Scratchwork: Lagrange Sum of Four Squares Theorem

Lagrange's equation says we can solve $x^2 + y^2 + z^2 + w^2 = n$ for any $n \in \mathbb{Z}$. Hasse's principle, says as a quadratic form we need only solve this equation in $\mathbb{Z}/p\mathbb{Z}$ for each prime p. This requires an infinite amount of work, since how do we find all roughly $10^6/6\log 10 \approx 70,000$ primes between $1 \times 10^6 < p < 2 \times 10^6$? The limiting space if we do all possible modular aritemtics is

$$\{x^2 + y^2 + z^2 + w^2 = n\}(X) \text{ with} X = \mathbb{R} \text{ or } \mathbb{Q} \text{ or } \mathbb{A}$$

Exact numbers can occur when we count rearrangements of things we've alread decided are separate and countable. Such as:

$$\binom{n}{2} = \big\{(m,n) : 0 \leq a < b < n\big\} = \frac{n \times (n-1)}{2} \in \mathbb{Z}$$

The left side is an integer because it always counts the number of things, but how did we know the right side counted anything?

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 &\in n + 7\mathbb{Z} \\ x^2 + y^2 + z^2 + w^2 &\in n + 13\mathbb{Z} \end{aligned}$$

These are two open sets in $\mathbb{Z}_7$ and $\mathbb{Z}_{13}$. A sample solution of both congruences in $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z}$ is

$$(0,0,0,1) \oplus (0,1,1,0) = (0,14,14,78)$$

I have not yet exploted such obvious things as $x^2 > 0$ always because now we are playing with p-adic integers. Here we have an isomorphism:

$$\mathbb{Z}/13\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \simeq \mathbb{Z}/91\mathbb{Z}$$

Called the Chinese remainder theorem.

References

[1] . . .