Theta Functions

John D Mangual

$$\theta(x;p) = (x;p)_\infty (px^{-1};p)_\infty = \exp\left(- \sum_{m\neq 0} \frac{x^m}{m(1-p^m)} \right)$$

another one

$$\theta(z;q) := (z;q)_\infty (q/z;q)_\infty = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} z^k q^{\binom{k}{2}}$$

the shifted factorials are defined by:

$$(z;q)_\infty = \prod_{i\geq 0} (1 - zq^i)$$

Let's see if

$$\binom{k}{2} = \frac{k(k-1)}{2} = \frac{k^2}{2} - \frac{k}{2}$$

Then it could be:

$$\theta(q^2; q) = \frac{1}{(q; q^2)} \sum_{k \in \mathbb{Z}} q^k q^{2\binom{k}{2}} = \frac{1}{(q; q^2)} \sum_{n \in \mathbb{Z}} q^{n^2}$$

Wikipedia has

$$\sum_{n \in \mathbb{Z}} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty$$

and we can set $a = b = q$:

$$\sum_{n \in \mathbb{Z}} q^{n^2} = (-q; q^2)_\infty (-q; q^2)_\infty (q^2; q^2)_\infty$$

This also seems odd we can try

$$\theta(q; q^2) = (q; q^2)_\infty (q; q)_\infty (q^2; q^2)_\infty = \sum_{n \in \mathbb{Z}} q^{n^2}$$

References

(1) Taro Kimura, Vasily Pestun **Quiver elliptic W-algebras** `arXiv:1608.04651`

(2) Wikipedia "Jacobi Triple Product", "Ramanujan Theta Function"

(3) Eric M. Rains, S. Ole Warnaar **Bounded Littlewood identities** `arXiv:1506.02755`