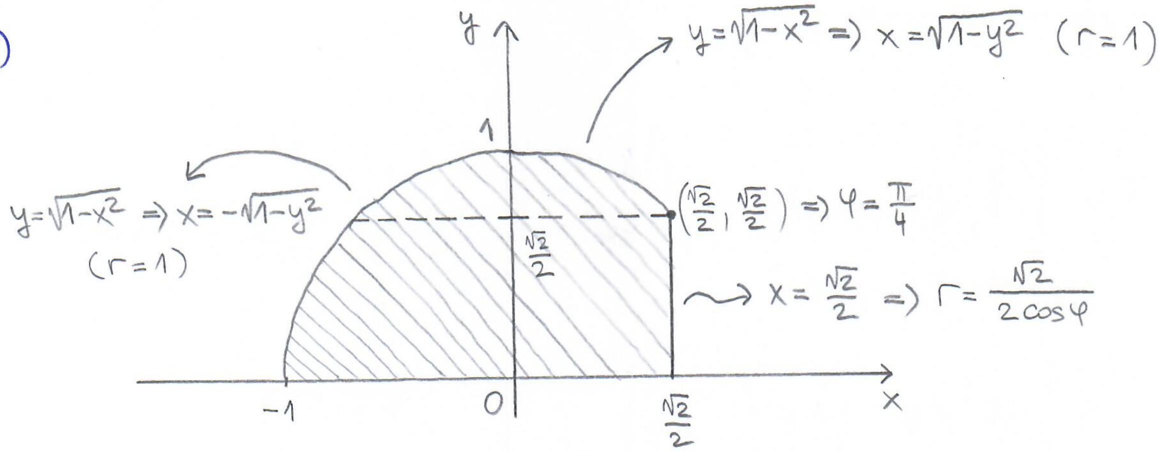
MATAN 2 - 6. vježbe

1. $$I=\int_{-1}^{\frac{\sqrt{2}}{2}}\int_{0}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

(a)



Područje integracije je dio jediničnog kruga u 1. i 2. kvadrantu „do pravca" $x=\frac{\sqrt{2}}{2}$. Zamjenom poretka integracije dijelimo područje integracije na dva dijela:

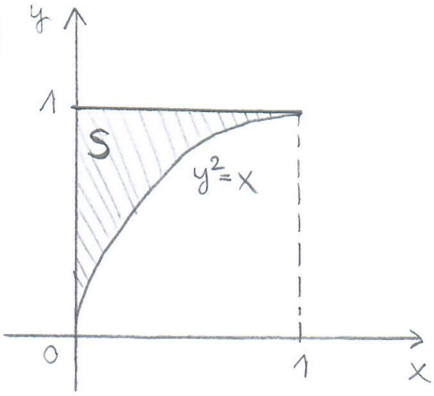
$$I=\int_{0}^{\frac{\sqrt{2}}{2}}\int_{-\sqrt{1-y^2}}^{\frac{\sqrt{2}}{2}} f(x,y)\,dx\,dy+\int_{\frac{\sqrt{2}}{2}}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

(b) U polarnim koordinatama ponovno dijelimo područje integracije na dva dijela:

$$I=\int_{0}^{\frac{\pi}{4}}\int_{0}^{\frac{\sqrt{2}}{2\cos\varphi}} f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi$$

$$+\int_{\frac{\pi}{4}}^{\pi}\int_{0}^{1} f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi.$$

2.



$$I = \iint\limits_S e^{\frac{x}{y}}\, dx\, dy$$

$$I = \int_0^1 \int_0^{y^2} e^{\frac{x}{y}}\, dx\, dy = \int_0^1 \left(y e^{\frac{x}{y}} \Big|_0^{y^2} \right) dx = \int_0^1 (y e^y - y)\, dy$$

$$= \int_0^1 y e^y\, dy - \left(\frac{1}{2} y^2 \Big|_0^1 \right) = \begin{bmatrix} u = y & \Rightarrow du = dy \\ dv = e^y\, dy & \Rightarrow v = e^y \end{bmatrix}$$

$$= y e^y \Big|_0^1 - \int_0^1 e^y\, dy - \frac{1}{2} = e - 0 - \left(e^y \Big|_0^1 \right) - \frac{1}{2}$$

