Linearna algebra - 5. auditorne vježbe

1. Hrvatska ratna mornarica obnavlja flotu kupovinom 18 novih brodova. Na tržištu postoje tri tipa brodova koji se razlikuju po broju putnika i količini tereta koju mogu prevesti. Tri tipa brodova mogu prevesti redom 70, 110 i 200 putnika te 210, 250 i 350 tona tereta. Koliko brodova pojedinog tipa moraju kupiti ako je poznato da žele ukupni kapacitet od 1880 putnika i 4420 tona tereta?

Označimo:

x_1 = broj potrebnih brodova prvog tipa

x_2 = broj potrebnih brodova drugog tipa

x_3 = broj potrebnih brodova trećeg tipa

Iz uvjeta zadatka dobivamo sustav tri linearne jednadžbe s tri nepoznanice:

$$\begin{cases} x_1 + x_2 + x_3 = 18 & \text{(izjednačavanjem broja brodova)} \\ 70x_1 + 110x_2 + 200x_3 = 1880 & \text{(izjednačavanjem broja putnika)} \\ 210x_1 + 250x_2 + 350x_3 = 4420 & \text{(izjednačavanjem količine tereta)} \end{cases}$$

Sustav rješavamo u matričnom obliku, Gaussovim eliminacijama:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ 70 & 110 & 200 & 1880 \\ 210 & 250 & 350 & 4420 \end{array}\right] \begin{array}{l} \\ |:10 \\ |:10 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ 7 & 11 & 20 & 188 \\ 21 & 25 & 35 & 442 \end{array}\right] \begin{array}{l} |\cdot(-7) \\ + \\ + \ |\cdot(-21) \end{array} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ 0 & 4 & 13 & 62 \\ 0 & 4 & 14 & 64 \end{array}\right] \begin{array}{l} \\ |\cdot(-1) \\ + \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ 0 & 4 & 13 & 62 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} + \ |\cdot(-1) \\ + \\ |\cdot(-13) \end{array} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 16 \\ 0 & 4 & 0 & 36 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \\ |:4 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 16 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} + \\ |\cdot(-1) \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\Rightarrow (x_1, x_2, x_3) = (7, 9, 2)$$

Dakle, treba kupiti 7 brodova prvog tipa, 9 brodova drugog tipa te 2 broda trećeg tipa.

2. Za balet "Orašar" prodano je 480 ulaznica po cijenama od 220, 180 i 150 kn. Ukupna zarada iznosi 90 080 kn, a najjeftinijih je ulaznica prodano tri puta manje nego svih ostalih zajedno. Koliko je kojih ulaznica prodano?

Označimo:

x_1 = broj prodanih ulaznica po cijeni od 220 kn

x_2 = broj prodanih ulaznica po cijeni od 180 kn

x_3 = broj prodanih ulaznica po cijeni od 150 kn

Iz uvjeta zadatka:

$$\begin{cases} x_1 + x_2 + x_3 = 480 & \text{(izjednačavanjem broja prodanih ulaznica)} \\ 220x_1 + 180x_2 + 150x_3 = 90\,080 & \text{(izjednačavanjem zarade)} \\ x_1 + x_2 - 3x_3 = 0 & \text{(posljednji uvjet)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 480 \\ 220 & 180 & 150 & 90\,080 \\ 1 & 1 & -3 & 0 \end{array}\right] \begin{array}{l} + \mid\cdot(-1) \\ + \mid\cdot(-220) \\ \end{array} \sim \left[\begin{array}{ccc|c} 0 & 0 & 4 & 480 \\ 0 & -40 & 810 & 90\,080 \\ 1 & 1 & -3 & 0 \end{array}\right] \mid : 4 \sim$$

