Homework: Fourier Transform

Summability kernel on the real line is a family of continous functions $\{k_\lambda\}$ on $\mathbb{R}$ (either discrete or continous parameter) λ doing three things:

- $\int k_\lambda(x)\,dx = 1$
- $||k_\lambda||_{L^1(\mathbb{R})} = O(1)$ as $\lambda \to \infty$
- $\lim_{\lambda\to\infty} \int_{|x|>\delta} dx = 0$ for all $\delta > 0$

These are formal ways of writing Dirac's δ function.

Fejér Kernel: $\mathbf{K}_\lambda(x) = \lambda \mathbf{K}(\lambda x)$ (rescaling) for $\lambda > 0$. with:

$$\mathbf{K}(x) = \frac{1}{2\pi}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right) = \frac{1}{2\pi}\int_{-1}^{1}(1-|\xi|)e^{i\xi x}\,d\xi$$

Thm Let $f \in L^1(\mathbb{R})$ and let $\{k_\lambda\}$ be a summability kernel of $\mathbb{R}$ then:

$$\lim_{\lambda\to\infty} ||f - k_\lambda * f||_{L^1(\mathbb{R}} = 0$$

Proof [*Repeats the proof of two previous arguments in Chapter I on summability kernels on $L^2(S^1)$ (Fourier analyss on the circle.).*]

Thm Let $f \in L^1(\mathbb{R})$ then

$$f = \lim_{\lambda\to\infty} \frac{1}{2\pi}\int_{-\lambda}^{\lambda}\left(1 - \frac{|\xi|}{\lambda}\hat{f}(\xi)\,e^{i\xi x}\,d\xi\right)$$

in the $L^1(\mathbb{R})$ norm.

Corollary ("uniqueness" theorem) Let $f \in L^1(\mathbb{R})$ and assume $\hat{f}(\xi) = 0$ for all $\xi \in \hat{\mathbb{R}}$ then $f \equiv 0$.

Thm The functions with compactly carried Fourier transform sform a dense subspace of $L^1(\mathbb{R})$.

All of these theorems exist and are correct and yet they are also templates.

Q What were we adding to get the term "summability kernel"?

Functions in $L^1(\mathbb{R})$ contain a lot of information (e.g. all MP3 music files could be modeled by such equations. All TV shows could be modeled by elements of $L^2(\mathbb{R}^2)$. We could have to explain how our model connects to the real world.

References

[1] Yithak Katznelson. **Introduction of Harmonic Analysis** Dover, 1968.