Lecture 4:
Decision making under uncertainty

Overview

- Decision making?
- Uncertainty models

- Objectives under uncertainty

- One-stage problems
  - Newsvendor problem

- Multi-stage problems
  - Decision trees
  - Stochastic programming

Decision making?

- Process of identifying and choosing alternatives based on values, preferences and beliefs of decision-maker

- What we have been doing so far…deterministic
- Decision making environments
  - Deterministic
  - Uncertain
    - No certainty about some aspects
  - Games – more than one decision-maker
    - Adversarial
    - Cooperative
  - Multi-objective – more than one objective

Sources of uncertainty

- Parametric (uncertainty and variability)
- Experimental (measurement error)
- Structural (model errors)
- Algorithmic (numerical errors)
- Interpolation (approximation of data-points)

Causes of uncertainty

- Natural limit – for example uncertainty principle
  - Inherent uncertainty
- Some uncertainty is epistemological
  - Roulette wheel

Uncertainty models and risk

- Types of uncertainty (depend on amount of available info)
  - **Stochastic**
    - Probability distributions available about uncertainty
  - **Robust**
    - Only ranges of possible values known
  - **Unknown unknowns** – unidentified risks

Objective under uncertainty

- Stochastic
  - Functions of probability distribution
  - Expected value (what about variance?)
  - Expected utility
- Robust
  - Not using the probability
  - Minimax (pessimistic)
  - Minimax regret (pessimistic)
  - Maxmax (optimistic)

One-stage problem

- The simplest situation of decision making under uncertainty
- Decision is made before the uncertainty realization
- The result is collected after the realization
- Famous <u>discrete</u> example of the **newsvendor problem**

News-vendor problem

News vendor sells newspapers at the corner of the street, and each day she must determine how many newspapers q to order. She pays the company c=*20¢* for each paper and sells the papers for b=*25¢* each. Newspapers that are unsold at the end of the day are worthless. She knows that each day she can sell between 6 and 10 papers, with each possibility being equally likely. Show how this problem fits into the state-of-the-world model.

News-vendor problem

Possible demands (states-of-the-world)

$$S=\{6,7,8,9,10\} \text{ with } p_i=0.2 \text{ for } i\in S$$

Possible actions for newsvendor

$$A=\{6,7,8,9,10\}$$

Return

Demand *i* and purchased *j*

$$r_{ij}=25j-20i$$

News-vendor problem

Papers Ordered	Papers Demanded				
	6	7	8	9	10
6	30¢	30¢	30¢	30¢	30¢
7	10¢	35¢	35¢	35¢	35¢
8	−10¢	15¢	40¢	40¢	40¢
9	−30¢	−5¢	20¢	45¢	45¢
10	−50¢	−25¢	0¢	25¢	50¢

Dominating actions

An action a_i is **dominated** by an action $a_{i'}$ if for all $s_j \in S$, $r_{ij} \leq r_{i'j}$ and for some state $s_{j'}$, $r_{ij'} < r_{i'j'}$.(for maximization)

- Actions A'={1,2,3,4,5,11,…} are all dominated by A={6,7,8,9,10} in our example.

Maximin criterion

- for each action the worst outcome is determined and the action with the best worst outcome is selected
  - $a^*=\text{argmax}_{i \in A}\{\min_{j \in S} r_{ij}\}$
- for situations with no probabilistic information or critical situations
  - Mitigation of the worst case - **pessimistic**
- in newsvendor example a*=6
  - earns a profit at least 30**¢** (an at most)

- for problems with minimization - minimax

Maximax criterion

- for each action the best outcome is determined and the action with the best worst outcome is selected
  - $a^*=\text{argmax}_{i\in A}\{\max_{j\in S} r_{ij}\}$
- for situations with no probabilistic information or non-critical situations
  - Opening potential for the best case - **optimistic**
- in newsvendor example a*=10
  - earns a profit at most 50**¢** (an at worst -50**¢**)

- for problems with minimization - minimin

Minimax regret

- Tends to **avoid disappointment** with hindsight (regret)
- $a^*=\text{argmin}_{i\in A}\,(\max_{j\in S}[\underbrace{\max_{k\in A}\{r_{kj}\}-r_{ij}}_{\textbf{Regret of action i in state j}}])$
- in newsvendor example a*=6 or 7
  - regret of at most 20¢

