Prime Number Theorem

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The Wiener-Ikehera Tauberian Theorem implies the Prime Number Theorem, as can be shown in various places.

None of those discussions really explain to me:

- connection to divergent series
- how we can proceed on our own

Norber Weiner's original argument is very easy to follow: Möbius inversion is what gives the series to start:

$$\sum_{m=1}^{\infty} x^m \log m = \sum_{n=1}^{\infty} \Lambda(n) \frac{x^n}{1-x^n}$$

and then set $x = e^{-\xi}$ and $\xi \to 0$ (or $x \to 1$).

The two limiting behaviors are rather diffrent

$$\frac{x^n}{1-x^n} = \begin{cases} 1/n\epsilon & \text{as } x \to 1 \\ \epsilon^n & \text{as } x \to 0 \end{cases}$$

and the cuttoff point is when $(1-\epsilon)^n$ is getting small (near $n = 1/\epsilon$)

$$\sum_{n=1}^{\infty} \Lambda(n) \frac{\xi e^{-n\xi}}{1-e^{-n\xi}} \approx \sum_{n=1}^{1/\xi} \frac{\Lambda(n)}{n}$$

Maybe if we take the derivative of both sides:

$$\sum_{n=1}^{\infty} \Lambda(n) \frac{d}{d(n\xi)} \left[\frac{n\xi}{1-e^{n\xi}} \right] \approx \sum_{n=1}^{1/\xi} \Lambda(n)$$

And clearly the two sides are approximate so we are done.

$$\frac{d}{du}\left(\frac{u}{1-e^u}\right) = \begin{cases} 1 & \text{as } u \to 0 \\ ue^{-u} & \text{as } u \to \infty \end{cases}$$

The left side can be shown to be 1 through "elementary" arguments.

$$\sum_{n=1}^{\infty} \Lambda(n) \frac{d}{d(n\xi)} \left[\frac{n\xi}{1-e^{n\xi}} \right] \approx \frac{1}{\xi} + O(\log \xi)$$

References

(1) David Vernon Widder **The Laplace Transform** Princeton University Press, 1948.

(2) Norbert Wiener **Tauberian Theorems** Annals of Mathematics Vol. 33, No. 1 , pp. 1-100

(3) G. H. Hardy **Divergent Series** Oxford University Press 1973

The argument in the previous section is wrong.[1]

It is very difficult to express in a vivid way - with images - why it is incorrect.

And in most situations it doesn't really matter. As for motivation, I can only speak for myself.

- why is Prime Number Theorem in a book on Laplace Transforms?
- why is Prime Number Theorem in a book in Divergent Series?

These lead me to neo-classical approaches - they will feel pretty modern to you![2]

One possibility is: **there is nothing new under the sun**. Everything is thinly dressed-up versions of the same problems since antiquity.

[1] I got really good at writing glib and suggestive proofs to hand in to graders. Who themselves are not always sure so they give you a check.

$$\sum_{n=1}^{1/\xi} a_n\, g(n\epsilon) \approx \sum_{n=1}^{1/\xi} a_n g(0)$$

Glib arguments can be acceptable, since we don't always have the time / resources to check all the cases.

[2] I tried to read the most modern papers first: there is Tao and Gowers and Green. However, that conversation presupposes knowledge I don't have and is written in language that I really don't like. They are pretty dreadful to read as are most papers in Analytic Number Theory as well as the people who write them! The "beauty of the primes" is just a marketing term.

And that's pretty much how we feel. So let's try take some examples from `hep-th` and `math-nt`.

Hopefully, also finish a real proof of PNT.[3] What could possibly go wrong with approximations like this?

$$\sum_{n=1}^{\infty} \Lambda(n) \frac{\xi e^{-n\xi}}{1 - e^{-n\xi}} \approx \sum_{n=1}^{1/\xi} \frac{\Lambda(n)}{n}$$

Maybe if we take the derivative of both sides:

$$\sum_{n=1}^{\infty} \Lambda(n) \, \frac{d}{d(n\xi)} \left[\frac{n\xi}{1 - e^{n\xi}} \right] \approx \sum_{n=1}^{1/\xi} \Lambda(n)$$

These should be satisfactory in any Physics or Engineering textbook.

[3]The terms "combinatorial" or "geometric" or "algebraic" or "elementary" have all been misleading and so I have often chosen to start from scratch.