Resurgence and Square-Tiled Surfaces

John D Mangual

1 Divergent Integrals

In (1) the Authors analyze a very classical integral and find a very new result. In particular, they look at the Bessel function:

$$Z(g) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx e^{-\frac{1}{2g}\sin^2 x} = \frac{\pi}{\sqrt{g}} e^{-\frac{1}{4g}} I_0(\frac{1}{4g})$$

Although the $\sin^2 x$ in the exponent looks exotic, this integral is exactly solvable. Lots of information can be found in the treatise by Watson (2).

The three authors of (1) ask about "strong coupling" and "weak coupling" approximations to $Z(g)$ which amount to the ranges $g \ll 1$ and $g \gg 1$ (also the intermediate regime $g \sim 1$. In particular, they remark that approximations for $Z(g)$ fail for small and large values of $g \in \mathbb{C}$:

- For $g \ll 1$ and $\arg g \neq 0$
- For $g \sim 1$ interpolating between small and large g
- For $g \gg 1$ away from the real line $\mathbb{R}$.

Therefore we want approximations of the Bessel functions that work for complex values of g near $g = 0$ and $g = \frac{1}{4\xi} = \infty$.

The asympotic series for $I_0(\frac{1}{4g})$ can be found in (2), along wth many other formulas.

1.1 Weak Coupling, i.e. $g \ll 1$

How do Cherman-Koroteev-Ünsal establish continuity between the strong- and weak-coupling regimes for $g \in \mathbb{C}$? Based on physics intuition (or that $\sin x$ has two critical points) they guess a form:

$$Z(g) \stackrel{?}{=} e^{S_0} \sum a_k g^k + e^{S_1} \sum b_k g^k$$

This is only partially true. We can certainly write down series of this type that work for individual $g \in \mathbb{C}$ but do not change **holomorphically** in g. In addition, the series near $g = 0$ are badly divergent:

$$Z(g)\Big|_{g=0} = \sqrt{2\pi} \sum \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)\Gamma(\frac{1}{2})}(2g)^k = \Phi_0(g)$$

A close look shows the coffieicents diverge like $a_k \sim k!$ which is not very meaningful. Additionally there is a "non-perturtabative" contribution from the other saddle point.

$$\cdots = \sqrt{2\pi} e^{-\frac{1}{2g}} \sum \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)\Gamma(\frac{1}{2})}(-2g)^k = e^{-\frac{1}{2g}}\Phi_1(g)$$

These are related to the hypergometric functions via a Laplace transform:

$$S\Phi(g) = \int_0^\infty dt e^{-t/g} \left[\sqrt{2\pi}\; {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 2t)\right]$$

Let's show Borel summation fails exactly in the case we know how to do $\arg\theta = 0$.

$$2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; 2t\right) = \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-2t\,x)}}$$

This hypergometric function has a branch cut singularity at $t = \frac{1}{2}$ so that the $\int_{\mathbb{R}}$ does not make any sense. Instead the authors of (1) show

$$\begin{aligned}(S_{0^+} - S_{0^-})\Phi_0(g) &= \lim_{\epsilon\to 0}\frac{\sqrt{2\pi}}{g}\int_0^\infty dt e^{-t/g}\left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; 2t + i\epsilon\right) - {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; 2t - i\epsilon\right)\right] \\ &= \cdots \\ &= 2ie^{-t/2g} S_0\Phi_1(g)\end{aligned}$$

(add more info) This is related to the discontinuities[1] in the funtions ${}_2F_1$.

