Scratchwork: Jacobi Triple-Product Formula

Comparing apples and oranges. There is lots of discussion of theta functions in the literature. How do we normalize our definitions? Here's an examplle from Integrable Systems:

$$\theta_1(u;p) = \theta_1(u) = 2p^{1/4}\sin u \prod_{n=1}^{\infty}\big(1-2p^{2n}\cos 2u + p^{4n}\big)(1-p^{2n})$$

These are called **elliptic theta functions**, and the branch of mathematics is called *integrable systems*. Here is anothe theta function:

$$\theta_4(u;p) = \theta_4(u) = \prod_{n=1}^{\infty}\big(1-2p^{2n-1}\cos 2u + p^{4n}\big)(1-p^{2n})$$

Why is this going to be so confusing. Here's another definition of theta function.

$$\begin{aligned}
\overline{\theta}_1(x|\tau) &= 2e^{i\pi\frac{\tau}{4}}\sin(x)\prod_{n=1}^{\infty}\big(1-e^{2ix}e^{\pi i\tau(2n)}\big)\big(1-e^{-\pi ix}e^{\pi i\tau(2n)}\big) \\
\overline{\theta}_4(x|\tau) &= \prod_{n=1}^{\infty}\big(1-e^{2ix}e^{\pi i\tau(2n-1)}\big)\big(1-e^{-\pi ix}e^{\pi i\tau(2n-1)}\big)
\end{aligned}$$

and the Jacobi theta function, which is missing a factor of an infinite product.

$$\theta_i(x|\tau) = \prod_{n=1}^{\infty}(1-e^{2\pi i\tau n})\theta_i(x|\tau)$$

The same discussion tells is there are some modlarity transformations available to us:

$$\begin{aligned}
\theta_1(z|\tau+1) &\overset{S}{=} \exp\left(\frac{\pi i}{4}\right)\theta_1(z|\tau) \\
\theta_1\left(z|-\frac{1}{\tau}\right) &\overset{T}{=} -i\sqrt{-\tau i}\exp\left(-\frac{1}{\pi i}\tau z^2\right)\theta_1(z\tau|\tau) \\
\theta_1\left(z|\frac{\tau}{1-\tau}\right) &\overset{STS}{=} \exp\left(\frac{\pi i}{4}\right)\sqrt{1-\tau}\exp\left(-\frac{1}{i\pi}(\tau-1)\right)\theta_1\big(z(\tau-1)|\tau\big)
\end{aligned}$$

These formulas are unfortunately, very complicated have an unclear meaning as stated and we don't know where they come from. Here's another source of examples:

$$\begin{aligned}
\Theta(z|\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi in^2\tau}e^{2\pi inz} \\
\theta(\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi in^2\tau}, \tau\in\mathbb{H} \\
\vartheta(t) &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, t>0
\end{aligned}$$

Between these three definitions, we get there is a symmetry within the set of perfect squares of integers $\square = \{n^2 : n\in\mathbb{Z}\}$.

Here's a fourth, more varied definition of theta function which follows the same idea:

$$\theta(z;u) = \sum_{m\in\mathbb{Z}^3} u(m)e(|m|z)$$

with u is a spherical harmonic of degree ℓ. The Fourier coefficients are given by:

$$a(n) = n^{\ell/2}r_3(n)W_u(n) \quad \text{with} \quad W_u(n) = \frac{1}{r_3(n)}\sum_{\xi\in V_n} u(\xi)$$

and we have that $\theta(z;u)$ is a holomorphic cusp form on $\Gamma_0(4)$.

Let's see a statement of the Jacobi Triple Product formula. Here's the exercise in Apostol's **Modular Function and Dirichlet Series in Number Theory**.

$$\theta(\tau) = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 \tau} = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}$$

abcd. Here's the exercise in **Conformal Field Theory**.

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Here's the exercise in Elias Stein's textbook on **Complex Analysis**.

$$\begin{aligned}
\Theta(z|\tau) &= \sum_{n=-\infty}^{\infty} e^{\imath i n^2\tau} e^{2\pi i n z} \\
\Pi(z|\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}) \\
\Theta(z|\tau) &= \Pi(z|\tau)
\end{aligned}$$

Θ is entire in $z \in \mathbb{C}$ and holomorphic in $\tau \in \mathbb{H}$.

- $\Theta(z+1|\tau) = \Theta(z|\tau)$
- $\Theta(z+\tau|\tau) = \Theta(z|\tau)e^{-\pi i \tau}e^{-2\pi i z}$
- $\Theta(z|\tau) = 0$ whenever $z \in \frac{1}{2}(1+\tau) + \mathbb{Z} + \tau\mathbb{Z}$.

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References

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