Reading: Quadratic Forms

Let's try to read Margulis again... inhomogeneous quadratic forms, we've seen thmem before, what are they doing here?

In year 628, Brahmagupta studies equations of the form $x^2 - ny^2 = c$ and gets unnervingly accurate estimates of $\sqrt{67}$. This was translated to Arabic in the year 773 and into Latin by the 12th century. Example:

$$\left|\sqrt{67} - \frac{48842}{5967}\right| < 2 \times 10^{-9} \quad \text{or} \quad 48842 \times 48842 - 67 \times 5967 \times 5967 = 1$$

The "chakravala" method also solves the 61 case:

$$(1766319049)^2 - 61 \times (226153980)^2 = 1$$

This is about 1000 years before Pell solves the equation with his name.

Gauss writes *Disquisiciones Arithmeticae* in 1801 (in Latin).

Grigori Margulis won the Fields Medal (in Mathematics) in 1978 and doesn't resolve the Oppenheim conjecture until 1986. He discusses Lebesgue measure on the sphere (is $ds^2 = dx^2 + dy^2 + dz^2$ on the sphere $s = 1$ the only rotationally invariant, finitely additive measure on the sphere).

So ... Quadratic forms, what are they doing here?

> **Thm** (1998) Let Q be a quadratic form with signature (p, q) with $p \geq 3$ and $q \geq 1$. Suppose Q is not proportional to a rational form. Then for any interval
>
> $$N_{Q,\Omega}(a, b, T) \sim \lambda_{Q,\Omega}(b - a)T^{n-2} \text{ as } T \to \infty$$
>
> where $n = p + q$ and $\lambda_{Q,\Omega}$ as in [prior equation].
>
> Theorem 1.1 fails if Q has signature $(2, 2)$ and $(2, 1)$. Example, $N_{Q,\Omega}(a, b, T) = T^{n-2}(\log T)^{1-\epsilon}$, however these rational forms are very well approximated by split rational forms.

These definitions are bit dense and it already lacks the spirit of the kid's examples on the top of the page. The object in question is a single quadratic form, Q and it's approximated by another quadratic form $Q' \approx Q$. By graduate school we learn that quadratic forms are dime-a-dozen to such an extent that entire classes of them can be mapped to one another.

The two prototypes they give us are:

- $(p, q) = (2, 2)$ or $Q(a, b, c, d) = ad - bc$ the "determinant" and the space is $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \times \mathrm{SO}(2)$
- $(p, q) = (2, 1)$ we let $x_1x_3 - x_2^2$ (a "hyperboloid of one sheet", a conic section) be the standard form on $\mathbb{R}^3$, we have that $SO(2, 1) \simeq \mathrm{SL}_2(\mathbb{R})$.

This language is generic and official and correct, it's just a little bit too generic for our taste. The rest of our work is to decide what this looks like when we give this to the kids...

My favorite examples are $Q = x_1^2 + \sqrt{2}\,x_2^2 - x_3^2 - \sqrt{3}\,x_4^2$ and $Q = x^2 + y^2 - \sqrt{2}\,z^2$. These examples are considered "known" by experts and yet my questions aren't answered and I don't have any way of presenting this to the kindergarten classroom.

The area of the parallelogram spanned by $(a,b),(c,d)$ is given by the determinant $ad - bc$. Let's try $a = x_1 + x_3$ and $b = x_1 - x_3$ and $c = \sqrt{2}x_2 - \sqrt{3}x_4$ and $d = \sqrt{2}x_2 + \sqrt{3}x_4$. By standard rules of algebra these work.

We even get determinants when we try to verify that two numbers $a, b, \in \mathbb{Z}$ are "relatively prime" (the standard school notion, convenient for classroom teaching) and the algorithm returns two other numbers $c, d \in \mathbb{Z}$ with $ad - bc = 1$. By the time it reaches Grigori Margulis, the ideas of "quadratic forms" and "Number Theory" are firmly separated, for example, "functional analysis" also has quadratic forms. Quite severely, what's **addition** ? As soon as we choose a setting, our questions are either too specific or too generic. Do you want the one or the many? Does your point of view matter?

