Theta Functions

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$$\theta(x;p) = (x;p)_\infty (px^{-1};p)_\infty = \exp\left(- \sum_{m \neq 0} \frac{x^m}{m(1-p^m)} \right)$$

another one

$$\theta(z;q) := (z;q)_\infty (q/z;q)_\infty = \frac{1}{(q;q)_\infty} \sum_{k \in \mathbb{Z}} z^k q^{\binom{k}{2}}$$

the shifted factorials are defined by:

$$(z;q)_\infty = \prod_{i \geq 0} (1 - zq^i)$$

Let's see if

$$\binom{k}{2} = \frac{k(k-1)}{2} = \frac{k^2}{2} - \frac{k}{2}$$

Then it could be:

$$\theta(q^2;q) = \frac{1}{(q;q^2)} \sum_{k\in\mathbb{Z}} q^k q^{2\binom{k}{2}} = \frac{1}{(q;q^2)} \sum_{n\in\mathbb{Z}} q^{n^2}$$

Wikipedia has

$$\sum_{n\in\mathbb{Z}} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

and we can set $a = b = q$:

$$\sum_{n\in\mathbb{Z}} q^{n^2} = (-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty$$

This also seems odd we can try

$$\theta(q;q^2) = (q;q^2)_\infty(q;q)_\infty(q^2;q^2)_\infty = \sum_{n\in\mathbb{Z}} q^{n^2}$$

It might parameterized in terms of two angles:

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$$

which has another triple product

$$\prod_{m=1}^{\infty} (1 - e^{2\imath m i \tau}) \left[1 + e^{(2m-1)\pi i \tau + 2\pi i z}\right] \left[1 + e^{(2m-1)\pi i \tau - 2\pi i z}\right]$$

Then $q = e^{2\pi i \tau}$ and $x = e^{2\pi i z}$:

$$\theta(0;q) = \prod (1-q^2)(1+q^{2m-1})(1-q^{2m+1})$$

This is a beautiful triple product but we have to write in terms of rising and falling factorials.

$$\sum_{n\in\mathbb{Z}} q^{n^2} = (-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty$$

The exponent formula looks like

$$\log(1-x) = \sum \frac{x^m}{m}$$

and the geometric series formula:

$$\sum p^{km} = \frac{1}{1-p^m}$$

If we put two of them together it says:

$$\sum_m \sum_k \frac{1}{m} x^m p^{km} = \sum_m \frac{1}{m} \frac{x^m}{1 - p^m}$$

This is very much the logarithm in the beginning of this article.

Part II

References

(1) Taro Kimura, Vasily Pestun **Quiver elliptic W-algebras** `arXiv:1608.04651`

(2) Wikipedia "Jacobi Triple Product", "Ramanujan Theta Function"

(3) Eric M. Rains, S. Ole Warnaar **Bounded Littlewood identities** `arXiv:1506.02755`