MATAN 2 - 3. vježbe

(1.) Definiramo f-ju $f(x,y) := xe^{-xy}$.

Želimo izračunati $f(1.05, 0.11)$, a znamo stvarnu vrijednost

$$f(1,0) = 1 \cdot e^{-0} = 1.$$

Zato koristimo linearnu aproksimaciju:

$$\frac{\partial f}{\partial x}(x,y) = e^{-xy} + x \cdot \left(-ye^{-xy}\right) = e^{-xy}(1-xy) \Rightarrow \frac{\partial f}{\partial x}(1,0) = 1$$

$$\frac{\partial f}{\partial y}(x,y) = -x^2 e^{-xy} \qquad \Rightarrow \frac{\partial f}{\partial y}(1,0) = -1$$

$$\Rightarrow f(1.05, 0.11) = f(1+0.05, 0+0.11)$$

$$\approx f(1,0) + 0.05 \frac{\partial f}{\partial x}(1,0) + 0.11 \frac{\partial f}{\partial y}(1,0)$$

$$= 1 + 0.05 - 0.11$$

$$= 0.94$$

Nap. Prava vrijednost izraza je

$$f(1.05, 0.11) = 0.9354666\ldots$$

pa je greška aproksimacije oko 0.5%.

Greška aproksimacije (tj. njena najveća moguća vrijednost) ovisi o drugim parcijalnim derivacijama f-je f.

2. Neka je $\tilde{z} = \tilde{z}(u,v)$ fja t.d.

$$z(x,y) = \tilde{z}(u(x,y), v(x,y)) = (\tilde{z} \circ \varphi)(x,y),$$

gdje je $\varphi(x,y) = (u(x,y), v(x,y))$. Prema lančanom pravilu slijedi

$$\nabla z(x,y) = \nabla \tilde{z}(\varphi(x,y)) \cdot \nabla \varphi(x,y)$$

$$= \begin{bmatrix} \frac{\partial \tilde{z}}{\partial u} & \frac{\partial \tilde{z}}{\partial v} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \tilde{z}}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \tilde{z}}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial \tilde{z}}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \tilde{z}}{\partial v}\frac{\partial v}{\partial y} \end{bmatrix}$$

Budući da je $\nabla z(x,y) = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$, slijedi

$$\frac{\partial z}{\partial x} = \frac{\partial \tilde{z}}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \tilde{z}}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial \tilde{z}}{\partial u} + e^{x+2y}\frac{\partial \tilde{z}}{\partial v},$$

$$\frac{\partial z}{\partial y} = \frac{\partial \tilde{z}}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \tilde{z}}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial \tilde{z}}{\partial u} + 2e^{x+2y}\frac{\partial \tilde{z}}{\partial v},$$

pa imamo

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial v} = xy\frac{\partial \tilde{z}}{\partial u} + xe^{x+2y}\frac{\partial \tilde{z}}{\partial v} + xy\frac{\partial \tilde{z}}{\partial u} + 2ye^{x+2y}\frac{\partial \tilde{z}}{\partial v}$$

$$= \underbrace{2xy}_{=u}\frac{\partial \tilde{z}}{\partial u} + \underbrace{(x+2y)}_{=\ln v}\underbrace{e^{x+2y}}_{=v}\frac{\partial \tilde{z}}{\partial v}$$

$$= 2u\frac{\partial \tilde{z}}{\partial u} + v\ln v\frac{\partial \tilde{z}}{\partial v}$$

③ $z = e^{x+y+z-1} \quad \Big| \frac{\partial}{\partial x}$

$$z_x = e^{x+y+z-1} \cdot (1 + z_x)$$

$$(1 - e^{x+y+z-1}) z_x = e^{x+y+z-1} \quad \Rightarrow \quad z_x = \frac{e^{x+y+z-1}}{1 - e^{x+y+z-1}}.$$

$z = e^{x+y+z-1} \quad \Big| \frac{\partial}{\partial y}$

$$z_y = e^{x+y+z-1} \cdot (1 + z_y)$$

$$(1 - e^{x+y+z-1}) z_y = e^{x+y+z-1} \quad \Rightarrow \quad z_y = \frac{e^{x+y+z-1}}{1 - e^{x+y+z-1}}$$

(uočimo da smo z_y mogli odrediti i bez računanja jer je zadana implicitna jednadžba simetrična u odnosu na x i y)

$$\Rightarrow dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{e^{x+y+z-1}}{1 - e^{x+y+z-1}} (dx + dy)$$