Fortunately, we expect all of these changes in perspective to be absorbed into the mathematics itself. So that could be why at Margulis' level the issues are a bit more flexible, more granular or more pliable. $8th$ grade is the maximum of the people we would like to engage here, beyond that it becomes another matter.

Let's try our example: $Q = x_1^2 + \sqrt{2}\,x_2^2 - x_3^2 - \sqrt{3}\,x_4^2$.

> **Thm** Let a_t and K as in Theorem 4.1 Let Λ be any lattice in $\mathbb{R}^4$. Then for any $i = 1, 3$ and any $\epsilon > 0$:
>
> $$\sup_{t>0} \int \alpha_i(a_t K \Lambda)^{2-\epsilon} < \infty$$
>
> Hence there exists a constant c dependending on ϵ and Λ such that for all $t > 0$ and $0 < \delta < 1$
>
> $$|\{k \in K : \alpha_i(a_t k \Lambda) > \frac{1}{\delta}\}| < c\,\delta^{2-\epsilon}$$
>
> Hint: [Chebyshev's Theorem]

These problems are not hard, but they are require attentive reading, good bookkeeping and a vivid imagination. We are telling you that the numbers described by Q get chaotic very quickly in a nice musical way, that approach randomness. What do we mean by "musical"? For example, these numbers could be eigenvectors of the wave equation

$$\Box = \frac{\partial^2}{\partial x_1^2} + \sqrt{2}\,\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \sqrt{3}\,\frac{\partial^2}{\partial x_4^2}$$

on some shape that we call a "torus". Now say the thing isn't exactly flat here.

What do these measurable sets look like? "Measurable" means we can measure it. How much difference? And "where"? These things are scattered about, yet it's more here and than there. So we can measure and quantify how they are the same and different and maybe it's good enough to tell them apart. What could $\Box$ be confused for?

And how do we explain this to kindergarteners?

Let G, H, K and $\{a_t\}$ be chosen as in [the previous section]

- $n = p + q = 2 + 2 = 4$. $G = SL_4(\mathbb{R}) \rtimes \mathbb{R}^4$ and $\Gamma = SL_4(\mathbb{Z}) \rtimes \mathbb{Z}^4$ is a lattice.[1] This is similar to the way $\mathbb{Z}$ is a lattice in $\mathbb{R}$. So we could examine a "number theory" of the $\mathbb{Z}$ that lives within $\mathbb{R}$.

[1] At this level we could even try to explain what $\mathbb{Z}$ is or what $\mathbb{R}$ is. The real world does not know what these objects are. By graduate level, Margulis could say any theory we could up with turns into $\mathbb{Z} \subseteq \mathbb{R}$. This just sounds like professor-speak.

- $H = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ and $K = SO(2) \times SO(2)$. The matrix $a_t = (b_t, b_t)$ is a 4×4 matrix:

$$a_{2t} = \begin{bmatrix} e^{-t} & & & \\ & e^{t} & & \\ & & e^{-t} & \\ & & & e^{t} \end{bmatrix}$$

  A great question here is why the exponential function is so important? It arises as the limit of compound interest $(1 + \frac{t}{n})^n \to e^t$ and also LCM($\{1, 2, \ldots, x\}) \asymp e^x$. Why are we using the exponential map to describe relatively prime numbers? Are we OK with that? There are multiplicative inverses:

$$\det \begin{bmatrix} 5 & \\ & \frac{1}{5} \end{bmatrix} = 5 \times \frac{1}{5} = 1$$

  As we run the GCD algorithm, the numbers get exponentially smaller, and the LCM algorithm the numbers get exponentially larger. Wouldn't we like to see more discussion of this ?

- We have that $SO(2,2) \simeq SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. The action $v \mapsto g_1 v g_2^{-1}$ leaves the area ("determinant") invariant.

