Tune-Up: Cauchy Riemann Equations

We learn in calculus class that $f(x) = x^2$ then $\frac{df}{dx} = 2x$. And this is a very special operation, since have a function and can turn it into another function. Let's remind ourselves how that one happened:

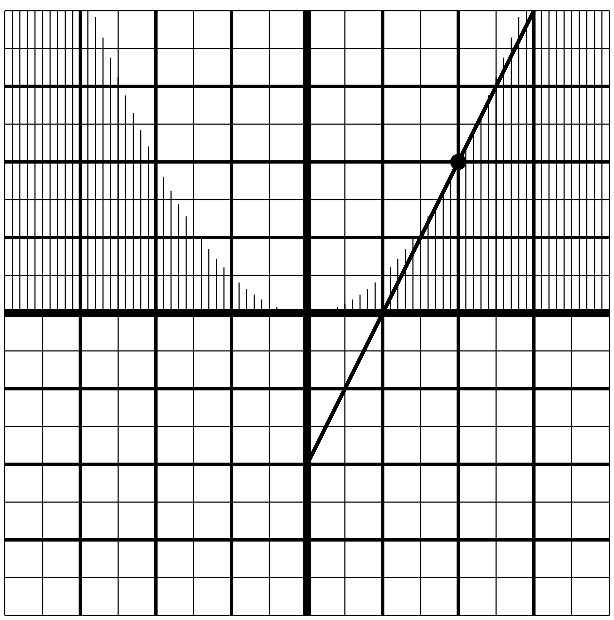
$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

Let's see if we can do anything about the expression in the limit:

$$\frac{\big(x^2 + 2x\,\Delta x + (\Delta x)^2\big) - x^2}{\Delta x} = \frac{\Delta x\big(2x + \Delta x\big)}{\Delta x} = 2x + \Delta x \to 2x + [0]$$

Notice the division works out perfectly. And for this class of functions it always does.

Let's draw the picture. This is easy enough to draw with only `tikz`.



Here we were able to *parameterize* the curve well enough to draw, let's see if we're so lucky in future circumstances.

In the mean time, let consider what happens when $\Delta x \neq 0$. Let's say $0 < \Delta x = 0.1 \ll 1$. Then let's take square:

$$(1 + 0.1)^2 = 1.21 \text{ yet } (1 + 0.1)^2 \approx 1 + 2 \times 1 \times 0.1 = 1.2$$

We are off by less than 1%. So this is a pretty good most of the time. We're going to be OK.

In fact, Taylor's theorem says that we're almost always going to be OK, no matter what function we choose:

$$f(x + \Delta x) \approx f(x) + \Delta x \cdot \frac{\Delta f}{\Delta x} + O(\Delta x^2)$$

Therefore line and parabola should do the job almost every single time. Any further doubts require the **Mean Value Theorem**.

The complex analysis textbook begins the same way. Let $f(z) = z^2$. Then

$$f(z) = z^2 \text{ then } \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = 2z + O(\Delta z)$$

We even have a sort of guarantee of how far our line is. Let's try to check and makes sure we understand what $|\Delta z| \to 0$ really means.

References

[1] Vladimir Zorich **Mathematical Analysis I** (Universitext) Springer, 2015.