Linearna algebra - 7. auditorne vježbe

1. Zadan je paralelogram s vrhovima $A(-3,-2,0)$, $B(3,-3,1)$, $C(5,0,2)$ i D. Odredite kut koji zatvaraju dijagonale tog paralelograma.

Neka je $D(d_1, d_2, d_3)$. Imamo

$$ABCD \text{ paralelogram } \Rightarrow \overrightarrow{AB} = \overrightarrow{DC}$$

$$\Rightarrow 6\vec{i} - \vec{j} + \vec{k} = (5-d_1)\vec{i} - d_2\vec{j} + (2-d_3)\vec{k}$$

$$\Rightarrow d_1 = -1,\ d_2 = 1,\ d_3 = 1.$$

Dakle, $D(-1,1,1)$. Sada su vektori dijagonala tog paralelograma

$$\overrightarrow{AC} = 8\vec{i} + 2\vec{j} + 2\vec{k}, \quad \overrightarrow{BD} = -4\vec{i} + 4\vec{j},$$

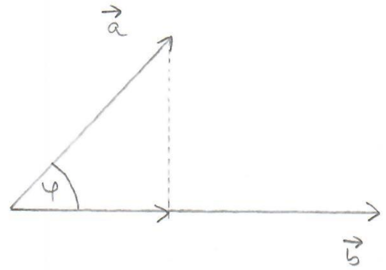
pa za kut među njima vrijedi

$$\cos\measuredangle(\overrightarrow{AC}, \overrightarrow{BD}) = \frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}|\cdot|\overrightarrow{BD}|} = \frac{8\cdot(-4) + 2\cdot 4 + 2\cdot 0}{\sqrt{8^2+2^2+2^2}\cdot\sqrt{(-4)^2+4^2}} = \frac{-24}{\sqrt{72}\cdot\sqrt{32}} = -\frac{1}{2}$$

$$\Rightarrow \measuredangle(\overrightarrow{AC}, \overrightarrow{BD}) = \frac{2\pi}{3}$$

2. Izračunajte skalarnu projekciju vektora $\mathbf{a} = 3\mathbf{i} - 12\mathbf{j} + 4\mathbf{k}$ u smjeru vektora $\mathbf{b} = (\mathbf{i} - 2\mathbf{k}) \times (\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$.

$$\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 1 & 3 & -4 \end{vmatrix} = 6\vec{i} + 2\vec{j} + 3\vec{k}$$



Tražena skalarna projekcija je

$$\pi_{\vec{b}}(\vec{a}) = |\vec{a}| \cos\varphi = \cancel{|\vec{a}|} \cdot \frac{\vec{a} \cdot \vec{b}}{\cancel{|\vec{a}|} \cdot |\vec{b}|}$$

$$= \frac{3 \cdot 6 - 12 \cdot 2 + 4 \cdot 3}{\sqrt{6^2 + 2^2 + 3^2}} = \frac{6}{7}.$$

3. Neka je $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 8\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$. Nađite vektor $\mathbf{c}$ takav da $\mathbf{c} \perp \mathbf{a}$, $\mathbf{c} \perp \mathbf{b}$, $|\mathbf{c}| = 12$ te takav da zatvara tupi kut s osi Oy. Koliko je oplošje paralelepipeda koji razapinju vektori $\mathbf{a}$, $\mathbf{b}$ i $\mathbf{c}$?

Iz $\vec{c} \perp \vec{a}$ i $\vec{c} \perp \vec{b}$ slijedi da postoji $\lambda \in \mathbb{R}$ takav da

$$\vec{c} = \lambda(\vec{a} \times \vec{b}) = \lambda \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ 8 & 5 & 2 \end{vmatrix} = \lambda(-\vec{i} + 2\vec{j} - \vec{k}).$$

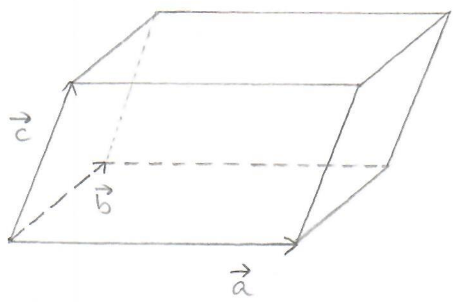
Nadalje,

$$12 = |\vec{c}| = |\lambda| \sqrt{(-1)^2 + 2^2 + (-1)^2} = |\lambda|\sqrt{6} \Rightarrow |\lambda| = 2\sqrt{6}.$$

