Item: **Cohomology of Arithmetic Groups**

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The Eichler-Shimura isomorphism let's you express weight-2 modular forms theory into cohology of $\mathsf{SL}_2(\mathbb{Z})$. This will take a lot of effort to unpack. One version I have found says these two are the same:

- $f(z)\,dz$ is a Γ-invariant form on $\mathbb{H}$
- $[\,f(z)dz\,] \in H^1(\mathbb{H}/\Gamma, \mathbb{C}) = H^1(\Gamma, \mathbb{C})$

but these are two different kinds of cohomology. One of them is a hyperbolic space $\mathbb{H}/\Gamma$ and the other is a group of 2×2 matrices $\Gamma \subseteq \mathsf{SL}_2(\mathbb{Z})$. How can we not have a complete understanding of both of these objects?

There's a trade-off between generality and our ability to supply details. I never told you what Γ was, and the entire textbook writes the discussion without naming a specific answer. How can they have the best possible answer? If, I decide to focus on one Γ, let's say $\Gamma_0(4) = \langle z \mapsto z+1, z \mapsto -\frac{1}{4z} \rangle$ maybe I will say things that don't generalize.

In between, would be story where I examine many possible Γ and a statement will be true in cases and not others (in many cases and not others). I might even be able to express this type of meta-logic using a small amount of category theory.

Let's find a modular form of weight 2. The first one I can think of is a theta-function raised to the 4th power:

$$\theta(z) = \left(\sum q^{n^2}\right)^4 = \sum r_4(n)\, q^n$$

and here $\Gamma_0(4) \neq \mathsf{SL}_2(\mathbb{Z})$. How do we know it is modular form of weight 2? This is a great example if we keep in mind the following recipe:

$$\mathsf{M}_2(\Gamma_0(N)) = \mathsf{S}_2(\Gamma_0(N)) \oplus \mathsf{E}_2(\Gamma_0(N))$$

for all congruence groups, not just $N=4$. This says every weight two modular form splits into to parts:

- Eisenstein series
- Cusp forms

The jargon gets worse and worse. Eichler-Shimura theory, Atkin-Lehmer theory. If I have an interestng number theory problem, maybe I can turn it into a modular forms problem:

$$\text{modular forms} \overset{?}{\neq} \text{number theory}$$

I cannot find any modular forms of weight 2 that are not Eisenstein series until $\Gamma_0(11)$

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References

(1) John Cremona **The L-functions and modular forms database project** `arXiv:1511.04289` `http://www.lmfdb.org/`

(2) William Stein **Modular Forms, A Computational Approach** Modular forms of Weight 2 `http://wstein.org/books/modform/modform/index.html`