Let Q_ξ be a quadratic form of signature $(2,2)$ quadratic form which is "Diophantine".

$$\limsup_{t\to\infty} \int_K \tilde{f}(a_t k : \Lambda_{\mathfrak{q}\xi})\nu(k)dk \leq \int_{G/\Gamma} \hat{f}(g)d\mu(g) \cdot \int_K \nu(k)\, dk$$

What are the shapes of these orbits? Can we draw them? Do they look nice? More notations:

- μ is the G invariant measure on G/Γ and ν is a continuous function on K

- There seem to be different types of transforms we can do, such as $f \to \hat{f}$ and $f \to \tilde{f}$ and these could be **functorial**. These are averages related to the lattice Λ_ξ and a group element associated to Q.

There is an infinite amount of notation when you try to do this numerically or "computationally". Final warning, this paper has a lot of setting specific to him and we are responsible to turning it into what we need. These formulas look great at the level of math professor, they are not suitable to us. The are guidelines.[2]

- What are the key features of Margulis' argument? And how do we feature them in a Kindergarten classroom? My best guess we are talking about features of the GCD and LCM algorithm in grade school, with the items spun around in some way.

- There are no pictures.

So anyway, all of this is grade school math in disguise. Interesting, but rather complicated features of grade school math.

[2]in fact a "guideline" is a literal piece of metal

10/17 Let's use our $(2,2)$ signature quadratic form. $Q(x) = x_1^2 + \sqrt{2}\,x_2^2 - x_3^2 - \sqrt{3}\,x_4^2$. Let's get a list of symbols.

- Q non-degenerate indefinite quadratic form on $\mathbb{R}^n$.
- $\xi \in \mathbb{R}^n$ be a vector we get a second (inhomogeneous) quadratic form $Q_\xi(x) = Q(x+\xi)$. What does "rational" and "irrational" quadratic form mean here?
- ν be a continuous function from the sphere $\{v \in \mathbb{R}^n : ||v|| = 1\}$ to $\mathbb{R}$.
  $\Omega = \{v \in \mathbb{R}^n : ||v|| < \nu(v/||v||)\}$ and let $T\Omega$ be the dilate of Ω by the constant (or operator) T.
- Let's count the number of points of the quadratic form within a certain range.

  $$N_{Q,\xi,\Omega}(a,b,T) = \#\big\{x \in \mathbb{Z}^n : x \in T\Omega \text{ and } a < Q_\xi(x) < b\big\}$$

- The inequality $||v|| < \nu(\frac{v}{||v||})$ does something like "polar coordinates" in high school, $\vec{v} = (||v||, \frac{v}{||v||})$ (as could be done by a manifold chart), and we are looking for points "inside" the shape we've created. Ω could be defined as the graph of the function, $\vec{u} = \frac{v}{||v||} \mapsto \nu(\vec{u})$. Over $\mathbb{R}$, perhaps this shape is an ellipsoid,

  $$\text{Vol}(\{x \in \mathbb{R}^n : x \in T\Omega \text{ and } a < Q_\xi < b\}) \sim \lambda_{Q,\Omega}(b-a)T^{n-2}$$

  The number could count how many independent eigenstates there are of a certain operator. Alex Eskin, Grigori Margulis and Shahar Mozes show that:

  $$N_{Q,\xi,\Omega}(a,b,T) \sim \lambda_{Q,\Omega}(b-a)T^{n-2}$$

  The lattice points in the a (re-scaled) region of $\mathbb{R}^n$, with "radii" within a certain range, scale linearly as the size of the interval, and as the $(n-2)$-th power of T.

While the statement is very difficult, it's not something I would argue with. We are told the theorem fails for Q with signature $(3,1)$ (such as $x^2+y^2+z^2-w^2$), $(2,2)$ (such as $x^2+y^2-z^2-w^2$).

As a sample, the freshmen analysis class has that $\mathbb{R}$ the limit of equivalence classes of Cauchy sequences $[\{x_n\}]$ with $x_n \in \mathbb{Q}$, which are ratios of integers $x_n = \frac{a}{b} \in \mathbb{Q}$. On the other than any pure rational number $\frac{m}{n}$ was probabiliy the limit of real numbers, or the limit of a many observations of which we attached an average (e.g. using an experiment).

