Miscellaneous Shing-Tung Yau Stuff

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Here is just a short review of Yau stuff...

ST Yau is prolific, teaches math at Harvard, for decades. Obviously knows his stuff. So all of his work, comes with a guarantee that it's going to work. There's not just one reading.

A good, well-crafted paper is like a prism, you put stuff in and you get stuff out. By necessity these reviews have to be superficial and I picked 4 of them, all of which tie to stuff I care about.

#1 Dynamics of D-Branes

I have no idea what these things really are. Yau offers a formalism and more importantly, enticing images. The lack of compelling images in the math literature.

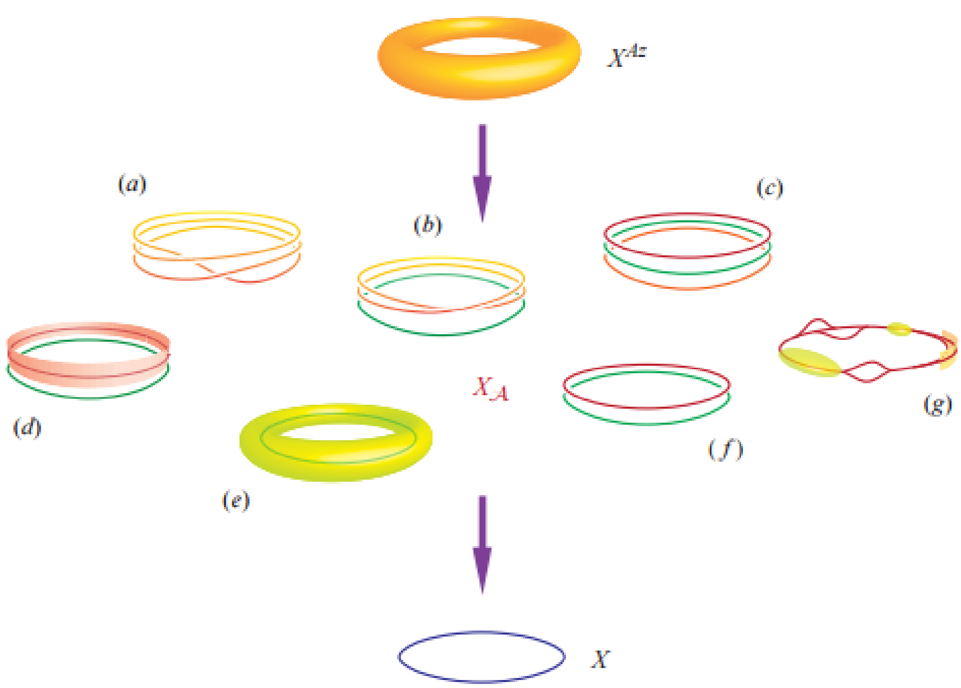


FIGURE 1-1. The noncommutative manifold X^{Az} has an abundant collection of C^∞-schemes as its commutative surrogates. See [L-Y8: FIGURE 2-1-1: caption] (D(13.1)) for more details.

