Examples: Quadratic Reciprocity

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Here is a theorem from the modern number theory literature (about 10 years old):[1]

> Let π be a unitary cuspical representation of $\mathrm{GL}_2(\mathbb{A}_F)$ and χ is a unitary character of $\mathbb{A}_F^\times/F^\times$ with finite conductor $\mathfrak{f}$.
>
> There is an $N > 0$ such that:
> $L(\frac{1}{2}, \pi\times\chi) \ll \mathrm{Cond}(\pi)^N \mathrm{Cond}_\infty(\chi)^N N(\mathfrak{f})^{1/2-\frac{1}{24}}$ and also a bound for thse other L-functions
> $L(\frac{1}{2}, \chi) \ll \mathrm{Cond}_\infty(\chi)^N N(\mathfrak{f})^{1/4-\frac{1}{200}}$

This is unfortunately written at such a level of abstraction that we have no idea what is going on.

I barely know what a modular form or an L-function is (though I am seeing them constantly)

[1] Akshay Venkatesh – Sparese Equdistribution Problems, Period Bounds and Subconvexity – *Annals of Mathematics* (2005)

Then the author says if we set $F = \mathbb{Q}$ this is a *subconvexity* result due to Burgess.

I know what "convex" means - a circle is convex. A square is convex. I don't know what **subconvex** means.

Burgess proved $L(\frac{1}{2}, \chi) \ll k^{\frac{7}{32}+\epsilon}$ and the "principal difficulty" (Burgess' words) is to show an estimate like this:

$$\sum_{x=1}^{k} \left| \sum_{y=1}^{h} \chi(x+y) \right|^{2r}$$

here χ is a Dirichlet charcter (such the Legendre symbol $(\frac{\cdot}{p})$.

There is nothing convex about this. And in a way it doesn't matter since we can write down the formula:

$$L(\frac{1}{2}, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}}$$

This is divergent if $\chi \equiv 1$ – the **trivial character** but what about other sequences of ± 1 ?

#1 - For every number there can be a Dirichlet character. Mod 3 we can set $\chi(2) = -1$ and then

$$L(\tfrac{1}{2}, \chi_3) = 0 + \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + 0 + \dots$$

and this symbol "$\ll$" is somewhat startling since are are not looking at any single Dirichlet characters, but *all* Dirichlet characters as $k \to \infty$.

Those numbers $L(\frac{1}{2}, \chi) \ll k^{\frac{7}{24}}$ (or whatever crazy exponent you will put).

#2 - Subconvexity bounds - admittedly not very convex - originate from

- the **Phragmen-Lindelöf** theorem and
- the **Hadamard three circles** theorem
- The **Maximum Modulus** theorem

as I found out flipping between various textbooks[2]

[2]If you know which book to look at, it is easy to get started. These are the resoures I found. Actually quite old.
Titchmarsh **Theory of Functions** (endlessly useful - I thought I knew it all already)
Iwaniec + Kowalski **Analytic Number Theory** The maximum modulus principle talks about **bounded holomorphic functions** if $|f(z) \leq M$ on C either:

- $|f(z)| < M$ at all interior points in D
- $f(z) \equiv M$ is a constant.

#3 - This might be the wrong idea but I would like to study the equation

$$x^2 + y^2 + z^2 = d$$

The number of three squares points is related to the

$$\#\{(x,y,z) : x^2 + y^2 + z^2 = n\} = \frac{24}{\pi}\sqrt{n}L(1, \chi_{n \text{ or } 4n})$$

These numbers should be very roughly evenly distributed on the sphere, but there are still many patterns which persist for large values $d \gg 1$.

And $L(1,\chi)$ is a slightly different series.

$$\sum \frac{\chi(n)}{n} < n^{\frac{1}{2}+\epsilon}$$

and if $\chi \equiv 1$ this series is the **Harmonic series** which is divergent.

Yet, I have to keep my L-functions straight. $L(\frac{1}{2},\chi)$ and $L(1,\chi)$. When is one appropriate and when is the other?

#4 - since we talk about Dirichlet characters, there must be talk of quadratic reciprocity. Let $(\frac{\cdot}{p})$ be the **Legendre symbol**

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a = x^2 \text{ mod } p \\ -1 & \text{if } a \neq x^2 \text{ mod } p \end{cases}$$

Then we have some reprocity between the two values:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$$

This is elementary discussed in Hardy's chapter **Fermat's Theorem and it's Consequences**

- what dynamical system can be used to prove QR?
- what permutation group action can be used to prove QR

This is Fermat's Little Theorem $a^p = a$ mod p.

#5 - I have no idea what Burgess bound is used for. Least quadratic residue?

References

(1) Jared Weinstein. **Reciprocity laws and Galois representations: recent breakthroughs** Bull. Amer. Math. Soc. 53 (2016), 1-39

(2) David A Cox. **Primes of the Form $x^2 + ny^2$: Fermat, Class Field Theory, and Complex Multiplication** Wiley, 2013.

(3) **A prime ideal $\mathfrak{p}$ decomposes in $\mathbb{Q}(\zeta_{24})/\mathbb{Q}(\sqrt{-6})$ iff it is generated by $\alpha \in 1 + 2\mathbb{Z}[\sqrt{-6}]$**
`http://mathoverflow.net/q/234570/1358`

(4) Roy L. Adler **Symbolic dynamics and Markov partitions** Bull. Amer. Math. Soc. 35 (1998), 1-56
`http://www.ams.org/journals/bull/1998-35-01/S0273-0979-98-00737-X/`