Examples: ABJM Theory

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There's not a whole lot to say until we write the formula:

$$\int \left(\prod_i e^{ik\pi(\theta_i^2 - \phi^2)} \right) \times \frac{\prod_{i \neq j} \big(2 \sinh \pi(\theta_i - \theta_j) 2 \sinh \pi(\phi_i -}{\prod_{i,j} (2 \cosh \pi(\theta_i - \phi_j))^2}$$

This integral could possibly be over $[0, 2\pi]^2$ but I am expecting it's over the real numbers:

$$(\theta, \phi) \in \mathbb{R}^N \times \mathbb{R}^N$$

since this the domain of integration of the reals. Here is the simplest one:

$$\int_{-\infty}^{\infty} d\theta \; e^{ik\pi\theta^2} = \sqrt{\frac{\pi}{8}}(1+i)$$

I have not done that yet. If we set $N = 1$ or $N = 2$ and this integral is totally feasible.

ABJM theory could refer to two things:

- The $U(N) \times U(N)$ Chern-Simons-matter theory with $\mathcal{N} = 6$ superconformal symmetry. The first $U(N)$ is at level k and the second $U(N)$ is at level $-k$.
- The $U(N) \times U(N)$ Gaussian matrix integral the CS-theory localizes to.

Both of these are called **ABJM theory**. This is rather confusing.

Here is another Chern-Simons formula:

$$Z_{\text{CS}}(N, k) = \frac{1}{\sqrt{N+k}} \prod_{\alpha>0} 2 \sin \frac{\pi\, \alpha \cdot \rho}{k+N}$$

where $\alpha_{ij} = e_i - e_j \in \mathbb{C}^N$ are vectors[1] and

$$\rho = \frac{1}{2} \sum_{\alpha>0} \alpha = \sum_{i=1}^{N} \left(\frac{N+1}{2} - i \right) e_i$$

and also $k \in \mathbb{Z}$ is a positive integer, and so is $n \in \mathbb{Z}$.

Therefore our Chern-Simons partition function Z_{CS} should be a number.

[1]This is called the **Cartan subalgebra** a term which should mean nothing right now.

My understanding is that any localization result of a supersymmetric gauge theory that is not **flat** is indebted to Vasily Pestun[2].

Flat spaces are things like $\mathbb{R}^4$ and even $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ which are **distorted** versions of flat 4-dimensional space.

The curved spaces being considered are remarkably simple $\mathbf{S}^3$ the 3-sphere and $\mathbf{S}^4$ the 4-sphere[3]

We get some hints from Pestun's original paper:

> equivariant Euler class of the infinite-dimension normal bundle to the localization locus

Do we know what this means? I don't. Maybe some complicated infinite dimensional object has been erected over our 4-sphere, S^4.

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[2] This is just looking at the citations. The 4-sphere is a curved four-dimensional space, I guess.

[3] In either case the procedure is the same:

- Do a very complicated algebra
- Claim a localization result
- Solve the integral

I will have a little bit to say about each of these steps. Here are some old ideas that may help:

- Invariant Theory, Spherical Harmonics
- Symlectic Geometry, ODE
- Integral Geometry

References

(1) Anton Kapustin, Brian Willett, Itamar Yaakov. **Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter**
`https://terrytao.wordpress.com/2009/08/23/determinantal-processes/`

(2) Ofer Aharony, Oren Bergman, Daniel Louis Jafferis, Juan Maldacena **N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals** `arXiv:0806.1218v4`