MATAN 2 - 8. vježbe

1. (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} = ?$

Rastavljamo opći član na parcijalne razlomke:

$$\frac{1}{n^2-1} = \frac{1}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1}$$

$$\Rightarrow \quad 1 = A(n+1) + B(n-1) = (A+B)n + (A-B)$$

$$\Rightarrow \begin{cases} A+B = 0 \\ A-B = 1 \end{cases} \quad \Rightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

Zato je n-ta parcijalna suma ovog reda jednaka

$$S_n = \frac{1}{2^2-1} + \frac{1}{3^2-1} + \ldots + \frac{1}{(n-1)^2-1} + \frac{1}{n^2-1}$$

$$= \frac{1}{2}\cdot\frac{1}{2-1} - \frac{1}{2}\cdot\frac{1}{2+1} + \frac{1}{2}\cdot\frac{1}{3-1} - \frac{1}{2}\cdot\frac{1}{3+1} + \ldots +$$

$$+ \frac{1}{2}\cdot\frac{1}{(n-1)-1} - \frac{1}{2}\cdot\frac{1}{(n-1)+1} + \frac{1}{2}\cdot\frac{1}{n-1} - \frac{1}{2}\cdot\frac{1}{n+1}$$

$$= \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \ldots + \cancel{\frac{1}{n-2}} - \frac{1}{n} + \cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right)$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{2} - \overset{\nearrow 0}{\frac{1}{n}} + \overset{\nearrow 0}{\frac{1}{n+1}}\right) = \frac{3}{4}$$

(b) $$\sum_{n=0}^{\infty} \frac{ch^2 n + 3^{n+1}}{4^{2n-1}} = ?$$

$$\sum_{n=0}^{\infty} \frac{ch^2 n + 3^{n+1}}{4^{2n-1}} = 4 \sum_{n=0}^{\infty} \frac{1}{4^{2n}} \left(ch^2 n + 3^{n+1} \right)$$

$$= 4 \sum_{n=0}^{\infty} \frac{1}{4^{2n}} \left(\frac{1}{4} \left(e^{2n} + 2 + e^{-2n} \right) + 3 \cdot 3^n \right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^2}{16} \right)^n + 2 \sum_{n=0}^{\infty} \left(\frac{1}{16} \right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{16 e^2} \right)^n + 12 \sum_{n=0}^{\infty} \left(\frac{3}{16} \right)^n$$

$$= \left[\begin{array}{l} \text{svi dobiveni geometrijski redovi} \\ \text{su konvergentni} \end{array} \right]$$

$$= \frac{1}{1 - \frac{e^2}{16}} + 2 \cdot \frac{1}{1 - \frac{1}{16}} + \frac{1}{1 - \frac{1}{16e^2}} + 12 \cdot \frac{1}{1 - \frac{3}{16}}$$

$$= \frac{16}{16 - e^2} + \frac{32}{15} + \frac{16e^2}{16e^2 - 1} + \frac{192}{13}$$

$$= \frac{3296}{195} + \frac{16}{16 - e^2} + \frac{16e^2}{16e^2 - 1}$$

2. (a) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n^2+2n}-\sqrt{n^2-n}\right)^n$

1. način

$$\left(\sqrt{n^2+2n}-\sqrt{n^2-n}\right)^n=\left(\sqrt{n^2+2n}-\sqrt{n^2-n}\right)^n\cdot\frac{\left(\sqrt{n^2+2n}+\sqrt{n^2-n}\right)^n}{\left(\sqrt{n^2+2n}+\sqrt{n^2-n}\right)^n}$$

$$=\left(\frac{n^2+2n-n^2+n}{\sqrt{n^2+2n}+\sqrt{n^2-n}}\right)^n=\Bigg(\underbrace{\frac{3n}{\sqrt{n^2+2n}+\sqrt{n^2-n}}}_{\to\frac{3}{2}>1}\Bigg)^{\overset{\nearrow\infty}{n}}\xrightarrow{n\to\infty}\infty$$

NUK $\Rightarrow$ red divergira

2. način

$$\sqrt[n]{\left(\sqrt{n^2+2n}-\sqrt{n^2-n}\right)^n}=\sqrt{n^2+2n}-\sqrt{n^2-n}\xrightarrow{n\to\infty}\frac{3}{2}>1$$

Cauchy $\Rightarrow$ red divergira

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+1}{2n-1}\right)^n$

$$\left(\frac{2n+1}{2n-1}\right)^n=\left(1+\frac{2}{2n-1}\right)^n=\left(1+\frac{1}{n-\frac{1}{2}}\right)^n$$

$$=\Bigg[\underbrace{\left(1+\frac{1}{n-\frac{1}{2}}\right)^{n-\frac{1}{2}}}_{\to e}\Bigg]^{\overset{\nearrow 1}{\frac{n}{n-\frac{1}{2}}}}\xrightarrow{n\to\infty}e$$

NUK $\Rightarrow$ red divergira

(uočimo i da u ovom slučaju Cauchyjev kriterij ne daje odluku)

