Scratchwork: Basic Theorems of Lebesgue Integration

9/20 Why do we need Riemann integration or even Lebesgue integration. Here's an integral formula:

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

In calculus class we just memorize a rule: $\int : x^n \to \frac{1}{n+1}x^{n+1}$ and apply the formula. Here's one more:

$$\int_0^x \cos x\,dx = \sin x$$

So there is a second rule $\int : \cos x \mapsto \sin x$. Once we buy into these one or two or a dozen rules, we can corner ourselves very quickly. **Ex**: Show that

$$\int_0^{2\pi} f(x)\,\big(\sin Nx\big)^2\,dx \to \pi f(x)$$

The formula looks right, we have that $0 < \sin^2 x < 1$ and it oscillates fairly evenly so the average should be $\frac{1}{2}$.

Riemann integration was already a formality, Lebesgue integration was an even bigger formality. Here's a common-sense looking theorem that requires Lebesgue integration.

Problem: Exchange integration and summation. Let f_k be a sequence of L^1 integrable functions such that $\sum_1^\infty |f_j| < \infty$. Then $\sum f_k$ converges (almost everywhere) to a function in L^1 and

$$\int \sum_{i=1}^\infty f_k(x)\,dx = \sum_{i=1}^\infty \int f_k(x)\,dx$$

If the function sequences look forbidding, let $f_n(x) = a_n\,\sin nx$ with $0 \le x \le 1$. We want to know if

$$\int \sum_{i=1}^\infty a_n\,\sin(2\pi n x)\,dx = \sum_{i=1}^\infty \frac{a_n}{n}\,\cos(2\pi n x)$$

There's even an easy choice $a_n = \frac{1}{\pi k}$. We obtain the sawtooth wave:

$$\sum_{m=1}^\infty \frac{\sin 2\pi m x}{\pi k} = \frac{1}{2} - \{x\}$$

It's the only one we ever study. Despite it's usefulness in music, I'm mostly concered about it's arithmetic and geometric properties. This "=" sign is really shaky too. At $x = \pi$

$$\sum_{m=1}^\infty \frac{0 \pm \epsilon}{\pi k} \approx 0 \pm \left(\frac{1}{2} - \epsilon\right)$$

Fortunately for us, nature (and certainly Mathemtical Physics) will offer us examples of natural processes that require Lebesgue theory to understand.

Problem When do limits and integrals converge?

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$$

If we look on the previous page of the Lebesgue theory textbook, we get conditions when this common-sense formula might work:

- f_n is a sequence of L^1 functins (Lebesgue integrable)[1] with $f_n \to f$ almost everywhere.
- There's a (non-negative) function $g \geq 0$ such that $|f_n| \leq g$ (almost everywhere).

Certainly the authors of these textbooks have lost their minds, and they are writing such a textbook for their health.

Thm with these conditions, the limit $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

If we ever want this common-sense result to whole we need to more results from Lebesgue integration theory:

- Dominated Convergence Theorem
- Fatou's Lemma
- Monotone Convergence Theorem

Lebesgue measure is impossible to construct. Half of probability (e.g. the Law of Large Numbers) is showing that a measure either converges to Lebesgue measure or to a point (or to a Gaussian centered at that point).

Example Here's a set that requires measure theory to even try to estimate the size of. Let $\alpha = \sqrt[4]{2}$ and consider the set $\{(m,n) : 0 < m^2 + \sqrt[4]{2}\,n^2 < X\}$. Here is a measure we could study:

$$\mu(x) = \sum 1_{m^2 + \sqrt[4]{2}\,n^2}(x)$$

There were two possible ways to write this set. Any major differences?

- $\{m^2 + \sqrt[4]{2}\,n^2 : 0 \leq m, n \leq N\}$
- $\{m^2 + \sqrt[4]{2}\,n^2 : 0 \leq m \leq M \; 0 \leq n \leq N\}$
- $\{(m,n) : 0 < m^2 + \sqrt[4]{2}\,n^2 < X\}$

The last set is a collection of pairs of integers (coordinates) and the first two sets are collections of real numbers.

