Wallis' Infinite Product for $\pi/2$

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The Gamma function

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

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The Sine Function

$$\sin x = x \times \left(1 - \frac{x^2}{\pi^2}\right) \times \left(1 - \frac{x^2}{4\pi^2}\right) \times \left(1 - \frac{x^2}{9\pi^2}\right)$$

From a (3rd Year) Complex Analysis Textbook

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Why is $n!$ like $\sin x$?

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$$(-\tfrac{1}{2})! = \Gamma\left(\frac{1}{2}\right)^2 = \pi$$

All Gamma functions can be written as factorial.

Why is $n!$ like $\sin x$?

$$x! = \lim_{n\to\infty} \frac{n!n^x}{(x+1)\dots(x+n)}$$

This is Euler's definition of Factorial. It works at $x = -\frac{1}{2}$

$$(-\frac{1}{2})! = \lim_{n\to\infty} \frac{n!n^x}{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\cdots\times\frac{2n-1}{2}} = \sqrt{\pi}$$

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$$(n+1)\times(n+2)\cdots\times(n+x) \stackrel{?}{\approx} n^x$$

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This is not quite right:

$$(n+1) \times (n+2) \cdots \times (n+x) = n^x \times (1+\frac{1}{n}) \times \cdots \times (1+\frac{x}{n})$$

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Euler's definition predicts:

$$\begin{aligned}(n+1)\times(n+2)\cdots\times(n+x) &= n^x\times(1+\frac{1}{n})\times\cdots\times(1+\frac{x}{n}) \\ &= n^x\times\frac{n!}{x!}\end{aligned}$$

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So when we see Stirling's formula is the π really an anomaly?

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Why is there this link between combinatorics and trigonometry?

Derivaton of Stirling's Formula

$$x! = \lim_{n\to\infty} \frac{n!n^x}{(x+1)\dots(x+n)}$$

Euler's - very reasonable - definition for $x \in \mathbb{R}$ also:

$$\log n! = \log 1 + \log 2 + \cdots + \log n = \int_1^n \log x \, dx$$

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I always recall this formula by trial-and-error

$$\int_1^n \log x \, dx = \Big(x \log x - x\Big)\Big|_{x=1}^{n} = n \log n - n$$

Easy Derivaton of Stirling's Formula

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$$\int_1^n \log x \, dx = \Big(x \log x - x\Big)\Bigg|_{x=1}^{n} = n \log n - n$$

if we take the exponent of both sides:

$$n! \approx \sqrt{2\pi n} \times \left(\frac{n}{e}\right)^n$$

The $\sqrt{n}$ factor is missing, causing must distress.

Miscellaneous Formulas

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

and the multiplication formula:

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

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and knowing this formula, we can set $z = \frac{1}{3}$:

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = \frac{2\pi}{\sqrt{3}}$$

and if we do the multiplication formula:

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{5}{6}) = 2^{1/3}\sqrt{\pi}\Gamma(\tfrac{2}{3})$$

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A more familiar connection between $n!$ and $\sin x$ . . .

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{(x)^{2n}}{2n!}+\sum_{n=0}^{\infty}(-1)^n\frac{(x)^{2n+1}}{2n+1!} = \cos x + i \sin x$$

Volume of a Sphere

$$\mathrm{Vol}(S^n) = \int_{S^n} 1\, dV = 2 \times \frac{\Gamma(\frac{1}{2})^n}{\Gamma(\frac{n}{2})}$$

I picked **surface area** instead of **volume** to make the formula neater.

Volume of a Sphere

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What is the Factorial / Gamma function doing here?

$$x_1^2 + \cdots + x_n^2 = 1$$

Shopping on MathWorld I found this formula possibly generated by computer:

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3} \cdot \sqrt{2+\sqrt{3}}$$

This is a lot more interesting!

Noam Elkies shows that this formula can also be derived from the multiplication formula[1]:

$$\Gamma(z)\Gamma(z+\frac{1}{3})\Gamma(z+\frac{2}{3}) = 2\pi \cdot 3^{\frac{1}{2}-3z}\Gamma(3z)$$

[1]Notice we never quite get away with a factor of $\sqrt{3^{1-6z}}$ this is similar to the doubling formula:

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

The left side has poles at $z = -1, -2, -3, \ldots$ as well as $z = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$ and the right side has poles at all the negative half integers $z \in -\frac{1}{2}\mathbb{N}$. While this kind of reasoning might make sense, it went under further scrutiny still.

