

Double-Shuffle

John D Mangual

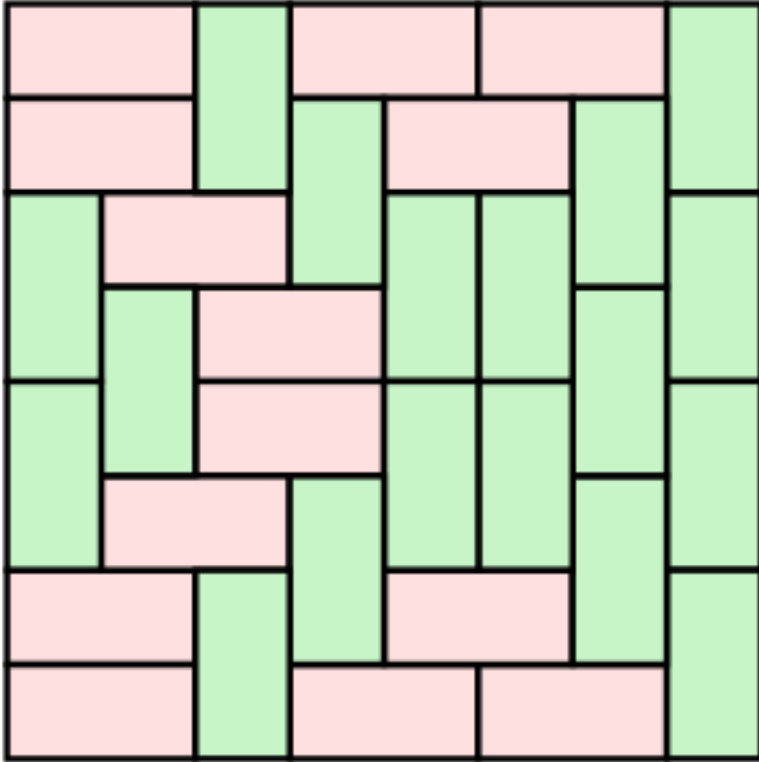
Many people like the formula $\zeta(2) = \frac{\pi^2}{6}$ and so do I. There are some great notes by Robin Chapman that offer about 15 proofs of that result.

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \left\{ 1 \leq m, n < N : \gcd(m, n) = 1 \right\} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{n^2} \right) = \frac{6}{\pi^2}$$

The classic “application” is that the probability of two numbers being relatively prime is $\frac{6}{\pi^2}$ which is about two-thirds.

There is nothing *round* about two numbers with no common factor. So why is there a π^2 ?

Another thing I liked to do is play dominoes¹. Here's an image of rectangle paved by dominoes randomly:



The number of such tilings is a trigonometric product:

$$\prod_{j=1}^{\frac{m}{2}} \prod_{k=1}^{\frac{n}{2}} \left(4 \cos^2 \frac{\pi j}{m+1} + 4 \cos^2 \frac{mk}{n+1} \right)$$

'Tis a bit of a miracle this number is even an integer².

¹Coding these on my computer has been awfully difficult. Everytime I think very hard, I forget and it is just like the first time.

²We are following the convention that we are iterating from $j = 1$ to $j = \frac{m}{2}$ (or possibly $\frac{m}{2} + 1$) and if $\frac{m}{2}$ is a fraction we just stop before.

This work of Kenyon and Wilson has a lot of pi in it.

