“Elementary” Proof(s) of PNT

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All these “basic” proofs require a lot of scratch-work. I will annotate the proofs in the left column.

1 “Prentious” Proof of Granville

$\boxed{\sum n^{-2} < \infty}$ and $\boxed{\sum n^{-1} = \infty}$

$\zeta(s) = \sum n^{-is}$ is analytic for $s = \sigma + it$ with $\sigma > 1$. And $\zeta(s)$ has a pole at $s = 1$. We'll show $\zeta(1+it) \neq 0$.

Then $\zeta(1+it+\Delta) \approx \zeta'(1+it)\Delta^r$

Let $\zeta(1+it) = 0$ have a zero of order $r \in \mathbb{N}$.

$$\boxed{\prod_{p \le x}\left(1 - \frac{1}{p^{1+it}}\right) \approx \frac{c}{(\log x)^r}}$$

Let $\Delta = \frac{1}{\log x}$ we can estimate the Euler product along the line $\sigma = 1$

Merten's theorem states the Euler product tends to zero a certain way. This is the fraction that you would get if you cross out all the primes $p \le x$ in the Sieve of Eratosthenes.

Mertens theorem states:

$$\prod_{p \le x}\left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}$$

This is going to lead to a contradiction since: $|1 - \frac{1}{p^{1+it}}| \ge 1 - \frac{1}{p}$ unless $\boxed{\mathbf{r = 1}}$ (simple pole)

$\mathbb{D}(p^{-2it}, 1; x) \le \mathbb{D}(p^{-it}, -1; x)$ is bounded

Δ is a **non-standard**[1] way of saying “0”

Let's show $p^{it} \approx -1$ and that $p^{2it} \approx (-1)^2 = 1$.

- Then $\zeta(1+2it+\Delta) \approx \dfrac{c}{\Delta}$ for $\Delta \ll 1$.
- Then $\zeta(1+2it) = \infty$ as we let $\Delta \to 0$.

It doesn't matter if standard or non-standard. As long as we stick to some choice.

[1] Just as Isaac Newton would have intended.

The symbol $A \asymp B$ means $c_1 A < B < c_2 A$. As functions of $\Delta \approx 0$. The strategy is to show:

$$\log \left[\zeta(1+\Delta+it) \times \prod_{p \leq x} \left(1 - \frac{1}{p^{1+it}}\right) \right] \ll 1$$

Then if $\log[\dots] \ll 1$ then $[\dots] \asymp 1$. IDK

$$\sum_{p>x} \left(1 - \frac{1}{p^{1+\Delta+it}}\right) \leq \sum_{p>x} \frac{1}{p^{1+\Delta}} + \sum \frac{1}{p^2}$$

Then a similar analysis for the small primes:

$$\sum_{p \leq x} \log \left[1 - \frac{1}{p^{1+it}}\right] - \sum_{p \leq x} \log \left[1 - \frac{1}{p^{1+\Delta+it}}\right]$$

Then use the triangle inequality or something.

$$\sum_{p \leq x} \frac{1}{|p^{1+it}|} \left(1 - \frac{1}{p^{\Delta}}\right) + \sum \frac{1}{p^2}$$

The next two $\ll$ that are kind of mysterious.

$$\sum_{p \leq x} \frac{1}{|p^{1+it}|} \left(1 - \frac{1}{p^{\Delta}}\right) \ll \Delta \sum_{p \leq x} \frac{\log p}{p} \ll \Delta \log x$$

This is a very different way of looking at the real number line $(\mathbb{R}, +, \cdot, >)$

$$\sum_{p>x} \frac{1}{p^{1+\Delta}} \ll \int_{u>x} \frac{du}{u^{1+\Delta} \log u} \ll \frac{x^{-\Delta}}{\Delta \log x}$$

Obviously $\boxed{2 \sum \frac{1}{p^2} \ll 1}$

The zeta function is asymptotic to a constant

$$\zeta(1+\Delta+it) \asymp \prod_{p \leq x} \left(1 - \frac{1}{p^{1+it}}\right)^{-1}$$

Let's separate into small primes $p \leq x$ and large primes $p \geq x$.

I forgot an important definition of $\mathbb{D}$.

$$\mathbb{D}(f, g; x) = \sum_{p \leq x} \frac{1 - \mathrm{Re}\big(f(p)\overline{g(p)} \big)}{p}$$

This might be the **Kullback Leibler distance** of information theory.

$$\mathbb{D}(f, g; x) + \mathbb{D}(g, h; x) \leq \mathbb{D}(f, h; x)$$

Thus we get a sort of triangle inequality.

$\mathbb{D}(f, g; x) = 0 \to f(p) = g(p)$ and $|f(p)| = 1$

2 Large Sieve Proof of Hildebrand

The "elementary" proofs of PNT are even harder than the "advanced" proofs. How I can simplify?

abc

def

3 Large Sieve Proof of Gallagher

Gallagher uses Large Sieve to prove Bombieri Vinogradov Inequality. Ommitted due to difficulty.

abc

def

References

[1] Andrew Granville *Pretentiousness in analytic number theory*
`https://eudml.org/doc/10868`

Andre Granville, Kannan Soundarajan.
Multiplicative number theory: The pretentious approach
`http://www.dms.umontreal.ca/~andrew/PDF/Book.To1.2.pdf`

[2] PX Gallagher *A Large Sieve Density Estimate near* $\sigma = 1$ Inventiones Math 11, 329-339 (1970) `https://eudml.org/doc/142061`