Papers Ordered	Papers Demanded 6	7	8	9	10
6	30 − 30 = 0¢	35 − 30 = 5¢	40 − 30 = 10¢	45 − 30 = 15¢	50 − 30 = 20¢
7	30 − 10 = 20¢	35 − 35 = 0¢	40 − 35 = 5¢	45 − 35 = 10¢	50 − 35 = 15¢
8	30 + 10 = 40¢	35 − 15 = 20¢	40 − 40 = 0¢	45 − 40 = 5¢	50 − 40 = 10¢
9	30 + 30 = 60¢	35 + 5 = 40¢	40 − 20 = 20¢	45 − 45 = 0¢	50 − 45 = 5¢
10	30 + 50 = 80¢	35 + 25 = 60¢	40 − 0 = 40¢	45 − 25 = 20¢	50 − 50 = 0¢

- for problems with minimization – maximin regret

Expected value criterion (EVC)

- Select the action with the best expected value
  - $a^*=\mathrm{argmax}_{i\in A}\ E[r_{ij}]$
- for situations with probabilistic information
  - non-critical situations

Papers Ordered	Expected Reward
6	$\frac{1}{5}(30 + 30 + 30 + 30 + 30) = 30¢$
7	$\frac{1}{5}(10 + 35 + 35 + 35 + 35) = 30¢$
8	$\frac{1}{5}(-10 + 15 + 40 + 40 + 40) = 25¢$
9	$\frac{1}{5}(-30 - 5 + 20 + 45 + 45) = 15¢$
10	$\frac{1}{5}(-50 - 25 + 0 + 25 + 50) = 0¢$

- Optimal solution is 6 or 7

Expected value criterion (EVC)

- Faster solution to newsvendor problem under EVC is using **critical fractile formula**
- Marginal value of ordering additional piece of newspaper with random demand *D* and cumulative distribution function *F*, selling price *b*, buying price *c*
  - h(q)=(b-c)*[1-F(q)]-c*F(q)
  - Buy additional as long as h(q)>0
- <u>Optimal q* is given by the formula:</u>

$$q^* = \min\{q \in N_0 | F(q) \geq \frac{b-c}{b}\} \text{ for discrete F}$$

  - If F is continuous and strictly increasing, inverse F^{-1} exists and the solution is: $q^* = F^{-1}(\frac{b-c}{b})$
- For our example q*=6 (critical fractile 1/5)

Multi-stage <u>stochastic</u> problems

- most often optimize expected value
  - computationally the easiest

- Modeling
  - often, Markov Decision Process
    - Markovian assumption

- Solving
  - Dynamic programming approaches
    - Concept of ***“value function”***
    - Decision trees
  - Stochastic programming

Decision trees

- People make series of decision at different points in time
  - Multistage decision problems
- Decision trees decompose large complex decision problem into several smaller ones made in stages
- Make sense only for reasonable number of stages
- Backward induction of DP is used for solving
- Sensitivity analysis over decisions
  - Expected Value of Sample Information (EVSI)
  - Expected Value of Perfect Information (EVPI)

Decision trees - example

Colaco currently has assets of $150,000 and wants to decide whether to market a new chocolate-flavored soda, Chocola.

Colaco has three alternatives:

1. Test Chocola locally, then utilize the results of the market study to determine whether or not to market it nationally
2. Immediately market Chocola nationally
3. Immediately decide not to market it nationally

In absence of market study, they believe there is 55% chance of national success with profit of 300,000$ and 45% change of national failure with loss of 100,000$.

With study (at a cost of 30,000$) there is 60% chance of local success and 40% of local failure. Local success implies 85% chance of national success, and local failure implies 10% chance of national success. If Colaco is risk-neutral, what strategy to follow?

