Proposal: Factorial

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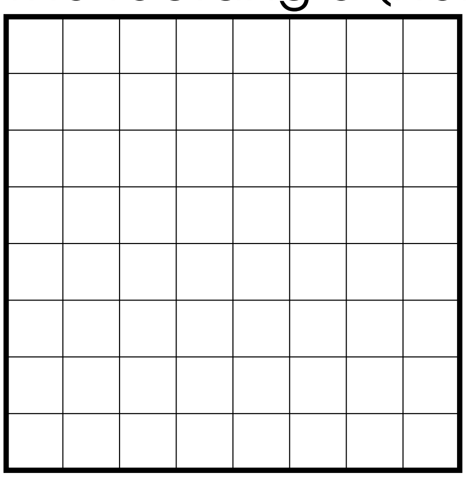
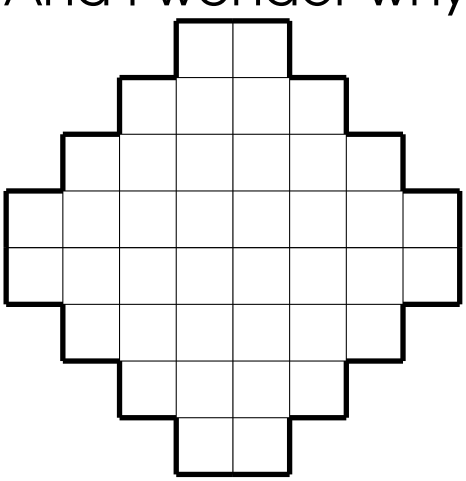
Some clever person turned the theory domino tilings into a fundamental object of mathematics and of nature. For a long time there were really two shapes being studied.

The rectangle (here an 8×8 square):



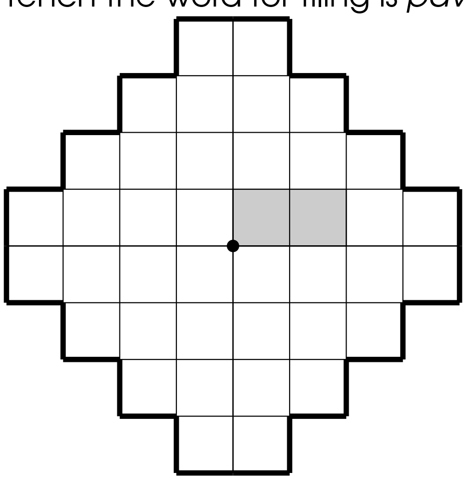
And I wonder why this particular shape is so essential:



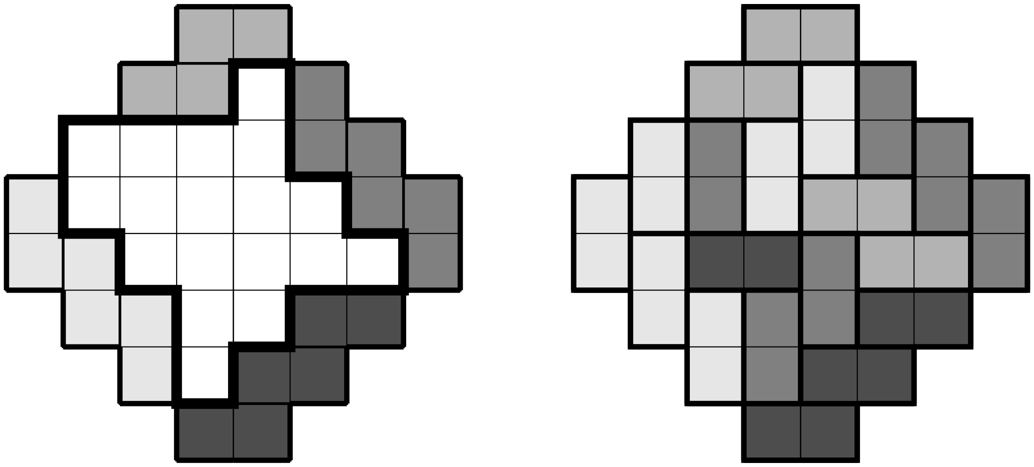
Pathetic Tutorial:

```
\begin{tikzpicture}[scale=0.75]

\foreach \a in {0,...,3}{

\draw[line width=2] (\a  ,4-\a)--(\a+1, 4-\a  );
\draw[line width=2] (\a+1,4-\a)--(\a+1, 4-\a-1);

\draw (   \a, 4-\a)--(   \a, \a-4);
\draw (-1*\a, 4-\a)--(-1*\a, \a-4);

\draw ( 4-\a,    \a)--(\a-4,    \a);
\draw ( 4-\a, -1*\a)--(\a-4, -1*\a);

\def \b {-1}
\def \c { 1}

\draw[line width=2] (\b*\a     ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a     );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

\def \b { 1}
\def \c {-1}

\draw[line width=2] (\b*\a     ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a     );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

\def \b {-1}
\def \c {-1}

\draw[line width=2] (\b*\a     ,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a     );
\draw[line width=2] (\b*\a+\b*1,\c*4-\c*\a)--(\b*\a+\b*1, \c*4-\c*\a-\c*1);

}

\end{tikzpicture}
```

