Examples: ABJM Theory

John D Mangual

There's not a whole lot to say until we write the formula:

$$\int \left(\prod_i e^{ik\pi(\theta_i^2 - \phi^2)} \right) \times \frac{\prod_{i \neq j} \left(2 \sinh \pi(\theta_i - \theta_j) 2 \sinh \pi(\phi_i - \right.}{\prod_{i,j} (2 \cosh \pi(\theta_i - \phi_j))^2}$$

This integral could possibly be over $[0, 2\pi]^2$ but I am expecting it's over the real numbers:

$$(\theta, \phi) \in \mathbb{R}^N \times \mathbb{R}^N$$

since this the domain of integration of the reals. Here is the simplest one:

$$\int_{-\infty}^{\infty} d\theta \; e^{ik\pi\theta^2} = \sqrt{\frac{\pi}{8}}(1+i)$$

I have not done that yet. If we set $N = 1$ or $N = 2$ and this integral is totally feasible.

ABJM theory could refer to two things:

- The $U(N) \times U(N)$ Chern-Simons-matter theory with $\mathcal{N} = 6$ superconformal symmetry. The first $U(N)$ is at level k and the second $U(N)$ is at level $-k$.
- The $U(N) \times U(N)$ Gaussian matrix integral the CS-theory localizes to.

Both of these are called **ABJM theory**. This is rather confusing.

Here is another Chern-Simons formula:

$$Z_{\text{CS}}(N,k) = \frac{1}{\sqrt{N+k}} \prod_{\alpha>0} 2\sin\frac{\pi\,\alpha\cdot\rho}{k+N}$$

where $\alpha_{ij} = e_i - e_j \in \mathbb{C}^N$ are vectors[1] and

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{i=1}^{N}\left(\frac{N+1}{2} - i\right)e_i$$

and also $k \in \mathbb{Z}$ is a positive integer, and so is $n \in \mathbb{Z}$.

Therefore our Chern-Simons partition function Z_{CS} should be a number.

[1]This is called the **Cartan subalgebra** a term which should mean nothing right now.

My understanding is that any localization result of a supersymmetric gauge theory that is not **flat** is indebted to Vasily Pestun[2].

Flat spaces are things like $\mathbb{R}^4$ and even $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ which are **distorted** versions of flat 4-dimensional space.

The curved spaces being considered are remarkably simple $\mathbf{S}^3$ the 3-sphere and $\mathbf{S}^4$ the 4-sphere[3]

We get hints from Pestun's original paper:

> equivariant Euler class of the infinite-dimension normal bundle to the localization locus

I don't know what this means. Maybe some complicated infinite dimensional object has been erected over our 4-sphere, S^4 ?

[2]This is just looking at the citations. The 4-sphere is a curved four-dimensional space, I guess.

[3]In either case the procedure is the same:

- Do a very complicated algebra
- Claim a localization result
- Solve the integral

I will have a little bit to say about each of these steps. Here are some old ideas that may help:

- Invariant Theory, Spherical Harmonics
- Symlectic Geometry, ODE
- Integral Geometry

The Chern-Simon's matter action is involved:

$$\frac{k}{4\pi}\int \mathrm{Tr}\big(A\wedge dA+\frac{2}{3}A^3-\overline{\chi}\chi+2D\sigma\big)$$

called a **multiplet** and a **superpotential**

$$W=\frac{2\pi}{k}\epsilon^{ab}\epsilon_{\dot{a}\dot{b}}\mathrm{Tr}(A_a B_{\dot{a}} A_b B_{\dot{b}})$$

and these are somehow going to localize.

Why do Path Integrals Localize to Gaussians[4]

An action depends on a function. By calculus of variations:

$$S(x+\delta x)=S(x)+\delta x\, S'(x)+\frac{(\delta x)^2}{2}S''(x)$$

Our job should be to set $\boxed{S'(x)=0}$ this is our critical point. Feynman writes the path integral:

$$Z=\sum e^{S(x)}$$

However our logic is the same, this sum should concentrate *near* the classical trajectory:

$$Z\approx e^{x_{\mathrm{cl}}}\sum_{x=x_{\mathrm{cl}}+t\delta x} e^{\frac{1}{2}t^2\, S''(x)}$$

[4]As soon as you hear the logic two things should happen:

- You should remember this has appeared in every single quantum mechanics textbook.
- Realize this is entirely bull-shit.

References

(1) Anton Kapustin, Brian Willett, Itamar Yaakov. **Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter** `https://terrytao.wordpress.com/2009/08/23/determinantal-processes/`

(2) Ofer Aharony, Oren Bergman, Daniel Louis Jafferis, Juan Maldacena **N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals** `arXiv:0806.1218v4`

(3) Richard Feynman **Path Integrals and Quantum Mechanics** Dover, 2010.