Proposal: Factorial

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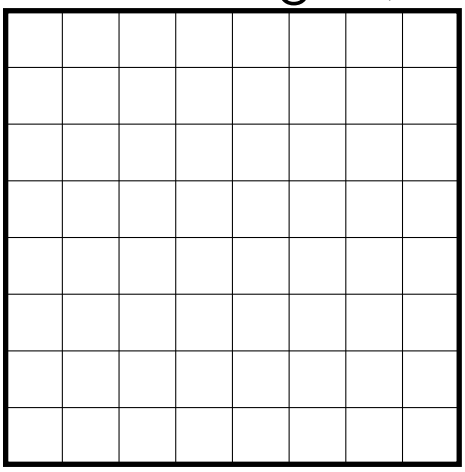
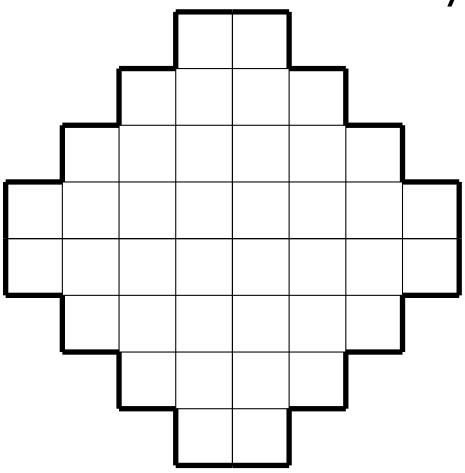
Some clever person turned the theory domino tilings into a fundamental object of mathematics and of nature. For a long time there were really two shapes being studied.

The rectangle (here an 8×8 square):



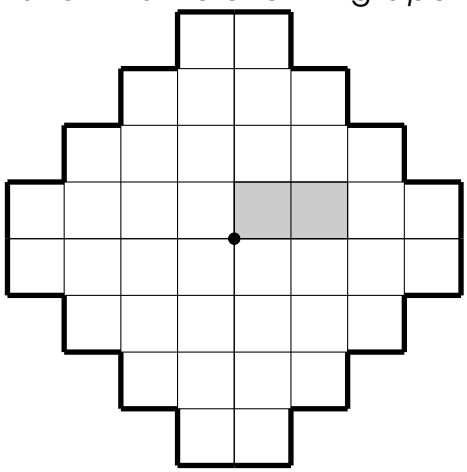
And I wonder why this particular shape is so essential:



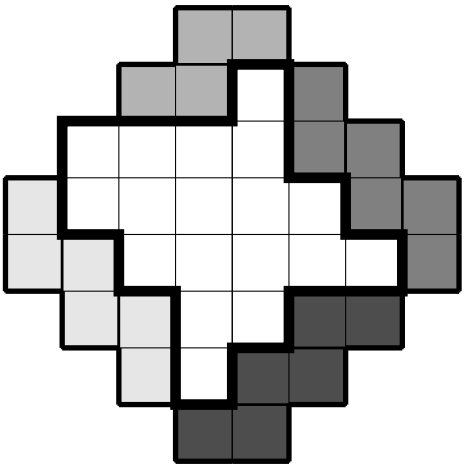
Pathetic Tutorial:

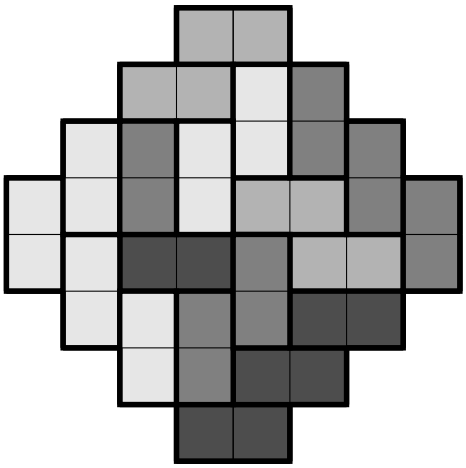
```
\begin{tikzpicture}[scale=0.75]

\foreach \a in {0,...,3}{

\draw[line width=2] (\a  ,4-\a)--(\a+1, 4-\a  );
\draw[line width=2] (\a+1,4-\a)--(\a+1, 4-\a-1);

\draw (   \a, 4-\a)--(   \a, \a-4);
\draw (-1*\a, 4-\a)--(-1*\a, \a-4);

\draw ( 4-\a,    \a)--(\a-4,    \a);
\draw ( 4-\a, -1*\a)--(\a-4, -1*\a);

\def \b {-1}
\def \c { 1}

\draw[line width=2] (\b*\a     ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a     );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

\def \b { 1}
\def \c {-1}

\draw[line width=2] (\b*\a     ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a     );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

\def \b {-1}
\def \c {-1}

\draw[line width=2] (\b*\a     ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a     );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

}

\end{tikzpicture}
```

In French the word for tiling is *pavage* – so literally we are **paving** the shapes with dominoes.

