# Scratchwork: Pythagoras Theorem

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ be vectors with $|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 1$. Then $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$ span the vector space $\mathbb{R}^3 \wedge \mathbb{R}^3 \simeq \mathbb{R}^3$.

$$|v \wedge w|^2 = |v_1 w_2 - v_2 w_1|^2 + |v_2 w_3 - v_3 w_1|^2 + |v_3 w_1 - w_3 v_1|^2$$

This looks an awful lot like the cross product with coordinates to me. $v = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $w = w_1 \mathbf{1} + w_2 \mathbf{j} + w_3 \mathbf{k}$.

$$v \times w = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix} = (v_1 w_2 - v_2 w_1)\mathbf{i} + (v_2 w_3 - v_3 w_2)\mathbf{j} + (v_3 w_1 - v_1 w_2)\mathbf{k}$$

Now we have the Pythagoras Theorem in 3-space again. Dually this can be read as De Gua's Theorem on areas of Triangles. The area of the *parallelogram* squared is te sum of the squares of the areas of the projetions. $\square$.

This could be more meaningful in four dimensions. Here, "four dimensions" could mean two objects in flat space:

$$\mathbb{R}^4 \simeq \mathbb{R}^2 \oplus \mathbb{R}^2$$

Then we could take the wedge of two 4-vectors - this would model $([pt] \oplus [pt]) \wedge ([pt] \oplus [pt])$. It's hard to count the six degrees of freedom at the moment:

$$\mathbb{R}^4 \vee \mathbb{R}^4 \simeq \mathbb{R}^6$$

What does Pythagoras' theorem say here? We have a Euclidean distance function on the set of pairs of points in the plane. Just two people standing in a room, with no restrictions.

$$|v \wedge w|^2 = (v_1 w_2 - v_2 w_1)^2 + (v_1 w_3 - v_3 w_1)^2 + (v_1 w_4 - v_4 w_1)^2 + (v_2 w_3 - v_3 w_2)^2 + (v_2 w_4 - v_4 w_2)^2 + (v_3 w_4 - v_4 w_3)^2$$

The expression $|v \wedge w|$ still measures the area of the parallelogram spanned by 4-vectors $\vec{v}$ and $\vec{w}$.

**Proposition** The vectors $A, B, C \in \mathbb{R}^2$ are collinear iff $A \wedge B + B \wedge C + C \wedge A = 0$.

## References


[1] ...