Nilsequences

John D Mangual

Terence Tao sees a lot of things, but he writes in an obfuscated way, and I think he misses a lot of things. About 10 years ago, I was introduced to the topic of **nilsequences** in a course of Dynamics and Number Theory. I did nothing with it. Let's read Terry's latest blog on this topic[1].

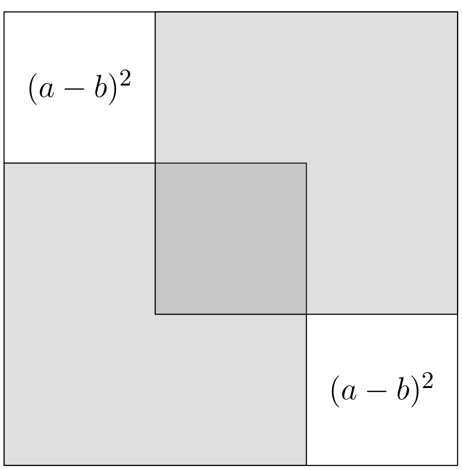
Part of it is like. . . where do fractions come from?

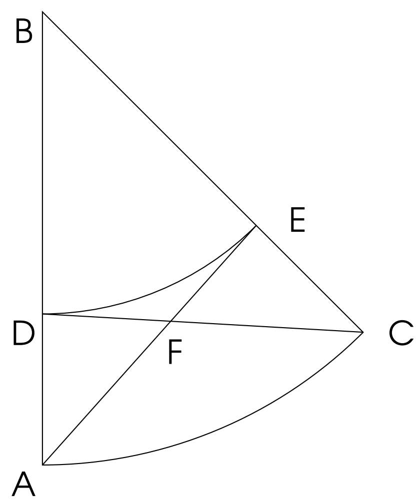
If we take scientific measurement, there's quite a bit error that obstructs us from observing the most delicate patterns. In fact, shielding us completely from finding them (or protecting us).

$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503$

If you examine the digits carefully[2] we can prove the decimals do not exhibit any pattern in the decimal expension. However, if we use continued fractions:

$\sqrt{2} = [1; 2, 2, 2, 2, 2, \dots]$ and this is a nicer system since we have exponential convergence of the the number. Here the error is 10^{-6} (microscopic).





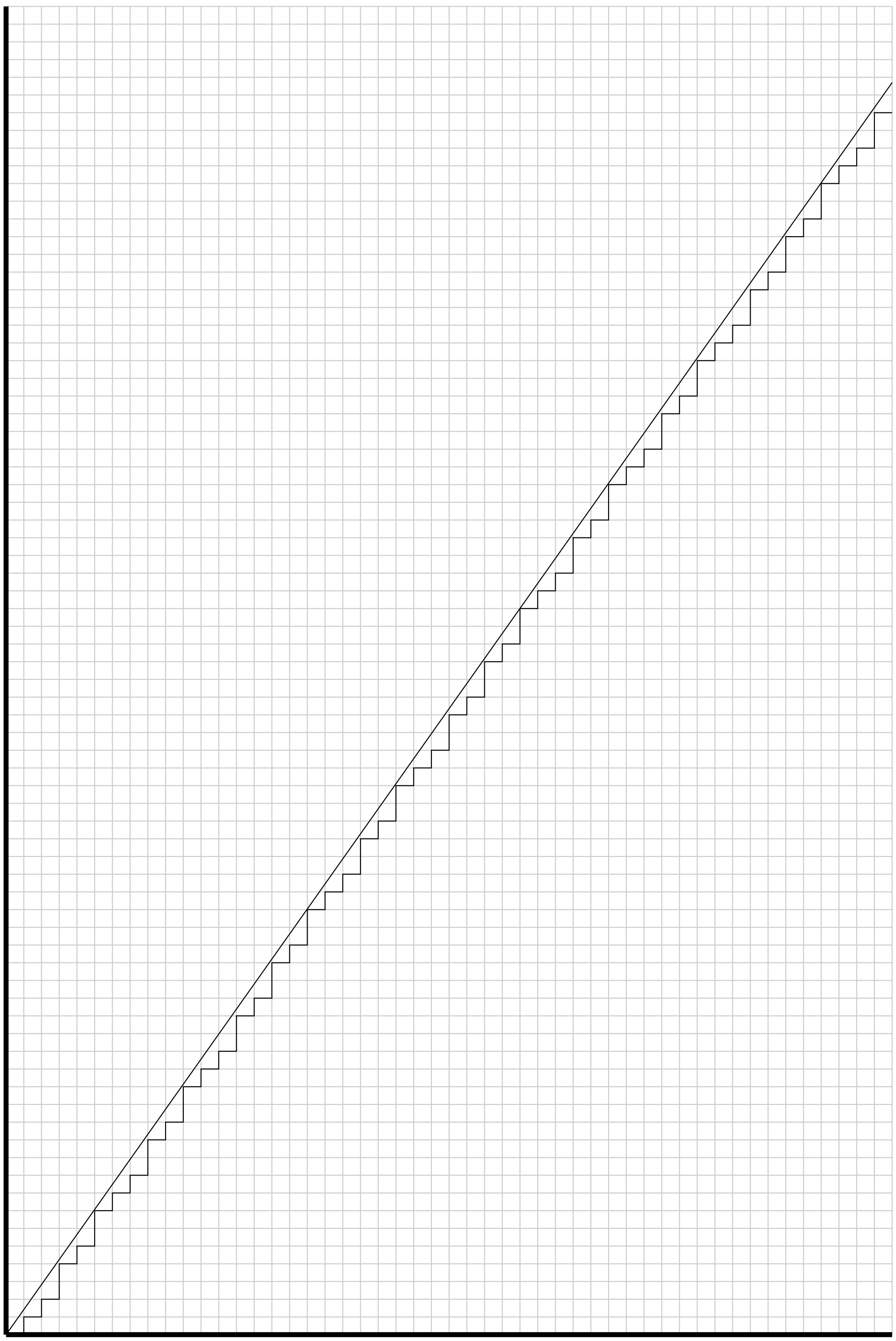
These pictures lead to two different proofs that $\sqrt{2} \notin \mathbb{Q}$. They are both geometric proofs, and argue by infinite descent. Therefore, there must have been an ètale cohomology.

I have no idea what ètale cohomology and the books do not simplify it enough for me. Forget it. One way to get meaningful numbers to arbitrary accuracy is to observe a dynamical system over time and take detailed measurements. And you will see.

[1] https://terrytao.wordpress.com/2017/04/28/notes-on-nilcharacters-and-their-symbols/

[2] http://www.gutenberg.org/files/129/129.txt

Using a minimimal amount of math we can define most (possibly all) nilsequences. The fractional part of multiples of $\sqrt{2}$ is nilsequence.



