Proposal: Fermat Two Squares

John D Mangual

The last number theory project I want to propose is the Fermat Two-Squares problem.

$$p = x^2 + y^2 \longleftrightarrow p \equiv 1 \text{ mod } 4$$

It took me years to decide these are separate projects, compared to the 3- and 4- squares projects.

- every positive number is the sum of four squares
- every prime number (with remainder of 1 after dividing by 4) is the sume of two perfect squares

Our perspective will be slightly advaned. If you go on Wikipedia, our prime number theorem is a type of **Fermat Descent**. Other examples are:

- $\sqrt{2}$ is not a fraction
- Vieté Jumping (e.g. $x^2 + y^2 + z^2 = 3xyz$)
- $a^2 + b^4 = c^4$ has no integer solutions

I am less interested in the last one. It is very specific and very technical. Hopefully I can find the specific name for the reason I don't like it very much.

In a way, all of my projects are the same...no matter which equation I pick it all becomes hard-core analysis and cohomology.

Here is the from the Wikipedia page on Fermat's Infinite Descent:

> To extend this to the case of an abelian variety, André Weil had to make more explicit the way of quantifying the size of a solution, by means of a **height function** – a concept that became foundational.
>
> To show that $A(Q)/2A(Q)$ is finite, which is certainly a necessary condition for the finite generation of the group $A(Q)$ of rational points of A, one must do calculations in what later was recognised as **Galois cohomology**. In this way, abstractly defined cohomology groups in the theory become identified with ***descents*** in the tradition of Fermat.

One example of Fermat decent is that the solutions to

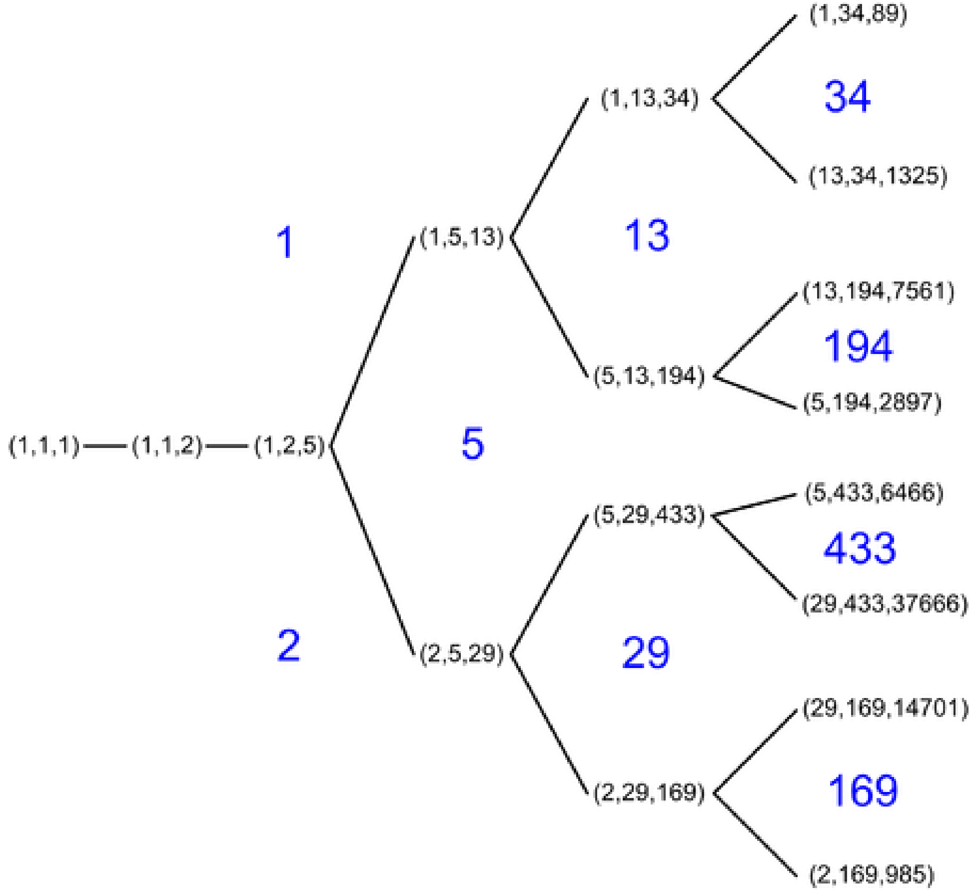
$$x^2 + y^2 + z^2 = 3xyz$$

form a tree. In the two squares case descent means that:

$$p = 102761 = 19 \times 320 + 320 \times 320$$

can be found in a systematic way. Our goal is to understand why his is an instance of Galois Cohomology.

Markoff Tree:



References

(1)