Cubic Pell Equation

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On Math.StackExchange I learned we can solve Pell's Equation purely from Pigeonhole Principle. Let's try[1]

$$x^2 - 17y^2 = 1$$

#1 Using Pigeonhole, there are infinely many pairs (x, y) with

$$x^2 - 17y^2 \leq 2\sqrt{17}$$

This stems from attempting to use the Euclidean algorithm to find the GCD of $\sqrt{17}$ and 1:

$$\sqrt{17} = 4 \times 1 + (\sqrt{17} - 4)$$

[1]Here is also

The number $\sqrt{17}-4$ is the remainder when we use the division algorithm. But this is getting ahead of ourselves.

The 18 multiples of $\sqrt{17}$ starting from 0 can be put into groups:

$$0, \sqrt{17}, 2\sqrt{17}, \ldots, 10\sqrt{17}$$

How do I compute the best integer approximations, I might square these:

$$0, 17, 68, \ldots, 4913$$

In the second case $68 = 8 \times 8 + 4$ so that $2\sqrt{7} - 8 < 1$. And one more:

$$9 \times 17 = 153 = 12 \times 12 + 9 \longrightarrow 3\sqrt{17} - 12 < 1$$

I know I shouldn't generalize, but probably we can take any multiple of $\sqrt{17}$ and find a number very close to it. One more:

$$10 \times 10 \times 17 = 1700 = 41 \times 41 + 19 \longrightarrow 10\sqrt{17} - 19 < 1$$

I forgot that in this case we get very lucky $4^2 - 17 \times 1^2 = -1$

I wanted to solve $x^2 - 17y^2 = 1$ (with $+1$ instead of -1)

Can we find two numbers (p, q) which give a very small remainder and $q < 10$?

$$0 < q\sqrt{17} - p < \frac{1}{10}$$

I think our initial guess still works. I hate when I get it on the first try. Not dramatic enough.

$$\sqrt{17} - 4 = \frac{(\sqrt{17} - 4)(\sqrt{17} + 4)}{\sqrt{17} + 4} = \frac{1}{\sqrt{17} + 4} < \frac{1}{8}$$

And worse of all it looks correct because like I said:

$$4^2 - 17 \times 1^2 = 1 \textbf{ not } -1$$

Technical point: in the words of Bill Clinton we have to meditate of the meaning of the word "is" ...

I just realized our example is not good enough. I asked for $\frac{1}{10}$ and not $\frac{1}{8}$. I went and looked at a calculator:

$$8\sqrt{17} = 32.984845\ldots \longrightarrow -8\sqrt{17} + 32 < \frac{1}{10}$$

What's funny about that…even if we asked $0 \leq q \leq 10$ we found a $q = -8$ which is out of our range[2]. This is why we can use absolute value sign:

$$\left|8\sqrt{17} - 32\right| < \frac{1}{10}$$

If we were really good, we'd do all the steps without a calculator. With a slide rule or something.

Lastly how bad is the error?

$$33 \times 33 - 8 \times 8 \times 17 = 1089 - 1088 = 1$$

[2]Acceptable just not what I originally had in mind

Solving Pell Equation with Pigeonhole

Let's try to approximate $\sqrt{19}$ as a fraction. We can list multiples of this number and see which one is nearest a whole number:

$$0, \sqrt{19}, 2\sqrt{19}, 3\sqrt{19}, \ldots 10\sqrt{19}$$

This is kind of like interlacing the perfect squares $\square = \{1, 4, 9, 16, \ldots$ and $19 \times \square$:

0, $1, 4, 9, 16,$ **19**, $25, 36, 49, 64,$ **76**, $81, 100, 121, 144, 169,$ **171**, $196, 225, 256$

$289,$ **304**, $324, 361, 400, 441,$ **475**, $484, 529, 576, 625, 676,$ **684**, $729, 784, 841,$

$900,$ **931**, $961, 1024, 1089, 1156,$ **1216**, $1225, 1296, 1369, 1444, 1521,$ **1539**

Interleaving these two squences of numbers, we can see nont of them are next to a square:

$$x^2 - 19y^2 \neq 1$$

So far. How are we going to generate an answer if none of these small numbers work?

$4^2 - 19 \times 1^2 = -3$

$31^2 - 19 \times 7^2 = 30$

We have no guarantee these differences should be small, but pigeonhole-principle has found us infinitely many.

Once we see that $0 < \sqrt{19} - 4 < 1$, let's save ourselves some time by just multiplying by this number instead.

$$\sqrt{19} - 4 = \frac{(\sqrt{19}-4)(\sqrt{19}+4)}{\sqrt{19}+4} = \frac{3}{\sqrt{19}+4} > \frac{1}{3}$$

So if I multiply this number by 4 I get a new solution slightly larger than 1.

$$4(\sqrt{19}-4) - 1 = 4\sqrt{19} - 17$$

In fact $4^2 \times 19 - 17^2 = 15$ which is rather large but still less than 19.

Finding units of $\mathbb{Z}[\sqrt[3]{2}]$ vis Pigeonhole

There's no continued fraction algorithm for cube roots. Some people think $\sqrt[3]{2}$ is a sequence that never repeats[3] – but nobody knows for sure.

Hale and Trotter print out the first 1000 digits (they fit neatly on a page), the paper is from the 1970's and the numbers are type-written. I don't know how they got a computer to do all of that... I can do it on my laptop with some effort.

There is a continued fraction algorithm due to Brun, which starts from the vector $(1, \sqrt[3]{2}, \sqrt[3]{4})$ and leads to approximate vectors in $\mathbb{Z}^3$ (up to a proportionate factor).

In a comuter simulation, the algorithm began repeating after 19 steps

[3] Think about that... how we we know a sequence avoids not just any pattern but all patterns. That seems like a rather odd thing.

but how do we know our Euclidean algorithm finished?

– insert discussion here –

References

(1) Math.StackExchange **What is your favorite application of the Pigeonhole Principle?** http://math.stackexchange.com/q/62565/4997