Problem: AMC 12A (2016)

John D Mangual

Problem What is the value of $\dfrac{11! - 10!}{9!}$?

(A) 99 (B) 100 (C) 110 (D) 121 (E) 132

Better get multiplying! And we don't have a calculator on this test! What is this?

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11$$

I know that $2^{10} = 1024$ or is it 512? Not important right now:

$$1 \times 2 \times 3 \times 4 = 24$$

and then multiply by $5 \times 6 = 30$ so that makes

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 24 \times 30 = 720$$

Just five moroe numbers left.

$$7 \times 8 = 56 \qquad 9 \times 10 \times 11 = 990$$

so what is 56×990 this is not getting us very far.

$11!$ is some very large number[1].

$$11! = 11 \times 10!$$

Then maybe we can subtract against the other $10!$

$$\frac{11! - 10!}{9!} = \frac{10! \times (11-1)}{9!} = 9 \times 10 = 90$$

That would be choice
and that was none of the choices ...

$$\frac{11 \times 10! - 10!}{9!} = (11-1) \times \frac{10 \times 9!}{9!} = 100$$

The answer is choice **(B)**.

Alterantive If we took remainders upon division modulo 10. Forget it.

$$\frac{11! - 10!}{9!}$$

$9!$ is a very complicated thing. Since $2 < 9$ and $5 < 9$ we have that 10 divides $9!$ – this is not relevant.

$$\frac{11 \times 10 \times 9! - 10 \times 9!}{9!} = 11 \times 10 - 10 = 100$$

[1]hopefully you know what the factorial symbol means

The peculiar thing about this problem is that we rarely need the last decimal place of a large number! If I said:

$$10! \approx 10^{10}$$

that might be convincing enough. But that is not right since:

- $1 < 10$
- $2 < 10$
- $3 < 10$
- $\ldots$
- $9 < 10$
- $10 = 10$

Then if we multiply all these inequalities together

$$1 \times 2 \times 3 \times \cdots \times 9 \times 10 < 10^{10}$$

So how bad is our estimate? That must wait for another time...

The other place this number might appear:

$6 = 3! = \#\left|\{ABC, ACB, BAC, BCA, CAB, CBA\}\right|$

The factorials $n!$ count the number or re-orderings of a word.

So how do we ascribe meaning to this?

$$\frac{\text{rearrange 11 letters} - \text{rearrange 10 letters}}{\text{rearrange 9 letters}}$$

Let's rearrange the letters $\{a, b, c, d, e, f, g, h, i, j, k\}$.

$$\text{any letter is not } a \,\Big|\, \text{any letter } \Big|\, \text{everything else}$$

and this is another reason why $\boxed{10! < 10^{10}!}$ is because we are not allowing repeats:

- **NO** AABBCCDDEEF
- **YES** ABCDEFGHIKJ

Scratchwork if I could find the last digit mod 10 I'd be done:

$$\frac{11! - 10!}{9!} \equiv x \text{ mod } 10$$

Then multiply both sides by $9!$ some very large perculiar number we don't fully understand.

$$11! - 10! \equiv 9!x \text{ mod } 10 \times 9!$$

Rather disappointly this does not yield anything new. This merely yields one of the solutions we've presented before.