

Numbers and Entropy

John Mangual

1 Random Sorting Networks

We argue that random sorting networks are a discretization of the geodesic flow on the n -sphere.

1.1 The Great Circle Conjecture

A sorting network is a shortest path from $(123 \dots n)$ to $(n \dots 21)$ in the S_n Cayley graph generated by transpositions $(i, i + 1)$.

$$d(\Delta_n) = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3} \dots (2n-3)^1}$$

This is also the number of staircase Tableaux, [?].

Ex Let ω_n be an n -element uniform sorting network. For each n there exists a random great circle $C_n \in \mathbb{S}_n$ such that

$$\|\omega_n - C_n\|_\infty < \alpha\sqrt{n}$$

The sphere is the intersection $\mathbb{S}_n = \{\mathbb{E}[x] = \frac{n(n+1)}{2}\} \cap \{\mathbb{E}[x^2] = \frac{(n+1)(2n+1)}{6}\}$ maybe center and rescale for symmetry.

1.2 Grassmanian

How can we parameterize great circles on the n -sphere S^n ? The circle has to start somewhere $x \in S^n$ and travel in some direction $T_x^1(S^n) \simeq S^{n-1}$. So I have that great circles in the sphere are indexed by

$$T^1(S^n) \approx S^n \times S^{n-1}$$

Here I use \approx to mean the tangent bundle is like the product, but the gluing maps are different. I can't think of the correct vector bundle term or notation.

Then rotation group $SO(n)$ acts on the space of great circles. Certainly stabilizer of this group action is the copy of $SO(2)$ fixing the circle itself. This leaves $SO(n)/SO(2)$; hopefully I have not missed any additional symmetries.

The Grassmanian of 2-planes in n dimensional space is $SO(n)/SO(2) \times SO(n-2)$ since not only the great circle can be fixed (and the plane it spans) but also the complementary subspace orthogonal to the plane.

1.3 Back to S_n

Ex All the points of S_n can embed into a sphere of center and radius:

$$\frac{1}{n} \sum_{i=0}^n i \times (1, 1, \dots, 1) = \frac{1}{n} \binom{n+1}{2} (1, 1, \dots, 1)$$

I am stumbling a bit on the radius computation:

$$\begin{aligned} \sum_{i=0}^n (i - \mathbb{E}[x])^2 &= \sum_{i=0}^n i^2 - 2\mathbb{E}[x] \sum_{i=0}^n i + n\mathbb{E}[x]^2 \\ &= \sum_{i=0}^n i^2 - n\mathbb{E}[x]^2 \\ &= \frac{n(n+1)(2n+1)}{6} - n \left[\frac{n+1}{2} \right]^2 \\ &= \frac{n(n^2-1)}{12} \end{aligned}$$

Not sure it matters 100% but mainly none of the authors of [?] take advantage of the symmetry of the Cayley graph problem they have.

Ex In fact the sorting network should map to a half a great arc around the sphere. How do we complete the other half? I think we just do it twice. Let σ be a word in $\langle (12), (23), \dots, (n-1, n) \rangle$

$$(12 \dots n) \xrightarrow{\sigma} (n \dots 21) \xrightarrow{\sigma} (12 \dots n)$$

We can do a bit more and only get partial rotations using the conjugation map.

$$\prod_{i=0}^k \prod_{i=k+1}^{\binom{n+1}{2}} \approx \prod_{i=k+1}^{\binom{n+1}{2}} \prod_{i=0}^k$$

Starting from any Random Sorting Network we can get $\binom{n+1}{2}$ “rotated” versions of the same network.

Is there a way to “parallel translate” the action of $(\sigma, (ij)) \in S_n \times \langle (12), (23), \dots, (n-1, n) \rangle$? There is some literature by Diaconis on “random walks on groups” in the case when you allow all transpositions. There some literature on nearest neighbor transposition random walks as well [?]. Here we have condition on the beginning and end of our process as well as how much time it should take and studying the stochastic process that emerges there.

The easiest way to discretize the sphere is to sort the elements. If $\sigma^{-1}(x)$ is increasing then $x \in \sigma$. I am nervous we are reproducing the “geometric” sorting networks also discovered by Omer Angel, but those live in \mathbb{R}^2 , so I don’t think he exploits the geometry of Lie groups in any way.

So there are S^{n-1} tangent directions to go half of which increase the inversion number and half do not. These two sets are convex - closed under averages $(x, y) \mapsto xt + y\sqrt{1-t^2}$, $t \in [0, 1]$. These are separated by a hyperplane.

Algorithm reasonably, any great circle determines a sequence of permutations by watching the order of the elements as $R_\theta \vec{x}$ evolves $\theta : 0 \rightarrow \pi$.

- Do we get all sorting networks this way?
- Do we get things that are not sorting networks?
- Are all sorting networks from $(12 \dots n)$ to $(n \dots 21)$ equally likely?

If we choose Haar measure on the Grassmanian $\text{Gr}(2, n) = SO(n)/(SO(2) \times SO(n-2))$ this uniformity should pass down to the sorting networks themselves.

Consider $i(\theta) = \text{inv}(R_\theta \vec{x})$ as a function of $\theta \in [0, 2\pi]$. The inversion number foliates the sphere into level sets along hyperplanes. Any other hyperplane is transverse to this foliation, and so the inversion number must have a unique maximum and minimum. Since the hyperplane cuts through both critical points $(12 \dots n)$ and $(n \dots 21)$ of this foliation, the inversion number increases monotonically from 0 to $\binom{n}{2}$, so these are always sorting networks. This rules out question #2.

The answer to the first question #1 is certainly **no**. Consider the sorting network:

$$\left[(12)(13)(14) \dots (1n) \right] \left[(23) \dots (2n) \right] \dots \left[(n-1, n) \right]$$

This curve makes successive right angles and so is definite **not** a great circle. This curve is also not typical. We conjecture that *typical* random sorting networks will have approximately great arc shapes.

2 Calculus of Variations

We now know that typical random sorting networks (but certainly not all of them) will be near great circles. How do we prove this?

$$\mathbb{P}(f) = \exp \left[- \int_0^\pi |f(t)|^2 dt \right]$$

Probably they need to minimize a functional such as the distance functional on the sphere to get a geodesic. Since this is a probability measure we probably just need to take the exponent.

References

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