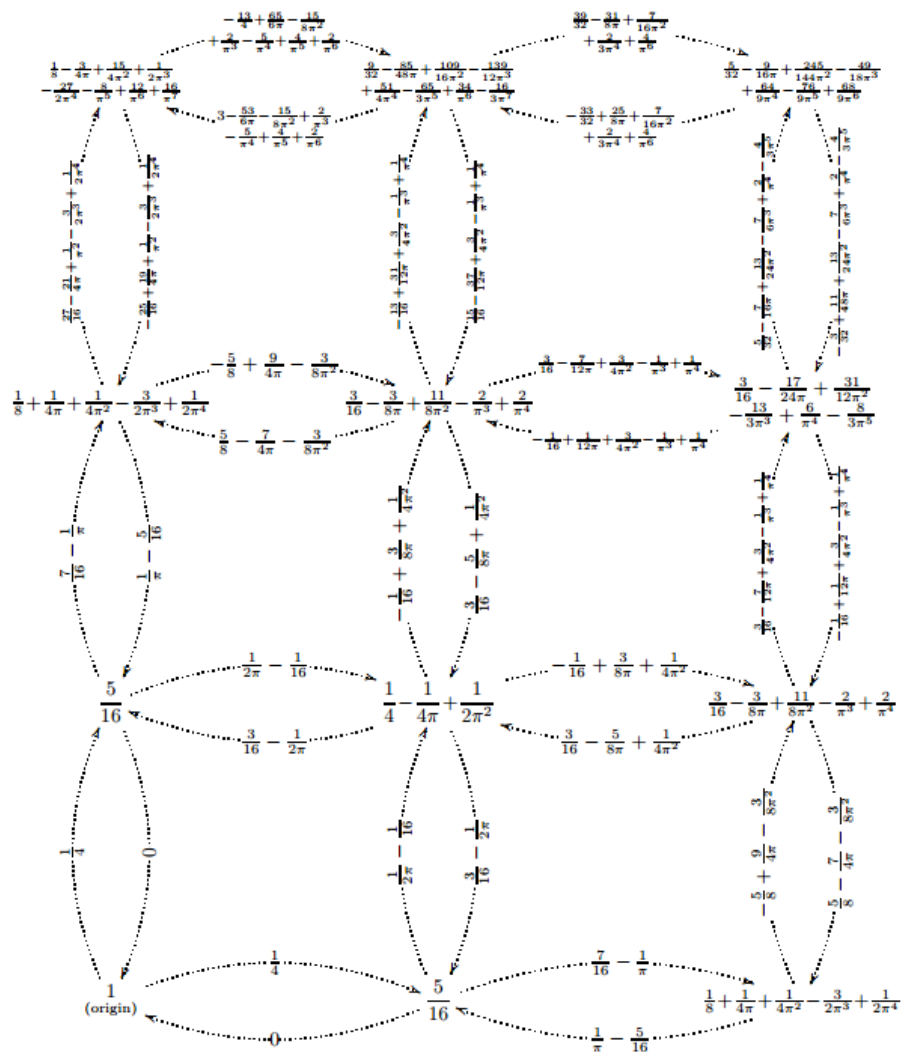


FIGURE 2. Intensity of loop-erased random walk on \mathbb{Z}^2 . The origin is at the lower-left, and directed edge-intensities as well as vertex-intensities of the LERW are shown. (See also Figure 9.)

These are related to the appearances of **patterns** in the dominos. The frequencies of these patterns are values in $\mathbb{Q}[\pi]$.

Why do formulas like \int tend to have π in them?

There is a nice theory of Maxim Kontsevich and Don Zagier which talks about **periods** - which are values of integrals.

- integrate
- a function
- over a space
- to get an answer

If that is too complicated. We are always adding things.

However it's very rare that we abstract what we are dealing with to reveal a simple problem at hand³.

What are the odds the train will be full? Can I get a good deal at the supermarket this week? Is my boyfriend cheating on me?

³There is just too much noise - uncertainty - that we can't account for. Noise which just gets bigger and bigger until it swallows up our whole argument. Real numbers are not pristine, they are filthy, living entities full of stuff.

There are two places that got me thinking about $\frac{\pi^2}{6}$

- The odds of two numbers being relatively prime
- The area of $e^x + e^{-x} = 1$ (it is also the area of a triangle)

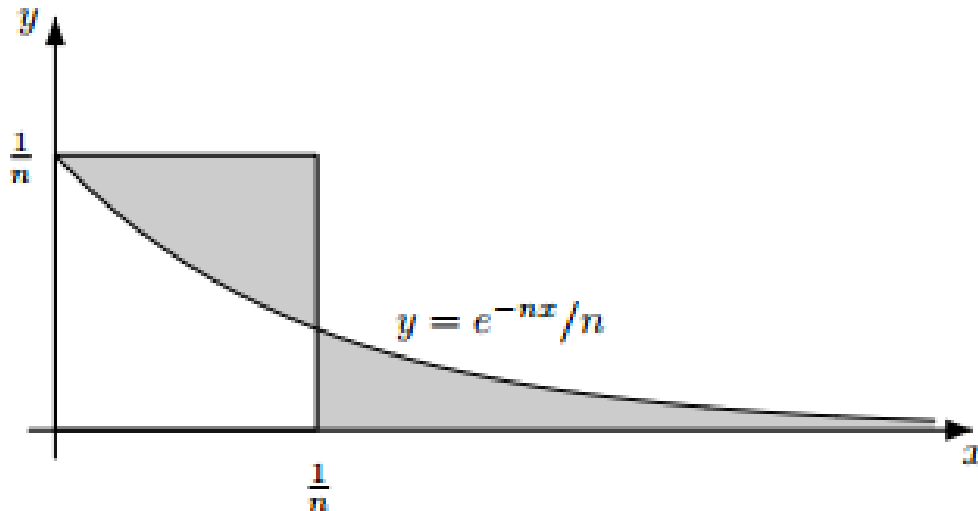
The first has to do with **Siegel-Veech** constants and the second is related to “amoebas” and **dominoes**.

