Scratchwork: Intersecton of Two Lines

In geometry class, we learn the Cramer rule for the intersection two lines.

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

And so the intersection of these two lines can be found with a **determinant** of a 2×2 matrix:

$$x = \frac{\left| \begin{array}{cc} c_1 & b_1 \\ c_2 & b_2 \end{array} \right|}{\left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right|} \quad \text{and} \quad y = \frac{\left| \begin{array}{cc} a_1 & c_1 \\ a_2 & c_2 \end{array} \right|}{\left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right|}$$

In a Linear Algebra course - or a Geometry course - one might check that $a, b, c \in \mathbb{R}$ means our solutions $(x, y) \in \mathbb{R}^2$. We don't have that for an integer problem $a, b, c \in \mathbb{Z}$ the solution remains in integers $(x, y) \in \mathbb{Z}^2$.

Since the $+$ and $\times$ operations we do aren't too fancy, we can do Linear Algebra over a field such as $K = \mathbb{Q}$ or $K = \mathbb{C}$. In addition, let's use a tiny bit of Exterior Algebra taken from a Geometry textbook.

Thm The points A, B and C are collinear if and only if $A \wedge B + B \wedge C + C \wedge A = 0$.

In our case, the equation has one line $\boxed{\text{Ax + By = C}}$. In that case, we can write the Cramer rule in an more condensed way:

$$Ax + By = C \to A \wedge (Ax + By) = (A \wedge B)y = (A \wedge C) \to y = \frac{A \wedge C}{A \wedge B}$$

And a similar formula for x. Is it okay to write the coordinate value of x and y as the ratio of two areas. The geometric objects look kind of funny but OK.

$$\text{[number]} = \frac{\text{[area]}}{\text{[area]}}$$

This is not outrageous. Pedeo gives a careful derivation of the wedge product of two vectors:

$$u \wedge v = (x_1 E_1 + x_2 E_2) \wedge (y_1 E_1 + y_2 E_2) = (x_1 y_2 - x_2 y_1)(E_1 \wedge E_2)$$

where $E_1, E_2 \in \mathbb{R}^2$ are unit vectors in the plane.

There are even more intersection formulas like this. Two planes in Four dimensions intersect (generically) at a point.

$$\mathbb{R}^2 \cap \mathbb{R}^2 = \{pt\} \text{ in } \mathbb{R}^4$$

Since all we're doing is linear algebra, this still could work over $\mathbb{Q}$ we'd have $\mathbb{Q}^2 \cdot \mathbb{Q}^2 = [pt] \subseteq \mathbb{Q}^4$. This is the beginnings of intersection theory and a lot of sheafy things could occur.

References

[1] Joe Harris **Algebraic Geometry: A First Course** (GTM #133) Springer, 1992.

[2] Gabriel Cramèr **Introduction à l'Analyse des Lignes Courbes Algbriques** `https://archive.org/details/bub_gb_HzcVAAAAQAAJ`

[3] Dan Pedoe. **Geometry: A Comprehensive Course** Dover, 1970.