Scratchwork: Averages

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In the process of trying to solve the **Averged Chowla Conjecture** two young mathematicians Radziłłand Mätomaki (accompanied by Tao) are producing novel range of averaging techniques. For years (since about high school) I have told myself I would learn to do inequalities.

These results are somewhat high-end[1] and we shall take a somewhat "compendious" approach and merely collect the results in a suggestive way. They are insterested in averages like this:

$$\sum_{X<n\leq 2X} \Lambda(n)\Lambda(n+h) = c\,X + o\,(\,X^{\frac{1}{2}+o(1)}\,)$$

these are related to the "Twin Primes" conjecture. They are interested in the Van Mangoldt function, which is basically $\log p$ whenever p is a prime:

$$\Lambda(n) = \left\{ \begin{array}{cc} \log p & n = p^k \\ 0 & n \neq p \end{array} \right.$$

These function definition has to do with unique factorization.

$$n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} \to \log n = k_1 \log p_1 + \dots k_m \log p_m \to \text{“}\log\text{”} n \stackrel{?}{=} \log p_1 + \cdots + \log p_m$$

I dislike **analytic** techniques because we don't . The proofs don't have to be elementary, they could be quite advanced, but they should look like number theory. I had hoped.

Conjectures like these are a little dry for my taste, but as a disciplined person I know they are very important.

$$c = 0 \text{ if } h \text{ is odd}$$

I can prove this part. If h is odd, then either one or the other of n or $n+h$ is odd. So either $\Lambda(n) = 0$ or $\Lambda(n+h) = 0$. E.g. either 6 is prime or 7 is prime, not both. If h is even, there is content:

$$\dots$$

This is classic Tao (who learned it from Erdős) to juxtapose the tautological and the impossible. We might wonder what kind of stuff is lurking around the number **2** ? They will also be interested in the divisor function. $d(n)$. E.g.

$$d(6) = \#\{1,2,3,6\} = d(2) \times d(3) = \#\{1,2\} \times \#\{1,3\} = 4$$

I've concluded this type of "hardness" is also protecting us? What if it was easy to open up our house? Our safety rests on somebody not reproducing the notches of our key – just a few bits of information.

[1]this is Wal-Mart vs Bergdorf Goodman

The value of c when h is even has to so with the twin primes constant:

$$c = 2 \times \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2} \Big) \times \prod_{p|h} \frac{p-1}{p-2}$$

Although…I might threaten to find structure there as well. If we shift the twin-primes constant just a bit, we get one of my favorite constants:

$$\frac{6}{\pi^2} = \prod_{p>2} \Big(1 - \frac{1}{p^2} \Big)$$

The rest of these conjectures as so schematic. You can't just say "for some polynomial" – this kind of numbr theory is a bit myopic in my taste.

Lastly, there are lots and lots of multiplicative or well-behaved number theoretic functions that could be derived from modular forms. These people reason about classes of modular forms "cuspidal" or "automorphic"[2] but we don't get any translation to classical number theory. Perhaps that's up to us.

Just to give you a taste. Here's the sum of squares function:

$$r_k(n) = \Big| \{ (a_1, \dots, a_k) \in \mathbb{Z}^k : a_1^2 + \dots + a_k^2 = n \} \Big|$$

It doesn't look that way, but it is multiplicative. This sum of squares formula can be expressed in terms of the divisor function.

$$r_2(n) = 4 \, \Big[\, d_{1(4)}(n) - d_{3(4)}(n) \, \Big]$$

and maybe there is a formula for $r_3(n)$ or $r_4(n)$. Maybe not? This would mean the sum-of-squares type functions have somthing that's not contained in the divisors of n. That could make sense too…

The estimates then become very meaningful. Some type of twin-prime type estimate; very difficult to obtain:

$$\sum_{X < n \leq 2X} d_2(n) d_2(n+h) = P_{2,2,h} \Big(\log X \Big) \, X + O(X^{\frac{2}{3} + o(1)})$$

For some anonymous polynomial logarithm. That ends my complaining. Onto formulas !

[2]or $E_k(\Gamma)$ or $H^1(\Gamma, \mathbb{C})$ or $L^2(G(Q) \backslash G(A))$ or what-have-you.

All of these equations they cite are new and look very different:

$$\int_T^{2T} |\zeta(\frac{1}{2} + it)|^6 \, dt \ll T^{\frac{5}{4}+\epsilon}$$

and for rather mysterious reasons they compare to averages of the divisor function *in a specific range*

$$\sum_{h \le H} \sum_{n \le X} d_3(n) d_3(n+h)$$

with $H = X^{\frac{1}{3}}$. Even though in theory, studying the zeta function is a kind of "multiplicative" Fourier analysis – the exponent map changes the Fourier transform to the Mellin transform, but I don't have much practice with the Mellin transform version.[3]

$$\int_{-\infty}^{\infty} e^{-sx} f(x) \, dx \longleftrightarrow \int_0^{\infty} x^{s-1} f(x) \, dx$$

Here's an amazing totally useless integral. It's what we're here for.

$$\int_0^X \left(\left| \sum_{m=1}^M a_m \, e(t\, \phi(m)) \right|^* \right)^4 dt \ll \frac{X \log^4 X}{Y} \int_0^Y \left(\left| \sum_{m=1}^M e(t\, \phi(m)) \right| \right)^4 dt$$

and there's a mean-value-theorm that I liked:

$$\int_{T_0}^{T_0+T} \left| \mathcal{D}[f](\frac{1}{2} + it) \right|^2 \ll \frac{T+X}{X} ||f||_{\ell^2} \text{ where } \mathcal{D}[f](s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

One thing that's different from their Fourier analysis vs the textbooks, is they are not integrating over all of $\mathbb{R}$ instead only on a segment $[0, X]$ or $[T, 2T]$ for specific, rather chaotic functions. This is kind of like a tennis player playing grass-court instead of hard-court.

References

(1) Kaisa Matomäki, Maksym Radziwiłł, Terence Tao **Correlations of the von Mangoldt and higher divisor functions I. Long shift ranges** `arXiv:1707.01315`

[3] for a brand-new perspective on the Mellin transform from AdS/CFT `https://arxiv.org/abs/1107.1499`