Linearna algebra - 10. auditorne vježbe

1. Neka je $A\colon \mathbb{R}^3 \to \mathbb{R}^4$ funkcija zadana formulom

$$A(x_1, x_2, x_3) = (x_1 - x_2 + x_3,\ 2x_1 - x_3,\ x_1 - 4x_2 + 2x_3,\ 3x_1 - x_2).$$

Dokažite da je A linearni operator, pronađite $\operatorname{Ker} A$, $\operatorname{Im} A$, $d(A)$, $r(A)$ te po jednu bazu za $\operatorname{Ker} A$ i $\operatorname{Im} A$.

Neka su $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ te $\alpha, \beta \in \mathbb{R}$ proizvoljni. Računamo

$$\begin{aligned}
A(\alpha\vec{x} + \beta\vec{y}) &= A(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) \\
&= \big((\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3),\ 2(\alpha x_1 + \beta y_1) - (\alpha x_3 + \beta y_3), \\
&\qquad (\alpha x_1 + \beta y_1) - 4(\alpha x_2 + \beta y_2) + 2(\alpha x_3 + \beta y_3),\ 3(\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2)\big) \\
&= \big(\alpha(x_1 - x_2 + x_3) + \beta(y_1 - y_2 + y_3),\ \alpha(2x_1 - x_3) + \beta(2y_1 - y_3), \\
&\qquad \alpha(x_1 - 4x_2 + 2x_3) + \beta(y_1 - 2y_2 + 2y_3),\ \alpha(3x_1 - x_2) + \beta(3y_1 - y_2)\big) \\
&= \alpha(x_1 - x_2 + x_3,\ 2x_1 - x_3,\ x_1 - 4x_2 + 2x_3,\ 3x_1 - x_2) \\
&\qquad + \beta(y_1 - y_2 + y_3,\ 2y_1 - y_3,\ y_1 - 4y_2 + 2y_3,\ 3y_1 - y_2) \\
&= \alpha A\vec{x} + \beta A\vec{y},
\end{aligned}$$

odakle po definiciji slijedi da je A linearni operator.

Odredimo jezgru od A: $\operatorname{Ker} A = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0}\}$. Imamo

$$A(x_1, x_2, x_3) = \vec{0} \iff \begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 - x_3 = 0 \\ x_1 - 4x_2 + 2x_3 = 0 \\ 3x_1 - x_2 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & -4 & 2 & 0 \\ 3 & -1 & 0 & 0 \end{array}\right] \begin{matrix} \\ +\,|\cdot(-2) \\ +\,|\cdot(-1) \\ +\,|\cdot(-3) \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right] \begin{matrix} +\,|\cdot(-1) \\ +\,|\cdot 3 \\ |\cdot 3 \\ + \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 0 \\ 0 & -7 & 0 & \vdots & 0 \\ 0 & -3 & 1 & \vdots & 0 \\ 0 & -7 & 0 & \vdots & 0 \end{bmatrix} \begin{matrix} | \cdot \frac{2}{7} \\ | \cdot (-\frac{3}{7}) \\ | \cdot (-1) \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & -7 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \begin{matrix} \Rightarrow x_1 = 0 \\ \Rightarrow -7x_2 = 0 \Rightarrow x_2 = 0 \\ \Rightarrow x_3 = 0 \\ \end{matrix}$$

Dakle, $\operatorname{Ker} A = \{(0,0,0)\}$ pa je defekt od A jednak $d(A) = 0$ (i A je injekcija).

Sada prema teoremu o rangu i defektu slijedi da je rang jednak $r(A) = \dim \mathbb{R}^3 - d(A) = 3$.