Our initial formula works upto introduction of **transseries parameters** which are piece-wise constant functions $\sigma_1(g), \sigma_2(g)$:

$$Z(g) = \sigma_1 e^{S_0}\sum a_k g^k + \sigma_2 e^{S_1}\sum b_k g^k$$

These locally constant functions are *not* holomorphic in g. Finally we achieve an answer:

$$Z(g) = \begin{cases} S_\theta\Phi_0(g) - ie^{\frac{1}{2g}}S_\theta\Phi_1(g) & \theta \in (0,\pi) \\ S_\theta\Phi_0(g) + ie^{\frac{1}{2g}}S_\theta\Phi_1(g) & \theta \in (-\pi, 0)\end{cases}$$

One of the transseries parameters is always constant $\sigma_1 = \mathbf{1}$ the other one jumps $\sigma_2 = \mathbf{1}\Big[\mathrm{Re}(g) > 0\Big] - \mathbf{1}\Big[\mathrm{Re}(g) < 0\Big]$.

In practice, computers cannot do infinite series. We truncate the series, and use Borel-Ecalle summation to interpolate between $g \ll 1$ and $g \gg 1$ in a manner that works throughout the complex plane $\mathbb{C}$.

[1] https://en.wikipedia.org/wiki/Pochhammer_contour

1.2 Strong Coupling

For the case of $g \gg 1$ instead try $\xi = \frac{1}{4g} \ll 1$ so we expand around $\xi = 0$.

$$Z_*(\xi) = \int_{-\pi}^{\pi} e^{\xi \cos\phi} d\phi = 2\pi I_0(\xi) = \int_C \frac{e^{-\xi x}}{\sqrt{(1+x)(1-x)}}\, dx$$

This "partition function" can be written as the integral of a cycle $\int_A \lambda$. It is also related to the arc-tangent differntiation identity:

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

This will guide us to generalizatons later on. For now we note that $Z_*(\xi) = \int_A \lambda$ is a holomorphic function of ξ. As a consequence of:

$$\frac{d}{dx}\left(e^{-\xi x}\sqrt{1-x^2}\right) = \frac{\xi x^2 - x - \xi}{\sqrt{1-x^2}} e^{-\xi x}$$

we can find a linear combination of $Z_*(\xi), Z_*'(\xi), Z_*''(\xi)$ which equals 0.

$$Z_* = aI_0(\xi) + bK_0(\xi)$$

We are ready to show the monodromy properties of the Bessel functions around $\xi = 0$.

$$K_0(e^{\pi i}\xi) = f(\xi)\log\xi - \pi i f(\xi) = K_0(\xi) - \pi i\, I_0(\xi)$$

This identity of Bessel functions lifts to relationships between the cycles. As ξ moves around the origin:

$$Z_A \mapsto Z_A - 2Z_B$$

This is reflecting a similar relation to the Lefschetz thimbles above, because they are related by Laplace transform. This machinery is older than the previous section and is related to Gauss-Manin connection.

2 Hypergeometric Functions

The theory of hypergeometric functions, Bessel functions, Fuchsian differential equations are rich and classical topics with many exapmles in the spirit of the example of resurgence in (1). We show in many cases, it is possible to find similar interpolations between strong and weak coupling using the same wall-crossing type analysis. Period integral relationships appear in numerous places, such as the work of Fricke-Klein[2] and always have relations to Moduli space and Hyperbolic geometry. Let's see if we can get any resurgence phenomena from there.

[2] `http://math.stackexchange.com/a/1276043/4997`

The basic input for the theory Cherman-Koroteev-Unsal is a hypergeometric function.

$$I(b_1,\dots,b_n)=\int\frac{dx}{(x-b_1)^{\mu_1}\dots(x-b_n)^{\mu_n}}$$

This function is a **period** of a multi-valued 1-form and is hypergeometric in all variables b_i, as studied by Curtis McMullen (3). Another example are the hypergeometric functions studied by Beukers.

$${}_2F_1(a,b,c;t)=\int_0^\infty\int_0^\infty\frac{z^a w^b}{\left[1+z+w+(1-t)\,zw\right]^c}\frac{dz}{z}\wedge\frac{dw}{w}$$

Surprisingly this is the same as ${}_2F_1$ in the previous section[3]; it's dangerous to get carried away, so we shall only stick with these two.

Our goal is to understand if the resurgence phenomena discuss above has analogus for these new classes of hypergeometric functions, and to understand better the role of hyperbolic lattices.

