Scratchwork: Mean Value Theorem

Do we understand the Mean Value Theorem at all? Let's try studying an example with sine.

$$f(b) + f(a) = (b-a)f'(c) \text{ with } b > c > a$$

This is used when we want to do a linear approxmation, which we usually write as:

$$f(x+h) \approx f(x) + f'(x) \times h$$

Do we know what these operations are $+$ and $\times$? And can we qualify this symbol $\approx$? And we still haven't set $f(x) = \sin x$.

$$\sin(\theta + h) \approx \sin\theta + h \times \cos\theta$$

and haven't told you that $\sin'\theta = \cos\theta$. It's just something that we religiously beleive. These are, for example, the x and y coordinate charts of a circle

$$x^2 + y^2 = 1 \text{ let's write } x = \cos\theta \text{ and } y = \sin\theta$$

as if we've never seen these objects in our lives.[1]

The mean value theorem let's use quantitative versions:

$$f'(x) \approx \frac{1}{2h}\big(f(x+h) - f(x-h)\big) + \frac{h^2}{2} \times \max_{[x-h,x+h]} |f''(c)|$$

Let's look at a table of exact values of sine:

- $\sin 72^\circ = \sqrt{\frac{1}{8}(5+\sqrt{5})}$
- $\sin 75^\circ = \frac{\sqrt{2}}{4}\big(\sqrt{3}+1\big)$ and $\cos 75^\circ = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$
- $\sin 78^\circ = \frac{1}{8}(\sqrt{30+6\sqrt{5}} + \sqrt{5} - 1)$

These angles are in increments of $3^\circ = \frac{\pi}{120}$ radians. And very innocently put in the values:

$$\begin{aligned}
\cos 75^\circ &= \frac{60}{\pi} \times (\sin 78^\circ - \sin 72^\circ) + (\frac{\pi}{120})^2 \times |\sin(?)| \\
\frac{\pi}{60} \times \frac{\sqrt{2}}{4} \times (\sqrt{3}-1) &= \Big[\frac{1}{8}(\sqrt{30+6\sqrt{5}} + \sqrt{5} - 1)\Big] - \Big[\sqrt{\frac{1}{8}(5+\sqrt{5})}\Big] \pm \frac{1}{2} \times (\frac{\pi}{120})^3
\end{aligned}$$

Not that $|\sin\theta| < 1$. Not quite Ramanujan level, but it's less than obvious.[2]

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[1] Later, we might try to replace this with another algebraic curve. It could even be birational to a circle. To an algebraic geometer this might be dull. Here's an example $(x,y) = (\cos 2\theta, \sin 3\theta)$ is an algebraic curve.

[2] There will be times when $|\sin\theta| < 1$ less than sufficient. See if we can find ways to bootstrap such inequalities? For the moment. . .

Here's a stencil for the second derivative. Matching stencil's to their estimates goes back to a time when we didn't have calculators at our disposal.

$$f''(x) \times 2h \approx f(x+h) - 2f(x) + f(x-h)$$

And our previous values of $\sin$ have already been computed:

$$3^\circ \times (-\sin 75^\circ) \approx \sin 72^\circ - 2\sin 75^\circ + \sin 78^\circ$$

and we have exact algebraic values for all three angles by some miracle.

$$\frac{\pi}{120} \times (-\frac{\sqrt{2}}{4}\sqrt{3}+1) = \sqrt{\frac{1}{8}(5+\sqrt{5}} - 2 \times (\frac{\sqrt{2}}{4}\sqrt{3}+1) + \frac{1}{8}(\sqrt{30+6\sqrt{5}} + \sqrt{5} - 1) \pm \frac{1}{6}\left(\frac{\pi}{120}\right)^3$$

and we can do some rearranging. These are mysterious classical identities one might have studied in the 1700's.

$$\left(2 - \frac{\pi}{120}\right) \times \left(\frac{\sqrt{2}}{4}\sqrt{3}+1\right) = \sqrt{\frac{1}{8}\left(5+\sqrt{5}\right)} + \frac{1}{8}\left(\sqrt{30+6\sqrt{5}} + \sqrt{5} - 1\right) \pm \frac{1}{6}\left(\frac{\pi}{120}\right)^3$$

How do we **check** such a computation? A computer uses a `float` data type which some kind of $\mathbb{R}$-like object build on the shift map $T : x \mapsto 10\,x$. And the edge cases can be handled by various types of rounding, such as $\{x\} = \min_{n \in \mathbb{Z}} |x - n|$. In order to punch these things to a calculator we'd replace $\sqrt{2}$ by $\sqrt{2} \pm 10^{-8}$ or an even more precise error.

These *may* be adequate for our job. There might even be very simple algebraically distinct expressions that agree to this decimal error. Enough to fool our calculators.

Ex. Do we know how to compute this floor function $\lfloor 10^4(\sqrt{2}+\sqrt{3}) \rfloor$?

References

[1] ...