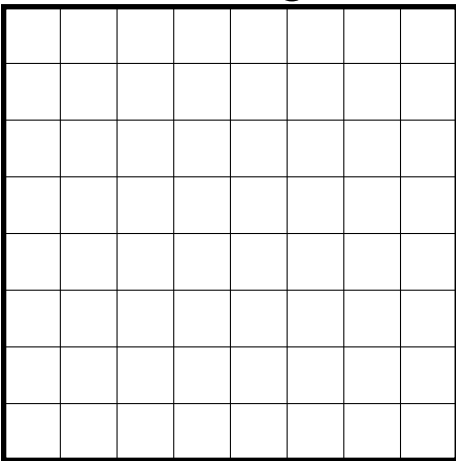


Proposal: Factorial

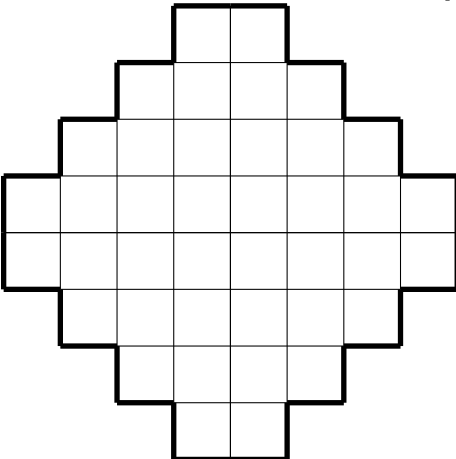
John D Mangual

Some clever person turned the theory domino tilings into a fundamental object of mathematics and of nature. For a long time there were really two shapes being studied.

The rectangle (here an 8×8 square):



And I wonder why this particular shape is so essential:



Pathetic Tutorial:

```

\begin{tikzpicture}[scale=0.75]
\foreach \a in {0,...,3}{
\draw[line width=2] (\a ,4-\a)--(\a+1, 4-\a );
\draw[line width=2] (\a+1,4-\a)--(\a+1, 4-\a-1);

\draw (\a ,4-\a)--(\a , \a-4);
\draw (-1*\a, 4-\a)--(-1*\a, \a-4);

\draw ( 4-\a, \a)--(\a-4, \a);
\draw ( 4-\a, -1*\a)--(\a-4, -1*\a);

\def \b {-1}
\def \c { 1}

\draw[line width=2] (\b*\a ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

\def \b { 1}
\def \c {-1}

\draw[line width=2] (\b*\a ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

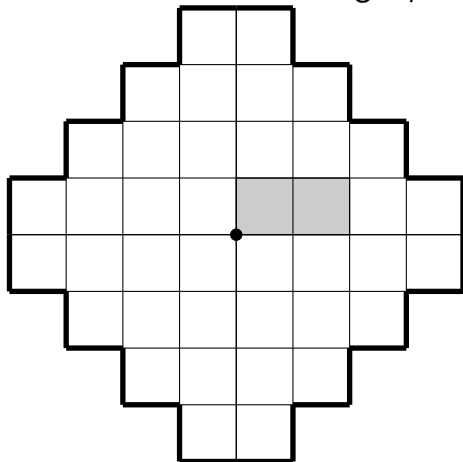
\def \b {-1}
\def \c {-1}

\draw[line width=2] (\b*\a ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

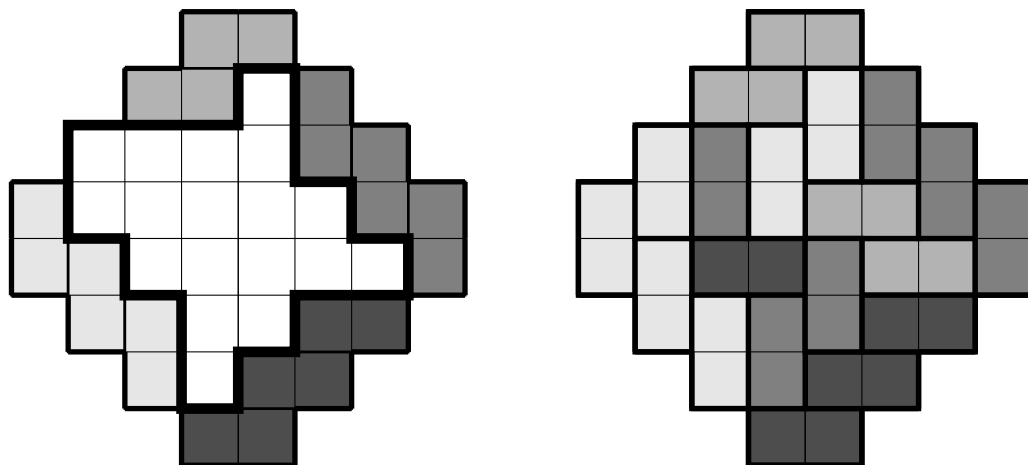
}
\end{tikzpicture}

