Examples: Quadratic Reciprocity

John D Mangual

Analytic number theory is not my expertise. Here we work out some softball examples related to quadratic reciprocity.

Here's a question: **Can permutations be used to prove Artin reciprocity** or even parts of Class Field Theory?

The proof of quadratic reciprocity seems like a random hodge-podge of techniques.[1] Can we unify some of these arguments using:

- Geometry of Numbers
- Pigeonhole Principle

Gauss in his *Disquiciones Arithmeticae* uses Pigeonhole to prove that $a^p \equiv 0 \mod p$.

[1] This is great for a first class when I was 15 years old it is not so great when you in graduate school are trying to learn Class Field Theory.

Any other applications?

Quadratic Reciprocity is stated in the number theory of textbook by **Hardy + Wright** in a Chapter called

Fermat's Theorem and its Consequences

after his discussion of other more advanced topics

- prime numbers
- Farey fractions
- irrational numbers
- congruences

I dislike prime number theory. Papers in that subject are quite tedious to read.

Fermat's little theorem says, e.g. $27|3^{27} - 1$:

$$a^p = a \mod p$$

a theorem that I really like is that $5 = 2^2 + 1^2$:

$$p = a^2 + b^2 \iff p = 4k + 1$$

For proof of Quadratic Reciprocity I always refer to

- John Conway, **The Sensual Quadratic Form**

and he will use a proof by Zolotarev, involving the permutation group.

So let's get started.

References

(1) Jared Weinstein. **Reciprocity laws and Galois representations: recent breakthroughs** Bull. Amer. Math. Soc. 53 (2016), 1-39

(2) David A Cox. **Primes of the Form $x^2 + ny^2$: Fermat, Class Field Theory, and Complex Multiplication** Wiley, 2013.

(3) **A prime ideal $\mathfrak{p}$ decomposes in $\mathbb{Q}(\zeta_{24})/\mathbb{Q}(\sqrt{-6})$ iff it is generated by $\alpha \in 1 + 2\mathbb{Z}[\sqrt{-6}]$** http://mathoverflow.net/q/234570/1358

(4) Roy L. Adler **Symbolic dynamics and Markov partitions** Bull. Amer. Math. Soc. 35 (1998), 1-56 http://www.ams.org/journals/bull/1998-35-01/S0273-0979-98-00737-X/