Notes: Monoids of Pisot Matrices

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1 Fully Subtractive Continued Fractions

Avila and Delecroix find **monoids** of 3×3 matrices such that the eigenvalues are **pisot**, in other words $|\lambda_1| > 1 > |\lambda_2|, |\lambda_3| > 0$. The roots of $x^3 = x + 1$ are examples of such a number. $x = 1.32471795\ldots$

The monoid is related to the **fully subtractive continued fraction algorithm** for 3 numbers. Starting from 3 matrices:

$$A = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right) \quad B = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array} \right) \quad C = \left(\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right)$$

The Pisot monoid is the regular language in $\{A, B, C\}^*$ which contains each of the letters A, B, C. In fact[1]:

$$\textbf{Pisot} = \{w : |w|_A \geq 1\} \cap \{w : |w|_B \geq 1\} \cap \{w : |w|_C \geq 1\}$$

where $|w|_A$ counts the instances of A in the language $\{A, B, C\}^*$. This monoid condition is very naturel since it insures all the matrix entries are ≥ 0:

$$ABC = \left(\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{array} \right)$$

Numerical computation returns characteristic polynomial $x^3 - 7x^2 + 5x - 1$ with eigenvalues $x = \boxed{6.222}$ and $x = \boxed{0.389 \pm 0.0974\, i}$ The largest eigenvalue is the Pisot number and it's always real, and now we have a whole monoid[2] of them.

1.1 Constructing the Eigenvalues by Hand

If was very instructive for me to do the computations by hand (and I wasn't near a computer at the time). At moments, jumping through all sorts of hoops to get an estimate. For starters how do we get the characteristic polynomial of a 3×3 matrix? For a 2×2:

$$x^2 - \mathrm{tr}(A)x + \det(A) = 0$$

There is a formula for the larger matrices. I think it's the last chance we get:

$$x^3 - \mathrm{tr}(A)x^2 - \left[\mathrm{tr}(A^2) - \mathrm{tr}(A)^2\right]x - \det(A) = 0$$

[1] http://cs.stackexchange.com/a/45187/3131

[2] **Monoid** is also a term in Category Theory and Functional Programming.

1.2 Minmax

In order to prove this very broad statement about eigenvalues, Avila and Delecroix observe the matrices A, B, C preserve the **cone** determined by the triangle inequalties.

$$x + y > z \text{ and } y + z > x \text{ and } z + x > y$$

The triangle inequalities determine a cone in $\mathbb{R}P^2$ (since we can multiply x, y, z by the same number, dialing the triangle. The transformation A gives another triangle.

$$x + (x + y) > (x + z) \text{ and } (x + y) + (x + z) > x \text{ and } (x + z) + x > (x + y)$$

In this new triangle $(x, x + y, x + z)$ the first coordinate is always the smallest.

If we set $x + y + z = 1$, the set of triangles is itself an equalateral triangle which can be divided into 4 pieces. The maps A, B, C map the big triangle to each of the 3 corners. The limit set of this iterated funciton system is called the **Sierpinski Gasket**, which the authors of [**?**] may have found too obvious to mention. The set of points not in $\{A, B\}^* \cup \{B, C\}^* \cup \{C, A\}^*$ form the **interior** of the gasket.

The triangle represents the set of possible largest eigenvectors of

$$X \in \{A, B, C\}^* \backslash \Big(\{A, B\}^* \cup \{B, C\}^* \cup \{C, A\}^* \Big)$$

If we multiply enough matrices together, the cone gets narrower and narrower converging to a single ray. This is the basis of **power iteration** eigenvalue method.

For any cone we can define an L^∞ norm:

$$||A^T||_\Lambda = \sup_{v \in \mathbb{P}(\Lambda)} \max_{\substack{||z|| \leq 1 \\ z \perp v}} ||A^T z||$$

This unusual matrix norm has its origins in Perron Frobenius theory. If we have any information about the first eigenvector, we would like to know the second eigenvalue:

$$\lambda_2 \leq \sup_{x \in v^\perp \backslash \{0\}} \frac{||Ax||}{||x||}$$

more importantly we can to check that $|\lambda_2| \leq 1$. Knowing that dominant eigenvector $v = (x, y, z)$ satisfies the triangle inequality, shomewhow we have to show $||Ax|| < ||x||$ for $x \perp v$.

1.3 Gershgorin Circles

I think Avila's proof is crazy! Let's try proving instead with Gershgorin circles!

References

[1] G. H. Hardy , Edward M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press; 2008.

[2] Harry Furstenberg. *On the Infinitude of Primes* American Mathematical Monthly, 62, (1955), 353.

[3] Idris Mercer. *On Furstenberg's Proof of the Infinitude of Primes* American Mathematical Monthly 116: 355-356

[4] Artur Avila, Vincent Delecroix. *Some monoids of Pisot matrices* `arXiv:1506.03692`