

Primes in Arithmetic Sequences, Entropy, Hyperbolic 3-Manifolds

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One way to show there are infinitely many primes is to show certain series diverges¹.

$$\sum_{p \in \mathcal{P}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots = \infty$$

By **fundamental theorem of arithmetic** $n = p_1^{a_1} \dots p_n^{a_n}$ and then

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p \in \mathcal{P}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots \right) \asymp \exp \left[\sum_{p \in \mathcal{P}} \frac{1}{p} \right] = \infty$$

Both of these sequences diverge. I left out a tiny bit of work.

¹I get bored just looking at such a series. “Music of the primes” my foot!

Another way is to consider the set of numbers not divisible by any prime², call it A .

$$\frac{1}{N}|A \cap \{1, \dots, N\}| = \frac{1}{N}|A_k \cap \{1, \dots, N\}| - O\left(\sum_{p > p_k} \frac{1}{p}\right)$$

The density³ of A is 0. Here A_k is the integers after crossing out multiples of the prime number p_k .

$$0 = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)$$

As we run over all primes, we cross out all natural numbers⁴.

There's not a whole lot more to say, but I thought this was a good test case. Remember:

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots = \infty$$

²Always watch this words “consider” and “let” in mth papers.

³By **fundamental theorem of arithmetic**, $A = \{1\}$, which has density 0.

⁴I do not like arguing by contradiction. In the future, it could matter which one I pick. Here it seems to be a matter of style. It is best we continue...

I wonder if there are other starting points for the infinitude of primes:

$$0 < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < 1$$

No obvious contradiction here. Maybe there are enough prime powers to fill up the number line⁵.

In order to prove Legendre's 3-squares theorem, we need to show every arithmetic sequence has infinitely many primes

$$\sum_{p \in \mathcal{P}_{a\mathbb{Z}+b}} \frac{1}{p}$$

Even though the proof uses L-series, we can get away with just $L(1)$. In fact we have written it.

⁵This is how I waste my day... with dead-end questions like these.

References

- (1) JP Serre **Course on Arithmetic** Springer-Verlag
- (2) Davenport **Multiplicative Number Theory** Springer-Verlag