```

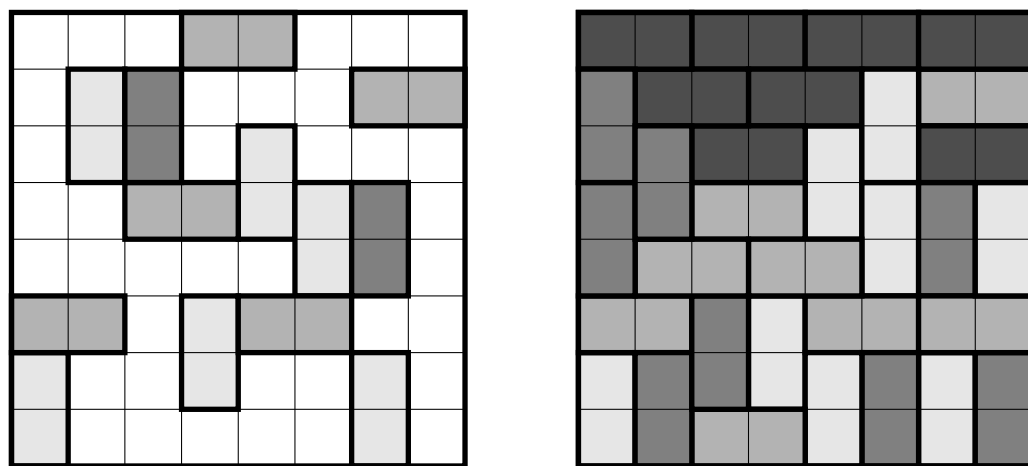
In French the word for tiling is *pavage* – so literally we are **paving** the shapes with dominoes.



Let's put two reasonable tilings on the board. One strategy for this shape is to start from the corners and work inwards. And in this case we get lucky: it always works.



And for rectangle the case is even clearer. There are no intermediate stages. I mean, if you put enough tiles down there can be some question as to whether you put yourself in a corner yet.



There is also the lovely John Conway game of "Domineering" which is not related but you also place dominoes on a checkerboard. See **Winning Ways for Your Mathematical Plays** (Vol I).

Exercise Fill the rest of the tiling.

Answer There is no solution, but if you move the tiles slightly (actually quite a bit here...) you can get a problem with an answer.

