Examples: L^p spaces

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L^2 spaces are somewhat natural because we have Pythagoras theorem:

$$\big|\big|(a,b)\big|\big|^2 = a^2 + b^2$$

I have trouble understanding the meaning of the L^p norms:

$$\big(a^p + b^p\big)^{1/p}$$

I have no interpretation for them. There's no picture I can draw.

There are a few starting points. Let $p, q \in (1, \infty)$ be related by

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then all measurable functions $f, g \geq 0$ satisfy:

$$\int f g \, d\mu \leq \big|\big|f\big|\big|_p \, \big|\big|g\big|\big|_q$$

so for some reason this relationship of fractions gets promoted to a relationship of measurable functions.

There is also Minkowski inequality:

$$||f+g||_p \leq ||f||_p + ||g||_p$$

and it's downhill from there. I have read in some places these have their origins in **convex gometry** in high dimensional space $\mathbb{R}^n$.

All the arguments I'm finding are pretty clumsy and not very geometric. Instead we drown in a morass of

- poor disorganized writing
- clumsy notation
- non-visual thinking

I am concluding this subject is simply too difficult. And going for my bike ride.

Bourgain's Conjecture says for $a_\xi \in \mathbb{C}$ and $\epsilon > 0$ and $p \geq 6$:

$$|| \sum_{\xi_1^2+\xi_2^2+\xi_3^2=N^2} a_\xi e(\xi \cdot x) ||_{L^p(\mathbb{T}^n)} \leq_\epsilon N^{\cdots} ||a_\xi||_{l^2\left(\{\xi_1^2+\xi_2^2+\xi_3^2=N^2\}\right)}$$

Some kind of Fourier series over points on the sphere is smaller than some average of the coefficients. I could even write the left side:

$$\left[\int_{[0,1]^3} \left| \sum_{\xi_1^2+\xi_2^2+\xi_3^2=N^2} a_\xi e(\xi \cdot x) \right|^p dx \right]^{1/p}$$

If I knew what these spaces where, I might have something intelligent to say here. And I can't even type the correct inequality symbol it's not "$\leq$". I don't know what Fourier restriction is. And Bourgain's methods are just not visual or explicit enough.

On the other hand if I'm this unhappy then I must certainly have something to way not in Bourgain-Demeter.

References

(1) Jean Bourgain, Ciprian Demeter. **Proof of the l^2 Decoupling Conjecture** `arXiv:1403.5335`