Scratchwork: Divergences

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It's time to get serious. I can nearly put it together myself.

How do I review this proof without it degenerating into some kind of recitation fo facts? One of my critiques of analytic number theory is that... it doesn't look like number theory. If I spend all this effort to prove the Prime Number Theory... I already believed it was true!

I started to look for arguments where the connection to prime factorization or to Geometry or Probablity or any other branch of Math.[1] Talking to professors, I'm pretty much out of luck. They are satisfied with the current arguments. They are professionals, I'm not.

Try to imagine if Hollywood or a Hip-Hop label adopted the garbled narrative style of a Math Textbook. To me, Mathematics has been a giant bait-and-switch. They sold me one result, I got completely another.

I'm trying not to do that to you.

Zagier proves PNT as a series of facts. He introduces 3 functions:

- $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$
- $\Phi(s) = \displaystyle\sum_{p>1} \frac{\log p}{p^s}$
- $\theta(s) = \displaystyle\sum_{p<x} \log p$

If you a sequence of numbers a_n here are three different numbers you could assign to that sequence:

$$\{a_n\}_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} a_n \text{ or } \{a_n\}_{n=1}^{\infty} \mapsto \sum_{p>1} a_p \text{ or } \{a_n\}_{n=1}^{\infty} \mapsto \sum_{p<x} a_p$$

One is a sum over *all integers* another is a sum over *all primes* and the other is over *primes less than* x. This is what Hardy saw: all these are just ways of assigning numerical averages to sequences of numbers. Some are better behaved than others.

He starts to prove a sequence of facts.

[1] If I express someone of my aggravation, and provide a partial demonstration / answer, that could be new.

#1 The Euler Product: this sum over integers n is also sum over primes p.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

This falls out of Unique Factorization, which calls out of the Euclidean Algorithm. Lurking in the background is the statment $\mathrm{Spec}(\mathbb{Z}) = \{$primes$\}$, in the event we choose to use scheme.

#2 If we subtract the pole at $s = 1$, this function extends holomorphically to $\mathrm{Re}(s) > 0$.

$$\zeta(s) - \frac{1}{s-1}$$

This statement is loaded. I never noticed it before. There is already "analytic continuation":

$$\zeta(\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots = \infty$$

We'll say, rather bluntly, that it equals infinity. Leaving more refined information later.
The numbers never get small enough. And we only subtract off a small amount:

$$\zeta(\frac{1}{2}) - \frac{1}{\frac{1}{2}-1} = \infty + 2$$

and perhaps this is why I dislike analytic number theory. We have to competely change our way of adding - without any mention at all - just because of this one small thing.

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{dx}{x^s} = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx < \infty$$

so for each term, we can subtract off the excess, and get a finite number.

#3 $\theta(x) = O(x)$. Could we be a little more obnoxious here?

$$\sum_{p<x} \log p = O(x)$$

if $f(x) = O(x)$ this means $f(x) \leq Cx$ and we don't care about the constant. Or how about this?

$$\frac{1}{x} \sum_{p<x} \log p = O(1)$$

which is saying this number is finite.[2] No surprises here.

#4 $\zeta(1+it) \neq 0$ and we have holomorphic function on $\mathrm{Re}(s) \geq 1$.

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{p>1} \frac{\log p}{p^s - 1} = \sum_{p>1} \frac{\log p}{p^s} + (\text{ finite number })(s)$$

[2]There are many theories to try to include "infinite numbers" but none of them have been successful. Even in the time I have left graduate school, there are peole trying and doing pretty well. And others, trying to remove all the divergences.

#5 → #6

$$\left[\int_1^\infty \frac{\theta(x)-x}{x^2}\,dx < \infty\right] \to [\theta(x) \sim x]$$

This notation is aggravating for a number of reasons. Usually I define the symbol θ to mean something else:

$$\theta(q) \neq \sum_{n=1}^{\infty} q^{n^2}$$

and I still don't know the preicse meaning of $A \sim B$ (instead of $A \asymp B$). Here it means $\frac{A}{B} \to 1$.

If I phrase it differently setting $\frac{1}{x}\theta(x) = f(x)$ and say:

$$\left[\int_1^\infty \frac{f(x)-1}{x}\,dx < \infty\right] \to \big[f(x) \sim 1\big]$$

I don't know if this is precise but I am a lot happier. That leaves us to prove item #5.

One great question here is why it was possible to write down the integral of $\theta(x)$. Between the $\int$ sign and the word "holomorphc" a lot of things have gotten swept under the rug. However, correctly.

#5 Tauberian theorems all have statements like "obviously that thing should converge" and then it does converge. And we could try to imagine functions where the implication does not hold.

