Tune-Up: Clock Arithmetic

Thm (2010) "Non-Conventional" ergodic theorem:

- $T : \mathbb{Z}^r \curvearrowright (X, \Sigma, \mu)$ commuting probability-preserving $\mathbb{Z}^r$ actions
- $(I_N)_{N \geq 1}$ Følner sequence of subsets of $\mathbb{Z}^r$.
- $(a_N)_{N \geq 1}$ sequence of points in $\mathbb{Z}^r$
- $f_1, f_2, \dots f_d \in L^\infty(\mu)$

The sequence of "non-conventional" ergodic averages converges in $L^2(\mu)$.

$$\frac{1}{|I_N|} \sum_{n \in I_n + a_N} \prod_{i=1}^{d} f_i \circ T_i^n$$

Let's try a re-phrasing of the theorem.

Thm (2010) The sequence x_n converges. $x_n \to x$.

Let's try with a couple of more details. We have the sequence of averages of other functions.

Thm (2010) The sequence of functions $\text{Avg}_1(f), \dots \text{Avg}_n(f)$ converges in the function space $L^2(\mu)$. Here $Avg_n = \sum_{i=0}^{n-1} T^i$ or "shuffle n times", we have separated the procedure from the thing it's acting on.

Ex $\mathbb{Q}^\times$ is a multiplicative group. I am being sloppy we could write $\mathbb{Q}^\times \simeq \mathbb{Z}^\infty$. This is a contradiction because we could write an equally true statement for any number field $F^\times \simeq \mathbb{Z}^\infty$. The term seems to be "loclly compact abelian group".

Ex $\times 2 \times 3$ these are commuting operations. The arithmetic makes it instantly clear even though

- "divide by 2 parts and shuffle"
- "divide by 3 parts and shuffle"

could be very different if we switch the operations.

d=1 This is the **Von Neumann ergodic theorem** which is already in the textbook[1].

Thm (Mean Ergodic Theorem) Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system.

$$\frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n$$

converges to a limit for any function $f \in L^2(\mu)$.

Here $U_T : L^2(\mu) \to L^2(\mu)$ is the operator associated to the measure-preserving transformation.

$U_T(f) = f \circ T$.

How do we use the ergodic theorem?

- mostly likely we will observe the dynamical system *first*
- Then we can guess the limiting average behavior(s).
  - chaotic
  - various patterns
- we can guess or prove that the behavior is ergodic or possiby mixing

Example A radio channel has an average sound that's just slightly over silent.

Example A highway is 50% occupied and half the cars are moving "left" and half the cars are moving "right".

Example The average color of a TV scrren is gray.

In colloquial speech we are comparing the two ways of averaging a dynamical system.

$$\text{[space average]} = \text{[time average]}$$

so if we observe a dynamical system over time we compute the average we could have also computed the average over space. Ergodicity is a spectral property so this really depends on the "color" of the "noise" that we are observing.[2]

[1]it's new textbook as of 2011

[2]Hence they are called "observables".

References

[1] Tim Austin **On the Norm Convergence of non-conventional ergodic averges** `arXiv:0805.0320`

[2] Manfred Einsiedler, Thomas Ward **Ergodic Theorem with a View Towards Number Theory** GTM#259. Springer, 2011.

[3] solenoids ...*Ergodic Theory and Dynamical Systems*