We want. . . even more nilsequences. In the Terry Tao blog we get two to get us started:

$$n \mapsto \left(\sqrt{2}n\{\sqrt{3}n\}\right) \text{ or } n \mapsto \left(\sqrt{2}n\{\sqrt{2}n\}\right)$$

Earlier, two professors Vitaly Bergelson and Alexander Leibman studied these kinds of sequences of numbers, to my satisfaction.

Green and Tao are looking for patterns in sequences of numbers. I can never produce a sequence of numbers that would require the techniques they are using. And yet, we kind of see them every day.

In empirical measurements; any time we try to use math to solve a real problem, things get "complicated" and maybe Green and Tao help us reason about that.

Tao's blog today was about **symbols** of nilsequences. I don't understand how you can write an entire theory of numbers and still not write down a single one. We can compute:

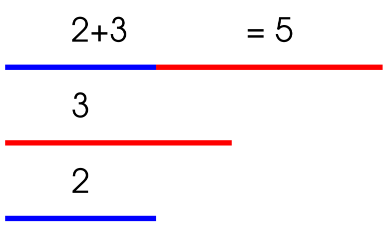
$$n \times \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n\,x & n\,y + \binom{n}{2}\,xz \\ 0 & 1 & n\,z \\ 0 & 0 & 1 \end{pmatrix}$$

The notation $n \times \cdot \equiv (\,\cdot\,)^n$ is short for multiplying a number by itself n times. Here x, y, z are elements of $\mathbb{R}$, but they could also be elements of $\mathbb{Z}$. And it's the number in the upper right corner we are intested. So, if you do the reduction right you get

$$n \mapsto (\{nx\}, \{ny + nx\lfloor nz \rfloor\}, \{nz\})$$

and I'm faking it slightly because I haven't done the arithmetic.

It helps to think like a cave man. What if there were no numbers? We could take two lengths and add them.



Tao - and Green, somewhat - have become suspect of the basic number operations. $+$, $\times$, $-$, $\div$ if you do it too much and then someone tells you that it's wrong. So, let's try new kind of number!

5/2 I would like to try to fill in the part about Galois cohomology. Leading to the side issues like dynamics and computabiity, so the scenery is very rich.

We are trying to show that $x^2 = 2$ has no solutions in rational number $x \in \mathbb{Q}$ or (clearing denominators) $a^2 = 2b^2$ has no solutions $a, b \in \mathbb{Z}$. There seems to be a way using **heights** and a way using **torsors** but every book reviews it differently. And hopefully we see what this cohomology nonsense is about.

#1 The "height" of a fraction $\frac{a}{b}$ is just the numerator plus the denominator $|a| + |b|$.

$$h(\frac{2}{3}) = |2| + |3| = 5$$

what is the hehgit of a fraction that could represent the $\sqrt{2}$? We could have that:

$$h(2) = h(\tfrac{2}{1}) = 2 = h(\tfrac{a^2}{b^2}) = a^2 + b^2$$

here I used the symbol "=" to mean the English word "but also" and $(a, b) = 1$ the numbers a and b are **relatively prime**. How can two integers, have a square that sums to two?

$$a^2 + b^2 = 1 \longrightarrow a = b = 1$$

That's not possible.

The find print is, we could find completions where $\sqrt{2}$ exists, such as $\mathbb{R}$ or $\mathbb{Q}_3$ or any $\mathbb{Q}_p$ when $\frac{p^2-1}{8} \in \mathbb{Z}$ is an even number[3].

#2 These math textbooks give really deep names to really basic concepts. Continuing with $\sqrt{2}$

$$x^2 - 2y^2 = (x - \sqrt{2}y)(x + \sqrt{2}y) = 1$$

One way to phrase this is that we have adjoint $\sqrt{2}$ to the ring of integers $\mathbb{Z}$ and obtained $\mathbb{Z}[\sqrt{2}]$. Bjorn Poonen defines an affine variety (an "equation"):

$$x^2 - 2y^2 = -3 \text{ over } \mathbb{A}_{\mathbb{Q}}$$

In his notation $\mathbb{A}_{\mathbb{Q}} = \mathrm{Spec}\ \mathbb{Q}$ is the "line" while $\mathbf{A}_{\mathbb{Q}}$ are the Adeles. I think $\mathrm{Spec}\ \mathbb{Q}$ is just $\mathbb{Q}$. So we are trying to solve this equation over fractions. We don't know if there are any solutions just yet.

He defines a **torus** – a term which throws me completely off because it is a hyperbola.

- $T = \{x^2 - 2y^2 = 1\} \subseteq \mathbb{A}^2_{\mathbb{Q}} = \mathrm{Spec}\ \mathbb{Q}[x, y]$
- there is some kind of multiplication: $\times : T \times T \to T$
- $(x_1, y_1) \times (x_2, y_2) = (x_1x_2 + 2y_1y_2, x_1y_2 + x_2y_1)$

The real definition of torus is the most obnoxious thing ever: it is a twist on $\mathbb{G}_m^n$ (as a group scheme), where $\mathbb{G}_m = \mathrm{Spec}\ \mathbb{Q}[t, t^{-1}]$ and m stands for "multiplication".

[3] https://mixedmath.wordpress.com/2012/08/23/an-elementary-proof-of-when-2-is-a-quadratic-residue/

This guy really emphasizes schemes and he considers $\mathrm{Spec}[\,\cdot\,]$ a lot (this is a **functor**)

We can look for different points on curves, and there will be a torus action:

$$x^2 - 2y^2 = -3 \quad \text{or} \quad x^2 - 2y^2 = 0$$

The last one might be a scheme and not a variety.[4] In both cases, $X(\mathbb{Q}) = \varnothing$ (there are no rational "points") in either case so. . . this is a non-trivial T-torsor.

For any fixed smooth algebraic group G (in our case, the solutions to $x^2 - 2y^2 = 1$ with multiplication and we will write it T) these three objects are thes same:

- T-torsor over $\mathbb{Q}$
- twists of $\mathbf{G}$ (possibly $\mathbf{G}$)
- $H^1(\mathbb{Q}, T)$

if $X = \{x^2 - 2y^2 = 0\}$ is a torsor it has a cohomology the cohomology class.

$$[\{x^2 - 2y^2 = 0\}] \in H^1\big(\mathbb{Q}, \{x^2 - 2y^2 = 1\}\big)$$

$\mathbf{G}$ is the **trivial torsor** (Grothendieck thought math centered around the notion of zero, which took thousands of years to develop. So we have all thse notations for zero-like concepts.).

