

## John D Mangual

- Lagrange Interpolation
- Bezier Curves

- passing through points
- weaving around points

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$
$$\int_0^1 f(x) dx = 1 \times \left| \{x \notin \mathbb{Q}\} \cap [0, 1] \right| + 0 \times \left| \{x \in \mathbb{Q}\} \cap [0, 1] \right| = 1$$

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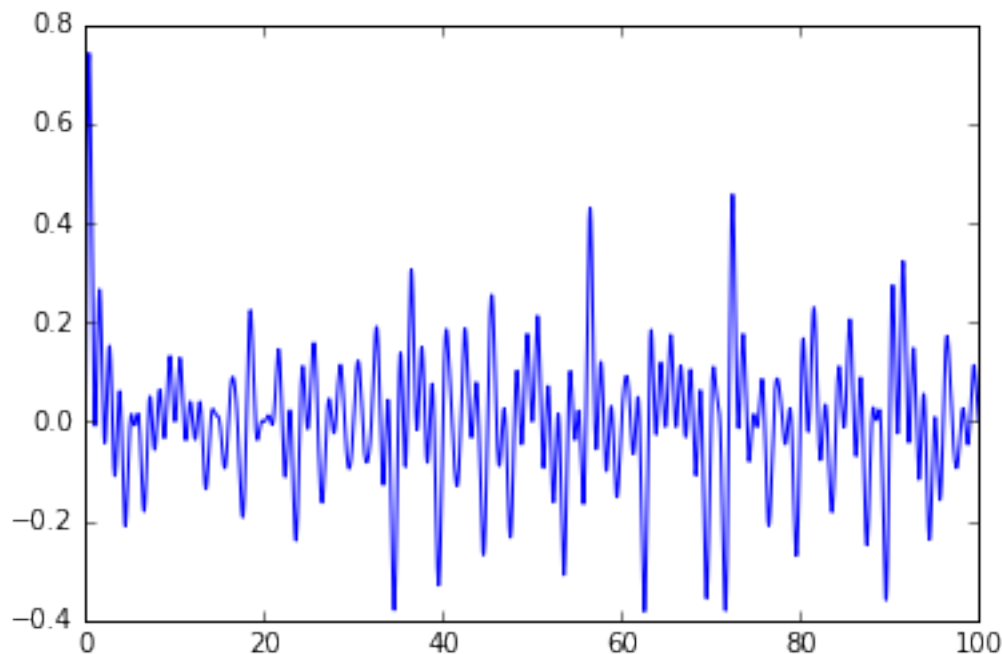
<sup>1</sup>Other times, theories comes out of the box, complete. I think such theories are unusuable, or they leave from for input from you and me. A comeback from that camp could be, “John, why are you so obsessed with this particular problem?” And I don’t have any good reason. Just because. Sometimes about me, and the time and place makes me interested. And I could be wrong!

This turns out to be one of the worse functions around, because it looks like 1 but isn't:

$$f(x) \approx 1 \quad \text{therefore} \quad \int f \approx \int 1$$

If I recall, we placed intervals around every single fraction  $\frac{a}{b} \in \mathbb{Q}$  we get an upper estimate for how large this integral could be, and that upper estimate  $\rightarrow 0$ .

How to deal with a function that often looks like  $f(x) \equiv 1$  but isn't?<sup>2</sup>



Here is a plot if we add all of the pure tones at all frequencies  $\frac{a}{b}$  with  $0 < a < b < 10$ . Theoretically we have found:

$$\frac{1}{28} \sum_{0 < a < b < 10} e^{2\pi i \frac{a}{b} t} \approx \frac{1}{c N^2} \sum_{0 < a < b < N} e^{2\pi i \frac{a}{b} t} = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

The constant in front is an oversimplification. The odds of two numbers being relatively primes is a famous one:

$$\phi(1) + \phi(2) + \cdots + \phi(N) = \#\{(a, b) : a < b \text{ and } \gcd(a, b) = 1\} \approx \frac{2}{\zeta(2)^2}$$

Analysis is the branch of math where we account for all the uses of the  $\approx$  symbol. What if I told you the function we just charted is  $\approx 0$ ?