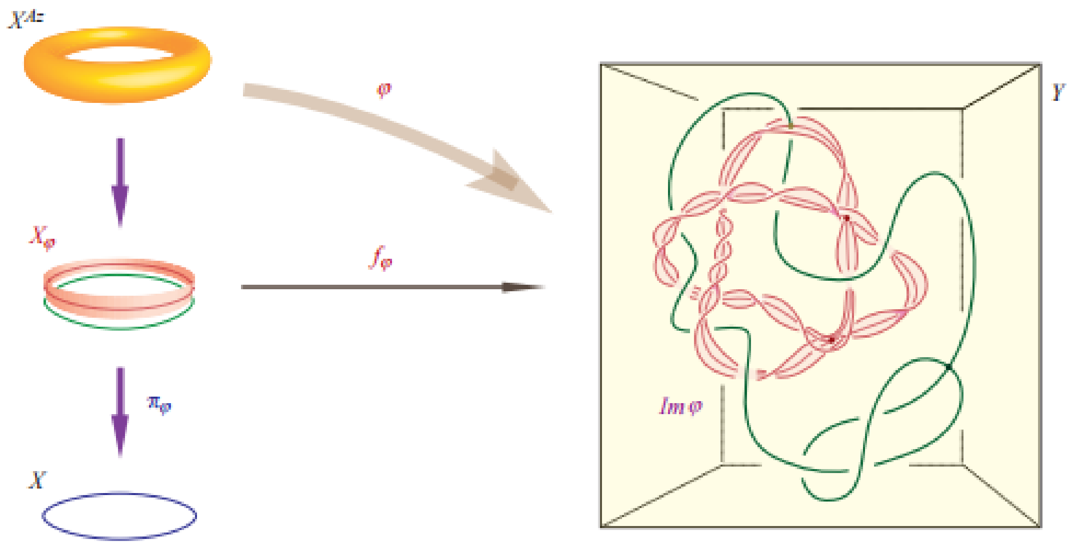


FIGURE 1-2. A map $\varphi : (X^{Az}, E) \to Y$ specifies a surrogate X_φ of X^{Az} over X. X_φ is a C^∞-scheme that may not be reduced (i.e. it may have some nilpotent fuzzy structure thereon). It on one hand is dominated by X^{Az} and on the other dominates and is finte and germwise algebraic over X.

