Scratchwork: Configuration Spaces and Étale Cohomology

One way to get representations of the Symmetric group S_n is to use configuration space, e.g. of Points or Lines.

$$\begin{aligned} X_n(\mathbb{C}) &= \big\{(z_1, \dots, z_n) \big| z_i \in \mathbb{C} \text{ and } z_i \neq z_j\big\} \\ X_n(\mathbb{C}) &= \big\{(L_1, \dots, L_n) \big| L_1 \text{ line in } \mathbb{C}^n \;\; L_1, \dots, L_n \text{ linearly independent}\big\} \end{aligned}$$

Before I even try to learn the main "new" results of the paper, the authors remark that it's easy to compute these homologies by hand:

$$H^0(\mathbb{C}\mathbf{P}^2) = \mathbb{Q} \text{ and } H^2(\mathbb{C}\mathbf{P}^2) = \mathbb{Q} \text{ and } H^4(\mathbb{C}\mathbf{P}^2) = \mathbb{Q}$$

and from this they will deduce the number of points in the counting formula:

$$|\mathbf{P}^2(\mathbb{F}_q)| = q^2 + q + 1$$

and this is true for any prime power q. A more interesting result about "twisted cohomology" is that

$$H^1\big(\mathsf{Conf}_n(\mathbb{C}); \wedge^2\mathbb{Q}^n\big) \simeq \mathbb{Q} \text{ for } n \geq 4$$

This is already an astonishing amount of information.

References

[1] Thomas Church, Jordan S. Ellenberg, Benson Farb. **Representation stability in cohomology and asymptotics for families of varieties over finite fields** `arXiv:1309.6038`