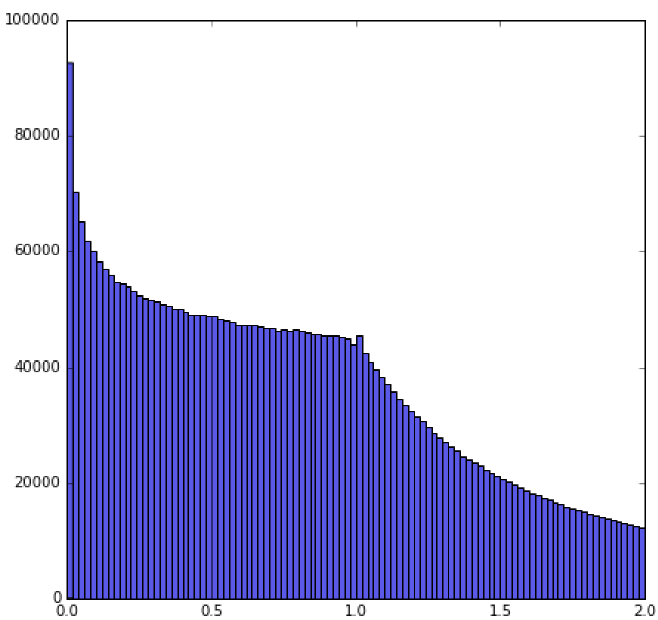
Stumbling Across the Prime Number Theorem

John D Mangual

A few weeks ago I read that ratios of prime numbers are dense in the real number line.

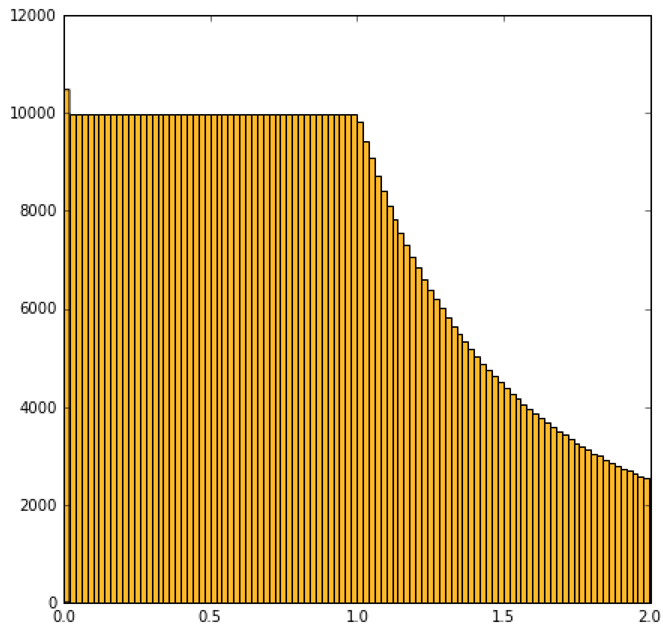
Using my script for primes I plotted a histogram of the values of $\frac{p}{q}$ for primes $0 < p < 2q < 1000$ (so there are about 10^6 values in all).

There seems to be a cutoff around $p = q$ and a long tail.



What if we include all integers? Our fractions have a nice flat line for $m < n$ and some type of curve for $n < m < 2n$.

The set is $\{\frac{m}{n} : 0 < n < 1000$ and $0 < \frac{m}{n} < 2\}$.



We should hope for a perfectly even histogram on the interval $[0,1]$ since our set of integrs include all fraction.

If we are lucky the fractions have density 1 in the set of real numbers.

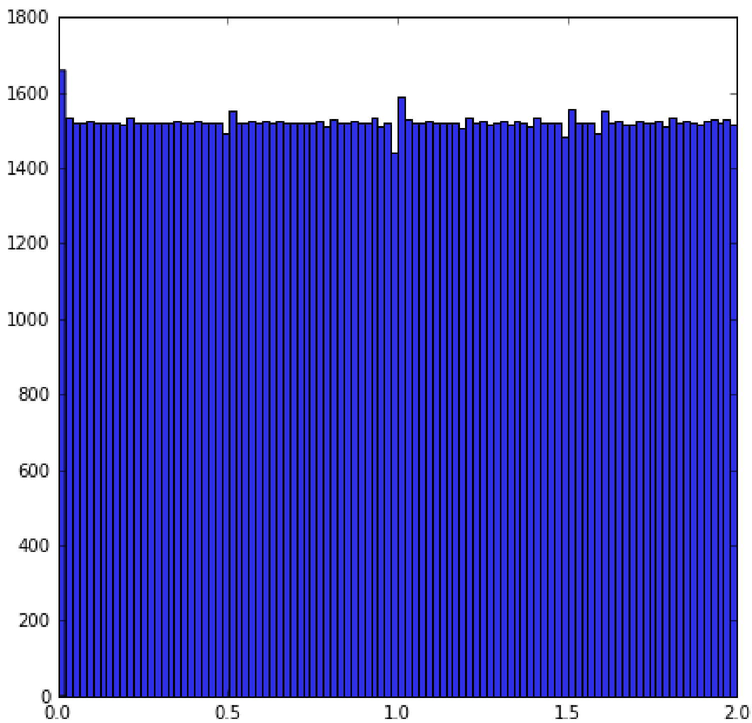
$$\overline{\mathbb{Q} \cap [0,1]} = [0,1]$$

We save the analysis of these two curves for another time. The density of fractions $\frac{p}{q} \approx \alpha$ is equivalent to the **prime number theorem**.

Restricting only to reduced fractions, $\frac{a}{b}$ relatively prime numbers have a perfectly even density

$$\mathcal{F} = \{\frac{a}{b} : 0 < a < 2b,\ \text{gcd}(a,b) = 1\}$$

These are the **Farey Fractions** - obviously these should have uniform density in the real number line $\mathbb{R}$ since they represent every fraction once and without repeat.



The Farey Fractions become uniformly even when the denominator is not too large:

$$\sum_{0 \leq \frac{a}{b} < 1} e^{2\pi i\, m \frac{a}{b}} \ll \sqrt{N}$$

If we prove the cancellation is faster, it is equivalent to showing the **prime number theorem**.

Ramanuja's Sum. If p is prime:

$$\sum_{0 \leq \frac{a}{q} < 1} e\left(h\frac{a}{q}\right) = \sum_{c|(h,q)} c\,\mu\left(\frac{q}{c}\right)$$

If $h = 1$ there is only one term on the right hand side:

$$\sum_{0 \leq \frac{a}{q} < 1} e\left(\frac{a}{q}\right) = \mu(q)$$

Here is the identity linking Ramanujan's sum to the Mobius averages:

$$\sum_{1}^{N} e(h\,x_n) = \sum_{d|h} dM(Q/d)$$

where $0 < x_n < 1$ ranges over all the Farey fractions $\mathcal{F}$.

I have not been able to find a proof of the prime number theorem in the book, but the results he's approving approach that of the **Riemann Hypothesis**. It might be possible to prove PNT using the techniques in the book[1].

References

(1) M. N. Huxley **Area, Lattice Points, and Exponential Sums** London Mathematical Society Monographs. Oxford University Press, 1996.

[1] I suspect that Weyl's Equidistribution theorem is strong enough to prove PNT. Karamata's Theorem can be used to lead to PNT. Both of these are consequences of Weierstrass Approximation Theorem...