What do we do with a hypergeometric function once we found it?

$${}_2F_1(\frac{1}{2},\frac{1}{2},1;2t+i\epsilon)-{}_2F_1(\frac{1}{2},\frac{1}{2},1;2t-i\epsilon)=\begin{cases}2\times{}_2F_1(\frac{1}{2},\frac{1}{2},1;2t)+O(\epsilon) & \text{if } 0<t<\frac{1}{2}\\ O(\epsilon) & \text{if } \frac{1}{2}<t<\infty\end{cases}$$

How do we know this? Let's use the definiton as contour integral:

$$\begin{aligned}\lim_{\epsilon\to 0}\ {}_2F_1(\frac{1}{2},\frac{1}{2},1;2t+i\epsilon) &= \lim_{\epsilon\to 0}\int_0^1\frac{dx}{\sqrt{x(1-x)(1-(2t\pm i\epsilon)\,x)}}\\ &= \pm\int_0^1\frac{dx}{\sqrt{x(1-x)(1-2t\,x)}}\\ &= \pm{}_2F_1(\frac{1}{2},\frac{1}{2},1;2t)\end{aligned}$$

It's important to notice the (analytic continuation of) the ${}_2F_1$ changes sign across the real axis. This explains why $\int_0^\infty$ can be replaced with $\int_{\frac{1}{2}}^\infty$ in this case. Next

$${}_2F_1(\frac{1}{2},\frac{1}{2},1;2t)={}_2F_1(\frac{1}{2},\frac{1}{2},1;1-2t)$$

This can be affected by change of variables $x\mapsto 1-x$ in the integral definining ${}_2F_1$

The last thing I want to verify is the "corrected" expansion, in terms of Bessel and Hypergeometric functions. the original formula

$$Z(g)=S_0\Phi_0(g)\mp ie^{-\frac{1}{2g}}S_{0\pm}\Phi_1(g)$$

[3] In fact ${}_2F_1(\frac{1}{2},\frac{1}{2},1;t)=\frac{1}{\pi^2}\int_0^1\int_0^1\frac{1}{1-t\,xy}\,\frac{dx}{\sqrt{x(1-x)}}\wedge\frac{dy}{\sqrt{y(1-y)}}$ (this needs to be cheked)

subsituting the original definitions we get some other integral formula to check

$$\frac{\pi}{g}e^{-\frac{1}{4g}}I_0(\frac{1}{4g}) = \lim_{\epsilon\to 0}\frac{\sqrt{2\pi}}{g}\int_0^\infty dt\, e^{-t/g}\left[{}_2F_1(\frac{1}{2},\frac{1}{2},1;2t+i\epsilon) \mp ie^{-\frac{1}{2g}}{}_2F_1(\frac{1}{2},\frac{1}{2},1;-2t+i\epsilon)\right]$$

It's hard to tell off the bad if it's true as written or needs to be fixed. This is schematic.

If we did it the "strong coupling" or "holomorphic" way...

$$Z_*(\xi) = \int_C \frac{e^{-\xi x}}{\sqrt{(1-x)(1+x)}}\,dx \approx e^{\xi}\int_C \frac{dx}{\sqrt{(1-x)(1+x)(1-2\xi x)}} + e^{-\xi}\int_C \frac{dx}{\sqrt{(1-x)(1+x)(1+2\xi x)}}$$

These poor heuristics look oddly like the hypergeometric laplace transform we just wrote.

In either case, we come short of our goal: a machine that takes in hypergeometric functions and churns out Bessel functions and vice versa. Obviousy these should be related to the Laplace transform or Borel summation.