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<sup>2</sup>One set of problems that plagued me was if we had two competing definition of limit, maybe one returns a number and the other does not. If we have two limiting procedures 1 and 2, maybe:

$$[\lim]_1 a_n = A \quad \text{implies} \quad [\lim]_2 a_n = A$$

Most of the time we don't really care how the limit procedure is defined. Who cares really?

$$N \gg 1 \quad \text{implies} \quad a_N \approx A$$

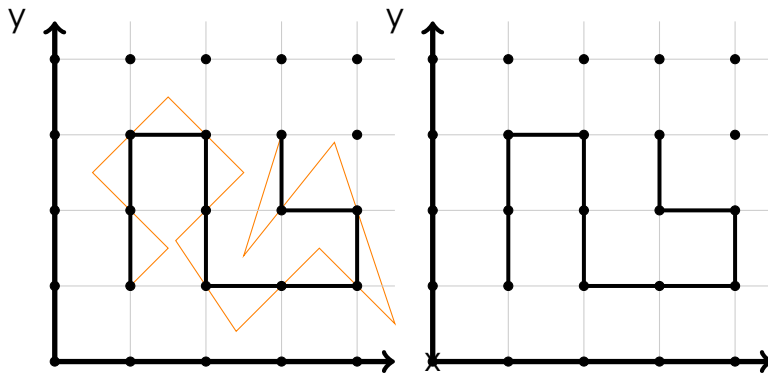
and the implication looks self-evident and most people won't question it. The only reason we remember the exception is because that particular conversation went on record.

When I read someone else's paper, a lot of my surprise stems from the author suggesting there is enough room in function space for something (very unlikely) to occur. Let's make a little problem set:

# 1 Show constant function on the rational numbers integrates to zero:

$$\int_{[0,1]} 1_{\mathbb{Q}} = 0$$

# 2 Find a curve that weaves around the obstacle course on page one:



With a pencil it's always possible to draw a curve that passes through all the dots. Our challenge is to find an equation that passes through all the loops. This is not so obvious it can always be done.

# 3 While I am remembering that my other example was going to be, we are going to solve the Prime Number Theorem.

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p \\ 0 & \text{not prime} \end{cases}$$

The prime number says that the density of primes is roughly  $\frac{1}{\# \text{ digits}}$  it can also be phrased as:

$$\sum_{n \leq x} \Lambda(n) = x + o(x)$$

where  $o(x)$  is a really small, unpredictable number<sup>3</sup> This won't be a review of number theory. Just ironing out one or two parts in a previous discussion. It's pretty hard.

# 4 Look at a real paper. There are one or two short papers of Bourgain that I try to get through from time to time.<sup>4</sup>

$$f(t) = \sum_{|x|,|y|,|z| < N} e^{2\pi i t (x^2 + y^2 - \sqrt{2}z^2)}$$

This is my favorite way to cause trouble and I get the sense this is the recommended way to start. A sense that comes from nowhere. .

<sup>3</sup>if you looked at my other projects, a great question to ask would be "how small", and "how unpredictable". We could ask even though  $o(x)$  is a small number, maybe it is bigger than 1 or 5 or 100.

<sup>4</sup>He does not write in a very forgiving way, and occasionally it's so bad, we may as well try to write the step ourselves.

All good project start from raw ingredients and tell a story.<sup>5</sup> For me pictures form an engine that start projects. Or it could be a memory or something that happened.