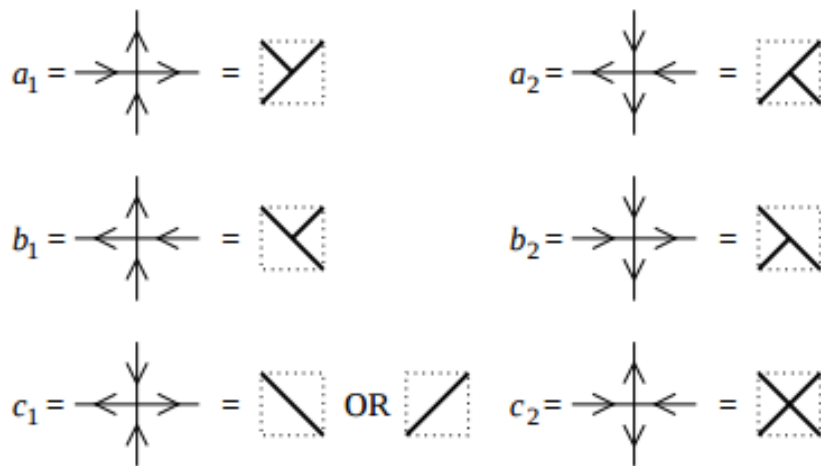
The number of tilings on the top looks atypical – just purely on a hunch. Where did that hunch come from? Is it right?

As with all hunches, it can be proven with hundreds of pages of equations this tiling is not likely to appear in a rectangles. We'll settle for somewhat simpler patterns.

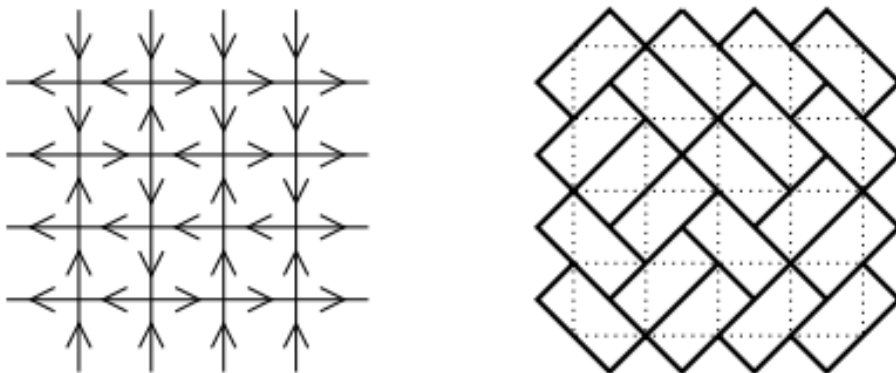
Next I have to address some redundancy. Every couple of years, it seems there were more and more people who came up with a similar theory and did not talk to each other.

- **domino tilings**
- dimer model
- perfect matchings
- **six-vertex model**
- Alternating Sign Matrix (**ASM**)
- XXZ spin chain
- Ising model

Domino tilings and six-vertex are closely related. You can see the family resemblance.



For every 6-vertex configuration there is a domino tiling, up to various factors of $\times 2$.



The Aztec Diamond is a natural shape because it **is** the square lattice.

Problem Session Let us compute some square-ice and domino tiling partition functions!

References

- (1) Paul Zinn-Justin **Six-Vertex Model with Domain Wall Boundary Conditions and One-Matrix Model** [arXiv:math-ph/0005008](#)
- (2) Noam Elkies, Greg Kuperberg, Michael Larsen, James Propp
Alternating-Sign Matrices and Domino Tilings (Part I)
Journal of Algebraic Combinatorics (1992) 1: 111. doi:10.1023/A:1022420103267
- (3) Patrick Ferrari, Herbert Spohn **Domino tilings and the six-vertex model at its free fermion point** [arXiv:cond-mat/0605406](#)
- (4) Sunil Chhita, Kurt Johansson **Domino statistics of the two-periodic Aztec diamond**
[arXiv:1410.2385](#)

Big Idea

The link between the domino tilings and the Ising model is “well known” but I can never find out all the details I want. And if I ask the question, it doesn’t have any merit.

Who are we to believe?

There are actually *two* Ising models. The statistical Ising model and the Ising Conformal Field Theories.

