q-Series

John D Mangual

Every day we are being slammed on arXiv with ever more complicated q-series. In order to attempt the infinite backlog today's example is **rarefied elliptic hypergeometric functions**.

They define the elliptic Gamma function:

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^jq^k}$$

for some numbers $|p|, |q| < 1$ and $z \in \mathbb{C}^*$. If that's not enough here is a second-order elliptic Gamma function:

$$\Gamma(z;p,q,t) = \prod_{j,k,l=0}^{\infty} (1 - zp^jq^kt^l)(1 - z^{-1}p^{j+1}q^{k+1}t^{l+1})$$

with $|t|, |p|, |q| < 1$. Why not stop there?

The **rarefied** elliptic hypergeometric function:

$$\left(-\frac{z}{\sqrt{pq}}\right)^{\frac{m(m-1)}{2}} \left(\frac{p}{q}\right)^{\frac{m(m-1)(2m-1)}{12}} \Gamma(zp^m; p^r, pq)\Gamma(zq^{r-m}; q^r, pq)$$

This is related to the Lens space a quotient of the 3-sphere:

$$S^3/\mathbb{Z}_r = \{|z_1|^2+|z_2|^2 = 1\}/(e^{2\pi i/r}z_1, e^{2\pi i/r}z_2) \sim (z_1, z_2)$$

and now we have mixed the topology and geometry of 3-manifolds with the classical Γ function.

Some facts about the Gamma function:

- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
- $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$
- $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2\pi}{\sqrt{3}}$

The special values formulas[1] are endless I am just getting started:

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3}\cdot\sqrt{2+\sqrt{3}}$$

References

(1) V P Spiridonov **Rarefied Ellipic Hypergeometric Fucnction** `arXiv:1609.00715`

(2) Christine Berkesch, Jens Forsgård, Mikael Passare **Euler–Mellin integrals and A-hypergeometric functions** `arXiv:1103.6273`

[1] http://mathworld.wolfram.com/GammaFunction.html