Scratchwork: Pell Equation over F

9/18 I always wondered what Pell Equation would look like over number fields other than $\mathbb{Q}$. Let's try:

$$x^2 - 2y^2 = 1 \text{ with } x, y \in \mathbb{Z}[\sqrt{3}]$$

This ring is a Euclidean domain. I haven't proved that. For example:

- Show that 5 and 17 are prime in $\mathbb{Z}[\sqrt{3}]$. Find the continued fraction of $\frac{17}{5}$ as a number in $\mathbb{Q}(\sqrt{3})$.
- We could write rational numbers in two different ways $\frac{a+b\sqrt{3}}{c+d\sqrt{3}} \in \mathbb{Q}(\sqrt{3})$ or $\frac{a}{b} + \frac{c}{d}\sqrt{3} \in \mathbb{Q}$ with $a, b, c, d \in \mathbb{Z}$.

Let's naïvely write the real and imaginary parts of that:

$$(x_1 + \sqrt{3}x_2)^2 - 2(y_1 + \sqrt{3}y_2)^2 = \big[(x_1^2 + 3x_2^2) - (y_1^2 + 3y_2^2)\big] + 2\sqrt{3}\big[x_1x_2 - y_1y_2\big] = 1$$

So this could read as a simultaneous equations, the intersection of two conic sections in 4 dimensions:

$$\begin{aligned} (x_1^2 + 3x_2^2) - (y_1^2 + 3y_2^2) &= 1 \\ x_1x_2 - y_1y_2 &= 0 \end{aligned}$$

This could also be read as a diophantine approximation problem. What's the best approximation of $\sqrt{2}$ in the number field $\mathbb{Q}(\sqrt{3})$?

$$\left| \frac{a}{b} - \sqrt{2} \right| < \frac{1}{\left| \, b\mathbb{Z}[\sqrt{3}] : \mathbb{Z}[\sqrt{3}] \, \right|^m} \text{ with}$$

I don't even know the correct exponent of m. This could be found using the Pigeonhole principle (the "Dirichlet principle").

$$\begin{aligned} (x_1^2 + 3x_2^2) - (y_1^2 + 3y_2^2) &= a \\ x_1x_2 - y_1y_2 &= b \end{aligned}$$

I'm not even sure how to draw these equations. If you have two quadrics C_1 and C_2 then the linear combination $\lambda C_1 + \mu C_2$ is also a quadratic. And this family is called a "divisor". For specific values of λ, μ degenerates into the intersection of two lines.

Before we do any of that, let's observe our Pell equation can be written as a 2×2 matrix:

$$\det \begin{vmatrix} x & 2y \\ y & x \end{vmatrix} = 1$$

Sadly, matrices and determinants are old-fashioned objects and need to be one away with. The determinant just the endomorphism of the exterior product of lattices anyway[1]. And a matrix is just an element of $V \otimes V^*$ with $V = \mathbb{Q}^2$.

[1] https://ncatlab.org/nlab/show/determinant

Next observe we can do a quadratic form in four variables, elements of a quaterion algebra:

$$\det \left[\begin{array}{cc} x_1 + x_2\sqrt{3} & y_1 - y_2\sqrt{3} \\ 2(y_1 + y_2\sqrt{3}) & x_1 - x_2\sqrt{3} \end{array} \right] = x_1^2 - 3x_2^2 - 2y_1^2 + 6y_2^2 = 1$$

As soon as you write the thing carefully, Pell equation becomes quite categorical. The matrices themselves become somwhat obsolete with no clear replacement.

Q: How many integer solutions does this equation have? $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ or more? We can solve:

$$\begin{aligned} a^2 - 3b^2 &= 1 \\ a^2 - 2b^2 &= 1 \end{aligned}$$

This leads us astray somewhat. I'd think the well-approximated ness of certain fraction algorithms can be turned into Pell-type equations.