Zbog pretpostavke da $\vec{c}$ zatvara tupi kut s osi Oy

$$\vec{c} \cdot \vec{j} = |\vec{c}| \cdot |\vec{j}| \underbrace{\cos \sphericalangle(\vec{c}, \vec{j})}_{<0} < 0 \Rightarrow 2\lambda < 0 \Rightarrow \lambda < 0,$$

pa slijedi: $\lambda = -2\sqrt{6}$ i $\vec{c} = 2\sqrt{6}(\vec{i} - 2\vec{j} + \vec{k})$.



Oplošje zadanog paralelepipeda je jednako dvostrukom zbroju površina paralelograma koje u parovima razapinju vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$:

$$\begin{aligned} O &= 2\left(|\vec{a} \times \vec{b}| + |\vec{b} \times \vec{c}| + |\vec{c} \times \vec{a}|\right) \\ &= 2\left(|-\vec{i} + 2\vec{j} - \vec{k}| + |\vec{b}| \cdot |\vec{c}| \sin\frac{\pi}{2} + |\vec{c}| \cdot |\vec{a}| \sin\frac{\pi}{2}\right) \\ &= 2\left(\sqrt{6} + \sqrt{8^2 + 5^2 + 2^2} \cdot 12 + 12\sqrt{3^2 + 2^2 + 1^2}\right) \\ &= 2\sqrt{6} + 24\sqrt{93} + 24\sqrt{14}. \end{aligned}$$

4. Neka je $\mathbf{a} = -2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mu\mathbf{i} + \lambda\mathbf{j} + 2\mathbf{k}$ i $\mathbf{c} = 4\mathbf{i} - 10\mathbf{j} + \mu\mathbf{k}$. Za koje su vrijednosti parametara $\lambda, \mu \in \mathbb{R}$ vektori $\mathbf{a}$, $\mathbf{b}$ i $\mathbf{c}$:

(a) kolinearni,

(b) komplanarni?

(a) Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su kolinearni ako su međusobno proporcionalni. Usporedivanjem trećih koordinata slijedi

$$\vec{b} = 2\vec{a} \Rightarrow 2\mu = 2\cdot(-2),\ \lambda = 2\cdot 5 \Rightarrow \mu = -2,\ \lambda = 10,$$

i za te vrijednosti μ i λ vidimo da je $\vec{c} = -\vec{a}$ i $\vec{c} = -2\vec{a}$.

(b) Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni ako i samo ako je njihov mješoviti umnožak jednak nuli:

$$0 = [\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} -2 & 5 & 1 \\ 2\mu & \lambda & 2 \\ 4 & -10 & \mu \end{vmatrix} \begin{matrix} | \cdot 2 \\ + \\ \end{matrix} = \begin{vmatrix} -2 & 5 & 1 \\ 2\mu & \lambda & 2 \\ 0 & 0 & \mu+2 \end{vmatrix} \leftarrow$$

$$= (\mu+2) \begin{vmatrix} -2 & 5 \\ 2\mu & \lambda \end{vmatrix} = -2(\mu+2)(\lambda+5\mu).$$

Dakle, vektori $\vec{a}$, $\vec{b}$, $\vec{c}$ su komplanarni ako je $\lambda = -5\mu$ ili $\mu = -2$, $\lambda \in \mathbb{R}$.

5. Zadani su vektori $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ i $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Odredite jedinični vektor $\mathbf{v}$ koji leži u ravnini razapetoj vektorima $\mathbf{b}$ i $\mathbf{c}$, a okomit je na vektor $\mathbf{a}$.