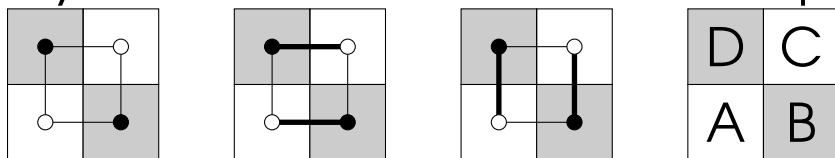
The bad news is, statistical models – or maybe, **lattice gauge theories**. Here’s a simple question: **are domino tilings an instance of lattice gauge theory?** The answer is likely “yes” but all of the responses are wanting.

The domino tilings like I have been showing you? Which lattice gauge theory was that? Domino tilings is very simple, lattice gauge theory rather involved.

+	+	+	+	+	+	+	+	+	+
+	-	+	-	-	-	-	-	-	+
+	-	-	+	+	+	+	+	-	+
+	+	-	+	-	-	-	-	+	+
+	+	+	+	+	+	+	+	+	+

The Ising model is even more simple-minded than the domino tilings. How could it be so advanced?

One way to count the domino tilings is through the Kasteleyn determinant. In the 2×2 square, there are 4 vertices:



There should be two tilings. Let's make sure Kasteleyn's method returns the answer **2**. He would draw a matrix:

$$Pf \left[\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 0 & 1 \\ B & 1 & 0 & 1 & 0 \\ C & 0 & 1 & 0 & 1 \\ D & 1 & 0 & 1 & 0 \end{array} \right] \stackrel{?}{=} Det \left[\begin{array}{c|cc} & C & D \\ \hline A & 1 & 1 \\ B & 1 & 1 \end{array} \right]$$

Since A is next to B, the entry A-B is 1. B and D are not next to each other. And then we compute not the Determinant, but the Pfaffian.

Hopefully the value of the Pfaffian is 2. The value of that determinant is 0. I guess I have to put a minus sign.

$$Pf \left[\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 0 & 1 \\ B & 1 & 0 & 1 & 0 \\ C & 0 & 1 & 0 & 1 \\ D & 1 & 0 & 1 & 0 \end{array} \right] = Det \left[\begin{array}{c|cc} & C & D \\ \hline A & 1 & -1 \\ B & 1 & 1 \end{array} \right] = 2$$

We are just making it up as we go along and this is the Kasteleyn orientation.

Without knowing why, this is just what we have to do in order to make sense, and this is called an "anomaly".

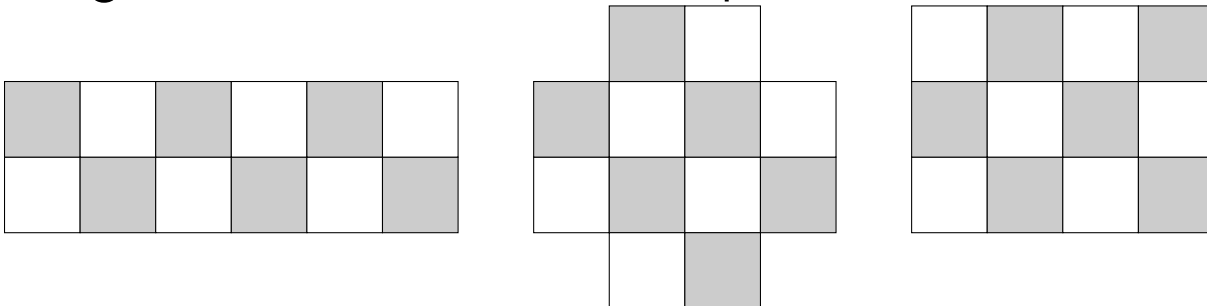
There should be ways to “glue” the patches together and the number of domino tilings will behave in an organized way.

As for all this Ising model stuff. As long as we know the Ising model and the domino tilings are the same theory.

