Scratchwork: Induced Representations

The quaternions are a number system defined by three rules of multiplication. They generalize complex numbers:

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \text{ and } \mathbf{i} \times \mathbf{j} = \mathbf{k} \text{ and } \mathbf{i} \times \mathbf{i} = -\mathbf{1}$$

These multiplications can we completed to form a group of order 8.

$\times$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$	$-\mathbf{j}$	$-\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$	$-\mathbf{j}$	$-\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{1}$	$-\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$	$-\mathbf{j}$	$\mathbf{k}$	$\mathbf{1}$	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{1}$	$-\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$
$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{i}$	$-\mathbf{j}$	$-\mathbf{k}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$-\mathbf{i}$	$-\mathbf{i}$	$\mathbf{1}$	$-\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$	$-\mathbf{k}$	$-\mathbf{j}$
$-\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$\mathbf{1}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$-\mathbf{k}$	$-\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{1}$

It looks like there are eight things being multiplied, so we made an 8×8 table. There are eight things being permuted in 8 possible ways:

$$\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}, -\mathbf{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\}$$

It may even be possible to whittle this down to four - with the inclusion of a minus sign $(-\mathbf{1})$.

$$\begin{aligned} -\mathbf{1} \times \mathbf{1} &= -\mathbf{1} \\ -\mathbf{1} \times \mathbf{i} &= -\mathbf{i} \\ -\mathbf{1} \times \mathbf{j} &= -\mathbf{j} \\ -\mathbf{1} \times \mathbf{k} &= -\mathbf{k} \end{aligned}$$

Cayley's Theorem says every group can be placed into a permutation group. We could call the elements of this group $\{1, 2, \ldots, 8\}$.

$$\mathbf{1} \mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right]$$

and now we replace with different rows of the multiplication table:

$$\begin{aligned} \mathbf{i} &\mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 7 & 6 & 1 & 8 & 3 \end{array} \right] \\ \mathbf{j} &\mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 5 & 2 & 7 & 4 & 1 & 6 \end{array} \right] \\ \mathbf{k} &\mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 6 & 1 & 8 & 3 & 2 & 5 \end{array} \right] \end{aligned}$$

The rule for $(-\mathbf{1})$ looks a little bit complicated. For the time being switch the first and second half.

$$\mathbf{1} \mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{array} \right]$$

There's even other ways of representing the quaternion group. Here's the more usual 2×2 matrices (in case you're scared of Quaternion objects).

$$\mathbf{1} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ \mathbf{i} \mapsto \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \ \mathbf{j} \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ \mathbf{k} \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

It could even be instructive to write out the full 8×8 matrices:

$$\mathbf{1} \rightarrow \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1\cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

That one was not too informative let's try the other three.

$$\mathbf{i} \rightarrow \left[\begin{array}{cccc|cccc} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \text{ and } \mathbf{j} \rightarrow \left[\begin{array}{cccc|cccc} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{array}\right] \text{ and } \mathbf{k} \rightarrow \left[\begin{array}{cccc|cccc} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{array}\right]$$

Do we lose any information by writing them as 4×4 matrices?

$$\mathbf{1} \rightarrow \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \text{ and } -\mathbf{1} \rightarrow \begin{bmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix} \text{ and } \mathbf{i} \rightarrow \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{bmatrix} \text{ and } \mathbf{j} \rightarrow \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

So we've now found three different representations of the quaternion algbra as matrices of various sizes 2×2 and 4×4 and 8×8. It seems like we can keep going...

References

[1] ...

