Scratchwork: Intersecton of Two Lines

In geometry class, we learn the Cramer rule for the intersection two lines.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

And so the intersection of these two lines can be found with a **determinant** of a 2×2 matrix:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

In a Linear Algebra course - or a Geometry course - one might check that $a, b, c \in \mathbb{R}$ means our solutions $(x, y) \in \mathbb{R}^2$. We don't have that for an integer problem $a, b, c \in \mathbb{Z}$ the solution remains in integers $(x, y) \in \mathbb{Z}^2$.

Since the $+$ and $\times$ operations we do aren't too fancy, we can do Linear Algebra over a field such as $K = \mathbb{Q}$ or $K = \mathbb{C}$. In addition, let's use a tiny bit of Exterior Algebra taken from a Geometry textbook.

Thm The points A, B and C are collinear if and only if $A \wedge B + B \wedge C + C \wedge A = 0$.

In our case, the equation has one line $\boxed{\text{Ax + By = C}}$. We can write the Cramer rule in an more condensed way:

$$Ax + By = C \to A \wedge (Ax + By) = (A \wedge B)y = (A \wedge C) \to y = \frac{A \wedge C}{A \wedge B}$$

And a similar formula for x. Is it okay to write the coordinate value of x and y as the ratio of two areas. The geometric objects look kind of funny but OK.

$$[\text{number}] = \frac{[\text{area}]}{[\text{area}]}$$

This is not outrageous. Pedoe gives a careful derivation of the wedge product of two vectors:

$$u \wedge v = (x_1E_1 + x_2E_2) \wedge (y_1E_1 + y_2E_2) = (x_1y_2 - x_2y_1)(E_1 \wedge E_2)$$

where $E_1, E_2 \in \mathbb{R}^2$ are unit vectors in the plane.

There are even more intersection formulas like this. Two planes in Four dimensions intersect (generically) at a point.

$$\mathbb{R}^2 \cap \mathbb{R}^2 = \{pt\} \text{ in } \mathbb{R}^4$$

Since all we're doing is linear algebra, this still could work over $\mathbb{Q}$ we'd have $\mathbb{Q}^2 \cdot \mathbb{Q}^2 = [pt] \subseteq \mathbb{Q}^4$. This is the beginnings of intersection theory and a lot of sheafy things could occur.

Algebraic geometry could be done over any field K, such as $K = \mathbb{Q}(i)$ or possibly $K = \mathbb{Q}(\sqrt{2})$. Let's check two ways of writing fractions in $\mathbb{Q}(i)$:

$$\frac{a_1}{b_1} + \sqrt{-1}\frac{a_2}{b_2} = \frac{c_1 + \sqrt{-1}d_1}{c_2 + \sqrt{-1}d_2} \in \mathbb{Q}(i)$$

What assures us that the first way of writing rationals it the same as the second one? There should be "highly algebraic" way connecting the two.

$$(a, b) \mapsto (c, d)$$

This map is called "birational" or something. Clearly they will repreent the same thing, the **affine plane** or $\mathbb{A}^1$.

If we wanted to change from one notation to the other we "just" clear denominators.

$$\frac{c_1 + \sqrt{-1}d_1}{c_2 + \sqrt{-1}d_2} = \frac{c_1 + \sqrt{-1}d_1}{c_2 + \sqrt{-1}d_2} \cdot \frac{c_2 - \sqrt{-1}d_2}{c_2 - \sqrt{-1}d_2} = \frac{(c_1c_2 + d_1d_2) + \sqrt{-1}(c_2d_1 - c_1d_2)}{c_2^2 + d_2^2}$$

Then we can try looking at less convient systems such as de Moivre's theorem:

$$p + \sqrt{-1}q = \sqrt{p^2 + q^2} \times \exp\big[i\theta\big] \text{ with } \theta = \tan^{-1}\frac{p}{q}$$

When we add two of these numbers, we have not left this domain.

$$\sqrt{p_1^2 + q_1^2}\,\mathsf{exp}(i\,\theta_1) + \sqrt{p_2^2 + q_2^2}\,\mathsf{exp}(i\,\theta_2) = \sqrt{(p_1 + p_2)^2 + (q_1 + q_2)^2}\,\exp\Big(i\tan^{-1}(\tan\theta_1 + \tan\theta_2)\Big)$$

and if we multipy them we get another bunch of trigonometric identities:

$$\sqrt{p_1^2 + q_1^2}\,\mathsf{exp}(i\,\theta_1) \times \sqrt{p_2^2 + q_2^2}\,\mathsf{exp}(i\,\theta_2) = \sqrt{(p_1p_2 - q_1q_2)^2 + (p_1q_2 + p_1q_2)^2}\,\mathsf{exp}\big[i\,\big(\theta_1 + \theta_2)\big]$$

What do we even mean "put into context"? Or "generlized"? Even this is going to be made very preicse, even mechancial, at the expense that we'll barely know what we're talking about.

Ex. Does playing off the field extenion $\mathbb{Q} \to \mathbb{Q}(i)$ against the operations $+$ and $\times$ have a name?

Ex. How do we "lift" GCD from $\mathbb{Q}$ to $\mathbb{Q}(i)$? Can this be made functorial?

Ex. How do these field extensions interact with the geometry of circles? Here are the change of variables formulas for the differentials:

$$dr = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} \quad (1)$$

$$d\theta = \frac{x\,dy - y\,dx}{x^2 + y^2} \quad (2)$$

$$(dr)^2 + (r\,d\theta)^2 \in \mathcal{O}_{\mathbb{A}^2}(-2) \quad (3)$$

This was taken on the Wikipedia article on the Levi-Civita connection and **parallel transport**. How do we compare information from two different points on the circle or sphere? $D : T_{(x,y)} \to T_{(x,y)}$? I need to write down the sheaf to keep track of the algebraic object I am using to imitate the geometry I am doing here. Real-world data, if only it had this much structure, right?

