Scratchwork: $\zeta(2)$ and Random Processes

9/15 Is there any connection between special values of zeta function probability? There's the statistic that the odds of two random numbers being relatively prime is $\zeta(2)^{-1} = \frac{6}{\pi^2} \approx \frac{2}{3}$. This could turn into a question about what other fundamental constants we know and why π is such an important number.

Q What are the odds of having gcd$(a, b) = 1$ for two numbers $a, b \in \mathbb{Z}[i]$? Here there is still a Euclidean algorithm and so the answer should be:

$$\zeta_{\mathbb{Z}[i]}(2) = \zeta(2)L(2, \chi_4) = \left(1 - \frac{1}{2^{-2}}\right) \prod_p \left(1 - \frac{1}{p^{-4}}\right) \prod_p \left(1 - \frac{1}{p^{-2}}\right)^2 = \sum_{m+in} \frac{1}{(m^2+n^2)^2}$$

and we can obtain a complete solution by compiling data from various sources. These can be considered Artin L-functions with a $\rho = triv$ and $\rho = sign$ part.

What's going to happen is we solve one problem and we inadvently solve many others, because that's real life.

Bourgade, Fujita and Yor offer a proof of the value of $\zeta(2)$ by examining the Cauchy distribution:

$$d\mu(x) = \frac{dx}{\pi(1+x^2)}$$

We also need the product of two Cauchy distributions. These integrals shouldn't be taken for granted:

$$d\nu(x) = \frac{2\log|x|}{\pi^2(x^2-1)}$$

Then we can take the absolute value of the Cauchy distribution and take the log of that

$$|\mathbb{C}_1| \overset{law}{=} e^{\frac{\pi}{2}C_1} \text{ with } \mathbb{E}\big(e^{i\lambda C_1}\big) = \mathbb{E}\big(e^{i\lambda \log|\mathbb{C}_1|}\big) = \frac{1}{\cosh\lambda} \tag{1}$$

The proof of the values of the zeta function is a little bit tautological:

$$1 = \mathbb{E}[1] = \frac{8}{\pi^2}\big(1 - \frac{1}{2}\big)\zeta(2)$$

and we know for a fact there is something (close to) random about the factorization of prime numbers. We feel like GCD is a more even-handed phenonenon to work with and there's something random about those too!

References

[1] Paul Bourgade, Takahiko Fujita, and Marc Yor **Euler's formulae for $\zeta(2n)$ and products of Cauchy variables** Electronic Communications in Probability Volume 12 (2007), paper no. 9, 73-80.

[2] Robin Chapman **Evaluating $\zeta(2)$** `https://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf`

[3] L Pace **Probabilistically Proving that $\zeta(2) = \frac{\pi^2}{6}$** American Mathematical Monthly, Vol. 118 Issue 7, 2011

[4] Lars Holst **Probabilistic proofs of Euler identities** Journal of Applied Probability Vol. 50, No. 4s, pp. 1206-1212