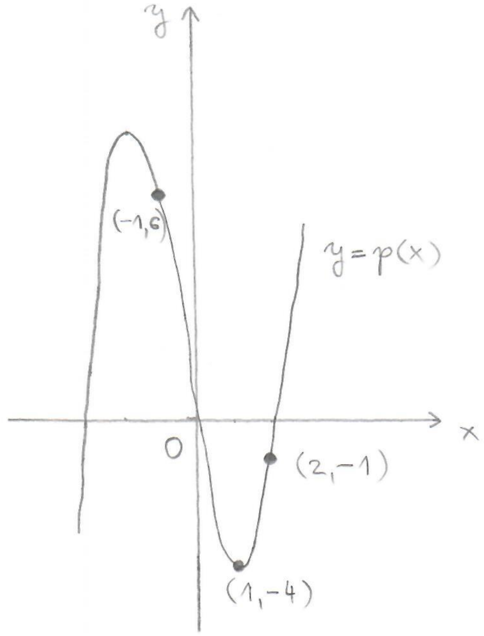
$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 120 \\ 0 & -40 & 810 & 90\,080 \\ 1 & 1 & -3 & 0 \end{array}\right] \begin{array}{l} \mid\cdot(-810) \\ + \\ + \mid\cdot 3 \end{array} \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 120 \\ 0 & -40 & 0 & -7120 \\ 1 & 1 & 0 & 360 \end{array}\right] \mid : (-40) \sim$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 120 \\ 0 & 1 & 0 & 178 \\ 1 & 1 & 0 & 360 \end{array}\right] \begin{array}{l} \\ \mid\cdot(-1) \\ + \end{array} \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 120 \\ 0 & 1 & 0 & 178 \\ 1 & 0 & 0 & 182 \end{array}\right]$$

$\Rightarrow (x_1, x_2, x_3) = (182, 178, 120)$

Prodane su 182 ulaznice po 220 kn, 178 ulaznica po 180 kn i 120 ulaznica po cijeni od 150 kn.

3. **(Lagrangeov interpolacijski polinom)** Odredite sve polinome stupnja manjeg ili jednakog tri kojima graf prolazi točkama $(1,-4)$, $(-1,6)$ i $(2,-1)$. Ukoliko dodamo još i uvjet da polinom treba biti normiran, postoji li takav polinom i je li jedinstven?



Polinom stupnja manjeg ili jednakog 3 je općenito oblika:

$$p(x) = ax^3 + bx^2 + cx + d, \quad a,b,c,d \in \mathbb{R}.$$

Tražimo polinome čiji grafovi prolaze zadanim točkama:

$$p(1) = -4 \Rightarrow a + b + c + d = -4$$
$$p(-1) = 6 \Rightarrow -a + b - c + d = 6$$
$$p(2) = -1 \Rightarrow 8a + 4b + 2c + d = -1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -4 \\ -1 & 1 & -1 & 1 & 6 \\ 8 & 4 & 2 & 1 & -1 \end{array}\right] \begin{array}{l} \cdot 1 \\ \cdot(-8) \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -4 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & -4 & -6 & -7 & 31 \end{array}\right] \begin{array}{l} \\ :2 \\ \end{array} \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -4 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -4 & -6 & -7 & 31 \end{array}\right] \begin{array}{l} \cdot(-1) \\ \cdot 4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -5 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -6 & -3 & 35 \end{array}\right] \begin{array}{l} \\ \\ :(-3) \end{array} \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -5 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & -\frac{35}{3} \end{array}\right] \cdot(-1) \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -5 \\ 0 & 1 & -2 & 0 & \frac{38}{3} \\ 0 & 0 & 2 & 1 & -\frac{35}{3} \end{array}\right] \begin{array}{l} \Rightarrow a = -5 - c \\ \Rightarrow b = \frac{38}{3} + 2c \\ \Rightarrow d = -\frac{35}{3} - 2c \end{array}$$

Stavljanjem $c = t$, $t \in \mathbb{R}$, dobivamo rješenje u parametarskom obliku

$$(a,b,c,d) = \left(-5-t, \ \frac{38}{3} + 2t, \ t, \ -\frac{35}{3} - 2t\right), \quad t \in \mathbb{R}.$$

