Tutorial : Ping-Pong Lemma

Half of these projects start by a conjectured relationship between two theorems. And it's not always correct. Today's pair is:

- Ping-Pong Lemma
- Sum-Product Theorem

I *think* that's correct. Let's state these results and being the long journey of connecting these thing to "real life".

#1 Ping-Pong Lemma I was able to find two examples of Ping-Pong Lemma having to with group matrices over the integers $\mathbb{Z}$ we can have

- Banach-Tarski paradox
- Establishing free groups

Here's a proposition:

> This is a Free group on two generators (no relations)
>
> $$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \subseteq SL_2(\mathbb{Z})$$
>
> Proof: use the ping-pong Lemma. Another example I found from Sarnak[1]
>
> $$\left\langle \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & -5 \end{pmatrix} \right\rangle \subseteq SL_4(\mathbb{Z})$$
>
> and these play generalized Ping-Pong. The reason we like the free group F_2 is becuase it's one of the few discrete groups we can understand. And I don't even think that's true, because if $A, B \in SO(3)$ that group is compact and we can measure:
>
> $$\phi : \langle A, B \rangle \to SO(3)$$
>
> we'd have that $\overline{\langle A, B \rangle} \subseteq SO(3)$ is a **dense** subgroup, so we can try to quantify (by whatever means we have availble) how quickly this is mixing.

In another direction we have the Banach-Tarski paradox. The groups used to contruct the paradox seem related to the Pythagorean theorem[2] He will use:

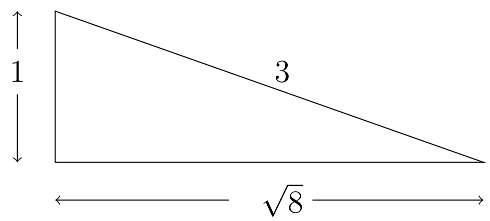
[1] `http://web.math.princeton.edu/sarnak/NotesOnThinGroups.pdf`

[2] https://stanford.box.com/shared/static/wesg27648yqomf4ar3mkmfwf89ptzqj4.pdf

This group is free:

$$\left\langle \begin{pmatrix} \frac{1}{3} & +\frac{2\sqrt{3}}{3} & 0 \\ -\frac{2\sqrt{3}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & +\frac{\sqrt{8}}{3} \\ 0 & -\frac{\sqrt{8}}{3} & \frac{1}{3} \end{pmatrix} \right\rangle \subseteq SO_3(\mathbb{R})$$

This group is related to the right triangle with side lengths 1 and $\sqrt{8}$



This tringle is oriented in various ways in three dimensional space.

Therefore... without stating the ping-pong Lemma we can imagine such a thing could be useful! Now for the statement:

Ping-Ping Lemma Let G be a group acting on a set X. Let H_1, H_2 be sub-groups and suppose we can find subsets $X_1, X_2 \subseteq X$ subset that we observe

$$\big[g \in H_1 \to g(X_2) \subseteq X_1\big] \text{ and } \big[g \in H_2 \to g(X_1) \subseteq X_2\big]$$

then our subgroup is free, $\langle H_1, H_2 \rangle = H_1 * H_2 \simeq F_2$.

One thing we notice is that ping-pong "works" (i.e. produces free groups or expander graphs?) only for 3×3 and not for 2×2. Another possiblity is that the group we are studying isn't free (there are relations). This is not a bad thing, $SL(2, \mathbb{Z})$ is not free, there's a relation:

$$SL_2(\mathbb{Z}) \simeq \big\langle S, T : S^2 = (ST)^3 = 1 \big\rangle \simeq \big(\mathbb{Z}/2\mathbb{Z}\big) * \big(\mathbb{Z}/3\mathbb{Z}\big)$$

and for a general number field, F, $SL_2(\mathcal{O}_F)$ for the ring of integers $\mathcal{O}_F$ can be an open problem. Therefore, we conclude, $SL(2, \mathbb{Z})$ can house rather complicated objects.

I should remark this is a case of the **Tits alternative**.

Veresion #2 $SO(3)$ contains a copy of the free group on two generators.

Our candidate free group will be the the standard 3-4-5 triangle. Since $3^2 + 4^2 = 5^2$ we have

$$\left\langle \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix} \right\rangle \subseteq SO_3(\mathbb{Q})$$

and we resort to 5-adic numbers to choose our ping-pong sets:

- $A_{\pm} = 5^{\mathbb{Z}} \cdot \{(x, y, z) \in \mathbb{Z}^3, x \pm 3y \in 5\mathbb{Z}, z \in 5\mathbb{Z}\}$
- $B_{\pm} = 5^{\mathbb{Z}} \cdot \{(x, y, z) \in \mathbb{Z}^3, z \pm 3y \in 5\mathbb{Z}, z \in 5\mathbb{Z}\}$

and our set is $X = A_- \cup A_+ \cup B_- \cup B_+ \cup \{(0, 1, 0)\}$ and this is our **ping-pong** set!

So our ping-pong style proofs involve finding a "tennis court" and playing a game!

#2 Sum-Product Theorem This is the dumbest-sounding theorem I have ever heard of.

Emmanuel Breuillard put the origin of approximate groups with the Sum-Product Theorem:

> Thm Let $\mathbb{F}_p$ be a finite field with p elements (where p is prime). They for every $\delta > 0$ there is an $\epsilon > 0$ (which does not depend on p) such that:
>
> $$|S \times S| + |S + S| \geq |S|^{1+\epsilon}$$
>
> for every subset $|S| \subset \mathbb{F}_p$ such that $p^\delta < |S| < p^{1-\delta}$.

For me it is not so easy to picture what this set S could look like. In a sense, we'll never be told what numbers δ or ϵ should be. It just says

> Give me $\epsilon \rightarrow$ find δ such that $S + S$ or $S \times S$ is growing. And this is only for sets which that do not have just a few elements or all of $\mathbb{F}_p$.

and approximate group will start here. I can't even turn this statement into a hypothesis to test. I can't connect it to other number theory facts I have learned. Very basic but I don't know what it means. Here's another, due to Helfgott:

> For every ϵ we can find a δ such that $|S \times S \times S| \geq |S|^{1+\epsilon}$ for every generating set S of $SL_2(\mathbb{Z}/p\mathbb{Z})$ such that $|S| < |SL_2(\mathbb{Z}/p\mathbb{Z})|^{1-\delta}$

Commendible. What do you mean this was only discovered in 2008? I said to myself the following: consider the expansion of sine:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^3}{5!} = x - \frac{x \times x \times x}{1 \times 2 \times 3} + \frac{x \times x \times x \times x \times x}{1 \times 2 \times 3 \times 4 \times 5}$$

How many instances of $+$ and $\times$ and $\div$ have we done? And if we are using floating-point arithmetic, we are losing accuracy every step of the way. What happens if we use a million of these?

That error will be pseudo-random, but maybe the result will be in tact.

I don't know if I could have articulated the result as nicely as Helfgott or Tao or Bourgain, but if definitely know if I had proposed such a thing I'd be laughed out of the room.[3]

#3 Do these two overlap? I guess not.

[3] In fact, yes.

#4 Approximate Groups What kinds of problems were being solved which led mathematicians to christen this idea with a name "approximate group"?

Arguably, sum-set problems are *ubuiquitous*, however nobody is tapping you on the shoulder telling you where to look, or that such an object is being in use. The theorems connecing them are not the modern one's but some classical counterpart.

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References

(1) Emmanuel Breulliard **A brief introduction to approximate groups** from: "Thin Groups and Superstrong Approximation" (MSRI Publications, **61**) 2013.