Scratchwork: Decoupling

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It seems very important we review the topic of **decoupling**...Many analytic number theorists seem to feel strongly this is the next big advance. Me, being a fair bit outside this circle, observe that *they* feel it's important. It's like when Lebron James wears a certain sneaker and *everyone* feels they gotta have it.

Even the statement of decoupling is very difficult to understand. We're greatful to Ms. Lillian Pierce for charging through impenetrable sections of the literature. We include Bourgain's own "study guide" and his original sources. We do not include any primary sources.

Why study? Like any subject there are "moments" of familiarity followed by mostly chaos. If it is too much effort you simply give up and move to the next project.

Here's one way to phrase the conjecture from Tao's blog:

$$\int_{[0,1]^2} \Big|\sum_{j=0}^{N} e(jx + j^2 y)\Big|^6 \, dx\, dy \ll N^{3+o(1)} \tag{☎}$$

as $N \to \infty$. It looks like a freshman Calculus exercise or something you will see on the Putnam exam. It does not look very modern. It looks very easy to compute. If I recall, getting bogged down with the sixth power:

$$|a+b+c|^6 = (a+b+c)^3\, \overline{(a+b+c)^3}$$

I am less satisfied when I learned which equation they were solving:

$$\begin{aligned} j_1 + j_2 + j_3 &= k_1 + k_2 + k_3 \\ j_1^2 + j_2^2 + j_3^2 &= k_1^2 + k_2^2 + k_3^2 \end{aligned}$$

If these equations look intuitive their solution certainly does not. Here's an exercise:

- Let $N = 5$ or $N = 100$. Evaluate the integral, ☎ to 10 decimal places.

At this level I really should qualify this term "evaluate" to mean, as an element of $\mathbb{R}$ find 10 decimal places. While I am reading (it may be some time) here's another way to look at this exercise:

- Which quadrant does $\sum_{j=0}^{N} e(jx+j^2y)$ live in? as a function of x and y ? Find the entropy of the corresponding partition of $[0,1]^2$ into those 4 sets.
- When does this function exhibit lots of "cancellation" (if we can define such a thing) ?
- Why is the 6th power important?

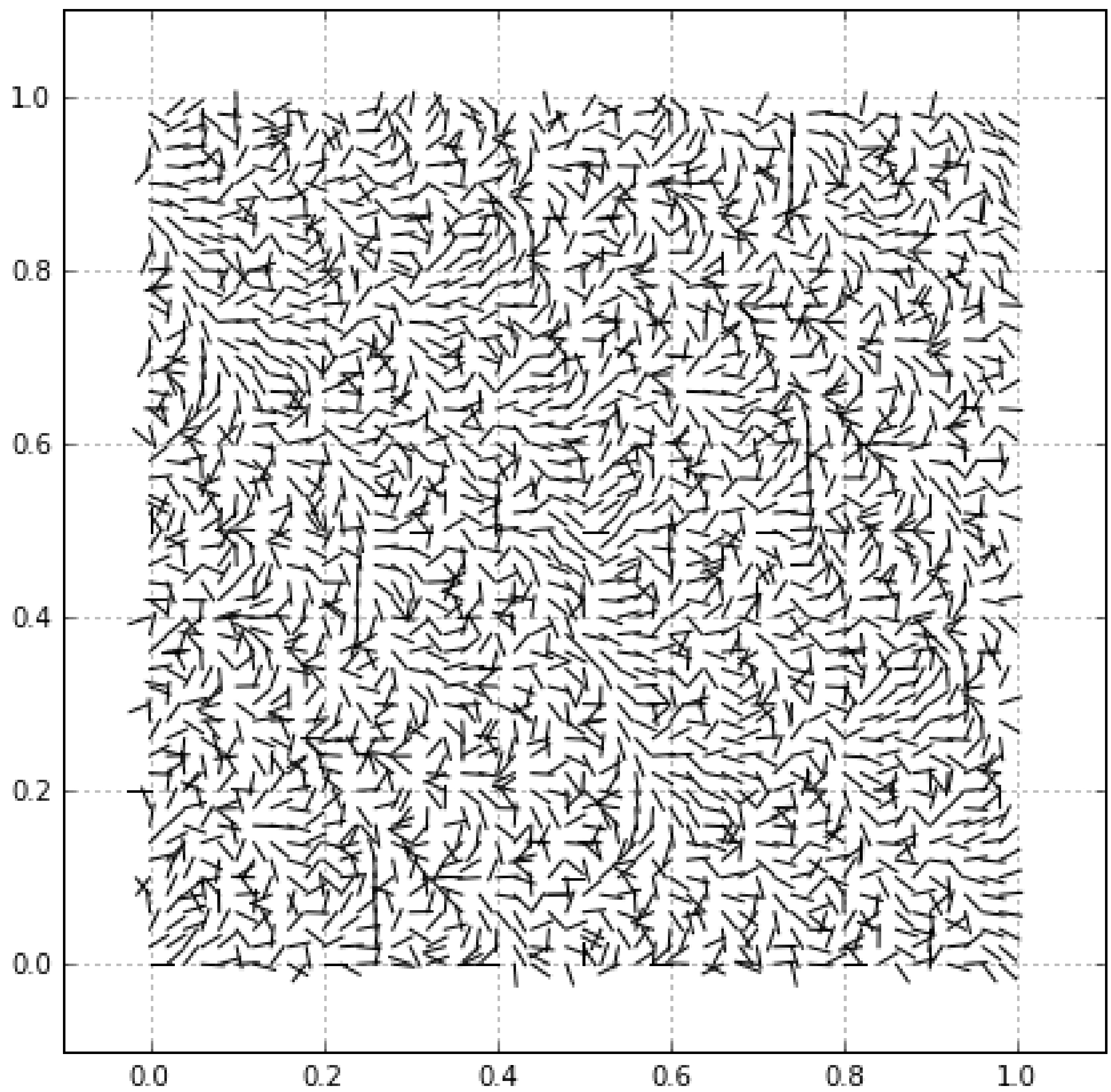
Math is the luxurious world were we can "evaluate" every function f to arbitrary accuracy at every "point" x or y. If I am skeptical of Tao's framework, or Bourgain's own devices, I can derive my own way of understanding this. Very simply: we can't use decimals anymore is there a better language to describe this cacncellation?[1]

