Scratchwork: Geometry of Numbers

As I read attempting to read Manjul Bhargava's papers, a few problems emerge:

- I don't know some of the groups definitions that he refers to
- I don't know what he is calling "geometry of numbers"
- I don't know what the big deal is about class numbers. Or why his contrubtions were so crucial.

Geometry of Numbers has been around since Hermann Minkowski, and there's even the book, written in academic 19th century German *Geometry der Zahlen*. And who knows? Perhaps that bounds have improved since then, ... there is the book of Cassels in the mid 20th century. This area - when I learned it - amounded to a "cute proof" that should be solve using more robust methods - and then Manjul won a Field's medal with it in 2014.[1]

The geometry of numbers proof (e.g. Davenport) that $n = a^2 + b^2 + c^2 + d^2$ uses lattices over $\mathbb{R}^4$. However the proof that $p = 4k+1 = a^2+b^2$ uses a mix of mod p arithmetic – over $(\mathbb{Z}/p\mathbb{Z})^2$ to define (an essentially random) lattice over $\mathbb{R}^2$. So, we are going to combine the two objects:

At this point p is still generic, so it's safe to say we are solving the "family" equations $x^2+y^2 = n$ **at all primes p**.

$$X = \{x^2 + y^2 - n = 0\} \to \big(\mathbb{Z}/p\mathbb{Z} \times \mathbb{R}\big)^2$$

and we are looking for lattices in this mixed geometirc object. And if we have $X(\mathbb{Z}/p\mathbb{Z})$ and $X(\mathbb{R})$ we could even have:

$$X(\mathbb{Z}_p) \to \cdots \to X(\mathbb{Z}/p^{k+1}\mathbb{Z}) \to X(\mathbb{Z}/p^k\mathbb{Z}) \to \cdots \to X(\mathbb{Z}/p\mathbb{Z})$$

While looking at the scheme $X(\mathbb{Z})$ at one place p is not enough if we look at a generic place p *as well as* $p = \infty$ we can have

$$X(\mathbb{Z}) \text{``} \simeq \text{''} X(\mathbb{Z}_p) \cap X(\mathbb{R})$$

and maybe this is sufficient.

References

[1] Manjul Bhargava ...

[1] 1) that is my personal opinion, and usually when you ask someone suddenly everyone changes their position ... 2) I have look at him in person, maybe one or two-odd times in my life.