In Chapter 1 of Titchmarsh, Tauber's Theorem was stated like this:

$$\left[\lim_{x\to 1} \sum_{n=0}^{\infty} a_n\, x^n = s\right] \overset{a_n = o(n^{-1})}{\to\to\to} \left[\sum_{n=0}^{\infty} a_n = s\right]$$

and it as the converse to **Abel's Theorem** which is why these are called Abelian and Tauberian theorems.

$$\left[\sum_{n=0}^{\infty} a_n = s\right] \to \left[\lim_{x\to 1} \sum_{n=0}^{\infty} a_n\, x^n = s \text{ (uniformly)}\right]$$

and we would confront an unknown series and we would apply one of these litmus tests and if we were careful it yielded a positive result.

$$\left[g(z) = \int_0^\infty f(t)e^{-zt}\,dt \textbf{ holomorphic} \text{ on } \mathrm{Re}(z) \geq 0\right] \to \left[g(0) = \int_0^\infty f(t)\,dt\right]$$

More belaboring the obvious. This is why we like Complex Analysis. At least in principles, theorem yield under $\mathbb{C}$ that wouldn't under $\mathbb{R}$. And this is our replacement for pages and pages of Tauberian theorems.

It remains to prove item #5, philosophize about the Cauchy-Riemann equations, meditate about many proofs whose statements are modeled after the Prime Number Theorem. This could be done later.

What could be done now, is to figure out why Sauzin talks about Zagier-Newman's paper in 2014 in this discussion on **resurgence** and **1-summability**. This is me taking something at face-value, when I shouldn't.

Zagier had taken Newman's short proof and made it even shorter:

$$\left[(\mathcal{L}\phi)(z) = \int_0^\infty e^{-zt}\phi(t)\,dt \text{ holo. } \mathrm{Re}(z) \geq 0 \right] \to \left[\int_0^T \phi(t)\,dt = (\mathcal{L}\phi)(0) \right]$$

I guess it's time to say something about holomorphic extension. The number $(\mathcal{L}\phi)(0)$ was not defined by writing an integral as it was for $(\mathcal{L}\phi)(z)$ with $\mathrm{Re}(z) > 0$. Here's one definition:[3]

$$g(0) = \frac{1}{2\pi i} \oint_C f(z)\, e^{zT} \left(1 + \frac{z^2}{R^2} \right) \frac{dz}{z}$$

and this integral is the same, regardless of the value of $T \in \mathbb{R}$. We could even set $T = 0$ and leave it out. On the other hand,

$$g_T(0) = \int_0^T f(t)\,dt = \oint_C f(z)\, e^{zT} \left(1 + \frac{z^2}{R^2} \right) \frac{dz}{z}$$

and now the statement looks really tautological, because we wish to show

$$\lim_{T\to\infty} \big| g(0) - g_T(0) \big| = 0$$

for some **bounded** and **locally integrable** function. These are big words, whose number-theory content I don't fully understand. Yet

$$f(t) = \theta(e^t) - e^t \quad \text{and} \quad g(z) = \frac{1}{z} \sum_{p>1} \frac{\log p}{p^s} - \frac{1}{z-1}$$

Hopefully I wrote the Laplace transform identities correctly:

- $\mathcal{L}(t\,\phi) = \dfrac{d\phi}{dz}$
- $(e^{ct}\,\mathcal{L}\phi) = \phi(z+c)$
- $\mathcal{L}(1 * \phi) = z^{-1}\phi(z)$
- $\mathcal{L}\left(\dfrac{d\phi}{dt} \right) = z\phi(z) - \phi(0)$

There's nothing "divergent" about this discussion, but I have to think about it some more.

For one thing I'd be concerned about writing an Laplace trasnform integral

$$\ldots$$

C is a semicircle contour $|z| = R$ and $\mathrm{Re}(z) > -\delta$, moved "infinitesially" over the real l ine.

[3] The contour C should contain the origin 0.

The Gamma function:

$$\Gamma(z+1)t^{-(z+1)} = \int_0^\infty x^z e^{-xt}\, dx$$

is the Laplace transform of a power function. Therefore if we write down an operator like:

$$g(0) = \frac{1}{2\pi i} \oint_C f(z)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

we might ask for the inverse Laplace transform of g and hopefully it will be something like f.

$$\sum_{n=0}^{\infty} a_n z^n \overset{\mathcal{L}^{-1}}{\to} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

The problem is unequivocally solved, but there are still maybe some interesting problems about how all the different part work. I am still reading. There's a Stirling series:

$$\log\left[\Gamma(z)\right] \sim \sqrt{2\pi z}(z/e)^z \sum \frac{B_{2k}}{2k(2k-1)} z^{-2k+1}$$

and this is this is also the resurgent expansion of $\log n!$ This is looking like the start of a very different discussion than the one by Newman+Zagier.

