Tune-Up: Holomorphic Functions

From the mathematician's point of view, Number Theory is so far along many of our questions should have thorough answers and still be quite accessible. From the academic point of view, these cases were complete decades ago. Nature might feel a bit differently, it's up to us to try them out.

Thm If $f \in \mathcal{S}(\mathbb{R})$, then $\sum\limits_{n \in \mathbb{Z}} f(x+n) = \sum\limits_{n \in \mathbb{Z}} \hat{f}(n)\, e^{-2\pi i n x}$.

In particular, setting $x = 0$ we have: $\sum\limits_{n \in \mathbb{Z}} f(n) = \sum\limits_{n \in \mathbb{Z}} \hat{f}(n)$.

Stein's book is somewhat stringent. The Schwartz space on $\mathbb{R}$ is the space of infinitely differentiable functions so that f and all it's derivatives are rapidly decreasing.

$$\sup_{x \in \mathbb{R}} |x|^k \, |f^\ell(x)| < \infty \text{ for every } k, \ell \geq 0$$

Example is the Gaussian function $f(x) = e^{-x^2}$. There should be a Fourier transform:

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$$

Foundation Hopefully we have some aggreement on the symbols $\frac{d}{dx}$ and $\int_{\mathbb{R}}$ or a choice of $f(x)$. How do we handle uncertainty in the choice of f?

Exercise (i) Both sides are continuous. We have that if $f \in \mathcal{S}(\mathbb{R})$ that

- show that $\sum\limits_{n \in \mathbb{Z}} f(x+n)$ is continuous.
- show that $\sum\limits_{n \in \mathbb{Z}} \hat{f}(n)\, e^{2\pi i\, nx}$ is continuous.

Exercise (i') Both sides are continuous. We have that if $f \in \mathcal{S}(\mathbb{R})$ that

- show that $\sum\limits_{n \in \mathbb{Z}} e^{-\pi(x+n)^2}$ is continuous.
- (optional) verify that $f(x) = e^{-\pi s x^2}$ and $\hat{f}(x) = \frac{1}{\sqrt{x}} e^{-\pi \xi^2/x}$. The Gaussian is it's own Fourier transform. In fact $\mathcal{F}^4 = I$.
- show that $\sum\limits_{n \in \mathbb{Z}} \dfrac{1}{\sqrt{x}}\, e^{-\pi n^2/x}\, e^{2\pi i\, nx}$ is continuous.
- Observe that $\sum\limits_{n \in \mathbb{Z}} e^{-\pi(x+n)^2} = \sum\limits_{n \in \mathbb{Z}} \dfrac{1}{\sqrt{x}}\, e^{-\pi n^2/x}\, e^{2\pi i\, nx}$ at least numerically.

Stein's book reviews the definition of "integral" only the definition of $\int_{[0,1]}$ and $\int_{\mathbb{R}}$ and why they are numbers. Even. . . what is "continuity"? What is $\mathbb{R}$? These are not discussed in calculus class. Contruction in $\mathbb{R}$ is done in the Analysis course.

The example in the book of Lighthill is:

$$1 + 2e^{-\pi} + 2e^{-4\pi} + 2e^{-9\pi} + \cdots = 1 + 0.0864278 + 0.0000070 + (10^{-12}) + \ldots$$

This is done at a time when there very few computers, and the estimates are to very high accuracy.[1] and everything else is left as "an exercise for the interested reader".

References

[1] Stein . . .

[2] Lighthill **Fourier Analysis and Generalized Functions** . . .

[3] Körner

[1]When do we need measurements of high accuracy?