Attempt at: the Arctic Circle Theorem

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Working backwards, I will try to prove the Artic Circle Theorem, then I will state the Arctic Circle Theorem and finally explain why it is important to me[1].

I wonder why dominos have height functions at all. One set of computer science notes shows perfect matchings as an instance of the max-flow problem.

The goal here is to keep the animals at bay, to keep the complexity at a manageable level.

[1]There are many proofs of the Arctic Circle Theorem using a wide range of techniques. Often the Aztec Diamond cases is considered "known" or "settled" which doesn't help our cases any. I am trying to find a proof that can be read from end to end without too much difficulty.

I recall speaking to David Speyer about the advances and proofs in the Aztec Diamond and yet – an expert in the field – he claimed he wasn't aware. This was very much confusing. He did also say that all the proofs he is seeing are very similar. There must be Kasteleyn formula and maybe the Smith determinant.

The proofs I have seen often involve difficult complex analysis... steepest descent and/or Riemann-Hilbert equations. The equations look a big mess and I am hoping there is away out. The only case of Riemann-Hilbert problem I understand are **Stirling Formula** $n! \approx \sqrt{2\pi n}(n/e)^n$ and the **Szegö** curve, $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = 0$ which is almost like $|xe^{1-x}| = 1$

One big turn-of-the-millenium mathematical idea is determinantal processes[2].

However, the Aztec Diamond domino shuffling can be turned into a determinantal process in various ways.

Kurt Johansson in 2000 writes about a few generalizations of the Plancherel Measure. All of these have to do with the permutation group, S_n.

$$\sum_{|\lambda|=n} (\chi_\lambda)^2 = n!$$

In a way all these permutations occur randomly (like shuffling a deck of cards), so the representations also occur randomly

$$\frac{1}{n!} \sum_{|\lambda|=n} \mathbb{P}[\chi = \lambda] = 1$$

Then, why do domino tilings of the Aztec Diamond constitute a measure on the representations of the permutation group[3].

[2] They were a canidate to help solve the Riemann Hypothesis (e.g. the Keating Snaith conjecture. Certainly they help you talk about the behavior of the Riemann zeta function on the critical line $\zeta(\frac{1}{2} + it)$.

[3] shuffling a deck of cards

And what would happen if I used a really crappy source of randomness. Do I get a convincing shape? LOL

I would like the minimum amount of difficuly that still constitute a proof[4]. Let me not stall any more:

$$\mathbb{P}[Z_r(h)] \asymp \prod_{1\leq i<j\leq r} \frac{h_j - h_i}{j-i} \prod_{1\leq i<j\leq n+1-r} \frac{k_j - k_i}{j-i}$$

I should be more careful... there's just a number which I am too lazy to type[5]

This is known as the Krawtchouk Ensemble. A lot is known about these equations (here they are discribing "zig-zag paths" in the tiling).

I would like to show as $n, r \to \infty$ with $r/n = k$, this approaches the GUE. This has been done before.

All I need to show is that if $\frac{h}{n} < \sqrt{1 - (\frac{r}{n})^2}$ the region is frozen... but we know more details.

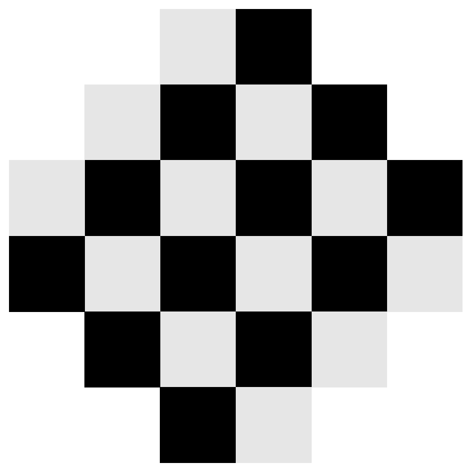
[4]A physicist emphasizes the result and skips all the details, the analyst is showing off their calculation skills. I am a geometer and emphasize shapes :-/

[5]$n \to \infty$ anyway, so why not just start now?

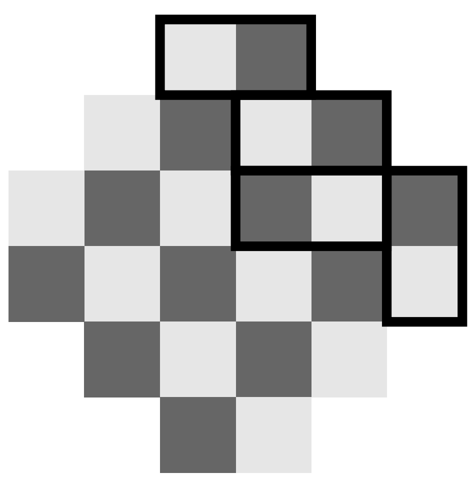
2 Plancherel Measure, Zig-Zag Paths

Domino Tilings is notoriously hard to code and to theorize about at the same time. All my scrawled pages are lost.

