Scratchwork: Theta Functions

William Duke's proof that the solutions to $n = a^2 + b^2 + c^2$ become equidistributed as $n \to \infty$ takes a quarter of a page:

- Goro Shimura shows there are many theta functions, each invariant under $\Gamma_0(4)$

$$\theta(z; u) = \sum_{m \in \mathbb{Z}^3} u(m)\, e(z|m|^2) = \sum_{n > 0} n^{\ell/2}\, r_3(n) \left[\frac{1}{r_3(n)} \sum_{\xi \in V_3(n)} u(\xi) \right] e(nz)$$

  one for each spherical Harmonic $u \in L^2(S^2)$. Here $|m|^2 = m_1^2 + m_2^2 + m_3^2$ and $V_3(n) = \#\{(a,b,c) : a^2 + b^2 + c^2 = m\}$.

- Henryk Iwaniec offers a bound for the Fourier coffients of cusp forms.

$$a_n \ll_{k,\epsilon} n^{k/2 - 2/7 + \epsilon}$$

  These are tending to zero if we fix a tolerance (ϵ) and a "weight" of modular form (k).

- Combining Iwaniec and Shimura's result[1] we obtain an estimate for the sphere averages

$$\frac{1}{r_3(n)} \sum_{\xi \in V_3(n)} u(\xi) \ll_{u,\epsilon} n^{-1/28 + \epsilon}$$

  This bound depends on the spherical harmonic (u) and the tolerance (ϵ). And we need $n \not\equiv 7 \pmod 8$.

There have been many surprises along the way learning this topic. And I have problems with a lot of this discussion, because these are professors talking to other professors. Slowly turning these objections into contributions of my own!

- $\theta(z)$ is $\Gamma_0(4)$ invariant not $SL(2,\mathbb{Z})$ invariant. As subgroups the index $[SL(2,\mathbb{Z}) : \Gamma_0(4)] = 6$. For the record $\Gamma_0(4) \simeq \langle z \mapsto z+1, z \mapsto -\frac{1}{4z} \rangle$ while $SL(2,\mathbb{Z}) \simeq \langle z \mapsto z+1, z \mapsto -\frac{1}{z} \rangle$. These are like continued fractions $a = [a_0; a_1, \dots, a_n]$ all of whose digits are multiples of 4.

- $\theta(z)$ is *not* a cusp form, but if $\deg u > 0$ then $\theta(z; u)$ **is** a cusp form. The Fourier series *is* the q-series as $q = e^{2\pi i t}$ is the change of variables.

$$\theta(z; u) = 0 + a_1\, q + a_2\, q^2 + \dots$$

- Even though we are studying equidistribution of solutions to an equation $f(x) = n$ we don't necessarily study the existence of one solution. After reading Serre's **Course on Arithemtic** (GTM #85) we learn that the existence of one solution is a rather deep problem. Lastly, I don't think it's "done" by the time we discuss equidistribution.

- I didn't know what the symbol "$\ll$" meant. $1 \ll 100$ and also $100 \ll 1$. Here's one more: $10^{100} \ll x$ since *eventually* $x > 10^{100}$. These are things we learn in beginning math classes, but sometimes a brand new symbol is introduced and we have to do it again.

[1] and an estimate of Siegel, which I haven't even looked at, $r_3(n) \gg_\epsilon \sqrt{n^{1-\epsilon}}$.

Then I find out all of this is slightly out of date. This one pricked my last nerve. I believe, in the process of verifying Duke's claim (leaning me through Iwaniec and Shimura and Walspurger and even more) we have made new lemmas. One of them is definitnely new.

OK. Here's a problem statement: let's try to find the constant to go with the $\ll$ sign:

$$\left[a_n \ll n^{k/2-2/7+\epsilon}\right] \to \left[a_n < C\, n^{k/2-2/7+\epsilon}\right]$$

The constant C is not known (probably because nobody cares) if we fix (ϵ) and (k).

Shimura's constructions are the start of the **theta correspondence**, and I'm choosing to use the out-of-date version that doesn't use any of Waldspurger's technology. Duke makes no use of the adeles, $\mathbb{A}$; any strategy involving them will be new. Iwaniec makes heavy use of Eisenstein series and it's rather mysterious:

$$[\text{theta functions}] \to [\text{Eisenstein series}]$$

Iwaniec says it works, making such a kind of map, risk-free, but do we really understand it? I don't immediately have a problem that is unkown about it. The more we unpack, the scarier it gets.

$$[\text{Kloosterman sums}] \quad || \quad [\text{Bessel functions}]$$

Do we really understand this? In the case of 4-squares, the proof involves the Weil conjectures, yet in 3-squares this just falls out of some very tricky averaging procedures, that I do not wish to replicate.[2]

This discussion has created some deep and lingering doubts: do I understand Weyl's equidistrubiton criterion *for spheres* or requirements for Poisson summation? Do I know what it was so important that $\theta(z;u)$ was a **holomorphic** cusp form, or how to get number-theoretic information out of that? Not really.

