Proposal: Oppenheim Conjecture

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On the one hand, the Oppenheim Conjecture is settled. For example, we can find $x, y, z \in \mathbb{Z}$ solving:

$$|x^2 + y^2 - \sqrt{3}z^2| < 10^{-k}$$

for any natural number $k \in \mathbb{N}$. This has been resolved by Grigori Margulis in the affirmative since 1988.

As a researcher, is there anything new to say here? The goal of this note is to state problems that are expected to be "open" - at least at the time of writing[1].

The most trivial density result I can think of is that rationals are dense in the reals:

$$\overline{\mathbb{Q}} = \mathbb{R}$$

This merely says that we can get arbitrarily close to any real number with a fraction.

This theorem doesn't tell us how much work it takes to ap-

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proximate a real number.

$$\left|\sqrt{2} - \frac{17}{12}\right| < \frac{1}{12 \times 12} = \frac{1}{144}$$

using small denominators we can get pretty good estimate.

Even a reasonable statement like showing fractions are equidistributed in the reals – that fractions are evenly space in the real numbers – has content.

$$0 < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{3}{4} < 1$$

Partly it's because the fractions are not perfectly evenly space, and we are claiming this behavior averages out for large denominators. For the Oppenheim conjecture I can make two objections.

#1 Margulis' proof is really complicated.

Marina Ratner's orbit closure theorem runs about 200 pages and plays a crucial role in the Oppenheim conjecture. In our case we would like to know if

- the orbit of $(1, 1, -\sqrt{3})$
- under the action of $SO(2, 1)$
- is dense in the space $SL(3, \mathbb{Z})\backslash SL(3, \mathbb{R})$

This orbit is spiraling around so much it actually hits the whole space... but I would like more details.

#2 We don't know how quickly the values of quadratic forms converge to all of $\mathbb{R}$

Jean Bourgain shows, in about 8 pages the largest number is not too big:

$$\min_{|x|<N \quad x \in \mathbb{Z}^3 \setminus \{0\}} |x^2 + y^2 - bz^2| \ll N^{-\frac{2}{5}+\epsilon}$$

In order to be correct this statment is full of legalese. This inequality will be true for **almost every** $b \in [\frac{1}{2}, 1]$.

A few things make me nervous about Bourgain's statement.

- The decay is slow: $10^{2/5} \approx 2.5$ and $100^{2/5} \approx 6.3$, which indicates how uneven these values of quadratic forms.
- We can't get rid of ϵ. It is wrong to say

  $$\min_{|x|<N} |Q(x)| < C \cdot N^{-2/5}$$

  is wrong. It's not totally wrong, and we could try to find the constant C and the exceptional values.
- Almost every b – this is an open invitation to find a quadrtic form where this inequality definitely **does not hold**.[2]

[2] One reason Bourgain's proof is so short is that he spreads his attention on many quadratic forms, telling us nothing about each individual case.

#3 Lastly, there are obvious generalizations to number fields.

$$x^2 + y^2 - \sqrt{2}z^2 \neq 0$$

definitely has a solution if we adjoin roots of unity:

$$z = \frac{\sqrt{x^2+y^2}}{\sqrt[4]{2}}$$

so there are solutions in a large enough ring, such as $\mathbb{Z}[\sqrt[4]{2}]$.

Thanks to work of Borel and Prasad we know the values of:

$$\left| x^2 + y^2 - i\sqrt{3}z^2 \right| \overset{?}{<} 10^{-k}$$

are dense in $\mathbb{C}$ if we let $x, y, z \in \mathbb{Z}[i]$. These numbers will also be dense in an ultrametric space such as:

$$\mathbb{Z}[i]_{1+i} = \lim_{\longleftarrow} \mathbb{Z}[i]/(1+i)^k\mathbb{Z}[i]$$

or we could define a hybrid metric such as

$$\mathbb{Z}[i]_\infty \times \mathbb{Z}[i]_{1+i} \times \mathbb{Z}[i]_3 \times \mathbb{Z}[i]_{1+2i} \times \mathbb{Z}[i]_7$$

results in these directions will certainly be new.

Maybe not profound, but certainly necessary in order to gain a basic understanding and slightly new. In fact, upon writing I am likely to be corrected on all of this.

References

(1) S.G. Dani **Diophantine approximation and dynamics of unipotent flows on homogeneous spaces** *(online)*

(2) Elon Lindenstrauss **Effective Estimates on Indefinite Ternary Forms**

(3) Jean Bourgain **A quantitative Oppenheim Theorem for generic diagonal quadratic forms** `arXiv:1604.02087`