Tune-Up: Categories

Let's read through this documentations.

Def Let $\mathcal{C}$ be a category. There exists an object $\mathbf{1}$ with

- There exists a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with $(X, Y) \mapsto X \otimes Y$.
- $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ (**associative**)
- $X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$

An infinite amount of sleight-of-hand follows.

Canonical isomorphisms:

$$\mathbf{1} \otimes (\mathbf{1} \otimes X) \longrightarrow (\mathbf{1} \otimes \mathbf{1}) \otimes X \longrightarrow \mathbf{1} \otimes X$$

Left Unit Constraint:

$$\begin{tikzcd}
\mathbf{1} \otimes (\mathbf{1} \otimes X) \arrow[r] \arrow[dr] & (\mathbf{1} \otimes \mathbf{1}) \otimes X \arrow[d] \\
 & \mathbf{1} \otimes X
\end{tikzcd}$$

Right Unit Constraint:

$$\begin{tikzcd}
X \otimes (\mathbf{1} \otimes \mathbf{1}) \arrow[r] \arrow[dr] & (X \otimes \mathbf{1}) \otimes \mathbf{1} \arrow[d] \\
 & X \otimes \mathbf{1}
\end{tikzcd}$$

Left and Right Unit Constraint

$$\begin{tikzcd}
\mathbf{1} \otimes X \arrow[r] \arrow[d] & X \arrow[d, "f"] & X \otimes \mathbf{1} \arrow[l] \arrow[d] \\
\mathbf{1} \otimes Y \arrow[r] & Y & Y \otimes \mathbf{1} \arrow[l]
\end{tikzcd}$$

Fact Check ?

- **Set** with Cartestian product is monoidal category with one-element set $\{*\}$ as the unit.
- R-**Mod** the category of modules over commutative ring R with tensor product $\otimes_R$
- k-**Vect** the category of abelian groups over a field K has the one-dimensional vector space as the "unit".
- **Ab** $\simeq$ $\mathbb{Z}$-**Mod** Abelian groups are the category of $\mathbb{Z}$-modules. Here $\otimes$ is the product.
- $\mathbb{C}[G]$-**Mod** $\simeq$ Rep$[G]$ is the category of representations of G or of $\mathbb{C}[G]$ modules.

References

[1]