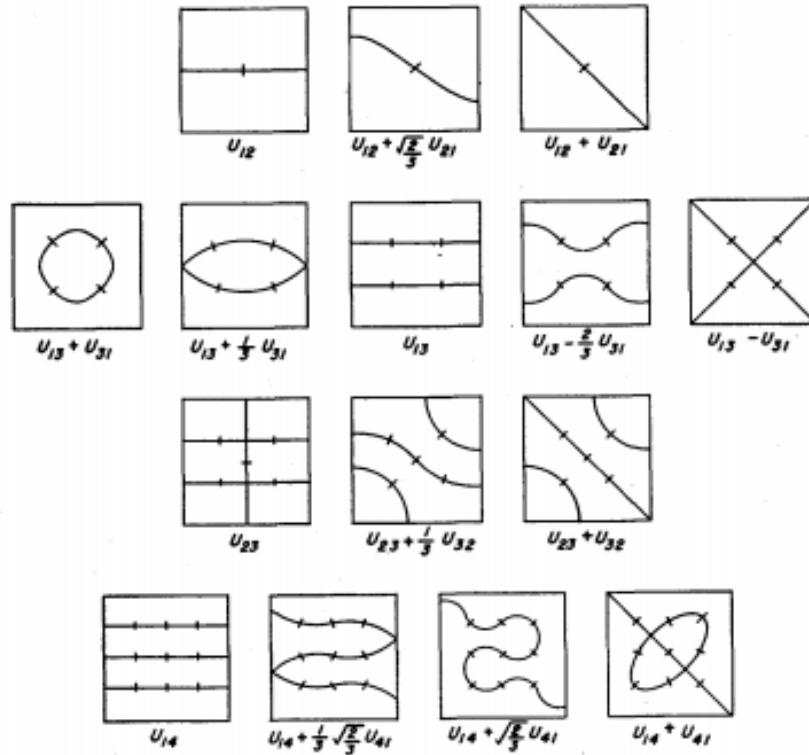


Figure 3. Nodal lines for a square membrane.

These pictures of the vibrating nodes (“notes” ♪♪) of a square drum were made are before computer graphics (the textbook was written in the 1930s). Drafting and technical drawing (without computers) seems like stupid and arcane technical skill, but our ability to make things of good quality is constrained by this. Maybe the textbooks fail to make the case.

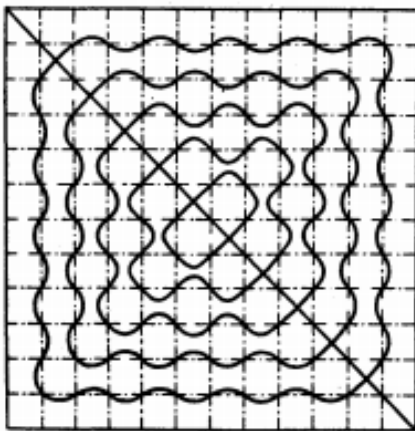


Figure 7

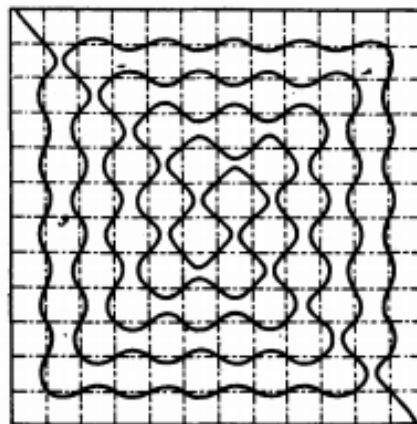


Figure 8

Generating these curves with computer is not so easy. The equation for both of these is:

$$f(x, y) = 0$$

<sup>5</sup>On day I got a textbook “Methods in Mathematical Physics (Vol 1)” The book is in German so maybe that’s why it was so cheap; it is available in English. These math-for-physicists textbooks are pretty dull. This is the middle of the 20th century and Math and Physics, which you’d assume were related had taken very different paths. Mathematicians were tired of calculations and instead proposed to build a world entirely of abstrct principles. Physicists needed to build things and used math whenever they needed it, or fabricated their own... Courant-Hilbert managed to write a textbook that assembles hundreds of unrelated techniques into a pretty-good story. And it’s still the story in the modern literature, more or less.

for some judiciously chosen function  $f$ . In the case of the square we can try sines and cosines:

$$f(x, y) = \sum_{m, n \in \mathbb{Z}} a_{m, n} \sin mx \sin ny$$

Maybe I need to include cosines as well. Different variations. Why stop at two? Maybe I can get *every possible pattern* using a level-set of bunch of sine and cosine.

Additionally, even when we have the equation, solving  $f(x, y) = 0$  is not very realistic, maybe solving  $f(x, y) < \epsilon$  where  $\epsilon = 10^{-3}$  or  $10^{-6}$  or however small you are interested in. And this becomes an algorithms problem.

