Linearna algebra - 13. auditorne vježbe

1. Odredite udaljenost i kut među vektorima

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

u $\mathcal{M}_2$ sa skalarnim produktom

$$\langle \mathbf{A} \mid \mathbf{B} \rangle = \operatorname{tr}(\mathbf{A}\mathbf{B}^\top).$$

$$d(A,B) = \|A-B\| = \sqrt{\langle A-B \mid A-B \rangle} = \sqrt{tr\left(\begin{bmatrix} 2 & -2 \\ 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ 5 & -1 \end{bmatrix}^T\right)}$$

$$= \sqrt{tr\left(\begin{bmatrix} 2 & -2 \\ 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ -2 & -1 \end{bmatrix}\right)} = \sqrt{tr\begin{bmatrix} 8 & -12 \\ -12 & 26 \end{bmatrix}}$$

$$= \sqrt{8+26} = \sqrt{34}$$

$$\langle A \mid B \rangle = tr(AB^T) = tr\left(\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\right) = tr\begin{bmatrix} -1 & 2 \\ 7 & -4 \end{bmatrix} = -5$$

$$\|A\| = \sqrt{\langle A \mid A \rangle} = \sqrt{tr\left(\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}\right)} = \sqrt{tr\begin{bmatrix} 1 & -3 \\ -3 & 13 \end{bmatrix}} = \sqrt{14}$$

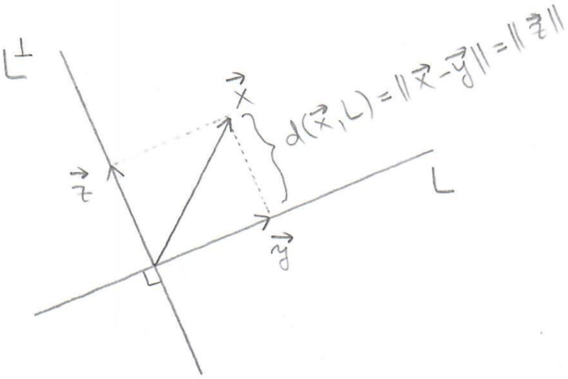
$$\|B\| = \sqrt{\langle B \mid B \rangle} = \sqrt{tr\left(\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\right)} = \sqrt{tr\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}} = \sqrt{10}$$

$$\Rightarrow \cos\measuredangle(A,B) = \frac{\langle A \mid B \rangle}{\|A\| \cdot \|B\|} = \frac{-5}{\sqrt{14}\cdot\sqrt{10}} = -\frac{\sqrt{35}}{14}$$

$$\Rightarrow \measuredangle(A,B) = \arccos\left(-\frac{\sqrt{35}}{14}\right)$$

2. Neka je $L = [\{(1,0,0), (0,-1,1), (5,-2,2)\}]$ potprostor od $\mathbb{R}^3$ i $\mathbf{x} = (1,1,1)$. Odredite ortogonalnu projekciju vektora $\mathbf{x}$ na potprostor L te udaljenost vektora $\mathbf{x}$ od L.

Uočimo da je $(5,-2,2) = 5\cdot(1,0,0) + 2\cdot(0,-1,1)$ pa je $L = [\{\underbrace{(1,0,0)}_{=:\vec{a}_1}, \underbrace{(0,-1,1)}_{=:\vec{a}_2}\}]$.



Za zadani vektor $\vec{x}$ postoje (jedinstveni) vektori $\vec{y} \in L$ i $\vec{z} \in L^\perp$ takvi da

$$\vec{x} = \vec{y} + \vec{z}.$$

Nadalje, budući da je $\vec{y} \in L$, postoje (jedinstveni) skalari $\alpha, \beta \in \mathbb{R}$ takvi da $\vec{y} = \alpha\vec{a}_1 + \beta\vec{a}_2$.

