Examples: Pythagorean Theorem

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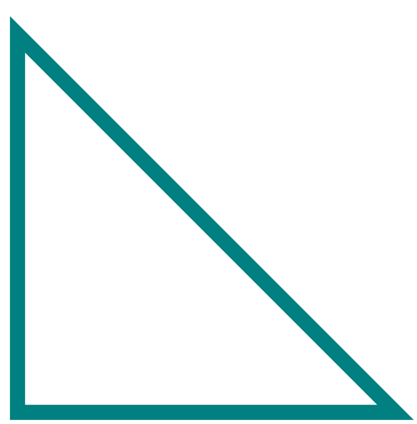
At this stage of the game, I have felt a need to review the Pythagorean theorem.

$$a^2 + b^2 = c^2$$

Sadly I can't think of any really good examples, but we do get our first irrational number:

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

This is the hypoteneus lenght of an isosceles right triangle:



and we can expend some effort showing that $\sqrt{2} \neq \frac{p}{q}$ for some intgers $p, q \in \mathbb{Z}$.

For something more contemporary, Alex Kontrovich shows us that all Pythagorean triples can be written:

$$x = u^2 - v^2,\ y = 2uv\ z = u^2 + v^2$$

and shows us a homomorphism $\mathsf{SL}_2(\mathbb{R}) \to \mathsf{SO}(2,1)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{a^2-b^2-c^2+d^2}{2} & ac-bd & \frac{a^2-b^2+c^2-d^2}{2} \\ ab-cd & bc+ad & ab+cd \\ \frac{a^2+b^2-c^2-d^2}{2} & ac+bd & \frac{a^2+b^2+c^2+d^2}{2} \end{pmatrix}$$

Several things jump out at me about this formula:

- We get all possible quadratic forms in a, b, c, d that are "invariant" or at least transform nicely[1] such as $a^2 - b^2 - c^2 + d^2$
- $SO(2,1)$ preserves $x^2 + y^2 - z^2 = 0$ which is the "light cone" in special relativity, and also the hyperboloid[2] of one sheet $x^2+y^2-z^2 = 1$.
- Since $\mathsf{SL}(2,\mathbb{Z})$ is generated by $z \mapsto z+1$ and $z \mapsto -\frac{1}{z}$, the Pythagorean triples form a "tree"
- It may be reasonable[3] to examine $\mathsf{SL}(\mathbb{Z}[i])$ or the equation $x^2 + y^2 - \sqrt{2}z^2 < 10^{-6}$.

[1] In mathematics "good" means "I just made up a word. Figure it out for yourself."

[2] These days one might consider $x^2 + y^2 - z^2 - w^2 = 1$. Especially if you are Juan Maldacena.

[3] How about something unreasonable: $\mathrm{GL}_2(\mathbf{A})$ if you know what **adèles** are.

Kontrovich's question (or possibly Hee Oh's) is how many factors in :

$$\frac{1}{2}xy = \frac{A}{6} = \frac{1}{6}(u+v)(u-v)\,uv$$

When does this Area have at ≤ 4 factors[4] ?

Just a reminder why quadratic quations are everywhere:

$$\begin{aligned} f(x+t) &= e^{t\,\frac{d}{dx}} f(x) \\ &= \left[1 + t\,\frac{d}{dx} + \frac{t^2}{2!}\frac{d^2}{dx^2} + \frac{t^3}{3!}\frac{d^3}{dx^3} + \dots\right] f(x) \\ &= f(x) + t f'(x) + (t^2/2) f''(x) + \dots \end{aligned}$$

If it happens that $f'(x) = 0$ then $\deg f \approx 2$:

$$f(x+t) = f(x) + \tfrac{1}{2} f''(x)\, t^2 + \dots$$

By the **Fundamental Theorem of Algebra** there should be ≈ 2 roots $f(x) = 0$ with $x \in \mathbb{C}$.

$$f(x+s, y+t) \approx f(x,y) + \frac{1}{2}\left[s^2\,\frac{\partial^2 f}{\partial x^2} + 2st\,\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + t^2\,\frac{\partial^2 f}{\partial y^2}\right]$$

In two dimensions.

[4]This is as close as we can get to prime. I am not going to give you an answer. How might we use such a Pythagorean triple? The proofs use the most difficult techniques in mathematics. I am curious what you can obtain with less. . .

How about some more exotic examples of the quadratic equation?

