MATAN 2 - 9. vježbe

1. (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{(n+1)\ln(n+1)}$

Cauchyjev kriterij:

$$\lim_{n\to\infty} \sqrt[n]{\left|(-1)^{n+1} \frac{(x-2)^n}{(n+1)\ln(n+1)}\right|} = \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt[n]{n+1}}_{\to 1} \cdot \underbrace{\sqrt[n]{\ln(n+1)}}_{\to 1}} |x-2|$$

$$= |x-2| < 1$$

$\Rightarrow$ radijus konvergencije reda je $R=1$ i red konvergira na $\langle 1, 3\rangle$

Rubovi:

- $x = 1$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{(n+1)\ln(n+1)} = -\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$$

Funkcija $f: [1, \infty\rangle \to [0, \infty\rangle$, $f(x) = \dfrac{1}{(x+1)\ln(x+1)}$ je neprekidna i padajuća te vrijedi:

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{(x+1)\ln(x+1)} = \begin{bmatrix} y = \ln(x+1) \\ dy = \dfrac{dx}{x+1} \end{bmatrix} = \int_{\ln 2}^{\infty} \frac{dy}{y}$$

$$= \ln y \Big|_{\ln 2}^{\infty} = \infty,$$

pa po integralnom kriteriju zadani red divergira.

• $x = 3$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1^n}{(n+1)\ln(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)\ln(n+1)}$$

Za niz $a_n := \dfrac{1}{(n+1)\ln(n+1)}$ vrijedi

1° $a_n > 0 \quad \forall n \in \mathbb{N}$

2° $n+1 < n+2 \Rightarrow (n+1)\ln(n+1) < (n+2)\ln(n+2)$

$$\Rightarrow \frac{1}{(n+1)\ln(n+1)} > \frac{1}{(n+2)\ln(n+2)}$$

$$\Rightarrow a_n > a_{n+1} \qquad \forall n \in \mathbb{N}$$

3° $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{(n+1)\ln(n+1)} = 0$

pa po Leibnizovom kriteriju zadani red konvergira.

$\Rightarrow$ područje konvergencije zadanog reda potencija je $I = \langle 1, 3]$

(b) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) x^n$

D'Alambertov kriterij:

$$\lim_{n\to\infty} \left| \frac{\sin\left(\frac{1}{n+1}\right) x^{n+1}}{\sin\left(\frac{1}{n}\right) x^n} \right| = \lim_{n\to\infty} \frac{\overbrace{\dfrac{\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}}}^{\to 1}}{\underbrace{\dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}}_{\to 1}} \cdot \overbrace{\frac{n}{n+1}}^{\to 1} |x| = |x| < 1$$

$\Rightarrow$ radijus konvergencije reda je $R = 1$ i red konvergira na $\langle -1, 1 \rangle$

Rubovi:

- $x = -1$

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$

Za niz $a_n := \sin\left(\frac{1}{n}\right)$ vrijedi

1° $0 < \frac{1}{n} < \frac{\pi}{2} \Rightarrow a_n = \sin\left(\frac{1}{n}\right) > 0 \quad \forall n \in \mathbb{N}$

2° $n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1} \Rightarrow a_n = \sin\left(\frac{1}{n}\right) > \sin\left(\frac{1}{n+1}\right) = a_{n+1}$

$\forall n \in \mathbb{N}$

3° $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\overset{\nearrow 0}{\frac{1}{n}}\right) = 0$

pa po Leibnizovom kriteriju zadani red konvergira.

- $x = 1$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

Budući da je

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \in \langle 0, \infty\rangle,$$

a red $\sum_{n=1}^{\infty} \frac{1}{n}$ divergira, prema usporednom kriteriju zadani red divergira.

$\Rightarrow$ područje konvergencije zadanog reda potencija je $I = [-1, 1\rangle$

(2.) $\displaystyle\sum_{n=1}^{\infty} (2\sqrt[n]{2}-1)^n (x^2+x)^n$

Cauchyjev kriterij:

$$\lim_{n\to\infty} \sqrt[n]{\left|(2\sqrt[n]{2}-1)^n (x^2+x)^n\right|} = \lim_{n\to\infty} (2\overset{\nearrow 1}{\sqrt[n]{2}}-1)|x^2+x| = |x^2+x| < 1$$

$\Rightarrow$ red konvergira za

$$|x^2+x| < 1$$

$$\Rightarrow \quad -1 < x^2+x < 1$$

$$\Rightarrow \begin{cases} x^2+x+1>0 \iff \left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0 \quad \checkmark \\ x^2+x-1<0 \Rightarrow x \in \left\langle \frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right\rangle \end{cases}$$

Na rubu, tj. za $x = \frac{-1\pm\sqrt{5}}{2}$, imamo $x^2+x=1$ te zadani red postaje

$$\sum_{n=1}^{\infty} (2\sqrt[n]{2}-1)^n \cdot 1^n = \sum_{n=1}^{\infty} (2\sqrt[n]{2}-1)^n.$$

No, za svaki $n \in \mathbb{N}$ imamo

$$(2\underbrace{\sqrt[n]{2}}_{\geq 1}-1)^n \geq 1^n = 1,$$

a budući da red $\displaystyle\sum_{n=1}^{\infty} 1$ divergira, prema usporednom kriteriju slijedi i da zadani red divergira.

