Reading: L-functions

Let's try to cobble together the exercise.

Example (Soundarajan, 2000)
The squares of Dirichlet L-functions at $s = \frac{1}{2}$ average out (roughly) to a polynomial Q of degree 3.

$$\sum_{0\leq d\leq X}{}^{*} L(\frac{1}{2}, \chi_{8d})^2 = XQ(\log X) + O(X^{\frac{5}{6}+\epsilon})$$

The cubes of Dirichlet L-functions at $s = \frac{1}{2}$ average out (roughly) to a polynomial R of degree 6.

$$\sum_{0\leq d\leq X}{}^{*} L(\frac{1}{2}, \chi_{8d})^3 = XR(\log X) + O(X^{\frac{11}{12}+\epsilon})$$

Example (Young, 2009)
Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function of compact support.
The weighted average of the squares of Dirichlet L-functions at $s = \frac{1}{2}$ average out (roughly) to a polynomial Q of degree 3.

$$\sum_{0\leq d\leq X}{}^{*} L(\frac{1}{2}, \chi_{8d})^2 = XP(\log X) + O(X^{\frac{1}{2}+\epsilon})$$

The weighted average of the cubes of Dirichlet L-functions at $s = \frac{1}{2}$ average out (roughly) to a polynomial R of degree 6.

$$\sum_{0\leq d\leq X}{}^{*} L(\frac{1}{2}, \chi_{8d})^3 = XR(\log X) + O(X^{\frac{3}{4}+\epsilon})$$

We do not even need to read the modern article. Study of the "mean value" .

Let's try to parse a few of the symbols here:

- What are "fundamental discriminants"? $\displaystyle\sum_{0\leq d\leq X}{}^{*}$
- The shape of the result is: $\sum_{0\leq d\leq X} f(d, X) = X(\log X)^3 + O(\sqrt{X})$, where $f : \mathbb{Z} \to \mathbb{R}$ is function on the integers (itself an average of other functions).
- What's wrong with $\zeta(\frac{1}{2}) \stackrel{?}{=} \sum \frac{1}{\sqrt{n}}$ (this is a divergent series), and also $L(\frac{1}{2}) = \sum \frac{\chi(n)}{\sqrt{n}} \neq 0$?
- Let's remind ourselves what are Dirichlet characters, $\chi_D(n) = (\frac{D}{n}) \ldots$
- Why are the exponents like $O(X^{\frac{3}{4}})$ or $O(X^{\frac{5}{6}})$ difficult to optimize? What kind of hard problem to they represent that we can no longer describe them with a polynomial formula or even estimate their size?

Fractals have fractional growth exponents.
These exponents represent classes of **arithmetic problems** that are difficult to predict and control.

These "analytic" formulas are blurry because they are averaging out highly chaotic arithemetic functions.

Let's just scrape off the exercises from the page:

Ex The following identity is almost surely true:

$$\sum_{ab=\ell} \left(\frac{a}{b}\right)^s = \prod_{p|\ell}(p^{-s}+p^s) = \prod_{p|\ell}(2+s^2\log^2 p + O(s^4)) = d(\ell)\left(1+\frac{s^2}{2}\sum_{p|\ell}\log^2 p + O(s^4)\right)$$

Ex Show that (note here that $\mathbb{Q}^\times$ is a **group** where we consider all possible fractions, while $(\mathbb{Q},+,\times)$ is the **field**):

$$\left(\frac{\Gamma(\frac{1}{4}+s)}{\Gamma(\frac{1}{4}}\right)^2\left(\frac{16}{\pi}\right)^s\Gamma_1(s)\frac{4^s+4^{-s}-\frac{5}{2}}{4^s}\zeta(2s)\zeta(2s+1) = \frac{1}{8s^2}+a_0+O(s)$$

What sequence of numbers might this correspond to? There are many, many answers here. For example, the Stirling formula $n! \asymp n^n e^{n\log n}$ suggests an answer. So many that we just look at equivalence classes, $[a]$. This would be a great time to review the concept of **Laurent series** as well as **Group Theory**.

Can we explain a bit what Prof. Soundarajan might have been doing?

Lemma (Heath-Brown, 1979) Let N and Q be positive integers and let $a_1,\dots,a_N$ be arbitrary complex numbers (e.g. $a_n = e^{\sqrt{2}n}$, exponents or "characters".) Let $S(Q)$ be a set of real primitive characters χ with conductor $\le Q$. Then

$$\sum_{\chi\in S(Q)}\left|\sum_{n\le N} a_n\chi(n)\right|^2 \ll_\epsilon (QN)^\epsilon(Q+N)\sum_{n_1n_2=\square}|a_{n_1}a_{n_2}|$$

for any $\epsilon>0$. Let M be any positive integer, and for each $|m|\le M$ write $4m=m_1m_2^2$ where m_1 is a fundamental discriminant, and m_2 is positive. Suppose the sequence a_n satisfies $|a_n|\ll n^\epsilon$. Then

$$\sum_{|m|\le M}\frac{1}{m_2}\left|\sum_{n\le N}a_n\left(\frac{m}{n}\right)\right|^2 \ll (MN)^\epsilon N(M+N)$$

For example, $a_n\in\mathbb{Q}^\times$ be some result of multiplications and divisions $a_n\in\{0,1,\dots,9,\times,\div\}^*$ (please write the automaton correctly) with $|a_n|\ll n^\epsilon$ (looks like $a_n\asymp 1$?)

We are throwing away increasing amounts of work to get these leading-term results.

- Where is the information hiding??
- What did Number Theory look like before 1980 that this result is an improvement?

There are many ways to generate number sequences for example, $\mathbb{Q}^\times\to\mathbb{R}$ such as $\phi(\frac{m}{n})=m$ is nowhere integrable, nowhere differentiable and violently chaotic. Arithemetic functions are dime-a-dozen yet we have failed to produce even one.

Thm Jutila (1981) this is simpler, no less profound result:

$$\sum_{|d|\le X} L(\frac{1}{2},\chi_d)\sim cX\log X$$

and for sum of squares

$$\sum_{|d|\le X}|L(\frac{1}{2},\chi_d)|^2\sim cX(\log X)^3$$

References

[1] Keiju Sono. **The Second Moment of Quadratic L-functions**
Journal of Number Theory 206 (2020) 194-230

[2] Kannan Soundarajan **Nonvanishing of Quadratic Dirichlet L-Functions at** $s = \frac{1}{2}$
Annals of Mathematics 152 (2) (2000) 447-488.