In the process I may have mixed up two kinds of problems:

- Drawing the curve (maybe with Bezier curves)
- Finding a function with reasonable level set (maybe with Lagrange Interpolation)

I think they're almost the same but after reading Wikipedia I might be wrong. When I started off in number theory, there are all these cute averaging statements, such as:

$$\frac{1}{r_3(n)} \sum_{x^2+y^2+z^2=n} \phi(x, y, z) \ll n^{-1/28}$$

All of these theorems rest on the interpolation results I'm trying to explore here... and perhaps all of these number theory problems are sources of ill-behaved functions I am outlining.

All the papers I have seen resort to horrible complicated rearrangements that deserve a better explanation.

# 1 Starting small, let's show that  $\int 1_{\mathbb{Q}} = 0$ . The rational numbers are dense but form a set of measure zero.

- $\overline{\mathbb{Q}} = \mathbb{R}$
- $\mu(\mathbb{Q}) = 0$

You could take for granted that you can always find a fraction close enough to your number

$$47\% = \frac{47}{100} \approx \frac{12}{25}$$

**12 in 25** sounds pretty good instead of **47 in 100**. Maybe one is more persuasive than the other.<sup>6</sup> Even though we say  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , finding any fraction is easy, finding a really good fraction can be some work. If  $\alpha \notin \mathbb{Q}$ :

$$\exists p, q \text{ such that } \left| \frac{p}{q} - \alpha \right| < \frac{1}{q^2}$$

Therefore density misses other more refined statements we could make about  $\overline{\mathbb{Q}}$ .

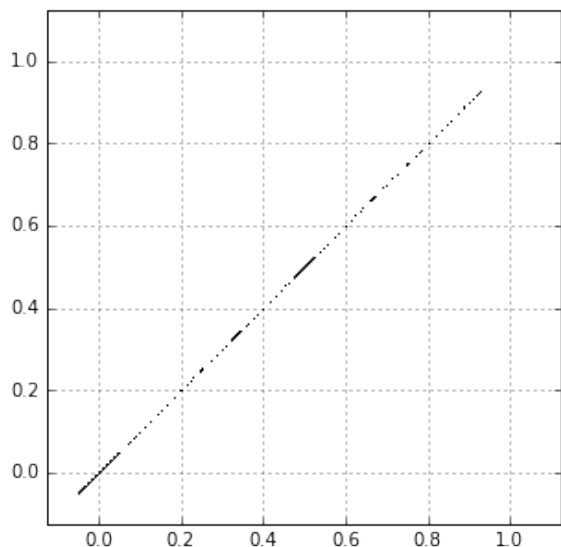
Let's walk through a few sample arguments. How did I first learn that  $\mathbb{Q}$  is a set of measure zero? This was self-evident to Stein. A point has measure zero, there are countably many fractions, and therefore  $\mathbb{Q}$  has measure zero.

$$\mu(\{pt\}) = 0 \text{ and } \mu(\mathbb{Q}) \leq \bigcup_{\frac{a}{b} \in \mathbb{Q}} \mu(\{|x - \frac{a}{b}| < \frac{\epsilon}{2^k}\}) = \epsilon \sum \frac{1}{2^k} = \epsilon \rightarrow 0$$

If we index the fractions one by one we can get some kind of geometric series.

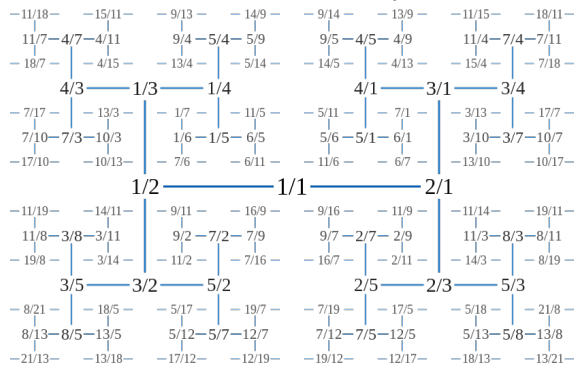
$$(\frac{a_1}{b_1}, \frac{1}{2}) \rightarrow (\frac{a_2}{b_2}, \frac{1}{2^2}) \rightarrow (\frac{a_3}{b_3}, \frac{1}{2^3}) \rightarrow (\frac{a_4}{b_4}, \frac{1}{2^4}) \rightarrow \dots (\frac{a_k}{b_k}, \frac{1}{2^k}) \rightarrow \dots$$

