

(2) Nancy Childress **Class Field Theory** (Universitext) Springer, 2009.

(3) Jared Weinstein **Reciprocity Laws and Galois Representations: Recent Breakthroughs** Bull. Amer. Math. Soc. 53 (2016), 1-39 https://doi.org/10.1090/bull/1515