Odredimo sliku od A: $\operatorname{Im} A = \{ A\vec{x} : \vec{x} \in \mathbb{R}^3 \}$. Elementi prostora $\operatorname{Im} A$ su vektori oblika

$$\begin{aligned} A(x_1, x_2, x_3) &= (x_1 - x_2 + x_3,\ 2x_1 - x_3,\ x_1 - 4x_2 + 2x_3,\ 3x_1 - x_2) \\ &= (x_1, 2x_1, x_1, 3x_1) + (-x_2, 0, -4x_2, -x_2) + (x_3, -x_3, 2x_3, 0) \\ &= x_1 \underbrace{(1,2,1,3)}_{=:\vec{f}_1} + x_2 \underbrace{(-1,0,-4,-1)}_{=:\vec{f}_2} + x_3 \underbrace{(1,-1,2,0)}_{=:\vec{f}_3}, \quad x_{1,2,3} \in \mathbb{R}. \end{aligned}$$

Budući da vrijedi $\vec{f}_1, \vec{f}_2, \vec{f}_3 \in \operatorname{Im} A$ (naime, $\vec{f}_1 = A(1,0,0)$, $\vec{f}_2 = A(0,1,0)$, $\vec{f}_3 = A(0,0,1)$), slijedi $\operatorname{Im} A = L(\vec{f}_1, \vec{f}_2, \vec{f}_3)$.

No, budući da je $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ tročlan skup, a $\dim \operatorname{Im} A = r(A) = 3$, taj skup mora biti i baza za $\operatorname{Im} A$.

Nap. Neka je V vektorski prostor i $\dim V = n$, $n \in \mathbb{N}$.

1) Svaki n-člani skup vektora iz V koji razapinje V je ujedno i baza za V.

2) Svaki n-člani skup vektora iz V koji je linearno nezavisan u V je ujedno i baza za V.

2. Odredite matricu operatora $T\colon \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (x+y,\ x-y,\ x)$, u paru kanonskih baza.

Znamo da su $(e) = \{(1,0),\ (0,1)\}$ i $(f) = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ redom kanonske baze za domenu i kodomenu od T.

Određujemo slike vektora kanonske baze za domenu:

$$T(1,0) = (1,1,1) = 1\cdot(1,0,0) + 1\cdot(0,1,0) + 1\cdot(0,0,1),$$

$$T(0,1) = (1,-1,0) = 1\cdot(1,0,0) - 1\cdot(0,1,0) + 0\cdot(0,0,1).$$

Zato je matrični prikaz od T u paru kanonskih baza:

$$T(f,e) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

3. Odredite matricu operatora $S\colon \mathbb{R}^3 \to \mathcal{M}_2$ zadanog s

$$S(x,y,z) = \begin{bmatrix} x-y & y-z \\ z-x & x+y+z \end{bmatrix}$$

u paru kanonskih baza.

Kanonska baza za domenu:

$$(e) = \{ e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1) \}.$$

Kanonska baza za kodomenu:

$$(E) = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Imamo

$$S(e_1) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = 1 \cdot E_{11} + 0 \cdot E_{12} - 1 \cdot E_{21} + 1 \cdot E_{22},$$

$$S(e_2) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = -1 \cdot E_{11} + 1 \cdot E_{12} + 0 \cdot E_{21} + 1 \cdot E_{22},$$

$$S(e_3) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = 0 \cdot E_{11} - 1 \cdot E_{12} + 1 \cdot E_{21} + 1 \cdot E_{22}.$$

Dakle,

$$S(E,e) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

4. Linearni operator $A\colon \mathcal{P}_3 \to \mathcal{P}_2$ preslikava kanonsku bazu $\{t^3,\ t^2,\ t,\ 1\}$ u skup $\{t^2+t+1,\ t^2+3t+5,\ -2t^2-4t-6,\ -2t^2+2\}$.

(a) Odredite matricu od A u paru kanonskih baza.

(b) Odredite rang i defekt od A.

