Examples: the Gamma Function

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A Let's show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. We can use the mirror formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

and if we set $z = 1 - z = \frac{1}{2}$ our number pops out:

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \frac{\pi}{\sin \frac{\pi}{2}} = \pi$$

Why stop there set $z = \frac{1}{3}$ and we have:

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = \frac{\pi}{\sin \frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

while shopping on MathWorld I found this formula, possibly generated by computer:

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3} \cdot \sqrt{2 + \sqrt{3}}$$

This is a lot more interesting!

Noam Elkies shows that this formula can also be derived from the multiplication formula[1]:

$$\Gamma(z)\Gamma(z+\frac{1}{3})\Gamma(z+\frac{2}{3}) = 2\pi \cdot 3^{\frac{1}{2}-3z}\Gamma(3z)$$

Let's follow Elkies' instructions at set $z = \frac{1}{24}$ and also $z = \frac{1}{8}$:

$$\Gamma(\frac{1}{8})\Gamma(\frac{11}{24})\Gamma(\frac{19}{24}) = 2\pi \cdot 3^{\frac{1}{8}}\Gamma(\frac{3}{8})$$

but also

$$\Gamma(\frac{1}{24})\Gamma(\frac{3}{8})\Gamma(\frac{17}{24}) = 2\pi \cdot 3^{\frac{5}{8}}\Gamma(\frac{1}{8})$$

and sure enough when you multiply the answer is:

$$\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})\Gamma(\frac{19}{24})\Gamma(\frac{17}{24}) = 4\pi^2\sqrt{3}$$

[1]Notice we never quite get away with a factor of $\sqrt{3^{1-6z}}$ this is similar to the doubling formula:

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

The left side has poles at $z = -1, -2, -3, \dots$ as well as $z = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$ and the right side has poles at all the negative half integers $z \in -\frac{1}{2}\mathbb{N}$. While this kind of reasoning might make sense, it went under further scrutiny still.

Yet if we set $z = 5/24$ and $z = 7/24$ into the mirror formula:

$$\Gamma(\frac{5}{24})\Gamma(\frac{19}{24}) = \frac{\pi}{\sin 5\pi/24}$$

and also

$$\Gamma(\frac{7}{24})\Gamma(\frac{17}{24}) = \frac{\pi}{\sin 7\pi/24}$$

and multiplying these we get:

$$\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{17}{24})\Gamma(\frac{19}{24}) = \frac{\pi^2}{\sin\frac{5\pi}{24}\sin\frac{7\pi}{24}}$$

and there is even more cancellation:

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})\Gamma(\frac{17}{24})\Gamma(\frac{19}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{17}{24})\Gamma(\frac{19}{24})} = 4\sqrt{3}\sin\frac{5\pi}{24}\sin\frac{7\pi}{24}$$

It remains to show that:

$$\sin\frac{5\pi}{24} = \sin\frac{7\pi}{24} = \sqrt[4]{2+\sqrt{3}}$$

I noticed immediately the action of:

- $z \mapsto 1 - z$
- $z \mapsto z + \frac{1}{3}$
- on $\frac{1}{24}\mathbb{Z}$

I never thought too much before of $\boxed{\frac{1}{24} + \frac{1}{3} = \frac{3}{8}}$ but here we are.

B the second more complicated solution – if we have no idea which cancelations might occur, we can just randomly use the triplication formula until we get someting that works:

$$\Gamma(\tfrac{1}{3}x)\Gamma(\tfrac{1}{3}x+1)\Gamma(\tfrac{1}{3}x+2) = \frac{2\pi}{3^{x-\frac{1}{2}}}\Gamma(x)$$

In a way, I don't worry too much about the letter Γ or the algebraic fractor of: $\frac{2\pi}{3^{x-\frac{1}{2}}}\Gamma(x)$.

I can just write a shorthand of brackets $[\cdot]$ so

$$\left[\tfrac{1}{3}x\right] \oplus \left[\tfrac{1}{3}x+1\right] \oplus \left[\tfrac{1}{3}x+2\right] \approx [x]$$

and I don't know which combination will work in advance so I keep writing them out:

$$\left[\frac{1}{24}\right] \oplus \left[\frac{3}{8}\right] \oplus \left[\frac{17}{24}\right] \approx \left[\frac{1}{8}\right] \quad (1)$$

$$\left[\frac{1}{8}\right] \oplus \left[\frac{11}{24}\right] \oplus \left[\frac{19}{24}\right] \approx \left[\frac{3}{8}\right] \quad (2)$$

Then we can add these two equations and conclude:

$$\left[\frac{1}{24}\right] \oplus \left[\frac{11}{24}\right] \oplus \left[\frac{17}{24}\right] \oplus \left[\frac{19}{24}\right] \approx [0]$$

These equations may wind up becoming faulty, but seem to do the bookkeeping for us. At least part of it.

C Can all Chowla-Selberg formulas be proven with careful use of the mirror + multiplication formulas?

$$\log \Gamma(x) = \left(\frac{1}{2} - x\right)(\gamma + log2) + (1-x)\log \pi - \frac{1}{2}\log \sin \pi x \sum$$

Then multiply both sides by Legendre symbol:

$$\sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \log \Gamma\left(\frac{n}{p}\right) = -(\log + 2\pi) \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) n + \sqrt{p} \sum_{n=1}^{\infty} \Bigg($$

and the Chowla-Selberg formula in a logarithmic form.

Here is an example:

$$\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4}) \int_0^1 x^{-3/4}(1-x)^{-1/4}(1-x/64)^{-1/4}\, dx$$

is equal to

$$\left[\frac{7\pi}{2} \times \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)}\right]^{1/2}$$

Chowla's paper deserves a more careful reading than this[2].

[2]Somehow we needed the Fourier series to prove the multiplication formula and mirror formulas. The Weierstrass product formula could yield a quick proof of

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

It would be quick if we proved the Weierstrass product formula. Fourier series also has the same fine print.

D For the moment we move on to the paper of Benedict Gross and David Rohrlich.

Chowla and Selberg's paper starts off with a zeta function:

$$Z(s) = \sum \frac{1}{(am^2 + bmn + cn^2)^s}$$

and they prove a functional equation relating $Z(s)$ and $Z(1-s)$ and somehow it leads to these Γ function identities.

I'll take this moment to mention the Wallis Product formula:

$$\frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)} \to 1$$

sorry if the relation to the formula for $\frac{\pi}{2}$ is not clear.

We've already shown the Chowla-Selberg formula and that $p = 7$ does not follow from the 7-multiplication and mirror formulas.

I guess the issue is resolved.

References

(1) MathOverflow **show that** $\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3} \cdot \sqrt{2+\sqrt{3}}$ `http://mathoverflow.net/q/249164`