Examples: $n!$ the Factorial

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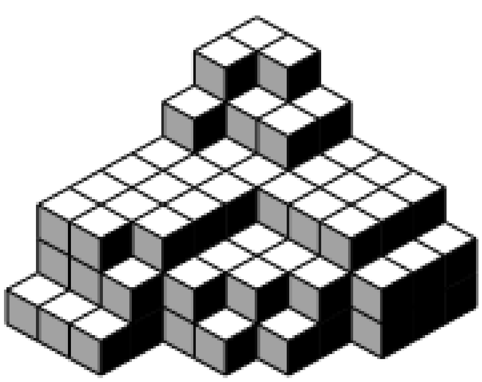
There are famous McMahon formulas and I could be mixing them up. Here is one, which counts literally *all* plane paritions:

$$\sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^n$$

My apology in advance for not having a good picture. They are take some work to draw. Here we take example from Mirjana Vuletic.



This is good but not really what I am looking for today.

I am looking for those partitions which fit inside an $a \times b \times c$ box. There is an exact number on Wikipedia:

$$\#\{boxes\} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$

and you might wonder why so much attention might be drawn to a simple equation like this:

- Inside Mathematics - literally thousands of papers and they keep coming
- Outside Mathematics - maybe a statistician or data analyst might find this structure relates to something in the real world[1]

Today we will do something totally useless and set $a = b = c = \frac{1}{2}$. How many ways to pack $1 \times 1 \times 1$ cubes into a box of side $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$?

The procedure for guessing a value of $(\frac{1}{2})!$ might stem from Euler's definition of factorial:

$$x! = \lim_{n\to\infty} \frac{n^x\, n!}{x \times (x+1) \times (x+2) \times \ldots (x+n)}$$

[1]Baseball, elections, the human genome, the weather, instagram, the radio... and the end of the day these are all tables of numbers and there exist procedures where these can processed in a similar ways.

Let's try to shift the product lattice by (x,y,z):

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{(i+j+k)+(x+y+z)-1}{(i+j+k)+(x+y+z)-2}$$

I don't even know what this number is. OK. Let

$$n!! = 0! \times 1! \times \cdots \times (n-1)!$$

and we can even define the q-factorial

$$[n]_q! := \prod_{i=1}^{n}\frac{1-q^n}{1-q}$$

The hyperfactorial (and the Barnes G-function)

$$\sum_{\pi} q^{|\pi|} = \frac{a!!_q b!!_q c!!_q (a+b+c)!!_q}{(a+b)!!_q (b+c)!!_q (c+a)!!_q}$$

and we can set $q = 1$ to get one McMahon formula. Or let $a,b,c \to \infty$ to get the other.[2]

And now we can set $a = b = c = \frac{1}{2}$, so that $(\frac{1}{2})! = \sqrt{\pi}$ and $(\frac{1}{2})!! = \ldots$ Euler never wrote a formula for the Barnes G-function.

Oh my this got complicated. Have an answer in a bit.

[2]Everything fits together in McMahon world. Everything is beautiful!

References

(1) Mirjana Vuletic **A generalization of MacMahon's formula** `arXiv:0707.0532`

(2) Paul Zinn-Justin **Six-Vertex, Loop and Tiling models: Integrability and Combinatorics** `arXiv:0901.0665`