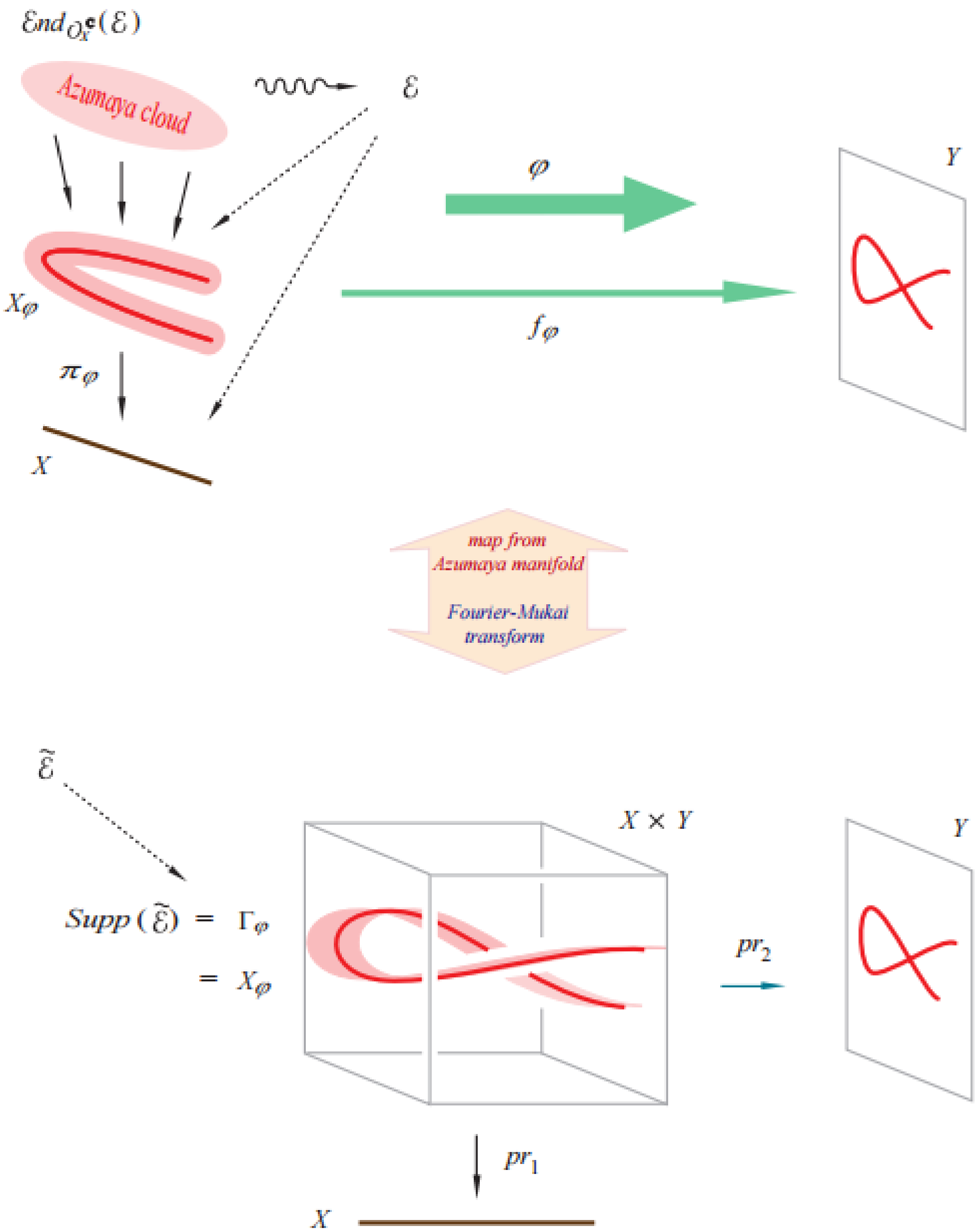


FIGURE 1-3. The equivalence between a map φ from an Azumaya manifold with a fundamental module $(X, \mathcal{O}_X^{Az} := \mathcal{E}nd_{\mathcal{O}_X^{\mathbb{C}}}(\mathcal{E}), \mathcal{E})$ to a manifold Y and a special kind of Fourier-Mukai transform $\tilde{\mathcal{E}} \in \mathcal{M}od^{\mathbb{C}}(X \times Y)$ from X to Y. Here, $\mathcal{M}od^{\mathbb{C}}(X \times Y)$ is the category of $\mathcal{O}_{X \times Y}^{\mathbb{C}}$-modules.

You must have better things to do than draw pictures, but I don't!

#2 The images are seductive. Hopefully not misleading!