Dakle, tvrdnju zadatka zadovoljavaju svi polinomi oblika

$$p_t(x) = (-5-t)x^3 + \left(\frac{38}{3} + 2t\right)x^2 + tx + \left(-\frac{35}{3} - 2t\right), \quad t \in \mathbb{R}.$$

Ukoliko dodamo i uvjet da p mora biti normiran (vodeći koeficijent mu mora biti jednak 1), imamo

$$-5-t=1 \implies t=-6,$$

dobivamo jedinstveno rješenje

$$p(x)=x^3+\left(\frac{38}{3}-12\right)x-6x+\left(12-\frac{35}{3}\right)=x^3+\frac{2}{3}x^2-6x+\frac{1}{3}.$$

4. Zadan je sustav linearnih jednadžbi:

$$\begin{cases} 2x_1 & + & x_2 & - & 3x_3 = 4 \\ -x_1 & + & 2x_2 & - & x_3 = 2 \\ x_1 & + & 3x_2 & - & 4x_3 = 6 \end{cases}.$$

(a) Dokažite da ovaj sustav ima beskonačno mnogo rješenja.

(b) Odredite jedno rješenje tog sustava za koje vrijedi:

i. sve koordinate rješenja su nenegativni realni brojevi,

ii. zbroj koordinata rješenja je veći od 4,

iii. prva i treća koordinata rješenja se podudaraju.

(a)

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 4 \\ -1 & 2 & -1 & 2 \\ 1 & 3 & -4 & 6 \end{array}\right] \begin{matrix} + \\ + \\ |\cdot 1 \;\; |\cdot(-2) \end{matrix} \sim \left[\begin{array}{ccc|c} 0 & -5 & 5 & -8 \\ 0 & 5 & -5 & 8 \\ 1 & 3 & -4 & 6 \end{array}\right] \begin{matrix} + \\ |\cdot 1 \\ \\ \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 5 & -5 & 8 \\ 1 & 3 & -4 & 6 \end{array}\right] \begin{matrix} \\ |:5 \\ \\ \end{matrix} \sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & \frac{8}{5} \\ 1 & 3 & -4 & 6 \end{array}\right] \begin{matrix} \\ |\cdot(-3) \\ + \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & \frac{8}{5} \\ 1 & 0 & -1 & \frac{6}{5} \end{array}\right] \begin{matrix} \Rightarrow x_2 = \frac{8}{5} + x_3 \\ \Rightarrow x_1 = \frac{6}{5} + x_3 \end{matrix}$$

Rang proširene matrice sustava je 2 pa sustav ima beskonačno mnogo rješenja koja ovise o $3-2=1$ parametru.

Stavljanjem $x_3 = t$, $t \in \mathbb{R}$, dobivamo rješenje

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} + t \\ \frac{8}{5} + t \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{8}{5} \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

(b) i.

$$\left.\begin{array}{lll} x_1 \geq 0 & \Rightarrow \frac{6}{5} + t \geq 0 & \Rightarrow t \geq -\frac{6}{5} \\ x_2 \geq 0 & \Rightarrow \frac{8}{5} + t \geq 0 & \Rightarrow t \geq -\frac{8}{5} \\ x_3 \geq 0 & \Rightarrow t \geq 0 & \end{array}\right\} t \geq 0$$

