Examples: ABJM Theory

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There's not a whole lot to say until we write the formula:

$$\int \left(\prod_i e^{ik\pi(\theta_i^2 - \phi^2)} \right) \times \frac{\prod_{i \neq j} \left(2\sinh \pi(\theta_i - \theta_j) 2 \sinh \pi(\phi_i - \right.}{\prod_{i,j} (2\cosh \pi(\theta_i - \phi_j))^2}$$

This integral could possibly be over $[0, 2\pi]^2$ but I am expecting it's over the real numbers:

$$(\theta, \phi) \in \mathbb{R}^N \times \mathbb{R}^N$$

since this the domain of integration of the reals. Here is the simplest one:

$$\int_{-\infty}^{\infty} d\theta \; e^{ik\pi\theta^2} = \sqrt{\frac{\pi}{8}}(1+i)$$

I have not done that yet. If we set $N = 1$ or $N = 2$ and this integral is totally feasible.

ABJM theory could refer to two things:

- The $U(N) \times U(N)$ Chern-Simons-matter theory with $\mathcal{N} = 6$ superconformal symmetry. The first $U(N)$ is at level k and the second $U(N)$ is at level $-k$.
- The $U(N) \times U(N)$ Gaussian matrix integral the CS-theory localizes to.

Both of these are called **ABJM theory**. This is rather confusing.

Here is another Chern-Simons formula:

$$Z_{\mathsf{CS}}(N,k) = \frac{1}{\sqrt{N+k}} \prod_{\alpha>0} 2\sin\frac{\pi\,\alpha\cdot\rho}{k+N}$$

where $\alpha_{ij} = e_i - e_j \in \mathbb{C}^N$ are vectors[1] and

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{i=1}^{N}\left(\frac{N+1}{2} - i\right)e_i$$

and also $k \in \mathbb{Z}$ is a positive integer, and so is $n \in \mathbb{Z}$.

Therefore our Chern-Simons partition function Z_{CS} should be a number.

[1]This is called the **Cartan subalgebra** a term which should mean nothing right now.

My understanding is that any localization result of a supersymmetric gauge theory that is not **flat** is indebted to Vasily Pestun[2].

Flat spaces are things like $\mathbb{R}^4$ and even $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ which are **distorted** versions of flat 4-dimensional space.

The curved spaces being considered are remarkably simple $\mathbf{S}^3$ the 3-sphere and $\mathbf{S}^4$ the 4-sphere[3]

We get hints from Pestun's original paper:

> equivariant Euler class of the infinite-dimension normal bundle to the localization locus

I don't know what this means. Maybe some complicated infinite dimensional object has been erected over our 4-sphere, S^4 ?

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[2] This is just looking at the citations. The 4-sphere is a curved four-dimensional space, I guess.

[3] In either case the procedure is the same:

- Do a very complicated algebra
- Claim a localization result
- Solve the integral

I will have a little bit to say about each of these steps. Here are some old ideas that may help:

- Invariant Theory, Spherical Harmonics
- Symlectic Geometry, ODE
- Integral Geometry

The Chern-Simon's matter action is involved:

$$S = \frac{k}{4\pi}\int \mathsf{Tr}\big(A \wedge dA + \frac{2}{3}A^3 - \overline{\chi}\chi + 2D\sigma\big) + W$$

called a **multiplet** and a **superpotential**

$$W = \frac{2\pi}{k}\epsilon^{ab}\epsilon_{\dot{a}\dot{b}}\mathsf{Tr}(A_a B_{\dot{a}} A_b B_{\dot{b}})$$

and we're somehow going to localize this.

Why Path Integrals Localize to Gaussians[4]

An action S depends on a function. By calculus of variations:

$$S(x+\delta x) = S(x) + \delta x\, S'(x) + \frac{(\delta x)^2}{2}S''(x)$$

Our job should be to set $\boxed{S'(x) = 0}$ this is our critical point. Feynman writes the path integral:

$$Z = \sum e^{S(x)}$$

However our logic is the same, this sum should concentrate *near* the classical trajectory:

$$Z \approx e^{x_{\mathsf{cl}}} \sum_{x = x_{\mathsf{cl}} + t\delta x} e^{\frac{1}{2}(t\,\delta x)^2\, S''(x_{\mathsf{cl}})}$$

[4]As soon as you hear the logic two things should happen:

- You should remember this has appeared in every single quantum mechanics textbook.
- Realize this is entirely bull-shit.

Pestun cites the **Duistermaat-Heckman** theorem[5]

$$\int_M \frac{\omega^n}{n!} e^{-\mu} = \sum_i \frac{e^{-\mu(x_i)}}{e(x_i)}$$

Here are some notations we might need:

- M is a symplectic compact manifold
- ω is a symplectic form
- $\dim M = 2n$ the number of dimensions
- μ is the moment map of the $U(1)$ action[6]
- x_i are the fixed points of the rotation
- $e(x_i)$ is the product of the **weight** of te $U(1)$ action on the tangent space at x_i

$$\int_M \alpha = \sum_i \frac{\pi^n \alpha_0(x_i)}{\sqrt{\det(\partial_\mu V^\nu(x_i))}}$$

This **Berline-Vergne-Atiyah-Bott** formula is quite robust. The low-end version of this formula is:

$$\int e^{t\, f(x)}\, dx \approx \frac{e^{t\, f(x_0)}}{\sqrt{f''(x_0)}}$$

Because $f(x) \approx f(x_0) + \frac{t^2}{2} f''(x_0)$ near $x \approx x_0$.

[5] This is a formula from **Symplectic Geometry** or closely related to **Hamiltonian Mechanics**.
[6] It is a **rotation**.

Some Hamiltonian Mechanics

Consider n particles with mass m_i and with phase space $X = \mathbb{R}^{2n}$ and total angular momentum:

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy_i$$

and consider the rotation in phase space:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with $\theta \in [0, 2\pi]$. This are n independent harmonic oscillators. The total energy of the n particles is:

$$\mu = \sum m_i(x_i^2 + y_i^2)$$

and mysteriously the Duistermaat Heckman formula produces a Gaussian integral:

$$\frac{1}{n!}\int_{\mathbb{R}^n} (dx^1 \wedge dy_1) \wedge \cdots \wedge (dx^n \wedge dy_n) e^{-\epsilon \sum (x_i^2 + y_i^2)}$$

The pair $X = \mathbb{R}^{2n}$ and $\omega = \sum(dx \wedge dy)$ could be the phase space of n harmonic oscillators.

And we achieve a Gaussian distribution.

$$= \frac{(2\pi)^n \exp(-\epsilon)}{\prod_{i=1}^{n} w_i}$$

The geometric object used here is the Euler class, rotation each particle (x^i, y_i) by an angle θ_i.

The Hamiltonian is $H = \mu$. The DH formula "quantizes" this rotation into n random walks (or n copies of the **free particle**).

References

(1) Anton Kapustin, Brian Willett, Itamar Yaakov. **Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter** `https://terrytao.wordpress.com/2009/08/23/determinantal-processes/`

(2) Ofer Aharony, Oren Bergman, Daniel Louis Jafferis, Juan Maldacena **N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals** `arXiv:0806.1218v4`

(3) Vasily Pestun, Maxim Zabzine. **Introduction to Localization in Quantum Field Theory.** `arXiv:1608.02953`

(4) Richard Feynman **Path Integrals and Quantum Mechanics** Dover, 2010.

(5) Vasily Pestun **Review of Localization in Geometry** `arXiv:1608.02954`