Dakle,

$$\vec{x} = \alpha\vec{a}_1 + \beta\vec{a}_2 + \vec{z}$$

te skalarnim množenjem ove jednakosti redom s vektorima $\vec{a}_1$ i $\vec{a}_2$ dobivamo

$$\begin{cases} \langle \vec{x} \mid \vec{a}_1\rangle = \alpha\langle \vec{a}_1 \mid \vec{a}_1\rangle + \beta\langle \vec{a}_2 \mid \vec{a}_1\rangle + \overbrace{\langle \vec{z} \mid \vec{a}_1\rangle}^{=0} \\ \langle \vec{x} \mid \vec{a}_2\rangle = \alpha\langle \vec{a}_1 \mid \vec{a}_2\rangle + \beta\langle \vec{a}_2 \mid \vec{a}_2\rangle + \underbrace{\langle \vec{z} \mid \vec{a}_2\rangle}_{=0} \end{cases}$$

$$\Rightarrow \begin{cases} 1 = \alpha\cdot 1 + \beta\cdot 0 & \Rightarrow \alpha = 1 \\ 0 = \alpha\cdot 0 + \beta\cdot 2 & \Rightarrow \beta = 0 \end{cases}$$

Dakle, ortogonalna projekcija vektora $\vec{x}$ na potprostor L je

$$\vec{y} = 1\cdot\vec{a}_1 = (1,0,0),$$

dok je udaljenost $\vec{x}$ od L jednaka

$$d(\vec{x}, L) = \|\vec{z}\| = \|\vec{x} - \vec{y}\| = \|(0,1,1)\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}.$$

3. U unitarnom prostoru $\mathbb{R}^3$ sa standardnim skalarnim produktom dani su vektori $\mathbf{a}_1 = (1,2,2)$, $\mathbf{a}_2 = (1,-2,0)$ i $\mathbf{a}_3 = (-1,0,1)$. Ispitajte jesu li ti vektori linearno nezavisni te ih ortonormirajte.

Zadane vektore ćemo ortonormirati koristeći Gram-Schmidtov postupak ortogonalizacije. Pritom su vektori linearno nezavisni ako i samo ako postupak možemo provesti do kraja.

$$\vec{e}_1 = \frac{1}{\|\vec{a}_1\|}\vec{a}_1 = \frac{1}{\sqrt{1^2+2^2+2^2}}(1,2,2) = \frac{1}{3}(1,2,2)$$

$$\vec{b}_2 = \vec{a}_2 - \langle \vec{a}_2 | \vec{e}_1 \rangle \vec{e}_1$$

$$= (1,-2,0) - \frac{1}{9}(1\cdot 1 + (-2)\cdot 2 + 0\cdot 2)(1,2,2)$$

$$= (1,-2,0) - \frac{1}{9}\cdot(-3)\cdot(1,2,2)$$

$$= (1,-2,0) + \frac{1}{3}(1,2,2) = \frac{2}{3}(2,-2,1)$$

$$\vec{e}_2 = \frac{1}{\|\vec{b}_2\|}\vec{b}_2 = \frac{1}{\cancel{\frac{2}{3}}\cdot\sqrt{2^2+(-2)^2+1^2}}\cdot\cancel{\frac{2}{3}}(2,-2,1) = \frac{1}{3}(2,-2,1)$$

$$\vec{b}_3 = \vec{a}_3 - \langle \vec{a}_3 | \vec{e}_2 \rangle \vec{e}_2 - \langle \vec{a}_3 | \vec{e}_1 \rangle \vec{e}_1$$

$$= (-1,0,1) - \frac{1}{9}((-1)\cdot 2 + 0\cdot(-2) + 1\cdot 1)(2,-2,1) - \frac{1}{9}(-1\cdot 1 + 0\cdot 2 + 1\cdot 2)(1,2,2)$$

$$= (-1,0,1) + \frac{1}{9}(2,-2,1) - \frac{1}{9}(1,2,2) = \frac{4}{9}(-2,-1,2)$$

$$\vec{e}_3 = \frac{1}{\|\vec{b}_3\|}\vec{b}_3 = \frac{1}{\cancel{\frac{4}{9}}\cdot\sqrt{(-2)^2+(-1)^2+2^2}}\cdot\cancel{\frac{4}{9}}(-2,-1,2) = \frac{1}{3}(-2,-1,2)$$

Dakle, Gram-Schmidtovim postupkom smo dobili ortonormiranu bazu

$$\left\{ \frac{1}{3}(1,2,2),\ \frac{1}{3}(2,-2,1),\ \frac{1}{3}(-2,-1,2) \right\}$$

te vidimo da su zadani vektori linearno nezavisni.