The Ising model has varied literature¹ - all over the place:

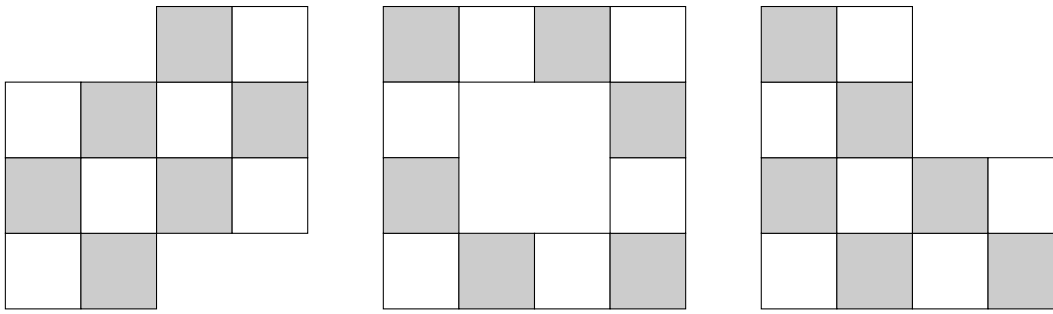
- statistical mechanics
- integrable systems
- TQFT
- lattice gauge theory
- Chern Simons Theory

and yet we only except one tractable model, the domino tilings. We need more examples:



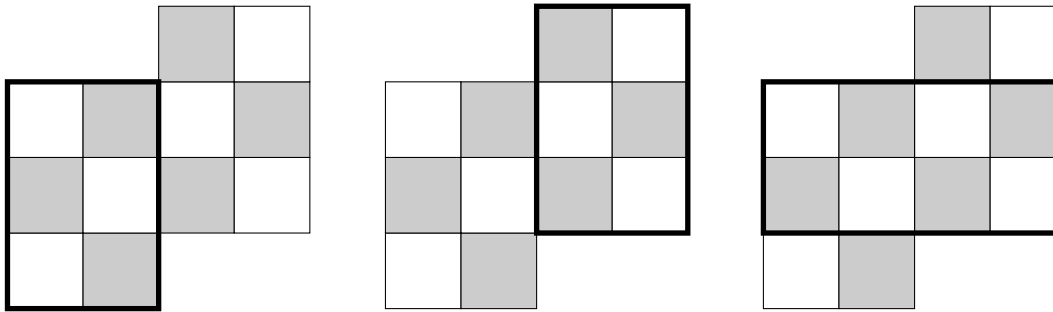
Can we glue partial answers together into a big picture? You can. In a very complicated way.

¹The dilemma these are 4 very complicated topics, none of which I know very well, and we are trying to make a comparison. Namely, the Ising model literature is all over the place, there should be variants of the domino tilings as well, where the answers still make a lot of sense.



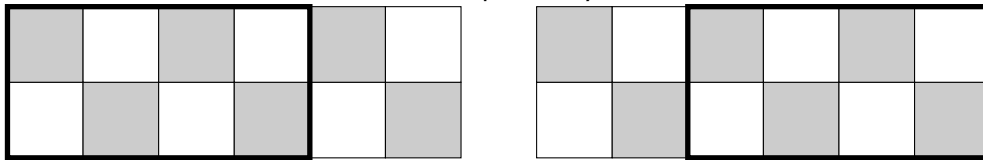
An aside for the grown-ups. The Kasteleyn determinants are put into a computer and with some effort the correct number will be returned. In a sense, we have already used a tiny bit of cohomology theory.

- All of these shapes are homeomorphic ("the same as") \mathbb{D} (all the rectangles are homeomorphic. The L-shapes are homeomorphic to rectangles). This is a really myopic way of doing things and obviously loses important information.
- Fiddling with these shapes is essential. The "new" exercise here is to use the patchwork technique to check how the overlapping works.

