Reminder: Group Theory

Let's get some help with computers. What are the groups of order $|G| = 400$? Here is the computer program.[1]

```
G := AllSmallGroups(400);;
List(G, g -> StructureDescription(g));
[ "C25 : C16", "C400", "C25 : C16", "C25 : Q16", "C8 x D50",
  "C25 : (C8 : C2)", "C25 : QD16", "D400", "C2 x (C25 : C8)",
  "C25 : (C8 : C2)", "C4 x (C25 : C4)", "C25 : (C4 : C4)",
  "C25 : (C4 : C4)", "C25 : ((C4 x C2) : C2)", "C25 : QD16", "C25 : D1
  "C25 : Q16", "C25 : QD16", "C25 : ((C4 x C2) : C2)", "C100 x C4",
  "C25 x ((C4 x C2) : C2)", "C25 x (C4 : C4)", "C200 x C2",
  "C25 x (C8 : C2)", "C25 x D16", "C25 x QD16", "C25 x Q16",
  "C25 : (C8 x C2)", "C25 : (C8 : C2)", "C4 x (C25 : C4)",
  "C25 : (C4 : C4)", "C2 x (C25 : C8)", "C25 : (C8 : C2)",
  "C25 : ((C4 x C2) : C2)", "C2 x (C25 : Q8)", "C2 x C4 x D50",
  "C2 x D200", "C25 : ((C4 x C2) : C2)", "D8 x D50",
  "C25 : ((C4 x C2) : C2)", "Q8 x D50", "C25 : ((C4 x C2) : C2)",
  ...
]
```

Notice there is is both $C_{25} \times QD_{16}$ and $C_{25} \ltimes QD_{16}$.

In more commom notation we can write with the symbols $\times$ (direct product) and $\ltimes$ (indirect product):

- $G = (C_5 \ltimes Q_8) \times D_{10}$
- $G = (C_5 \ltimes C_5) \ltimes (C_4 \times C_4)$
- $G = C_2 \times ((C_5 \times C_5) \ltimes C_8)$
- $G = D_8 \times ((C_5 \times C_5) \ltimes C_2)$

Once we have these explicit descriptions of groups, we look at the representation theory of finite groups. Example, by linear algebra:

$$\mathsf{dim}\ \mathsf{Ind}_H^G(\mathbf{1}_H) = [G : H] = \mathsf{dim}(V)$$

[1] https://math.stackexchange.com/questions/4108993/the-221-groups-of-order-g-400
https://www.gap-system.org/

At least theoretically we can relate the induced representation from the subgroups and the character theory of the subgroups.

$$\sum_{\rho} (\dim \rho)^2 = |G|$$

This is called **Maschke's Theorem**. We have 500 examples to check dimensions of representation and Harmonic analysis of finite groups.[2]

[2]The category is called Mod_G so that functorial properties there could be related to character theoretic formulas here.

Approximate Groups Does "commutativity" matter? We've been studying the equation $ab = ba$. It certainly works for numbers $2 \times 3 = 3 \times 2$ and we can describe when it fails:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

so we have lots of examples of non-commutativity.

Lemma Let G be an aribtary group. Let $A \subset G$ be a subset with $|A^2| \leq K|A|$. Then $|A^{-1}A| \leq K^2|A|$ and $|AA^{-1}| \leq K^2 A$.

In my experience the proofs not very exciting. Counting he possibilities on both sides and making sure both sides are equal. Here's the counter-example the book has provided: Let H be a finite group and $G = H * \langle x \rangle$, the free-product of H and the infinite cyclic group one generatory (basically $\mathbb{Z}$). Set $A = H \cup \{x\}$. Then

$$|A^2 \leq 3|A| \tag{$*$}$$

but $HxH \subseteq A^3$ and yet $|HxH| = |H|^2 \asymp |A|^2$.

This is their instance of **small tripling**. We could imamgine $A \subseteq \mathbb{Z}$ then:

$$A = A + A + A \text{ or } |3A| \leq 3|A|$$

so once we throw away the exact relation $A + A = A$ (such as the arithmetic progression or a **subgroup** or **coset** of $\mathbb{Z}$) we get to consider small-doubling or small-tripliing moves.

Ex $(A \cup \{1\} \cup A^{-1})^2$ is an $O(K^9)$-approximate group.

representation theory[3] Just a reminder that $\int$ is just a short-hand of $\sum$ which just means "$+$".

$$\int_G f(xg)\, d\mu(x) = \int_G f(x) d\mu(x)$$

we need Haar measure on group of choice $G = \mathbb{R}/\mathbb{Z}$ or $G = \mathbb{R}^d$ or (non-abelian) $G = \mathsf{SO}(3)$ (the space of rotations of the sphere). We could get the approximate-groups by setting $f = 1_A$!

? Where did we get these invariant measures and perfect lattices and spheres?

Thm (Peter-Weyl) Let G be a compact topological group with Haar measure μ. Then the regular representation of G on the space $L^2(G, \mu)$ decomposes to Hilbert spae direct sum:

$$L^2(G, \mu) = \bigoplus_\rho M(\rho)$$

of isotypic components of the finite dimensional unitary representations, each $M(\rho)$ being isomorphic to $\dim(\rho)$ copies of ρ.

Q What does it measn that $\mathsf{SO}(3)$ is a **compact** topological group?

References

[1]

[3] https://people.math.ethz.ch/~kowalski/representation-theory.pdf