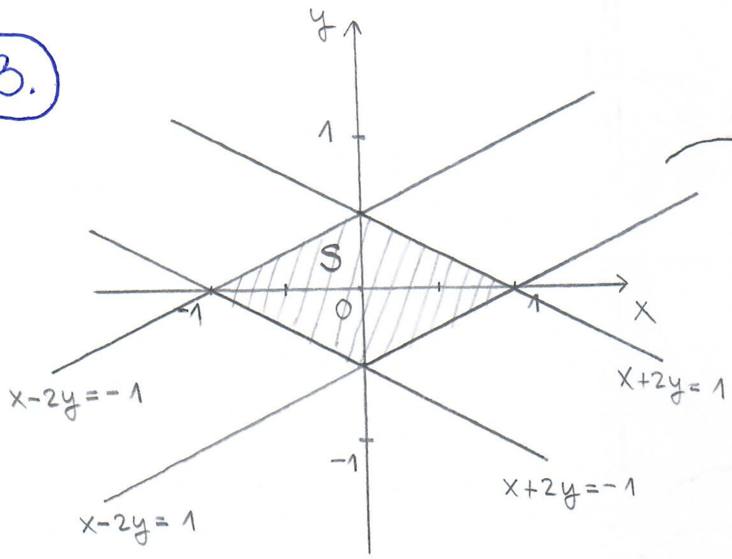
$$= e - 0 - e + 1 - \frac{1}{2} = \frac{1}{2}$$

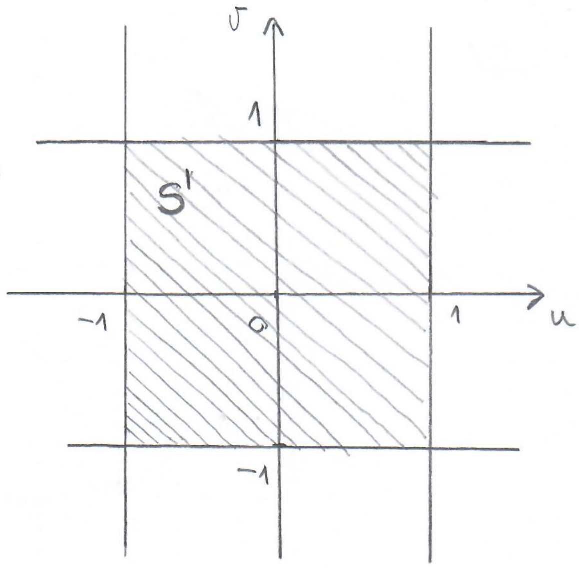
Nap. Zamjenom poretka integracije dobivamo

$$I = \int_0^1 \int_{\sqrt{x}}^1 e^{\frac{x}{y}}\, dy\, dx,$$

što je integral koji ne možemo elementarno izračunati (integral $\int e^{\frac{1}{t}}\, dt$ se ne može prikazati pomoću konačno mnogo elementarnih funkcija).

3.





Uvedimo zamjenu varijabli $u = x+2y$, $v = x-2y$ (uočimo da tada područje integracije postaje kvadrat $S' = [-1,1]^2$). Imamo

$$\begin{aligned} u &= x+2y \\ v &= x-2y \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \frac{1}{2}u + \frac{1}{2}v \\ y &= \frac{1}{4}u - \frac{1}{4}v \end{aligned}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{4}$$

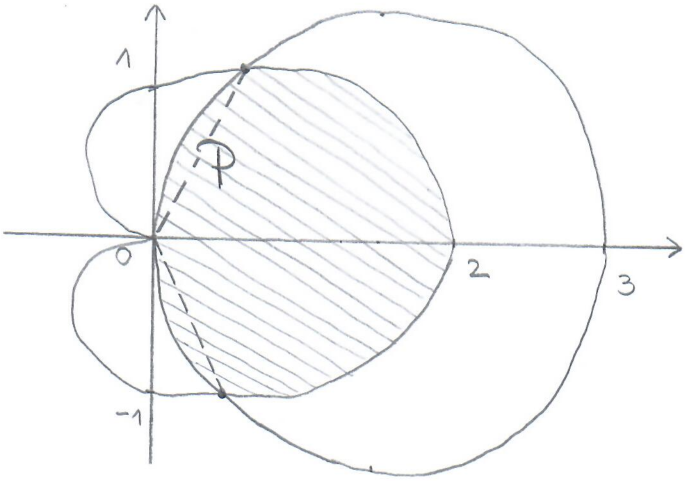
$$\Rightarrow \iint_S (x+2y)^{10}(x-2y)^8 \, dx\,dy = \iint_{S'} u^{10} v^8 \cdot |J| \, du\,dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} \frac{1}{4} u^{10} v^8 \, du\,dv = \frac{1}{4}\left(\int_{-1}^{1} u^{10}\,du\right)\left(\int_{-1}^{1} v^8\,dv\right)$$

