

Tune-Up: Lagrange 3-Squares Theorem

Problem Statement For a rational two-dimensional subspace L of \mathbb{R}^4 the discriminant of L is the square of the co-volume of $L \cap \mathbb{Z}^4$ in L :

$$\text{disc}(L) = \text{vol}(L/(L \cap \mathbb{Z}^4)^2) \in \mathbb{N}$$

For any $D \in \mathbb{N}$ let \mathcal{R}_D be the finite set of rational planes of discriminant D (which is a subset of the real Grassmanian $\text{Gr}_{2 \times 4}(\mathbb{R})$).

Theorem(Folklore, 1930s ?) \mathcal{R}_D is non-empty iff

$$D \in \mathcal{D} := \{D \in \mathbb{N} : D \not\equiv 0, 7, 12, 15, \pmod{16}\}$$

This is analogous to Legendre's theorem of sums of three squares, and works in the early 20th century on the sums of four squares.

Let's read their conjecture.

Conjecture(2019) The normalized counting measure on the finite set

$$\mathcal{J}_D = \{(L, [L(\mathbb{Z})], [L^+(\mathbb{Z})], [\Lambda_{a_1(L)}], [\Lambda_{a_2(L)}] : L \in \mathcal{R}_D\} \subseteq \text{Gr}(\mathbb{R}) \times \mathcal{X}_2^4$$

equidistributes to the uniform probability measure on $\text{Gr}_{2 \times 4}(\mathbb{R}) \times \mathcal{X}_2^4$ as $D \rightarrow \infty$ with $D \in \mathbb{D}$. That is:

$$\frac{1}{\mathcal{J}_D} \sum_{x \in \mathcal{J}_D} \delta_x \rightarrow m_{\text{Gr}_{2 \times 4} \times \mathcal{X}_2^4}$$

in the weak*-topology where $m_{\text{Gr}_{2 \times 4}}$ is the probability measure obtained from an $SO(4)$ invariant measure on $\text{Gr}_{2 \times 4}(\mathbb{R})$ and an $SL_2(\mathbb{R})$ -invariant measure on \mathbb{H}^2

There's an infinite amount of geometric background here.

- E.g. what does weak* convergence of measures look like?
- $SO(4)$ -invariant measure is usually given careful discussion in Lie algebra textbooks for some reason.

Here's what they actually prove (with congruence conditions). Resting on this conjecture (like a table) we could try to do a lot of numerical experimentation.

Theorem (Equidistribution) Let p, q be any two odd primes. The normalized counting measure on the (finite set) \mathcal{J}_D equidistributes to uniform probability measure on $\mathrm{Gr}_{2 \times 4}(\mathbb{R}) \times \mathcal{X}_2^4$ as $D \in \mathbb{D}$ goes off to infinity while D has the additional conditions $-D \in (\mathbb{F}_p^\times)^2$ and $-D \in (\mathbb{F}_q^\times)^2$.

These innocent counting-measure problems have been argued using modular forms (in particular Maass forms). Results of this type have only been around since the 1950's.

Results due to Maass himself show that $(L, L(\mathbb{Z}))$ are equidistributed for rational planes with equidistribution up to D .

- $L(\mathbb{Z}) := L \cap \mathbb{Z}^4$

Exercise Let $L = \{\frac{3}{5}x + \frac{4}{7} + \frac{5}{11}z + \frac{6}{13}w = 0\} \cap \{\frac{1}{7}x + \frac{2}{11} + \frac{6}{5}z + \frac{7}{13}w = 0\} \subseteq \mathbb{Q}^4$.

- Solve the equation in rational numbers \mathbb{Q} .
- Is $L(\mathbb{Z}) = \emptyset$?
- Find the co-volume $[L : (L \cap \mathbb{Z}^4)]$.

This one exercise should already illustrate the complexity of the equation involved, before we ask about a limiting statement.

References

- [1] Manfred Einsiedler, Elon Lindenstrauss **Joinings of Higher Rank Torus Actions on Homogeneous Spacs** arXiv:1502.05133
- [2] Manfred Einsiedler, Manuel Luethi, Nimish Shah **Primitive Rational Points on Expanding Horocycles in Products of the Modular Surface with the Torus** arXiv:1901.03078
- [3] Manny Aka, Manfred Einsiedler, Andreas Wieser **Planes in Four-Space and Four Associated CM Points** arXiv:1901.05833