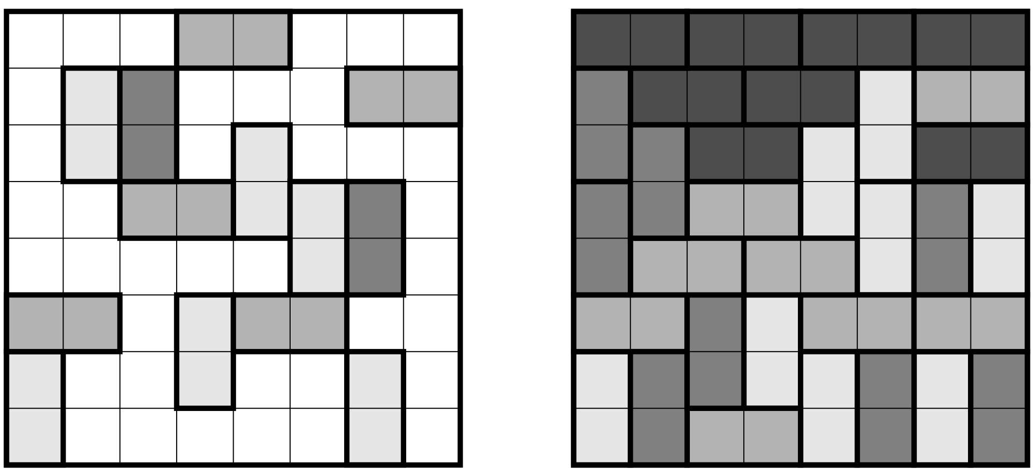
In French the word for tiling is *pavage* – so literally we are **paving** the shapes with dominoes.



Let's put two reasonable tilings on the board. One strategy for this shape is to start from the corners and work indwards. And in this case we get lucky: it always works.



And for rectangle the case is even clearer. There are no intermediate stages. I mean, if you put enough tiles down there can be come question as to whether you put yourself in a corner yet.



There is also the lovely John Conway game of "Domineering" which is not related but you also place dominoes on a checkerboard. See **Winning Ways for Your Mathematical Plays** (Vol I).

Exercise Fill the rest of the tiling.

Answer There is no solution, but if you move the tiles slightly (actually quite a bit here...) you can get a problem with an answer.

