Linearna algebra - 3. auditorne vježbe

1. Izračunajte determinantu matrice

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}.$$

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \begin{matrix} \\ +\ 1\cdot 2 \\ +\ 1\cdot 1 \end{matrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} =$$

$$= 1 \cdot \begin{vmatrix} 0 & -5 \\ 3 & 2 \end{vmatrix} = 0 \cdot 2 - 3 \cdot (-5) = 15$$

2. Zadana je matrica

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & a & 0 & 0 \\ a & 0 & a & a \\ a & 0 & a & 1 \end{bmatrix}.$$

Izračunajte determinantu matrice $\mathbf{A}$. Za koje vrijednosti parametra $a \in \mathbb{R}$ vrijedi $\det \mathbf{A} = 0$?

$$\det A = \begin{vmatrix} a & a & a & a \\ a & a & 0 & 0 \\ a & 0 & a & a \\ a & 0 & a & 1 \end{vmatrix} = \begin{vmatrix} 0 & a & a & a \\ 0 & a & 0 & 0 \\ a & 0 & a & a \\ a & 0 & a & 1 \end{vmatrix} \leftarrow =$$

$$+1\cdot(-1)$$

$$= a \begin{vmatrix} 0 & a & a \\ a & a & a \\ a & a & 1 \end{vmatrix} = a \begin{vmatrix} 0 & a & a \\ a & 0 & a \\ a & 0 & 1 \end{vmatrix} = a \cdot \underbrace{(-1)}_{=(-1)^{1+2}} \cdot a \cdot \begin{vmatrix} a & a \\ a & 1 \end{vmatrix}$$

$$1\cdot(-1)\ +$$

$$= -a^2(a-a^2) = -a^3(1-a)$$

$$\det A = 0 \iff -a^3(1-a) = 0 \iff a \in \{0, 1\}$$

3. Neka su **A** i **B** kvadratne matrice reda 5 te neka je $\det \mathbf{A} = 2$ i $\det \mathbf{B} = 3$. Izračunajte

$$\det\left[(\mathbf{AB})^{-1}(5\mathbf{A})(\mathbf{BA})^{\top}\right].$$

Koristimo svojstva determinante:

1° Binet-Cauchyjev teorem: $\det(AB) = \det A \cdot \det B$

2° $\det(A^{-1}) = \dfrac{1}{\det A}$ (posljedica Binet-Cauchyjeva teorema)

3° $\det(\alpha A) = \alpha^n \det A$ za $\alpha \in \mathbb{R}$ i $A \in M_n$

4° $\det(A^T) = \det A$

Imamo

$$\det\left[(AB)^{-1}(5A)(BA)^T\right] \overset{1^\circ}{=} \det\left((AB)^{-1}\right)\det(5A)\det\left((BA)^T\right)$$

$$\overset{2^\circ,3^\circ,4^\circ}{=} \frac{1}{\cancel{\det(AB)}} \cdot 5^5 \det A \cdot \cancel{\det(BA)}$$

$$= 5^5 \det A$$

$$= 2 \cdot 5^5$$

$$= 6250$$

4. Neka je n neparan broj i $\mathbf{A}$ kvadratna matrica reda n za koju vrijedi $\mathbf{A}^\top = -\mathbf{A}$. Dokažite da $\mathbf{A}$ nije invertibilna matrica.

$$A^T = -A \qquad / \det$$

$$\Rightarrow \det(A^T) = \det(-A)$$

$$\Rightarrow \det A = \underbrace{(-1)^n}_{= -1 \text{ (n neparan)}} \det A$$

$$\Rightarrow \det A = -\det A$$

$$\Rightarrow 2 \det A = 0$$

$$\Rightarrow \det A = 0$$

$\Rightarrow$ A nije invertibilna (vrijedi: A invertibilna $\Leftrightarrow \det A \neq 0$)

5. Zadan je niz matrica koje na glavnoj dijagonali imaju trojke, neposredno ispod glavne dijagonale jedinice, neposredno iznad glavne dijagonale dvojke, a svugdje ostalo nule:

$$\mathbf{A}_1 = \begin{bmatrix} 3 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}, \ \mathbf{A}_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \ \mathbf{A}_4 = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \ldots$$

Za $n \in \mathbb{N}$ označimo $D_n = \det \mathbf{A}_n$. Korištenjem kofaktorske formule za izračun determinante odredite a i b u rekurzivnoj formuli

$$D_n = aD_{n-1} + bD_{n-2}.$$

$$D_n = \begin{vmatrix} 3 & 2 & 0 & \cdots & 0 \\ 1 & 3 & 2 & \cdots & 0 \\ 0 & 1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 3 \end{vmatrix}_{(n)}$$

$$= 3 \cdot \underbrace{\begin{vmatrix} 3 & 2 & \cdots & 0 \\ 1 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{vmatrix}_{(n-1)}}_{=D_{n-1}} - 2 \cdot \begin{vmatrix} 1 & 2 & 0 & \cdots & 0 \\ 0 & 3 & 2 & \cdots & 0 \\ 0 & 1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 3 \end{vmatrix}_{(n-1)}$$

$$= 3D_{n-1} - 2 \cdot 1 \cdot \underbrace{\begin{vmatrix} 3 & 2 & \cdots & 0 \\ 1 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{vmatrix}_{(n-2)}}_{=D_{n-2}} = 3D_{n-1} - 2D_{n-2}$$

$$\Rightarrow D_n = 3D_{n-1} - 2D_{n-2} \qquad (a = 3, \ b = -2)$$

6. Zadane su matrice

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Riješite matričnu jednadžbu

$$(\mathbf{X}+\mathbf{A})^2 = \left[(\mathbf{X}+\mathbf{A})^{-1}\mathbf{X}^{-1}\right]^{-1} + \mathbf{B}.$$

$$(X+A)(X+A) = \left\{\left[X(X+A)\right]^{-1}\right\}^{-1} + B$$

$$X^2 + XA + AX + A^2 = X(X+A) + B$$

$$\cancel{X^2} + \cancel{XA} + AX + A^2 = \cancel{X^2} + \cancel{XA} + B$$

$$A^{-1}\cdot \Big| \quad AX = B - A^2$$

$$X = A^{-1}(B - A^2) = A^{-1}B - A$$

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} \begin{matrix} + \\ \\ 1\cdot(-1) \end{matrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -4 \\ -4 & -1 & 6 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -4 \\ -4 & -1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 1 \\ 6 & -4 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & -1 \\ -2 & 1 & 0 \\ 5 & -5 & -2 \end{bmatrix}.$$

7. Zadana je matrica

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Riješite matričnu jednadžbu

$$\mathbf{X}^{-1} = \mathbf{A} - \mathbf{X}^{-1}\mathbf{A} + 2\mathbf{I}.$$

$$X^{-1} + X^{-1}A = A + 2I$$

$$X^{-1}(I+A) = A+2I \qquad \Big| \cdot (I+A)^{-1}$$

$$X^{-1} = (A+2I)(I+A)^{-1} \qquad \Big/^{-1}$$

$$X = (I+A)(A+2I)^{-1}$$

$$\det(A+2I) = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} = -6+1 = -5$$

$$(A+2I)^{-1} = -\frac{1}{5}\begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T = -\frac{1}{5}\begin{bmatrix} 3 & -3 & 1 \\ -6 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \cdot \left(-\frac{1}{5}\right)\begin{bmatrix} 3 & -3 & 1 \\ -6 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8 & -3 & 1 \\ -6 & 6 & -2 \\ -1 & 1 & 3 \end{bmatrix}.$$