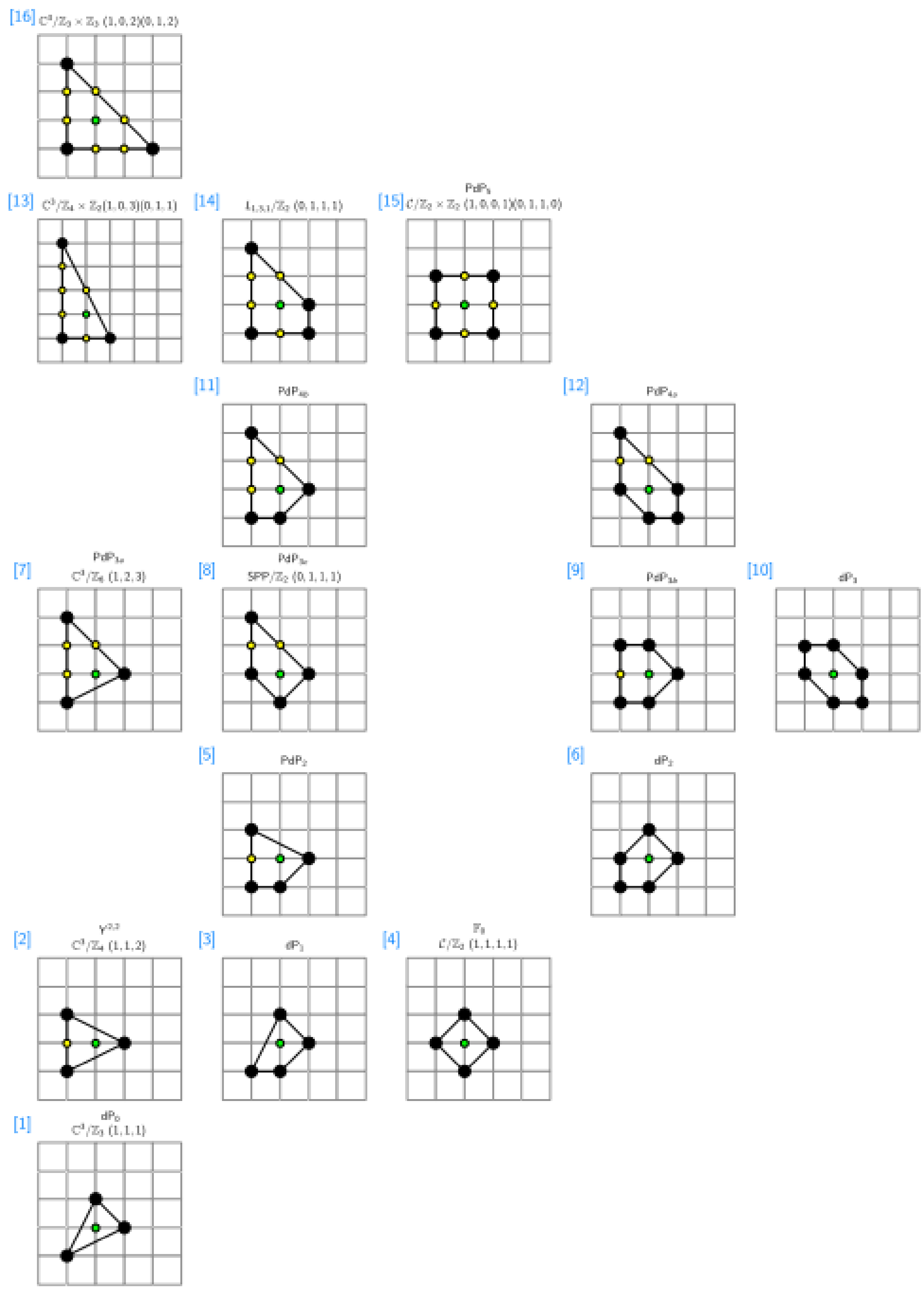


Figure 1: The 16 inequivalent reflexive polytopes Δ_2 in dimension 2. We see that, in particular, this includes the toric del Pezzo surfaces, numbers 1, 4, 3, 6, 10, which are the 5 smooth ones. For the naming of these polytopes we refer to the Calabi-Yau cone $\mathcal{X} = \mathcal{C}(X(\Delta))$. The middle 4 are self-dual while the other 6 polar dual pairs are drawn mirror-symmetrically about this middle line.

		Brane Configuration										T-Duality	D-Brane Probe
a)		0	1	2	3	4	5	6	7	8	9	2 times $\longleftrightarrow$	D3 $\perp$ CY3
	D5	×	×	×	×	·	×	·	×	·	·		
	NS5	×	×	×	×	—	Σ	—	—	·	·		
b)		0	1	2	3	4	5	6	7	8	9	3 times $\longleftrightarrow$	D1 $\perp$ CY4
	D4	×	×	·	×	·	×	·	×	·	·		
	NS5	×	×	—	—	—	Σ	—	—	·	·		
c)		0	1	2	3	4	5	6	7	8	9	4 times $\longleftrightarrow$	D(-1) $\perp$ CY5
	D3	·	×	·	×	·	×	·	×	·	·		
	NS5	—	—	—	—	Σ	—	—	—	·	·		

Table 3: The various brane configurations for brane tilings and how, under T-duality, they map to D-branes probing affine Calabi-Yau cones in various dimensions. (a) Brane tilings where D5-branes are suspended between a NS5-brane that wraps a holomorphic surface Σ. The D5 and NS5-branes meet in a T^2 inside Σ. Under thrice T-duality, the D5-branes are mapped to D3-branes probing CY3; (b) Brane brick models where D4-branes are suspended between a NS5-brane that wraps a holomorphic 3-cycle Σ. The D4 and NS5-branes meet in a T^3 inside Σ. Under T-duality, the D4-branes are mapped to D1-branes probing CY4; (c) Brane hyper-brick models where Euclidean D3-branes are suspended between a NS5-brane that wraps a holomorphic 4-cycle Σ. The D3 and NS5-branes meet in a T^4 inside Σ. Under T-duality, the D3-branes are mapped to D(-1)-branes probing CY5.

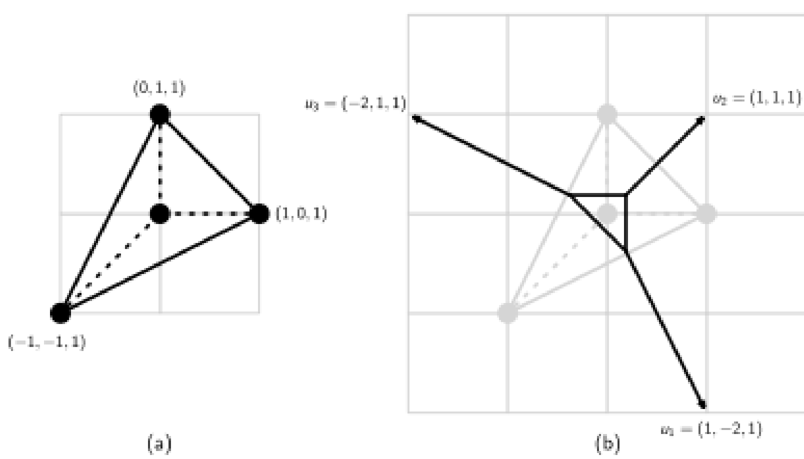


Figure 5: (a) Toric diagram and (b) dual (p, q)-web diagram for $\mathbb{C}^3/\mathbb{Z}_3$. The vectors $u_{1,2,3}$ are the outer normals to the fan and furnish the (p, q)-charges.