$$= \left[\begin{array}{l}\text{integriramo parne fje} \\ \text{na simetričnom intervalu}\end{array}\right] = \frac{1}{4}\cdot 4\left(\int_0^1 u^{10}\,du\right)\left(\int_0^1 v^8\,dv\right)$$

$$= \left(\frac{1}{11}u^{11}\Big|_0^1\right)\left(\frac{1}{9}v^9\Big|_0^1\right) = \frac{1}{11}\cdot\frac{1}{9} = \frac{1}{99}$$

4.



$r = 1 + \cos\varphi \rightarrow$ kardioida

$r = 3\cos\varphi \Rightarrow r^2 = 3r\cos\varphi$

$\Rightarrow x^2 + y^2 = 3x$

$\Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

$\hookrightarrow$ kružnica $k\left(\left(\frac{3}{2}, 0\right), \frac{3}{2}\right)$

Odredimo presjek zadanih krivulja

$$\left.\begin{array}{l} r = 1 + \cos\varphi \\ r = 3\cos\varphi \end{array}\right\} \Rightarrow 1 + \cos\varphi = 3\cos\varphi \Rightarrow \cos\varphi = \frac{1}{2} \Rightarrow \varphi = \pm\frac{\pi}{3}$$

Da bismo izračunali zadanu površinu, podijelit ćemo područje integracije na tri dijela:

$$P = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_0^{1+\cos\varphi} r\,dr\,d\varphi + \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} \int_0^{3\cos\varphi} r\,dr\,d\varphi + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{3\cos\varphi} r\,dr\,d\varphi$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2} r^2 \Big|_0^{1+\cos\varphi}\right) d\varphi + \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} \left(\frac{1}{2} r^2 \Big|_0^{3\cos\varphi}\right) d\varphi + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{2} r^2 \Big|_0^{3\cos\varphi}\right) d\varphi$$

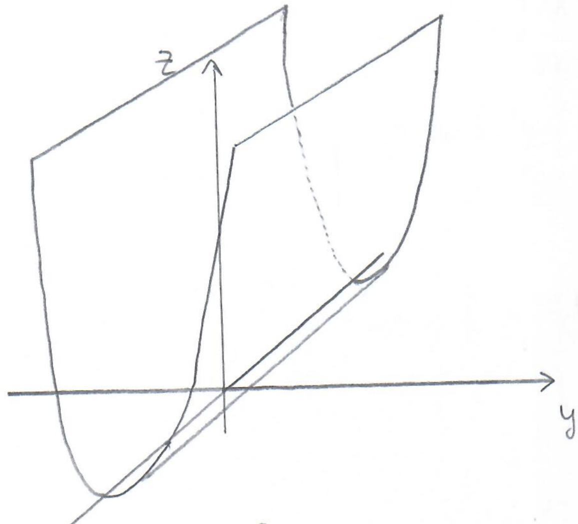
$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left(1 + 2\cos\varphi + \cos^2\varphi\right) d\varphi + \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} \frac{9}{2}\cos^2\varphi\, d\varphi + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{9}{2}\cos^2\varphi\, d\varphi$$

$= \Big[$ uočimo da u zadnja dva integrala integriramo istu parnu fju po međusobno simetričnim intervalima $\Big]$ $= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(1 + 2\cos\varphi + \frac{1}{2}(1 + \cos 2\varphi)\right) d\varphi + 9\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\varphi)\, d\varphi$

$$= \frac{1}{2}\left(\varphi + 2\sin\varphi + \frac{1}{2}\left(\varphi + \frac{1}{2}\sin 2\varphi\right)\right)\Bigg|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} + \frac{9}{2}\left(\varphi + \frac{1}{2}\sin 2\varphi\right)\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