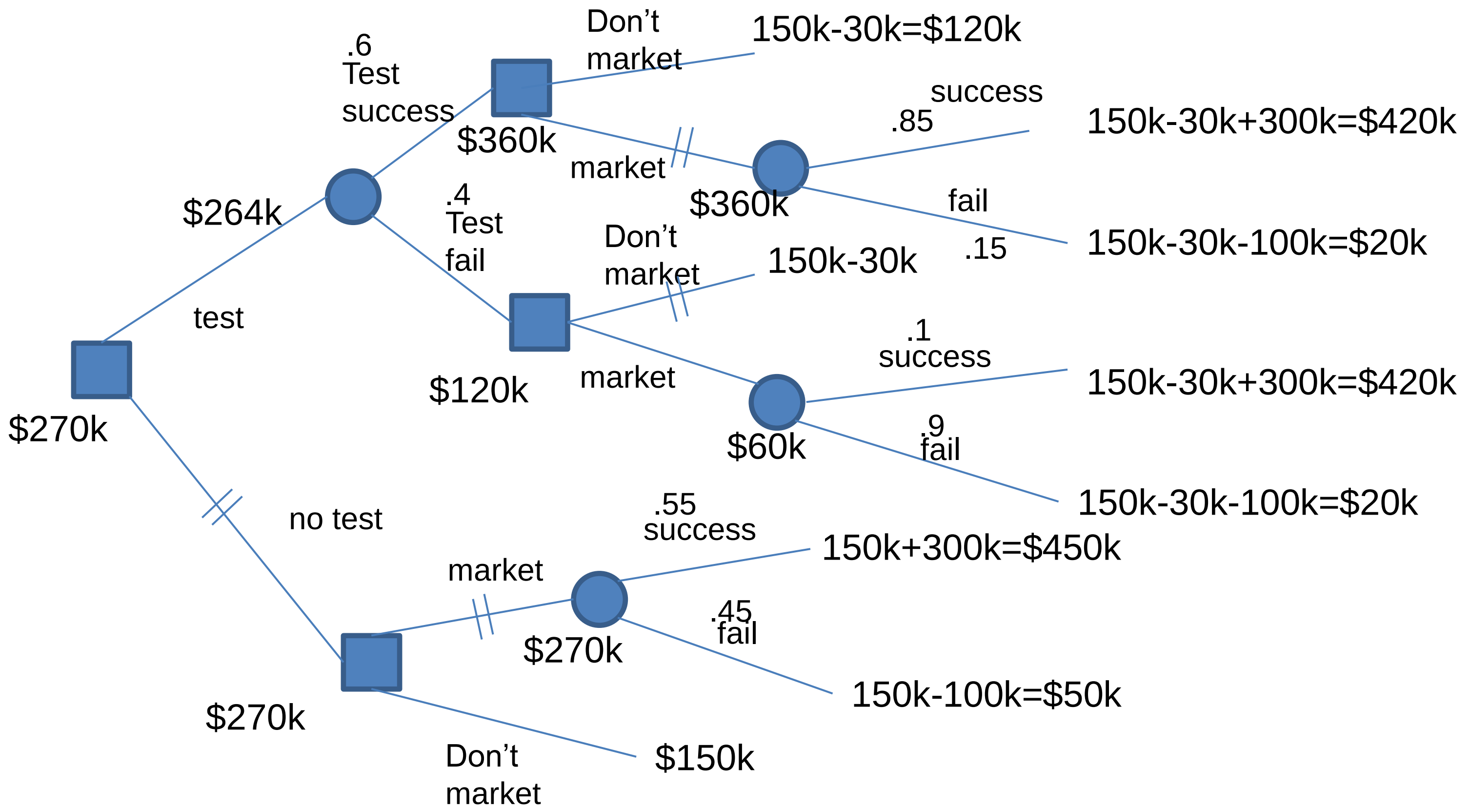
Decision trees - example

- Decision tree if created in forward pass

- Calculations are done in backward pass (folding back the tree)
  - Instantiation of backwards induction algorithm based on dynamic programming
  - For finite horizon MDPs

- The solution gives strategy or policy that for each state (decision fork) defines the optimal decision

Decision trees - example



- Each square is **decision node** (fork)
- Each circle is **event node** (fork)
- **Terminal branch** is a branch with no forks

- Backward operations:
  - Terminal branch – read deterministic utility
  - Event node – calculate expected utility
  - Decision node – pick decision from optimal sub-branch