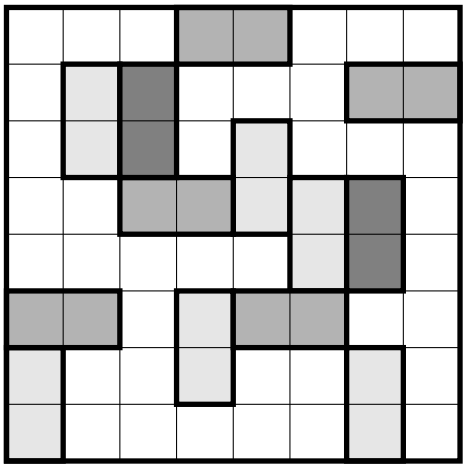


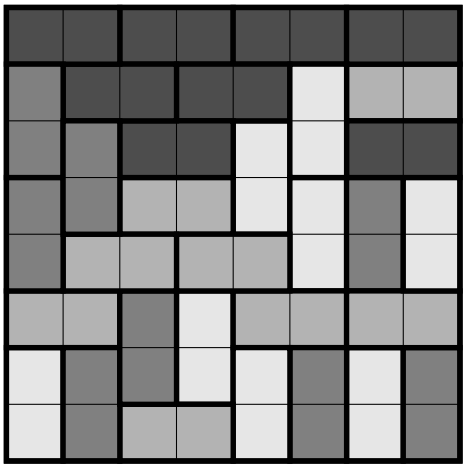
Let's put two reasonable tilings on the board. One strategy for this shape is to start from the corners and work indwards. And in this case we get lucky: it always works.





And for rectangle the case is even clearer. There are no intermediate stages. I mean, if you put enough tiles down there can be come question as to whether you put yourself in a corner yet.





There is also the lovely John Conway game of "Domineering" which is not related but you also place dominoes on a checkerboard. See **Winning Ways for Your Mathematical Plays** (Vol I).

Exercise Fill the rest of the tiling.

Answer There is no solution, but if you move the tiles slightly (actually quite a bit here...) you can get a problem with an answer.

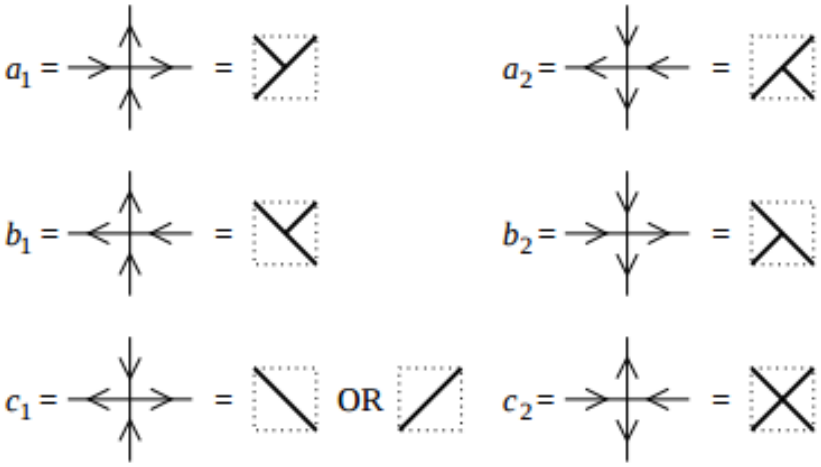
The number of tilings on the top looks atypical – just purely on a hunch. Where did that hunch come from? Is it right?

As with all hunches, it can be proven with hundreds of pages of equations this tiling is not likely to appear in a rectangles. We'll settle for somewhat simpler patterns.

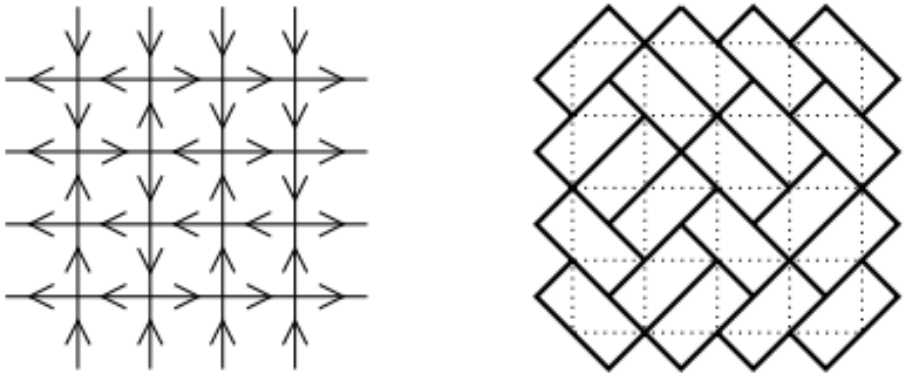
Next I hav to address some redundancy. Every couple of years, it seems there were more and more people who came up with a similar theory and did not talk to each other.