Nadalje,

$$z_x = z_y = -1 + \frac{1}{1 - e^{x+y+z-1}}$$

$$\Rightarrow z_{xx} = -\frac{1}{(1 - e^{x+y+z-1})^2} \cdot (-e^{x+y+z-1}) \cdot \underbrace{(1 + z_x)}_{\frac{1}{1 - e^{x+y+z-1}}}$$

$$= \frac{e^{x+y+z-1}}{(1 - e^{x+y+z-1})^3}.$$

$$z_{yy} = -\frac{1}{(1 - e^{x+y+z-1})^2} \cdot (-e^{x+y+z-1}) \cdot (1 + z_y)$$

$$= \frac{e^{x+y+z-1}}{(1 - e^{x+y+z-1})^3}$$

$$z_{xy} = \left[\begin{array}{l}\text{uočimo da je zbog } z_x = z_y \text{ račun} \\ \text{isti kao i za } z_{xx}, \text{ tj. } z_{yy}\end{array}\right] = \frac{e^{x+y+z-1}}{(1-e^{x+y+z-1})^3}$$

$$\Rightarrow d^2 z = d(dz)$$

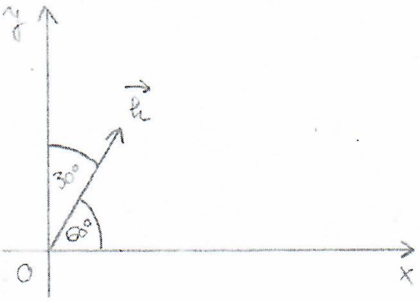
$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)dy$$

$$= \frac{\partial^2 z}{\partial x^2}(dx)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}(dy)^2$$

$$= \frac{e^{x+y+z-1}}{(1-e^{x+y+z-1})^3}\left((dx)^2 + 2\,dx\,dy + (dy)^2\right)$$

~~MATAN 2 – 4. vježbe~~

4. $f(x,y) = x^2 - xy - 2y^2$



Neka je $\vec{h}$ jedinični vektor koji s osi apscisa zatvara kut od $60°$.

Imamo

$$\vec{i}\cdot\vec{h} = \|\vec{i}\|\cdot\|\vec{h}\|\cos 60° = \frac{1}{2},$$

$$\vec{j}\cdot\vec{h} = \|\vec{j}\|\cdot\|\vec{h}\|\cos(90°-60°) = \frac{\sqrt{3}}{2}.$$

Nadalje,

$$\frac{\partial f}{\partial x}(x,y) = 2x - y \implies \frac{\partial f}{\partial x}(P) = 0$$

$$\frac{\partial f}{\partial y}(x,y) = -x - 4y \implies \frac{\partial f}{\partial y}(P) = -9$$

$$\implies \frac{\partial f}{\partial \vec{h}}(P) = \nabla f(P)\cdot\vec{h} = -9\vec{j}\cdot\vec{h} = -\frac{9\sqrt{3}}{2}.$$

5. (a)

$$\frac{\partial u}{\partial x}(x,y,z) = 2x - yz \implies \frac{\partial u}{\partial x}(1,2,1) = 0$$

$$\frac{\partial u}{\partial y}(x,y,z) = -2y - xz \implies \frac{\partial u}{\partial y}(1,2,1) = -5$$

$$\frac{\partial u}{\partial z}(x,y,z) = 2z - xy \implies \frac{\partial u}{\partial z}(1,2,1) = 0$$

$$\implies \frac{\partial u}{\partial \vec{s}}(1,2,1) = \nabla u(1,2,1)\cdot\frac{1}{\|\vec{s}\|}\vec{s} = -5\vec{j}\cdot\frac{1}{\sqrt{3}}(\vec{i}-\vec{j}+\vec{k}) = \frac{5}{\sqrt{3}}$$

(b) Za usmjerenu derivaciju od u u smjeru jediničnog vektora $\vec{h}$ vrijedi

$$\left|\frac{\partial u}{\partial \vec{h}}(1,1,1)\right| = |\nabla u(1,1,1)\cdot \vec{h}| \overset{CSB}{\leq} \|\nabla u(1,1,1)\|\cdot \underbrace{\|\vec{h}\|}_{=1}$$

$$\Rightarrow -\|\nabla u(1,1,1)\| \leq \frac{\partial u}{\partial \vec{h}}(1,1,1) \leq \|\nabla u(1,1,1)\|$$

Znamo da se jednakost postiže u slučaju kada su $\nabla u(1,1,1)$ i $\vec{h}$ kolinearni, tj. postoji $\lambda \in \mathbb{R}$ t.d.

$$\vec{h} = \lambda \nabla u(1,1,1) \qquad / \|\cdot\|$$

$$\Rightarrow \quad 1 = |\lambda| \cdot \|\nabla u(1,1,1)\|$$

$$\Rightarrow \quad |\lambda| = \frac{1}{\|\nabla u(1,1,1)\|} = \frac{1}{\|\vec{i}-3\vec{j}+\vec{k}\|} = \frac{1}{\sqrt{11}}$$

Imamo dvije mogućnosti:

1° za $\lambda_1 = \frac{1}{\sqrt{11}}$ imamo $\vec{h}_1 = \frac{1}{\sqrt{11}}(\vec{i}-3\vec{j}+\vec{k})$ i

$$\frac{\partial u}{\partial \vec{h}_1}(1,1,1) = \|\nabla u(1,1,1)\| = \sqrt{11}$$

postiže svoj maksimum,

2° za $\lambda_2 = -\frac{1}{\sqrt{11}}$ imamo $\vec{h}_2 = -\frac{1}{\sqrt{11}}(\vec{i}-3\vec{j}+\vec{k})$ i

$$\frac{\partial u}{\partial \vec{h}_2}(1,1,1) = -\|\nabla u(1,1,1)\| = -\sqrt{11}$$

postiže svoj minimum.