4. **(Parsevalova jednakost)** Neka su $\mathbf{x}$ i $\mathbf{y}$ ortonormirani vektori iz unitarnog prostora U nad $\mathbb{R}$. Dokažite da za sve $\alpha, \beta \in \mathbb{R}$ vrijedi

$$\|\alpha\mathbf{x} + \beta\mathbf{y}\|^2 = \alpha^2 + \beta^2.$$

$$\|\alpha\vec{x} + \beta\vec{y}\|^2 = \langle \alpha\vec{x} + \beta\vec{y} \mid \alpha\vec{x} + \beta\vec{y} \rangle$$

$$= \langle \alpha\vec{x} \mid \alpha\vec{x} + \beta\vec{y} \rangle + \langle \beta\vec{y} \mid \alpha\vec{x} + \beta\vec{y} \rangle$$

$$= \langle \alpha\vec{x} \mid \alpha\vec{x} \rangle + \langle \alpha\vec{x} \mid \beta\vec{y} \rangle + \langle \beta\vec{y} \mid \alpha\vec{x} \rangle + \langle \beta\vec{y} \mid \beta\vec{y} \rangle$$

$$= \alpha^2 \underbrace{\langle \vec{x} \mid \vec{x} \rangle}_{=\|\vec{x}\|^2 = 1} + \alpha\beta \underbrace{\langle \vec{x} \mid \vec{y} \rangle}_{=0} + \beta\alpha \underbrace{\langle \vec{y} \mid \vec{x} \rangle}_{=0} + \beta^2 \underbrace{\langle \vec{y} \mid \vec{y} \rangle}_{=\|\vec{y}\|^2 = 1}$$

$$= \alpha^2 \cdot 1 + \beta^2 \cdot 1$$

$$= \alpha^2 + \beta^2$$

5. Nađite ortonormiranu bazu u kojoj je matrica

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

dijagonalna.

Uočimo da je matrica A simetrična (pa tražena baza uistinu postoji).

Karakteristični polinom od A je

$$\kappa_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} \overset{|\cdot(-1)\, +}{=} \begin{vmatrix} \lambda+1 & -(\lambda+1) & 0 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix}$$

$$= (\lambda+1) \begin{vmatrix} 1 & -1 & 0 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} \overset{|\cdot 1\, +}{=} (\lambda+1) \begin{vmatrix} 1 & 0 & 0 \\ -1 & \lambda-1 & -1 \\ -1 & -2 & \lambda \end{vmatrix}$$

$$= (\lambda+1) \begin{vmatrix} \lambda-1 & -1 \\ -2 & \lambda \end{vmatrix} = (\lambda+1)(\lambda^2-\lambda-2) = (\lambda+1)^2(\lambda-2)$$

$\Rightarrow$ svojstvene vrijednosti od A su $\lambda_1 = -1$ i $\lambda_2 = 2$

Tražimo pripadne svojstvene vektore (i ortonormiramo ih):

1° $\lambda_1 = -1$

$$(-I-A)\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \overset{|\cdot(-1)\,+}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = -x_2 - x_3$$

$$\Rightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad x_{2,3} \in \mathbb{R}$$

