Unipotent Flows

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The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **also known as** $(x, y) \mapsto (x + y, y)$

Ratner's Theorem says[1]

Let G be a Lie group, $\Gamma < G$ be a discrete subgroup, and $H < G$ be a subgroup isomorphic to $\mathrm{SL}(2, \mathbb{R})$.

Then any H-invariant and ergodic probability measure μ on $X = \Gamma \backslash G$ is homogeneous.

i.e. there is

- a closed connected subgroup $L < G$ containing H such that μ is L-invariant and
- some $x_0 \in X$ such the L-orbit $x_0 L$ is closed and supports μ

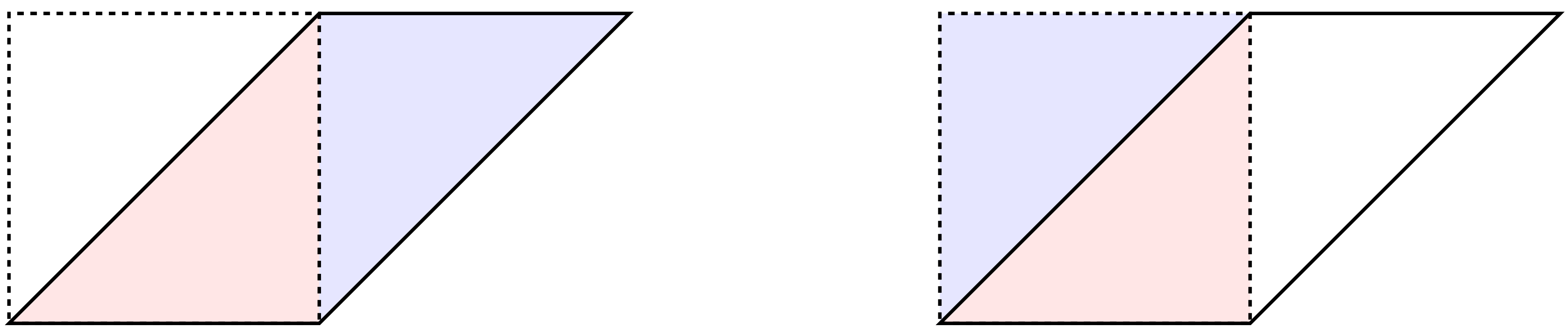
That means μ is an L-invariant volume measure on $x_0 L$

[1] Somtimes in a very difficult theorem we start of great with a clear discussion with lots of illustration. Then after some point we decide everything is "technical" and go for dozens of pages which basically can be omitted. This statement is due to Manfred Einsiedler, whose exposition is a simplification of a simplification of a simplification... The work is endless.

Why Learn Ratner's Theorem?

There are integers (a, b, c) such that $|a^2 + b^2 - \sqrt{2}c^2| < \epsilon$
Why can this have arbitrarily small numbers? This is proven by Dani and Margulis[2]

Ratner's Theorem on the equidistribution of the horocycle flow seems to do with the shear of a rhombus.



This in turn has to do with Euclid's Elements. The shear preserves area, these two rectangles have equal area. Just take a pair of scissors and cut.

[2] Although I find a closed proof of Bourgain of certain cases in 2016

Our goal is to try to express some of these “entropy” considerations in simple language. And I need to re-write all of these statements since I am not an expert on Ratner Theorem.

Instances of $(x,y) \to (x+y,y)$ in number theory

In one paper I found an isomorphism from PGL_2 to SO_3:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ad-bc)^{-1} \begin{bmatrix} ad+bc & i(ac+bd) & bd-ac \\ -i(ab+cd) & \frac{a^2+b^2+c^2+d^2}{2} & i\frac{a^2-b^2+c^2-d^2}{2} \\ -(ab-cd) & i\frac{-a^2-b^2+c^2+d^2}{2} & \frac{a^2-b^2-c^2+d^2}{2} \end{bmatrix}$$

Over which field is this an isomorphism? Certainly not $\mathrm{PGL}_2(\mathbb{R}) \not\simeq \mathrm{SO}_3(\mathbb{R})$. Instead[3] we could try the p-adic numbers $\mathbb{Q}_p \overset{\lim}{=\!\!\longrightarrow} \mathbb{Q}[\frac{1}{p^k}]$ (did I type that correctly?) What if I plug in $a=b=d=1$ and $c=0$?

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & i & 1 \\ -i & \frac{3}{2} & -\frac{i}{2} \\ -1 & \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

$\sqrt{-1} \in \mathbb{Q}_5$ and $\sqrt{-1} \notin \mathbb{Q}_3$ but there is still an isomorphism.

[3]The source I am reading has k to be a field of characteristic 0 and k^s is a fixed separable closer. So if $k = \mathbb{Q}$ the fractions, and $k^s \subset \overline{\mathbb{Q}}$ are the separable elements of the algebraic closure. This is a humungous field instead I pick anything with $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}[x,y]/(x^2-2, y^2+1)$

Instances of $(x, y) \to (x + y, y)$ in number theory

This wedge product notation is rather modern, instead of determinants:

$$\xi \mapsto 1 \wedge \xi \wedge \xi^2 : C/A \to \wedge^3 C$$

This is a map from a cubic ring modulo a principal ideal domain. If

$$\phi(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

with the $\mathrm{GL}_2(A)$ action:

$$\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}, \phi \right) (x, y) = \frac{1}{ps - qr} \phi(px + ry, qx + sy)$$

so if I plug in $p = q = s = 1$ and $r = 0$:

$$a(x + ty)^3 + b(x + ty)^2 y + c(x + ty)y^2 + dy^3$$

is a unipotent flow on the space of cubic forms.

Mixing of $(x, y) \to (x + y, y)$ in Number Theory

Both flows I have shown on the previous page are *mixing*. How mixing can adding $x + ty$ be? How do we quantify the mixiness of adding $+1$?

References

(1) Evan M. O'Dorney **A remarkable identity in class numbers of cubic rings.** arXiv:1608.00166

(2) Jack A. Thorne **Arithmetic invariant theory and 2-descent for plane quartic curves** arXiv:1607.08816

(3) Manjul Bhargava **Higher composition laws II: On cubic analogues of Gauss composition** http://annals.math.princeton.edu/2004/159-2/p09

(4) Manjul Bhargava **The density of discriminants of quintic rings and fields** arXiv:1005.5578