

# Scratchwork: Intersection of Two Lines

In geometry class, we learn the Cramer rule for the intersection two lines.

$$\begin{aligned}a_1x + b_1y &= c_1 \\a_2x + b_2y &= c_2\end{aligned}$$

And so the intersection of these two lines can be found with a **determinant** of a  $2 \times 2$  matrix:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

In a Linear Algebra course - or a Geometry course - one might check that  $a, b, c \in \mathbb{R}$  means our solutions  $(x, y) \in \mathbb{R}^2$ . We don't have that for an integer problem  $a, b, c \in \mathbb{Z}$  the solution remains in integers  $(x, y) \in \mathbb{Z}^2$ .

Since the  $+$  and  $\times$  operations we do aren't too fancy, we can do Linear Algebra over a field such as  $K = \mathbb{Q}$  or  $K = \mathbb{C}$ . In addition, let's use a tiny bit of Exterior Algebra taken from a Geometry textbook.

**Thm** The points  $A$ ,  $B$  and  $C$  are collinear if and only if  $A \wedge B + B \wedge C + C \wedge A = 0$ .

In our case, the equation has one line  $\boxed{Ax + By = C}$ . We can write the Cramer rule in an more condensed way:

$$Ax + By = C \rightarrow A \wedge (Ax + By) = (A \wedge B)y = (A \wedge C) \rightarrow y = \frac{A \wedge C}{A \wedge B}$$

And a similar formula for  $x$ . Is it okay to write the coordinate value of  $x$  and  $y$  as the ratio of two areas. The geometric objects look kind of funny but OK.

$$[\text{number}] = \frac{[\text{area}]}{[\text{area}]}$$

This is not outrageous. Pedoe gives a careful derivation of the wedge product of two vectors:

$$u \wedge v = (x_1E_1 + x_2E_2) \wedge (y_1E_1 + y_2E_2) = (x_1y_2 - x_2y_1)(E_1 \wedge E_2)$$

where  $E_1, E_2 \in \mathbb{R}^2$  are unit vectors in the plane.

There are even more intersection formulas like this. Two planes in Four dimensions intersect (generically) at a point.

$$\mathbb{R}^2 \cap \mathbb{R}^2 = \{pt\} \text{ in } \mathbb{R}^4$$

Since all we're doing is linear algebra, this still could work over  $\mathbb{Q}$  we'd have  $\mathbb{Q}^2 \cdot \mathbb{Q}^2 = [pt] \subseteq \mathbb{Q}^4$ . This is the beginnings of intersection theory and a lot of sheafy things could occur.

Algebraic geometry could be done over any field  $K$ , such as  $K = \mathbb{Q}(i)$  or possibly  $K = \mathbb{Q}(\sqrt{2})$ . Let's check two ways of writing fractions in  $\mathbb{Q}(i)$ :

$$\frac{a_1}{b_1} + \sqrt{-1} \frac{a_2}{b_2} = \frac{c_1 + \sqrt{-1}d_1}{c_2 + \sqrt{-1}d_2} \in \mathbb{Q}(i)$$

What assures us that the first way of writing rationals is the same as the second one? There should be "highly algebraic" way connecting the two.

$$(a, b) \mapsto (c, d)$$

This map is called "birational" or something. Clearly they will represent the same thing, the **affine plane** or  $\mathbb{A}^1$ .

If we wanted to change from one notation to the other we "just" clear denominators.

$$\frac{c_1 + \sqrt{-1}d_1}{c_2 + \sqrt{-1}d_2} = \frac{c_1 + \sqrt{-1}d_1}{c_2 + \sqrt{-1}d_2} \cdot \frac{c_2 - \sqrt{-1}d_2}{c_2 - \sqrt{-1}d_2} = \frac{(c_1c_2 + d_1d_2) + \sqrt{-1}(c_2d_1 - c_1d_2)}{c_2^2 + d_2^2}$$

Then we can try looking at less convenient systems such as de Moivre's theorem:

$$p + \sqrt{-1}q = \sqrt{p^2 + q^2} \times \exp[i\theta] \text{ with } \theta = \tan^{-1} \frac{p}{q}$$

When we add two of these numbers, we have not left this domain.

$$\sqrt{p_1^2 + q_1^2} \exp(i\theta_1) + \sqrt{p_2^2 + q_2^2} \exp(i\theta_2) = \sqrt{(p_1 + p_2)^2 + (q_1 + q_2)^2} \exp\left(i \tan^{-1}(\tan \theta_1 + \tan \theta_2)\right)$$

and if we multiply them we get another bunch of trigonometric identities:

$$\sqrt{p_1^2 + q_1^2} \exp(i\theta_1) \times \sqrt{p_2^2 + q_2^2} \exp(i\theta_2) = \sqrt{(p_1p_2 - q_1q_2)^2 + (p_1q_2 + p_2q_1)^2} \exp[i(\theta_1 + \theta_2)]$$

What do we even mean "put into context"? Or "generalized"? Even this is going to be made very precise, even mechanical, at the expense that we'll barely know what we're talking about.

**Ex.** Does playing off the field extension  $\mathbb{Q} \rightarrow \mathbb{Q}(i)$  against the operations  $+$  and  $\times$  have a name?

**Ex.** How do we "lift" GCD from  $\mathbb{Q}$  to  $\mathbb{Q}(i)$ ? Can this be made functorial?

**Ex.** How do these field extensions interact with the geometry of circles? Here are the change of variables formulas for the differentials:

$$dr = \frac{x dx + y dy}{\sqrt{x^2 + y^2}} \quad (1)$$

$$d\theta = \frac{x dy - y dx}{x^2 + y^2} \quad (2)$$

$$(dr)^2 + (r d\theta)^2 \in \mathcal{O}_{\mathbb{A}^2}(-2) \quad (3)$$

This was taken on the Wikipedia article on the Levi-Civita connection and **parallel transport**. How do we compare information from two different points on the circle or sphere?  $D : T_{(x,y)} \rightarrow T_{(x,y)}$ ? I need to write down the sheaf to keep track of the algebraic object I am using to imitate the geometry I am doing here. Real-world data, if only it had this much structure, right?

In algebraic geometry, we can localize the tangent sheaf at a point to obtain  $\mathcal{O}(-2)_{(x,y)}$ . This gives us the place to discuss the information we are caring about.

**Example** Even more tensor product identities to ponder (here we work over  $\mathbb{Z}$  can we please generalize these to a number field  $F$  ? )

$$\begin{aligned}\mathbb{Z}_a \otimes \mathbb{Z}_b &= \mathbb{Z}_{\gcd(a,b)} \\ \mathbb{Q}(i) \otimes \mathbb{R} &= \mathbb{R}^2 \\ \overline{\mathbb{Q}} &= \mathbb{R} \\ \overline{\mathbb{Q}(\sqrt{2}, \sqrt{3})} &= \mathbb{R}\end{aligned}$$

We have  $p$ -adic completions of various kinds, and tensor products, closures in Zariski topology. All of these are known to fit in a large categorical context. It's so tempting to rattle off these definitions as if we know something.

## References

- [1] Joe Harris **Algebraic Geometry: A First Course** (GTM #133) Springer, 1992.
- [2] Gabriel Cramèr **Introduction à l'Analyse des Lignes Courbes Algébriques**  
[https://archive.org/details/bub\\_gb\\_HzcVAAAAQAAJ](https://archive.org/details/bub_gb_HzcVAAAAQAAJ)
- [3] Dan Pedoe. **Geometry: A Comprehensive Course** Dover, 1970.