#3 I can't do much more than admire. The formulas conjure powerful memories and response and yet I can't take a more tangible action.

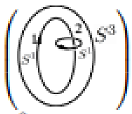
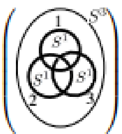
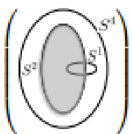
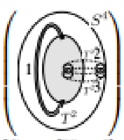
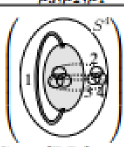
(i). Path-integral linking invariants; Quantum statistic braiding data	(ii). Group-cohomology cocycles distinguished by the braiding in (i)	(iii). TQFT actions **S** characterized by the spacetime-braiding in (i)
	2+1D	
Z  $= Z[S^3; \text{Hopf}[\sigma_1, \sigma_2]]$ $= \mathcal{S}_{\bar{\sigma}_1 \sigma_2}$	$\exp\left(\frac{2\pi \mathrm{i}\, p_{IJ}}{N_I N_J}\, a_I(b_J + c_J - [b_J + c_J])\right)$	$\int \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I + \frac{p_{IJ}}{2\pi} A^I \wedge \mathrm{d}A^J$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + \mathrm{d}\eta^I.$
Z  $= Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3];$ Also $Z[T^3_{xyt}; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$	$\exp\left(\frac{2\pi \mathrm{i}\, p_{123}}{N_{123}}\, a_1 b_2 c_3\right)$	$\int \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I + c_{123} A^1 \wedge A^2 \wedge A^3$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + \mathrm{d}\eta^I + 2\pi \bar{c}_{IJK} A^J g^K$ $-\pi \bar{c}_{IJK} g^J \,\mathrm{d}g^K.$
	3+1D	
Z  $= \mathrm{L}^{(S^2, S^1)}_{\mu\sigma}$	1	$\int \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + \mathrm{d}\eta^I.$
Z  $= Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_3, \mu_2]], \mu_1]]$ $= \mathrm{L}^{\text{Tri}}_{\mu_3, \mu_2, \mu_1}$	$\exp\left(\frac{2\pi \mathrm{i}\, p_{IJK}}{(N_{IJ} \cdot N_K)}(a_I b_J)(c_K + d_K - [c_K + d_K])\right)$	$\int \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I + \sum_{I,J} \frac{N_I N_J\, p_{IJK}}{(2\pi)^2 N_{IJ}} A^I \wedge A^J \wedge \mathrm{d}A^K$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + \mathrm{d}\eta^I + \epsilon_{IJ} \frac{N_I N_J\, p_{IJK}}{2\pi N_{IJ}} dg^J \wedge A^K,$ here K is fixed.
Z  $= Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]];$ Also $Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1]$	$\exp\left(\frac{2\pi \mathrm{i}\, p_{1234}}{N_{1234}} a_1 b_2 c_3 d_4\right)$	$\int \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I + c_{1234} A^1 \wedge A^2 \wedge A^3 \wedge A^4$ $A^I \to A^I + \mathrm{d}g^I,$ $N_I B^I \to N_I B^I + d\eta^I - \pi \bar{c}_{IJKL} A^J A^K g^L$ $+\pi \bar{c}_{IJKL} A^J g^K dg^L - \frac{\pi}{3} \bar{c}_{IJKL} g^J dg^K dg^L.$

