Getting Good at Inversions

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My apologies in advance that I did not draw lots of pictures!

$$w = \frac{az+b}{cz+d}$$

If $ad - bc \neq 0$ this map is invertible and maps circles to circles.

$$\left| \frac{z-p}{z-q} \right| = k$$

represents a circle where p and q are **inverse** points[1].

[1] These terms from Eucldean Geometry deserve a lot of reflection! The goal of this note is to jot down one simple hard-to-find formula.

Trouble is we don't know the radius and center. The only book I could find with a formula is Titchmarsh's **Theory of Functions** which has a nice section on "conformal" maps.

$$\left| \frac{w - \frac{ap+b}{cp+d}}{w - \frac{aq+b}{cq+d}} \right| = k \left| \frac{cq+d}{cp+d} \right|$$

The image of a circle under a fractional linear transformation[2] is another circle.

A circle of radius R centered at p has p and ∞ as inverse points. Perhaps we can write p and $q \gg 1$ as the inverse points.

$$\left| \frac{z-p}{z/q-1} \right| = kq = R$$

so additionally we could have $k \ll 1$.

[2]The image of a (very tiny) cirle under any conformal map is another circle.

So what is the limit? Setting $q = \infty = \frac{1}{0}$ we get:

$$\left| \frac{w - \frac{ap+b}{cp+d}}{w - \frac{a}{c}} \right| = R \left| \frac{1}{p + \frac{d}{c}} \right|$$

This doesn't look right at all.

There is a circle for each $k \neq 1$

$$\left| \frac{z-p}{z-q} \right| = k$$

The radius and center of the circle is:

$$\left| z - \frac{p - k^2 q}{1 - k^2} \right| = \frac{k}{|1 - k^2|} \; |p - q|$$

References

(1) Davide Gaiotto, Peter Koroteev **On Three Dimensional Quiver Gauge Theories and Integrability** `arXiv:1304.0779`

(2) Titchmarsh **Theory of Functions** `https://archive.org/details/TheTheoryOfFunctions`