Sum of 3 Squares Theorem, Hasse Principle, Banach-Tarski Paradox

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Scaling back, a much more ambitious project, we try to address three problems from number theory to measure theory:

- Lagrange showed $n = a^2 + b^2 + c^2$ iff $n \neq 4^a(8k+7)$

One quicky way to solve this eq is to solve in congruences:

- Solve equation in 2-adic numbers $n = a^2 + b^2 + c^2$, so $n \not\equiv 0 \mod 4$
- Solve $n = a^2 + b^2 + c^2 \mod p$ for all $p > 2$
- Solve $n = a^2 + b^2 + c^2$ in $\mathbb{R}$ (this just says $n > 0$)

Hasse-Minkowski principle tells us this is sufficient, but why does Hasse-Minkowski principle work at all?

How do we know the Hasse-Minkowski principle?

Case $n = a^2 + b^2 + c^2$

Reading's Serre's *Course on Arithmetic* we can solve in $\mathbb{Q}$:

$$a^2 + b^2 + c^2 - n\,d^2 = 0$$

We are able to find an $x \neq 0$ in $\mathbb{Q}$ solving two quadratic eqs:

$$a^2 + b^2 = x = c^2 - n\,d^2$$

This works because $x \in \mathbb{Q}$ not just $x \in \mathbb{Z}$, we'd better write

$$a^2 + b^2 - x\,e^2 = c^2 - n\,d^2 - x\,f^2 = 0$$

Once we have solved for $x \in \mathbb{Q}$ we solve $(a, b, c), (d, e, f) \in \mathbb{Q}^3$

$\circ \circ \circ$ This is reduction from 4 variables to 3 variables

How do we know the Hasse-Minkowski principle?

The only case $a^2 + b^2 + c^2 - n\,d^2 = 0$

Reading's Serre's *Course on Arithmetic* we can solve in $\mathbb{Q}$

Reduce 4 variables to 3...

Why is solving all congruences $n = a^2 + b^2 + c^2 \bmod p$ enough?

References

(1) JP Serre **Course on Arithmetic** Springer-Verlag