Scratchwork: Infinite Products

9/24 Here, let's try working backwards from the answer. These numbers tend to be infinite products "regularized" in some way. Here is one:

$$Z = \prod_{n\in\mathbb{Z}} \prod_{j=0}^{\infty} \frac{j + |m| + 1 - \frac{r}{2} - \frac{1}{\tau}(2\pi i n - i\alpha)}{j + |m| + 0 + \frac{r}{2} + \frac{1}{\tau}(2\pi i n - i\alpha)}$$

Exercise Here's another (possibly equivalent?) infinite product. Let, $q = e^{-\tau}$ and $z = e^{i\tau\alpha}$:

$$Z = e^{-i\pi m^2/2}\big(q^{1-r/2}\, z^{-1}\big)^{|m|/2} \prod_{j=0}^{\infty} \frac{1 - q^{1-r/2+|m|/2+j}\ z^{-1}}{1 - q^{0+r/2+|m|/2+j}\ z}$$

Hint Have to consult a textbook:

- $(z;q) := \prod_{j=0}^{\infty}(1 - zq^j)$ (this is a notation)
- $\prod_{n\in\mathbb{Z}}(2\pi i n + z) = e^{-z/2}(1 - e^z)$

These numbers were obtained from a highly symmetric infinite dimensional object, but all we have is an ambiguous product of numbers. These computations are done rather expeditiously and my goal is just to check the logic for my own personal interest and to look for any missed opportunities.

> **Thm** Given an sequence $\{a_n\}$ of complex numbers with $|a_n| \to \infty$ as $n \to \infty$, there exists an entire function f that vanishes at $z = a_n$ and nowhere lse. Any other such entire function is of the form $f(z)e^{g(z)}$.
>
> **Thm** Suppose that f is entire and has growth order ρ_0. let k be the integer so that $k \le \rho_0 < k+1$. If $a_1, a_2, \ldots$ are the (non-zero) zeros of f then:
>
> $$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n)$$
>
> where P is a olynomial of degree $\le k$ and m is the order of the zero of f at $z = 0$.

I am not here to check their logic. Surely their answers are 100% correct. If you have a specific function it could be easier to prove it from scratch:

$$\sin \pi z = z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) \quad \text{and} \quad \pi \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

> **Thm** (Jensen) Let Ω be an open set that contains the closure of a disc D_R and suppose that f is holomorhic in Ω, $f(0) \neq 0$ and f vanishes nowhere on the circle C_R. If $z_1, \ldots, z_N$ denote the zeros of f inside the disc. Then
>
> $$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta$$

Stein's textbook is one of the few places I know that tells you a straight story. His proofs earmark steps you might take with less rigorous calculations to yield just a little more.

References

[1] **Localization techniques in quantum field theories**
`https://arxiv.org/abs/1608.02952`
`https://arxiv.org/src/1608.02952/anc/LocQFT.pdf`

[2] Vladimir Zorich **Mathematical Analysis I, II** (Universitext) Springer, 2015, 2016.

[3] Eberhard Freitag **Complex Analysis I, II** (Universitext) Springer, 2009, 2011.

[4] Elias Stein **Analysis I / II** Fourier Analysis / Complex Analysis (Princeton Lectures in Analysis) Princeton University Press, 2003.

[5] Tudor Dimofte, Davide Gaiotto, Sergei Gukov **3-Manifolds and 3d Indices** `arXiv:1112.5179`

9/26 Even more random formulas because we don't know any better. Here is the ABJM partition function:

$$Z(N,k) = \frac{1}{N!}\int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\prod_{i<j}\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2}\exp\left[\frac{ik}{4\pi}(\mu^2-\nu^2)\right]$$

A few remarks about this integral:

- it is called the ABJM partition function on the 3-sphere S^3 with gauge group $SU(N)$ – there could be other partition function on other three-manifolds
- This matrix integral arises as an infinite-dimensional path integral - localization reduces this to a finite dimensional "matrix integral" - a 0-dimensional path integral (i.e. over a point).
- Although the formula doesn't specify, we're guessing the domain of integration is $(\mu,\nu)\in\mathbb{R}^N\times\mathbb{R}^N$.
- What is the value of this integral? E.g. $Z(2,k)$?

If we evaluate the thing for ourselves, it's very unlikely that we'll repreat too many of the existing computations.

Q What was the infinite dimensional object we integrated over?

Here are some values of the ABJM partition function (as found by Putrov and Yamazaki)

$$Z\Big(N=1,k=1\Big)=\frac{1}{4} \text{ and } Z(2,1)=\frac{1}{16\pi} \text{ and } Z(3,1)=\frac{-3+\pi}{64\pi}$$

Here another evaluation of ABJM partition functions for $U(2)_k\times U(2)_{-k}$:

$$Z\Big(N=2,k=1\Big)=\frac{1}{\pi} \text{ and } Z(2,2)=\frac{1}{2\pi^2} \text{ and } Z(3,2)=\frac{1}{3}-\frac{1}{\pi} \text{ and } Z(5,2)=\frac{10-8\sqrt{5}}{25}-\frac{1}{\pi}$$

It looks like up top $Z(\cdot,6)$ has been computed. Regardless, given the effort to produce these numbers it's worth looking into the intermediate objects that were constructed. Mainly, for now these numbers are needed to validate that $\mathbb{R}^N\times\mathbb{R}^N$ is the correct domain of integration.

References

[1] Pavel Putrov, Masahito Yamazaki **Exact ABJM Partition Function from TBA** `arXiv:1207.5066`

[2] Yasuyuki Hatsuda, Sanefumi Moriyama, Kazumi Okuyama
Exact Results on the ABJM Fermi Gas `arXiv:1207.4283`
Instanton Effects in ABJM Theory from Fermi Gas Approach `arXiv:1211.1251`

[3] Kazumi Okuyama **A Note on the Partition Function of ABJM theory on S^3** `arXiv:1110.3555`