Neka je $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$. Budući da su vektori $\vec{v}$, $\vec{b}$ i $\vec{c}$ komplanarni,

$$0 = [\vec{v}, \vec{b}, \vec{c}] = \begin{vmatrix} v_1 & v_2 & v_3 \\ 1 & 2 & -1 \\ 1 & 1 & -2 \end{vmatrix} = -3v_1 + v_2 - v_3 .$$

Budući da su $\vec{v}$ i $\vec{a}$ okomiti,

$$0 = \vec{v} \cdot \vec{a} = 2v_1 - v_2 + v_3 .$$

Dakle,

$$\left.\begin{cases} -3v_1 + v_2 - v_3 = 0 \\ 2v_1 - v_2 + v_3 = 0 \end{cases}\right\} + \Rightarrow \ v_1 = 0, \quad v_2 = v_3 .$$

Konačno, iz uvjeta $|v| = 1$ slijedi:

$$v_1^2 + v_2^2 + v_3^2 = 1$$

$$\Rightarrow \quad 2v_2^2 = 1$$

$$\Rightarrow \quad v_2 = v_3 = \pm \frac{1}{\sqrt{2}} .$$

Dakle, postoje dva takva tražena vektora,

$$\vec{v}_1 = \frac{1}{\sqrt{2}} (\vec{j} + \vec{k}) \quad i \quad \vec{v}_2 = -\frac{1}{\sqrt{2}} (\vec{j} + \vec{k}).$$

6. Pojednostavnite izraze:

(a) $\mathbf{i} \times (\mathbf{j}+\mathbf{k}) - \mathbf{j} \times (\mathbf{i}+\mathbf{k}) + \mathbf{k} \times (\mathbf{i}+\mathbf{j}+\mathbf{k})$,

(b) $(\mathbf{a}+\mathbf{b}+\mathbf{c}) \times \mathbf{c} + (\mathbf{a}+\mathbf{b}+\mathbf{c}) \times \mathbf{b} + (\mathbf{b}-\mathbf{c}) \times \mathbf{a}$,

(c) $2\mathbf{i} \cdot (\mathbf{j} \times \mathbf{k}) + 3\mathbf{j} \cdot (\mathbf{i} \times \mathbf{k}) + 4\mathbf{k} \cdot (\mathbf{i} \times \mathbf{j})$.

(a) $\vec{i} \times (\vec{j}+\vec{k}) - \vec{j} \times (\vec{i}+\vec{k}) + \vec{k} \times (\vec{i}+\vec{j}+\vec{k}) =$

$$= \vec{i}\times\vec{j} + \vec{i}\times\vec{k} - \vec{j}\times\vec{i} - \vec{j}\times\vec{k} + \vec{k}\times\vec{i} + \vec{k}\times\vec{j} + \underbrace{\vec{k}\times\vec{k}}_{=\vec{0}}$$

$$= \vec{i}\times\vec{j} + \cancel{\vec{i}\times\vec{k}} + \vec{i}\times\vec{j} - \vec{j}\times\vec{k} - \cancel{\vec{i}\times\vec{k}} - \vec{j}\times\vec{k}$$

$$= 2(\vec{i}\times\vec{j} - \vec{j}\times\vec{k})$$

$$= 2(\vec{k} - \vec{i})$$

(b) $(\vec{a}+\vec{b}+\vec{c})\times\vec{c} + (\vec{a}+\vec{b}+\vec{c})\times\vec{b} + (\vec{b}-\vec{c})\times\vec{a} =$

$$= \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \underbrace{\vec{c}\times\vec{c}}_{=\vec{0}} + \vec{a}\times\vec{b} + \underbrace{\vec{b}\times\vec{b}}_{=\vec{0}} + \vec{c}\times\vec{b} + \vec{b}\times\vec{a} - \vec{c}\times\vec{a}$$

$$= \vec{a}\times\vec{c} + \cancel{\vec{b}\times\vec{c}} + \cancel{\vec{a}\times\vec{b}} - \cancel{\vec{b}\times\vec{c}} - \cancel{\vec{a}\times\vec{b}} + \vec{a}\times\vec{c}$$

$$= 2(\vec{a}\times\vec{c})$$

(c) $2\vec{i}\cdot(\vec{j}\times\vec{k}) + 3\vec{j}\cdot(\vec{i}\times\vec{k}) + 4\vec{k}\cdot(\vec{i}\times\vec{j}) =$