Zato je područje konvergencije zadanog reda funkcija

$$I = \left\langle \frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right\rangle.$$

3. (a) $\sum_{n=1}^{\infty} \frac{n^2}{8^n} = ?$

Krenimo od formule za sumu geometrijskog reda:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \Big/ \frac{d}{dx}$$

$$\Rightarrow \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad \Big/ \cdot x$$

$$\Rightarrow \sum_{n=1}^{\infty} n x^{n} = \frac{x}{(1-x)^2} \quad \Big/ \frac{d}{dx}$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{1+x}{(1-x)^3} \quad \Big/ \cdot x$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^{n} = \frac{x(1+x)}{(1-x)^3}$$

Sve ove jednakosti vrijede (barem) za $|x| < 1$. Posebno, uvrštavanjem $x = \frac{1}{8}$ u posljednju nejednakost dobivamo

$$\sum_{n=1}^{\infty} \frac{n^2}{8^n} = \frac{\frac{1}{8}\left(1+\frac{1}{8}\right)}{\left(1-\frac{1}{8}\right)^3} = \frac{72}{343}$$

(b) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,5^{2n+1}} = ?$$

Ponovno krećemo od formule za sumu geometrijskog reda:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x|<1$$

$$\Leftrightarrow \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} \qquad |-x^2|<1 \Leftrightarrow |x|<1$$

$$\Leftrightarrow \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \qquad / \int dx$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \operatorname{arctg} x + C$$

Da bismo odredili konstantu C, uvrstimo $x=0$ (za koji gornja jednakost svakako mora vrijediti):

$$0 = \operatorname{arctg} 0 + C \Rightarrow C = 0.$$

Dakle,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \operatorname{arctg} x, \qquad |x|<1.$$

Posebno, za $x=\frac{1}{5}$ imamo

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,5^{2n+1}} = \operatorname{arctg}\frac{1}{5}.$$

4. (a) $f(x) = \frac{x-7}{9-x}$, $x_0 = 7$

$$f(x) = (x-7) \cdot \frac{1}{9-x} = (x-7) \cdot \frac{1}{2-(x-7)}$$

$$= (x-7) \cdot \frac{1}{2} \cdot \frac{1}{1-\frac{x-7}{2}} = (x-7) \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x-7}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (x-7)^{n+1}$$

(b) $f(x) = \frac{2x-4}{x^2-4x-5}$, $x_0 = 0$

Rastavljamo funkciju f na parcijalne razlomke:

$$\frac{2x-4}{x^2-4x-5} = \frac{2x-4}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$

$$\Rightarrow \quad 2x-4 = A(x+1) + B(x-5) = (A+B)x + (A-5B)$$

$$\Rightarrow \left\{ \begin{array}{l} A+B = 2 \\ A-5B = -4 \end{array} \right\} - \quad \Rightarrow \quad 6B = 6 \quad \Rightarrow \quad B = 1, \; A = 1$$

$$f(x) = \frac{1}{x-5} + \frac{1}{x+1} = -\frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}} + \frac{1}{1-(-x)}$$

$$= -\frac{1}{5} \sum_{n=0}^{\infty} \left(-\frac{x}{5}\right)^n + \sum_{n=0}^{\infty} (-x)^n$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^{n+1} x^n + \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=0}^{\infty} \left((-1)^n + \left(-\frac{1}{5}\right)^{n+1} \right) x^n$$

(c) $f(x)=\frac{2x-4}{x^2-4x-5}, \quad x_0=2$

$$f(x)=\frac{2(x-2)}{(x^2-4x+4)-9}=\frac{2(x-2)}{(x-2)^2-9}=-\frac{2}{9}(x-2)\cdot\frac{1}{1-\left(\frac{x-2}{3}\right)^2}$$

$$=-\frac{2}{9}(x-2)\sum_{n=0}^{\infty}\left(\frac{x-2}{3}\right)^{2n}=-\frac{2}{9}(x-2)\sum_{n=0}^{\infty}\frac{1}{9^n}(x-2)^{2n}$$

$$=\sum_{n=0}^{\infty}-\frac{2}{9^{n+1}}(x-2)^{2n+1}$$

(d) $f(x)=\sin x, \quad x_0=\pi$

$$f(x)=\sin x=-\sin(x-\pi)=\left[\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},\ x\in\mathbb{R}\right]$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x-\pi)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$$

(e) $f(x)=\sqrt{1+x}, \quad x_0=4$

$$f(x)=\sqrt{1+x}=\sqrt{5+(x-4)}=\sqrt{5}\cdot\sqrt{1+\frac{x-4}{5}}=\left[(1+x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n,\ |x|<1\right]$$

$$=\sqrt{5}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(\frac{x-4}{5}\right)^n=\sqrt{5}+\frac{\sqrt{5}}{10}(x-4)+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{2^n}\cdot\frac{(2n-3)!!}{n!}\cdot\frac{1}{5^{n-\frac{1}{2}}}(x-4)^n$$

(f) $f(x)=\ln(1+x), \quad x_0=3$

$$f(x)=\ln(1+x)=\ln(4+(x-3))=\ln 4+\ln\left(1+\frac{x-3}{4}\right)$$

$$=\left[\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n,\ |x|<1\right]=\ln 4+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\cdot 4^n}(x-3)^n$$