Scratchwork: "Locally Compact Abelian" Groups

This harmonic analysis jargon "Locally Compact Abelian" groups appears basically all over number theory in a very abstract form. Here are the two examples I can think of immediately: $\mathbb{R}$ and $\mathbb{Q}_p$. Harmonic analysis is generalization of Fourier Analysis to abstract groups, such as $SO(3)$ the rotations of a three-dimensional object and many other kinds of symmetry.[1]

In our case, there is an isomorphism (as group) in Number Theory

$$\mathbb{Q}^\times = \prod_p p^{\mathbb{Z}} \quad \text{and} \quad \mathbb{Q}(i)^\times = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathbb{Z}}$$

In fact all the $F^\times$ are isomorphic for any number field F. That cannot be right. On the left the primes are $p \in \mathbb{Z}$ and on the right $\mathfrak{p} \in \mathbb{Z}[i]$ and for little more than grade school arithmetic, we have that primes $p \in 4\mathbb{Z}+1$ factor into $p = \mathfrak{p}_1\mathfrak{p}_2$ E.g. $13 = (2+3i) \times (2-3i)$ or $17 = (4+i) \times (4-i)$. Therefore both number fields share the primes in $p \in 4\mathbb{Z}+3$. So we need one extra thing to distinguish between $\mathbb{Q}^\times$ and $\mathbb{Q}(i)^\times$, that is a **topology**.

Have you seen Fourier analysis on $\mathbb{Q}^\times$ or $\mathbb{Q}(i)^\times$? These not compact, since we can find Cauchy sequences $x \to \sqrt{2}$ or $\sqrt{3} \notin \mathbb{Q}^\times$. So how can we define a derivative $f'(x) = 2x$.

$$f'(x) = \lim_{|\epsilon| \to} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

Therefore we need to find a space where this derivative makes any kind of sense at all. I think we have some freedom here. Why is this thing $f'(x)$ even a number?

Since I didn't study Munkres' textbook carefully (and all the pathological but nifty examples) we just settle for asking, which number $x \in \mathbb{Q}^\times$ or $x \in \mathbb{Q}(i)^\times$ solve $x \approx 0$? Which numbers are close to zero? And our number theory tool is called **strong approximation**. So there is in fact some ambiguity in the statement $x \to \sqrt{2}$ in a manner that is important to Number Theory.[2]

References

[1]

[1] When do we study a group action? Usually this means the object was not a priori symmetric! A cube can be rotated in three-dimensional space by $SO(3)$ elements, and we can study the remainder that's leftover.

[2] Don't just call their bluff, there should hopefully be a good reason.

01/24/19 One very easy "topological" group, or "Locally Compact Abelian" group should be $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$, the rational numbers modulo the perfect squares. Why such a thing. Let's consider a variety of some kind:

$$V_1 = \{x^2 + y^2 = a\} \text{ and } V_2 = \{x^2 + y^2 = ab^2\}$$

The map $(x, y) \mapsto (bx, by)$ takes the small circle to the larger circle, so we call them *birational equivalent*.[3] So we can say that $V(a) \equiv V(ab^2)$ for $a, b \in \mathbb{Q}$. And more basically, $a \equiv ab^2$. Therefore our conic sections (mostly circles) are equivalence classes $[a] \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$.

What kind of observables can we study here? I'd like to count the rational points on this circle of a given height. For example:

$$\{x^2 + y^2 = 1\}$$

The rational points here correspond to **Pythagorian triples** or to **right triangles**. It's very unlikely that you've seen a proof of Pythagoras theorem. There are several hundred and they all say slightly different things about triangles. We would like the points in this circle of bounded height:

$$\left\{(a, b, c) \in \mathbb{Q} : \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \text{ and } |a|, |b|, |c| < N \text{ and } (a, b) = (a, c) = 1\right\}$$

We would have to "solve" the equation - and at this level it's harder and hard to means - and then we have to do GCD computations over and over. And we could let $f(N)$ be the number of points in this set.

Let's try one more. Here's a circle which clearly has points over $\mathbb{R}$ and yet is vacuous over $\mathbb{Q}$:

$$x^2 + y^2 = 3$$

No rational points whatsover. Since $\{0^2, 1^2\} + \{0^2, 1^2\} = \{0, 1, 2\}$ using a tiny amount of sumset theory. We've shown that $3 \notin \square$. We have shown a circle in $\mathbb{R}^2$ that has successfully avoided all of $\mathbb{Q}^2$. Nothing particularly wrong with that, but maybe we can quantify how close to points in $\mathbb{Q}^2$ we can get.

References

[1] Manfred Einsiedler, Thomas Ward. **Ergodic Theory with a view towards Number Theory** (Graduate Texts in Mathematics #259). Springer, 2011.

[2] Philippe Gille, Tamás Szamuely. **Central Simple Algebras and Galois Cohomology** (Cambridge Studies in Advanced Mathematics #165). Cambridge Uniersity Press, 2017.

[3] Pierre Guillot. **A Gentle Course in Local Class Field Theory** Cambridge University Press, 2018.

[3] Algebraic geometry is such that every time we make such a call, it merely becomes an invitation to check an endless hierarchy of exceptions. And it's OK. There are plenty of authoritive resources at varying levels.