Scratchwork: Torus Orbits

A common step in contemprary number theory literature is to re-define the problem as a group theory problem... Especially using "torus orbits".

Linear algebra works over any field, K. So we just say "let K be a field..." and if we have a single plane $\mathbb{R}^m \subseteq \mathbb{R}^n$ we know more-or-less they all behave the same. In fact, rational planes $\mathbb{Q}^m \subseteq \mathbb{Q}^n$ are not all identical.

$$X(\mathbb{Q}) = \{x^2 + y^2 = 1\} \subseteq \mathbb{Q}^2$$

This is a $\mathbb{Q}$-torus. Are there any rational points on this circle? I can name $(0, \pm 1)$ and $(\pm 1, 0)$. Here are two more:

$$(\tfrac{3}{5}, \tfrac{4}{5}), (\tfrac{5}{13}, \tfrac{12}{13}) \in X(\mathbb{Q})$$

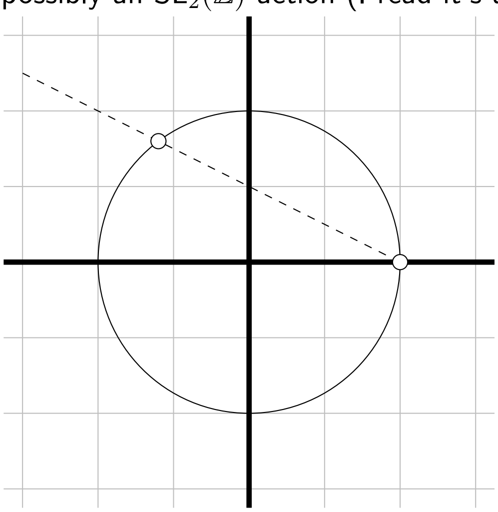
Since the circle was defined using algebra, we can say it is a **variety**. In fact, $X(\mathbb{Q})$ forms a group:

$$\left(\frac{3}{5} + i\,\frac{4}{5}\right)^2 = \left(\frac{3^2 - 4^2}{25}\right) + 2 \times \left(\frac{3 \times 4}{25}\right) i = -\frac{7}{25} + i\,\frac{24}{25}$$

Or we culd use 2×2 matrices. There's a way find a 2×2 matrix solution to $x^2 + 1 = 0$. E.g. $(x, y) \mapsto (-y, x)$.

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}^2 = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} \\ \\ \frac{24}{25} & \frac{7}{25} \end{bmatrix}$$

This tell us the Pythagorean triples form a *multiplicative group*, but also we are looking for a $\mathbb{Q}$-action, and possibly an $\mathrm{SL}_2(\mathbb{Z})$ action (I read it's actually a $\Gamma(2)$ action.)



Notice we've used a tiny bit of degree theory $[circle] \cdot [line] = 2$. This **is** an instance of cohomology[1] and we don't bother giving it a fancy name. This is called **stereographic projection**, I think?

$$\begin{aligned} \big|(1,0) + (t, mt)\big|^2 = (t+1)^2 + mt^2 &= 1 \\ t &= (1,0), (\tfrac{1-m^2}{m^2+1}, -\tfrac{2m}{m^2+1}) \end{aligned}$$

[1] intersection theory

with $m = \frac{a}{b} \in \mathbb{Q}$. These become $[a^2 - b^2 : 2ab : a^2 + b^2]$ over $\mathbb{Z}$ as a proportion. Multiplying by the slope $m \mapsto qx$ could lead to a reasonable group action of $\mathbb{Q}^\times$ on $\mathbb{Q}$.

Ex Let's try solving $x^2 + y^2 + z^2 = n$ as a torus orbit of some kind. This exercise is worked out in a handful of places.[2] I'm going to ask for the automorphic representation π of $\mathsf{SL}_2(\mathbb{A})$ for a spherical harmonic $\phi \in L^2(\mathsf{SO}_3)$.

[2] My standard for well-known is that everybody knows it. There are only 20000 practicing mathematicians in the US, maybe a tenth of those are number theorists. So...