Here is an estimate of the Laplacian I have always liked:

$$\Delta \approx \frac{1}{h^2(w+2)} \begin{bmatrix} 1 & w & 1 \\ w & -4(w+1) & w \\ 1 & w & 1 \end{bmatrix}$$

This construction is connected to the Vernose map (or Veronese embedding[5]):

$$(u:v:w) \in \mathbb{P}^2 \mapsto (u^2 : v^2 : w^2 : uv : vw : uw \in \mathbb{P}^5$$

these are called Sobel masks. I lost my copy of **Robot Vision**. Here we recover:

$$\frac{1}{8h} \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} = \Delta + \frac{h^2}{12}\Delta^2 + O(h^4)$$

as $h \to 0$. The idea is to approximate a linear functional[6] as a sum of points:

$$\Delta f(\vec{v}) \approx \sum_{P \in \square} w_i \, f(\vec{v} + \vec{P})$$

That's the (truly terrible) approximation I always use[7].

[5] Recall that $5+1 = 2 \times (2+1)$.

[6] Not function, function*al*...

[7] These are refined versions of Taylor's Theorem. Or the mean value theorem $f(b) - f(a) \approx f'(c)(b-a)$ for $a < c < b$.

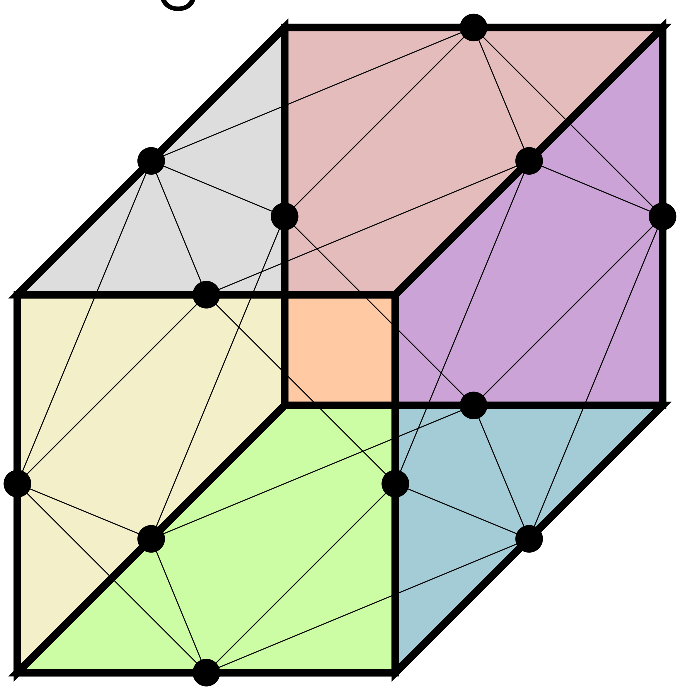
The Veronese map is used to lift approximations of ∇ to other higher-order operators. They are a special case of:

$$(x, y) = (\cos\theta, \sin\theta) \mapsto (x^2 - y^2, 2xy) = (\cos 2\theta, \sin 2\theta)$$

this is the **Harmonic** Veronese map. The same paper also show integral version:

$$\frac{1}{4\pi}\int_{S^2} F \cdot dS \approx \frac{1}{12}\sum_{P \in M} F(P)$$

where $M = (\pm 1, \pm 1, 0)$ or $(\pm 1, 0, \pm 1)$ or $(0, \pm 1, \pm 1)$ forming the vertices of a **snub cube**.



Janos Kollar gives us a bunch of exotic new Veronese embeddings.

For now we simply note this quadratic identity:

$$\frac{1}{4}(u^2+v^2+(a+b)w^2)^2-\frac{1}{4}(u^2-v^2+(a-b)w^2)^2 = (uv)^2+a(vu$$

and if $a > 0$ and $b > 0$ this has equivalent form:

$$\frac{1}{4}(|A|^2 + |B|^2)^2 - \frac{1}{4}(|A|^2 - |B|^2)^2 = |AB|^2$$

and $A = u + iw\sqrt{a}$ and $B = v + jw\sqrt{b}$ are elements of $\mathbb{H}$ (the Quaternions).

Here are some embeddings of $\mathbb{R}P^2$ into the sphere of radius 1:

- $(u : v : w) \mapsto$
  $\frac{\sqrt{3}}{u^2+v^2+w^2}\Big(uv : vw : uw : \frac{1}{2}(u^2-v^2) : \frac{1}{\sqrt{12}}(u^2+v^2-2w)\Big)$

- $(u : v : w) \mapsto$
  $\frac{2}{u^2+v^2+2w^2}\Big(uv : vw : uw : w^2 : \frac{1}{2}(u^2 - v^2)\Big)$

these are in the unit sphere in 5 dimensions

$$(u : v : w) \mapsto \begin{pmatrix} u^2 & uv & uw \\ uv & v^2 & vw \\ uw & vw & w^2 \end{pmatrix}$$

Leaving the circles, sheaves, ruled surfaces for another time...