Waldspurger Formula

Another, equally vague, proof of equidistribution comes from the study of torus integrals, but what kind of torus? As a collection of points on the sphere $G_d = \{(a,b,c) : a^2+b^2+c^2 = d\} \subseteq X = S^2$ we could assign a probability measure for each number $d \in \mathbb{Z}$, and compute averages with respect to this probability measure:

$$\int_{S^2} \phi\, \mu_d = \frac{1}{r_3(n)} \sum_{(a,b,c)\in G_d} \phi\left(\frac{a}{\sqrt{d}}, \frac{b}{\sqrt{d}}, \frac{c}{\sqrt{d}}\right)$$

There's some measure μ_d that magically when you integrate against it returns the sphere averages. The equidstirbution statement now looks very terse:

$$\lim_{d\to\infty} \mu_d = \mu$$

where μ is the Lebesgue measure on the 2-sphere $X = S^2$. To me, Lebesgue measure means that despite equidistribution, interesting sets may occur on the way as $d \to \infty$. Weak-* convergence measns:

$$W(\phi, d) := \int_X \phi\, \mu_d \to 0 \text{ as } |d| \to \infty$$

as we let ϕ range over an fixed orthogonal basis of continuous functions[3] $\phi \in L^2_0(X,\mu)$. In our case, these are **spherical harmonics**.

[2] at this time

[3] no step functions!!

The Weyl averages could be written as an "integral" over a two-sided quotient of Torus.

$$W(\phi, d) = \int_{T_d(\mathbb{Q})\backslash T_d(\mathbb{A})/K_{T_d}} \phi(z_d.t)\, dt$$

The exericse would be to understand what this torus could be; The paper I was reading had a typo. The torus is described using "restriction of scalars". Let $K = \mathbb{Q}(\sqrt{d})$:

$$T_d := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)/\mathbb{G}_m$$

which measures how much bigger $\mathbb{G}_m$ is over K than over Q. And it would be great if we knew a definition of $\mathbb{G}_m$. In fact, it's not just a group it's a **functor** from the category from (the oppososite category) of $\mathbb{Q}$-schemes to the category of groups:

$$\begin{array}{rcl} \mathrm{Res}_{K/\mathbb{Q}} : (\mathbb{Q}\text{-}\textbf{scheme})^{\mathsf{op}} & \to & \textbf{Group} \\ \mathrm{Res}_{K/\mathbb{Q}}\, X(S) & = & X(S \times_{\mathbb{Q}} K) \end{array}$$

where $\mathbb{G}_m$ is the multiplicative group. And then I need a definition of K_{T_d}.

...

If we keep going, there is the Waldspurger formula finally:

$$|W(\varphi, d)|^2 = c_{\varphi,d}\, \frac{L(\pi, \frac{1}{2})\; L(\pi \times \chi_d, \frac{1}{2})}{L(\chi_d, 1)^2 \sqrt{d}}$$

where φ is a "new cuspform" - the L^2 normalized newvector in some automorphic representation π. And where π' [sic] is the GL_2 automorphic respresentation corresponding to π by the Jacquet-Langlands correspondence, and $c_{\varphi,d}$ is a number.

The Waldspurger formula is used in conjunction with a **subconvexity bound**

$$\left[L(\pi \times \chi_d, \frac{1}{2}) \ll_\pi |d|^{1/2-\delta} \right] \text{ therefore } \left[W(\phi, d) \to 0 \text{ as } d \to \infty \right]$$

My problem with this reductionist point of view is that someone has to be responsible for placing these abstract results back into a context.[4] Fortunately, there is one and Akshay Venkatesh and his colleagues have developed the sparse equidistribution framework for solving this type of problem.

Unfortunately we have left out a lot of details. As long as we focus on the basics, there are several textbooks that have emerged since 2005. I like representation theory and geometry so there are at least two textbooks[5]

- Anton Deitmar **Automorphic Forms** Universitext, 2013.
- Franoise Dal'Bo **Geodesic and Horocyclic Trajectories** Universitext, 2011.
- Akshay Venkstesh **Sparse equidistribution problems, period bounds, and subconvexity** `arXiv:math/0506224` Annals of Mathematics, 172 (2010), 989-1094

[4] `https://en.wikipedia.org/wiki/Reductionism`

[5] Thus as research topics trickle down into graduate-level teaching, first textbooks written for new, cutting-edge courses may make their way into Universitext.

Comprehension Check Is $\theta(z,u) = \sum_{m\in\mathbb{Z}^3} u(m)e(z|m|^2)$ an automorphic form? What is the representation?

Getting Ahead of Ourselves Are solutions to $n = a^2 + b^2 + c^2$ points on a quaternion Shimura variety?