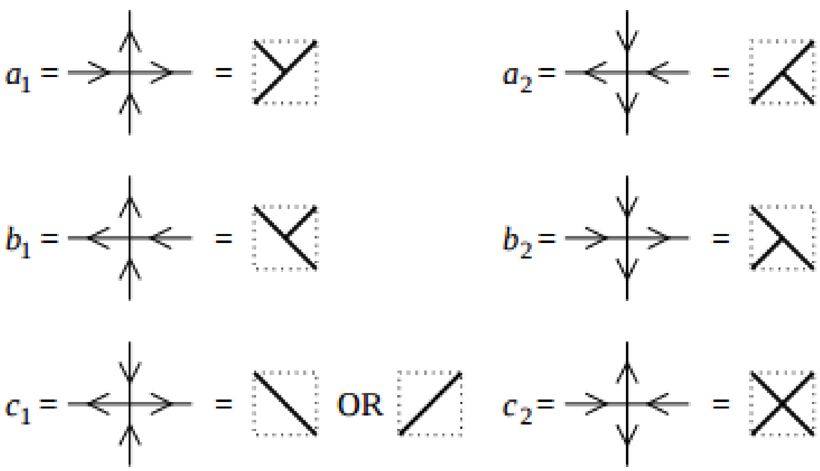
The number of tilings on the top looks atypical – just purely on a hunch. Where did that hunch come from? Is it right?

As with all hunches, it can be proven with hundreds of pages of equations this tiling is not likely to appear in a rectangles. We'll settle for somewhat simpler patterns.

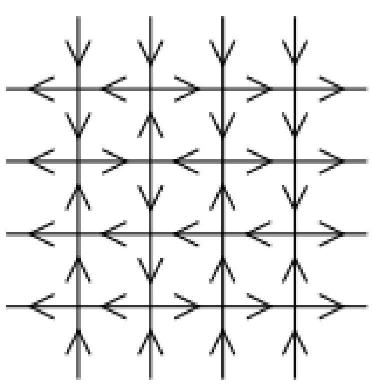
Next I hav to address some redundancy. Every couple of years, it seems there were more and more people who came up with a similar theory and did not talk to each other.

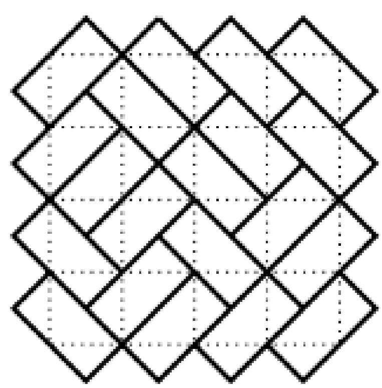
- **domino tilings**
- dimer model
- perfect matchings
- **six-vertex model**
- Alternating Sign Matrix (**ASM**)
- XXZ spin chain
- Ising model

Domino tilings and six-vertex are closely related. You can see the family resemblance.



For every 6-vertex configuration there is a domino tiling, up to various factors of $\times 2$.





The Aztec Diamond is a natural shape because it **is** the square lattice.

Problem Session Let us compute some square-ice and domino tiling partition functions!

References

(1) Paul Zinn-Justin **Six-Vertex Model with Domain Wall Boundary Conditions and One-Matrix Model** `arXiv:math-ph/0005008`

(2) Noam Elkies, Greg Kuperberg, Michael Larsen, James Propp
Alternating-Sign Matrices and Domino Tilings (Part I)
Journal of Algebraic Combinatorics (1992) 1: 111. `doi:10.1023/A:1022420103267`

(3) Patrick Ferrari, Herbert Spohn **Domino tilings and the six-vertex model at its free fermion point** `arXiv:cond-mat/0605406`

(4) Sunil Chhita, Kurt Johansson **Domino statistics of the two-periodic Aztec diamond** `arXiv:1410.2385`

Big Idea

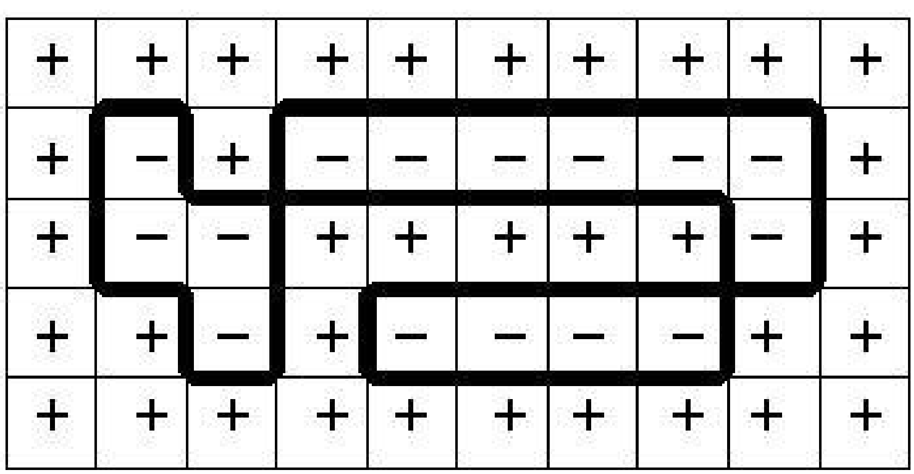
The link between the domino tilings and the ising model is "well known" but I can never find out all the details I want. And if I ask the question, it doesn't have any merit.