Možemo uzeti svojstvene vektore $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ i $\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ te ih

ortonormirati Gram-Schmidtovim postupkom:

$$\vec{e}_1 = \frac{1}{\|\vec{x}_1\|}\vec{x}_1 = \frac{1}{\sqrt{(-1)^2+1^2+0^2}}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$\vec{b}_2 = \vec{x}_2 - \langle \vec{x}_2 | \vec{e}_1 \rangle \vec{e}_1 = \begin{bmatrix}-1\\0\\1\end{bmatrix} - \frac{1}{2}\left((-1)\cdot(-1)+0\cdot 1+1\cdot 0\right)\begin{bmatrix}-1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$\vec{e}_2 = \frac{1}{\|\vec{b}_2\|}\vec{b}_2 = \frac{1}{\cancel{\frac{1}{2}}\sqrt{(-1)^2+(-1)^2+2^2}}\cdot\cancel{\frac{1}{2}}\begin{bmatrix}-1\\-1\\2\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$2^\circ \ \lambda_2 = 2$

$$(2I - A)\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c}2 & -1 & -1 & 0\\ -1 & 2 & -1 & 0\\ -1 & -1 & 2 & 0\end{array}\right] \begin{array}{l} +\,|\cdot 2 \\ +\,|\cdot(-1) \\ \end{array} \sim \left[\begin{array}{ccc|c}0 & -3 & 3 & 0\\ 0 & 3 & -3 & 0\\ -1 & -1 & 2 & 0\end{array}\right] \begin{array}{l} + \\ |\cdot 1 \\ |\cdot \frac{1}{3} \ + \end{array} \sim$$

$$\sim \left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 0 & 3 & -3 & 0\\ -1 & 0 & 1 & 0\end{array}\right] \begin{array}{l} \\ \Rightarrow x_2 = x_3 \\ \Rightarrow x_1 = x_3 \end{array} \qquad \Rightarrow \vec{x} = x_3\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad x_3 \in \mathbb{R}$$

Možemo uzeti svojstveni vektor $\vec{x}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ – on je već okomit na vektore $\vec{e}_1, \vec{e}_2$ (svojstveni vektori pridruženi različitim svojstvenim vrijednostima simetrične matrice su međusobno ortogonalni) pa ga je dovoljno samo normirati

$$\vec{e}_3 = \frac{1}{\|\vec{x}_3\|}\vec{x}_3 = \frac{1}{\sqrt{1^2+1^2+1^2}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Dakle, A se dijagonalizira u ortonormiranoj bazi $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ i vrijedi

$$A = \begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}^T.$$

6. Neka je $\mathbf{v}$ jedinični vektor u $\mathbf{R}^3$. Tada je $\mathbf{v}^\top \mathbf{v} = 1$. Neka je zadana i matrica $\mathbf{H} = \mathbf{I} - 2\mathbf{v}\mathbf{v}^\top$.

(a) Dokažite da vrijedi $\mathbf{H}^2 = \mathbf{I}$.

(b) Dokažite da je matrica $\mathbf{H}$ simetrična.

(c) Dokažite da je matrica $\mathbf{H}$ ortogonalna.

(a)
$$
\begin{aligned}
H^2 &= (I - 2\vec{v}\vec{v}^T)^2 \\
&= (I - 2\vec{v}\vec{v}^T)(I - 2\vec{v}\vec{v}^T) \\
&= I - 2\vec{v}\vec{v}^T - 2\vec{v}\vec{v}^T + 4\vec{v}\underbrace{\vec{v}^T\vec{v}}_{=1}\vec{v}^T \\
&= I - \cancel{4\vec{v}\vec{v}^T} + \cancel{4\vec{v}\vec{v}^T} \\
&= I
\end{aligned}
$$

(b)
$$
\begin{aligned}
H^T &= (I - 2\vec{v}\vec{v}^T)^T = I^T - 2(\vec{v}\vec{v}^T)^T = I^T - 2(\vec{v}^T)^T\vec{v}^T \\
&= I - 2\vec{v}\vec{v}^T = H
\end{aligned}
$$

$\Rightarrow$ H je simetrična

(c)
$$H \cdot H^T \overset{(b)}{=} H \cdot H = H^2 \overset{(a)}{=} I$$

$$H^T \cdot H \overset{(b)}{=} H \cdot H = H^2 \overset{(a)}{=} I$$

$\Rightarrow$ H je ortogonalna