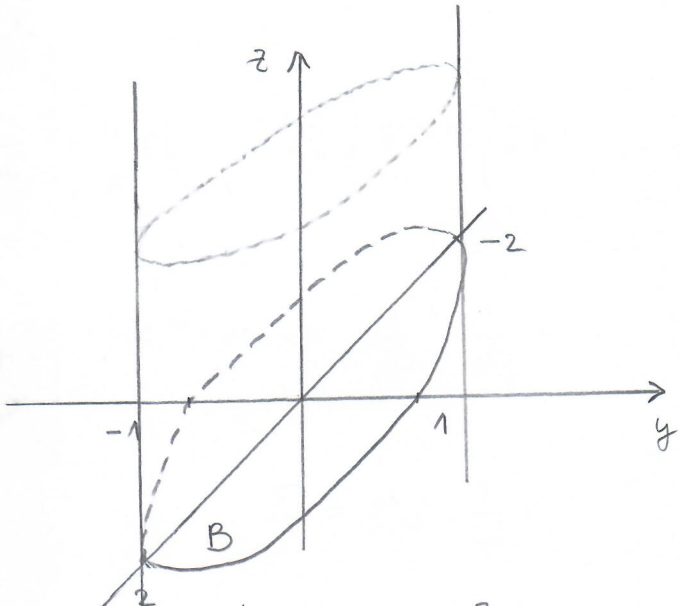
$$= \frac{1}{2}\left(\frac{2\pi}{3} + 2\cdot 2\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{2\pi}{3} + \frac{1}{2}\cdot 2\cdot\frac{\sqrt{3}}{2}\right)\right) + \frac{9}{2}\left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{4}$$

5.



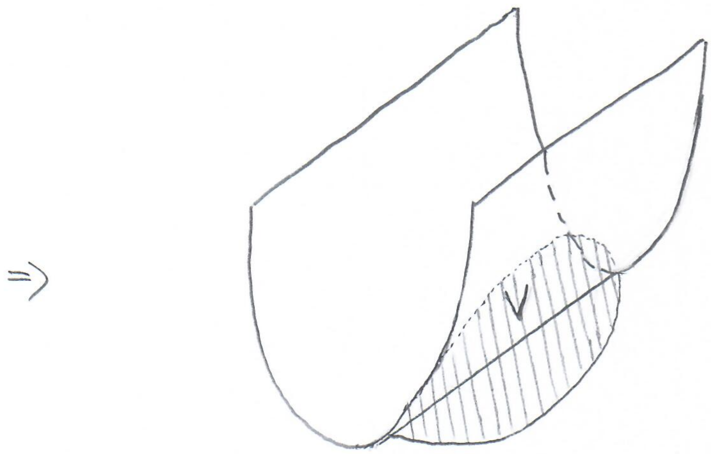
$z = y^2$

parabolički cilindar



$x^4 + 4y^2 = 4 \Rightarrow \frac{x^2}{2^2} + y^2 = 1$

eliptički cilindar



Tražimo volumen tijela koji iz eliptičkog cilindra „isijeca" zadani parabolički cilindar. Taj volumen je jednak integralu funkcije $f(x,y) = y^2$ po bazi B eliptičkog cilindra (što je elipsa s poluosima duljina 2 i 1):

$$V = \iint_B y^2 \, dx \, dy = \left[\begin{array}{ll} \text{Eliptičke koordinate:} & \\ x = 2r\cos\varphi & r \in [0,1] \\ y = r\sin\varphi & \varphi \in [0, 2\pi] \\ J = 2 \cdot 1 \cdot r = 2r & \end{array} \right]$$

$$= \int_0^{2\pi} \int_0^1 r^2 \sin^2\varphi \cdot 2r \, dr \, d\varphi = 2 \int_0^{2\pi} \sin^2\varphi \int_0^1 r^3 \, dr \, d\varphi$$

$$= 2\int_0^{2\pi} \sin^2\varphi \left(\frac{1}{4} r^4 \Big|_0^1\right) d\varphi = \frac{1}{2}\int_0^{2\pi} \sin^2\varphi \, d\varphi$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{2}\left(1 - \cos 2\varphi\right) d\varphi = \frac{1}{4}\left(\varphi - \frac{1}{2}\sin 2\varphi\right)\Big|_0^{2\pi}$$

$$= \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$