Examples: WKB Approximation

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We hit the ground running a bit here[1] - why are we drawing all these curves in the first place. I'll draw a few in a moment.

The **WKB approximation** says if the guage where:

$$\phi = \left(\begin{array}{cc} \lambda & 0 \\ 0 & -\lambda \end{array} \right)$$

there are two independent $\mathcal{A}$-flat sections of the form:

$$\psi^1 \sim \left(\begin{array}{c} e^{-\frac{R}{\zeta}\int^z \lambda} \\ 0 \end{array} \right), \psi^2 \sim \left(\begin{array}{c} 0 \\ e^{-\frac{R}{\zeta}\int^z \lambda} \end{array} \right)$$

therefore in the limit as $\zeta \to 0$ we obtain **essential singularities**, which behave like $e^{1/z}$ near $z = 0$.

[1] I have been encouraged to look at this topic by various sources and have had the privilege to meet Davide Gaiotto during my one-time visit to Perimeter and Andrew Neitzke at various conferences. Since the papers were written about 7 years ago. These read somewhat like Harry Potter, and they are quite lengthy. The reward, potentially is a new look at holomorphic functions and complex analysis (at least for me.

Locally the essential singularity looks like a spiral. Let's solve:

$$e^{1/z} \in \mathbb{R}$$

then $z = Re^{i\theta}$,

$$z(t) = z_0 e^{t\, e^{i\theta}}$$

so the generic WKB curve is a **logarithmic spiral**.

$$x^K + \sum_{k=2}^{K} u_k(z) x^{K-k} = 0$$

the coefficients polynomials of suitable degree[2]

This is a polynomial in x of degree K, but let's use some earth-shattering language and say that Σ it is K-fold cover of C, and that coefficients are sections of the sheaf $u_k \in H^0(C, K^{\otimes k})$. Which in turn which define a curve $\Sigma \subset T^*C$.

And we will set $K = 2$.

[2]There are lecture notes of Nigel Hitchen or Simon Donaldson – one of them – for undergraduates and he gives you the Riemann existence theorem. I didn't think to much of this but it's **rare** for any good explanation of this result.

I can say a tiny bit more the "Coulomb branch" of this theory is:

$$\mathcal{B} = \bigoplus_{k=1}^{r} H^0(C, K^{\otimes d_k})$$

so an element of the Coulomb branch is the as choosing these polynomial cofficients.

I feel these singularities are interesting in their own right, but we have indicated other motives.

Our choice of lie group $G = SU(2)$ or quiver $A = A_2$. The central charge is the integral of this differential:

$$Z = \frac{1}{\pi} \otimes_\gamma \lambda$$

yet this theory is also described by a case of Hitchin's equations:

$$\begin{aligned} F + R^2[\phi, \overline{\phi}] &= 0 \qquad (1) \\ \overline{\partial}_A \phi := \big(\partial_{\overline{z}}\phi_z + [A_{\overline{z}}, \phi_z)]\big) d\overline{z} \wedge dz &= 0 \qquad (2) \\ \partial_A \phi := \big(\partial_z \phi_{\overline{z}} + [A_z, \phi_{\overline{z}})]\big) dz \wedge d\overline{z} &= 0 \qquad (3) \end{aligned}$$

and the solutions to this equation can be combined into a single connection:

$$\mathcal{A} = \frac{R}{\zeta}\phi + A + R\zeta\overline{\phi}$$

this is much analogous to how the **real** and **imaginary** parts combine to form a **complex number**. These equations involve twistor theory which for now is just a parameter $\zeta \in \hat{\mathbb{C}}$ and especially $\zeta \in 0, \infty$ but also $\zeta \in \hat{\mathbb{C}}$.

Then near $\zeta = 0$ we know the singular behavior of $\mathcal{A}$ is determined by the zeros and polesof λ, but putting all this physics aside our singularities are of the type:

$$e^{\frac{R}{\zeta}\int^{z}\sqrt{\frac{p(x)}{q(x)}}\,dz}$$

and which obviously has a bunch of essential singuarlties the tool we shall use to evaluate these are **resurgence analysis** otherwise known as “steepest descent”.

– TODO –

- what is a 2D wall-crossing formula?
- what is a 4D wall-crossing formula?
- how the WKB curves predict such formulas?
- how to obtain **flat-surfaces** and the **siegel-veech** constants.

References

(1) Davide Gaiotto, Gregory W. Moore, Andrew Neitzke. **Wall-Crossing in Coupled 2d-4d Systems.** `arXiv:1103.2598v1`

(2) Davide Gaiotto, Gregory W. Moore, Andrew Neitzke. **Wall-crossing, Hitchin Systems, and the WKB Approximation.** `arXiv:0907.3987`

References

(1) Alex Eskin, Howard Masur, Anton Zorich **Moduli Spaces of Abelian Differentials: The Principal Boundary, Counting Problems and the Siegel–Veech Constants** `math/0202134`

(2) David Sauzin. **Introduction to 1-summability and resurgence.** `arXiv:1405.0356`

(3) Tom Bridgeland, Ivan Smith. **Quadratic differentials as stability conditions.** `1302.7030`

(4) Kohei Iwaki, Tomoki Nakanishi **Exact WKB analysis and cluster algebras** `1401.7094`

(5) David Aulicino **The Cantor-Bendixson Rank of Certain Bridgeland-Smith Stability Conditions** `1512.02336`