Examples: Pell Equation

John D Mangual

Ex #1 Consider $1, \sqrt[3]{2}, \sqrt[3]{4}$ generating $\mathbb{Z}[\sqrt[3]{2}]$. What happens if we multiply by $\sqrt[3]{2}$?

$$\begin{array}{rcl} \sqrt[3]{2} \cdot 1 &=& a + b\sqrt[3]{2} + c\sqrt[3]{4} \\ \sqrt[3]{2} \cdot \sqrt[3]{2} &=& 2c + a\sqrt[3]{2} + b\sqrt[3]{4} \\ \sqrt[3]{2} \cdot \sqrt[3]{4} &=& 2b + 2c\sqrt[3]{2} + a\sqrt[3]{4} \end{array}$$

Here is the equation for units (vectors of size 1)

$$\det \begin{bmatrix} a & 2c & 2c \\ b & a & 2c \\ c & b & a \end{bmatrix} = a^3 + 2b^3 + 4c^3 - 6abc = 1$$

This is very much like the Pell equation.

Ex # 2 In the quadratic case, $K = \mathbb{Q}[\sqrt{2}]$. The equation is:

$$\det \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} = a^2 - 2b^2 = 1$$

and remember the continued fraction of the square root of two

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\ldots}} = [1; 2, 2, 2, \ldots] = [1; \overline{2}]$$

Does there exixt an analogous shorthand for Ex #1

Can we solve $\sqrt[3]{2} = \ldots$ in a similar way?

References

(1) ...