Who are we to believe?

There are actually *two* Ising models. The statistical Ising model and the Ising Conformal Field Theories.

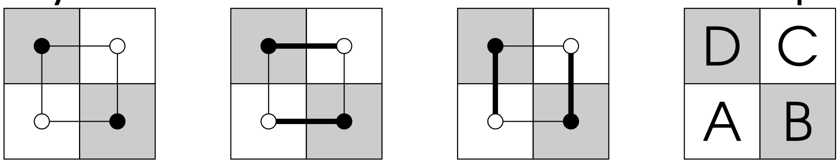
The bad news is, statistical models – or maybe, **lattice gauge theories**. Here's a simple question: are domino tilings an instance of lattice gauge theory? The answer is likely "yes" but all of the responses are wanting.

The domino tilings like I have been showing you? Which lattice gauge theory was that? Domino tilings is very simple, lattice gauge theory rather involved.



The Ising model is even more simple-minded than the domino tilings. How could it be so advanced?

One way to count the domino tilings is through the Kasteleyn determinant. In the 2×2 square, there are 4 vertices:



There should be two tilings. Let's make sure Kateleyn's method returns the answer **2**. He would draw a matrix:

$$Pf \begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \stackrel{?}{=} Det \begin{array}{cc} & \begin{matrix} C & D \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

Since A is next to B, the entry A-B is 1. B and D are not next to each other. And then we compute not the Determinant, but the Pfaffian.

Hopefully the value of the Pfaffian is 2. The value of that determinant is 0. I guess I have to put a minus sign.

$$Pf \begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} = Det \begin{array}{cc} & \begin{matrix} C & D \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{array} = 2$$

We are just making it up as we go along and this is the Kasteleyn orientation.

Without knowing why, this is just what we have to do in order to make sense, and this is called an "anomaly".

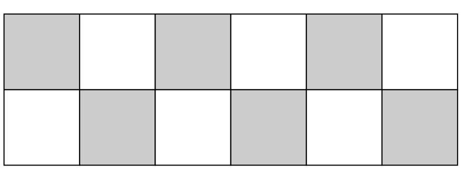
There should be ways to "glue" the patches together and the number of domino tilings will behave in an organized way.

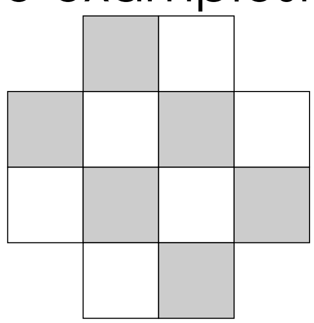
As for all this Ising model stuff. As long as we know the ising model and the domino tilings are the same theory.

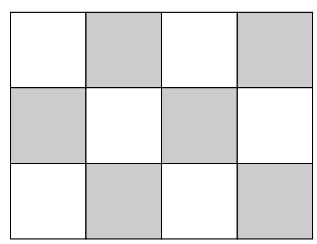
The ising model has varied literature[1] - all over the place:

- statistical mechanics
- integrable systems
- TQFT
- lattice gauge theory
- Chern Simons Theory

and yet we only except one tractable model, the domino tilings. We need more examples:

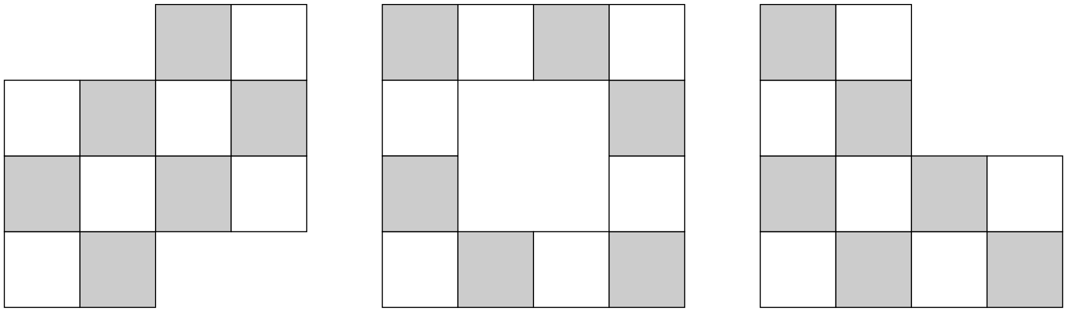






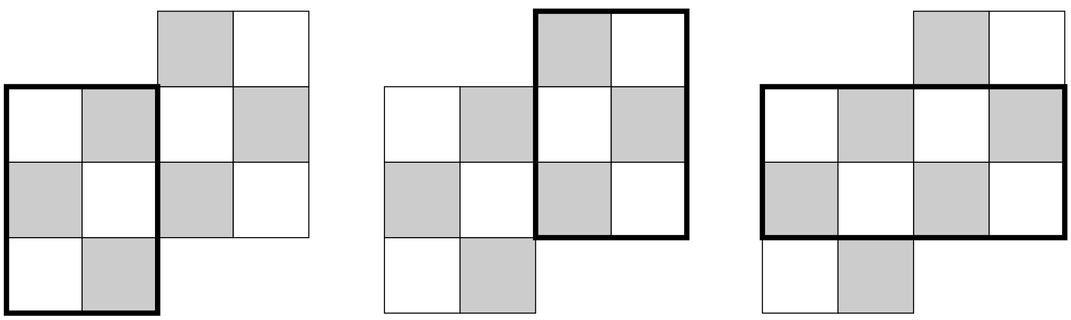
Can we glue partial answers together into a big picture? You can. In a very complicated way.

[1] The dilemma these are 4 very complicated topics, none of which I know very well, and we are trying to make a comparison. Namely, the ising model literature is all over the place, there should be variants of the domino tilings as well, where the answers still make a lot of sense.



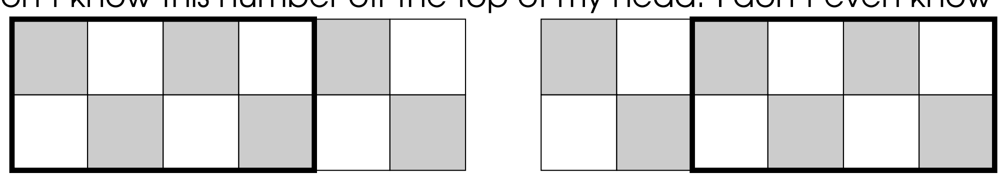
An aside for the grown-ups. The Kasteleyn determinants are put into a computer and with some effort the correct number will be returned. In a sense, we have already used a tiny bit of cohomology theory.

- All of these shapes are homeomorphic ("the same as") $\mathbb{D}$ (all the rectangles are homomeomorphic. The L-shapes are homeomorphic to rectangles). This is a really myopic way of doing things and obviously loses important information.
- Fiddling with these shapes is essential. The "new" exercise here is to use the patchwork technique to check how the overlapping works.