The second case is clear since we look for patterns in the most boring case we can think of - dominos in a rectangle.

The first could have a variety of interpretations. The gcd function, square-free numbers, rectangular billiards, and on and on⁴.

⁴Conspicuously absent is the study of the zeta function itself. The Riemann Hypothesis isn't playing a huge role here (nor should it).

As I learned by Mikael Passare...



Any triangle of side lengths $(1, A, B)$ must satisfy triangle inequality:

$$1 < A + B, A < 1 + B, B < 1 + A$$

Then let $A = e^{-x}$ and $B = e^{-y}$ then we have 3 eqs:

$$1 < e^{-x} + e^{-y}, e^{-x} < 1 + e^{-y}, e^{-y} < 1 + e^{-x}$$

Let $A = \cos \alpha$ and $B = \cos \beta$. In fact the Jacobian of this transformation $(\alpha, \beta) \rightarrow (x, y)$ is 1. There there is a 2-1 map between real and imaginary solutions of $1 + z + w$ - it is a **Harnack Curve**.

Thewe two regions have the same area:

- $1 < A + B$
- $A < 1 + B$
- $B < 1 + A$

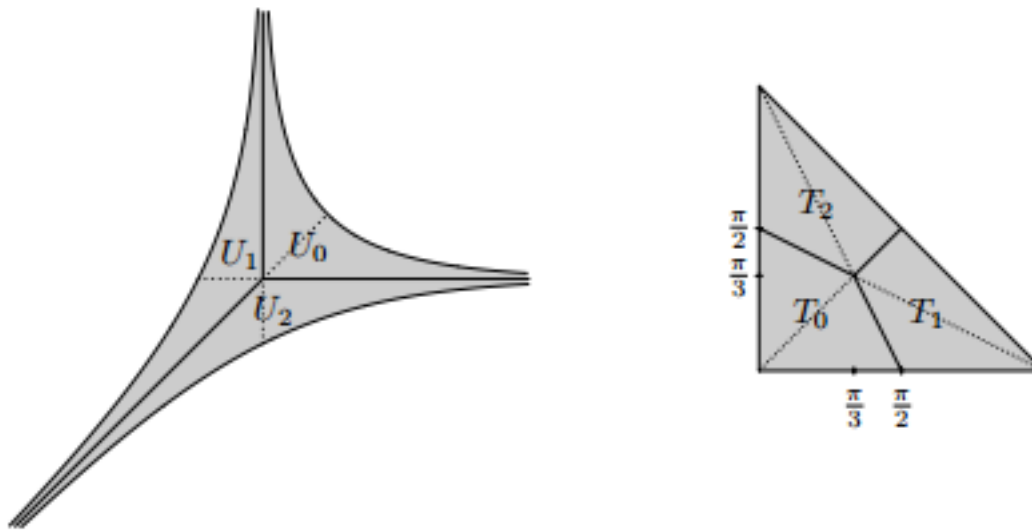


FIGURE 4. All six subsets have equal area.

- $1 < e^{-x} + e^{-y}$
- $e^{-x} < 1 + e^{-y}$
- $e^{-y} < 1 + e^{-x}$

Multiple Zeta Values

The Drinfeld-Zagier approach to multi-zeta values looks as such:

$$\zeta(2) = \sum_{m \geq 2} \frac{1}{m^2} = \int_{1 > t_1 > t_2 > 0} \frac{dt_1}{t_1} \frac{dt_2}{1 - t_2}$$

This is an exercise in Freshman analysis - which was so long ago!

$$\dots = \int_0^1 \frac{dt}{t} \log(1 - t) = \int_0^1 \frac{dt}{t} \sum_{n=1}^{\infty} \frac{t^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We still have to show $\zeta(2) = \frac{\pi^2}{6}$. This is fine... but I have not made connection to the other two formula I just showed you⁵.

I have an idea. Let $t = e^{-x}$. Then $dt = -e^{-x} dx = -t dx$:

$$\dots = \int_0^{\infty} dx \log(1 - e^{-x})$$

⁵A link to 15 proofs. <http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf>

References

- (1) JP Serre **Course on Arithmetic** Springer-Verlag
- (2) Davenport **Multiplicative Number Theory** Springer-Verlag
- (3) Kentaro Ihara, Masanobu Kaneko and Don Zagier **Derivation and double shuffle relations for multiple zeta values**
- (4) Maxim Kontsevich, Don Zagier **Periods**
- (5) Maxim Kontsevich **Exponential Integral** (Online Course) <https://youtu.be/tM25X6AI5dY>