$$= 2[\vec{i},\vec{j},\vec{k}] + 3[\vec{j},\vec{i},\vec{k}] + 4[\vec{k},\vec{i},\vec{j}]$$

$$= 2\underbrace{[\vec{i},\vec{j},\vec{k}]}_{=1} - 3[\vec{i},\vec{j},\vec{k}] + 4[\vec{i},\vec{j},\vec{k}]$$

$$= 2 - 3 + 4$$

$$= 3$$

7. Odredite nužne i dovoljne uvjete na parametre $\alpha, \beta \in \mathbb{R}$ tako da vektori $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b} = \mathbf{i}+\alpha\mathbf{j}+\beta\mathbf{k}$ i $\mathbf{c} = \mathbf{i} + \alpha^2\mathbf{j} + \beta^2\mathbf{k}$ čine bazu prostora V^3. Zapišite vektor $\mathbf{d} = 4\mathbf{i} + (\alpha+1)^2\mathbf{j} + (\beta+1)^2\mathbf{k}$ u toj bazi.

Vektori $\vec{a}, \vec{b}, \vec{c}$ čine bazu prostora V^3 ako i samo ako su nekomplanarni, tj. ako i samo ako je

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \neq 0.$$

Računamo

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{matrix} \\ \mid\cdot(-1) \, + \\ + \, \mid\cdot(-1) \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & \beta-1 \\ 0 & \alpha^2-1 & \beta^2-1 \end{vmatrix} = \begin{vmatrix} \alpha-1 & \beta-1 \\ (\alpha-1)(\alpha+1) & (\beta-1)(\beta+1) \end{vmatrix}$$

$$= (\alpha-1)(\beta-1)\begin{vmatrix} 1 & 1 \\ \alpha+1 & \beta+1 \end{vmatrix}$$

$$= (\alpha-1)(\beta-1)(\beta-\alpha).$$

Dakle, $\{\vec{a}, \vec{b}, \vec{c}\}$ je baza prostora V^3 ako i samo ako je $(\alpha-1)(\beta-1)(\beta-\alpha) \neq 0$.

Sada za zadani vektor $\vec{d}$ tražimo skalare $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{R}$ takve da

$$\varphi_1\vec{a} + \varphi_2\vec{b} + \varphi_3\vec{c} = \vec{d}$$

$$\Leftrightarrow \begin{cases} \varphi_1 + \varphi_2 + \varphi_3 = 4 \\ \varphi_1 + \alpha\varphi_2 + \alpha^2\varphi_3 = (\alpha+1)^2 \\ \varphi_1 + \beta\varphi_2 + \beta^2\varphi_3 = (\beta+1)^2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & \alpha & \alpha^2 & \alpha^2+2\alpha+1 \\ 1 & \beta & \beta^2 & \beta^2+2\beta+1 \end{array}\right] \begin{matrix} \\ \mid\cdot(-1) \, + \\ + \, \mid\cdot(-1) \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \alpha-1 & (\alpha-1)(\alpha+1) & (\alpha-1)(\alpha+3) \\ 0 & \beta-1 & (\beta-1)(\beta+1) & (\beta-1)(\beta+3) \end{array}\right] \begin{matrix} \\ \mid : (\alpha-1)\neq 0 \\ \mid : (\beta-1)\neq 0 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \alpha+1 & \alpha+3 \\ 0 & 1 & \beta+1 & \beta+3 \end{array}\right] \begin{array}{l} + \\ |\cdot(-1) \\ |\cdot(-1) \\ + \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -\alpha & 1-\alpha \\ 0 & 1 & \alpha+1 & \alpha+3 \\ 0 & 0 & \beta-\alpha & \beta-\alpha \end{array}\right] |:(\beta-\alpha) \neq 0 \quad \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\alpha & 1-\alpha \\ 0 & 1 & \alpha+1 & \alpha+3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} + \quad |\cdot\alpha \\ + \\ |\cdot(-\alpha-1) \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \Rightarrow \varphi_1 = 1 \\ \Rightarrow \varphi_2 = 2 \\ \Rightarrow \varphi_3 = 1 \end{array}$$

Dakle, $\vec{d} = \vec{a} + 2\vec{b} + \vec{c}$.