Theta Functions and Chiral Dirac Fermions

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A spinor is a spin-$\frac{1}{2}$ representation of $SU(2)$. A spinor field on the Torus is a map ψ from the torus $S^1 \times S^1$ to that representation. So the torus spinor ψ is – somewhat oversimplifying – a vector in $\mathbb{C}^2$ with entries whch are functions of θ, ϕ which are the **angles**:

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

Geometry without visualization is a moot point, but these are difficult to draw and sadly we must continue. This vector should behave a certain way under the action of $SU(2)$ which here is really the 2×2 unitary matrices. And boundary conditions:

$$\psi(a + 2\pi, b) = -e^{2\pi i\theta}\psi(a, b) \quad \text{and} \quad \psi(a + 2\pi, b) = -e^{-2\pi i\phi}\psi(a, b)$$

The "chiral Dirac operator with Twisted boundary conditions" is related to the Hamiltonian:

$$H = \sum_{n\in\mathbb{Z}} :b^\dagger_{n+\theta-\frac{1}{2}} b_{n+\theta-\frac{1}{2}}: + \left(\frac{\theta^2}{2} - \frac{1}{24}\right)$$

The $b, b^\dagger$ are called the raising and lowering operators:

$$\{b^\dagger_m, b_n\} = \delta_{m,n} \text{ and } \{b_m, b_n\} = \{b^\dagger_m, b^\dagger_n\} = 0$$

These are like infinitely many copies of the Harmonic oscillator, $x, \frac{d}{dx}$ one for each element of $\mathbb{Z}$. Or maybe like taking e^{imt} for the various Fourier modes.

This deserves a more careful write-up, but the string theory paper doesn't way much more than this. Quickly moves to other Riemann surfaces, which we might need. How does $\mathrm{SL}_2(\mathbb{Z})$ act on this theta function?

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+7n^2}$$

Confusingly this is function of z even though the lattce is $\mathbb{Z} + \sqrt{-7}\,\mathbb{Z}$.

The determinant of the chiral Dirac operator is the theta function.

$$\begin{aligned}\mathrm{Det}(\theta,\phi) &= \mathrm{Tr}\, g q^{H} \\ &= e^{2\pi i\,\theta\phi} q^{\frac{\theta^2}{2}-\frac{1}{24}} \prod (1+q^{n+\phi-\frac{1}{2}}\, e^{2\pi i\phi})(1+q^{n-\phi-\frac{1}{2}}\, e^{2\pi i\phi})\end{aligned}$$

Sometimes, confusingly, I hear about the Cauchy-Riemann operator. What physicists are calling "spin structure" and "twisted chiral Fermion" looks awfully like the $SL(2,\mathbb{Z})$ action on the cosets of $\Gamma_0(4)$ which there are exactly 6 of them.

References

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