Linearna algebra - 6. auditorne vježbe

1. Odredite dvije različite linearne kombinacije vektora $\mathbf{a} = (1,3)$, $\mathbf{b} = (2,7)$ i $\mathbf{c} = (1,5)$ koje su jednake vektoru $\mathbf{v} = (0,1)$.

Tražimo skalare $\alpha, \beta, \gamma \in \mathbb{R}$ takve da vrijedi

$$\alpha \vec{a} + \beta \vec{b} + \gamma \vec{c} = \vec{v}$$

$$\Leftrightarrow \quad (\alpha + 2\beta + \gamma,\ 3\alpha + 7\beta + 5\gamma) = (0, 1).$$

Pripadni sustav linearnih jednadžbi rješavamo Gaussovim eliminacijama:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & 7 & 5 & 1 \end{array}\right] \begin{array}{l} | \cdot (-3) \\ + \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] \begin{array}{l} + \\ | \cdot (-2) \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right] \begin{array}{l} \Rightarrow \alpha = -2 + 3\gamma \\ \Rightarrow \beta = 1 - 2\gamma \end{array}$$

Stavljanjem $\gamma = t$, $t \in \mathbb{R}$, dobivamo jednoparametarsko rješenje

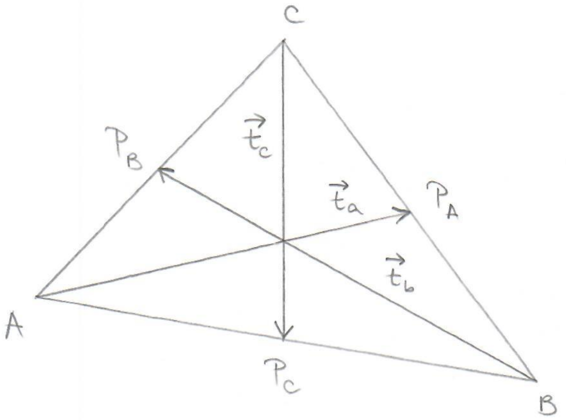
$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Odabirom različitih vrijednosti parametra t dobivamo različite tražene linearne kombinacije. Na primjer, za $t = 0$ i $t = 1$ imamo

$$\vec{v} = -2\vec{a} + \vec{b} = \vec{a} - \vec{b} + \vec{c}.$$

2. Neka su $\mathbf{t}_a$, $\mathbf{t}_b$ i $\mathbf{t}_c$ vektori težišnica proizvoljno odabranog trokuta. Dokažite

$$\mathbf{t}_a + \mathbf{t}_b + \mathbf{t}_c = \mathbf{0}.$$



Vrijedi:

$$\vec{t_a} = \overrightarrow{AB} + \overrightarrow{BP_A} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC},$$

$$\vec{t_b} = \overrightarrow{BC} + \overrightarrow{CP_B} = \overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA},$$

$$\vec{t_c} = \overrightarrow{CA} + \overrightarrow{AP_C} = \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB}.$$

Zbrajanjem ovih jednakosti dobivamo

$$\vec{t_a} + \vec{t_b} + \vec{t_c} = \frac{3}{2}\Big(\underbrace{\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}}_{=\vec{0}}\Big) = \vec{0}.$$

(Geometrijski: vektori težišnica proizvoljno odabranog trokuta se podudaraju s vektorima stranica nekog drugog trokuta.)

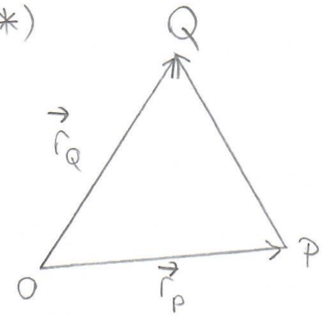
3. (a) Trokuti $A_1B_1C_1$ i $A_2B_2C_2$ imaju redom težišta T_1 i T_2. Dokažite da vrijedi

$$\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\overrightarrow{T_1T_2}.$$

(b) U trokutu ABC su A', B' i C' redom polovišta stranica $\overline{BC}$, $\overline{CA}$ i $\overline{AB}$. Dokažite da trokuti ABC i $A'B'C'$ imaju zajedničko težište.

(a) Općenito, ako je T težište trokuta ABC, onda za radijus-vektore točaka T, A, B i C vrijedi

$$\vec{r}_T = \frac{1}{3}(\vec{r}_A + \vec{r}_B + \vec{r}_C). \quad (*)$$



$$\overrightarrow{PQ} = \vec{r}_Q - \vec{r}_P$$

Sada imamo:

$$\begin{aligned}
\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} &= (\vec{r}_{A_2} - \vec{r}_{A_1}) + (\vec{r}_{B_2} - \vec{r}_{B_1}) + (\vec{r}_{C_2} - \vec{r}_{C_1}) \\
&= (\vec{r}_{A_2} + \vec{r}_{B_2} + \vec{r}_{C_2}) - (\vec{r}_{A_1} + \vec{r}_{B_1} + \vec{r}_{C_1}) \\
&\overset{(*)}{=} 3\vec{r}_{T_2} - 3\vec{r}_{T_1} \\
&= 3(\vec{r}_{T_2} - \vec{r}_{T_1}) = 3\overrightarrow{T_1T_2}.
\end{aligned}$$

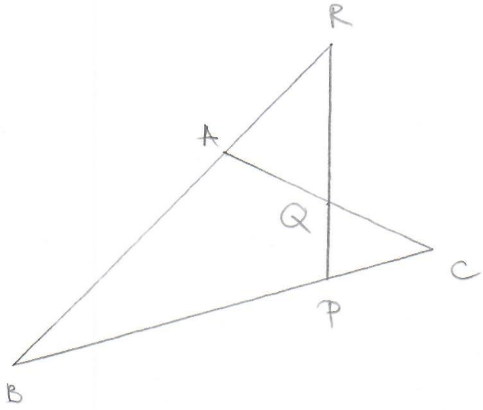
(b) Neka su T, T' redom težišta trokuta ABC i $A'B'C'$. Prema (a) podzadatku

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 3\overrightarrow{TT'},$$

a budući da su $\overrightarrow{AA'}$, $\overrightarrow{BB'}$ i $\overrightarrow{CC'}$ vektori težišnica trokuta ABC, prema prethodnom je zadatku njihov zbroj jednak nul-vektoru.

