Some Interesting Formulas Involving the GCD

John D Mangual

Sometimes, when I read a String Theory paper, I try to find a verifiable statement. Here is one I found in a paper:

$$I^{\mathcal{N}=1^*}(N,1,0) = N \sum_{d|N} 1 = N\sigma_0(N)$$

This number is called a **superconformal index** and it also equals:

$$I^{\mathcal{N}=1^*}(N,N,n) = \sum_{d|N} \sum_{l=1}^{N} \gcd(d,l)$$

I was heckled on MathOverflow for posting such an elementary formula. It's not mine, it's his.

Perhaps the general formula can show us the pattern:

$$I^{\mathcal{N}=1^*}(N,m,n) = \frac{N}{m}\sum_{d|N}\sum_{l=1}^{\gcd(d,m)} \gcd\left(\gcd(d,m), n + \frac{ld}{\gcd(d,m)}\right)$$

This formula is later shown to be equal to:

$$I^{\mathcal{N}=1^*}(N,m,n) = \sum_{d|N}\sum_{t=0}^{d-1}\gcd\left(N\frac{d}{m}, N\frac{m}{d}, N\left(\frac{t}{m}+\frac{m}{d}\right)\right)$$

In order for this equation to make sense, I eventually found $m|N$ and $d|N$ - I hope I guessed correctly.

By **Möbius inversion** we should have:

$$\frac{N}{m}\sum_{l=1}^{\gcd(d,m)} \gcd\left(\gcd(d,m), n + \frac{ld}{\gcd(d,m)}\right) = \sum_{t=0}^{d-1}\gcd\left(N\frac{d}{m}, N\frac{m}{d}, N\left(\frac{t}{m}+\right.\right.$$

These seem rather tedious to verify and their meaning unclear.

A starting point could be the Bezout theorem that:

$$\gcd(a, b) = \min_{x,y\in\mathbb{Z}} |ax + by|$$

Buried in the paper is his original statements about lattices when explain the appearance of **GCD** everywhere.

The inputs seem to be a Lie algebra (such as $\mathfrak{so}(N)$) plus a 4-manifold such as $\mathbb{R}^4$ or $\mathbb{R}^3 \times S^1$

Algebra	Theory	On $\mathbb{R}^4$	On $\mathbb{R}^3 \times S^1$
$\mathfrak{a}_{N-1}$	$(\mathrm{SU}(N)/\mathbb{Z}_m)_n$	N	$I^{\mathcal{N}=1}(N,m,n)$
$\mathfrak{b}_{N\geq 2}$	$\mathrm{Spin}(2N+1)$	$2N-1$	$2N-1$
	$\mathrm{SO}(2N+1)_+$		$2(2N-1)$
	$\mathrm{SO}(2N+1)_-$		$2N-1$
$\mathfrak{c}_{N\geq 2}$	$Sp(2N)$	$N+1$	$N+1$
	$(Sp(2N)/\mathbb{Z}_2)_+$		$\begin{cases} 2(N+1) & \text{for even } N \\ \frac{3}{2}(N+1) & \text{for odd } N \end{cases}$
	$(Sp(2N)/\mathbb{Z}_2)_-$		$\begin{cases} N+1 & \text{for even } N \\ \frac{3}{2}(N+1) & \text{for odd } N \end{cases}$
$\mathfrak{d}_{N\geq 3}$	$\mathrm{Spin}(2N)$	$2(N-1)$	$2(N-1)$
	$\mathrm{SO}(2N)_+$		$4(N-1)$
	$\mathrm{SO}(2N)_-$		$2(N-1)$
$\mathfrak{d}_{N \text{ odd}}$	$(\mathrm{Spin}(2N)/\mathbb{Z}_4)_n$	$2(N-1)$	$4(N-1)$
$\mathfrak{d}_{N\equiv 2 \bmod 4}$	$\mathrm{Sc}(2N)_\pm$ and $\mathrm{Ss}(2N)_\pm$	$2(N-1)$	$3(N-1)$
	$(\mathrm{SO}(2N)/\mathbb{Z}_2)_{\substack{00\\00}}$		$5(N-1)$
	$(\mathrm{SO}(2N)/\mathbb{Z}_2)_{\substack{11\\11}}$		$3(N-1)$
	$(\mathrm{SO}(2N)/\mathbb{Z}_2)_{\substack{10\\00}}$		$4(N-1)$
	$(\mathrm{SO}(2N)/\mathbb{Z}_2)_{\substack{01\\11}}$		$2(N-1)$
$\mathfrak{d}_{N\equiv 0 \bmod 4}$	$\mathrm{Sc}(2N)_+$ and $\mathrm{Ss}(2N)_+$	$2(N-1)$	$4(N-1)$
	$\mathrm{Sc}(2N)_-$ and $\mathrm{Ss}(2N)_-$		$2(N-1)$
	$(\mathrm{SO}(2N)/\mathbb{Z}_2)_{\substack{00\\00}}$		$5(N-1)$
	$(\mathrm{SO}(2N)/\mathbb{Z}_2)_{\substack{11\\11}}$		$3(N-1)$
	$(\mathrm{SO}(2N)/\mathbb{Z}_2)_{\substack{01\\00}}$		$3(N-1)$
	$(\mathrm{SO}(2N)/\mathbb{Z}_2)_{\substack{00\\10}}$		$3(N-1)$
$\mathfrak{e}_6$	E_6	12	12
	$(E_6/\mathbb{Z}_3)_n$		20
$\mathfrak{e}_7$	E_7	18	18
	$(E_7/\mathbb{Z}_2)_n$		27
$\mathfrak{e}_8$	E_8	30	30
$\mathfrak{f}_4$	F_4	9	9
$\mathfrak{g}_2$	G_2	4	4

Table 2. The Pure $\mathcal{N}=1$ Indices on a Circle. The notations are as in [9].

References

(1) arXiv:1606.01022 The Arithmetic of Supersymmetric Vacua. Antoine Bourget, Jan Troost. physics.hep-th.

(2) arXiv:1511.03116 On the N=1* Gauge Theory on a Circle and Elliptic Integrable Systems. Antoine Bourget, Jan Troost. physics.hep-th.

(3) arXiv:1506.03222 Counting the Massive Vacua of N=1* Super Yang-Mills Theory. Antoine Bourget, Jan Troost. physics.hep-th.

(4) arXiv:1305.0318 Reading between the lines of four-dimensional gauge theories. Ofer Aharony, Nathan Seiberg, Yuji Tachikawa. WIS/03/13-APR-DPPA, UT-13-15, IPMU13-0081. physics.hep-th.