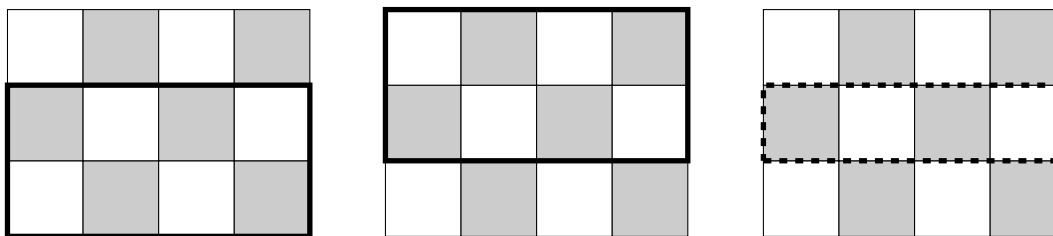


A 2×3 rectangle can be paved 3 ways by dominoes. And a 2×4 rectangle can be paved 5 ways. And a 2×2 rectangle can be paved 2 ways. And we could try to build the whole shape.

I don't know this number off the top of my head. I don't even know the 3×4 rectangle.



So again $Z(2 \times 4) = 5$ and $Z(2 \times 2) = 2$ and we should patch these together to obtain $Z(2 \times 6) = 13$. Is that right?



The $3 \times n$ or $4 \times n$ cases deserve attention and could be pedagogical. We will have more observables to think about soon. And one should already be thinking about whether domino tilings are the only cases we can reasonably solve. That answer might be yes ...

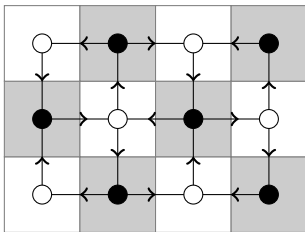
Let's try to evaluate the 3×4 case. The Kasteleyn matrix should be 6×6 :

B	D	E	A
F	C	C	F
A	E	D	B

I do **not** like matrix notation because the indices are themselves two-dimensionals A-,B-,C-,D-,E-,F- have information all around them.

$$\det \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = 0$$

This is just like last time. Determinant is zero². Except I am not sure where to put the signs $+1$ and -1 . If I recall the orientation comes from picking one of the tilings.

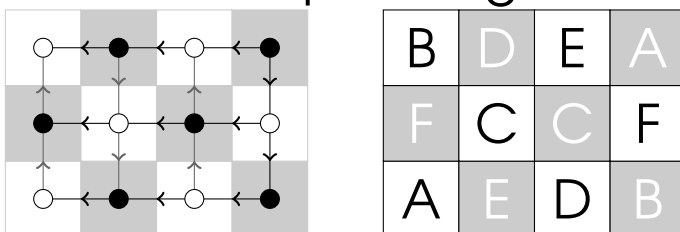


It was pretty exhausting to type this. Because there is a lot of structure that goes into drawing a very simple shape.

²This could be that rectangle has trivial homology. There is rel

$$\det \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} = 2 \neq 11$$

Time to read a bit more on Kasteleyn orientations. OK It's done with a spanning tree.



As you move around the square, there should be either

- 3 \leftarrow , 1 \rightarrow
- 1 \leftarrow , 3 \rightarrow

and in some cases these are called “IRF” models (interaction around face). I also have a round face.

$$\det \begin{bmatrix} 0 & 0 & 0 & 0 & +1 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & +1 & -1 & +1 & -1 \\ 0 & +1 & +1 & 0 & -1 & 0 \\ +1 & 0 & -1 & -1 & 0 & 0 \\ -1 & +1 & -1 & 0 & 0 & 0 \end{bmatrix} = 11$$



Can the physics literature offer more interesting tiling models? Lots of overloaded terms here:

- domino tiling
- dimer model

The dimer model is more general (since any planar graph has a dual tiling model), but maybe not all tiling problems come from a dimer model (the shapes can be more varied).

