Scratchwork: Multiplication

Example Furstenburg shows in 1967, that the only closed infinite subset of $\mathbb{R}/\mathbb{Z}$ invariant under $S : (\cdot) \mapsto \cdot \times a$ and $T : (\cdot) \mapsto \cdot \times b$ is all of $\mathbb{R}/\mathbb{Z}$ itself. For any irrational $\theta \notin \mathbb{Q}$,

$$\overline{\{a^k \times b^\ell \times \theta : k, \ell \geq 0\}} = \mathbb{R}/\mathbb{Z}$$

I don't know how such a basic point could be under argument in this first place. No non-expert would even question such a thing. It takes a small bit of anaysis and topology to *state* that the closure of an infinite set is something. And then we need to consult a resource on "commuting automorphisms", here $\theta \mapsto \theta \times a$ and $\theta \mapsto \theta \times b$.

For example, Let's find k and ℓ such that $\left|2^k \times 3^\ell \times \sqrt{2} - \sqrt{3}\right| < 10^{-2}$. These questions (even to me) seem like novelties, and each single case can be solved with a computer, given enough time and resources.

Example Weyl's Law (say for $H = \partial_x^2 + \sqrt{n}\,\partial_y^2$) gives an estimate for the distribution of eigenvalues (in $\mathbb{R}$):

$$\#\{j : \lambda_j < X\} = \#\{(a,b) : a^2 + \sqrt{n}\,b^2 < X\} \sim \frac{\pi}{4\sqrt{n}} X$$

We notice the distribution is approximately a line. Perhaps we could find the "doubling constant":

$$A + A \approx k\,A$$

The paper introduces all sorts of interesting measurable sets. Starting with $\mu(\mathbb{Q}) = 0$ and yet $\overline{\mathbb{Q}} = \mathbb{R}$. The paper also computes the gaps:

$$\delta_{min}(N) = \min(\{\lambda_{i+1} - \lambda_i : 1 \leq i \leq N\})$$

Here the λ_i are the sorted vales of $a^2 + \sqrt{n}\,b^2 \in \mathbb{Z}[\sqrt{n}]$. Notice we do not even need all of $\mathbb{R}$ to define this thing.

Let's even take a step further and remind ourselves that the numbers $\overline{\{a^2 + b^2 - \sqrt{2}c^2 : a, b, c \geq 0\}} = \mathbb{R}$ and we could have an exercise:

$$\left|a^2 + b^2 - \sqrt{2}c^2\right| < 10^{-6}$$

Finding such integers one time, is certainly tractable. I think the difficulty is showing this can *always* happen to arbitrary accuracy that requires the Ratner theory. In other words, that there is something complicated about these numbers.

We have a dilemma, no matter how many times we use a computer, no matter how much data we have, we are no closer to a proof. This problem and many others, originated in this use of computers, where $\mathbb{R} \approx 2^{-N}\,\mathbb{Z}$ where we only get $N = 20$ or 30 decimal places and the addition is truncated towards the end. Maybe we can find other measurable subsets $X \subseteq \mathbb{R}$ that show why this elementary result might offer such difficulty.

If we don't have enough decimal places, the approximate relation $a^2 + b^2 \approx \sqrt{2}c^2$ becomes an equality $a^2 + b^2 = \sqrt{2}c^2$ or $\sqrt{2} = \frac{a^2+b^2}{c^2} \in \mathbb{Q}$. This is a standard geometric exercise from Euclid that $\sqrt{2} \notin \mathbb{Q}$.

Let's settle for a timeline.

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References

[1] Proofs that $\sqrt{2} \notin \mathbb{Q}$
https://math.stackexchange.com/q/2382318
https://math.stackexchange.com/q/451700

Proof that $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$
https://math.stackexchange.com/q/452078
https://math.stackexchange.com/q/457382

[2] ...

[3] ...