In algebraic geometr, we can localize the tangent sheaf at a point to obtain $\mathcal{O}(-2)_{(x,y)}$. This gives us the place to discuss the information we are caring about.

Example Even more tensor product identities to ponder (here we work over $\mathbb{Z}$ can we please generalize these to a number field F ?)

$$\begin{aligned} \mathbb{Z}_a \otimes \mathbb{Z}_b &= \mathbb{Z}_{\mathsf{gcd}(a,b)} \\ \mathbb{Q}(i) \otimes \mathbb{R} &= \mathbb{R}^2 \\ \overline{\mathbb{Q}} &= \mathbb{R} \\ \overline{\mathbb{Q}(\sqrt{2}, \sqrt{3})} &= \mathbb{R} \end{aligned}$$

We have $\mathfrak{p}$-adic completitions of various kinds, and tensor prodcts, closures in Zariksi topology. All of these are known to fit in a large categorical context. It's so tempting to rattle off these definitions as if we know something.

Example Other quick and easy things we can do is to write out the intersection of two lines as a wedge product. How do wedge products occur?

$$a \wedge b = \frac{1}{2}\big(a \otimes b - b \otimes a\big)$$

This enable us to write down the wedge product as the alternating sum of two tensor products. This recovers two identities:

$$a \wedge a = 0 \quad \text{and} \quad a \wedge b = -(b \wedge a)$$

Our analogy-making machine may have an interpretation for one side and not the other. How comfortable are we with an object such as

$$\mathbb{Z} \wedge \mathbb{Z} = \Big\{ \tfrac{1}{2}(a \otimes b - b \otimes a) : a, b, \in \mathbb{Z} \Big\}$$

This could help us describe some of nuance in a GCD computation or something. In the middle a proof of the Chebyshev Theore or the Prime Number Theorm - when an analogous object no longer seems to exist - will we lose interest in such a neat generalization?

Example Let's work through the fact that $- \otimes \mathbb{R}$ as is an additve right-exact functor, where R is a module. This might even be worked out in an algebra textbook, such as Hungerford.

E.g. $R = \mathbb{Z}[x]$. All I can get for the moment is that $- \otimes \mathbb{Z}[x]$ is a functor. Let's try:

$$1 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \overset{\times\left(\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}\right)}{\longrightarrow} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow (\mathbb{Z} \oplus \mathbb{Z}) / \Big((1,2)\mathbb{Z} \oplus (2,1)\mathbb{Z} \Big) \longrightarrow 1$$

This could be written more succinctly as complex numbers, if I admit the identification (as $\mathbb{Z}$-modules): $\mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z}[i] \simeq \mathbb{Z}[x]/(x^2+1)$.

$$1 \longrightarrow \mathbb{Z}[i] \overset{\times(2+i)}{\longrightarrow} \mathbb{Z}[i] \longrightarrow \mathbb{Z}[i] / \big(2+i\big)\mathbb{Z}[i] \longrightarrow 1$$

It might be less than obvious there's even more isomorphisms, here between a 2D and 1D object:

$$\mathbb{Z}[i] / \big(2+i\big)\mathbb{Z}[i] \simeq \mathbb{Z}/5\mathbb{Z}$$

I hope this works... I have now three objects in the category of $\mathbb{Z}$-modules, see if you can find them.

$$\big(\mathbb{Z}[i] / \big(2+i\big)\mathbb{Z}[i]\big) \otimes \mathbb{Z}[x] \simeq \big(\mathbb{Z}/5\mathbb{Z}\big) \otimes \mathbb{Z}[x]$$

The isomorphism could be written $\leftrightharpoons$ or something like that. It has a $\rightarrow$ and a $\leftarrow$. Does anyone buy this?

Let's revisit Cramer rule with complex numbers. We are trying to solve a system of equation:

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

And so the intersection of these two lines can be found with a **determinant** of a 2×2 matrix:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

If $a, b, c \in \mathbb{C}$, we're still in variety-land $\ldots \mathbb{A}^1(\mathbb{C})$ is an affine variety.

$$x = \frac{C \wedge B}{A \wedge B} = \frac{(C_1 + iC_2) \wedge (B_1 + iB_2)}{(A_1 + iA_2) \wedge (B_1 + iB_2)} = \frac{[(C_1 \wedge B_1) - (C_2 \wedge B_2)] + i[(C_1 \wedge B_2) + (C_2 \wedge B_1)]}{[(A_1 \wedge B_1) - (A_2 \wedge B_2)] + i[(A_1 \wedge B_2) + (A_2 \wedge B_1)]}$$

This is a complex number and therefore it has a real and imaginary part.

$$x = \frac{\big[(C_1 \wedge B_2) + (C_2 \wedge B_1)\big]\big[(A_1 \wedge B_1) - (A_2 \wedge B_2)\big] + \big[(C_1 \wedge B_1) - (C_2 \wedge B_2)\big]\big[(A_1 \wedge B_2) + (A_2 \wedge B_1)\big]}{\big[(A_1 \wedge B_1) - (A_2 \wedge B_2)\big]^2 + \big[(A_1 \wedge B_2) + (A_2 \wedge B_1)\big]^2}$$

while this looks impressive, it also indicates a lack of understanding. First we make a mess, and then we do understanding.

References

[1] Joe Harris **Algebraic Geometry: A First Course** (GTM #133) Springer, 1992.

[2] Gabriel Cramèr **Introduction à l'Analyse des Lignes Courbes Algbriques** https://archive.org/details/bub_gb_HzcVAAAAQAAJ

[3] Dan Pedoe. **Geometry: A Comprehensive Course** Dover, 1970.