Scratchwork: Farey Fractions

Here's a nice question about Farey Fractions:

18

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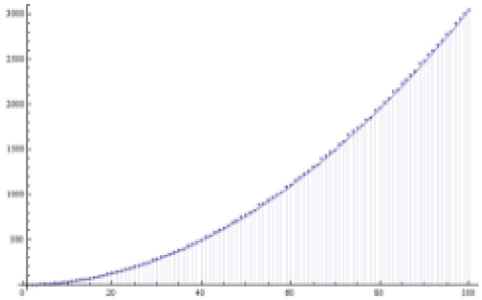
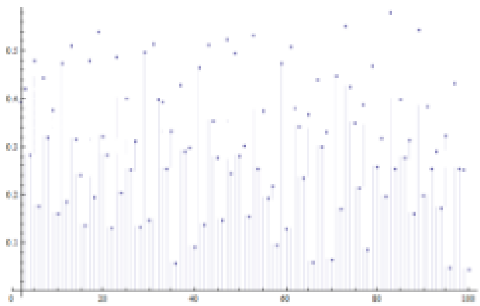
The number theory identity $\phi(1) + \phi(2) + \cdots + \phi(n) \approx \frac{3n^2}{\pi^2}$ can be interpreted as counting relatively prime pairs of numbers $0 \leq \{x, y\} \leq n$.



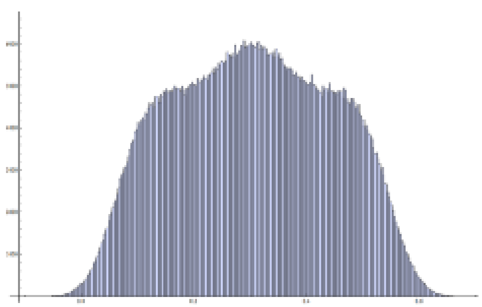
Has anyone studied the distribution of error term?

$$\frac{1}{n}\left[\sum_{k=1}^{n} \phi(k) - \frac{3n^2}{\pi^2}\right]$$

It looks like white noise:



The histogram has a distinctive shape, maybe hard to prove. I suspect it's the Gaussian Unitary Ensemble (a Hermite polynomial times a Gaussian).



Similar questions:

Question concerning the arithmetic average of the Euler phi function:

averages of Euler-phi function and similar

st.statistics nt.number-theory

What's good or bad about this question? These "elementary" questions tend to be the most applicable. If you think of a "number" you're probably thinking of $\mathbb{Z}$. However, if we're more empirical, that object behaves like a number with a few common-sense exceptions. Well, we have exited the realm of $\mathbb{Z}$ - it is some other object. If we continue into the pristine world of number theory where everything is known to infinite accuracy, the **visible points** of $\mathbb{Z}^2$ might be part of a family of sets of points for each number field $K/\mathbb{Q}$, or maybe there is variant of Euler ϕ function associated to modular form.

If we remain in $\mathbb{Z}$ we are asking for a push towards the Riemann Hypothesis. There's no rush. However, in the vaguely-titled **On the error term of a lattice counting problem** they considere the Farey Fractions:

$$\mathcal{F}(T) = \{\frac{a}{b} : (a,b) \in \mathbb{Z}^2, 0 \leq a \leq b \leq T, \gcd(a,b) = 1\}$$

The subset of the Farey Fractions he chooses to measure is rather specfic. Less than $\frac{1}{2}$

$$\mathcal{I}(T) = \mathcal{F}(T) \cap [0, \frac{1}{2})$$

For each Farey Fraction, we define a subset rather close to 1:

$$\mathcal{C}_{a,b}(T) = \mathcal{F}(T) \cap [1 - a^2/b^2, 1]$$

and we define some kind of counting measure as the sum over all these fractions:

$$C(T) = \sum_{a/b \in \mathcal{I}(T)} \#\mathcal{C}_{a,b}(T)$$

He tells you an interpretation of these fractions: s the number of similarity classes of semi-stable arithmetic planar lattices of height at most T. And there's a lot of number theory based on that, using dynamial systems. What was his result?

$$C(T) = \frac{3}{8\pi^4} T^4 + O(T^3 \log T)$$

I've always wanted to "interpret" these error terms. At least the constant, $3/8\pi^4$ I feel I understand better. Maybe not even that. They improve it to:

$$C(T) = \frac{3}{8\pi^4} T^4 + O\big(T^3 (\log T)^{2/3} (\log\log T)^{4/3}\big)$$

and they proceed to do whatever transformatins they are going to do.

What is a fraction? Is it a proportion? Is it the direction of a ray in space? If the fraction is 63% can we get a way with saying "two-thirds"? Etc. These fractions are generated by some kind of **process** and modeling that process could lean to an argment that feels more concrete. Have we pushed towards the deeper issues?

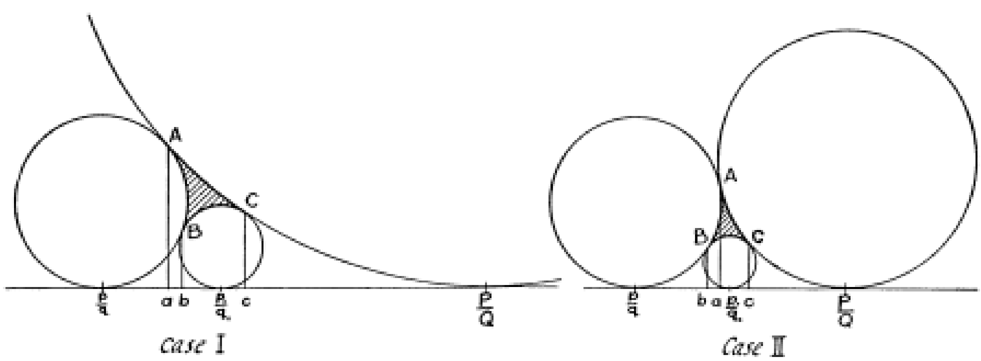


FIG. 3

From 1938, showing Markov's theorem that $\alpha \notin \mathbb{Q}$ implies that $|\alpha - p/q| < 1/\sqrt{5}q^2$ has infinitely many solutions. There just happens to be enough "room" in configuration space.

References

[1] MathOverflow **Error to sum of Euler phi-functions** `https://mathoverflow.net/q/95836/1358`

[2] Noam D. Elkies and Curtis T. McMullen **Gaps in $\sqrt{n}$ mod 1 and Ergodic Theory** Duke Math. J. Volume 123, Number 1 (2004), 95-139.

[3] Lester Ford **Fractions** American Mathematical Monthly. Vol. 45, No. 9 (Nov., 1938), pp. 586-601.

[4] Olivier Bordellès, Florian Luca, Igor E. Shparlinski **On the error term of a lattice counting problem** Journal of Number Theory Volume 182, January 2018, Pages 19-36