We could try to read broader equivalence relations . Do we know what "$\in$" means here, or "$\leq$" ? The continuous function ν can be drawn as a hypersurface in $\mathbb{R}^n \times \mathbb{R}$ diffeomorphic to $S^n \times \mathbb{R}$. So there there are many degrees of freedom here.

Let's settle for the $(2,2)$ case. We are referred to their result from 2005, it was already "quantitative":

$$\#\Big(\big\{x \in \mathbb{Z}^n\big\} \cap \big\{x : ||x|| < T\,\nu(\tfrac{x}{||x||})\big\} \cap \big\{x : a < Q(x) = x_1^2 + \sqrt{2}\,x_2^2 - \sqrt{3}\,x_3^2 - x_4^2 < b\big\}\Big) \sim \lambda_{Q,\Omega}(b-a)T^{n-2}$$

An idiotic example, is $\nu \equiv 1$, and Ω is a perfect sphere and $T\Omega$ is a sphere, and $\lambda_{Q,\Omega} \overset{?}{=} 1$ (requires proof?).

This seems to be the framework for a range of number theory problems. Proving increasingly self-evident claims. Statements, achieved with much difficulty. All the items quite vague and open-ended. The result is something like a number.

Two annals of mathematics papers we should be suspicious of succinct statement.

References

[1] Alex Eskin, Grigory Marguils, Shahar Mozes.
Quadratic Forms of Signature (2,2) and Eigenvalue Spacings on Rectangular 2-Tori
Annals of Mathematics, 161 (2005), 679-725
Upper Bounds and Asymptotics in a Quantitative Version of the Oppenheim Conjecture
Annals of Mathematics, 147 (1998), 93-141

[2] Grigori Margulis, Amir Mohammadi.
Quantitative Version of the Oppenheim Conjecture Inhomogeneous Quadratic Forms
Duke Math. J. Volume 158, Number 1 (2011), 121-160.

[3] . . .

10/20 Alex Eskin, Grigory Margulis and Shahar Mozes in 2005. The Oppenheim conjecture - proven by Margulis - shows that for a non-degenerate indefinite irrational quadratic form Q in $n \geq 3$, the set $Q(\mathbb{Z}^n)$ is dense.

- 1998 a quantative version is proven (except for signature $(2, 2)$ and $(2, 1)$)
- 2005 signature $(2, 2)$ and $(2, 1)$ are proven – if we rule out the exceptional cases

Here are some the keywords given by the jstor computer. [rectangles] [signatures] [lattices] [sine function] [eigenvalues] [vectors] [Mathematical theorems] [integers] [Diophantine sets] [coefficients]

- Let ρ be a continuous positive function on the sphere $\{v \in \mathbb{R}^n : ||v|| = 1\}$
  Ex: what is the definition of a **norm** or **metric** ?
- $\Omega = \{v \in \mathbb{R}^n : ||v|| < \rho(\frac{v}{||v||})\}$ and $T\Omega$ is the rescaling of Ω by the operator $T : x \mapsto ax$.

$$\begin{aligned} \#\{x \in \mathbb{Z}^n : x \in T\Omega \text{ and } a < Q(x) < b\} &\sim \lambda_{Q,\Omega}(b-a)T^{n-2} \\ \text{Vol}\{x \in \mathbb{R}^n : x \in T\Omega \text{ and } a < Q(x) < b\} &\sim \lambda_{Q,\Omega}(b-a)T^{n-2} \end{aligned}$$

If the signature is $(2, 2)$ or $(2, 1)$ we have to rule out certain cases and the statement is still true.

