Tune-Up: Topological Groups

Number theory offers general results about numbers as true, how do we turn these into more actionable statements?

$$\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z}$$

This result doesn't say anything about **decimal representation** of numbers or what **additive structur** is. Since when do we care about 20 digit numbers to exact precision and still have addition?

Let K be complete with respect to a discrete valuation. There are homomorphism(s):

- $\mathcal{O} \to \mathcal{O}/\mathfrak{p}^n$
- $\mathcal{O}/\mathfrak{p} \leftarrow \mathcal{O}/\mathfrak{p}^2 \leftarrow \mathcal{O}/\mathfrak{p}^3 \leftarrow \dots$
- $\mathcal{O} \to \lim\limits_{\longleftarrow} \mathcal{O}/p^n$

This gives us a whole slew of exotic **number systems** as **projective limits** of rings of various kinds.

$$\lim_{\longleftarrow} \mathcal{O}/\mathfrak{p}^n = \left\{ (x_n) \in \prod_{n=1}^{\infty} \mathcal{O}/\mathfrak{p}^n : \lambda_n(x_{n+1}) = x_n \right\}$$

Proposition The canonical mappings are isomorphisms and homeomorphisms:

- $\mathcal{O} \quad \to \lim\limits_{\longleftarrow} \mathcal{O}/\mathfrak{p}^n$
- $\mathcal{O}^\times \to \lim\limits_{\longleftarrow} \mathcal{O}^\times/U^{(n)}$

The difference between $\mathbb{Z}_{10}$ possibly as $\lim\limits_{\longleftarrow} \mathbb{Z}/10^n\mathbb{Z} \simeq \lim\limits_{\longleftarrow} \mathbb{Z}/2^n\mathbb{Z} \times \lim\limits_{\longleftarrow} \mathbb{Z}/5^n\mathbb{Z}$ and $\mathbb{R}$ is …They have very different shapes, they are connected very different.

References

[1] Terence Tao, Van Vu. **Additive Combinatorics** (Cambridge Advanced Studies in Mathematics #105) Cambridge University Press, 2006.

[2] Atiyah McDonald's