Notes: Quasi-Coherent Sheaves

A quasi-coherent sheaf on a ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules where every point in X has an open neighborhood with an exact sequence:

$$\mathcal{O}_X^{\oplus I}|_U \to \mathcal{O}_X^{\oplus J}|_U \to \mathcal{F}|U \to 0$$

In order to be called a coherent sheaf there are two more restrictions:

- $\mathcal{F}$ is of finite type on $\mathcal{O}_X$; every point in X has an open neighborhood U in X such there is a surjective morphism $\mathcal{O}_X^n|U \to \mathcal{F}|_U$ for some natural number $n \in \mathbb{N}$.
- for any open set $U \subseteq X$ and any morhism $\phi : \mathcal{O}_X^n|_U \to \mathcal{F}_U$ of $\mathcal{O}_X$ modules, the kernel is of finite type.

Notice we haven't consulted a textbook yet. We could rewind and look at the definition of a vector bundle.

A real vector bundle consist of:

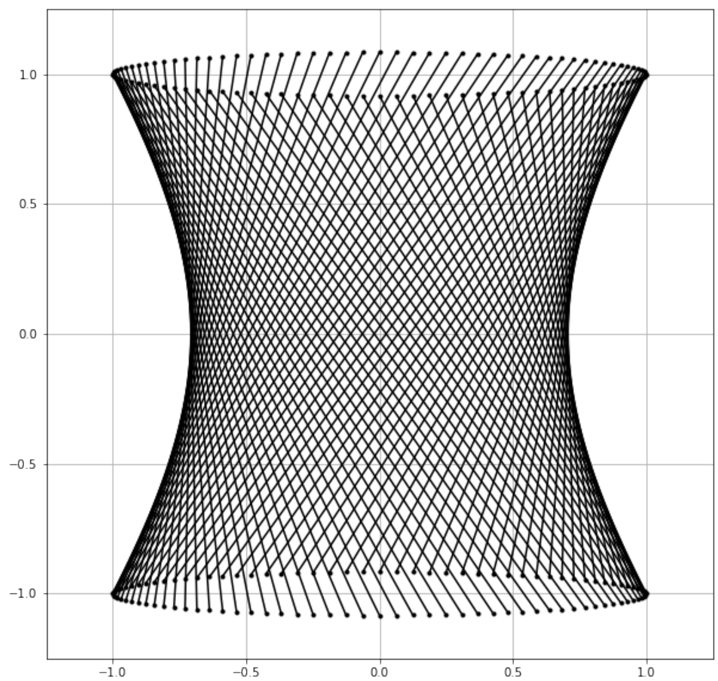
- topological space X (base space) and E (total space)
- continous surjection $\pi : E \to X$ (bundle projection)
- for every $x \in X$ there is structure of finite dimenaional vector space on the fiber $\pi^{-1}(x)$.

Example, the Möbius band is a bundle over the circle S^1.

When we move to quasi-coherent sheaves, perhaps we are sacrificing the third item.

- $\varphi_U : U \times \mathbb{R}^k \to \pi^{-1}(U)$
- $\varphi_V : V \times \mathbb{R}^k \to \pi^{-1}(V)$
- $\varphi_U^{-1} \circ \varphi_V : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ (composite function)
- The transition functions form a Čech cocyle:
  - $g_{UV} : U \cap V \to \mathsf{GL}(\mathbb{R}^k)$
  - $g_{UU} \equiv I$ (the identity function)
  - $g_{UV} g_{VW} g_{WU} = I$ (cocycle condition) whenver $U \cap V \cap W = \varnothing$.

So there's a vague but not perfect way of moving information around. At least we get the two Wikipedia pages to match.



References

[1] Wikipedia "Coherent Sheaf" https://en.wikipedia.org/wiki/Coherent_sheaf

[2] Wikipedia "Vector Bundle" https://en.wikipedia.org/wiki/Vector_bundle

[3] . . .