Scratchwork: Symmetric Polynomials

The quaternions are a number system defined by three rules of multiplication. They generalize complex numbers:

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \text{ and } \mathbf{i} \times \mathbf{j} = \mathbf{k} \text{ and } \mathbf{i} \times \mathbf{i} = -\mathbf{1}$$

These multiplications can we completed to form a group of order 8.

$\times$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$	$-\mathbf{j}$	$-\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$	$-\mathbf{j}$	$-\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{1}$	$-\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$	$-\mathbf{j}$	$\mathbf{k}$	$\mathbf{1}$	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{1}$	$-\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$
$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{i}$	$-\mathbf{j}$	$-\mathbf{k}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$-\mathbf{i}$	$-\mathbf{i}$	$\mathbf{1}$	$-\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$	$-\mathbf{k}$	$-\mathbf{j}$
$-\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$\mathbf{1}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$-\mathbf{k}$	$-\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{1}$

It looks like there are eight things being multiplied, so we made an 8×8 table. There are eight things being permuted in 8 possible ways:

$$\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}, -\mathbf{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\}$$

It may even be possible to whittle this down to four - with the inclusion of a minus sign $(-\mathbf{1})$.

$$\begin{aligned} -\mathbf{1} \times \mathbf{1} &= -\mathbf{1} \\ -\mathbf{1} \times \mathbf{i} &= -\mathbf{i} \\ -\mathbf{1} \times \mathbf{j} &= -\mathbf{j} \\ -\mathbf{1} \times \mathbf{k} &= -\mathbf{k} \end{aligned}$$

Cayley's Theorem says every group can be placed into a permutation group. We could call the elements of this group $\{1, 2, \ldots, 8\}$.

$$\mathbf{1} \mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right]$$

and now we replace with different rows of the multiplication table:

$$\begin{aligned} \mathbf{i} &\mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 4 & 7 & 6 & 1 & 8 & 3 \end{array} \right] \\ \mathbf{j} &\mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 5 & 2 & 7 & 4 & 1 & 6 \end{array} \right] \\ \mathbf{k} &\mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 6 & 1 & 8 & 3 & 2 & 5 \end{array} \right] \end{aligned}$$

The rule for $(-\mathbf{1})$ looks a little bit complicated. For the time being switch the first and second half.

$$\mathbf{1} \mapsto \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{array} \right]$$

There's even other ways of representing the quaternion group. Here's the more usual 2×2 matrices (in case you're scared of Quaternion objects).

$$\mathbf{1} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ \mathbf{i} \mapsto \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \ \mathbf{j} \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ \mathbf{k} \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

It could even be instructive to write out the full 8×8 matrices:

$$\mathbf{1} \to \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

That one was not too informative let's try the other three.

$$\mathbf{i} \to \left[\begin{array}{cccc|cccc} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \text{ and } \mathbf{j} \to \left[\begin{array}{cccc|cccc} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{array}\right] \text{ and } \mathbf{k} \to \left[\begin{array}{cccc|cccc} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{array}\right]$$

Do we lose any information by writing them as 4×4 matrices?

$$\mathbf{1} \to \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \text{ and } -\mathbf{1} \to \begin{bmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix} \text{ and } \mathbf{i} \to \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{bmatrix} \text{ and } \mathbf{j} \to \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

So we've now found three different representations of the quaternion algbra as matrices of various sizes 2×2 and 4×4 and 8×8. It seems like we can keep going...

References

[1] ...