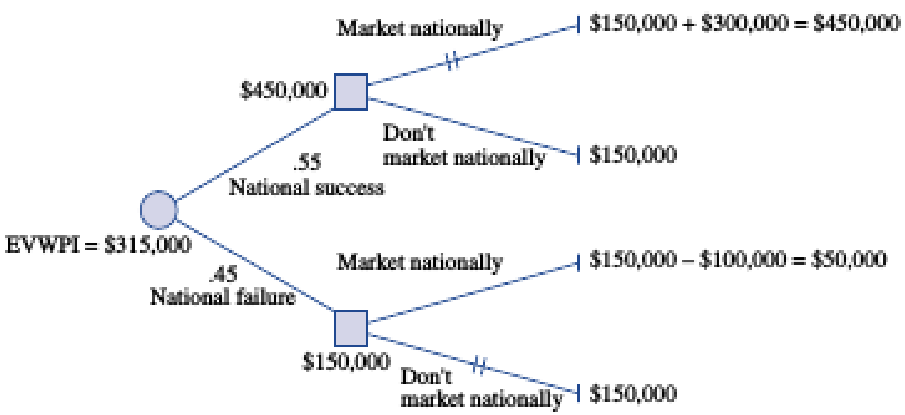
The optimal strategy is not to test locally and then to market nationally.

Expected value of sample information

- Measuring value of sample information using decision trees
  - Sensitivity analysis wrt. utilization of testing
- Expected value **with** sample information (EVWSI)
  - Profit if acting optimally and test market is costless
  - In Chocola example EWSI=264,000+30,000=294,000 (the best branch with costless testing)
- Expected Value with original value (EVWOI)
  - Value of problem if there is no testing available
- Expected value **of** sample information (EVSI)
  - EVSI=EVWSI-EVWOI
  - In Chocola example, EVSI=294,000 – 270,000=24,000$
  - Chocola would pay for such test only if it costs up to EVSI=24,000$

Expected value of perfect information (EVPI)

- Similar to sample information but perfect information tells the outcome **before** making decision
  - Sample information only created more skewed (informative) forecast
- EVPI=EVWPI-EVWOI
  - for Cholaco EVPI=315,000-270,000=45,000$



- EVPI is an upper bound on EVSI

Bayes' rule and decision trees

- Different states of world result in different rewards
- s_i , i=1,…,n – possible states of the world
- $p(s_i)$ - **prior probabilities** of states of the world
  - Before any action
- Buying information (for example experiments) might give more knowledge about state of the world
  - enable better decisions
- o_j, j=1,…,m – possible outcomes of the experiment
- if the decision maker is given conditional probabilities $p(s_i|o_j)$
  - after the experiment we get outcome o_k
    - new probability of the states is given by posterior distribution $p(s_i|o_k)$

- if given likelihoods $p(o_j|s_i)$ [stats from previous test]
  - Calculate posterior probabilities using Bayes' formula

$$p(s_i|o_j) = \frac{p(o_j|s_i) * p(s_i)}{p(o_j)}$$

- $p(o_i)$ – marginal probability of outcomes – needed for normalization

Decision trees?

- Good for discrete problems
  - with small dimensionality
  - with small branching factor

- What about problems with continuous and/or vector decision variables with high dimensionality?
  - Stochastic programming
  - Robust programming
    - When no stochastic data available

Two-stage stochastic problem - Farmer

- raising wheat and corn on 300 ha
- at least 200 t of wheat and 240 t of corn for cattle
- they can be sold for 170$/t and 150$/t and purchased for 40% higher price
- yield of each culture depends on weather:

	Below p=1/3	Average p=1/3	Above p=1/3
wheat	2 t/ha	2.5 t/ha	3 t/ha
corn	2.8 t/ha	3 t/ha	3.2 t/ha

Use linear programming to determine a production schedule to minimize the sum of production and inventory costs during the next four quarters.

Two stages:

1. Decide on planting (proactive – before information)
2. Decide how to deal with the outcome (reactive – after information), recourse
    - Sell excess/buy deficit (for cattle)

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Linear stochastic programming

- Constraints are linear
- Objective must be linear
  - Expected value/utility
  - Minimax
    - Can be expressed as linear function + constraints

Two-stage stochastic problem – Farmer - solution

- Check the notebook “Farmer – Stochastic programming”