Like good librarians we have compiled all of our resources. Now it's time to do some math!

The first thing I do is draw a chart. While a single chart does not "prove" a theorem - in fact they can be very misleading - they can provide intuition. What is "proof" anyway? The public can be convinced of something completely false by a single chart, rather than be presented with the full argument.

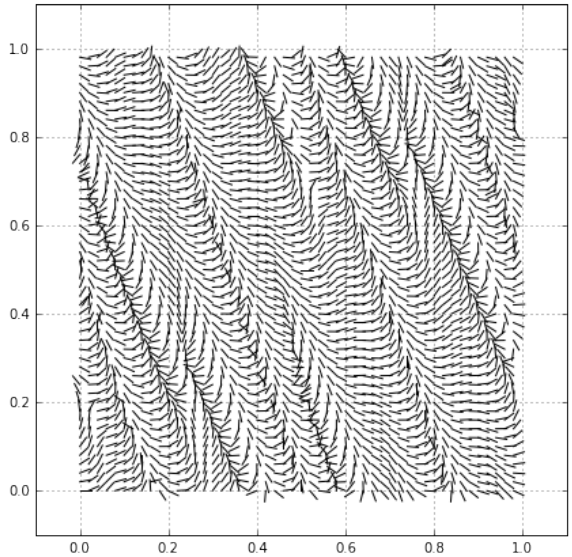
Here "proof" will mean there is a community of people who call themselves **mathematicians** who are more convinced of the verification of a certain class of statements after the discussion than before.

If I set $dx = dy = 0.1$ and set $N = 10$ I can draw the chart.



[1] One that's more concrete than what is being provided?

For comparison the case for $N = 4$ is relatively smooth. And we are letting $N \to \infty$ so the chaotic picture seems more accurate. It means that no matter how finely we place the grid, $\epsilon\mathbb{Z}^2$ our function oscillats so wildly as to simply tear apart at the fabric of space-time.



I will (eventually) argue that $\mathbb{R}^2$ is wild enough to handle any such partition if we write our grid in a different, rather unique way, that is suited to the problem that I have not discovered yet.

Secondly I claim that these charts are **new!** and that nobody in the world has visualized these functions before in quite this manner, at least in regards to the Vinogradov Mean Value conjecture. None of the argument we'll read will use the visual inuition that I am calling for.

References

(1) Lillian B. Pierce **The Vinogradov Mean Value Theorem** (after Wooley, and Bourgain, Demeter and Guth) `arXiv:1707.00119`

(2) Terence Tao
Decoupling and the Bourgain-Demeter-Guth proof of the Vinogradov main conjecture `https://terrytao.wordpress.com/2015/12/10/decoupling-and-the-bourgain-demeter-guth-proof-of-`
The two-dimensional case of the Bourgain-Demeter-Guth proof of the Vinogradov main conjecture `https://terrytao.wordpress.com/2015/12/11/the-two-dimensional-case-of-the-bourga`

(3) Jean Bourgain, Ciprian Demeter. **A study guide for the l^2 Decoupling Theorem** `arXiv:1604.06032`