# Tutorial : ABJM Theory

New matrix models associated to Chern-Simon's matter theory were found about 10 years ago. Especially ABJM theory:

- how did ABJM theory happen?
- check for consistency

Consistent or not, this has not stopped a wave of results up to now. Yet when I ask around, people are quick to point out contradictions in the setup.

It was significant that a supersymmetric Chern-Simon's matter theory was found. With a minimal knowledge of supersymmetry, I looked at the matrix model definition. This is what happens *after* localization:

$$Z = \frac{1}{\mathcal{W}} \int da \, e^{-4i\pi^2 \mathsf{Tr}(a^2)} \frac{\det_{\mathsf{Ad}} 2\sinh(\pi a)}{\det_R 2\cosh(\pi a)}$$

this is a finite-dimensional matrix integral. The Lie-group machinery is very ecumenical.

- if we find a supersymmetric Chern-Simon's theory with gauge group $G$
- and the "chiral multiplet" is in the representation $R \oplus R^*$
- the partition function is the finite dimensional integral $Z$
- it's an integral over the Cartan $\mathcal{W}$. If $G$ is compact (e.g. $G = SU(2)$) then $\mathcal{W}$ is the maximal torus.
- Tr is an invariant inner product on the Lie algebra $\mathfrak{g}$.
- Ad is the **adjoint** representation of $G$
- R is whatever representation we have chosen.
- $\det_R f(a) = \prod_\rho (f \circ \rho)(a)$ where the product is over the weights $\rho$ of the representation $R$.

While I respect Kapustin and his colleagues in terms of general correctness, if we want to find the number these matrix integrals evaluate too, these answers are still much too schematic.

At this point, I should consult a Lie groups textbook such as Fulton-Harris and find many of my answers there.

Resources on ABJM theory are not hard to find. What remains is to express our doubts in such as way as to not bring down the whole discussion. A totally rigorous discussion of ABJM theory should involve one about Chern-Simon's theory which simply doesn't exist. Therefore, we gently point out gaps in the argument.

Kapustin-Willet-Yaakov work out the answer for us in the case of ABJM theory and at least a few people are calculating with it:

$$Z = \int d\lambda \, d\mu \left( \prod_i e^{-ik\pi\,(\lambda_i^2 - \mu_i^2)} \right) \frac{\prod_{i<j} 2\sinh(\mu_i - \mu_j) \prod_{i<j} 2\sinh(\lambda_i - \lambda_j)}{\prod_{i \neq j} \left[ 2\cosh(\lambda_i - \mu_j) \right]^2}$$

Now we are left with the opposite problem. Where did this integral come from? We have two choices:

- supersymmetric gauge theory
- nothing

The authors compare their results to other physics calculations, which merely creates more room for confusion. Following their own recipe they have a little bit to say about how their integral was constructed:

- $G = U(N) \times U(N)$
- two chrial multiplets in the $(N, \overline{N})$ representation.
- two chrial multiplets in the $(N, \overline{N})$ representation (dual).
- $\mathrm{Tr} = \frac{k}{4\pi}(\mathrm{tr} - \hat{\mathrm{tr}})$

As mathematicians, I don't have to provide just one proof or definition. There's always talk about "the mathematical definition" we can have many.

**Conjecture #1** $Z$ is finite.

This conjecture is false, but at lease we have a provable statement. Up to now we have sound physical reasoning, but it's also speculative. The integral might converge:

$$Z = \int d\lambda \, d\mu \left( \prod_i e^{-ik\pi\,(\lambda_i^2 - \mu_i^2)} \right) \frac{\prod_{i<j} 2\sinh(\mu_i - \mu_j) \prod_{i<j} 2\sinh(\lambda_i - \lambda_j)}{\prod_{i \neq j} \left[ 2\cosh(\lambda_i - \mu_j) \right]^2} = \left[ \int d\lambda \left( e^{-ik\pi\,|\lambda|^2} \right) \right]^2$$

This looks oddly like the Gauss sum from number theory or the Fresel integral (which converges) and I've even found resources for that.

Lastly, I don't have the resources to understand chiral multiplets at this time (these are representations of superalgebra). By the Hilbert-Nullstellensatz or results of this kind we can study two kinds of geometric objects:

- maps of the object organized into charts
- the rings of observables living on the object

At a crucial step, the 3-manifold in Kapustin-Willet-Yaakov switches from flat space $\mathbb{R}^3$ to the 3-sphere $S^3$. This is excellent physical intuition but mathematically it's a nightmare.

If we find an observable like a chiral multiplet on a sphere, the underlying geometric object must be rather interesting: it's some kind of enriched $S^3$. What is it? I will just make something up:

$$S^3 \Big( \times U(N) \times U(\overline{N}) \Big)$$

Maybe I should say it is a vector space with an action of $U(N)$. I can never hope to solve the Chern-Simon's equations, what could this object be then? There needs to be some other way of thinking about these Chern-Simons-matter integrals that doesn't involve so much havy machinery.

While that is going on, I noticed the preponderance of infinite produts – that we just ignore:

$$\det(bosons) = \prod_{i<j} \prod_{\ell=1}^{\infty} \Big( (\ell+1)^2 + (\lambda_i - \lambda_j)^2 \Big)^{2\ell(\ell+2)}$$

Hopefully I typed it correctly. I don't know what a 1-loop determinant is and I have no plans of figuring it out. So I have to find some other way to understand this matrix integral that captures some of the localization process.

This seems to have been a big topic in 2011-12, so we are quite behind. Marcos Marino helps us along. First of all he concedes, details are missing in the research paper version.

- There is a thing called Chern-Simon's matter theory. And it can be made to be supersymmetric:

$$S = -\int d^3x \mathrm{Tr}\left(A \wedge dA + \frac{2i}{3}A^3 - \overline{\lambda}\lambda + 2D\sigma\right)$$

- There is a procedure to find the "partition" function of CS, involving a "saddle-point" approximation to the Feynman (or Dirac) path-integral: $\sum e^{iS}$ and finding the 1-loop approximation. This is called the "perturbative" approach.

$$Z(M) = \frac{1}{\text{vol}(\mathcal{G})}\int [\mathcal{D}A]\, e^{ikS} = \sum_{pt} Z^{(pt)}(M)$$

  There's no guarantee any of these expansions will converge to the exact answer. Chern-Simon's theory is "exactly solvable" and I don't know the reason they stop at 1-loop.

$$Z^{pt}(M) = Z^{pt}_{\text{1-loop}}(M)\left[\sum_{\ell=1}^{\infty} S_\ell^{pt}\, k^{-\ell}\right]$$

  From Mathematical point of view, there's no reason this **semiclassical approximation** is guaranteed to work. Or that this is the only acceptable way. Reasoning about the correctness of these procedures is not likely to make us very popular...

- Sometimes we can get an exact answer (computed by other means)

$$Z(S^3) = \frac{1}{(k+N)^{\frac{N}{2}}}\prod_{j=1}^{N-1}\left(\sin\frac{\pi j}{k+N}\right)^{N-j}$$

  and if we get $k \gg 1$ we get an approximate formula:

$$Z_{\text{1-loop}}(S^3) = k^{-\frac{N}{2}}\prod_{j=1}^{N-1}\left(\frac{2\pi j}{k}\right)^{N-j} = (2\pi)^{\binom{N}{2}} k^{-\frac{1}{2}N^2} G_2(N+1)$$

Do we buy the reasoning presented here? There are certainly problems with it, but it does yield us answer, which is better than other situation.

Every time, Marino or Kapustin or Maldacena uses a formula, can we do something else?

# References


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