$\sqrt{2} \notin \mathbb{Q}$ how can our torsor have any points?

We have to examine the **separable closure** $\mathbb{Q}_s$ – I have never seen so many numbers in my life. Hopefully if we adjoin enough numbers, $\sqrt{2}$ is in there. Maybe

$$\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[x]/(x^2 - 2)$$

will be enough.

I haven't really said what "cohomology" is. Any function $\mathbb{Q} \to T$ is an element of cohomology and the "class" is just a way of declaring two of these two be the same. These equivalences form a hierarchy.

I am thinking the end result here is not a "proof" but an exhibition of how all the machinery is connected.

I guess "torsor" was just another wod for "equation". The idea was there should be a local to global principle:

$$H^1(\mathbb{Q}, G) \xrightarrow{1:1} \prod_{p \in \text{prime}} H^1(\mathbb{Q}_p, G)$$

the real idea is that if you want to solve the equation in $\mathbb{Q}$ you should try to solve the equation in $\mathbb{Q}_p$ for all prime numbers p (and also $\mathbb{R}$). A cohohomology class $[X] \in H^1(\mathbb{Q}, G)$ is just an equation and $[X] = 0$ means we can solve it.

Having exceeded the time cap for this approach, we pick a new one.[5].

[4]The book is written in such a formal way, it's hard to understand. Can't say I'm doing a great job.

[5]We'll review it more later. There is Neukirch's **Cohomology of Number Fields** (which is like the sequel to his textbook "Algebraic Number Theory" and Jean-Pierre Serre **Galois Cohomology** and the question is always the same: how can we get these frameworks or theories to work for us?

5/4 Christophe Soulé had me sold on this book for two reasons: he promises a theory of infinite descent using **Arakelov geometry**. As usual, not a single equation in sight.

What blew me away was how, in passing, he writes down almost every thought on renormalization I've had; again only in the most general terms, leaving "exercises" to the reader.

He does write **(A)** $\infty! = \sqrt{2\pi}$ and also **(B)** $\infty! = e^{-\zeta'(0)}$ leading to a kind of regularization:

$$1 \times 2 \times 3 \times 4 \times \cdots = \sqrt{2\pi}$$

and the discussion quickly gets rather pretentious. Yet, for the time being, we see that he is right.

There is an extensive literature of zeta-functions. There's a zeta function for Schemes. Bjorn Poonen gives us:

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p = \text{prime}} \frac{1}{1 - p^{-s}} = \prod_{\mathfrak{m} \in \text{maximal}} \frac{1}{1 - \#(\mathbb{Z}/\mathfrak{m})^{-s}} = \prod_{P \in \mathrm{Spec}\mathbb{Z}} \frac{1}{1 - \#\mathbb{Q}(P)^{-s}}$$

So $\mathbb{Z}$ is now the scheme $X = \mathrm{Spec}\mathbb{Z}$. In that section of the book he uses étale cohomology[6] The formulas that Soulé outlines for us a zeta function for any scheme X.

It seems to be on me, what I want for an "explicit computation". Seems to be kind of a game. We have an endless supply of zeta functions. Let's write down one or two:

$$\zeta(s)^2 = \left(\sum_{n \geq 1} n^{-s} \right)^2 = \sum_{n \geq 1} d(n)\, n^{-s} \neq \sum_{n \geq 1} r_2(n)\, n^{-s}$$

here $r_2(n)$ is the number of way of writing n as the sum of two squares. $d(n)$ is # of divisors.

$$50 = 5 \times 5 + 5 \times 5 = 7 \times 7 + 1 \times 1 = 1 \times 1 + 7 \times 7$$

considerations like these lead to a very rich theory of modular forms; and there is all kinds of stuff. I am wrong. Sum of two squares is it's own zeta function. I think it's

$$\zeta_{\text{Spec}\mathbb{Z}[i]}(s) = \left(\sum_{a,b} (a^2 + b^2)^{-s} \right) = \sum_{n \geq 1} r_2(n)\, n^{-s} = \prod_{p = 4k+1} \cdots \times \prod_{p = 4k+3} \cdots$$

Exercise (i.e. I have no idea right now) find an appropriate Dirichlet series for the sum of two squares.

Discussion Already did it. There are two kinds of primes in $\mathbb{Z}[i]$, $p = 4k+1$ which split into $p = a^2 + b^2 = (a+bi)(a-bi)$ and $p = 4k+1$ which do not split. This factorization result is not free, it uses a geometry of numbers leads into class field theory.

$$\left(\sum_{a,b,c,d} (a^2 + b^2 + c^2 + d^2)^{-s} \right) = \sum_{n \geq 1} r_4(n)\, n^{-s}$$

There are lots of interesting zeta-functions and a rich complicated literature. I think it's our choice to pick an encoding back into "elementary" number theory (if possible)

[6] to prove the Hasse principle (local-global principle) we were using Galois cohomology. It's tough to keep these straight.

Soulé has three approaches for understanding $\infty! = \sqrt{2\pi}$.

#1 Stirling's formula let's us estimate the number of rearrangements of things:

$$n! = 1 \times 2 \times 3 \times \cdots \times \times n = (n/e)^n \sqrt{2\pi n}$$

This formula can arise in two ways: gambling and estimation.

- $\binom{n}{k}$ the probability of flipping n coints and getting k heads and $n-k$ tails
- $f(x+\epsilon) = \sum f^{(n)}(x)\frac{\epsilon^n}{n!}$ this is called **Taylor Series**; it's just the best fit line. we do it mentally

this is a low-fat way of getting all sorts of dualities. Not rarefied enough for certain crowd but will have to suffice. Eventually I will return to the Terry Tao blog – he has all kinds of yummy low-fat tricks for us to learn even if he writes in a somewhat obfuscated way that I don't like.

Taking the logarithm of the Stirling series we get an informative identity:

$$\log(n!) = \log 1 + \log 2 + \cdots + \log n = n \log n - n + \log \sqrt{2\pi} + O(n^{-1})$$

We should already be suspicious of Soulé's approach. One of the inputs to his discussion is a derivaton Stirling formula:

- where did that $\sqrt{2\pi}$ come from?
- our estimate works within $O(n^{-1})$ precision. How good is that really?

#2 Another way is using "the functional equation" of the zeta function.[7] As eveyone knows:

$$\pi^{s/2}\Gamma(s/2)\zeta(s) = \pi^{(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$$

so far so good. There is symmetry around $\mathrm{Re}(s) = \frac{1}{2}$. And take derivaive of zeta function:

$$-\zeta'(s) = \sum_{n\geq 1} \log n \; n^{-s} = \Bigg|_{s=0} \sum_{n \geq 1} \log n$$

So...why is $\sqrt{2\pi}$ a reasonable number here? There is an estimate around $s = 1$ (again not for free)

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

This is a way of expressing that $\zeta(1) = \infty$ with a little information about what happens nearby.

