Scratchwork: Approximate Groups

Does the mathematical definition of "group" capture our notion of symmetry? Let's find an Abstract Algebra textbook and see what definition they gave us.

A **group** is an ordered pair of a set G and one binary operation on that set G such that

- the operation is associative
- there is an identity element
- every element $x \in G$ has an inverse

Symmetry of what? A physical objec definitely has a symmetry. But also we were saying two things are interchangeable. 10 people walking in a room without collision. Are the people interchangeable? Man $\leftrightarrow$ Woman? Adult $\leftrightarrow$ Child? It all depends... In that case, we can try to measure how much our situation fails to be symmetric.

What does "associative" mean here? I usually remember it as $a \times (b \times c) = (a \times b) \times c$. The book says "**associativity** avoids unseemly proliferations of products". Can we even describe the symmetries in question? It's the same if it goes this way or that way. Set inclusion can be a very hard problem... just ask anyone who loses their keys. When is $x \in G$?

Next we consider that we can **never** write down a number exactly. How can we write down the elements of our group if they're not exact.

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 1^\circ & -\sin 1^\circ \\ 0 & \sin 1^\circ & \cos 1^\circ \end{bmatrix} \approx \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-(\frac{\pi}{360})^2 & -\frac{\pi}{360} \\ 0 & \frac{\pi}{360} & 1-(\frac{\pi}{360})^2 \end{bmatrix} \approx$$

We have neither the time nor the space to evaluate these exactly. The price we pay is that we have to hope something covers the cost. Let's try writing a different approximate 90° rotation. Here we use $\cos\theta \approx 1-\theta^2$ and $\sin\theta \approx \theta$ (in radians).

$$R_{90^\circ} \approx \begin{bmatrix} \frac{1}{10} & -1+\frac{1}{10^2} & 0 \\ 1-\frac{1}{10^2} & \frac{1}{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{99}{100} & 0 \\ \frac{99}{100} & \frac{1}{10} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is no longer a rotation. It changes the angle of the frame slightly and in an unpredictable way. So we are really testing our ability to measure things... in this case what two matrices are "close" or "nearby" or "almost" or "good enough".

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{99}{101} & -\frac{20}{101} \\ 0 & \frac{20}{101} & \frac{99}{101} \end{bmatrix}$$

And that is how we'll build our approximate groups for now. We can just define rotations of angle $\theta = \tan^{-1}\frac{1}{10}$ or the angle of our choice. $A = R_{x,90^\circ} \times R_{y,\pm\theta}$ and there is our approximate group.

Here's the definition given in 2013 for an approximate group:

Let $K \geq 1$ be a parameter, G be a group and $A \subseteq G$ be a finite subset. We sa that A is a K-approximate subgrop of G if

- $1 \in A$
- A is symmetric $A = A^{-1}$
- there is a symmetric set X of size at most K such that $AA \subset XA$.

Since I can't seem to quite get Breuillard-Tao's definition perfectly correct, the example is my own. In our case $G = \mathsf{SO}_3(\mathbb{R})$. K is just a number that allows us to describe how much error we can tolerate of this symmetry.

Proposition There is an absolute constant $C > 0$ such that, given a finite set A in an ambient group G and parameter $K \geq 1$, the following conditions are roughly equivalent:

- $|AA| \leq K|A|$
- $|AAA| \leq K|A|$
- $\{(a,b,c,d) \in A \times A \times A \times A \big| ab = cd\}| \geq |A|^3/K$
- $\{(a,b) \in A \times A : ab \in A\}| \leq |A^2|/K$
- A is a K-approximate group of G.

It doesn't seem terribly deep to show these approximate group definitions are the same and we were able to generate our own examples. The equations we have to solve look really really dumb. We are good to go!

References

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[3] Wikipedia "Double Coset" `https://en.wikipedia.org/wiki/Double_coset`

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[6] Gregory T. Lee **Abstract Algebra: An Introductory Course** (Springer Undergraduate Mathematics Series) Springer, 2018.

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The thing is so clear... what kind of objects can we explicitly model that both can be calculated and yet only have approximate symmetry. Natural objects definintely have approximate symmetry and then we gently nudge them over to $\mathbb{Z}$ or the ideal number system of our choice. Within Mathematics, approximate symmetry can also occur. Let's try to read off a few examples from papers.

