Scratchwork: Decimals

At this point in my mathematical training, I take for granted that $\mathbb{R}$ is the number system that we all use. That $\mathbb{R}^2$ is the Euclidean plane. In order to represent numbers in $\mathbb{R}$ we should use decimals. Yet, when we solve equations we use Taylor series expansions or Fourier expansions or something less common. And finally, we turn our answer into a decimal representation of a number in $\mathbb{R}$.

We write in the decimal system base 10. We spend a few years learning a few idosyncracies of these basic operaions. Even type-setting decimal addiion and multiplication can be a chore.

$$\begin{array}{cccc} & 4 & {}^{1}2 & 3 \\ + & 7 & 5 & 8 \\ \hline 1 & 1 & 8 & 1 \end{array}$$

A modern *dynamical systems* view point is that we are studying the dynamics of the map $T : a \to (10 \times a)\%1$ on the real number line $\mathbb{R}/\mathbb{Z}$. We multiply by 10 and then remove the integer part. This requires an input the definition of a function: $f(x) = x\%1$ or sometimes written $\{x\}$ with some bit of fastidiousness

$$f(x) = \{x\} \stackrel{?}{=} \min_{n \in \mathbb{Z}} |x - n|$$

This definition is wrong since it returns $\{\frac{5}{4}\} = \frac{1}{4}$ but also $\{\frac{7}{4}\} = \frac{1}{4}$, since $\frac{7}{4} - 2 = -\frac{1}{4}$. So a careful definition of $f(x)$ is missing.

Exercise Find a correct definition of $f(x) = \{x\}$.

At the moment all I have is this annoying defintion: min A with $A = \{x - n : n \in \mathbb{Z}$ and $x > n\}$. And later we could ask what "$a > b$" even means? And "$n \in A$"? At some point we'll become too lazy to even check.

An insepction of the properties of $\mathbb{R}$ like this happens when we get stuck. In addition to base $b = 10$ we could have binary $b = 2$ with digits $\{0, 1\}$, so that $15_{10} = 1111_2$. There is even base systems for irrational base, so we could have silver ratio base $b = 1 + \sqrt{2}$ or Golden ratio base $b = \frac{1+\sqrt{5}}{2}$.

The map $a \mapsto \big[(1 + \sqrt{2}) \times a\big]\%1$ would have two outcomes:

- $0 < (1 + \sqrt{2}) \times a < 1$ so that $0 < a < \sqrt{2} - 1$.
- $1 < (1 + \sqrt{2}) \times a < 2$ so that $\sqrt{2} - 1 < a < 1$.

Then we have a partition of $[0, 1)$ that behaves nicely under the dynamical system T just described. We have a binary decimal system with digits $\{0, 1\}$ just as before but with some unusual properties. So what exceptional number shall we give? Let's try 3:

$$1 + \sqrt{2} < \mathbf{3} < (1 + \sqrt{2})^2 = 1 + 2 + 2 \times \sqrt{2} = 3 + 2\sqrt{2}$$

So this number would have two digits before the decimal place, $3 = 1\_.\_\_\_\ldots$ I'm not even sure if the decimal terminates.The next digit would be:

$$3 - (1+\sqrt{2}) = 2 - \sqrt{2} \overset{?}{<} 1 + \sqrt{2}$$

and we'd like to do this without peeking... without reverting to the decimal system, as any calculator does.[1]

Ex Using Pythagoras theorem we can find that $5^2 + 12^2 = 13^2$ what happens if we write them out in decimals. First as fractions: $(\frac{5}{13})^2 + (\frac{12}{13})^2 = 1$. Then let's try out the decimals:

- $\frac{5}{13} = 0.\overline{384615}_{\,10} = \frac{384615}{9999999}$ (this fraction is exact)
- $\frac{12}{13} = 0.\overline{923076}_{\,10} = \frac{923076}{9999999}$

There's a long division problem using $\div$ if we try to find the repeating decimal.

The fraction equality is *exact* $5 \times (10^7 - 1) = 13 \times 384615$. Also if we were to use decimals:

$$(0.384615)^2 + (0.923076)^2 = 0.999998000001 \approx 1 - 10^{-6}$$

The first equality might be exact. Do we really wish to use the shift-map $T : x \mapsto 10 \times x$ here?

This could motivate us to find result such as Fermat's Little Theorem, that $p \,\big|\, a^p - a$ or now we have a dynamical system $T : b \mapsto a \times b$, and now it says that T^p as a fixed point or that $T^p - T$ has a non-trivial kernel (in Linear Algebra-speak). E.g.

$$\left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right]^p - \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right] \equiv 0 \pmod{p}$$

Is that correct? If we diagonalize this matrix we get two algebraic numbers, $x^2 - (1+4)x + (1 \times 4 - 2 \times 3) = 0$ giving $x = \frac{5 \pm \sqrt{33}}{2}$. Then we are asking if 7 "divides" $(\frac{5\pm\sqrt{33}}{2})^7 - (\frac{5\pm\sqrt{33}}{2})$ and things like that.

Our use of the quadratic formula starts to look bad. Our symbol $\sqrt{x}$ means $f^{-1}(x)$ with $f(x) = x^2$. The notation $\sqrt{a}$ is the thing that solves $x^2 = a$. We are looking for the number that solves $x^2 - 5x - 2 = 0$.

Note Why do we need to inspect $\mathbb{R}$ so carefully? Tayor's Theorem is going to call for infinitely many steps involvin $+$ and $-$ and $\times$ and $\div$:

$$f(x+\epsilon) = f(x) + \epsilon \times f'(x) + \frac{\epsilon^2}{2} \times f''(x) + \ldots$$

This is also a dynamical system. Trivially, $T : x \mapsto x + 1$ so that $T^\epsilon f(x) = f(x + \epsilon)$, so we are moving the thing slightly to the left.

References

[1] Michael Coornaert. **Topological Dimension and Dynamical Systems** (Universitext) Springer, 2015.

[2] Michael Field. **Essential Real Analysis** (Springer Undergraduate Texts in Analysis) Springer, 2017.

[3] Manfred Einsiedler, Thomas Ward. **Ergodic Theory: with a view towards Number Theory** GTM #259 Springer, 2011.

[4] Steve Smale. **The Fundamental Theorem of Algebra and Complexity Theory** Bulletin of the American Matematical Society, Vol. 4 No. (1) 1-36.

[1] And at some point we could inspect the how our calculators implement the decimal system.