Scratchwork: "Locally Compact Abelian" Groups

This harmonic analysis jargon "Locally Compact Abelian" groups appears basically all over number theory in a very abstract form. Here are the two examples I can think of immediately: $\mathbb{R}$ and $\mathbb{Q}_p$. Harmonic analysis is generalization of Fourier Analysis to abstract groups, such as $SO(3)$ the rotations of a three-dimensional object and many other kinds of symmetry.[1]

In our case, there is an isomorphism (as group) in Number Theory

$$\mathbb{Q}^\times = \prod_p p^{\mathbb{Z}} \quad \text{and} \quad \mathbb{Q}(i)^\times = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathbb{Z}}$$

In fact all the $F^\times$ are isomorphic for any number field F. That cannot be right. On the left the primes are $p \in \mathbb{Z}$ and on the right $\mathfrak{p} \in \mathbb{Z}[i]$ and for little more than grade school arithmetic, we have that primes $p \in 4\mathbb{Z}+1$ factor into $p = \mathfrak{p}_1\mathfrak{p}_2$ E.g. $13 = (2+3i) \times (2-3i)$ or $17 = (4+i) \times (4-i)$. Therefore both number fields share the primes in $p \in 4\mathbb{Z}+3$. So we need one extra thing to distinguish between $\mathbb{Q}^\times$ and $\mathbb{Q}(i)^\times$, that is a **topology**.

Have you seen Fourier analysis on $\mathbb{Q}^\times$ or $\mathbb{Q}(i)^\times$? These not compact, since we can find Cauchy sequences $x \to \sqrt{2}$ or $\sqrt{3} \notin \mathbb{Q}^\times$. So how can we define a derivative $f'(x) = 2x$.

$$f'(x) = \lim_{|\epsilon| \to} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

Therefore we need to find a space where this derivative makes any kind of sense at all. I think we have some freedom here. Why is this thing $f'(x)$ even a number?

Since I didn't study Munkres' textbook carefully (and all the pathological but nifty examples) we just settle for asking, which number $x \in \mathbb{Q}^\times$ or $x \in \mathbb{Q}(i)^\times$ solve $x \approx 0$? Which numbers are close to zero? And our number theory tool is called **strong approximation**. So there is in fact some ambiguity in the statement $x \to \sqrt{2}$ in a manner that is important to Number Theory.[2]

References

[1]

[1] When do we study a group action? Usually this means the object was not a priori symmetric! A cube can be rotated in three-dimensional space by $SO(3)$ elements, and we can study the remainder that's leftover.

[2] Don't just call their bluff, there should hopefully be a good reason.