Linearna algebra - 4. auditorne vježbe

1. U ovisnosti o parametru $a \in \mathbb{R}$ odredite rang matrice

$$\begin{bmatrix} 2a-1 & a & 1 \\ a & a & 1 \\ 1 & 1 & a \end{bmatrix}.$$

$$\begin{bmatrix} 2a-1 & a & 1 \\ a & a & 1 \\ 1 & 1 & a \end{bmatrix} \begin{matrix} + \\ +\ \mid\cdot(-a) \\ \mid\cdot(1-2a) \end{matrix} \sim \begin{bmatrix} 0 & 1-a & 1+a-2a^2 \\ 0 & 0 & 1-a^2 \\ 1 & 1 & a \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & a \\ 0 & 0 & 1-a^2 \\ 0 & 1-a & 1+a-2a^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & a \\ 0 & 1-a & 1+a-2a^2 \\ 0 & 0 & 1-a^2 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & a \\ 0 & 1-a & (1-a)(1+2a) \\ 0 & 0 & (1-a)(1+a) \end{bmatrix}$$

Doveli smo matricu u trokutasti oblik odakle rang možemo odrediti s obzirom na vrijednosti elemenata na glavnoj dijagonali:

1° za $1-a=0$, tj. $a=1$ imamo matricu

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i rang je jednak 1

2° za $a=-1$ imamo

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

i rang je jednak 2

3° za $a \neq \pm 1$ imamo

$$\begin{bmatrix} 1 & 1 & a \\ 0 & 1-a & (1-a)(1+2a) \\ 0 & 0 & (1-a)(1+a) \end{bmatrix} \begin{matrix} \\ \Big| : (1-a) \neq 0 \\ \Big| : (1-a^2) \neq 0 \end{matrix} \sim \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & 1+2a \\ 0 & 0 & 1 \end{bmatrix}$$

i rang je jednak 3

2. Neka su $\mathbf{A}$ i $\mathbf{B}$ kvadratne matrice reda 3. Neka $\mathbf{A}$ ima rang jednak 1, a $\mathbf{B}$ rang jednak 2. Koje su moguće vrijednosti ranga matrice $\mathbf{A}+\mathbf{B}$? Za svaku vrijednost nađite primjere matrica $\mathbf{A}$ i $\mathbf{B}$.

Općenito, rang matrice reda 3 može biti 0, 1, 2 ili 3.

Tvrdimo da je $r(A+B) \neq 0$.

Naime, u suprotnom

$$r(A+B) = 0 \Rightarrow A+B = 0 \Rightarrow A = -B \Rightarrow r(A) = \underbrace{r(-B)}_{=r(B)} \Rightarrow 1 = 2.$$

Kontradikcija.

Za ostale vrijednosti ranga određujemo odgovarajuće primjere matrica A i B za koje se te vrijednosti postižu:

1° $r(A+B) = 1$

$$A = E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A+B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2° $r(A+B) = 2$

$$A = E_{11}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A+B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3° $r(A+B) = 3$

$$A = E_{11}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A+B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Dakle, $r(A+B) \in \{1, 2, 3\}$.

3. Zadana je blok matrica $\mathbf{A}$ tipa 6×6 čiji su blokovi matrice $\mathbf{B}$, $\mathbf{C}$ te nul-matrice:

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{B} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C} \end{array}\right], \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 3 & -9 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Nađite inverz matrica $\mathbf{A}$, $\mathbf{B}$ i $\mathbf{C}$ te inverze umnožaka $\mathbf{BC}$ i $\mathbf{B}^2$.

Vrijedi

$$A^{-1} = \left[\begin{array}{c|c} B^{-1} & 0 \\ \hline 0 & C^{-1} \end{array}\right], \quad (BC)^{-1} = C^{-1} B^{-1}, \quad (B^2)^{-1} = B^{-1} \cdot B^{-1}$$

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{matrix}}_{B} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \end{array}\right] \begin{matrix} \\ + \\ |\cdot 1 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} |\cdot 1 \\ + \\ |\cdot 1 \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{array}\right] |:2 \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \begin{matrix} + \\ |\cdot(-1) \\ |\cdot(-1) \\ + \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \sim$$

$$\sim \left[\begin{array}{c|c} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & \underbrace{\begin{matrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}}_{B^{-1}} \end{array}\right]$$

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 3 & -9 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix}}_{C} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \end{array}\right] \begin{matrix} + \; |\cdot 9 \\ + \\ |\cdot(-2) \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 9 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} + \\ |\cdot(-3) \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 15 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & -3 & 15 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{matrix}}_{C^{-1}}$$

$$(BC)^{-1} = \begin{bmatrix} 1 & -3 & 15 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 13 & 17 & 19 \\ -1 & -3 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(B^2)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 3 \\ -1 & -1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

4. Za koje $\lambda \in \mathbb{R}$ su vektori

$$\begin{bmatrix} 1 \\ \lambda+1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ \lambda \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 11 \\ -6 \\ 0 \end{bmatrix}$$

linearno nezavisni?

