Scratchwork: Mean Value Theorem + Bifurcation Theory

Do we know the implicit function theorem?

Thm Suppose that $F : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ with $(\lambda, x) \to F(\lambda, x)$ is a C^1 function (with one derivative) solving:

$$F(0,0) = 0 \text{ and } \frac{\partial F}{\partial x}(0,0) \neq 0$$

There are constants $\delta > 0$ and $\eta > 0$ and a C^1 function:

- $\psi : \{\lambda : ||\lambda|| < \delta\} \to \mathbb{R}$
- $\psi(0) = 0$ and $F(\lambda, \psi(\lambda)) = 0$ for $||\lambda|| < \delta$

If there is a $(\lambda_0, x_0) \in \mathbb{R}^k \times \mathbb{R}$ such that $||\lambda_0|| < \delta$ and $|x_0| < \eta$ and solved $F(\lambda_0, x_0)$ then $x_0 = \psi(\lambda_0)$.

What do bifurcations describe? And what do they look like?

Ex Co-dimension one vector field dependeing on two parameters.

$$\dot{x} = \lambda_1 + \lambda_2 x + x^2$$

Ex Here is a perturbation of the vector field $f(x) = x^2$ - on $T_1(\mathbb{R})$:

$$\dot{x} = \lambda^2 + 2a\lambda x + x^2$$

How does a mere shifting of the arrows change the qualitative flow of the vector field?

Ex How about this two-parameter bifurcation of the vector field $f(x) = \frac{1}{6}x^3$:

$$\dot{x} = \mu_1 + \mu_2^2\, x + \mu_2 \frac{1}{2}x^2 - \frac{1}{6}x^3$$

Bifurctions like these should matter a lot because we can try to compute Euler charactistics this way (e.g. the Poincaré-Hopf theorem). Or conversely (and more practical) we are confronted with a large complicated information, and the Euler characteristic is the only think we know and we can obtain an invariant.

Q What happens in a more serious setting? In gauge theory we might consider maps from $\mathbb{R}^4$ to $U(1)$ or to another group like $G = SU(2)$. I don't think I have ever seen a differential equation solved in this setting. Can bifurcations occur there?

The only vector bundle on $\mathbb{R}^4$ is the trivial vector bundle $\mathbb{R}^4 \times U(1)$. However, there could be other 4-manifolds where other structures can happen. We could start by consulting a differential geometry book, or even a K-theory text.

References

[1] Jack Hale, Hüseyin Koçak **Dynamics and Bifurcations** (Texts in Applied Mathematics) Springer, 1991.

[2] David Tong **Lectures on Gauge Theory** `http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html`

[3] Mark J.D. Hamilton **Mathematical Gauge Theory** (Universitext) Springer, 2017.