Tune-Up: Fubini Theorem

What is additive combinatorics and can we extract more common-sense statements from them? For the time being, these are stated in a somewhat sophisticated manner.

These two exercises, require Lebesgue integrals (that's how they are stated).

Ex #1 Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Show that:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x) f(x+t) dx dt \leq ||f||^2_{L^1(\mathbb{R})}$$

Hint: Fubini Theorem. Here, "$\int$" is Lebesgue integral.

Ex #2 For any $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} f(x) f(x+t) dx \leq ||f||^2_{L^2(\mathbb{R})}$$

Hint: Cauchy-Schwartz inequality.

The author is at Yale (this is the same department as Grigori Margulis and Richard Kenyon, for example.) Here's the toy example have about the autocorrelation of functions.

Proposition Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ Then

$$\left|\left| \int_{\mathbb{R}} f(x) f(x+t) dt - \chi_{[-1,1]}(t) \right|\right|_{L^2(\mathbb{R})} \geq \frac{3}{10}$$

Hint: The Fourier transform is unitary.

Analogy A set $A \subset \mathbb{Z}$ is a difference basis with respect to n if

$$\{1, 2, \ldots, n\} \subseteq A - A$$

What is the minimum size of A? This is an example due to Hungarian mathematicians Redei and Renyi. Here is the trivial estimate:

$$n = \#\{1, 2, \ldots, n\} \leq \#((A - A) \cap \mathbb{N}) \leq \binom{|A|}{2} \leq \frac{|A|^2}{2}$$

therfore the size of the mimimum set is $|A| \geq \sqrt{2n}$.

Exercise (Half-of-Fubini) Let $f \in L^1(\mathbb{R})$, then

$$\int_0^1 \int_{\mathbb{R}} f(x) f(x+t) dx dt \leq 0.5 ||f||^2_{L^1(\mathbb{R})}$$

What are the reasonable choices of f ? This type of number theory is describing patterns that we find spread out in physical space, and frequency space and interactions.

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References

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[3] Elias Stein **Real Analysis: Measure Theory, Integration and Hilbert Spaces** (Princeton Lectures in Analysis, III) Princeton University Press, 2005.

[4] Yitzak Katznelson **Harmonic Analysis** Dovr Publications, 1976.