MATAN 2 - 2. vježbe

1. (a)

$$\lim_{(x,y)\to(5,1)} \frac{xy}{x+y} = \left[\begin{array}{l} \text{fja } (x,y)\mapsto \frac{xy}{x+y} \text{ je} \\ \text{neprekinuta u točki} \\ (5,1) \end{array}\right] = \frac{5\cdot 1}{5+1} = \frac{5}{6}$$

(b) Promatramo dvije restrikcije, tj. krivulje po kojima se „približavamo" točki $(0,0)$:

1° pravac $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2\cdot 0}{x^4+3\cdot 0} = 0$$

2° pravac $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{x^2\cdot x^2}{x^4+3x^4} = \lim_{x\to 0} \frac{x^4}{3x^4} = \frac{1}{4}$$

Budući da $0 \neq \frac{1}{4}$, zadani limes ne postoji.

(c)

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} = \left[\begin{array}{l} \text{polarne koordinate} \\ x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right] = \lim_{r\to 0} \frac{r^2\sin^2\varphi}{r^2\cos^2\varphi + r^2\sin^2\varphi}$$

$$= \lim_{r\to 0} \frac{\cancel{r^2}\sin^2\varphi}{\cancel{r^2}\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{=1}} = \lim_{r\to 0}\sin^2\varphi = \sin^2\varphi$$

Dobivena vrijednost ovisi o kutu φ (koji može biti proizvoljan) pa vidimo da zadani limes ne postoji.

2. način

Možemo promatrati i restrikcije na pravce oblika $y=kx$, $k\in\mathbb{R}$:

$$\lim_{(x,kx)\to(0,0)} \frac{k^2x^2}{x^2+k^2x^2} = \lim_{x\to 0} \frac{k^2\cancel{x^2}}{\cancel{x^2}(k^2+1)} = \frac{k^2}{k^2+1}$$

Dobivena vrijednost ovisi o k pa tako za različite odabire pravaca dobivamo različite vrijednosti „limesa". Dakle, zadani limes ne postoji.

2. Da bi f bila neprekinuta u $(0,0)$, mora vrijediti

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

Zato određujemo

$$\lim_{(x,y)\to(0,0)} f(x,y) = (\alpha+1) \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = \left[\begin{array}{l}\text{polarne}\\ \quad\text{koordinate}\end{array}\right]$$

$$= (\alpha+1) \lim_{r\to 0} \frac{r^2(\cos^2\varphi - \sin^2\varphi)}{\sqrt{r^2\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}}} = (\alpha+1)\lim_{r\to 0} r \underbrace{\cos(2\varphi)}_{\substack{\in[-1,1],\\ \text{tj. omeđeno}}} = 0$$

Dakle,

$$0 = f(0,0) = \beta - 3 \implies \beta = 3,$$

dok je $\alpha\in\mathbb{R}$ proizvoljan.

3. $g(x,y) = \operatorname{arctg}\left(\frac{y}{x}\right)$

$$\frac{\partial g}{\partial x}(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial g}{\partial y}(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 g}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = \frac{y}{(x^2+y^2)^2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 g}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = -\frac{x}{(x^2+y^2)^2} \cdot 2y = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 g}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Sada imamo

$$\frac{\partial^2 g}{\partial x^2}(x,y) + \frac{\partial^2 g}{\partial y^2}(x,y) = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0,$$

$$\left(\frac{\partial g}{\partial x}(x,y)\right)^2 - \left(\frac{\partial g}{\partial x}(x,y)\right)^2 = \frac{y^2}{(x^2+y^2)^2} - \frac{x^2}{(x^2+y^2)^2} = \frac{\partial^2 g}{\partial x \partial y}(x,y).$$

4. Vektor normale na zadanu plohu u točki (x_0, y_0, z_0) jest

$$\vec{n} = \left(\frac{\partial f}{\partial x}(x_0,y_0),\ \frac{\partial f}{\partial y}(x_0,y_0),\ -1\right),$$

gdje je $z = f(x,y) = -\frac{1}{2}x^2 + y^2 + \frac{1}{2}$. Dakle,

$$\vec{n} = (-x_0,\ 2y_0,\ -1).$$

Prema uvjetima zadatka, ovaj vektor mora biti kolinearan s vektorom normale zadane ravnine, $\vec{n}_\pi = (3,-2,1)$, tj. mora postojati $\lambda \in \mathbb{R}$ t.d.

$$\vec{n} = \lambda \vec{n}_\pi$$

$$\Leftrightarrow \begin{cases} -x_0 = 3\lambda & \Rightarrow x_0 = 3 \\ 2y_0 = -2\lambda & \Rightarrow y_0 = 1 \\ -1 = \lambda \end{cases} \Rightarrow z_0 = -\frac{1}{2}\cdot 9 + 1 + \frac{1}{2} = -3$$

Dakle, jedina takva točka je $(3, 1, -3)$.

5. Neka je (x_0, y_0, z_0) proizvoljna točka na zadanoj plohi.
Stavljanjem $z = f(x,y) = 3x^2 + y^2$, jednadžba tangencijalne ravnine na plohu u toj točki glasi

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)\cdot(y - y_0)$$

$$z - z_0 = 6x_0(x - x_0) + 2y_0(y - y_0)$$

Ukoliko pretpostavimo da točka $(x, y, z) = (1, 1, 0)$ leži u toj ravnini, uvrštavanjem u jednadžbu dobivamo

$$0 - z_0 = 6x_0(1 - x_0) + 2y_0(1 - y_0)$$

$$-z_0 = 6x_0 - 6x_0^2 + 2y_0 - 2y_0^2$$

$$-z_0 = 6x_0 + 2y_0 - 2\underbrace{(3x_0^2 + y_0^2)}_{= z_0 \text{ jer } (x_0, y_0, z_0) \text{ leži na plohi}}$$

$$-z_0 = 6x_0 + 2y_0 - 2z_0$$

$$\Rightarrow 6x_0 + 2y_0 - z_0 = 0$$

Dakle, sve tražene točke leže u ravnini

$$\pi \ldots\ 6x + 2y - z = 0.$$