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} +\right) + \cdots \geq 1 + 1 + 1 + \dots$$

Regularization is kind of a disaster. The complex plane $\mathbb{C}$ is comparatively well-behaved compared to the infinite dimensional object actually being studied.

There's also a switch between $\zeta(0)$ and $\zeta(1)$ which is OK because of this mirror.

[7] Later, we will find a whole class of L-series which satisfy a function equation. These sequences of are not the easiest to calculate. Again, I think the hard part is linking these modular forms to the sequences of numbers we care about.

Using facts from a good complex analysis textbook. He shows that:

$$\begin{array}{ccccc} & & & & \gamma = -\Gamma'(1) \\ & & & & \downarrow \\ \Gamma(s+1) = \sqrt{\pi}2^s\Gamma(\frac{s+1}{2})\Gamma(s+\frac{1}{2}) & \to & \frac{\Gamma'(1)}{\Gamma(1)} = \log 2 + \frac{1}{2}\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + \frac{1}{2}\frac{\Gamma'(1)}{\Gamma(1)} & \to & -\frac{1}{2}\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = 2 + \frac{\gamma}{2} \end{array}$$

This is how I think of derivations: organized in two dimensions. In the modern language these are expressed in terms of injective maps, exact sequences, modules and cohomology. I hate it so much!

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \qquad \to \qquad \zeta(1-s) = \frac{1}{s} + \gamma + O(s)$$

This one is much clearer: subsitute $s \mapsto 1-s$. And we do these gymnastics, but now we organize into a careful way. The only factorial identity we haven't used yet is the most basic one:

$$s\,\Gamma(s) = \Gamma(s+1) \xrightarrow{\frac{d}{dx}\log(\cdot)} \frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} + \frac{\Gamma'(s+1)}{\Gamma(s+1)} \xrightarrow{s\approx 1} \frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} + \frac{\Gamma'(1)}{\Gamma(1)} + O(s)$$

To me this makes more sense that the equations are being chain togeter this way. Lastly:

$$\frac{\zeta'(0)}{\zeta(0)} = \log 2 + \log\pi \text{ and } \zeta(0) = -\frac{1}{2} \text{ so that } \zeta'(0) = -\frac{1}{2}\log 2\pi$$

This is my crude protoype for group cohomology: writing a bunch of equations in an organized way (using $\times$ and $+$).

#3 (Divergent Integrals)

The last template Soulé gives us has to do with theta functions and Mellin transforms:

$$\zeta(2s) = \frac{1}{\Gamma(s)}\int_0^\infty \underbrace{\Big[\sum_{n=1} e^{-n^2 t}\Big]}_{=\Theta(t)} t^{s-1}\,dt \xrightarrow{\frac{d}{ds}\big|_{s=0}} 2\zeta'(0) \text{“}=\text{”} \int_0^\infty \Theta(t)\left(\frac{t^s}{\Gamma(s)}\right)'_{s=0}\frac{dt}{t} \text{“}=\text{”} \int_0^\infty \Theta(t)\frac{dt}{t}$$

Poisson summation is related to the map $t \mapsto -\frac{1}{t}$ in $SL(2,\mathbb{Z})$.

$$\Theta(t) = \sqrt{\tfrac{\pi}{t}}\Theta(\tfrac{\pi}{t}) - \frac{1}{2} \overset{t\approx 0}{=} \tfrac{1}{2}\sqrt{\tfrac{\pi}{t}} - \tfrac{1}{2} + O(\sqrt{t})$$

The ((derivative of the Mellin transform near 0)) of $\Theta(t)$ is divergent and we can pick out the bad parts:

$$\int_\epsilon^\infty \Theta(t)\frac{dt}{t} = \# + a_0\log\epsilon + \frac{a_{-1}}{\sqrt{\epsilon}}$$

Regularization of infinite products involves doing stuff like this; just a bit old-fashioned yet lots of uncharted territory.[8]

[8] It's not hard to write an infinite product we'd be curious about (that is not just recycling the example we just did), and maybe it's important to a Math or Physics professor somewhere.

We still haven't shown $\sqrt{2} \notin \mathbb{Q}$ is irrational. ...

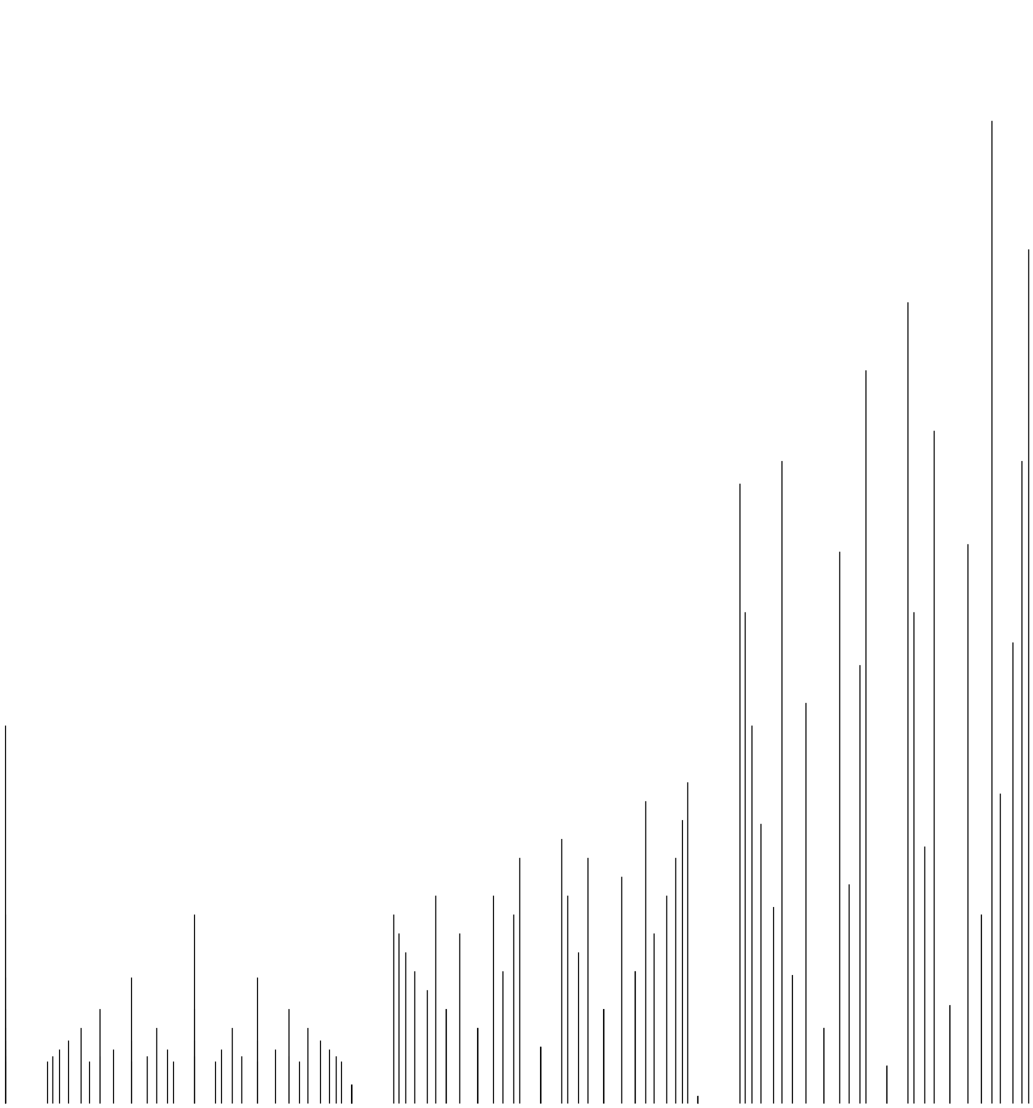
Just trying to get a sense of what this machinery is like and why it's there.

