

Tutorial : Ping-Pong Lemma

Half of these projects start by a conjectured relationship between two theorems. And it's not always correct. Today's pair is:

- Ping-Pong Lemma
- Sum-Product Theorem

I *think* that's correct. Let's state these results and being the long journey of connecting these thing to "real life".

#1 Ping-Pong Lemma I was able to find two examples of Ping-Pong Lemma having to with group matrices over the integers \mathbb{Z} we can have

- Banach-Tarski paradox
- Establishing free groups

Here's a proposition:

This is a Free group on two generators (no relations)

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \subseteq SL_2(\mathbb{Z})$$

Proof: use the ping-pong Lemma. Another example I found from Sarnak¹

$$\left\langle \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & -5 \end{pmatrix} \right\rangle \subseteq SL_4(\mathbb{Z})$$

and these play generalized Ping-Pong. The reason we like the free group F_2 is because it's one of the few discrete groups we can understand. And I don't even think that's true, because if $A, B \in SO(3)$ that group is compact and we can measure:

$$\phi : \langle A, B \rangle \rightarrow SO(3)$$

we'd have that $\overline{\langle A, B \rangle} \subseteq SO(3)$ is a **dense** subgroup, so we can try to quantify (by whatever means we have available) how quickly this is mixing.

In another direction we have the Banach-Tarski paradox. The groups used to construct the paradox seem related to the Pythagorean theorem² He will use:

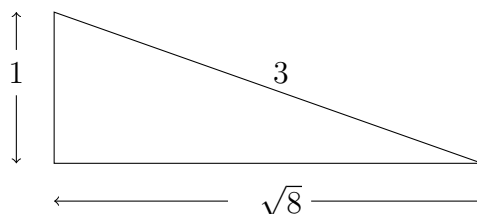
¹<http://web.math.princeton.edu/sarnak/NotesOnThinGroups.pdf>

²<https://stanford.box.com/shared/static/wesg27648yqomf4ar3mkmfwf89ptzqj4.pdf>

This group is free:

$$\left\langle \begin{pmatrix} \frac{1}{3} & +\frac{2\sqrt{3}}{3} & 0 \\ -\frac{2\sqrt{3}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & +\frac{\sqrt{8}}{3} \\ 0 & -\frac{\sqrt{8}}{3} & \frac{1}{3} \end{pmatrix} \right\rangle \subseteq SO_3(\mathbb{R})$$

This group is related to the right triangle with side lengths 1 and $\sqrt{8}$



This triangle is oriented in various ways in three dimensional space.

Therefore... without stating the ping-pong Lemma we can imagine such a thing could be useful! Now for the statement:

Ping-Ping Lemma Let G be a group acting on a set X . Let H_1, H_2 be sub-groups and suppose we can find subsets $X_1, X_2 \subseteq X$ subset that we observe

$$[g \in H_1 \rightarrow g(X_2) \subseteq X_1] \text{ and } [g \in H_2 \rightarrow g(X_1) \subseteq X_2]$$

then our subgroup is free, $\langle H_1, H_2 \rangle = H_1 * H_2 \simeq F_2$.

One thing we notice is that ping-pong “works” (i.e. produces free groups or expander graphs?) only for 3×3 and not for 2×2 . Another possibility is that the group we are studying isn’t free (there are relations). This is not a bad thing, $SL(2, \mathbb{Z})$ is not free, there’s a relation:

$$SL_2(\mathbb{Z}) \simeq \langle S, T : S^2 = (ST)^3 = 1 \rangle \simeq (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$$

and for a general number field, F , $SL_2(\mathcal{O}_F)$ for the ring of integers \mathcal{O}_F can be an open problem. Therefore, we conclude, $SL(2, \mathbb{Z})$ can house rather complicated objects.

#2 Sum-Product Theorem ...

#2 Approximate Groups ...

#3 Do these two overlap? ...

References

(1) ...