An Inversion Problem

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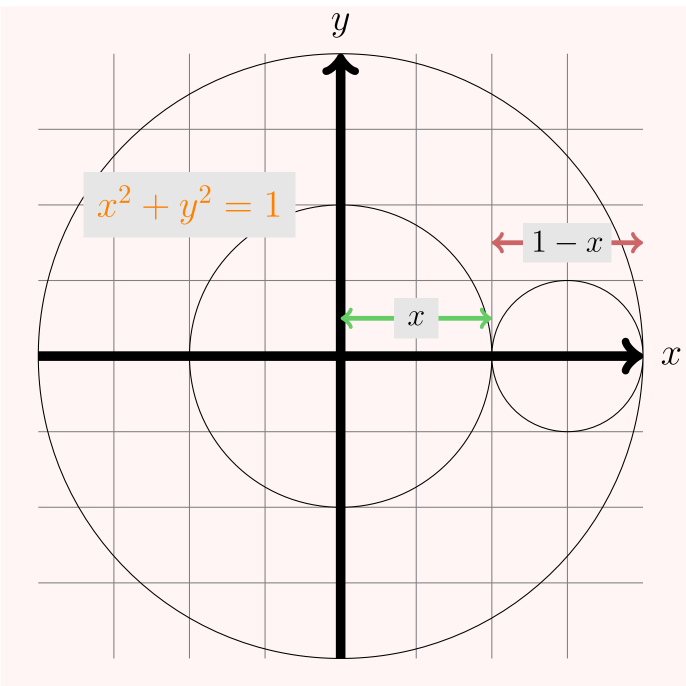
I ask around for a solution to an inversion problem. Everyone could show me **how** to solve it but nobody wanted put the solution[1]



Let $a = \frac{1}{3}$. I would like the image of these circles under the map:

$$z \mapsto \frac{z - a}{\overline{a}z - 1}$$

[1] I didn't ask "how would you solve it" – I was asking for an explicity answer, with a center and a radius. Nobody wanted to. If you do it neatly takes about a page (or less). If you don't know algebra it takes 10 pages and you get nowhere. This was a skill in textbook in 19th century – and in fact all of my resources come from that time period.

A - the Easy Way

This particular layout of circle is symmetric about the x axis — so we might[2] find an easier solution!

The image of $x^2 + y^2 = \frac{1}{4}$ is itself a circle, symmetric about the x axis.

- $z = \frac{1}{2} \mapsto w = \frac{\frac{1}{2}-a}{\frac{1}{2}\overline{a}-1}$ and $z = -\frac{1}{2} \mapsto w = \frac{-\frac{1}{2}-a}{-\frac{1}{2}\overline{a}-1}$
- The center should be midway between them:
$$\frac{1}{2}\left(\frac{\frac{1}{2}-a}{\frac{1}{2}a-1} + \frac{-\frac{1}{2}-a}{-\frac{1}{2}a-1}\right) = \frac{3}{4} \times \frac{a}{1-\frac{1}{4}a^2}$$
- and the radius should be half the difference
$$\frac{1}{2}\left(\frac{\frac{1}{2}-a}{\frac{1}{2}a-1} - \frac{-\frac{1}{2}-a}{-\frac{1}{2}a-1}\right) = \frac{1}{2} \times \frac{-1+a^2}{1-\frac{1}{4}a^2}$$

The circle $(x-\frac{3}{4})^2 + y^2 = \frac{1}{4^2}$ has diameter $z = \frac{1}{2}$ and $z = 1$:

- $z = 1 \mapsto w = \frac{1-a}{a-1} = -1$ and easy computation:
$$R = \frac{1}{2}\left(\frac{\frac{1}{2}-a}{\frac{1}{2}a-1} + 1\right) \qquad C = \frac{1}{2}\left(\frac{\frac{1}{2}-a}{\frac{1}{2}a-1} - 1\right)$$

[2]The small circle $|z-3| = \frac{1}{4}$ is moving in between $|z| = 1$ and the image of $|z| = \frac{1}{2}$. What happens (under inversion) if I rotate this figure? My question is what the image of the circle is under the map $z \mapsto \frac{z-a}{\overline{a}z-1}$ and also $a = e^{i\theta}\frac{1}{3}$ and $\theta \in [0, 2\pi]$.

B - from Old Textbooks

If p and q are inverse points on a Circle[3] that circle takes the form:

$$\frac{|z-p|}{|z-q|} = k$$

and the map $z \mapsto f(z) = \frac{az+b}{cz+d}$ maps perfectly:

$$\frac{|z-f(p)|}{|z-f(q)|} = k$$

This should feel awkard how to express the simple eq:

$$|z| = \frac{1}{2}$$

which says that $0 \leftrightarrow \infty$ are inverses. Instead try:

$$\frac{|z-\frac{1}{4}|}{|z-1|} = \frac{1}{2}$$

and for the other circle $|z-\frac{3}{4}| = \frac{1}{2}$ a new formula:

$$\frac{|z-\frac{5}{8}|}{|z-0|} = \frac{3}{8}$$

and hopefully our images agree.

[3]This could be Ptolemy's Theorem since we are discussing power of a point. I look at Titchmarsh's textbook on analysis and feel two ways: why are physicists skipping important steps even when they are very doable? Why is the geometry limited to only one chapter? Why not take a geometric approach to Hadamard's theorem or Lebesgue Theory?

C - nnnooooooooooo!!!!

we just want a bit of detailed balanace making sure the algebra checks out.

just because it is equation you think it is correct?

LOL

D - LOL

References

(1) Curtis McMullen. **Uniformly Diophantine Fixed Numbers in a Real Quadratic Field**

(2) Jean Bourgain, Alex Kontorovich. **Beyond Expansion II: Traces of Thin Semigroups** arXiv:1310.7190v1