Lastly we talk a little bit about wedge products. By coincicenc we have that:

$$\mathbb{R}^3 \wedge \mathbb{R}^3 \simeq \mathbb{R}^3$$

what does that mean? If we can count to three e_1, e_2, e_3 we can also count pairs of numbers (also up to three):

$$e_1 \wedge e_2, \quad e_2 \wedge e_3, \quad e_3 \wedge e_1$$

Pythagoras Theorem reads that:

$$||(a, b, c)||^2 = (a, b, c) \cdot (a, b, c) = a^2 + b^2 + c^2$$

Suppose we take the wedge of two vectors and find it's norm:

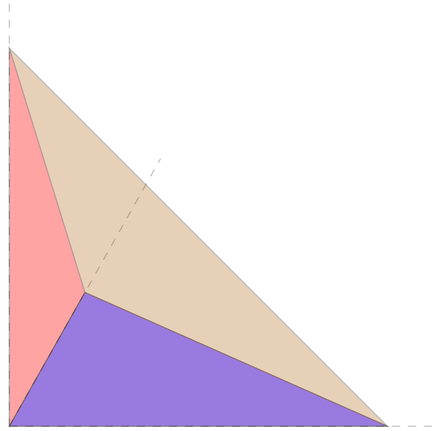
$$(a_1, b_1, c_1) \wedge (a_2, b_2, c_2) = (b_1 c_2 - b_2 c_1, c_1 a_2 - a_2 c_1, a_1 b_2 - b_2 a_1)$$

The area of the parallelogram[8] spanned by these two vectors should be the sum of squares:

$$A = (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - a_2 c_1)^2 + (a_1 b_2 - b_2 a_1)^2$$

These are the areas of the shadows of a 2D surface (a parallelogram) into the 3 coordinate axes, **which I must draw now**!

[8] quadrilateral?



Perspective drawing not so easy!

$$A^2_{\text{ABC}} = A^2_{\color{blue}{\text{ABO}}} + A^2_{\color{green}{\text{ACO}}} + A^2_{\color{red}{\text{BCO}}}$$

Finally since $a \wedge b = -(b \wedge a)$ we can try to arrange three wedges into an anti-symmetri matrix:

$$(a \wedge b) + (b \wedge c) + (c \wedge a) = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

now if we comine them we three plain-old vectors x, y, z notice we can do so meaningfully:

$$(x + y + z) + (a \wedge b + b \wedge c + c \wedge a) = \begin{bmatrix} x & a & b \\ -a & y & c \\ -b & -c & z \end{bmatrix}$$

I am slightly weary of my own calculation. In 4 dimensions we could have:

$E = e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k}$ yet $B = b_1\,\mathbf{j}\wedge\mathbf{k} + b_2\,\mathbf{k}\wedge\mathbf{i} + b_3\,\mathbf{i}\wedge\mathbf{j}$

and these two vectors fit in a 4×4 matrix:

$$(E, B) = \left[\begin{array}{cccc} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & b_1 & b_2 \\ -e_2 & -b_1 & 0 & b_3 \\ -e_3 & -b_2 & -b_3 & 0 \end{array} \right]$$

Semantically, what is the difference between **and** vs **yet** ?

More importantly we embed a space and it's wedge:

$$\mathbb{R}^3 \oplus \left(\mathbb{R}^3 \wedge \mathbb{R}^3\right) \subseteq \mathbb{R}^{4\times 4}$$

References

(1) Terence Tao. **Determinantal Processes** `https://terrytao.wordpress.com/2009/08/23/determinantal-processes/`

(2) Alex Kontrovich, Hee Oh **Almost prime Pythagorean triples in thin orbits** `arXiv:1001.0370`

(3) Alexander Belyaev, Boris Khesin, Serge Tabachnikov. **Discrete spherical means of directional derivatives and Veronese maps.** `arXiv:1106.3691`

(4) Janos Kollar **Quadratic solutions of quadratic forms** `arXiv:1607.01276`

(5) Wikipedia **De Gua's Theorem** and **Pythagorean Theorem** §**Generalizations**