Fibonacci Numbers

John D Mangual

I've been trying to figure out why in the middle of a rather serious paper, Curtis McMullen writes about the Fibonacci numbers. $F_0 = 0, F_1 = 1$:

$$F_n = \mathrm{tr}_{\mathbb{Q}}^K \left[\epsilon^m / \sqrt{D} \right] \text{ so that } F_m \asymp \epsilon^m$$

Or we can write in the usual, recursive bunny-rabit format:

$$F_{m+1} = tF_m - nF_{m-1}$$

here t and n are the **trace** and the **norm** of the number field unit ϵ. The number field is just adjoining the $\sqrt{D}$ to the rational numbers:

$$K = \mathbb{Q}(\sqrt{D})$$

Then ϵ is a unit in this quadratic field and it satisfies a quadratic equation:

$$\epsilon^2 - t\,\epsilon + n = 0 \text{ with } t = \mathrm{tr}_{\mathbb{Q}}^K(\epsilon) \text{ and } n = \mathrm{N}_{\mathbb{Q}}^K(\epsilon) = \pm 1$$

McMullen does something (standard) but a bit freaky, lifing the field $\mathbb{Q}(\sqrt{D})$ off the number line into two dimensions:

$$\mathbb{Z}[\epsilon] = \mathbb{Z} \oplus \mathbb{Z}\,\epsilon = (1, \epsilon)$$

The actions of ϵ and $\sqrt{D}$ get promoted to 2×2 matrices:

$$\epsilon \sim U = \left(\begin{array}{cc} 0 & -n \\ 1 & t \end{array} \right) \text{ and } \sqrt{D} \sim S = 2U - tI = \left(\begin{array}{cc} -t & -n \\ 2 & t \end{array} \right)$$

The Fibonacci identity lifts to one for 2×2 matrices:

$$U^m = f_m\,U - n\,f_{m-1}\,I$$

Curtis McMullen's objectives were to prove three results:

#1 Any real quadratic field $\mathbb{Q}(\sqrt{d})$ contains infinitely many periodic continued fractions $x = \overline{[a_0, \ldots, a_{p-1}]}$ with $1 \leq a_i \leq M_d$.

#2 For any fundamental geodesic $\gamma \subset \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ there is a compact subset of M that contains infinitely many primitive, closed geodesics whose lengths are multiples of $L(\gamma)$. (A geodesic γ is primitive if it's indivisible in $\pi_1(M)$.

The meaning of the word "compact" here in this context exceeds my geometric intuition:

- Complete geodesics lying in Z form a closed set $G(Z)$ of measure zero.
- Geodesics of length $m\,L(\gamma)$ become uniformly distributed in $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ as $m \to \infty$.
- Most geodesics whose lengths are multiples $m\,L(\gamma)$ are not contained in Z, but infinitely many of them are.
- Hausdorff dimension of $G(Z)$ can be made arbitrarily close to 2 if Z is large enough.

#3 In any real quadratic field K, there are infinitey many ideal classes with $\delta(I) > \delta_K > 0$.

- $\delta(I) = \frac{N^*(I)}{\det(I)}$ "packing density"
- $\det(I) = \sqrt{\mathrm{disc}(I)}$
- $N^*(I) = \min\left\{|N_{\mathbb{Q}}^{K}(x)| : x \in I \text{ and } N_{\mathbb{Q}}^{K}(x) \neq 0\right\}$

#2a (extension of result #2 for Bianchi groups)
For any fundamental geodesic in the hyperbolic orbifold $\mathbb{H}/\mathrm{SL}_2(\mathcal{O}_d)$, there is a compact set that contains infinitely many closed primitive geodesics whose lengths are multiples of $L(\gamma)$.

McMullen's main construction:

Start with matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ such that

- $A^2 = I$
- $\mathrm{tr}(A) = 0$
- $\mathrm{tr}(A^\dagger U) = \pm 1$

and let $L_m = U^m + U^{-m} A$. Then for all $m \geq 0$:

- $|\det(L_m)| = f_{2m}$ is a Fibonacci number
- The lattice $[L_m]$ is fixed by U^{2m}
- $L_{-m} = L_m\, A$
- $\big|\big|U^i L_m U^{-i}\big|\big|, \big|\big|U^i L_m U^{-i}\big|\big| \leq C\sqrt{|\det L_m|}$

This discussion has not used any modular forms, even though Duke's equidistribution theorem was proven using the elusive **Maass forms**. Not sure what gives.

Likely: many open questions about the geometry of the hyperbolic surfaces and 3-manifolds.

References

(1) Curtis McMullen
Uniformly Diophantine numbers in a Fixed Real Quadratic Field
http://www.math.harvard.edu/~ctm/papers/home/text/papers/cf/cf.pdf
Horocycles in Hyperbolic 3-manifolds
http://www.math.harvard.edu/~ctm/papers/home/text/papers/horo/horo.pdf

(2) Alex Brandts, Tali Pinsky, Lior Silberman
Volumes of hyperbolic three-manifolds associated to modular links arXiv:1705.04760

These references are more advanced, but also they are using the theory of modular forms, which I have deliberately left out of the previous discussion. Wouldn't it be nice to find mmore elementary stuff to go here?

References

(1) Paul D. Nelson **Quantum variance on quaternion algebras, I** arXiv:1601.02526

(2) Paul D. Nelson **Quantum variance on quaternion algebras, I** arXiv:1702.02669