- $X \subseteq \mathbb{P}^N$ is a bunch of equations
- $\hat{X} \in \widehat{CH}(\mathbb{P}^{N-d})$ is an "arithmetic cycle" or whatever
- $\hat{c}_1(\mathcal{O}(1)) \in \widehat{CH}(\mathbb{P}^N)$ is the first arithmetic chern class
- $f : X \mapsto \operatorname{Spec}\mathbb{Z}$ is the projection map
- The Faltings height is: $h(X) = f_*(\hat{X} \cdot \hat{c}_1(\overline{\mathcal{O}(1)})^d) \in \mathbb{R} = \widehat{CH}^1(\operatorname{Spec}\mathbb{Z})$

This height in the cases where it matters will be the same as the square-root height:

$$h(X) = \log(x_0^2 + x_1^2 + \cdots + x_n^2) \text{ with } (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^{N+1}$$

at this point you are referred to the proof on page 4.



My attempt at drawing heights. My failure to draw them represent the various structural constants of the Farey fractions.

5/6 For the rest of this blog we will learn about nilsequences:

- find examples we care about such as $\{n\sqrt{2}\}$ or $\{n\sqrt{2}\lfloor n\sqrt{3}\rfloor\}$
- explain why nilsequences are natural in the context of **all** dynamical systems

In a way, we'll be like janitors since these professors don't really talk about specific dynamical systems or optimize their constants in any way. But wouldn't you like that? Just a cleaner rendition of the song already being played.

Here is the definition of nilsequence from the blog:

> A nilmanifold of step at most d is a quotient G/Γ, where G is a connected, simply connected nilpotent Lie group of step at most d (thus, all $d+1$-fold commutators vanish) and Γ is a discrete cocompact lattice in G. A basic nilsequence of degree at most d is a sequence of the form $n \mapsto F(g^n g_0 \Gamma)$, where $g_0\Gamma \in G/\Gamma$, $g \in G$, and $F : G/\Gamma \to \mathbf{C}^m$ is a continuous function.

Rearranging the text on the page is sufficient to make a clearer definition. Again:

> A nilmanifold of step at most d is a quotient G/Γ, where:
>
> - G is a connected, simply connected nilpotent Lie group of step at most d (thus, all $d+1$-fold commutators vanish) and
> - Γ is a discrete cocompact lattice in G.
>
> A basic nilsequence of degree at most d is a sequence of the form $n \mapsto F(g^n g_0 \Gamma)$, where
>
> - $g_0\Gamma \in G/\Gamma$,
> - $g \in G$, and
> - $F : G/\Gamma \to \mathbf{C}^m$ is a continuous function.

Merely having rearranged the text we have a chance at this definition.[9] if we understand the words **nilpotent**, **co-compact**, **continuous**. This last one is a bit treacherous, since "contintuity" is something we take for granted, why should it require a definition.

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

The co-compact lattice will be setting integer values instead of real numbers:

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

and the nil-manifold will be G/Γ (using the right-action). What about our contiuous function F ?

[9] Math has these air-tight defintiions that you don't see in Law or Physics or Business or anywhere. Either it is, or it isn't. We don't have a notion of "kind of true" in stead approximating with these rigid, brittle definitions. Up to you, whether we are being particular, or legalistic or no-nonsense.

Our choice of continuous function is to project onto the third coordinate:

$$\left(\begin{array}{ccc} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array}\right) \overset{\simeq}{\mapsto} (x,y,z) \overset{\text{projection}}{\mapsto} z$$

Our continuous function has to behave nicely with the group structure:

$$\left(\begin{array}{ccc} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{ccc} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc} 1 & x_1 + x_2 & y_1 + y_2 + x_1 y_2 + y_1 x_2 \\ 0 & 1 & z_1 + z_2 \\ 0 & 0 & 1 \end{array}\right)$$

This equality has the same information as just some crazy addition rule:

$$(x,y,z)\,(x,y,z) = (x_1 + x_2, x_1 y_2 + y_1 z_2, z_1 + z_2)$$

This is almost but not quite three copies of the circle $S^1 \times S^1 \times S^1$ it has been twisted by a co-cycle. At least, we can multiply by n and get some type of sequence:

$$(x,y,z)^n = (nx, ny + \binom{n}{2} xz, nz) \mapsto ny + \binom{n}{2} xz$$

and this is what arithmetic in G is like. What about modular arithmetic in G/Γ? I'm just getting to that:

...

I asked about nilmanifolds and an angry Daryl Cooper responded there are **nilmanifolds** and **solvomanifolds**, which are iterated circle bundles over the torus. I asked if all nilmanifolds could be found using matrix multiplication, as in this nice example. Experts say **no** but no examples.