- **domino tilings**
- dimer model
- perfect matchings
- **six-vertex model**
- Alternating Sign Matrix (**ASM**)
- XXZ spin chain
- Ising model

Domino tilings and six-vertex are closely related. You can see the family resemblance.



For every 6-vertex configuration there is a domino tiling, up to various factors of $\times 2$.



The Aztec Diamond is a natural shape because it **is** the square lattice.

Problem Session Let us compute some square-ice and domino tiling partition functions!

References

(1) Paul Zinn-Justin **Six-Vertex Model with Domain Wall Boundary Conditions and One-Matrix Model** arXiv:math-ph/0005008

(2) Noam Elkies, Greg Kuperberg, Michael Larsen, James Propp
Alternating-Sign Matrices and Domino Tilings (Part I)
Journal of Algebraic Combinatorics (1992) 1: 111. doi:10.1023/A:1022420103267

(3) Patrick Ferrari, Herbert Spohn **Domino tilings and the six-vertex model at its free fermion point** arXiv:cond-mat/0605406

(4) Sunil Chhita, Kurt Johansson **Domino statistics of the two-periodic Aztec diamond** arXiv:1410.2385

Big Idea

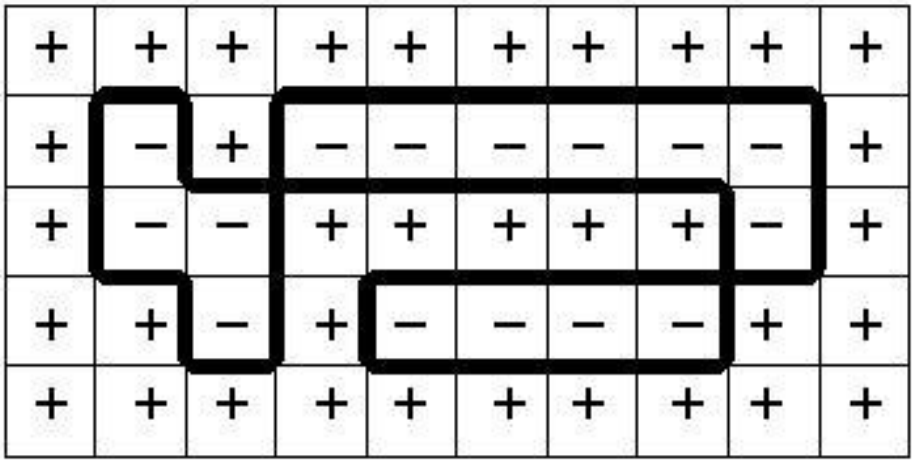
The link between the domino tilings and the ising model is "well known" but I can never find out all the details I want. And if I ask the question, it doesn't have any merit.

Who are we to believe?

There are actually *two* Ising models. The statistical Ising model and the Ising Conformal Field Theories.

The bad news is, statistical models – or maybe, **lattice gauge theories**. Here's a simple question: are domino tilings an instance of lattice gauge theory? The answer is likely "yes" but all of the responses are wanting.

The domino tilings like I have been showing you? Which lattice gauge theory was that? Domino tilings is very simple, lattice gauge theory rather involved.



The Ising model is even more simple-minded than the domino tilings. How could it be so advanced?

This is my attempt to be "contemporary" and "advanced".

References

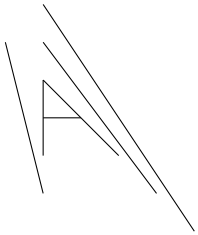
(1) Davide Gaiotto, Anton Kapustin **Spin TQFTs and fermionic phases of matter** arXiv:1505.05856

(2) Nathan Seiberg, Edward Witten **Gapped Boundary Phases of Topological Insulators via Weak Coupling** arXiv:1602.04251

(3) Davide Gaiotto, Anton Kapustin, Zohar Komargodski, Nathan Seiberg **Theta, Time Reversal, and Temperature** arXiv:1703.00501

(4) Werner Krauth **Statistical Mechanics: Algorithms and Computations** (Oxford Master Series in Physics), OUP, 2006.

The most amazing typo ever:



```
\begin{tikzpicture} [scale=0.5]
\foreach \a in {0,...,5}{
    \draw (\a, 5-\a)--(1, \a + 1);
}
\end{tikzpicture}
```