Scratchwork: Pythagorean Triples over $\mathbb{Z}[i]$

The equation $x^2 + y^2 = 1$ defines what we might call a **variety**. Here it's just a circle. Our decision to use Cartesian coordinates, algebra and equations, to describe Euclidean geometry, leads to all sorts of complications. Attempts to finalize what we might call a variety leads to all sorts of difficult **commutative algebra** and ultimately **schemes**.

Taking for granted a minute that circles are a meaningful concept and that we should use algebra and geometry, we observe the variety $X = \{x^2 + y^2 - 1 = 0\}$ has a rational paramterization.

$$t \in \mathbb{Q} \longrightarrow (x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \in \mathbb{Q}^2$$

And we can think of fractions as pairs of integers $\frac{m}{n} \in \mathbb{Q}$ or as pairs of integers up to proportion[1] $[m : n] \in P\mathbb{Z}^2$. Therefore we can solve an equation over integers: $a^2 + b^2 = c^2$ over $\mathbb{Z}$:

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2) \in \mathbb{Z}^3$$

The machinery of algebra tells us that all we required is that $K = \mathbb{Q}$ is a field. Therefore we could also try to solve the Pythagoras equation over $\mathbb{Z}[i]$ or $\mathbb{Z}[\sqrt{2}]$ and solve it in the same way.

What happens if we write the thing down we could write $a = a_1 + ia_2$ etc. and find:

$$(a_1 + ia_2)^2 + (b_1 + ib_2)^2 = (c_1 + ic_2)^2$$

and we learned we could seprate the real and imaginary components and find two equations:

$$\begin{aligned} (a_1^2 - a_2^2) + (b_1^2 - b_2^2) &= (c_1^2 - c_2^2) \\ 2a_1a_2 + 2b_1b_2 &= 2c_1c_2 \end{aligned}$$

which is the intersection of two conic sections. And here because of the structure of $\mathbb{Z}[i]$ we can expect a spectactular reduction. Is that all?

Let's find a few solutions. The algebra is meaningful if we can find a few number solutions. Let $m = i$ and $n = 1 + i$. Then:

$$\begin{array}{rcccl} m^2 - n^2 &=& (i)^2 - (1+i)^2 &=& -1 - 2i \\ 2mn &=& i \times (1+i) &=& -1 + i \\ m^2 + n^2 &=& (i)^2 + (1+i)^2 &=& -1 + 2i \end{array}$$

Then, using the same algebra identity we used over $\mathbb{Z}$ we obtain another Pythagoras formula:

$$(1 + 2i)^2 + (1 - i)^2 = (1 - 2i)^2$$

What kind of shape is this? Is it a right triangle? What is a "complexified" right triangle?

[1] Yet another word we can scrutinize. What are we calling "proportionate"?

Also we can encode $i = \sqrt{-1}$ as a 2×2 matrix:

$$\sqrt{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Does our Pythagoras equation look any better as relation on 2×2 matrices?

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^2 + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^2$$

Exact matrix identities like this should be pretty rare. Pythagoras theorem is a machine that consistently produces equations like this. The square-root operator has no guarantee of falling in the integer 2×2 matrices: $\sqrt{A} \overset{?}{\in} \mathsf{GL}_2(\mathbb{Z})$.

Are there other 2×2 matrices such that $A^2 = 1$? I can find an invertible matrix $B \in \mathsf{GL}_2(\mathbb{Z})$ and we always obtain:

$$(BAB^{-1})(BAB^{-1}) = (BA)(B^{-1}B)(A^{-1}B^{-1}) = (BA)I_{2\times 2}(BA)^{-1} = I_{2\times 2}$$

If these matrices were perfectly associative, if the transformations we were considering perfectly mapped two variables into two other variables, if A^{-1} is exactly the inverse of A, then our formula always works. Let's try:

$$B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \in \mathsf{GL}_2(\mathbb{Z}) \text{ and } B^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \in \mathsf{GL}_2(\mathbb{Z})$$

Then we can multiply the matrices:

$$BAB^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \cdots = \begin{bmatrix} 17 & -24 \\ 12 & -17 \end{bmatrix} \in \mathsf{GL}_2(\mathbb{Z})$$

In the process of computing a square root, for 2×2 matrices, we obtain approximate square roots of integers over $\mathbb{Z}$:

$$1 = 17 \times 17 - 12 \times 24 = \left(17^2 - 2 \times 12^2\right)$$

and these were the types of considerations that I'm hoping can motivate theory of modular forms and dynamical systems to other people.

Using numPy we can multiply 2×2 matrices as a dot product operation:

```
Python 2.7.12 (default, Dec  4 2017, 14:50:18)
[GCC 5.4.0 20160609] on linux2
Type "help", "copyright", "credits" or "license" for more information.
>>> import numpy as np

>>> B = np.array([[3,4],[2,3]])
>>> A = np.array([[0,-1],[1,0]])
>>> C = np.dot(B,A)

>>> D = np.dot(C, np.linalg.inv(B))
>>> D
array([[ 18., -25.],
       [ 13., -18.]])
>>> D.astype(int)
array([[ 17, -24],
       [ 12, -17]])
>>> np.dot(D,D)
array([[ -1.00000000e+00,  -2.84217094e-14],
       [ -1.06581410e-14,  -1.00000000e+00]])
```

Using Sympy we can obtain the same answer, staying within the realm of integers.

```
Python 2.7.12 (default, Dec  4 2017, 14:50:18)
[GCC 5.4.0 20160609] on linux2
Type "help", "copyright", "credits" or "license" for more information.
>>> from sympy import *
>>> B = Matrix([[3,4],[2,3]])
>>> A = Matrix([[0,-1],[1,0]])
>>> B*A*B**(-1)
Matrix([
[18, -25],
[13, -18]])
```

Our computers can never really behave like $\mathbb{R}$ or $\mathbb{Z}$ but for the moment 10^{-14} scale errors do not qualitatively change our answers.

We do not even scratch the surface. We notes that computing $\sqrt{2}$ motivates finding a circle on the modular surface $\mathsf{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ or even $\Gamma_0(4)\backslash\mathbb{H}$. And then we can study Maa. forms or theta functions.