Here is a nilmanifold that might not be in Terry Tao's paper. Oops, this is $S^1 \times S^1 \times S^1 \times S^1 = (S^1)^4$.

$$\left(\begin{array}{cc|cc} 1 & 0 & x & y \\ 0 & 1 & z & w \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \times \left(\begin{array}{cc|cc} 1 & 0 & x & y \\ 0 & 1 & z & w \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & x+x & y+y \\ 0 & 1 & z+z & w+w \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Experts say there is no structure theory of simply connected nilpotent Lie groups and have referred me to a book. Let me continue to make blind conjectures:

$$(x,y,z)(x,y,z) = (x+x, y+y+n\,xz, z+z)$$

Maybe if I do a commutator calculation:

$$(x,y,z)+(x,y,z)-(x,y,z)-(x,y,z) = (x+x, y+y+n\,xz, z+z)-(x+x, y+y+n\,xz, z+z) = (0, n\left(x_1 z_2 - x_2 z_1\right), 0)$$

For there to be a lack of any structure theory of these nilmanifolds, the world of bracket polynomials must be pretty wild:

$$n \in \mathbb{Z} \mapsto \left(\{n\sqrt{2}\}, \{\{n\sqrt{2}\}^2 \lfloor n\sqrt{3} \rfloor\}\right) \in \mathbb{R}^2$$

or any other example we can think of. Which nilpotent Lie group did this come from?

Let's try to build and extension. Maybe it will help to think like a topologist (an expert in geometry). What are the circle bundles over $S^1 \times S^1$? The fundamental group of the 2-torus is:

$$\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$$

I can find a bunch of circle bundles by specifying two whole numbers (m, n).

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2 + m\, x_1 z_2 + n\, x_2 z_1, z_1 + z_2)$$

Let's verify using some algeba this multiplication is associative:

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) \times (x_3, y_3, z_3)$$

If we multiply the first two

$$(x_1 + x_2, y_1 + y_2 + m\, x_1 z_2 + n\, x_2 z_1, z_1 + z_2) \times (x_3, y_3, z_3)$$

mmmm

$$(x_1 + x_2 + x_3, y_1 + y_2 + y_3 + m\, x_1 z_2 + n\, x_2 z_1 + m(x_1 + x_2) z_3 + n x_3 (z_1 + z_2), z_1 + z_2 + z_3)$$

alternatively if we multiply the second two:

$$(x_1, y_1, z_1) \times (x_2 + x_3, y_2 + y_3 + m\, x_2 z_3 + n\, x_3 z_2, z_2 + z_3)$$

and we do the other multiplication:

$$(x_1 + x_2 + x_3, y_1 + y_2 + y_3 + m\, x_2 z_3 + n\, x_3 z_2 + m x_1 (z_2 + z_3) + n(x_2 + x_3) z_1, z_1 + z_2 + z_3)$$

the Heisenberg construction goes much further than we thought. It's so hard to believe that works.

This is the same arithmetic we learned in grade school. It looks like we have added bells and whistles but we haven't.

Matrix multiplication looks bit different through. That last cross term:

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + x_2 & y_1 + y_2 + m x_1 z_2 + n x_2 z_1 \\ 0 & 1 & z_1 + z_2 \\ 0 & 0 & 1 \end{pmatrix}$$

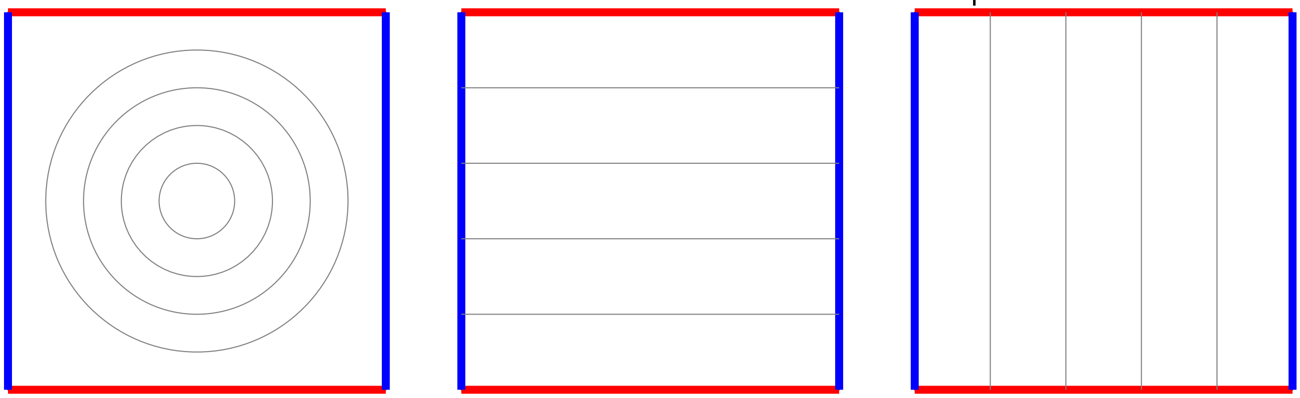
Tao's paper becomes a typographic nightmare, but we will manage. The last column:

$$\begin{pmatrix} 1 & \color{red}{\blacksquare} & \color{blue}{\blacksquare} \\ 0 & 1 & \color{green}{\blacksquare} \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \color{red}{\blacksquare} & \color{blue}{\blacksquare} \\ 0 & 1 & \color{green}{\blacksquare} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \color{red}{\blacksquare} + \color{red}{\blacksquare} & \color{blue}{\blacksquare} + \color{blue}{\blacksquare} + m \color{red}{\blacksquare}\ \color{green}{\blacksquare} + n \color{green}{\blacksquare}\ \color{red}{\blacksquare} \\ 0 & 1 & \color{green}{\blacksquare} + \color{green}{\blacksquare} \\ 0 & 0 & 1 \end{pmatrix}$$

It seems too good to be true. And that's why I sat on this project for 5 years!

5/8 This is my blog, and I defend my interest in *drawing* a nilmanifold.

What does “draw” mean? It means to find an embedding (or immersion) into the two dimensional plane (a sheet up paper). In Junior high someone gave me a science text-book and show me how a donut can be drawn in to this shape:



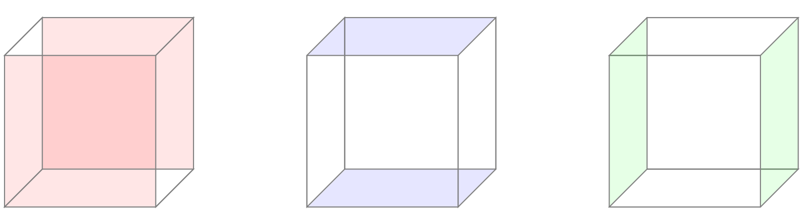
In topology textbooks, they tell you it’s a theory of shape which allows for distortion, but we were not told (nor did we ask) *how much* disortion is allowed. And the example only shows a little bit of distortion.

The torus is the arena for a lot of mathematics. Maybe it’s complicated in some way, recognizing that a shape is the torus, or mapping it to the square we just showed.

Three dimensional torus

As a data visualzation problem, we have to use coordinates and we have to choose good coordinates and nobody tells us ahead of time which coordinates are “good”. Therefore topology textbooks (if we’re lucky) are fully of drawings of squares that we mentally link together. And the worst, there are no pictures at all.