#1 Let $\Box = \{x^2 : x \in \mathbb{Z}\}$. Another way, $\Box = f(\mathbb{Z})$ for $f(x) = x^2$ is a "map".

In 2016, some estimates were obtained about gaps in the sequence of numbers $\Box + \sqrt{d}\Box$ with $d \in \mathbb{Z}$.

Proposition: just notice that by Weyl's law (which they do not prove for us), if we place the numbers $x^2 + \alpha\, y^2$ in order:

$$\#\{j : \lambda_j \leq X\} = \{\#(m,n) : m^2 + \alpha\, n^2 \leq X\} \sim \frac{\pi}{4\sqrt{\alpha}} X$$

Weyl's law is applicable here because this is eigenvalues of Hamiltonian

$$H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ and } E \in \{m^2 + \alpha n^2\}$$

We need to say this is quantum free particle Hamilontian on a rectangle of aspect ratio $1 \times \alpha \in \mathbb{R} \otimes \mathbb{R}$. Their paper will not even discuss the eigenfunctions.

Thm If the squared aspect ratio is a quadratic irrationality of the form $\alpha = \sqrt{r}$ with $r \in \mathbb{Q}$ then

$$\delta(\alpha, N) = \min\{\lambda_{i+1} - \lambda_i : 1 \leq i < N\} \ll \frac{1}{N^{1-\epsilon}}$$

Next, if we type this with a compuer is there a formula for λ_n ? Certainly, $\lambda = m^2 + \alpha\, n^2$ but now the $(m,n) \in \mathbb{Z}^2$ are badly scrambled and yet not totally random. Let's take our best guess. Let $N \asymp X$:

$$\{\lambda_i : 1 \leq i < N\} \asymp \{(m,n) : m^2 + \alpha\, n^2 \leq X\} \asymp \mathbb{Z}$$

We've walked right into it! We've walked into a nice project. :-) These numbers behave more-or-less like the real number line $\mathbb{Z}$ where each point has been moved around slightly.[1] We can find numbers $(m_1, n_1), (m_2, n_2) \in \mathbb{Z}$ with

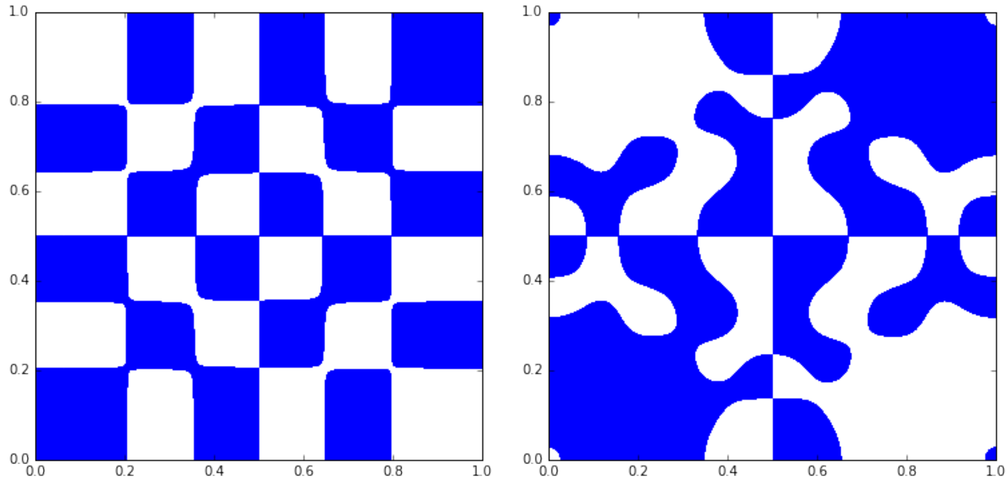
$$\left|(m_1^2 - m_2^2) - \alpha\,(n_1^2 - n_2^2)\right| \ll \frac{1}{N^{1-\epsilon}}$$

What has changed about our numbers are the local statistical properties. What is going to happen to operations like $+$ and $\times$? What happens to Taylor's formula $f(x + \frac{1}{N}) \approx f(x) + \frac{1}{N} f'(x)$? Here $x = m^2 + \alpha\, n^2$. Numbers don't just magically stop working because we failed to round things. You'll continue too if things aren't quite perfect.