[2] Sir William Rowan Hamilton
Elements of quaternions `https://archive.org/details/elementsofquater00hamirich`
Lectures on quaternions `https://archive.org/details/lecturesonquater00hami`

9/13 At this moment, we're going to invoke the machinery of Group Representations. We know for a fact there are 4 irreducible one-dimensional group representations.

$$\begin{array}{rcl}\mathbf{1} & \overset{\phi}{\mapsto} & 1 \in \mathbb{C} \\ \mathbf{i} & \mapsto & \pm 1 \text{ or } \pm i \in \mathbb{C} \\ \mathbf{j} & \mapsto & \pm 1 \text{ or } \pm i \in \mathbb{C} \\ \mathbf{k} & \mapsto & \phi(\mathbf{i}) \times \phi(\mathbf{k})\end{array}$$

and one more representation as 2×2 matrices, which are defined over $\mathbb{C}$ as well.

$$\mathbf{1} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{i} \mapsto \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \mathbf{j} \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{k} \mapsto \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and then we are told, in a sense, we have all the group representations we will ever need.

$$|Q_8| = 8 = 2 \times 2 + 4 \cdot (1 \times 1) = \dim(2 \times 2) + 4 \cdot \dim(1 \times 1)$$

Here's what Wikipedia has to say about Schur's Lemma:

> In mathematics, Schur's lemma is an elementary but extremely useful statement in representation theory of groups and algebras. In the group case it says that if M and N are two finite-dimensional irreducible representations of a group G and ϕ is a linear transformation from M to N that commutes with the action of the group, then either ϕ is invertible, or $\phi = 0$. An important special case occurs when $M = N$ and ϕ is a self-map.

To call Schur's lemma "elementary" is to risk missing an opportunity, I think. Nope I don't believe it for second.

How can I cast doubt? Here's a brand new representation I just made up, using polynomials. Any 2×2 matrix becomes a map linear map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : (x, y) \mapsto (ax + by, cx + dy)$$

This can be done to polynomials as well. Let's only allow quadratic terms x^2, xy, y^2. Then:

$$\begin{array}{rcl} x^2 & \mapsto & (ax + by)^2 \\ xy & \mapsto & (ax + by)(cx + dy) \\ y^2 & \mapsto & (cx + dy)^2 \end{array}$$

This linear map preserves the vector space of polynomials $\mathbb{C}[x^2, xy, y^2]$. So we have a three-dimensional representation of the quaternions.

$$\mathbf{i} \mapsto \begin{bmatrix} x^2 & \mapsto & -x^2 \\ xy & \mapsto & -xy \\ y^2 & \mapsto & -y^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{j} \mapsto \begin{bmatrix} x^2 & \mapsto & y^2 \\ xy & \mapsto & -xy \\ y^2 & \mapsto & x^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{k} \mapsto \begin{bmatrix} x^2 & \mapsto & y^2 \\ xy & \mapsto & xy \\ y^2 & \mapsto & x^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Schur's Lemma tells us our 3×3 representation is a direct sum of 1D representations $\mathbb{C}[x^2, xy, y^2] \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$.

Let's explicitly compute the matrices for cubic polynomials:

$$\mathbf{i} \;\mapsto\; \left[\begin{array}{ccc} x^3 & \mapsto & -i\,x^3 \\ x^2y & \mapsto & -i\,x^2y \\ xy^2 & \mapsto & -i\,xy^2 \\ y^3 & \mapsto & -i\,y^3 \end{array}\right] = \left[\begin{array}{cc|cc} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ \hline 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{array}\right]$$

$$\mathbf{j} \;\mapsto\; \left[\begin{array}{ccc} x^3 & \mapsto & -y^3 \\ x^2y & \mapsto & xy^2 \\ xy^2 & \mapsto & -x^2y \\ y^3 & \mapsto & x^3 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{k} \;\mapsto\; \left[\begin{array}{ccc} x^3 & \mapsto & i\,y^3 \\ x^2y & \mapsto & -i\,xy^2 \\ xy^2 & \mapsto & i\,x^2y \\ y^3 & \mapsto & -i\,x^3 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ \hline 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{array}\right]$$

We are finding the **symmetric product** of the two representations. In notation:

$$\text{Sym}(\mathbf{2}) \simeq \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \text{ and } \mathbf{2} \otimes \mathbf{2} \simeq \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$$

Even if the groups are comparatively straightforward, their *plethysms* are not.