Here is 3-dimensional torus:



It is a box, where the top and the bottom of the box are the same. Or the left and the right of the box are the same. Using coordinates:

- $(x, y, z) \equiv (x + 1, y, z)$
- $(x, y, z) \equiv (x, y + 1, z)$
- $(x, y, z) \equiv (x, y, z + 1)$

For Heisenberg nilmanifold we get two of the three identificatons:

- $(x, y, z) \equiv (x, y, z) + (1, 0, 0) = (x + 1, y, z)$
- $(x, y, z) \equiv (x, y + 1, z)$ instead $(x, y, z) \equiv (x, y, z) + (0, 1, 0) = (x, y + 1, z)$
- $(x, y, z) \equiv (x, y, z) + (0, 0, 1) = (x, y, z + 1)$

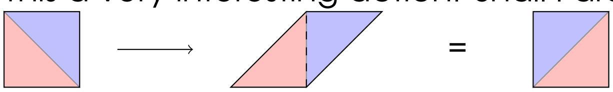
We got exactly the same identifications as before. If the rule is:

$$(x,y,z) + (x,y,z) = (x+x, y+y+xz, z+z)$$

and we want to see the action of the "shift map" the faces... the bottom face is the st

$$(x,y,0) \sim (x,y,0) + (0,0,1) = (x+0, y+0+x, 0+1) = (x, x+y, 1)$$

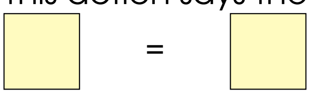
This a very interesting action. Shall I draw it for you?



The other two actions are less intresting:

$$(x,y,0) \sim (x,y,0) + (1,0,0) = (x+1, y+0+0, 0+0) = (x+1, y, 0)$$

This action says the square is equal to itself:



and the last one says that:

$$(x,y,0) \sim (x,y,0) + (0,1,0) = (x+0, y+1+0, 0+0) = (x, y+1, 0)$$

which is the same. If you are looking at the top and bottom of the box, hopefully moving left and right has the trivial action. All that is intersting is the vertical axis.

The left and right faces are: $(0,y,z)$ and the story is identical.[10] That remains is $(x,0,z)$

$$(x,0,z) \sim (x,0,z) + (1,0,0) = (x+1, 0+0+z, z+0) = (x+1, z, z)$$

and also

$$(x,0,z) \sim (x,0,z) + (0,0,1) = (x+0, 0+0+x, z+1) = (x, x, z+1)$$

and also

$$(x,0,z) \sim (x,0,z) + (0,1,0) = (x+0, 0+1+0, z+0) = (x, 1, z)$$

The first two are rather hard to picture. Moving my box around, it has gotten more slanted. I would like to say "only the third equation matters".

In really formal language, we are asking about the action of $\mathbb{Z}^3$ (or the integer Heisenberg group $\mathbb{Z}^2 \ltimes \mathbb{Z}$) on the first homology group H_1 of the cube or something like that.

Sorry I can't picture it right now.

[10] These words "trivial" and "identical" have to be watchd carefully. These words are all synonyms for "zero" and shift meaning from sentence to sentence.

Liebman shows a way to get nilsequences very quickly. (X, μ, T) is a measure-preserving dynamical system. $(h_0, \dots, h_d)$ are integers. $f_0, \dots, f_d$ are $L^\infty(X)$ are functions. Here is your nilsequence:

$$n \mapsto \int_X f_0(T^{h_0 n}x) \dots f_d(T^{h_d n}x)\, d\mu(x)$$

This is a bit abstract (even for me) so I might choose

- $h_0 = 0$, $h_1 = 1$, $h_2 = 2$, everything else $h_k = 0$ for $k > 3$
- $f_0 = f_1 = f_2 = 1_U$ with some realistic measurable set $U \subseteq X$.

The nilsequence (a bit simplified is) counting how many times a trajectory falls in a given set as well as, the same trajectory at twice it's speed:

$$n \mapsto \int_X 1_U(x)\, 1_U(T^n x)\, 1_U(T^{2n}x)\, d\mu(x) = \Big\{ x \in U \Big\} \cap \Big\{ T^n(x) \in U \Big\} \cap \Big\{ T^{2n}(x) \in U \Big\}$$

It would be really nice to make a choice of X and T (and μ).

A Let $X = S^1 = [0,1]/(0 \sim 1)$ and $U = [0, \frac{1}{2}]$ and the measure-preserving map is $x \mapsto x + \sqrt{2}$. Our nilsequence is:

$$n \mapsto \begin{cases} 1 & \text{if } \{n\sqrt{2}\} \in [0, \frac{1}{2}] \text{ and } \{2n\sqrt{2}\} \in [0, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}$$

Not a terribly exciting nilsequence but lots of stuff going on there already.

B How about the space of polynomials of degree $d \leq 3$. We could write a section $p(x) = ax^3 + bx^2 + cx + d$ and the map is $x \mapsto x + 1$, so that:

- $d \mapsto d$
- $cx \mapsto cx + c$
- $bx^2 \mapsto bx^2 + 2bx + b$
- $ax^3 \mapsto ax^3 + 3ax^2 + 3ax + a$

and if we put it together there is someting like the horocycle flow:

$$(a, b, c, d) \mapsto \big(a, (b + 3a), (3a + 2b + c), (a + b + c + d)\big)$$

The horocycle flow has to to with the map $x \mapsto x + 1$ or the 2×2 matrix map:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the shift map acts on various spaces of functions.

C Here's a bit of a magic trick example I like from Cellarosi and Sinai. Consider $x \mapsto x + 1$ acting simultanously on all $\mathbb{Z}/p^2\mathbb{Z}$.

$$\mathbb{G} = \prod_{\text{primes}} \mathbb{Z}/p^2\mathbb{Z}$$

and we're considering the map $x \mapsto x + 1$ on each factor. I'm guessing that: $\mathbb{G} \not\simeq \mathbb{Z}$, but our shift map really has to do with takeing a step $n \mapsto n + 1$ so that $3 \mapsto 4$ and $4 \mapsto 5$ and as long as we have avoided 0 for all primes p that number n is square-free.

I think we really hit on a recipe for creating (somewhat manageable) dynamical systems. Take a geometric map (even something stupid like $x \mapsto x+1$) and observe the action on a the space of functions. And this is going to be wonderful.

Yet, if we are using the inverse Gowers theorems these number sequences must be truly nasty defying all sorts of detection methods. The horocycle or geodesic flow, the morse flow – are certainly the candidate dynamical systems. Yet they look so benign, how can they be capable of generating such sequences?