There are several enumerations of the fractions we can use, such as the Farey Fractions or the Calkin-Wilf tree or the Stern-Brocot Tree. The textbook says **any** enumeration of fractions. Personally, I drew a picture:

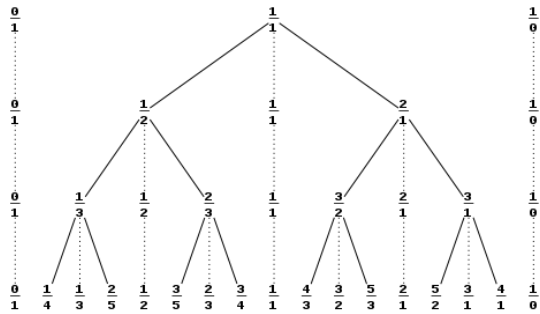


<sup>6</sup>Here we use a tiny bit of projective space  $\frac{a}{b} = [a : b] = \mathbb{Q}P^1$  and maybe even a tiny bit of the Galois cohomology of the field  $\mathbb{Q}$ . That is my theory what all these  $H^1$ 's are doing there.

## The Calkin-Wilf tree (Wikipedia)



## The Stern-Brocot tree



I really needed to make the countability of the rationals in a tangible, almost deadpan kind of way. Even there is some research you can do:

- Carlo Carminati, Giulio Tiozzo **A canonical thickening of  $\mathbb{Q}$  and the dynamics of continued fractions** [arXiv:1004.3790](https://arxiv.org/abs/1004.3790)

I have to look to see if there is more. But now we can appreciate how abstract real analysis can be: we are taking inventory of something infinitely complex and we're just saying it has been done.

Flipping through my old Real Analysis textbook I found some gems.<sup>7</sup> I can't look at this picture and not see Dehn's theorem, that if the length and the width are rational, then every single rectangle must have rational length and width.

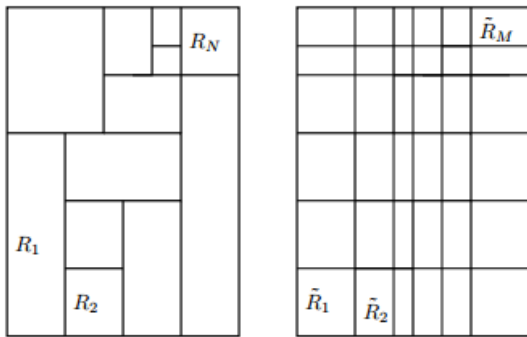


Figure 2. The grid formed by the rectangles  $R_k$

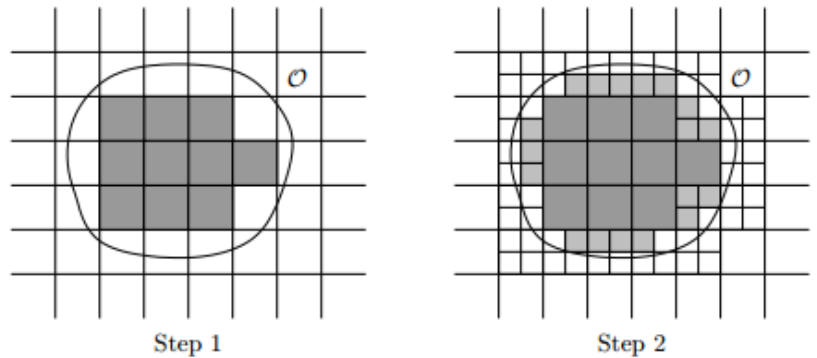


Figure 3. Decomposition of  $O$  into almost disjoint cubes

I could have spent an entire semester drawing ever more complicated shapes and putting rectangles on them.



<sup>7</sup>At my school, real analysis at the level of Lebesgue Theory was taught to 3rd year undergraduates who were pretty serious about math. I failed it. And passed the second time with a B. These days I almost agree with their assessment because I look at the textbook and still don't really get it.