Here's a function we might study with Lebesgue theory:

$$f(x) = \big|\{(m,n) : 0 < m^2 + \sqrt[4]{2}\,n^2 < X\}\big| - \frac{\pi}{4\sqrt[4]{2}} X \approx 0$$

we've used nothing but familiar household objects and the equations describe a natural thing.

References

[1] Gerald B. Folland **Real Analysis: Modern Techniques and their Applications**. Wiley, 1999.

[2] Valentin Blomer, Jean Bourgain, Maksym Radziwiłł, Zeev Rudnick **Small gaps in the spectrum of the rectangular billiard** `arXiv:1604.02413`

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[1] Did we ever compute a Lebesgue integral in our lives? So why we bother talking about Lebesgue integrable

We've been fiddling with the Prime Number Theorem. It starts to get really confusing. We are trying to estimate:

$$A(x) = \sum_{n<x} \Lambda(n) \quad \text{so that} \quad \alpha(s) = \int_1^\infty x^{-s}\, dA(x) = \sum_{n=0}^\infty \Lambda(n)\, n^{-s}$$

This is a Mellin transform. We are using the Riemann Stieltjes integral, since there are lots of jumps. $d[1_{x<0}] = \delta_0(x)$ e.g. using theory of distributions. The numbers are primes:

$$\Lambda(n) = \begin{cases} \log p & x = p^k \\ \\ 0 & \text{otherwise} \end{cases}$$

Originally we have an exponential series insted of a Dirichlet series and we are using a Laplace transform:

$$\alpha(s) = \int_0^\infty e^{-xs}\, da(x) = \sum_{n=0}^\infty \Lambda(n)\, e^{-ns} \text{ with } a(x) = A(e^x) = \sum_{n<e^x} \Lambda(n) \qquad (*)$$

The theoretical version deals with this version in the middle. We could even have power series, the most normal-looking:

$$a(x) = \sum_{n=0}^\infty \Lambda(n)\, x^n$$

The reason why the Van Mangoldt function was so important is because it was the GCD of the first n numbers:

$$e^{\sum_{n<x} \Lambda(n)} = \mathsf{LCM}(\{1, 2, \ldots, x\})$$

and Λ is the Möbius transform of the logarithm. We can turn one into the other by summing over the divisors:

$$\log n = \sum_{m|n} \Lambda(n) \text{ or } \log = \Lambda * 1 \quad \text{and} \quad \Lambda(n) = \sum_{m|n} \log(m) \text{ or } \Lambda = \log *1$$

Now we read the proof. We need a "copy" of the exponential function e^x (we're even afraid to call it that... $E(x)$) that is compactly supported in Foureir space. The exponent function is definitely not.

$$E(x) = \begin{cases} e^x & x < 0 \\ 0 & x > 0 \end{cases}$$

This function has a discontinuity at zero the right-limit $E(0^-) = 1$ and left-limit $E(0^+) = 0$. This function has a Fourier transform:

$$\hat{E}(t) = \int_{-\infty}^0 e^{x-tx}\, dx = \frac{1}{1-t}$$

The reason that we need a compactly-supported cousin of the expoenntial function the pole at $t = 1$.

- $\Delta_T(x) = T\left(\dfrac{\sin \pi T x}{\pi T x}\right)^2$ Fejer Kernel over $\mathbb{R}$.
- $J_T(x) = \dfrac{3T}{4}\left(\dfrac{\sin \pi \frac{T}{2} x}{\pi \frac{T}{2} x}\right)^4$ Jackson Kernel over $\mathbb{R}$

One way to think of all these function is that these things all approxiate boxes of area 1:

$$\Delta_T(x) \approx J_T(x) \approx 1_{|x|<\frac{1}{T}} \times T$$

These Kernels arise from doubts as to whether the Fourier series of a function converges to itself. No engineer in his right mind questions that, but we have a counter-example on the previous page.