References

(1) Terence Tao
The Euler-Maclaurin formula, Bernoulli numbers, the zeta function, and real-variable analytic continuation `http://bit.ly/2cyNB3H`

(2) GH Hardy **Divergent Series** Oxford University Press, 1949/1973.

(3) EC Titchmarsh **Theory of Functions** `https://archive.org/details/TheTheoryOfFunctions`

(4) Simon Rubinstein-Salzedo **Could Euler have conjectured the prime number theorem?** `arXiv:1701.04718`

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(8) David Sauzin **Introduction to 1-summability and resurgence** `arXiv:1405.0356`

(9) Don Zagier **On Newman's Short Proof of the Prime Number Theorem** `http://people.mpim-bonn.mpg.de/zagier/files/doi/10.2307/2975232/fulltext.pdf`

7/10 To keep things interesting, it's time to include more examples of problems that can be solved with technique. I doubt anyone really understands it the first time around but the algebraic number theory textbooks lift statements about $\mathbb{Q}$ and $\mathbb{Z}$ and $\zeta(s)$ to

- all number fields
- arithmetic progressions
- Grossencharakters (I don't know what they are but it sounds really big !)

Any "lift" that behaves this nicely is called a **functor**. We can write L-functions for very deep objects such as varieties and schemes and motives, cuspital automorphic representations, etc. Finally these things can be read in terms of **category theory**.[4]

All of their solutions will have the same spirit: they will solve the problem but either something terrifying was swept under the rug - in a relatively safe correct way, mind you - or the proof was write in such advanced language it is no longer possible to corroborate if their argument is correct. One is left with more questions than answers.

Part of why contributing to math seems to be difficult is the criterion of "novelty".[5] After much digging and probing, the exercise I had proposed, to show the equidistribution of prime numbers in $\mathbb{Z}[i]$ is very carefully and beautifully done in Edmund Hlawka's textbook.

This doesn't prelude me finding yet another technique that solves this style of problem. There's a field called **Additive Combinatorics** that I am really enjoying. However, which theorem do I use? They don't really deal with Prime Number Theorem – in their group, they will use an example called Szemeredi Theorem about arithmetic progressions in sequences of numbers. Prime numbers very sparse compared to arithmetic progressions (or any sequence of numbers that occurs with some "regularity"). If I'm not mistaken we'd strengthen Szemeredi's Theorem and start to strip away all the extraneous numbers until we get arithemtic sequences of primes.

And that's just one idea. When we are ready we are free to write our own. There are many other fish in the sea.

These theorems seem to be about the same strength:

- Hardy Littlewood Tauberian Theorem
- Weierstrass Approximation Theorem
- Hairy Ball Theorem (from Topology)
- Gambler's Ruin

and the Prime Number Theorm will evade all of these. So I tried to have a glimpse of

[4]My understanding is category theory can be taught to pre-schoolers. It's the type of reasoning we do before there is any concept of symbol or number. We just look at objects and connections between them. We manipulate them with our hands. In a Mathematician's mind these are the basic object. In practice, there's still a lot of work to be done. Mostly it looks like nonsense.

[5]What if we were looking for tall things? 5-foot-6 is hardly considered tall, it is the typical height. If someone is 6 foot, they are pretty tall. Even in a small town, the tallest person might be over 6-foot-6. The tallest people in the world well over 7 feet and suffer from genetic disorders. These things are also to subject to verification and to dispute. Perhaps being first is not the best thing? Or very meaningful? Sultan Kosen (Turkey) and I share the same birthdya of 10 December.

I've begun to speculate these other theorems are of the same strength:

- Wiener-Ikehara Tauberian Theorem
- Euler-Maclaurin Summation Formula
- The Heat Kernel

I currently do not have the technical resources or follow-through to solve it. Until then we have Newman + Zagier.

7/10 Time to get off the soapbox. Let's widen the panorama a bit with Edmund Hlawka's discussion bit. He did exactly what I was going to do: he read Newman's proof and solved it for $K = \mathbb{Z}[i]$. Perhaps I can try $\mathbb{Z}[\sqrt{-5}]$ which has a non-trivial class number $h = 2$.

$$6 = 2 \times 3 = (1 - \sqrt{5}) \times (1 + \sqrt{-5})$$

Or $\mathbb{Z}[i]$. Maybe let's try to write down an L-function that counts

$$\sum_{n \geq 0} \frac{\#\left\{(a,b) : a^2 + b^2 = n \text{ and } 0 < \tan^{-1}(\frac{b}{a}) < \frac{\pi}{3}\right\}}{n^s}$$

This blog is full of errors. I doubt even if I wrote this one correctly. Off we go...

References

(1) Simon Rubinstein-Salzedo **Could Euler have conjectured the prime number theorem?** `arXiv:1701.04718`

(2) Andrew Granville, Adam J Harper, K. Soundararajan **A more intuitive proof of a sharp version of Halász's theorem** `arXiv:1706.03755`

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