Continued fractions? Addition by 1? Really? I already know where to look, and every time I forget why this is the case.

Maybe Tao is advocating the opposite approach. We have a sequence of numbers and we wish to find arithmetic sequences within. Here's a sample:

$$\left\{ x^2 + y^2 - \sqrt{2} z^2 : |x|, |y|, |z| < N \right\}$$

is equidistributed on the reals. Therefore the density is something like

$$a_N = \sum_{|x|,|y|,|z| \leq N} e^{2\pi i n \left(x^2 + y^2 - \sqrt{2} z^2 \right)} \times 1_{|x^2+y^2-\sqrt{2}z^2|<1} = 0 + o(1)$$

Using Weyl equidistribution, know the main term is zero. We can ask, how big is that that remainder term? How often is it larger than 10% of it's average? Without fail these noise / error terms will be difficult to analye and these discrepancy-type questions could invite the use of a nilsequence.

$$\left\{ N : \frac{|a_N|}{\frac{1}{N} \sum_{m<N} |a_m|} \geq 2 \right\}$$

Almost every project I have involves a main term and an error term. And I am always stuck because I can't show the error term is $\to$ zero. That's every single project I have.

5/9 I don't know if this falls into the nilsequence framework, but we can study two matrices:

$$A = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix} \text{ so that } A, B \in SO(3, \mathbb{Q})$$

How do you propose generating a dynamical system from the free group on two generators $F = \langle A, B \rangle$. I guess we can study the sequence of matrices iterating powers of:

$$T = 0.5 \times A + 0.5 \times B$$

The result is a sequence of 3×3 matrices we can punch into a calculator:

$$I, \frac{1}{2}(A+B), \frac{1}{4}(A^2 + AB + BA + B^2), \frac{1}{8}(A^3 + \dots + B^3), \text{etc.}$$

Clearly the averages will tend to the identity matrix and return where we started. It takes forever. My question is really about the result of multiplying several of these matrices at random:

$$A \times A \times A \ , \ A \times A \times B \ , \ A \times B \times A \ , \ \dots \ , \ B \times A \times A \ , \ \dots \ , \ B \times B \times B$$

The sequence of matrices T^n is like shuffling a deck of cards, and we expect convergence to randomness to happen very quickly.

This is exactly the the type of probem I don't want. Maybe it's OK. There must be a way of turning this into a sequence of numbers.

Another recipe could be to start with sequences of numbers on $\mathbb{R}$ (so they will already be ordered) and we can turn them into :

- $\{x^2 + y^2 - \sqrt{3}z^2 : |x|, |y|, |z| < N\} \subset \mathbb{R}$ for large enough N these numbers will fill up any interval.
- $\{\frac{a}{b} + \frac{c}{d}\sqrt{2} : |a|, |b|, |c|, |d| < N\}$ we know that $\overline{\mathbb{Q}} = \mathbb{R}$ the fractions are dense in $\mathbb{R}$ so if we adjoin $\sqrt{2}$ the numbers should be even more dense $\overline{\mathbb{Q}(\sqrt{2})} = \mathbb{R}$
- $\{2^k 3^l \frac{m}{N} : 0 < k, l < 3 \log N\}$ it could be any sequence of numbers we know they're going to be evenly distributed and we can ask the same question.

Really, what is being looked for are these more number-theoretic sequences of numbers. And the nilsequences will describe ever possible pattern that can occur – even ones we've never thought of

- if $p = 4k+1$ then $p = a^2 + b^2$, we want the set $\left\{p : \theta_1 < \tan^{-1}(\frac{b}{a}) < \theta_2\right\}$
- Let $\vec{v} \in S^2$, $\left\{n : n = x^2 + y^2 + z^2 \text{ and } \cos^{-1}\left(\frac{(x,y,z)}{n} \cdot \vec{v}\right) < 0.05\right\}$ integers that can be written as the sum of three squares and also point (close to) a specific direction in space

Who knows? Maybe these will have nice nilsequene properties?

Experts have already earmarked special dynamical systems (such as interval exchange transformations, horocycle flow, geodesic flow, toral automorphisms) ...

Again we pick one out of a hat. Let f, g be two Hecke eigenforms of weight k on $\mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$ they are expected to be approximately perpendicular on the line segment $[0, 1] + iy$.:

$$\int_0^1 y^k f(x+iy)\overline{g(x+iy)}dx \to 0$$

This is an open problem. There are values of f and values of g, they have an inner product. Off you go.

To remind myself, the horocycle flow contains these modular forms problems because the horocycle flow can act on space of functions. So, this geometry prolem and this analysis problem have thes same information.

And we continue to make stuff up. Let $\mathrm{PSL}(2, \mathbb{Z}[i])$ be some kind of Bianchi group. Our 3-manifold could be:

$$X = \mathrm{PSL}(2, \mathbb{Z}[i])\backslash\mathrm{PSL}(2, \mathbb{C})$$

our matrix could be conjugate to the $x \to x+1$ shifting map. And we study the closure:

$$\overline{\left\{ \left(\begin{array}{cc} 1 & s \\ 0 & 1 \end{array} \right) x : s \in \mathbb{R} \right\}} \subseteq \mathrm{PSL}(2, \mathbb{C}) \text{ or } X$$

This part of Dynamics is aggravating to me. You know examples like these are everywhere and yet you can't point to this problem or that problem and say this is what you're looking for. And it's always the one staring you in the face.

Here's a reading list. I will leave in the Class Field Theory book since even though don't need it any way (we are solving over $\mathbb{Z}$), in fact we may need it anyway.

References

(1) Nicole Berline, Ezra Getzler, Michéle Vergne. **Heat Kernels and Dirac Operators** Grundlehren der mathematischen Wissenschaften, Vol 298. (Springer, 1992)

(2) Nancy Childress **Class Field Theory** (Universitext). Spinger, 2009.

(3) Ben Green, Terence Tao, Tamar Ziegler. **An inverse theorem for the Gowers $U^{s+1}[N]$-norm** `arXiv:1009.3998`

(4) Francesco Cellarosi, Yakov G. Sinai **Ergodic Properties of Square-Free Numbers** `arXiv:1112.4691`

(5) Ben Green **Approximate algebraic structure** `arXiv:1404.0093`

(6) Curtis T. McMullen, Amir Mohammadi and Hee Oh **Horocycles in hyperbolic 3-manifolds** `http://math.harvard.edu/~ctm/papers/home/text/papers/horo/horo.pdf`

(7) W. T. Gowers **Generalizations of Fourier analysis, and how to apply them** `arXiv:1608.04127`

(8) Ho Chung Siu, Kannan Soundararajan **Restriction of Hecke eigenforms to horocycles** `arXiv:1705.02080`