One more attempt. Like an engineer we write out the formula:

$$\frac{\sqrt{2\pi}}{g}\int_0^\infty dt\, e^{-t/g}\int_0^1 \frac{dx}{\sqrt{x(1-x)}}\left[\frac{1}{\sqrt{1-(2t+i\epsilon)x}} - \frac{ie^{-\frac{1}{2g}}}{\sqrt{1-(-2t+i\epsilon)x}}\right]$$

Let's change the order of integration. The inner integral is Laplace transform.

$$\frac{\sqrt{2\pi}}{g}\int_0^1 \frac{dx}{\sqrt{x(1-x)}}\int_0^\infty dt\, e^{-t/g}\left[\frac{1}{\sqrt{1-(2t+i\epsilon)x}} - \frac{ie^{-\frac{1}{2g}}}{\sqrt{1-(-2t+i\epsilon)x}}\right] \propto \int_0^1 \frac{e^{-\frac{x}{2g}}\,dx}{\sqrt{x(1-x)}}$$

Even though change of order is *unjustified* we do it anyway and get the formula.

2.1 Asymptotic Series

The book of Watson and Churchill does mention the asymptotic series. In Chapter 7 we get all Bessel functions:

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\sum_{m=0}^\infty \frac{(-1)^m\binom{\nu}{m}}{(2z)^m} + e^{(\nu+\frac{1}{2})\pi i}\frac{e^{-z}}{\sqrt{2\pi z}}\sum_{m=0}^\infty \frac{\binom{\nu}{m}}{(2z)^m}$$

Although the book makes no claim on how the series *converge*. Let's provide one more derivation:

$$\int_0^\infty dt\, e^{-tz}\int_0^1 \frac{dz}{\sqrt{z(1-z)}}\left[\frac{1}{\sqrt{1-(2t+i\epsilon)z}} + \frac{ie^{-z/2}}{\sqrt{1-(-2t+i\epsilon)z}}\right]$$

It's not obvious the 1st and 2nd terms are complex conjugates. First we switch order of integration:

$$\int_0^1 \frac{dz}{\sqrt{z(1-z)}} \int_0^\infty dt\, e^{-tz} \left[\frac{1}{\sqrt{1-(2t+i\epsilon)z}} + \frac{ie^{-z/2}}{\sqrt{1-(-2t+i\epsilon)z}} \right]$$

The inner two terms should then be what we are looking for, but we don't have:

$$\int_0^\infty dt\, e^{-tz} \left[\frac{1}{\sqrt{1-(2t+i\epsilon)z}} + \frac{ie^{-z/2}}{\sqrt{1-(-2t+i\epsilon)z}} \right] = e^{-z}$$

Instead we go back to first step:

$$\int_0^1 \frac{dz}{\sqrt{z(1-z)}} \left[\frac{ie^{-z/2}}{\sqrt{1-(-2t+i\epsilon)z}} \right] = \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \left[\frac{ie^{-z/2}}{\sqrt{1-(1+2t-i\epsilon)z}} \right]$$

Adding back the original measure, it equals

$$\int_0^1 \frac{dz}{\sqrt{z(1-z)}} \int_0^\infty dt\, e^{-tz} \left[\frac{1}{\sqrt{1-(2t+i\epsilon)z}} + \frac{ie^{-z/2}}{\sqrt{1-(1+2t-i\epsilon)z}} \right]$$

Then we shift only the second term by one half:

$$\int_0^1 \frac{dz}{\sqrt{z(1-z)}} \int_0^\infty dt\, e^{-tz} \left[\frac{1}{\sqrt{1-(2t+i\epsilon)z}} + \frac{i}{\sqrt{1-(2t-i\epsilon)z}} \right]$$

Now the first and second term are genuinely complex conjugates. The integrand is zero for $t < \frac{1}{2}$.

$$\int_0^1 \frac{e^{-z/2}dz}{\sqrt{z(1-z)}}$$

This is now equal to the Bessel function, $I_0(z)$. It takes a while and has to do wit the Riemann surface corresponding to $w = \sqrt{z(1-z)}$ which is a sphere, S^2.

2.2 Other Riemann Surfaces

It is also no coincidence that $\frac{1}{2} + \frac{1}{2} + 1 = 2$, represnting the total branching. This was explained by Deligne and Mostow **– add more –**

3 Physics

We d not discuss matrix integrals at this time.

References

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