Neka tražena rješenja su

$$\left(\frac{6}{5}, \frac{8}{5}, 0\right), \left(\frac{9}{5}, \frac{7}{5}, \frac{1}{5}\right), \left(\frac{11}{5}, \frac{13}{5}, 1\right) \text{ itd.}$$

ii. $x_1 + x_2 + x_3 > 4$

$$\frac{6}{5} + t + \frac{8}{5} + t + t > 4$$

$$\frac{14}{5} + 3t > 4$$

$$3t > \frac{6}{5} \Rightarrow t > \frac{2}{5}$$

Neka tražena rješenja su:

$$\left(\frac{9}{5}, \frac{11}{5}, \frac{3}{5}\right), \left(\frac{11}{5}, \frac{13}{5}, 1\right), \left(2, \frac{12}{5}, \frac{4}{5}\right) \text{ itd.}$$

iii. $x_1 = x_3$

$$\frac{6}{5} + \cancel{t} = \cancel{t}$$

$$\frac{6}{5} = 0$$

Takva rješenja ne postoje.

5. U ovisnosti o parametru $\lambda \in \mathbb{R}$ riješite sustav:

$$\begin{cases} \lambda x_1 + x_2 + x_3 = 1 \\ x_1 + \lambda x_2 + x_3 = \lambda \\ x_1 + x_2 + \lambda x_3 = \lambda^2 \end{cases}.$$

$$\left[\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & \lambda \\ 1 & 1 & \lambda & \lambda^2 \end{array}\right] \begin{array}{l} + \\ |\cdot(-\lambda) \\ |\cdot(-1) \\ + \end{array} \sim \left[\begin{array}{ccc|c} 0 & 1-\lambda^2 & 1-\lambda & 1-\lambda^2 \\ 1 & \lambda & 1 & \lambda \\ 0 & 1-\lambda & \lambda-1 & \lambda^2-\lambda \end{array}\right]$$

Razlikujemo slučajeve:

1° $\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 1 - x_2 - x_3$$

Stavljanjem $x_2 = t$, $x_3 = s$, $t, s \in \mathbb{R}$, dobivamo dvoparametarsko rješenje:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-t-s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad t, s \in \mathbb{R}.$$

2° $\lambda \neq 1$

Nastavljamo s Gaussovim eliminacijama:

$$\left[\begin{array}{ccc|c} 0 & (1-\lambda)(1+\lambda) & 1-\lambda & (1-\lambda)(1+\lambda) \\ 1 & \lambda & 1 & \lambda \\ 0 & 1-\lambda & -(1-\lambda) & -\lambda(1-\lambda) \end{array}\right] \begin{array}{l} |:(1-\lambda) \neq 0 \\ \\ |:(1-\lambda) \neq 0 \end{array} \sim$$

$$\sim \left[\begin{array}{ccc|c} 0 & 1+\lambda & 1 & 1+\lambda \\ 1 & \lambda & 1 & \lambda \\ 0 & 1 & -1 & -\lambda \end{array}\right] \begin{array}{l} + \quad |\cdot(-1-\lambda) \\ + \\ |\cdot(-\lambda) \end{array} \sim \left[\begin{array}{ccc|c} 0 & 0 & 2+\lambda & (1+\lambda)^2 \\ 1 & 0 & 1+\lambda & \lambda(1+\lambda) \\ 0 & 1 & -1 & -\lambda \end{array}\right]$$

Ponovno razlikujemo slučajeve:

$2.1^\circ \quad \lambda = -2$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right] \rightsquigarrow 0 = 1$$

U ovom slučaju sustav nema rješenje.

(Formalno, rang matrice sustava je 2, a rang proširene matrice sustava je 3 pa je narušen uvjet Kronecker-Capellijevog teorema.)

$2.2^\circ \quad \lambda \neq -2$

$$\left[\begin{array}{ccc|c} 0 & 0 & 2+\lambda & (1+\lambda)^2 \\ 1 & 0 & 1+\lambda & \lambda(1+\lambda) \\ 0 & 1 & -1 & -\lambda \end{array}\right] \begin{array}{l} \Big| \cdot \left(-\frac{1+\lambda}{2+\lambda}\right) \\ + \\ + \quad \Big| \cdot \frac{1}{2+\lambda} \end{array} \sim$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 2+\lambda & (1+\lambda)^2 \\ 1 & 0 & 0 & -\frac{1+\lambda}{2+\lambda} \\ 0 & 1 & 0 & \frac{1}{2+\lambda} \end{array}\right] \begin{array}{l} \Rightarrow x_3 = \frac{(1+\lambda)^2}{2+\lambda} \\ \Rightarrow x_1 = -\frac{1+\lambda}{2+\lambda} \\ \Rightarrow x_2 = \frac{1}{2+\lambda} \end{array}$$

