Examples: Theta Functions and Poisson Summation

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On the one hand, there are many people who know all sorts of things about theta functions. David Mumford wrote 3 volumes. They are going to my working examples of modular forms.[1]

Ex #1 Show that if $u(x)$ is a spherical harmonic of degree ℓ the generalized theta function

$$\theta(z;u) = \sum_{m\in\mathbb{Z}^3} u(m) e^{z|m|^2}$$

is a holomorphic cusp form for $\Gamma_0(4)$ of weight $3/2+\ell$ as long as $\ell \geq 1$. This will already keep me busy for some time. The coefficients of this Fourier series are averages over the sphere:

$$\frac{1}{r_3(n)} \sum_{\xi_1^2+\xi_2^2+\xi_3^2=n} u(\xi) \ll n^{-1/28}$$

Ex #2 Show Duke's bound for any exponent n^α with $\alpha < 0$. Could be $\frac{1}{28}$ or $\frac{1}{100}$... Anything.

If I knew what a $\Gamma_0(4)$ holomorphic cusp form was, I might be able to credit Goro Shimura's 1973 paper. However, he has also written some recent graduate level texts with chaptrs on half-integer weight modular forms.

The fraction $\frac{1}{28}$ falls out of difficult calculations involving Kloosterman sums, which I am not going to cite or reproduce. Therefore, I might expect these bounds to be slightly worse – if I achieve anything at all. Instead, I will emphasize elementary[2] computations.

There are a few easy cases $u(\xi) = 1$ is the spherical harmonic of degree 0 – it is excluded from discussion. In fact, $\ell \geq 1$ and $n \gg 1$ should be very large. Yet:

$$\frac{1}{r_3(n)} \sum_{\xi_1^2+\xi_2^2+\xi_3^2=n} 1 = 1$$

and $\xi(x,y,z) = x$ is an odd function. So that $x^2+y^2+z^2=n$ should have a solution (x,y,z) and $(-x,y,z)$. Therefore, any odd function such as $u(x,y,z)=x$ should have a vanishing average.

The first non-trivial case – and we're a far cry from proving Ex #1 or Ex #2 – is $u(x) = x_1^2$ (notice $u(x) = x_1x_2$ also vanishes). This is not to be understimated that we can switch signs.

$$(\pm x_1)^2 + (\pm x_2)^2 + (\pm x_3)^2 = n$$

However let's not to forget to try far more basic approaches such as the Hasse Princple.[3]

[1] There is also an excellent discussion of Theta Functions and String Theory and Riemann surfaces by the Verlinde brothers, some of which is formalized by Beauville. There's even a page or two in Feynman's **Statistical Mechanics**.

[2] Duke and Iwaniec also felt there arguments were elementary. I am sure Duke and Shimura are correct.

[3] Or the mean value theorem! Let $f:\mathbb{R}^3\to\mathbb{R}$ show that $|\frac{1}{6}\sum_{k\in\{1,2,3\}} f(x\pm dx_k) - f(x)| \ll \partial f(x)$ (this is ambiguous)

References

(1) Emil Artin and George Whaples **Axiomatic characterization of fields by the product formula for valuations** Bull. Amer. Math. Soc. Volume 51, Number 7 (1945), 469-492.