# 3 There are two approaches I can think of two the prime number theorem that should be written:

- Real Analysis
- Ergodicity of the Horocycle Flow

and I can put this up here, because if someone beats me to it, I will gladly take their existing argument simplify it and settle for the worse constant.

The starting point is a bit arbitrary, an artifact of how we have developed and do arithmetic over the centuries. . . just the way you and me do:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p \\ 0 & \text{not prime} \end{cases}$$

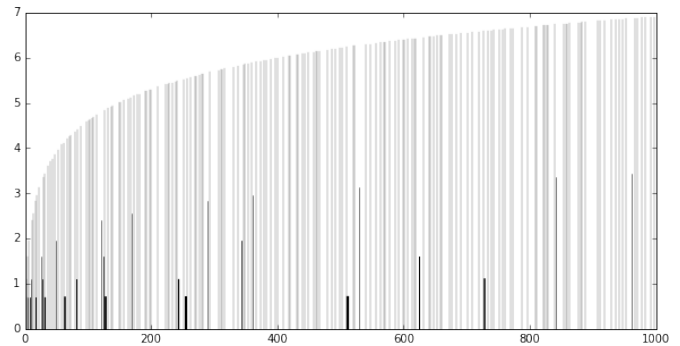
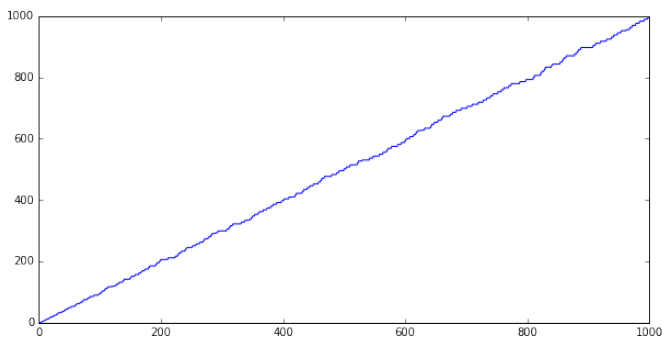
The prime number says that the density of primes is roughly  $\frac{1}{\# \text{ digits}}$  it can also be phrased as:

$$\sum_{n \leq x} \Lambda(n) = x + o(x)$$

With a little bit of work, this function can be charted, just like any other. We'd like to be able to say the derivative of the second equation is the first one:

$$\frac{d}{dx} \sum_{n \leq x} \Lambda(n) = \frac{d}{dx} (x + o(x)) = 1 + o(1) \quad \text{but} \quad \frac{d}{dx} o(x) \stackrel{?}{=} o(1)$$

So our instinct – mine at least – really falls apart on this problem. Unfortunately, I can try to get other instinct and it might look weird, but at least it's more accurate. For one thing: the line is not perfectly straight:

