Scratchwork: Symmetric Polynomials

Where do symmetric polynomials come from? The starting points are almost too obvious to even mention.

Ex. Find a cubic polynomial $f(x) = x^3 \times ax^2 + bx + c$ such that $f(0) = 1$ and $f(1) = 2$, $f(2) = 3$.

These constraints leads to simultaneous equations for the number a, b, c:

$$\begin{array}{ccccccccc} & & & & & & c & = & 1 \\ 1 & + & a & + & b & + & c & = & 2 \\ 8 & + & 4a & + & 2b & + & c & = & 3 \end{array}$$

A polynomial is just a made-up device that mathematicians use to solve equations anwyay. Why might such a thing be natural? if you're a believer in the Newton Leibniz calculus, there was the Taylor series expansion from 1715 or so:

$$f(x+a) = f(x) + f'(x)a + f''(x)\frac{a^2}{2} + f'''(x)\frac{a^3}{6} \dots$$

As long as you have enough derivatives. We are going to extrapolate nearby values basic on what we know at a single point, with itzero knowledge of f.

So we have matrix equation:

$$\left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{array}\right] \left[\begin{array}{c} a \\ b \\ c \end{array}\right] = \left[\begin{array}{r} 1 \\ 1 \\ -5 \end{array}\right]$$

We seem to be on the right track. Cramér's rule first appears in 1750 but we've likely had simultaneous equations before that. First of all there is a single equation:

$$c = 1 \quad \text{and} \quad \left[\begin{array}{cc} 1 & 1 \\ 4 & 2 \end{array}\right] \left[\begin{array}{c} b \\ c \end{array}\right] = \left[\begin{array}{r} 0 \\ -6 \end{array}\right]$$

We only have two variables. So let's just solve them:

$$a = \frac{\left|\begin{array}{rc} 0 & 1 \\ -6 & 2 \end{array}\right|}{\left|\begin{array}{rc} 1 & 1 \\ 4 & 2 \end{array}\right|} = \frac{6}{-2} = -3 \quad \text{and} \quad b = \frac{\left|\begin{array}{cr} 1 & 0 \\ 4 & -6 \end{array}\right|}{\left|\begin{array}{cr} 1 & 1 \\ 4 & 2 \end{array}\right|} = \frac{-6}{-2} = 3$$

and now like good students we should restate our answer. The polynomial should be:

$$f(x) = x^3 - 3x^2 + 3x + 1$$

Additionally, we observe that Taylor series motivates order of operations:

$$f(3) = 1 \times (3 \times 3 \times 3) - 3 \times (3 \times 3) + 3 \times (3) + 1$$

This is just a sketch of how the grade school operations could have emerged.

We can talk for a moment about the formulas of Viete.

$$(x-a)(x-b)(x-c) = x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$$

Then if we have a cubic equation which is very difficult to solve, we can still get information about the average behavior of the numbers:

$$x^3-3x^2+3x+1$$

and perhaps we can find the sum of the squares of the roots:

$$a^2+b^2+c^2 = (a+b+c)^2-2\times(ab+bc+ca) = (-3)^2-2\times 3 = 3$$

These things hide in front of your face. They're almost too obvious to state.[1]

Ex. Are the roots of $f(x)$ all real numbers? (1 real) + (2 imaginary)? Find $a^4+b^4+c^4$.

Ex. Derive Taylor's formula to 4th order. $(x, y) = (0, 0)$ is a point on the lemniscate: $(x^2+y^2)^2-2(x^2-y^2) = 0$. What are some nearby points? Can we get an exact answer?[2]

[1] is there a technical term for this??

[2] Gabriel Cramér "introuction a L'Analyse des Lines Courbes Algebraiques" https://archive.org/details/bub_gb_gtKvSzJPOOAC