Ex Is $\text{Sym}^3(\mathbf{2})$ isomorphic to $\mathbf{2} \oplus \mathbf{2}$ or $\mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}$ or $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$?

Ans Our ability to draw lines find the 2×2 matrices ourselves suggests the answer is $\boxed{\mathbf{2} \otimes \mathbf{2}}$.

This game of Tetris goes on forever. What we really have done is taken a representation of $SU(2)$ and found the induced representation. These are called "spin" and they are indexed by the half-integers (we could write $\frac{1}{2}\mathbb{Z}$ or $\mathbb{Z}$). This is fairly old hat.

The quaternions were also a subgroup of S_8 and we permuted 8 objects. This is called the regular representation (after we remove the trivial part). **Ex** Prove or disprove $\mathbf{8} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}$.

There's an infinite group inside $SU(2)$. Let's try $1 = (3/5)^2 + (4/5)^2$. Then we can generate quaternions $a = \frac{3}{5} + \frac{4}{5}\mathbf{i}$ or $b = \frac{3}{5} + \frac{4}{5}\mathbf{j}$ or $c = \frac{3}{5} + \frac{4}{5}\mathbf{k}$. These could be turned into 2×2 matrices:

$$\left[\begin{array}{cc} \frac{3}{5} + \frac{4}{5}\,i & 0 \\ 0 & \frac{3}{5} + \frac{4}{5}\,i \end{array}\right] \text{ or } \left[\begin{array}{cc} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{array}\right] \text{ or } \left[\begin{array}{cc} \frac{3}{5} + \frac{4}{5}\,i & 0 \\ 0 & \frac{3}{5} + \frac{4}{5}\,i \end{array}\right]$$

and we can't stop there because all of this is well-trodden territory. The two representations are called $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. How about we turn these into the 3×3 matrices we found before.

$$b = \frac{3}{5} + \frac{4}{5}\mathbf{k} \;\mapsto\; \frac{3}{5}\left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right] + \frac{4}{5}\left[\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc} -\frac{3}{5} & 0 & \frac{4}{5} \\ 0 & \frac{1}{5} & 0 \\ -\frac{4}{5} & 0 & -\frac{3}{5} \end{array}\right]$$

and the complex numbers get projected to a line? These non-commuting matrices generate an infinite group.

Somewhat more strict:

$$\begin{array}{rcl} e_1 & \mapsto & \frac{3}{5}e_1 - \frac{4}{5}e_2 \\ e_2 & \mapsto & \frac{4}{5}e_1 + \frac{3}{5}e_2 \end{array}$$

$$e_1 \otimes e_2 \quad \mapsto \quad \left(\tfrac{3}{5}e_1 - \tfrac{4}{5}e_2\right) \otimes \left(\tfrac{4}{5}e_1 + \tfrac{3}{5}e_2\right) = \frac{12}{25}\big(e_1 \otimes e_1\big) + \frac{9}{25}\big(e_1 \otimes e_2\big) - \frac{16}{25}\big(e_2 \otimes e_1\big) - \frac{12}{25}\big(e_2 \otimes e_2\big)$$

and in addition – since we have a symmetric product we really want:

$$\frac{1}{2}\big[(e_1 \otimes e_2) + (e_2 \otimes e_1)\big] \mapsto \frac{12}{25}\big(e_1 \otimes e_1\big) + \frac{7}{25}\big[\big(e_1 \otimes e_2\big) + \big(e_2 \otimes e_1\big)\big] - \frac{12}{25}\big(e_2 \otimes e_2\big)$$

Shows you how long it's been since I've done this kind of algebra...

References

[1] William Fulton **Representation Theory: A First Course** (GTM #129) Springer, 1991.

[2] Benjamin Steinberg **Representation Theory of Finite Groups** (Universitext) Springer, 2012.

[3] Ben Green **What is...an Approximate Group** Notices of the AMS. Volume 59 Number 5, 2012.