Q: It's unknown if the continued fraction of the $\sqrt[3]{2}$ has any patterns. There might be other continued fraction algorithms, e.g. over vectors $(1, \sqrt[3]{2}, \sqrt[3]{4})$ and this would be related to Pell equation $a^3+2b^3+4c^3-6abc = 1$.

We needed Pell equation in order to estimate square roots outside of our number field, e.g. find $x \in \mathbb{Q}(\sqrt{2})$ such that

$$\left| x - \sqrt{2+\sqrt{3}} \right| < \epsilon$$

This is not quite the statement of well-approximatedness. We need two integers $a + b\sqrt{3}$ with $a, b \in \mathbb{Z}$:

$$\min_{a,b \in \mathbb{Z}[\sqrt{2}]} (b_1^2 - 3b_2^2)^{1} \times \left| \left(\frac{a_1 + a_2\sqrt{3}}{b_1 + b_2\sqrt{3}} \right)^2 - \sqrt{2+\sqrt{3}} \right|$$

The 1 is a guess this is just the size of the lattice $|b_1^2 - 2b_2^2| = \big[(b_1 + \sqrt{2}b_2)\mathbb{Z}[\sqrt{3}] : \mathbb{Z}[\sqrt{3}]\big]$.

Q The number of elements of $\mathbb{Z}[\sqrt{3}]$ with norm $< m$ is roughly m. That's not true. There are infinitely many points in the regin $|x^2-3y^2| < m$. In that case since $x^2-3y^2 = 1$, we can count points in $\mathbb{Z}[\sqrt{3}]^\times/(2+\sqrt{3})$

Ex Using guess and check. Let's write the primes $p = 2, 3, 5, 7, 11, 13, 17, 19$. We have that $-11 = 1 \times 1 - 3 \times (2 \times 2)$ and $13 = 4 \times 4 - 3 \times (3 \times 3)$ and then $(11) = (1 + \sqrt{2}) \times (1 + \sqrt{2})$ and (13) so yeah. Then we can write

$$\mathbb{A}_F = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \mathbb{Q}_7 \times \mathbb{Q}_{1+\sqrt{2}} \times \mathbb{Q}_{1-\sqrt{2}} \times \dots$$

as part of the the Adeles.[2] Quadratic reciprocity or Fermat's Little Theorem or Geometry of Numbers or Pigeonhole Principle. This variety of techniques should lead to a variety of problems to solve.

Q Solve $|x^2 - 2|_{1+\sqrt{2}} < \frac{1}{11}$ and $|x^2 - 3|_{1-\sqrt{2}} < \frac{1}{11}$ in $\mathbb{Q}(\sqrt{3})$.

Q Why don't we use the Hasse principle to solve the Pell equation?

[2]It seems somewhat hypocritical to prove these results via Geometry of Numbers and to find these numbers by exhaustic search. Even if we do that… **search problems** are a well-studied genre of computer science problem. Ignoring all structore over $\mathbb{Z}$ - which wasn't too realistic anyway, we are evaluating a function $f(x, y) = x^2 - 3y^2$ over a 2D array

References

[1] Jurgen Neukirch. **Algebraic Number Theory** (Grundlehren der mathematischen Wissenschaften) Springer, 1999.

[2] Vladimir Platonov, Andrei Rapinchuk, Rachel Rowen **Algebraic Groups and Number Theory** Academic Press, 1993.

[3] JWS Cassels. **An Introduction to Diophantine Approximation** (Cambridge Tracts in Mathematics and Mathematical Physics, No. 45) Cambridge University Press, 1957.

What happens when $F = \mathbb{Q}(\sqrt{5})$ and the class number is $\mathbf{2}$ and there's no Euclidean algorithm. How about $F = \mathbb{Q}(\sqrt{14})$ which is Euclidean but not norm-Euclidean? [3].

Q Solve $x^2 \approx 2$ in $\mathbb{Q}(\sqrt{5})$ or $\mathbb{Q}(\sqrt{8})$ or $\mathbb{Q}(\sqrt{14})$.

[3] `https://math.stackexchange.com/q/1234305/4997`