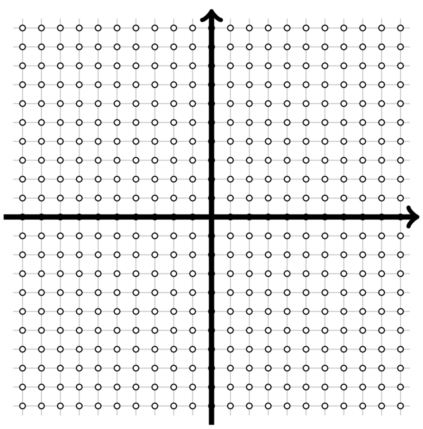
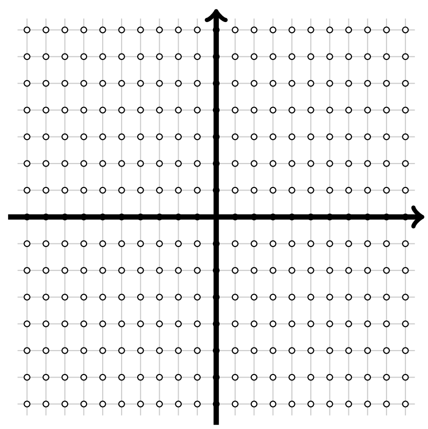
Scratchwork: Prime Numbers in $\mathbb{Z}(\sqrt{2})$ or $\mathbb{Z}[i]$

Let's start by drawing a grid:



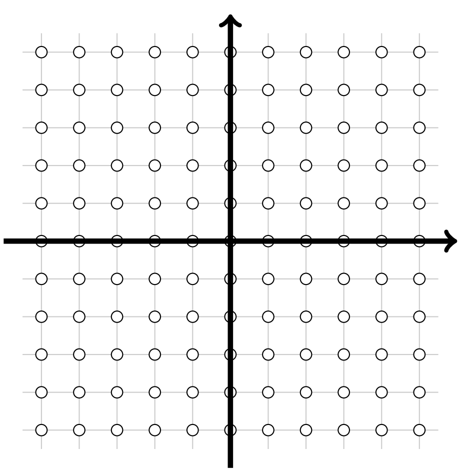


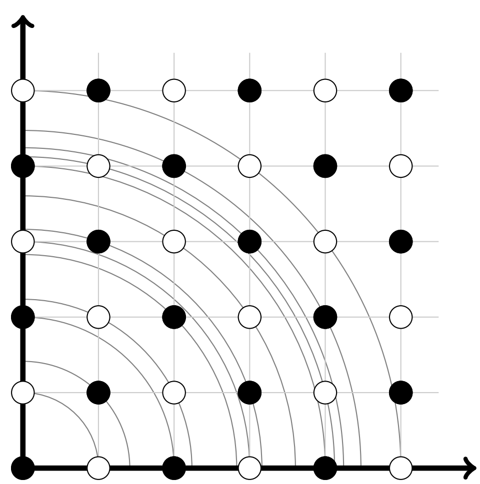
The first grid represents $\mathcal{O}_K = \mathbb{Z}[i]$ where $K = \mathbb{Q}(\sqrt{-1}) \simeq \mathbb{Q}[x]/(x^2+1)$.

The second grid represents $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$ where $K = \mathbb{Q}(\sqrt{2}) \simeq \mathbb{Q}[x]/(x^2-2)$.

Our goal is to strip away these grid to obtain the set of prime numbers in both settings. These are well-studied problems with detailed algebraic solutions, the goal is to solve these examples for ourselves.

In the case of $\mathbb{Z}[i]$ all that seems to matter is the first quadrant. The group action $\times\sqrt{-1}$ is, for the moment a 90° rotation counterclockwise. We have that $(1) < (1+i) < (2) < (1+2i)$ with nothing in between. These are **ideals** and we have $(1+i) = (1-i)$. We have that (2) is **not** prime. $(2) = (1+i)^2 = (1+i)(1-i)$





We know that $(1+2i)\big|(5)$ and we even have that $5 = (1+2i)\times(1-2i)$. At first glance $(3) \approx (1+2i) \approx 2\times(1+i)$. The algebra is slightly misleading here since it makes things that are close together (approximate over $\mathbb{R}$) look very distant. This could be useful to us.

Ex. $(1+2i) \stackrel{?}{=} (1-2i)$