Roth's Theorem

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Roth's Theorem

2 Poincare Recurrence

The idea should be that point-wise ergodic theorem is an extension of Poincare recurrence, which is an extension of the Pigeon-hole principle.

von Neumann ergodic theorem Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system, and let P_T denote the orthogonal projection to the closed subspace

$$\{g \in L^2_\mu : U_T g = g\} \subseteq L^2_\mu$$

Then for any $f \in L^2_\mu$

$$\frac{1}{N} \sum_{n=0}^{N-1} U_T^n f \longrightarrow_{L^2_\mu} P_T f$$

None of these proofs are very illuminating 'til we observe specific T, X, $\mathcal{B}$, μ and f.

Proof #1 The spectral theorem says a unitary operator $U : H \to H$ can be expressed as $U = \int_{S^1} \lambda \, d\mu(\lambda)$, since unitary operators have eigenvalues in the unit circle $S^1 = \{z : |z| = 1\}$. Then

$$\frac{1}{N} \sum_{n=0}^{N-1} U^n |v\rangle = \left[\int_{S^1} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n \, d\mu(\lambda) \right] |v\rangle$$

for any $|v\rangle \in H$. By geometric series and dominated convergence:

$$\left[\mu(\{1\}) + \int_{S^1 \backslash \{1\}} \frac{1}{N} \frac{\lambda^N - 1}{\lambda - 1} \, d\mu(\lambda) \right] |v\rangle \to \mu(\{1\}) |v\rangle$$

The RHS is the projection $P_T |v\rangle \in H^U$ to the subspace with eigenvalue 1. $\square$

Two proofs involving pre-compactness and the Banach-Alaoglu theorem I will skip. A "slick" geometric proof of Riesz is enough for me.

Proof #1 The proof argues that $\{U_T g - g : g \in L^2_\mu\}^\perp = \{U_T g = g : g \in L^2_\mu\}$ ✓

If $U_T f = f$ then by unitarity

$$\langle f | U_T g - g \rangle = \langle U_T f | U_T g \rangle - \langle f | g \rangle = 0$$

If $\langle f | U_T g - g \rangle \equiv 0$ so that f is perpendcular to all "noise" then

$$\langle U_T g | f \rangle = \langle g | U_T^* f \rangle = \langle g | f \rangle$$

for all states $g \in L^2_\mu$. That means $U_T^* f = f$ and $U_T f = f$.

We have shown the Hilbert space of states L^2_μ splits in two parts.

$$L^2_\mu = \{U_T g = g : g \in L^2_\mu\} \oplus \overline{\{U_T g - g : g \in L^2_\mu\}}$$

Given any unitary unitary operator, any function splits into a signal and a noise: $f = P_T f + h$. The claim is the noise averages out to 0 in the L^2_μ norm.

This is obvious by telescoping sum if $h = U_T g - g$.

$$\Big|\Big| \frac{1}{N} \sum_{n=0}^{N-1} U_T^n (U_T g - g) \Big|\Big|_2 = \frac{1}{N} \Big|\Big| U_T g - g \Big|\Big|_2 = 0$$

The noise is in the closure. $h = \lim (U_T g_i - g_i)$ The same argument before has two steps:

$$\Big|\Big| \frac{1}{N} \sum_{n=0}^{N-1} U_T^n h \Big|\Big|_2 \le \underbrace{\Big|\Big| \frac{1}{N} \sum_{n=0}^{N-1} U_T^n (h - h_i) \Big|\Big|_2}_{<\epsilon} + \underbrace{\Big|\Big| \frac{1}{N} \sum_{n=0}^{N-1} U_T^n h_i \Big|\Big|_2}_{<\epsilon} < 2\epsilon$$

The noise converges in the L^2_μ norm, as $N \to \infty$ and $i \to \infty$. □

The space of noise $\overline{\{U_T g - g : g \in L^2_\mu\}}$ has many interesting effects that get averages away in the L^2_μ limit, but we still see them pointwise. It was very instructive to see the ergodic averages $\frac{1}{N} \sum f$ convergence. Away from a measure 0 set the convergence is pointwise, but that difference is dramatic.

3 Physics

We do not discuss matrix integrals at this time.

References

(1) Enrico Bombieri, Alfred J. van der Poorten. **Continued Fractions of Algebraic Numbers** 1995.

(2) S. Lang and H. Trotter, **Continued fractions for some algebraic numbers**, J. Reine Angew. Math. 255 (1972), 112-134. https://eudml.org/doc/151239