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{10^n}{n \cdot 4^{2n+1}}$

Za sve $n \geq 3$ imamo

$$\frac{10^n}{n \cdot 4^{2n+1}} \leq \frac{10^n}{4^{2n+1}} = \frac{1}{4} \cdot \left(\frac{10}{16}\right)^n,$$

a budući da je $\displaystyle\sum_{n=3}^{\infty} \frac{1}{4} \cdot \left(\frac{10}{16}\right)^n$ konvergentan geometrijski red,

prema usporednom kriteriju zadani red konvergira.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n}$

$$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \cancel{(n+1)} \cdot \frac{1}{\cancel{n+1}} \cdot \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \xrightarrow{n\to\infty} \frac{1}{e} < 1$$

D'Alambert $\Rightarrow$ red konvergira

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$

$a_n := \ln\left(1+\frac{1}{n}\right)$

1° $1+\frac{1}{n} > 1 \Rightarrow a_n = \ln\left(1+\frac{1}{n}\right) > 0 \quad \forall n \in \mathbb{N}$

2° $\frac{1}{n} > \frac{1}{n+1} \Rightarrow 1+\frac{1}{n} > 1+\frac{1}{n+1} \Rightarrow a_n > a_{n+1} \quad \forall n \in \mathbb{N}$

3° $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \ln\left(1+\underbrace{\frac{1}{n}}_{\to 0}\right) = \ln 1 = 0$

Leibniz $\Rightarrow \displaystyle\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$ konvergira

(f) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

$$\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \ln\left(1+\frac{1}{n}\right)^{n} \xrightarrow{n\to\infty} \ln e = 1 \in \langle 0, \infty)$$

Budući da red $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ divergira, prema usporednom kriteriju i zadani red divergira.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(3^n)|}{3^n}$

Za svaki $n \in \mathbb{N}$ imamo

$$\frac{|\sin(3^n)|}{3^n} \leq \frac{1}{3^n},$$

a budući da je $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ konvergentan geometrijski red, prema usporednom kriteriju zadani red konvergira.

(h) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^3(2n)}$

Promotrimo funkciju

$$f : [2, \infty) \to \langle 0, \infty), \quad f(x) = \frac{1}{x \ln^3(2x)}.$$

Ona je neprekidna i padajuća na čitavoj svojoj domeni i vrijedi

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x \ln^3(2x)} dx = \begin{bmatrix} u = \ln(2x) \\ du = \frac{1}{x} dx \end{bmatrix}$$

$$= \int_{\ln 4}^{\infty} \frac{1}{u^3}\, du = -\frac{1}{2} \cdot \frac{1}{u^2}\Big|_{\ln 4}^{\infty} = \frac{1}{2} \cdot \frac{1}{\ln^2 4} - 0 < \infty,$$

pa prema integralnom kriteriju slijedi da zadani red konvergira.

(i) $$\sum_{n=1}^{\infty} \frac{2n}{(n+1)^{5/2} + n^2 + 1}$$

$$\frac{\frac{2n}{(n+1)^{5/2} + n^2 + 1}}{\frac{1}{n^{3/2}}} = \frac{2n^{5/2}}{(n+1)^{5/2} + n^2 + 1} \xrightarrow{n \to \infty} 2 \in \langle 0, \infty \rangle,$$

a budući da red $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ konvergira $\left(\frac{3}{2} > 1\right)$, prema usporednom kriteriju slijedi i da zadani red konvergira.

(j) $$\sum_{n=1}^{\infty} \operatorname{arctg}(n) \sin\left(\frac{1}{n}\right)$$

Imamo

$$\frac{\operatorname{arctg}(n) \sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \overset{\frac{\pi}{2}}{\overset{\nearrow}{\operatorname{arctg}(n)}} \cdot \overset{1}{\overset{\nearrow}{\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}}} \xrightarrow{n \to \infty} \frac{\pi}{2} \in \langle 0, \infty \rangle,$$

a budući da red $\sum_{n=1}^{\infty} \frac{1}{n}$ divergira, prema usporednom kriteriju i zadani red divergira.

(k) $\sum_{n=0}^{\infty} \operatorname{arctg} 2^{-n}$

$$\lim_{n\to\infty} \frac{\operatorname{arctg} 2^{-n}}{2^{-n}} = \left[\lim_{x\to 0} \frac{\operatorname{arctg} x}{x} = 1 \right] = 1 \in \langle 0, \infty \rangle ,$$

a budući da je $\sum_{n=0}^{\infty} 2^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ konvergentan geometrijski red, prema usporednom kriteriju slijedi i da zadani red konvergira.

(l) $\sum_{n=1}^{\infty} \sin\left(\sqrt{n^3+1} - \sqrt{n^3}\right)$

Najprije uočimo

$$\sqrt{n^3+1} - \sqrt{n^3} = \left(\sqrt{n^3+1} - \sqrt{n^3}\right) \cdot \frac{\sqrt{n^3+1} + \sqrt{n^3}}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}} \xrightarrow{n\to\infty} 0 .$$

Zato slijedi

$$\lim_{n\to\infty} \frac{\sin\left(\sqrt{n^3+1} - \sqrt{n^3}\right)}{\frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}} = 1 \in \langle 0, \infty \rangle$$

pa zadani red prema usporednom kriteriju dijeli konvergenciju s redom $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}$.

Nadalje, zbog

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}} = \frac{1}{2} \in \langle 0, \infty \rangle$$

prema usporednom kriteriju slijedi da red $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$

konvergira jer dijeli konvergenciju s konvergentnim redom $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

Zato i zadani red konvergira.