Dakle, $\overrightarrow{TT'} = \vec{0}$ pa se točke T i T' podudaraju.

4. U trokutu ABC točka Q je polovište stranice $\overline{CA}$, a točke P i R su takve da vrijedi $\overrightarrow{BP} = 3\overrightarrow{PC}$ i $2\overrightarrow{RA} = \overrightarrow{AB}$. Dokažite da su točke P, Q i R kolinearne (leže na istom pravcu) i nađite omjer $|PR| : |QR|$.



Točke P, Q i R su kolinearne ako i samo ako su vektori $\overrightarrow{PQ}$ i $\overrightarrow{QR}$ kolinearni, tj. postoji $\lambda \in \mathbb{R}$ takav da $\overrightarrow{PQ} = \lambda \overrightarrow{QR}$.

Imamo

$$\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CQ} = \frac{1}{4}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA},$$

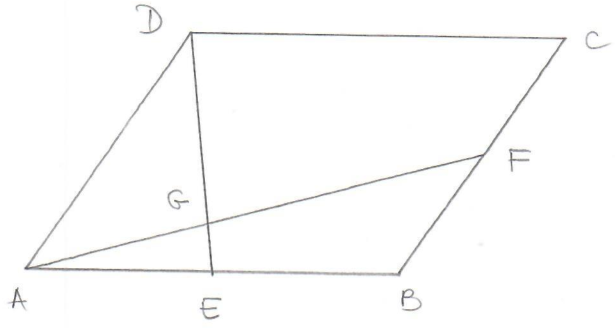
$$\overrightarrow{QR} = \overrightarrow{QA} + \overrightarrow{AR} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{BA}$$

$$= \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{CA}\right) = \frac{1}{2}\overrightarrow{BC} + \overrightarrow{CA} = 2\underbrace{\left(\frac{1}{4}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA}\right)}_{=\overrightarrow{PQ}}$$

$\Rightarrow \overrightarrow{PQ} = \frac{1}{2}\overrightarrow{QR} \Rightarrow$ točke P, Q i R su kolinearne

Budući da je $|PQ| : |QR| = 1 : 2$, imamo $|PR| : |QR| = 3 : 2$.

5. Neka su E i F redom polovišta stranica $\overline{AB}$ i $\overline{BC}$ paralelograma $ABCD$, a G sjecište dužina $\overline{AF}$ i $\overline{DE}$. U kojem omjeru točka G dijeli dužine $\overline{AF}$ i $\overline{DE}$?



Označimo $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{AD}$.

Nadalje, neka su $\lambda, \mu \in \mathbb{R}$ takvi da

$$\overrightarrow{AG} = \lambda \overrightarrow{AF}, \quad \overrightarrow{DG} = \mu \overrightarrow{DE}.$$

Zapisat ćemo vektor $\overrightarrow{AG}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$ na dva načina:

1° iz trokuta ABF slijedi

$$\overrightarrow{AG} = \lambda \overrightarrow{AF} = \lambda\left(\overrightarrow{AB} + \overrightarrow{BF}\right) = \lambda\left(\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}\right) = \lambda \vec{a} + \frac{1}{2}\lambda \vec{b},$$

2° iz trokuta ADG i ADE dobivamo

$$\overrightarrow{AG} = \overrightarrow{AD} + \overrightarrow{DG} = \overrightarrow{AD} + \mu \overrightarrow{DE} = \overrightarrow{AD} + \mu\left(\overrightarrow{DA} + \overrightarrow{AE}\right) = \overrightarrow{AD} + \mu\left(-\overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB}\right)$$

$$= \vec{b} + \mu\left(-\vec{b} + \frac{1}{2}\vec{a}\right) = \frac{1}{2}\mu \vec{a} + (1-\mu)\vec{b}.$$

Dakle,

$$\overrightarrow{AG} = \lambda \vec{a} + \frac{1}{2}\lambda \vec{b} = \frac{1}{2}\mu \vec{a} + (1-\mu)\vec{b}$$

$$\Rightarrow \left(\lambda - \frac{1}{2}\mu\right)\vec{a} + \left(\frac{1}{2}\lambda + \mu - 1\right)\vec{b} = \vec{0}.$$

No, vektori $\vec{a}$ i $\vec{b}$ su linearno nezavisni (jer nisu kolinearni) pa njihova linearna kombinacija može iščezavati samo na trivijalan način. Zato

$$\begin{cases} \lambda - \frac{1}{2}\mu = 0 \\ \frac{1}{2}\lambda + \mu = 1 \end{cases} \Rightarrow \lambda = \frac{2}{5}, \ \mu = \frac{4}{5}$$

Dakle, $\overrightarrow{AG} = \frac{2}{5}\overrightarrow{AF}$ i $\overrightarrow{DG} = \frac{4}{5}\overrightarrow{DE}$ pa je

$$|AG| : |GF| = 2:3, \quad |DG| : |GE| = 4:1.$$