Given a quadratic form, and any shape, we get an **approximate-line**. How does this thing differ from a more evenly-spaced line, such as our ideal $\mathbb{Z}$? If we set $A + B = \{a + b : a \in A, b \in B\}$ then we can write $\mathbb{Z} + \mathbb{Z} = \mathbb{Z}$. However, $Q(\mathbb{Z}^n) \approx \mathbb{R}$ and $Q(\mathbb{Z}^n) + Q(\mathbb{Z}^n) \neq \mathbb{Q}(\mathbb{Z}^n)$ and yet also $Q(\mathbb{Z}^n) + Q(\mathbb{Z}^n) \approx \mathbb{R}$. What would we like to see, then?

And we have two different measures on theset of points, the counting measure over $\mathbb{Z}$ and distance measure $\mathbb{R}$. In fact, the way we construct Lebesgue measure is to declare that $\mu([a, b]) = b - a$ and say that μ behaves nicely under unions and intersections.

. . .

We have a statement, and almost a modern approach to the thing that still retains the spirit of the thing for the kids. The next step is we demolish the entire thing. These are circles or circular shapes in Euclidean geometry. The notion of $\mathbb{Q}$, assumes that we know a particular number to all digits accuracy.

How did that occur? What is a number anyway?

References

[1] Ben Green **What is . . . an approximate group?** Notices of the AMS, **59** (5).

11/10/19 Quadratic forms and what are they doing here, at this level? Let $Q(\vec{x}) = x_1^2 + \sqrt{2}x_2^2 - x_3^2 - \sqrt{3}x_4^2$. Then we have that $\overline{Q(\mathbb{Z}^4)} = \mathbb{R}$. This result of 1998 gives us a quantative reading of the Oppenheim conjecture. Let's try $\{|x^2| < T\} \cap \mathbb{Z}^4$ and we want to count the number of points in the interval as $T \to \infty$.

$$V^Q_{(a,b)}(\mathbb{R}) = \{x \in \mathbb{R}^n : a < Q(x) < b\}$$

Our expectation is that the number of points we have is *linear* with the size of the interval and *quadratic* in the size of the sphere, T This is a type of **counting measure** and we are asking about it's convergence to, say Lebesgue measure.

$$V_{(a,b)}(\mathbb{Z}) \cap T\Omega \sim c_{Q,\Omega}(b-a)T^2$$

The theorem is that we can prove our expectations are correct for a constand c that changes with the quadratic form Q and shape Ω. The main result is that the constants and the rates of growth are always the same:

$$\begin{aligned} \text{Vol}(\{V_{(a,b)}(\mathbb{R}) \cap T\Omega\}) &\sim \lambda_{Q,\Omega}(b-a)T^2 \\ |V_{(a,b)}(\mathbb{Z}) \cap T\Omega| &\sim \lambda_{Q,\Omega}(b-a)T^2 \end{aligned}$$

The growth rate along the integers and along the real numbers are the same. Except in the signature $(2,2)$ or in $(2,1)$ which is the example which I have chosen.[3] An example of a quadratic form with signature $(3,1)$ is $Q(\vec{x}) = x_1^2 + \sqrt{2}x_2^2 + x_3^2 - \sqrt{3}x_4^2$.

Now we have seen the breadth and the scope of the proof of the Oppenheim conjecture and the statement of the quantitative oppenheim conjecutre. How do these counting problems get turned into a discussion of the Ratner flow?

Let f be a bounded function vanishing outside a compact subset:

$$f(x) = \begin{cases} 1 & |x| < T \text{ and } a < Q(x) < b \\ 0 & \text{otherwise} \end{cases}$$

Hopefully the intersection of a sphere and an ellipse is compact. Both closed and bounded. Next we want to count all the lattice points that solve this equation:

$$\tilde{f}(g) = \sum_{v \in \mathbb{Z}^4} f(gv) = \Big|_{g=1} |V_{(a,b)}(\mathbb{Z}) \cap T\Omega|$$

Now we have that our counting problem can be moved around with our elements of g. For any bounded function f of "compact support" we have:

$$\tilde{f}(\Delta) < c\alpha(\Delta)$$

a single bound for all functions, all counting problems of this type.

[3]These numbers were most likely *eigenvalues* of the Laplacian on the torus, which is why they have to be so exact. $\mathbb{R}$ is the set of spectra of operators.