Tražimo $\lambda \in \mathbb{R}$ za koje matrica

$$\begin{bmatrix} 1 & 2 & 1 \\ \lambda+1 & 1 & 11 \\ -1 & \lambda & -6 \\ 1 & 3 & 0 \end{bmatrix}$$

ima rang 3.

$$\begin{bmatrix} 1 & 2 & 1 \\ \lambda+1 & 1 & 11 \\ -1 & \lambda & -6 \\ 1 & 3 & 0 \end{bmatrix} \begin{array}{l} \cdot 1 \quad \cdot(-1) \\ + \\ + \end{array} \sim \begin{bmatrix} 1 & 2 & 1 \\ \lambda+1 & 1 & 11 \\ 0 & \lambda+2 & -5 \\ 0 & 1 & -1 \end{bmatrix} \begin{array}{l} + \,|\cdot 1 \end{array} \sim \begin{bmatrix} 1 & 3 & 1 \\ \lambda+1 & 12 & 11 \\ 0 & \lambda-3 & -5 \\ 0 & 0 & -1 \end{bmatrix} \begin{array}{l} + \,|\cdot 1 \\ + \,|\cdot 11 \\ + \,|\cdot(-5) \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 \\ \lambda+1 & 12 & 0 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{array}{l} |\cdot(-4) \\ + \\ \\ |\cdot(-1) \end{array} \sim \begin{bmatrix} 1 & 3 & 0 \\ \lambda-3 & 0 & 0 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ za $\lambda = 3$ matrica ima rang 2, dok za $\lambda \neq 3$ ima rang 3

Dakle, zadani su vektori linearno nezavisni za $\lambda \neq 3$.

5. U ovisnosti o parametru $\lambda \in \mathbb{R}$ odredite rang matrice $\mathbf{A}$ reda $n+1$:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 1 & \cdots & 1 \\ 1 & 1 & 2-\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n-\lambda \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 1 & \cdots & 1 \\ 1 & 1 & 2-\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n-\lambda \end{bmatrix} \begin{matrix} \mid\cdot(-1) \\ + \\ +\mid\cdot(-1) \\ \\ +\mid\cdot(-1) \end{matrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -\lambda & 0 & \cdots & 0 \\ 0 & 0 & 1-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)-\lambda \end{bmatrix}$$

Uočimo da je

$$\det A = -\lambda(1-\lambda)\cdot\ldots\cdot((n-1)-\lambda).$$

Dakle za $\lambda \notin \{0,1,2,\ldots,n-1\}$ je determinanta različita od nule pa je matrica A regularna, tj. punog ranga. Dakle, u ovom je slučaju rang od A jednak $n+1$.

Za $\lambda \in \{0,1,2,\ldots,n-1\}$ je matrica A ekvivalentna matrici

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(n jedinica na glavnoj dijagonali, svugdje ostalo nule)

pa je u ovom slučaju rang od A jednak n.

6. Izračunajte inverz matrice $\mathbf{A}$ reda n:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

$$[A \,\vdots\, I] \sim \left[\begin{array}{cccccc|cccccc} 1 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \overset{+}{1\cdot(-1)} \sim$$

$$\sim \left[\begin{array}{cccccc|cccccc} 1 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \sim$$

$\sim$ [u svakom idućem koraku nastavljamo oduzimati $(i+1)$-vi redak od i-tog kako bismo poništili jedinicu u i-tom retku iznad glavne dijagonale] $\sim$

$$\sim \left[\begin{array}{c|cccccc} & 1 & (-1)^1 & (-1)^2 & \cdots & (-1)^{n-2} & (-1)^{n-1} \\ & 0 & 1 & (-1)^1 & \cdots & (-1)^{n-3} & (-1)^{n-2} \\ I & 0 & 0 & 1 & \cdots & (-1)^{n-4} & (-1)^{n-3} \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & 0 & \cdots & 1 & (-1)^1 \\ & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & (-1)^1 & \cdots & (-1)^{n-1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{= A^{-1}}$$