Scratchwork: Multiplication

Example Furstenburg shows in 1967, that the only closed infinite subset of $\mathbb{R}/\mathbb{Z}$ invariant under $S : (\cdot) \mapsto \cdot \times a$ and $T : (\cdot) \mapsto \cdot \times b$ is all of $\mathbb{R}/\mathbb{Z}$ itself. For any irrational $\theta \notin \mathbb{Q}$,

$$\overline{\{a^k \times b^\ell \times \theta : k, \ell \geq 0\}} = \mathbb{R}/\mathbb{Z}$$

I don't know how such a basic point could be under argument in this first place. No non-expert would even question such a thing. It takes a small bit of anaysis and topology to *state* that the closure of an infinite set is something. And then we need to consult a resource on "commuting automorphisms", here $\theta \mapsto \theta \times a$ and $\theta \mapsto \theta \times b$.

For example, Let's find k and ℓ such that $\left|2^k \times 3^\ell \times \sqrt{2} - \sqrt{3}\right| < 10^{-2}$. These questions (even to me) seem like novelties, and each single case can be solved with a computer, given enough time and resources.

Example Weyl's Law (say for $H = \partial_x^2 + \sqrt{n}\,\partial_y^2$) gives an estimate for the distribution of eigenvalues (in $\mathbb{R}$):

$$\#\{j : \lambda_j < X\} = \#\{(a,b) : a^2 + \sqrt{n}\, b^2 < X\} \sim \frac{\pi}{4\sqrt{n}} X$$

We notice the distribution is approximately a line. Perhaps we could find the "doubling constant":

$$A + A \approx k\, A$$

The paper introduces all sorts of interesting measurable sets. Starting with $\mu(\mathbb{Q}) = 0$ and yet $\overline{\mathbb{Q}} = \mathbb{R}$. The paper also computes the gaps:

$$\delta_{min}(N) = \min(\{\lambda_{i+1} - \lambda_i : 1 \leq i \leq N\})$$

Here the λ_i are the sorted vales of $a^2 + \sqrt{n}\, b^2 \in \mathbb{Z}[\sqrt{n}]$. Notice we do not even need all of $\mathbb{R}$ to define this thing.

Let's even take a step further and remind ourselves that the numbers $\overline{\{a^2 + b^2 - \sqrt{2}c^2 : a, b, c \geq 0\}} = \mathbb{R}$ and we could have an exercise:

$$\left|a^2 + b^2 - \sqrt{2}c^2\right| < 10^{-6}$$

Finding such integers one time, is certainly tractable. I think the difficulty is showing this can *always* happen to arbitrary accuracy that requires the Ratner theory. In other words, that there is something complicated about these numbers.

We have a dilemma, no matter how many times we use a computer, no matter how much data we have, we are no closer to a proof. This problem and many others, originated in this use of computers, where $\mathbb{R} \approx 2^{-N}\mathbb{Z}$ where we only get $N = 20$ or 30 decimal places and the addition is truncated towards the end. Maybe we can find other measurable subsets $X \subseteq \mathbb{R}$ that show why this elementary result might offer such difficulty.

If we don't have enough decimal places, the approximate relation $a^2 + b^2 \approx \sqrt{2}c^2$ becomes an equality $a^2 + b^2 = \sqrt{2}c^2$ or $\sqrt{2} = \frac{a^2+b^2}{c^2} \in \mathbb{Q}$. This is a standard geometric exercise from Euclid that $\sqrt{2} \notin \mathbb{Q}$.

abcd

Example For a generic irrational point, $x \notin \mathbb{Q}$, the orbit under $S : \theta \mapsto \{a \times \theta\}$ and $T : \theta \mapsto \{b \times \theta\}$ as elements of $\mathbb{R}/\mathbb{Z}$ is still a set of measure zero, yet it is dense in $\mathbb{R}/\mathbb{Z}$.

$$\overline{\{a^k \times b^\ell \times x : k, \ell \geq 0\}} = \mathbb{R}/\mathbb{Z}$$

This is just like when $\mu(\mathbb{Q}) = 0$ and yet $\overline{\mathbb{Q}} = \mathbb{R}$, in its entirety. Let's see a quantitative statement of this result:

> Let a, b be multiplicative independent. Suppose $\alpha \in \mathbb{R}/\mathbb{Z}$ be diophantine-generic; there exists k such that
> $$\left|\alpha - \frac{p}{q}\right| \geq q^{-k} \text{ with } q \geq 2, \quad p, q \in \mathbb{Z}$$
> Then $\{a^k \times b^\ell \times \alpha : 0 < k, \ell < N\}$ is $(\log\log N)^{-\kappa}$ dense in $\mathbb{R}/\mathbb{Z}$ for some constant $\kappa > 0$.

Even if we restrict to a single map, our entire decimal system is based on the map $T : \theta \mapsto 10 \times \theta$. Here is a set of measure zero, where every other decimal is zero and the other numbers are in $\{1, 2, \dots, 9\}$.

$$A = \{0.x_1\, 0\, x_2\, 0\, x_3\, 0 \cdots : 1 \leq x_i \leq 9\} \text{ can we show that } A + A = \mathbb{R}/\mathbb{Z}$$

We have that $\mu(A) = 0$ can we show that $\mu(A + A) > 0$? Or let's construct a measurable function. What happens if we stop after N digits?

$$A_n = \{0.x_1\, 0\, x_2\, 0\, x_3\, 0\, \dots\, 0\, x_n : 1 \leq x_i \leq 9\} \text{ with } \mu(A_n) = \left(1 - \frac{1}{10}\right)^n \times \frac{1}{10^n}$$

Then we could write down a measurable function by adding functions supported on these subsets in various ways:

$$f(x) = \sum \mathbf{1}_{A_n}(x) \quad \text{and} \quad g(x) = \sum_{n \text{ squarefree}} 2^n \mathbf{1}_{A_n}(x)$$

I have not checked for convergence. The next question would be $\mu(f^{-1}([0, \frac{1}{2}])$ this set is measurable.

What happens if we try to take derivatives of these functions? We could try to find a limit $\frac{1}{\epsilon}(f(x + \epsilon) - f(x))$ for these functions which are roughly straight lines and have that $f'(x) \approx 1$.

Given a dynamical system $T : X \to X$ we could define a smaller subset of the orbit: $\{T^n x : n \text{ squarefree}\}$ and since the density of square-free numbers in $\mathbb{Z}$ is roughly $\frac{6}{\pi^2} \approx \frac{2}{3}$, we could have a shift-map related to these partial orbits. This might already have a name, like an "factor" or "return-map".

The measurable sets in this paper of $\times a \times b$ are

$$A_{M,N} = \left\{\{5^k \times 7^\ell \times \sqrt{2}\} : 0 < k < M, 0 < \ell < N\right\} + [-\epsilon, \epsilon]$$

for small enough $0 < \epsilon \ll 1$. The paper seems to be concerned with deviations from uniformity (since that's what we expect) and attempting to quantify that. Here's a sample. What can we say about this series?

$$h(x) = \sum_{M,N \geq 0} (-1)^{M+N} 1_{\mathbf{A}_{M,N}}(x)$$

and any other imaginable statistic. What is the correct averaging factor?

References

[1] Proofs that $\sqrt{2} \notin \mathbb{Q}$
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https://math.stackexchange.com/q/451700

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https://math.stackexchange.com/q/452078
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[2] Commuting Automorphisms
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[3] ...