Let's follow Elkies' instructions at set $z = \frac{1}{24}$ and also $z = \frac{1}{8}$:

$$\Gamma(\frac{1}{8})\Gamma(\frac{11}{24})\Gamma(\frac{19}{24}) = 2\pi \cdot 3^{\frac{1}{8}}\Gamma(\frac{3}{8})$$

but also

$$\Gamma(\frac{1}{24})\Gamma(\frac{3}{8})\Gamma(\frac{17}{24}) = 2\pi \cdot 3^{\frac{5}{8}}\Gamma(\frac{1}{8})$$

and sure enough when you multiply the answer is:

$$\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})\Gamma(\frac{19}{24})\Gamma(\frac{17}{24}) = 4\pi^2\sqrt{3}$$

Yet if we set $z = 5/24$ and $z = 7/24$ into the mirror formula:

$$\Gamma(\frac{5}{24})\Gamma(\frac{19}{24}) = \frac{\pi}{\sin 5\pi/24}$$

and also

$$\Gamma(\frac{7}{24})\Gamma(\frac{17}{24}) = \frac{\pi}{\sin 7\pi/24}$$

and multiplying these we get:

$$\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{17}{24})\Gamma(\frac{19}{24}) = \frac{\pi^2}{\sin\frac{5\pi}{24}\sin\frac{7\pi}{24}}$$

There is even more cancellation:

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})\Gamma(\frac{17}{24})\Gamma(\frac{19}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{17}{24})\Gamma(\frac{19}{24})} = 4\sqrt{3}\sin\frac{5\pi}{24}\sin\frac{7\pi}{24}$$

It remains to show that:

$$\sin\frac{5\pi}{24} = \sin\frac{7\pi}{24} = \sqrt[4]{2+\sqrt{3}}$$

I noticed immediately the action of:

- $z \mapsto 1 - z$
- $z \mapsto z + \frac{1}{3}$
- on $\frac{1}{24}\mathbb{Z}$

I never thought too much before of $\boxed{\dfrac{1}{24} + \dfrac{1}{3} = \dfrac{3}{8}}$ but here we are.

B the second more complicated solution

randomly use the triplication formula until we get someting that works:

$$\Gamma(\tfrac{1}{3}x)\Gamma(\tfrac{1}{3}x+1)\Gamma(\tfrac{1}{3}x+2) = \frac{2\pi}{3^{x-\frac{1}{2}}}\Gamma(x)$$

In a way, I don't worry too much about the letter Γ or the algebraic fractor of: $\frac{2\pi}{3^{x-\frac{1}{2}}}\Gamma(x)$.

I can just write a shorthand of brackets $[\,\cdot\,]$ so

$$\left[\tfrac{1}{3}x\right] \oplus \left[\tfrac{1}{3}x+1\right] \oplus \left[\tfrac{1}{3}x+2\right] \approx \left[x\right]$$

I don't know which combination will work in advance so I keep writing them out:

$$\left[\frac{1}{24}\right] \oplus \left[\frac{3}{8}\right] \oplus \left[\frac{17}{24}\right] \approx \left[\frac{1}{8}\right] \tag{1}$$

$$\left[\frac{1}{8}\right] \oplus \left[\frac{11}{24}\right] \oplus \left[\frac{19}{24}\right] \approx \left[\frac{3}{8}\right] \tag{2}$$

Then we can add these two equations and conclude:

$$\left[\frac{1}{24}\right] \oplus \left[\frac{11}{24}\right] \oplus \left[\frac{17}{24}\right] \oplus \left[\frac{19}{24}\right] \approx \left[0\right]$$

These equations may wind up becoming faulty, but seem to do the bookkeeping for us. At least part of it.

C Can all Chowla-Selberg formulas be proven with careful use of the mirror + multiplication formulas?

$$\log \Gamma(x)$$

Here we use Fourier expansion instead of Riemann sum:

$$\left(\frac{1}{2} - x\right)(\gamma + \log 2) + (1 - x)\log \pi - \frac{1}{2}\log \sin \pi x \sum_{n=1}^{\infty} \frac{\log n}{n\pi} \sin 2\pi n x$$

Then multiply both sides by Legendre symbol:

$$\sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \log \Gamma \left(\frac{n}{p}\right) = -(\log +2\pi) \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) n + \sqrt{p} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \frac{\log n}{n\pi}$$

and the Chowla-Selberg formula in a logarithmic form.

Here is an example:

$$\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4}) \int_0^1 x^{-3/4}(1-x)^{-1/4}(1-x/64)^{-1/4}\,dx$$

is equal to

$$\left[\frac{7\pi}{2} \times \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)}\right]^{1/2}$$

Chowla's paper deserves a more careful reading than this[2].

[2]Somehow we needed the Fourier series to prove the multiplication formula and mirror formulas. The Weierstrass product formula could yield a quick proof of

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

It would be quick if we proved the Weierstrass product formula. Fourier series also has the same fine print.

References

(1) MathOverflow **show that** $\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3}\cdot\sqrt{2+\sqrt{3}}$ `http://mathoverflow.net/q/249164`