(a) Ako su $(e)=\{t^3, t^2, t, 1\}$ i $(f)=\{t^2, t, 1\}$ redom kanonske baze za $\mathcal{P}_3$ i $\mathcal{P}_2$, onda je

$$A(f,e)=\begin{bmatrix} 1 & 1 & -2 & -2 \\ 1 & 3 & -4 & 0 \\ 1 & 5 & -6 & 2 \end{bmatrix}.$$

(b) Rang od A jednak je rangu matrice $A(f,e)$:

$$\begin{bmatrix} 1 & 1 & -2 & -2 \\ 1 & 3 & -4 & 0 \\ 1 & 5 & -6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 1 & 2 & -4 & 0 \\ 1 & 4 & -8 & 2 \end{bmatrix} \sim$$

$1\cdot(-1)$ $+$ $+$ $1\cdot(-1)$ $1\cdot(-1)$ $+$ $+$ $1\cdot(-1)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 2 & -4 & 2 \\ 0 & 4 & -8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1\cdot(-2)$ $+$

$\Rightarrow r(A)=$

Prema teoremu o rangu i defektu

$$d(A)=\dim \mathcal{P}_3 - r(A) = 4-2=2.$$

5. Preslikavanje $P\colon \mathcal{M}_n \to \mathcal{M}_n$ je zadano formulom

$$P(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top).$$

(a) Dokažite da je P linearan operator.

(b) Odredite jezgru i defekt od P.

(c) Odredite sliku i rang od P.

(a)
$$\begin{aligned} P(\alpha A+\beta B) &= \frac{1}{2}\left((\alpha A+\beta B)+(\alpha A+\beta B)^T\right) \\ &= \frac{1}{2}\left(\alpha A+\beta B+\alpha A^T+\beta B^T\right) \\ &= \alpha\cdot\frac{1}{2}(A+A^T)+\beta\cdot\frac{1}{2}(B+B^T) \\ &= \alpha P(A)+\beta P(B) \end{aligned}$$

(transponiranje je linearan operator)

za sve $A, B \in \mathcal{M}_n$ i $\alpha, \beta \in \mathbb{R}$ $\Rightarrow$ P je linearan operator

(b) Vrijedi

$$A \in \operatorname{Ker} P \iff P(A)=0 \iff \frac{1}{2}(A+A^T)=0 \iff A^T=-A,$$

tj. jezgru od P čine sve antisimetrične matrice reda n.

Budući da je svaka takva matrica oblika

$$\begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n} \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & -a_{3n} & \cdots & 0 \end{bmatrix},$$

dimenzija prostora $\operatorname{Ker} P$ je jednaka

$$(n-1)+(n-2)+(n-3)+\ldots+2+1 = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2} = d(P).$$

(c) Najprije uočimo da za svaku matricu $A \in M_n$ vrijedi

$$P(A)^T = \left[\frac{1}{2}(A+A^T)\right]^T = \frac{1}{2}(A+A^T) = P(A),$$

tj. $P(A)$ je simetrična matrica. Dakle,

$$\operatorname{Im} A \subseteq \{\text{simetrične matrice reda } n\}.$$

Obratno, neka je $B \in M_n$ proizvoljna simetrična matrica. Tada imamo

$$P(B) = \frac{1}{2}(B+B^T) = \frac{1}{2}(B+B) = B,$$

tj. $B \in \operatorname{Im} P$. Dakle,

$$\{\text{simetrične matrice reda } n\} \subseteq \operatorname{Im} A,$$

tj. vidimo da sliku od A čine upravo sve simetrične matrice reda n.

Budući da je svaka takva matrica oblika

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{bmatrix},$$

(simetrična matrica je u potpunosti određena vrijednostima elemenata na glavnoj dijagonali i u gornjem trokutu)

dimenzija prostora $\operatorname{Im} P$ je jednaka

$$n + (n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n+1)}{2} = r(P).$$

Alternativno, rang smo mogli izračunati i koristeći teorem o rangu i defektu

$$r(P) = \dim M_n - d(P) = n^2 - \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1).$$