TABLE II. Examples of topological orders and their topological invariances in terms of our data in the spacetime dimension $d+1$D. Here some explicit examples are given as Dijkgraaf-Witten twisted gauge theory [36] with finite gauge group, such as $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times \mathbb{Z}_{N_4} \times \dots$, although our quantum statistics data can be applied to more generic quantum systems without gauge or field theory description. The first column shows the path integral form which encodes the braiding process of particles and strings in the spacetime. In terms of spacetime picture, the path integral has nontrivial linkings of worldlines and worldsheets. The geometric Berry phases produced from this adiabatic braiding process of particles and strings yield the measurable quantum statistics data. This data also serves as topological invariances for topological orders. The second column shows the group-cohomology cocycle data ω as a certain partition-function solution of Dijkgraaf-Witten theory, where ω belongs to the group-cohomology group, $\omega \in \mathcal{H}^{d+1}[G, \mathbb{R}/\mathbb{Z}] = \mathcal{H}^{d+1}[G, \mathrm{U}(1)]$. The third column shows the proposed continuous low-energy field theory action form for these theories and their gauge transformations. In 2+1D, A and B are 1-forms, while g and η are 0-forms. In 3+1D, B is a 2-form, A and η are 1-forms, while g is a 0-form. Here $I, J, K \in \{1, 2, 3, \dots\}$ belongs to the gauge subgroup indices, $N_{12\dots u} \equiv \gcd(N_1, N_2, \dots, N_u)$ is defined as the greatest common divisor (gcd) of $N_1, N_2, \dots, N_u$. Here $p_{IJ} \in \mathbb{Z}_{N_{IJ}}, p_{123} \in \mathbb{Z}_{N_{123}}, p_{IJK} \in \mathbb{Z}_{N_{IJK}}, p_{1234} \in \mathbb{Z}_{N_{1234}}$ are integer coefficients. The $c_{IJ}, c_{123}, c_{IJK}, c_{1234}$ are quantized coefficients labeling distinct topological gauge theories, where $c_{12} = \frac{1}{(2\pi)} \frac{N_1 N_2\, p_{12}}{N_{12}}$, $c_{123} = \frac{1}{(2\pi)^2} \frac{N_1 N_2 N_3\, p_{123}}{N_{123}}$, $c_{1234} = \frac{1}{(2\pi)^3} \frac{N_1 N_2 N_3 N_4\, p_{1234}}{N_{1234}}$. Be aware that we define both $p_{IJ\dots}$ and $c_{IJ\dots}$ as constants with *fixed-indices* $I, J, \dots$ without summing over those indices; while we additionally define $\bar{c}_{IJ\dots} \equiv \epsilon_{IJ\dots} c_{12\dots}$ with the $\epsilon_{IJ\dots} = \pm 1$ as an anti-symmetric Levi-Civita alternating tensor where $I, J, \dots$ are *free indices* needed to be Einstein-summed over, but $c_{12\dots}$ is fixed. The lower and upper indices need to be summed-over, for example $\int \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I$ means that $\int \sum_{I=1}^{s} \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I$ where the value of s depends on the total number s of gauge subgroups $G = \prod_i^s \mathbb{Z}_{N_i}$. The quantization labelings are described and derived in [25, 43].

#4 Unfinished business. How can you not look at this table of formulas and feel something?