Exercise 1: Draw an Aztec Diamond



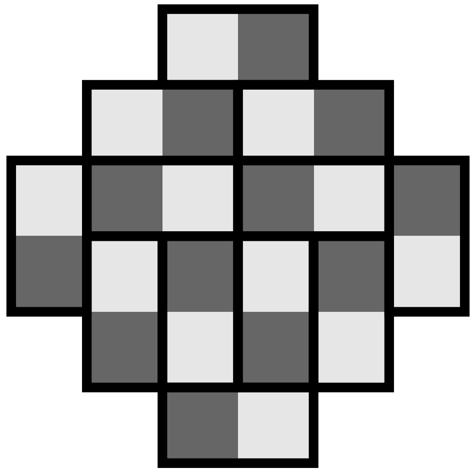
Exercise 2: Draw some dominoes



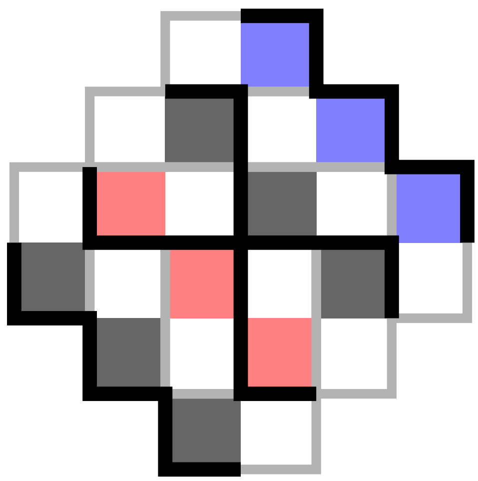
How to encode one domino? The middle line is at (1,0) and (1,1) so we will encode it as `1-0-1`, and we specify that it is horizontal[6].

[6] We'll always get stuck since there's no language for having one item in **two** places. Maybe it's not so bad. This would be a redundant way to store this information on a computer, but hey we're drawing. This is a separate problem!

Exercise 3: finish the domino tiling



Exercise 4: draw a zig-zag path[7]



Exercise 5: prove limiting behavior of zig-zag

[7] I am stressing the colors a bit.

The Krawtchouk ensemble is the correct ensemble for the Aztec Diamond problem. We only focus on $p = \frac{1}{2}$:

$$\mathbb{P}_{\mathsf{Kr}}[(h_1, \ldots h_n)] = \frac{1}{Z} \times \frac{1}{2^{\binom{N}{2}}} \times \Delta_N(h) \times \prod_{j=1}^{N} \binom{K}{h_j}$$

where the normalization Z is a number:

$$\frac{N!}{0! \times 1! \times 2! \times \cdots \times (N-1)!} \times \frac{1}{2^{\binom{N}{2}}} \times \prod_{j=1}^{N} \binom{K}{j}$$

the meaning of these number would be clear to a professor Combinatorics[8] these are related to the Krawtchouk polynomials.

Johansson attributes the circular shape o Timo Sepalainen. We settle for this summary by Persi Diaconis and Robert Griffiths

> Orthogonal polynomials for the multinomial distribution of N balls dropped into d boxes are called multivariate Krawtchouk polynomials.

These ball-and-urn models are commonplace.

[8] or various branches of Computer Science

The limit shape seems to be known from the theory of **first-passage percolation**. It would not hurt to read the book by Grimmett.

$$\mu(x,y) = \begin{cases} y & \text{if } x < y \\ y + (\sqrt{x/2} + \sqrt{y/2})^2 & \text{if } x \geq y \end{cases}$$

The earliest proof by Jockusch-Shor-Propp uses the TASEP model[9]

The most important is that we have a formula for all possible $\vec{h}$ on a given row (and later, the joint probability density over all rows).

Johansson says it is an easy analysis from here.

[9] which can be converted to this type of "first-passage percoltion". More on this once I figure it out.

2 - Determinantal Processes

The surprising fact that[10]:

$$\mathbb{P}(h) = \frac{1}{Z_M\, 2^{\binom{M}{2}}} \prod_{1 \leq i < j \leq M} (h_i - h_j)^2 \prod_{k=1}^{M} \binom{M}{k}$$

I could not figure out the ball-and-urn model this is associated to this combination formula.

Diaconis-Griffiths point out the Krawtchouk polynomials are related to bosonic Fock space from Quantum Field Theory[11]

On the other hand if $k, M \to \infty$ with $k/M \in \mathbb{R}$

$$\frac{1}{2^k}\binom{M}{k} \to e^{(\frac{k}{M})^2}$$

or something like that – De Moivre Laplace limit theorem.

[10]The surprising fact that there is any formula at all ...