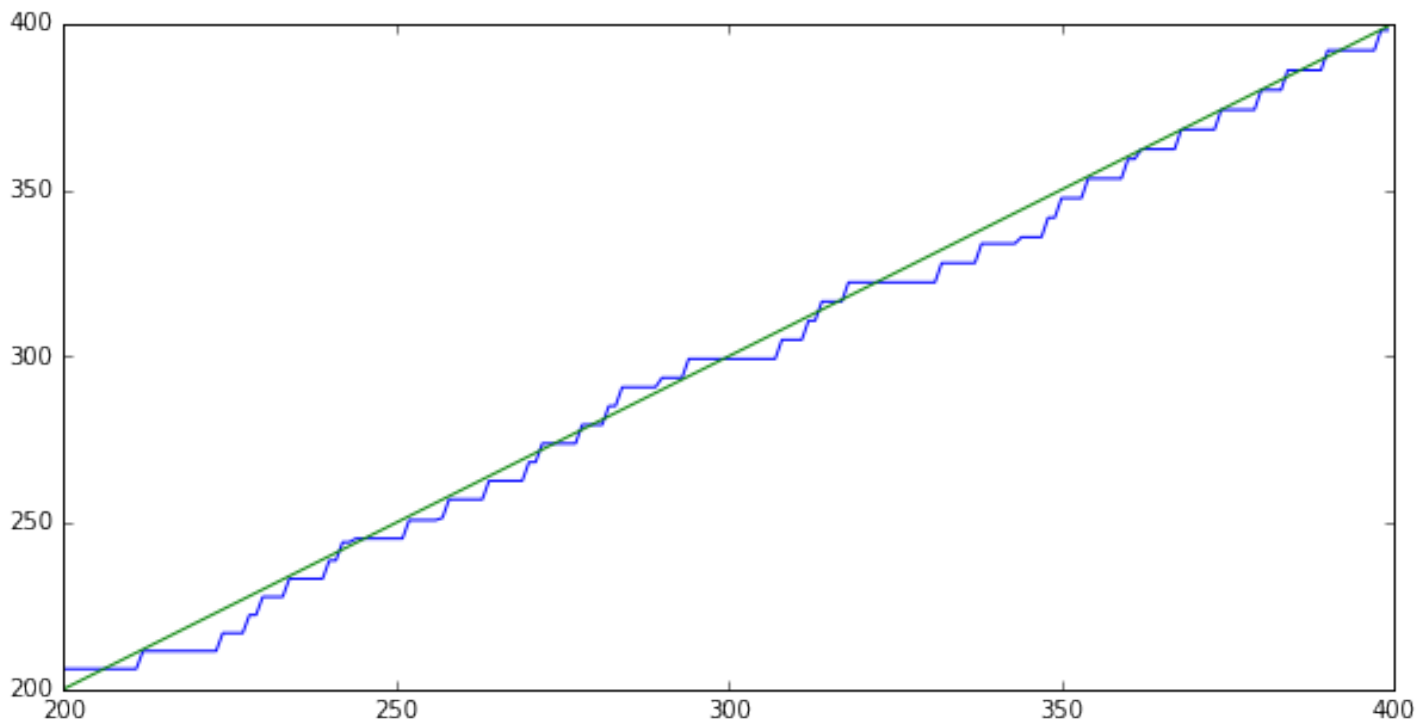


This took several years to state the problem in a visual way that I felt comfortable with. It exists in the literature. Always after you find it, suddenly the paper is right there:

- Don Zagier **The First Million Prime Numbers**<sup>8</sup>

I really want to drive the point home, how decidedly not straight these lines are.

<sup>8</sup><http://people.mpim-bonn.mpg.de/zagier/files/doi/10.1007/BF03039306/fulltext.pdf>



The prime number theorem is important because there are many other statements that I care about that are equivalent to PNT, so that it has become a sort of “gatekeeper” for me. And I can’t exactly say I’ve overcome, but we have a reasonable shot of pinning the last details here: ... ..

For now I’m in a hurry I’ll state some variants of Prime Number Theorem.

- the angle of a prime number in  $\mathbb{Z}[i]$  is equidistributed as  $p \gg 1$ . This one doesn’t look like PNT at all, it starts from noticing that  $5 = 2^2 + 1^2$  but  $7 \neq \square + \square$ .
- the Sato-Tate conjecture
- the Chebotarev Density Theorem

Despite their profound modernity, all of these are slightly old fashioned. Here I tend to check the most recent arXiv articles to see what the latest conjecture is.<sup>9</sup>

I think by the 1920’s, 30’s and 40’s I started to find really good, convincing proofs of PNT. Parts of it were still being settled the 80’s and 90’s and up to now. I heard the version I am about to give, has really bad constants.

If we wanted to we could sequester the Prime Number Theory to five very basic pages:

- Don Zagier **On Newman’s Short Proof of the Prime Number Theorem** American Mathematical Monthly (1997)

<sup>9</sup>And experts in other fields are not always sure where the frontier is. You and I should feel free to settle here, or if you know a lot, to take a brand new position! Here’s one:

- Kaisa Matomäki, Maksym Radziwiłł, Terence Tao **An averaged form of Chowla’s conjecture** arXiv:1503.05121

The time is certainly approaching!

# References

- (1) Richard Courant, David Hilbert. **Methoden der mathematischen Physik** (Satz I) Springer, 1937.
- (2) Elias Stein, Rami Shakarchi. **Analysis III: Real Analysis - Measure Theory, Integration and Hilbert Space** (Princeton Lectures in Analysis) Princeton University Press, 2004.

