MATAN 2 - 4. vježbe

1. Ako je $I(\alpha)=\int_{\varphi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx$, gdje su fje φ, ψ i f neprekinuto diferencijabilne, onda je I diferencijabilna i

$$I'(\alpha)=f(\psi(\alpha),\alpha)\psi'(\alpha)-f(\varphi(\alpha),\alpha)\varphi'(\alpha)+\int_{\varphi(\alpha)}^{\psi(\alpha)}\frac{\partial f}{\partial\alpha}(x,\alpha)\,dx.$$

Zato

$$F(\alpha)=\int_0^{\sin\alpha}\frac{\operatorname{tg}(\alpha x)}{x}\,dx$$

$$\Rightarrow F'(\alpha)=\frac{\operatorname{tg}(\alpha\sin\alpha)}{\sin\alpha}\cdot\cos\alpha-0+\int_0^{\sin\alpha}\frac{1}{\cancel{x}}\cdot\frac{1}{\cos^2(\alpha x)}\cdot\cancel{x}\,dx$$

$$=\operatorname{tg}(\alpha\sin\alpha)\operatorname{ctg}\alpha+\int_0^{\sin\alpha}\frac{1}{\cos^2(\alpha x)}\,dx$$

$$=\operatorname{tg}(\alpha\sin\alpha)\operatorname{ctg}\alpha+\left(\frac{1}{\alpha}\operatorname{tg}(\alpha x)\right)\Big|_0^{\sin\alpha}$$

$$=\left(\frac{1}{\alpha}+\operatorname{ctg}\alpha\right)\operatorname{tg}(\alpha\sin\alpha)$$

2. Definiramo fju $f(x,y)=e^x\cos y$. Želimo aproksimirati $f(1.05, 0.11)$, a znamo $f(1,0)=e$. Imamo

$$\frac{\partial f}{\partial x}(x,y)=e^x\cos y\Rightarrow\frac{\partial f}{\partial x}(1,0)=e \qquad \frac{\partial^2 f}{\partial y^2}(x,y)=-e^x\cos y$$

$$\frac{\partial f}{\partial y}(x,y)=-e^x\sin y\Rightarrow\frac{\partial f}{\partial y}(1,0)=0 \qquad \Rightarrow\frac{\partial^2 f}{\partial y^2}(1,0)=-e$$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=e^x\cos y\Rightarrow\frac{\partial^2 f}{\partial x^2}(1,0)=e \qquad \frac{\partial^2 f}{\partial x\partial y}(x,y)=-e^x\sin y$$

$$\Rightarrow\frac{\partial^2 f}{\partial x\partial y}(1,0)=0$$

$$\Rightarrow f(1.05, 0.11) = f(1+0.05, 0+0.11)$$

$$\approx f(1,0) + \frac{1}{1!}\left[\frac{\partial f}{\partial x}(1,0)\cdot 0.05 + \frac{\partial f}{\partial y}(1,0)\cdot 0.11\right]$$

$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(1,0)\cdot 0.05^2 + 2\frac{\partial^2 f}{\partial x \partial y}(1,0)\cdot 0.05\cdot 0.11 + \frac{\partial^2 f}{\partial y^2}(1,0)\cdot 0.11^2\right]$$

$$= e + 0.05e + \frac{1}{2}e\cdot(0.05^2 - 0.11^2)$$

$$= 1.0452e = 2.841148\ldots$$

Stvarna vrijednost je $f(1.05, 0.11) = 2.8403797\ldots$

3 $f(x,y) = x^2y + 3y^2 - 2xy - 2x^2 + 6x - 11y + 9$

Vrijedi $f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^n f(x_0,y_0).$

Računamo

$f_x(x,y) = 2xy - 2y - 4x + 6 \quad \Rightarrow f_x(1,2) = 2$

$f_y(x,y) = x^2 + 6y - 2x - 11 \quad \Rightarrow f_y(1,2) = 0$

$f_{xx}(x,y) = 2y - 4 \quad \Rightarrow f_{xx}(1,2) = 0$

$f_{xy}(x,y) = 2x - 2 \quad \Rightarrow f_{xy}(1,2) = 0$

$f_{yy}(x,y) = 6 \quad \Rightarrow f_{yy}(1,2) = 6$

$f_{xxy}(x,y) = 2 \quad \Rightarrow f_{xxy}(1,2) = 2$

Budući da sve ostale parcijalne derivacije iščezavaju, slijedi:

$$f(x,y) = \frac{1}{0!}f(1,2) + \frac{1}{1!}\left[f_x(1,2)(x-1) + f_y(1,2)(y-2)\right]$$

$$+ \frac{1}{2!}\left[f_{xx}(1,2)(x-1)^2 + 2f_{xy}(1,2)(x-1)(y-2) + f_{yy}(1,2)(y-2)^2\right]$$

$$+ \frac{1}{3!}\cdot 3f_{xxy}(1,2)(x-1)^2(y-2)$$

$$= 1 + 2(x-1) + 3(y-2)^2 + (x-1)^2(y-2)$$

4. $f(x,y) = e^{x-y}(x^2-2y^2)$

$$\frac{\partial f}{\partial x}(x,y) = e^{x-y}(x^2-2y^2) + 2xe^{x-y} = 0 \qquad \Big| : e^{x-y} > 0.$$

$$\frac{\partial f}{\partial y}(x,y) = -e^{x-y}(x^2-2y^2) - 4ye^{x-y} = 0 \qquad \Big| : (-e^{x-y}) < 0$$

$$\Rightarrow \begin{cases} x^2-2y^2+2x = 0 \\ x^2-2y^2+4y = 0 \end{cases} \Big) \overset{+}{\,| \cdot (-1)} \qquad \Rightarrow x = 2y$$

Uvrstimo u npr. prvu jednadžbu

$$4y^2 - 2y^2 + 4y = 0$$

$$y^2 + 2y = 0$$

$$\boxed{\begin{matrix} y_1 = 0 \\ x_1 = 0 \end{matrix}} \qquad \boxed{\begin{matrix} y_2 = -2 \\ x_2 = -4 \end{matrix}} \quad \text{stacionarne točke}$$

Nadalje računamo

$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^{x-y}(x^2-2y^2) + 2xe^{x-y} + 2e^{x-y} + 2xe^{x-y}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = -e^{x-y}(x^2-2y^2) - 4ye^{x-y} - 2xe^{x-y}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = e^{x-y}(x^2-2y^2) + 4ye^{x-y} - 4e^{x-y} + 4ye^{x-y}$$

U stacionarnim točkama Hesseove matrice od f glase

$$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \qquad \begin{matrix} \Delta_1 = 2 > 0 \\ \Delta_2 = -8 < 0 \end{matrix} \qquad \Rightarrow (0,0) \text{ je sedlasta točka za } f$$

(f nema lokalni ekstrem u $(0,0)$)

$$H_f(-4,-2) = \begin{bmatrix} -6e^{-2} & 8e^{-2} \\ 8e^{-2} & -12e^{-2} \end{bmatrix} \qquad \begin{matrix} \Delta_1 = -6e^{-2} < 0 \\ \Delta_1 = 8e^{-2} > 0 \end{matrix} \qquad \Rightarrow H_f(-4,-2) < 0$$

$\Rightarrow (-4,-2)$ je lokalni maksimum za f