Scratchwork: Decimals

At this point in my mathematical training, I take for granted that $\mathbb{R}$ is the number system that we all use. That $\mathbb{R}^2$ is the Euclidean plane. In order to represent numbers in $\mathbb{R}$ we should use decimals. Yet, when we solve equations we use Taylor series expansions or Fourier expansions or something less common. And finally, we turn our answer into a decimal representation of a number in $\mathbb{R}$.

We write in the decimal system base 10. We spend a few years learning a few idosyncracies of these basic operaions. Even type-setting decimal addiion and multiplication can be a chore.

$$\begin{array}{cccc} & 4 & {}^{1}2 & 3 \\ + & 7 & 5 & 8 \\ \hline 1 & 1 & 8 & 1 \end{array}$$

A modern *dynamical systems* view point is that we are studying the dynamics of the map $T : a \to (10 \times a)\%1$ on the real number line $\mathbb{R}/\mathbb{Z}$. We multiply by 10 and then remove the integer part. This requires an input the definition of a function: $f(x) = x\%1$ or sometimes written $\{x\}$ with some bit of fastidiousness

$$f(x) = \{x\} \overset{?}{=} \min_{n \in \mathbb{Z}} |x - n|$$

This definition is wrong since it returns $\{\frac{5}{4}\} = \frac{1}{4}$ but also $\{\frac{7}{4}\} = \frac{1}{4}$, since $\frac{7}{4} - 2 = -\frac{1}{4}$. So a careful definition of $f(x)$ is missing.

Exercise Find a correct definition of $f(x) = \{x\}$.

At the moment all I have is this annoying defintion: min A with $A = \{x - n : n \in \mathbb{Z} \text{ and } x > n\}$. And later we could ask what "$a > b$" even means? And "$n \in A$"? At some point we'll become too lazy to even check.

An insepction of the properties of $\mathbb{R}$ like this happens when we get stuck. In addition to base $b = 10$ we could have binary $b = 2$ with digits $\{0, 1\}$, so that $15_{10} = 1111_2$. There is even base systems for irrational base, so we could have silver ratio base $b = 1 + \sqrt{2}$ or Golden ratio base $b = \frac{1+\sqrt{5}}{2}$.

The map $a \mapsto \big[(1 + \sqrt{2}) \times a\big]\%1$ would have two outcomes:

- $0 < (1 + \sqrt{2}) \times a < 1$ so that $0 < a < \sqrt{2} - 1$.
- $1 < (1 + \sqrt{2}) \times a < 2$ so that $\sqrt{2} - 1 < a < 1$.

Then we have a partition of $[0, 1)$ that behaves nicely under the dynamical system T just described. We have a binary decimal system with digits $\{0, 1\}$ just as before but with some unusual properties. So what exceptional number shall we give? Let's try 3:

$$1 + \sqrt{2} < \mathbf{3} < (1 + \sqrt{2})^2 = 1 + 2 + 2 \times \sqrt{2} = 3 + 2\sqrt{2}$$

So this number would have two digits before the decimal place, $3 = 1\_.\_\_\_\ldots$ I'm not even sure if the decimal terminates.The next digit would be:

$$3 - (1+\sqrt{2}) = 2 - \sqrt{2} \overset{?}{<} 1 + \sqrt{2}$$

and we'd like to do this without peeking...without reverting to the decimal system, as any calculator does.[1]

References

[1] Michael Coornaert. **Topological Dimension and Dynamical Systems** (Universitext) Springer, 2015.

[2] Michael Field. **Essential Real Analysis** (Springer Undergraduate Texts in Analysis) Springer, 2017.

[3] Manfred Einsiedler, Thomas Ward. **Ergodic Theory: with a view towards Number Theory** GTM #259 Springer, 2011.

[1]And at some point we could inspect the how our calculators implement the decimal system.