Theta Functions

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$$\theta(x;p) = (x;p)_\infty (px^{-1};p)_\infty = \exp\left(-\sum_{m \neq 0} \frac{x^m}{m(1-p^m)}\right)$$

another one

$$\theta(z;q) := (z;q)_\infty (q/z;q)_\infty = \frac{1}{(q;q)_\infty} \sum_{k \in \mathbb{Z}} z^k q^{\binom{k}{2}}$$

the shifted factorials are defined by:

$$(z;q)_\infty = \prod_{i \geq 0} (1 - zq^i)$$

Let's see if

$$\binom{k}{2} = \frac{k(k-1)}{2} = \frac{k^2}{2} - \frac{k}{2}$$

Then it could be:

$$\theta(q^2; q) = \frac{1}{(q; q^2)} \sum_{k \in \mathbb{Z}} q^k q^{2\binom{k}{2}} = \frac{1}{(q; q^2)} \sum_{n \in \mathbb{Z}} q^{n^2}$$

Wikipedia has

$$\sum_{n \in \mathbb{Z}} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty$$

and we can set $a = b = q$:

$$\sum_{n \in \mathbb{Z}} q^{n^2} = (-q; q^2)_\infty (-q; q^2)_\infty (q^2; q^2)_\infty$$

This also seems odd we can try

$$\theta(q; q^2) = (q; q^2)_\infty (q; q)_\infty (q^2; q^2)_\infty = \sum_{n \in \mathbb{Z}} q^{n^2}$$

It might parameterized in terms of two angles:

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}$$

which has another triple product

$$\prod_{m=1}^{\infty}(1-e^{2\imath m i\tau})\left[1 + e^{(2m-1)\pi i\tau + 2\pi i z}\right]\left[1 + e^{(2m-1)\pi i\tau - 2\pi i z}\right]$$

Then $q = e^{2\pi i\tau}$ and $x = e^{2\pi i z}$:

$$\theta(0;q) = \prod(1-q^2)(1+q^{2m-1})(1-q^{2m+1})$$

This is a beautiful triple product but we have to write in terms of rising and falling factorials.

$$\sum_{n\in\mathbb{Z}} q^{n^2} = (-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty$$

The exponent formula looks like

$$\log(1-x) = \sum\frac{x^m}{m}$$

and the geometric series formula:

$$\sum p^{km} = \frac{1}{1-p^m}$$

If we put two of them together it says:

$$\sum_m \sum_k \frac{1}{m} x^m p^{km} = \sum_m \frac{1}{m} \frac{x^m}{1 - p^m}$$

This is very much the logarithm in the beginning of this article.

Part II

So one big problem I will have with a lot of elliptic index paper with θ functions eveywhere is the normalization. And their endless obsession with modular invariance[1]

Uh... so before I get into that we rewind to 2003 before a lot of this paper and read through Appendix A of Nekrasov-Okounkov:

$$\gamma_\hbar(x;\Lambda) = \frac{d}{ds}\bigg|_{s=0}\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty \frac{dt}{t}t^s\frac{e^{-tx}}{(e^{\hbar t}-1)(e^{-\hbar t}-1)}$$

This is a mouth-ful but notice right away this is a **Mellin transform** and also this is **zeta regularization**. The Nekrasov partition function is *badly* divergent (as most physics formulas are) and here is one way to fix it.

However, these gentlement have used a very common idea in number theory. Here is a baby example:

$$\sqrt{1}-\sqrt{2}+\sqrt{3}-\sqrt{4}+\cdots = (\sqrt{1}-2\sqrt{2}+\sqrt{3})+(\sqrt{2}-2\sqrt{3}+\text{√}$$

Uh... hopefully I remember later[2]

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[1] If the object is invariant under $\mathrm{SL}_2(\mathbb{Z})$ or a congruence subgroup $[\Gamma_0(N) : \mathrm{SL}_2(\mathbb{Z})] = N$ or a *non-congruenc* group. There are many possibilities that Nekrasov and Shatashvili do not account for ('cuz they're not interested).

[2] but **you** and read http://math.stackexchange.com/q/1896464/4997

Nekrasov and Okounkov state this really is zeta-function regularization so we have

$$\gamma_\hbar(0; \Lambda) = -\frac{1}{12}$$

and even some instance of the volumes of the unitary groups:

$$\log\left(\text{Vol } U(N)\right) = \gamma_1(N; 1)$$

and the other functions $\gamma_{\epsilon_1, \epsilon_2}$ are embellishments[3].

These $\gamma_\hbar$ satisfy a second-difference equation:

$$\gamma_\hbar(x - \hbar, \Lambda) - 2\gamma_\hbar(x, \Lambda) + \gamma_\hbar(x + \hbar, \Lambda) = \log\left(\frac{x}{\Lambda}\right)$$

Theres so many logs floating around but I really want to talk about this Λ:

$$\sum[\Lambda(n) - 1]\frac{e^{-ny}}{1 - e^{-ny}} \sim -\frac{2\gamma}{y}$$

Then by the Hardy-Littlewood Tauberian theorem (for Lambert series)[4]:

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n} = -2\gamma$$

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[3]and a nuisiance to read – painful on the eyes

[4]which we will argue is the same kind of regularization as Nikita Nekrasov uses

If this thing converges at all the coefficients must have been small:

$$\sum_{n \leq x} [\Lambda(n) - 1] = o(x)$$

and this is very much equivalent to the Prime Number Theorem.

Here the Λ in question is the Van Mangold function:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

and we regularize the Lambert sum to a normal finite sum

$$y \sum_{n=1}^{\infty} \frac{(\Lambda(n) - 1)e^{-ny}}{1 - e^{-ny}} \approx \sum_{n=1}^{\infty} \frac{(\Lambda(n) - 1)}{n} e^{-ny} \rightarrow -2\gamma$$

if we let $y \rightarrow 0$

Part III – Review with some more details[5]

Let me digress on the values of the Riemann Zeta function. Here is a formula for $\zeta(2)$:

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \int_{1>t_1>t_2>0} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2}$$

Then there's the formula by Eugenio Calabi

$$\frac{3}{8}\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \int_0^1 \int_0^1 \frac{dx\,dy}{1-(xy)^2}$$

There is another dissimilar looking formula:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^{\infty} \log(1-e^{-x})$$

Even before trying our hand at the double-shuffle identities, or relating $\zeta(2)$ to famous constants[6] how to transform the first formula into the third?

I dug up the fist formula from a paper of Zagier, the second in a paper of Elkies and a third in a paper by Passare[7]

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[5]So as you might guess the theme is ambiguity in the literature. The confusion as an outider to see the same classical formula in 20 different places – each one with their opinion how it should be developed (or ignored completely).

[6]Why stop at π, there's the Glaisher constant and the Euler-Masceroni number and the Twin Prime Constant etc.

[7]and there is more... Papers flying everywhere!

References

(1) Taro Kimura, Vasily Pestun **Quiver elliptic W-algebras** `arXiv:1608.04651`

(2) Wikipedia "Jacobi Triple Product", "Ramanujan Theta Function"

(3) Eric M. Rains, S. Ole Warnaar **Bounded Littlewood identities** `arXiv:1506.02755`

(4) GH Hardy **Divergent Series** textthttps://archive.org/details/DivergentSeries

(5) David Vernon Widder **The Laplace Transform** `https://archive.org/details/laplacetransform031816`