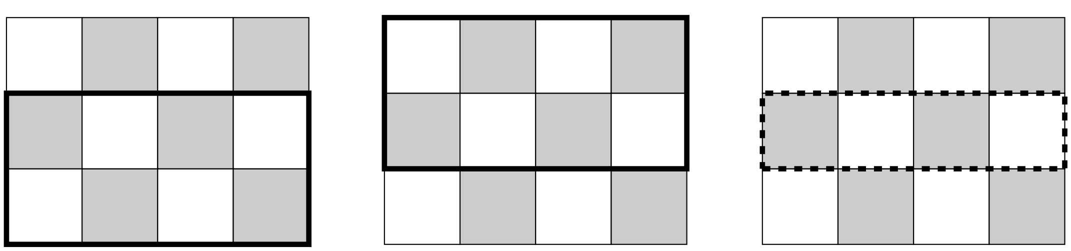


A 2×3 rectangle can be paved 3 ways by dominoes. And a 2×4 rectangle can be paved 5 ways. And a 2×2 rectangle can be paved 2 ways. And we could try to build the whole shape.

I don't know this number off the top of my head. I don't even know the 3×4 rectangle.



So again $Z(2\times4) = 5$ and $Z(2\times2) = 2$ and we should patch these together to obtain $Z(2\times6) = 13$. Is that right?



The $3 \times n$ or $4 \times n$ cases deserve attention and could be pedagogical. We will have more observables to think about soon. And one should already be thinking about whether domino tilings are the only cases we can reasonably solve. That answer might be yes . . .

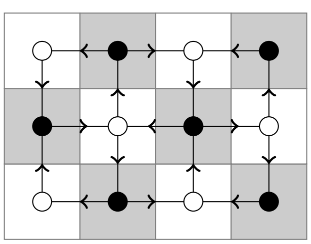
Let's try to evaluate the 3×4 case. The Kasteleyn matrix should be 6×6:

B	D	E	A
F	C	C	F
A	E	D	B

I do **not** like matrix notation because the indices are themselves two-dimensionals A-,B-,C-,D-,E-,F- have information all around them.

$$\det \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = 0$$

This is just like last time. Determinant is zero[2]. Except I am not sure where to put the signs $+1$ and -1. If I recall the orentation comes from picking one of the tilings.

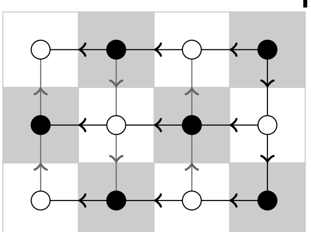


It was pretty exhausting to type this. Because there is a lot of structure that goes into drawing a very simple shape.

[2]This could be that rectangle has trivial homology. There is rel

$$\det \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} = 2 \neq 11$$

Time to read a bit more on Kasteleyn orientations. OK It's done with a spanning tree.



B	D	E	A
F	C	C	F
A	E	D	B

As you move around the square, there should be either

- 3 $\leftarrow$, 1 $\rightarrow$
- 1 $\leftarrow$, 3 $\rightarrow$

and in some cases these are called "IRF" models (interaction around face). I also have a round face.

$$\det \begin{bmatrix} 0 & 0 & 0 & 0 & +1 & +1 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & +1 & -1 & +1 & -1 \\ 0 & +1 & +1 & 0 & -1 & 0 \\ +1 & 0 & -1 & -1 & 0 & 0 \\ -1 & +1 & -1 & 0 & 0 & 0 \end{bmatrix} = 11$$

$\square$

References

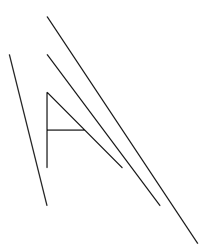
(1) Davide Gaiotto, Anton Kapustin **Spin TQFTs and fermionic phases of matter** `arXiv:1505.05856`

(2) Nathan Seiberg, Edward Witten **Gapped Boundary Phases of Topological Insulators via Weak Coupling** `arXiv:1602.04251`

(3) Davide Gaiotto, Anton Kapustin, Zohar Komargodski, Nathan Seiberg **Theta, Time Reversal, and Temperature** `arXiv:1703.00501`

(4) Werner Krauth **Statistical Mechanics: Algorithms and Computations** (Oxford Master Series in Physics), OUP, 2006.

The most amazing typo ever:



```
\begin{tikzpicture} [scale=0.5]
\foreach \a in {0,...,5}{
    \draw (\a, 5-\a)--(1, \a + 1);
}
\end{tikzpicture}
```