2 Midterm 2 Problem 5

a) (2pts)Based on the rotational symmetry along the x-axis, the electric field is pointing at the $+x$ direction.

(2pts)Cut the anulus into thin rings with radius $r \sim r + \mathrm{d}r$. The charge carried by the thin ring is

$$\mathrm{d}Q = \sigma 2\pi r \mathrm{d}r$$

with σ the charge density

$$\sigma = \frac{Q}{\pi(R_1^2 - R_2^2)}$$

(3pts)The electric field from such a thin ring is

$$\mathrm{d}E_x = \frac{\mathrm{d}Q}{4\pi\epsilon_0} \frac{a}{(\sqrt{a^2 + r^2})^3}$$

(3pts)The total electric field is

$$E_x(a) = \int \mathrm{d}E_x = \frac{2\pi a\sigma}{4\pi\epsilon_0} \int_{R_2}^{R_1} \frac{rdr}{(\sqrt{a^2 + r^2})^3} = \frac{Qa}{2\pi\epsilon_0(R_1^2 - R_2^2)} \left(\frac{1}{\sqrt{a^2 + R_2^2}} - \frac{1}{\sqrt{a^2 + R_1^2}} \right)$$

b) (5pts)Take the electric potential at ∞ as 0. Then the electric potential at P is

$$\begin{aligned} V(a) &= -\int_{+\infty}^{a} E_x(a')da' = \int_{a}^{+\infty} E_x(a')da' \\ &= \frac{Q}{2\pi\epsilon_0(R_1^2 - R_2^2)} \int_a^\infty \left(\frac{a'}{\sqrt{a'^2 + R_2^2}} - \frac{a'}{\sqrt{a'^2 + R_1^2}} \right) \mathrm{d}a' \end{aligned}$$

(3pts)The integral should be evaluated carefully,

$$\int_a^\infty \left(\frac{a'}{\sqrt{a'^2 + R_2^2}} - \frac{a'}{\sqrt{a'^2 + R_1^2}} \right) \mathrm{d}a' = \lim_{L\to\infty} \int_a^L \left(\frac{a'}{\sqrt{a'^2 + R_2^2}} - \frac{a'}{\sqrt{a'^2 + R_1^2}} \right) \mathrm{d}a'$$

$$= \lim_{L\to\infty} -(\sqrt{a^2 + L^2} - \sqrt{a^2 + R_1^2}) + (\sqrt{a^2 + L^2} - \sqrt{a^2 + R_2^2}) = \sqrt{a^2 + R_1^2} - \sqrt{a^2 + R_2^2}$$

so that

$$V(a) = \frac{Q}{2\pi\epsilon_0(R_1^2 - R_2^2)} (\sqrt{a^2 + R_1^2} - \sqrt{a^2 + R_2^2})$$

c) (1pt)The electric field is 0.

(2pts)If the anulus is a conductor, then the charges are actually accumulated just on the edge of the anulus. Then we are talking about the electric field of two charged circles, which are both zero. Thus, the total electric field is zero.

(or 2pts)The symmetry argument is also accepted. The **E** must be along the x-axis based on x-axis rotational symmetry. And the system has a reflection symmetry against the $y - z$-plane. The electric field at the origin must be zero, otherwise it doesn't follow the reflection symmetry.