I reiterate, this was figured out in 2016.

[1] There was no reason *a priori* to hold each number perfectly at $\mathbb{Z}$ in the first place. The first thing we do is mentally force things into the number line.

Random waves should have arithemtic statistical properties. Here are two examples. Clearly they should exist. They are easy to draw. It's learning to use them. If I write a formula $f(x,y) = \sum a_{m,,n} \sin mx \sin ny$ how to the numbers $a_{m,n}$ match with the pictures?



#2 Random spherical harmonics go down similarly. Some authoritative person says solutions to $a^2+b^2+c^2 = n$ equidistribute on the sphere S^2, what happens to all the information in the middle?

The main theorems of the paper:

Theorem Let l be even and $\mathcal{B}_l$ be the orthonormal basis of $\mathcal{H}_l$ eigenfunctions. Then for any $\epsilon > 0$ we have he lower bound for the total number of domains amongst members of $\mathcal{B}_l$.

$$\sum_{\phi \in \mathcal{B}_l} \mathcal{N}(\phi) \gg_\epsilon l^{\frac{5}{4}-\epsilon}$$

Corollary (The number of nodal domains on average). With everything as in the Theorem, we have for the expected number of nodal dimains amongst members of $\mathcal{B}_l$, for any $\epsilon > 0$

$$\mathbb{E}_{\mathcal{B}_l}\big(\mathcal{N}(\phi)\big) = \frac{1}{|\mathcal{B}_l|} \sum_{\phi \in \mathcal{B}_l} \mathcal{N}(\phi) \gg_\epsilon l^{\frac{1}{4}-\epsilon}$$

Theorem (The number of nodal domains grows uniformly). Assuming the generalied Lindelöf hypothesis, for any $\epsilon > 0$ we have

$$\mathcal{N}(\phi) \gg_\epsilon l^{\frac{1}{12}-\epsilon}$$

where l is even and $\phi \in \mathcal{H}_l^{\mathcal{O}^\times}$ is a Hecke eigenfunction.

This was published in 2015 in Communications in Mathematical Physics.

- $Z(f) = \{v \in S^2 : f(v) = 0\} = f^{-1}(0)$ that's it, the pre-image of zero.
  The number of nodal domains (which here is a collection of circles) is $\mathcal{N}(f)$.

- How does $\mathcal{N}(f)$ change as f varies? The theorem of Courant[2] says $\mathcal{N}(f) \le (l+1)^2$. This is called an "upper bound" in Mathematics since we are saying that $A \le B$ for two numbers A and B. Magee proves a lower bound, for a very specific class of eigenfunctions ϕ which are just very special functions $\phi : S^2 \to \mathbb{R}$.

[2]Courant and Hilbert *Mathods of Mathematical Physics, Volume 1*, 1966.

- The term "Hecke operator" in this context means, $\mathcal{O}(m) := \{\gamma \in \mathcal{O} : n(\gamma) = m\}$ is a collection of rings of integers. And here's operator in Hilbert space (a matrix):

$$T_m : L^2(S^2) \to L^2(S^2) \text{ with } [T_m f](x) = \sum_{\gamma \in \mathcal{O}(m)} f(\gamma . x)$$

  These are called "Brandt matrices" as discussed by Eichler in 1973. These eigenfunctions might be easy to compute, since $L^2(S^2)$ for the sphere decomposes by the degree of the polynomial:

$$L^2(S^2) = \overline{\bigoplus_l} \mathcal{H}_l$$

  where $\mathcal{H}_l$ is the space of $(2l+1)$-dimensional functions on S^2 – the degree l homogeneous harmonic polynomials on $\mathbb{R}^3$. This is the eigenspace of the spherical Laplacian $\Delta = \Delta_{S^2}$ with eigenvalue $l \times (l+1)$.

  And these operators are related to the Hecke operators on $\Gamma_0(2)\backslash\mathbb{H}$ in hyperbolic space (and to automorphic functions).

At the moment, the theorem has no truth value – it is sometimes true and someimes false – since we cannot state the correct function $\phi \in L^2(S^2)$ in the space of sphiercal harmonics.

References

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