U ovom slučaju sustav ima jedinstveno rješenje

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1+\lambda}{2+\lambda} \\ \frac{1}{2+\lambda} \\ \frac{(1+\lambda)^2}{2+\lambda} \end{bmatrix}.$$

6. U ovisnosti o parametrima $a, b \in \mathbb{R}$ riješite sustav:

$$\begin{cases} x_1 \quad\quad\quad\quad - x_4 = 2 \\ x_2 - x_3 + x_4 = 3 \\ ax_3 = 1 \\ x_2 + bx_4 = 0 \end{cases}.$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & a & 0 & 1 \\ 0 & 1 & 0 & b & 0 \end{array}\right] \begin{array}{l} \\ | \cdot (-1) \\ + \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & a & 0 & 1 \\ 0 & 0 & 1 & b-1 & -3 \end{array}\right]$$

Razlikujemo slučajeve:

1° $a = 0$

Iz treće jednadžbe bi tada slijedilo $0 = 1$ pa u ovom slučaju sustav nema rješenja.

2° $a \neq 0$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & a & 0 & 1 \\ 0 & 0 & 1 & b-1 & -3 \end{array}\right] |: a \neq 0 \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 1 & 0 & \frac{1}{a} \\ 0 & 0 & 1 & b-1 & -3 \end{array}\right] \begin{array}{l} + \\ | \cdot 1 \\ | \cdot (-1) \\ + \end{array} \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & \frac{3a+1}{a} \\ 0 & 0 & 1 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & b-1 & -\frac{3a+1}{a} \end{array}\right]$$

Ponovno razlikujemo slučajeve:

2.1° $b = 1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & \frac{3a+1}{a} \\ 0 & 0 & 1 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 0 & -\frac{3a+1}{a} \end{array}\right]$$

2.1.1° $a \neq -\frac{1}{3}$

Iz posljednje jednadžbe bi tada slijedilo $0 = -\frac{3a+1}{a}$ pa ni u ovom slučaju sustav nema rješenja.

2.1.2° $a = -\frac{1}{3}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow x_1 = 2 + x_4 \\ \Rightarrow x_2 = -x_4 \\ \Rightarrow x_3 = -3 \\ \end{array}$$

Stavljanjem $x_4 = t$, $t \in \mathbb{R}$, u ovom slučaju dobivamo jednoparametarsko rješenje:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

2.2° $b \neq 1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & \frac{3a+1}{a} \\ 0 & 0 & 1 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & b-1 & -\frac{3a+1}{a} \end{array}\right] \begin{array}{l} \\ \\ \\ \mid : (b-1) \neq 0 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & \frac{3a+1}{a} \\ 0 & 0 & 1 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 1 & -\frac{3a+1}{a(b-1)} \end{array}\right] \begin{array}{l} + \\ + \\ \\ \mid \cdot 1, \ \mid \cdot (-1) \end{array} \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{2ab-5a-1}{a(b-1)} \\ 0 & 1 & 0 & 0 & \frac{(3a+1)b}{a(b-1)} \\ 0 & 0 & 1 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 1 & -\frac{3a+1}{a(b-1)} \end{array}\right]$$

U ovom slučaju sustav ima jedinstveno rješenje:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2ab-5a-1}{a(b-1)} \\ \frac{(3a+1)b}{a(b-1)} \\ \frac{1}{a} \\ -\frac{3a+1}{a(b-1)} \end{bmatrix}.$$