Scratchwork: Pythagorean Triples over $\mathbb{Z}[i]$

The equation $x^2 + y^2 = 1$ defines what we might call a **variety**. Here it's just a circle. Our decision to use Cartesian coordinates, algebra and equations, to describe Euclidean geometry, leads to all sorts of complications. Attempts to finalize what we might call a variety leads to all sorts of difficult **commutative algebra** and ultimately **schemes**.

Taking for granted a minute that circles are a meaningful concept and that we should use algebra and geometry, we observe the variety $X = \{x^2 + y^2 - 1 = 0\}$ has a rational paramterization.

$$t \in \mathbb{Q} \longrightarrow (x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \in \mathbb{Q}^2$$

And we can think of fractions as pairs of integers $\frac{m}{n} \in \mathbb{Q}$ or as pairs of integers up to proportion[1] $[m : n] \in P\mathbb{Z}^2$. Therefore we can solve an equation over integers: $a^2 + b^2 = c^2$ over $\mathbb{Z}$:

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2) \in \mathbb{Z}^3$$

The machinery of algebra tells us that all we required is that $K = \mathbb{Q}$ is a field. Therefore we could also try to solve the Pythagoras equation over $\mathbb{Z}[i]$ or $\mathbb{Z}[\sqrt{2}]$ and solve it in the same way.

What happens if we write the thing down we could write $a = a_1 + ia_2$ etc. and find:

$$(a_1 + ia_2)^2 + (b_1 + ib_2)^2 = (c_1 + ic_2)^2$$

and we learned we could seprate the real and imaginary components and find two equations:

$$\begin{array}{rcl} (a_1^2 - a_2^2) + (b_1^2 - b_2^2) &=& (c_1^2 - c_2^2) \\ 2a_1a_2 + 2b_1b_2 &=& 2c_1c_2 \end{array}$$

which is the intersection of two conic sections. And here because of the structure of $\mathbb{Z}[i]$ we can expect a spectactular reduction. Is that all?

[1] Yet another word we can scrutinize. What are we calling "proportionate"?