Tutorial : Ping-Pong Lemma

Half of these projects start by a conjectured relationship between two theorems. And it's not always correct. Today's pair is:

- Ping-Pong Lemma
- Sum-Product Theorem

I *think* that's correct. Let's state these results and being the long journey of connecting these thing to "real life".

#1 Ping-Pong Lemma I was able to find two examples of Ping-Pong Lemma having to with group matrices over the integers $\mathbb{Z}$ we can have

- Banach-Tarski paradox
- Establishing free groups

Here's a proposition:

> This is a Free group on two generators (no relations)
>
> $$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \subseteq SL_2(\mathbb{Z})$$
>
> Proof: use the ping-pong Lemma. Another example I found from Sarnak[1]
>
> $$\left\langle \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & -5 \end{pmatrix} \right\rangle \subseteq SL_4(\mathbb{Z})$$
>
> and these play generalized Ping-Pong. The reason we like the free group F_2 is becuase it's one of the few discrete groups we can understand. And I don't even think that's true, because if $A, B \in SO(3)$ that group is compact and we can measure:
>
> $$\phi : \langle A, B \rangle \to SO(3)$$
>
> we'd have that $\overline{\langle A, B \rangle} \subseteq SO(3)$ is a **dense** subgroup, so we can try to quantify (by whatever means we have availble) how quickly this is mixing.

In another direction we have the Banach-Tarski paradox. The groups used to contruct the paradox seem related to the Pythagorean theorem[2] He will use:

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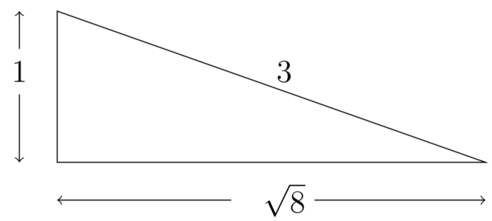
[1] `http://web.math.princeton.edu/sarnak/NotesOnThinGroups.pdf`

[2] https://stanford.box.com/shared/static/wesg27648yqomf4ar3mkmfwf89ptzqj4.pdf

This group is free:

$$\left\langle \begin{pmatrix} \frac{1}{3} & +\frac{2\sqrt{3}}{3} & 0 \\ -\frac{2\sqrt{3}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & +\frac{\sqrt{8}}{3} \\ 0 & -\frac{\sqrt{8}}{3} & \frac{1}{3} \end{pmatrix} \right\rangle \subseteq SO_3(\mathbb{R})$$

This group is related to the right triangle with side lengths 1 and $\sqrt{8}$



This tringle is oriented in various ways in three dimensional space.

Therefore... without stating the ping-pong Lemma we can imagine such a thing could be useful! Now for the statement:

Ping-Ping Lemma Let G be a group acting on a set X. Let H_1, H_2 be sub-groups and suppose we can find subsets $X_1, X_2 \subseteq X$ subset that we observe

$$\big[g \in H_1 \to g(X_2) \subseteq X_1\big] \text{ and } \big[g \in H_2 \to g(X_1) \subseteq X_2\big]$$

then our subgroup is free, $\langle H_1, H_2 \rangle = H_1 * H_2 \simeq F_2$.

One thing we notice is that ping-pong "works" (i.e. produces free groups or expander graphs?) only for 3×3 and not for 2×2. Another possiblity is that the group we are studying isn't free (there are relations). This is not a bad thing, $SL(2, \mathbb{Z})$ is not free, there's a relation:

$$SL_2(\mathbb{Z}) \simeq \big\langle S, T : S^2 = (ST)^3 = 1 \big\rangle \simeq \big(\mathbb{Z}/2\mathbb{Z}\big) * \big(\mathbb{Z}/3\mathbb{Z}\big)$$

and for a general number field, F, $SL_2(\mathcal{O}_F)$ for the ring of integers $\mathcal{O}_F$ can be an open problem. Therefore, we conclude, $SL(2, \mathbb{Z})$ can house rather complicated objects.

I should remark this is a case of the **Tits alternative**.

Veresion #2 $SO(3)$ contains a copy of the free group on two generators.

Our candidate free group will be the the standard 3-4-5 triangle. Since $3^2 + 4^2 = 5^2$ we have

$$\left\langle \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix} \right\rangle \subseteq SO_3(\mathbb{Q})$$

and we resort to 5-adic numbers to choose our ping-pong sets:

- $A_\pm = 5^{\mathbb{Z}} \cdot \{(x, y, z) \in \mathbb{Z}^3, x \pm 3y \in 5\mathbb{Z}, z \in 5\mathbb{Z}\}$
- $B_\pm = 5^{\mathbb{Z}} \cdot \{(x, y, z) \in \mathbb{Z}^3, z \pm 3y \in 5\mathbb{Z}, z \in 5\mathbb{Z}\}$

and our set is $X = A_- \cup A_+ \cup B_- \cup B_+ \cup \{(0, 1, 0)\}$ and this is our **ping-pong** set!

So our ping-pong style proofs involve finding a "tennis court" and playing a game!

#2 Sum-Product Theorem This is the dumbest-sounding theorem I have ever heard of.

#2 Approximate Groups . . .

#3 Do these two overlap? I guess not.

References

(1) . . .