Examples: Halasz Inequality

John D Mangual

In the nice paper of Radiziwill and Motomaki they talk about the "pretentious number theory". Two multiplicative functions pretend as measured by:

$$\mathbb{D}(f,g;x)^2 = \sum_{p \leq x} \frac{1 - \mathrm{Re}\big[\, f(p)\overline{g(p)}\, \big]}{p}$$

and this distance measure satisfies triangle inequality

$$\mathbb{D}(f,h;x) \leq \mathbb{D}(f,g;x) + \mathbb{D}(g,h;x)$$

however this is **not** a metric since this distince function is sometimes negative.[1]

This pretending metric $\mathbb{D}$ is defined for any functions f, g or $h : \mathbb{N} \to \{z \in \mathbb{C} : |z| \leq 1\}$ mapping to the unit disk.

[1]Somewhat like the Kullback-Liebler divergence from information theory. Except - there is no triangle inequality; could be the Fisher information metric?

Granville and Soundarajan are usually looking to find if a multiplicative function is pretending to be something it's not.

$$\mathbb{D}\Big(\mu(p), p^{it}; x\Big)^2 = \sum_{p \le x} \frac{1 - \cos\big(t\ \log p\big)}{p}$$

Here $\mu(x)$ is the **Möbius function** which is multiplicative on $\mathbb{N}$ and $\mu(p) = -1$ for all primes.

This distance is a trigonometric series over t and we can ask about the miminum value, for $-T < t < T$. Call it M.

The **Halasz-Montgomery-Tenenbaum** theorem says

$$\frac{1}{x}\Big|\sum_{n \le x} f(n)\Big| \ll (1+M)\, e^{-M} + O\left(\frac{1}{\sqrt{T}}\right)$$

$\log p$ is an interesting choice of frequencies

- $\log n$ is not equidistributed mod 1 (these are the numbers after the decimal point) yet
- $\frac{1}{N}\sum_{n \le N} e^{2\pi i \log n} \to 0$ we conclude $e^{2\pi i\ \log n} \to 0$

Weyl's Theorem (Satz 21) says:

Let $\lambda_1, \lambda_2, \ldots$ be a sequence of real numbers. If we can find two numbers c, ϵ such that

$$\left|\lambda_{n+\frac{n}{(\log n)^{1+\epsilon}}} - \lambda_n\right| \geq c$$

Then for x away from a set of measure[2] 0 the sequence

$$\lambda_1 x \; \lambda_2 x \; \ldots$$

is equidistributed mod 0.

This is a very funny condition for Weyl to put.

$$\lambda_{n\left(1+\frac{1}{(\log n)^{1+\epsilon}}\right)} - \lambda_n \geq c$$

E.g. $\lambda_n = a^n$ with $a > 1$. Hardy write this the conclusion as[3] an average tending to zero:

$$\frac{1}{N}\sum_{n=1}^{N} e^{\lambda_n x} = o(1)$$

and that $\{\lambda_n x\}$ is equidistributed **except** at set of measure 0, i.e. "never".

E.g. $\lambda_n = \log n$ we got $\frac{1}{N}\sum e^{2\pi i\, x \log n} = O(1)$ a.e.[4]

[2] the German says "mass"

[3] without ever saying $N \to \infty$ or somthing precise like that

[4] "almost everywhere" I think $x = 1$ works, but maybe you need $x = 1.00001$ or something. j/k $x = 1$ works.

I got caught up on this reading Kupers and Neiderreiter. One has this result.

$$\frac{1}{N}\sum e^{2\pi i\, x_n} = o(1) \longrightarrow n\big|x_{n+1} - x_n\big| \to \infty$$

Then if $x_n = \log n$ are function does not grow fast enough:

$$n(\log(n+1) - \log n) \approx n \times \frac{1}{n} = 1 < \infty$$

and this sequence is known to diverge.

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\ \log n} \not\to 0 \text{ it is } O(1)$$

This average spirals around 0 forever. To prove this theorem Kuipers-Neiderreiter cite a "well-known Tauberian Theorem"

If $n|x_{n+1} - x_n| = \infty$ then by triangle inequality and $1 + x < e^x$ we have:

$$|e^{2\pi i x_{n+1}} - e^{2\pi i x_n}| \leq 2\pi|x_{n+1} - x_n| = O(1/n)$$

and we know the Weyl average must be zero

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i x_n} = 0$$

a well-known theorem shows $e^{2\pi i x_n} \to 0$.

The well-known theorem could be:

$$a_n = O(\tfrac{1}{N}) \text{ and } \lim_{x\to 1} \sum_{n=0}^{\infty} a_n x^n = s \longrightarrow \sum_{n=0}^{\infty} a_n = s$$

this is equivalent to the prime number theorem. Example we might consider are:

- $a_n = e^{2\pi i\, x_{n+1}} - e^{2\pi i\, x_n}$ and $|x_{n+1} - x_n| = O(\frac{1}{n})$
- $a_n = e^{2\pi i\ \log(n+1)} - e^{2\pi i\ \log n}$ (and $\frac{1}{N}\sum e^{2\pi i\ \log n} \neq 0$)
- $a_n = \Lambda(n)$ (the van Mangoldt function)

There is lots of trigonometry here but the shape not very geometrical and awfully hard to visualize.

Halasz Theorem (again)

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \ll (1+M)e^{-M} + O(1/\sqrt{T})$$

if we solve the trigonometry problem of finding the minimum for $|t| < T$.

$$M = \mathbb{D}(f, p^{it}; x) = \sum_{p<x} \frac{1 - f(p)p^{-it}}{p}$$

References

(1) Kaisa Matomäki, Maksym Radziwiłł **Multiplicative functions in short intervals** arXiv:1501.04585

(2) Andrew Granville , Kannan Soundararajan **Decay of mean-values of Multiplicative Functions** arXiv:math/9911246

(3) Hermann Weyl **Über die Gleichverteilung von Zahlen mod. Eins**. Mathematische Annalen 77 (3): 313-352 (1916)

(4) http://math.stackexchange.com/q/1940261/4997

(5) Kuipers, Niederreiter **Uniform Distribution of Sequences** Dover, 2006.

(6) Antoni Zygmund **Trigonometric Series**, Cambridge Mathematical Library. CUP, 2003.