Type	$f(z_0, z_1, z_2, z_3)$	μ
I	$z_0^a + z_1^b + z_2^c + z_3^d$	$\mu = (a-1)(b-1)(c-1)(d-1)$
II	$z_0^a + z_1^b + z_2^c + z_2 z_3^d$	$\mu = (a-1)(b-1)[c(d-1)+1]$
III	$z_0^a + z_1^b + z_2^c z_3 + z_2 z_3^d$	$\mu = (a-1)(b-1)cd$
IV	$z_0^a + z_0 z_1^b + z_2^c + z_2 z_3^d$	$\mu = [a(b-1)+1][c(d-1)+1]$
V	$z_0^a z_1 + z_0 z_1^b + z_2^c + z_2 z_3^d$	$\mu = ab[c(d-1)+1]$
VI	$z_0^a z_1 + z_0 z_1^b + z_2^c z_3 + z_2 z_3^d$	$\mu = abcd$
VII	$z_0^a + z_1^b + z_1 z_2^c + z_2 z_3^d$	$\mu = (a-1)[bc(d-1)+b-1]$
VIII	$z_0^a + z_1^b + z_1 z_2^c + z_1 z_3^d + z_2^p z_3^q,$ $\frac{p(b-1)}{bc} + \frac{q(b-1)}{bd} = 1$	$\mu = \frac{(a-1)[b(c-1)+1][b(d-1)+1]}{b-1}$
IX	$z_0^a + z_1^b z_3 + z_2^c z_3 + z_1 z_3^d + z_1^p z_2^q,$ $\frac{p(d-1)}{bd-1} + \frac{qb(d-1)}{c(bd-1)} = 1$	$\mu = \frac{(a-1)d[c(bd-1)-b(d-1)]}{d-1}$
X	$z_0^a + z_1^b z_2 + z_2^c z_3 + z_1 z_3^d$	$\mu = (a-1)bcd$
XI	$z_0^a + z_0 z_1^b + z_1 z_2^c + z_2 z_3^d$	$\mu = abc(d-1) + a(b-1) + 1$
XII	$z_0^a + z_0 z_1^b + z_0 z_2^c + z_1 z_3^d + z_1^p z_2^q$ $\frac{p(a-1)}{ab} + \frac{q(a-1)}{ac} = 1$	$\mu = \frac{(a(c-1)+1)(ab(d-1)+a-1)}{a-1}$
XIII	$z_0^a + z_0 z_1^b + z_1 z_2^c + z_1 z_3^d + z_2^p z_3^q$ $\frac{p(a(b-1)+1)}{abc} + \frac{q(a(b-1)+1)}{abd} = 1$	$\mu = \frac{[ab(c-1)+a-1][ab(d-1)+a-1]}{a(b-1)+1}$
XIV	$z_0^a + z_0 z_1^b + z_0 z_2^c + z_0 z_3^d + z_1^p z_2^q + z_2^r z_3^s$ $\frac{p(a-1)}{ab} + \frac{q(a-1)}{ac} = 1 = \frac{r(a-1)}{ac} + \frac{s(a-1)}{ad}$	$\mu = \frac{[a(b-1)+1][a(c-1)+1][a(d-1)+1]}{(a-1)^2}$
XV	$z_0^a z_1 + z_0 z_1^b + z_0 z_2^c + z_2 z_3^d + z_1^p z_2^q$ $\frac{p(a-1)}{ab-1} + \frac{qb(a-1)}{c(ab-1)} = 1$	$\mu = \frac{a[c(d-1)(ab-1)+b(a-1)]}{a-1}$
XVI	$z_0^a z_1 + z_0 z_1^b + z_0 z_2^c + z_0 z_3^d + z_1^p z_2^q + z_2^r z_3^s$ $\frac{p(a-1)}{ab-1} + \frac{qb(a-1)}{c(ab-1)} = 1 = \frac{r(a-1)}{ac} + \frac{s(a-1)}{ad}$	$\mu = \frac{a[c(ab-1)-b(a-1)][d(ab-1)-b(a-1)]}{b(a-1)^2}$
XVII	$z_0^a z_1 + z_0 z_1^b + z_1 z_2^c + z_0 z_3^d + z_1^p z_2^q + z_0^r z_2^s$ $\frac{p(a-1)}{ab-1} + \frac{qb(a-1)}{d(ab-1)} = 1 = \frac{r(b-1)}{ab-1} + \frac{sa(b-1)}{c(ab-1)}$	$\mu = \frac{[c(ab-1)-a(b-1)][d(ab-1)-b(a-1)]}{(a-1)(b-1)}$
XVIII	$z_0^a z_2 + z_0 z_1^b + z_1 z_2^c + z_1 z_3^d + z_2^p z_3^q$ $\frac{p(a(b-1)+1)}{abc+1} + \frac{qc[a(b-1)+1]}{d(abc+1)} = 1$	$\mu = \frac{ab[abc(d-1)+c(a-1)+d]}{a(b-1)+1}$
XIX	$z_0^a + z_0 z_1^b + z_2^c z_1 + z_2 z_3^d$	$\mu = abcd$

Table 2. The Milnor number of the Hypersurface singularities.

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References

(1) Chien-Hao Liu, Shing-Tung Yau. **Dynamics of D-branes II. The standard action — an analogue of the Polyakov action for (fundamental, stacked) D-branes** `arXiv:1704.03237`

(2) Yang-Hui He, Rak-Kyeong Seong, Shing-Tung Yau. **Calabi-Yau Volumes and Reflexive Polytopes** `arXiv:1704.03462`

(3) Pavel Putrov, Juven Wang, Shing-Tung Yau. **Braiding Statistics and Link Invariants of Bosonic/Fermionic Topological Quantum Matter in 2+1 and 3+1 dimensions** `arXiv:1612.09298`

(4) Dan Xie, Shing-Tung Yau **4d N=2 SCFT and singularity theory Part I: Classification** `arXiv:1510.01324`