[11]Feynman himself may not have known this or many `hep-th` style physicists. We get very lucky here there is a formula for the Kernel for this determintal process. An adequate discussion (but certainly not the final word) of Determintal Process theory is in this blog `https://terrytao.wordpress.com/2009/08/23/determinantal-processes/` What if we do not have a Kernel?

The "reproducing kernel" should be the sum of Krawtchouk polynomials

$$Q(x,y) = \sum Q_n(x) Q_n(y)$$

adding over the Krawtchouk polynomials of giving M.

The answer is the binomial theorem that's all I'm gonna say[12]

3 - Candidates for Generalization:

- the Topological Vertex
- the Hilbert Scheme of Points
- Donaldson-Thomas invariants
- Black Hole microstate counting

it is much harder - in my opinion - to gain solid understanding of the original case. All 4 of these theories are missing crucial details as well.

For these reasons we resume a study of the basic case.

[12] :/

4 - New ideas in Random Walk

N balls dropped into two boxes with probabilities $p, 1-p$ satisfy the binomial distribution:

$$m(k,p) = \binom{N}{k} p^k (1-p)^{N-k}$$

The orthogonal polynomials with respect to this measure:

$$u_1^k u_1^l + u_2^k u_2^k = \delta_{kl}$$

As functions of $p \in [0,1]$ these are the **Krawtchouk polynomials**.

4 - New ideas in Random Walk (please ignore previous page)

Mathworld says: Krawtchouk polynoials are:

$$\sum_{k=1}^{N}(-1)^{n-k}\binom{N-x}{n-k}\binom{x}{k}p^{n-k}q^{k}$$

these polynials are orthogonal with respect to the binomial distribution:

$$w(x)=\binom{N}{x}p^{x}(1-p)^{N-x}$$

umm... that's all Mathworld says

4 - New ideas in Random Walk (please un-disregard page 10)

Here is a version by Koornwinder[13]

$$\sum_{x=0}^{N} \psi_m(x)\psi_n(x) \cdot \binom{N}{x} p^x (1-p)^{N-x} \asymp \mathbf{1}_{m=n}$$

and the constant of proportionality is:

$$||\psi_m(\,\cdot\,; p, N)||^2 = \left[\binom{N}{n} \left(\frac{p}{1-p} \right)^n \right]^{-1}$$

The easiest way I have found is to let X take a random walk on the number $0, 1, \ldots, N$ which has binomial distribution. Then

$$\mathbb{E}[\psi_m(X)\,\psi_n(X)] = \mathbf{1}_{m=n}\, n!^2 \binom{N}{n} (pq)^n$$

[13] Diaconis, being an expert statistician is speaking generally... I have one case in mind. It seems they are also looking for a generalized ball-and-box model – they find one but it's too general.

4 - New ideas in Random Walk

Let $u(\xi) = \xi - p$, so that $1 + u(0) = 1 - p$ and $1 + u(1) = p$ then we have a formula in terms of permutations

$$K_n(X; N, p) = n! \sum_{\sigma \in S_N} u(\xi_{\sigma(1)}) \dots u(\xi_{\sigma(n)})$$

and in addition to the K's there are the Q's:

$$Q_n(X) = \binom{N}{n}^{-1} \sum_{\sigma \in S_N} v(\xi_{\sigma(1)}) \dots v(\xi_{\sigma(n)})$$

with $v(\xi) = 1 - \xi/p$ These are sums of permutations of iid random variables[14] $u(\xi), v(\xi)$

References

(1) Kurt Johansson **Discrete orthogonal polynomial ensembles and the Plancherel measure.** `math/9906120` Annals of Mathematics (2) 153 (2001), No. 2, 259–296.

(2) Benjamin J. Fleming, Peter J. Forrester. **Interlaced particle systems and tilings of the Aztec diamond.** `arXiv:1004.0474`

(3) Manuel Fendler, Daniel Grieser. **A new simple proof of the Aztec diamond theorem.** `arXiv:1410.5590`

(4) Frédéric Bosio, Marc A. A. Van Leeuwen. **A bijection proving the Aztec diamond theorem by combing lattice paths.** `arXiv:1209.5373`

(5) David E. Speyer **Variations on a theme of Kasteleyn, with application to the totally nonnegative Grassmannian** `arXiv:1510.03501`

(6) Persi Diaconis, Robert Griffiths. **An introduction to multivariate Krawtchouk polynomials and their applications** `arXiv:1309.0112`

(7) Tom Koornwinder **Krawtchouk Polynomials, a Unification of Two Different Group Theoretic Interpretations** `SIAM J. Math. Anal., 13(6), 10111023.`

(8) Richard Feynman **Statistical Mechanics**

[14] Identical Independendently Distributed, these are taken from **Exchangeable pairs of Bernoulli random variables, Krawtchouk polynomials, and Ehrenfest urns** `http://statweb.stanford.edu/ cgates/PERSI/papers/DG11.pdf` These are not well-known.