Prime Number Theorem: A Conclusion

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Let $G = SU(n)$ be a lie grou and $\mathfrak{g} = \mathfrak{su}(n)$ be the Lie algebra. I am trying to "evaluate" the product:

$$\Delta = \prod_{\alpha \in \mathfrak{R}_+} 2 \sin \left(\frac{i\alpha}{2} \right)$$

as described in the first page of this paper by Gukov and Pei.

* $\mathfrak{R}_+$ is th set of positive roots of $\mathfrak{g}$

Then is the above product finite? Aren't roots vectors? So what does it mean to take $\sin \alpha$?

* The positive roots are indexed by $0 < j < k < n$ and take diagonal matrices to complex

numbers:

$$\alpha_{jk} : \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_n \end{bmatrix} \mapsto (\lambda_j - \lambda_k)$$

Then if $f = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in (S^1)^{n-1}$ is just such a diagonal matrix the product they are discribing is just the Vandermonde matrix:

$$\Delta(f) = \left[\prod_{\alpha \in \mathfrak{R}_+} 2 \sin\left(\frac{i\alpha}{2}\right) \right](f) = \prod_{j<k} 2 \sin\left(\frac{i(\lambda_i - \lambda_j)}{2}\right)$$

That looks like Vandermonde matrix. They define:

$$\theta(f) = \frac{\Delta(f)^2}{|F|}$$

where $|F|$ counts the kernel:

$$F = \ker\left[\xi \mapsto \left(\eta \mapsto e^{(n+k)\langle \zeta, \eta \rangle}\right)\right] \stackrel{?}{=} \{0\}$$

$F \subseteq T \simeq U(1)^{n-1}$ does not depend on choice of diagonal matrix $f \in T$. And I think $|F| = 1$.

Multiplication by $(n + k)$ would mean that 0 would have $n+k$ pre-images and $(n+k)^n$ pre-images on the maximal torus.

What a complicated object!

References

(1) Jørgen Ellegaard Andersen, Sergei Gukov, Du Pei **The Verlinde formula for Higgs bundles** arXiv:1608.01761