The physics papers I raise more distinctions. There's distinction between:

- high-energy physics (such as Chern Simons Theory)
- condensed matter physics (such as TQFT, symmetry protected topological phases, ising models or free fermions)

This is consistent with using Gauge Theory to describe the theory of everything. Here we are using Gauge Theory to describe the dominoes. In the case we find the ising model there is one more distinction:

- ising statistical model (state-sum model)
- ising conformal field theory (continuum limit)

Hopefully that is enough. I am not sure I care which Chern-Simons Theories have tiling models as low-energy limits, as long as I know that Kasteleyn determinants are discrete Dirac operators.

$$\#\{\text{tilings}\} = \det K \approx \det \mathcal{D}$$

I think the number of tilings is the factor of the Dirac determinant which gets thrown away ("renormalized").

Here is a really basic one if we define $\Delta f = \frac{1}{\Delta x}[f(x + \Delta x) - f(x)]$ then we discrete approximation:

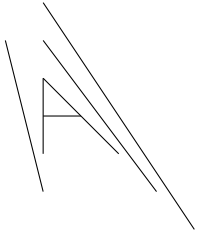
$$\frac{d}{dx} \approx \frac{1}{\Delta x} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ & 0 & \ddots & -1 \\ & & 0 & 1 \end{bmatrix}$$

In the same way, all these circulant matrices are discretizations of differential operators (maybe Dirac operators, for fermions) or whatever complicated thing that solves Chern-Simons Theory.

References

- (1) Davide Gaiotto, Anton Kapustin **Spin TQFTs and fermionic phases of matter** arXiv:1505.05856
- (2) Nathan Seiberg, Edward Witten **Gapped Boundary Phases of Topological Insulators via Weak Coupling** arXiv:1602.04251
- (3) Davide Gaiotto, Anton Kapustin, Zohar Komargodski, Nathan Seiberg **Theta, Time Reversal, and Temperature** arXiv:1703.00501
- (4) Edward Witten **Fermion Path Integrals And Topological Phases** arXiv:1508.04715
- (5) Nathan Seiberg, Edward Witten **Gapped Boundary Phases of Topological Insulators via Weak Coupling** arXiv:1602.04251
- (6) Werner Krauth **Statistical Mechanics: Algorithms and Computations** (Oxford Master Series in Physics), OUP, 2006.

The most amazing typo ever:



```
\begin{tikzpicture} [scale=0.5]
\foreach \a in {0,...,5}{
  \draw (\a, 5-\a)--(1, \a + 1);
}
\end{tikzpicture}
```

The gauge theory view on dimers, while natural, is also extremely confusing to use. So, just some final remarks.

#1 Where do height functions come from?

Height functions come from cohomology.

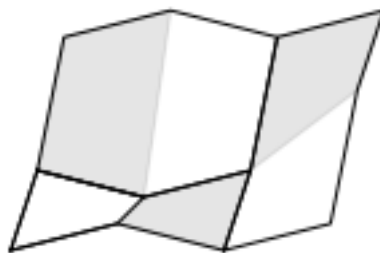
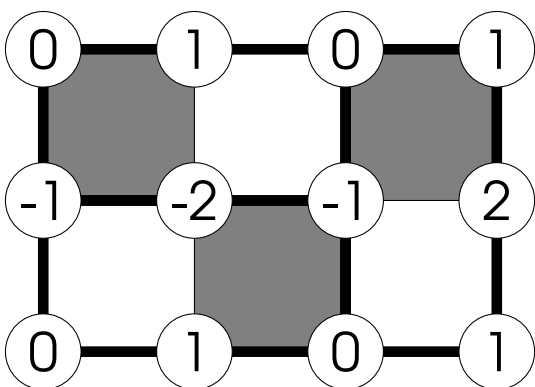
#2 Cohomology of what? No idea.

There are a few candidates. They all work kind of OK.

#3 Here is a hint from Seiberg-Witten '16:

The **representation of the braid group** that we have just described is very close to one that arises in **Ising conformal field theory** in **two** dimensions, or in the corresponding **Chern-Simons** topological field theory in **three** dimensions.

Even the term “height function” clearly means these tilings should have been look at as 3D objects.



With the height function the tiling is distorted quite a bit.