MATAN 2 - 10. vježbe

1. $xyy'^2 - (x^2+y^2)y' + xy = 0 \quad \Big| : x^2$

$$\frac{y}{x} \cdot y'^2 - \left(1 + \left(\frac{y}{x}\right)^2\right) y' + \frac{y}{x} = 0$$

Supstitucija: $z = \dfrac{y}{x}$

$$\Rightarrow y = zx \quad \Big/ \frac{d}{dx}$$

$$\Rightarrow y' = z'x + z$$

Jednadžba postaje:

$$z(z'x+z)^2 - (1+z^2)(z'x+z) + z = 0$$

$$z\left((z'x)^2 + 2zz'x + z^2\right) - \left(z'x + z + z^2z'x + z^3\right) + z = 0$$

$$zz'^2x^2 + \cancel{2}z^2z'x + \cancel{z^3} - z'x - \cancel{z} - \cancel{z^2z'x} - \cancel{z^3} + \cancel{z} = 0$$

$$zz'^2x^2 + z^2z'x - z'x = 0 \quad \Big| : x$$

$$zz'^2x + z^2z' - z' = 0$$

$$z'(zz'x + z^2 - 1) = 0$$

$z' = 0$

$z = C \qquad C \in \mathbb{R}$

$$\boxed{y = Cx, \quad C \in \mathbb{R}}$$

$zz'x = 1 - z^2$

$$\frac{z\,dz}{1-z^2} = \frac{dx}{x} \quad \Big/ \int$$

$$-\frac{1}{2}\ln|1-z^2| = \ln|x| + \ln C \qquad C>0$$

$$\ln|1-z^2| = \ln\frac{1}{Cx^2} \qquad C>0$$

$$1 - z^2 = \frac{1}{Cx^2} \qquad C \neq 0$$

$$1 - \frac{y^2}{x^2} = \frac{C}{x^2} \qquad C \neq 0$$

$$\boxed{x^2 - y^2 = C, \quad C \neq 0}$$

2.

$$y' = \frac{y+2}{x+1} + tg\,\frac{y-2x}{x+1}$$

$$y' = \frac{y+2}{x+1} + tg\left(\frac{y+2}{x+1} - 2\right)$$

Supstitucija: $z = \dfrac{y+2}{x+1}$

$$\Rightarrow y+2 = z(x+1) \quad \Big/ \frac{d}{dx}$$

$$\Rightarrow y' = z'(x+1) + z$$

Jednadžba postaje:

$$z'(x+1) + \cancel{z} = \cancel{z} + tg(z-2)$$

$$z'(x+1) = tg(z-2)$$

$$\frac{dz}{tg(z-2)} = \frac{dx}{x+1} \quad \Big/ \int$$

$$\int \frac{\cos(z-2)}{\sin(z-2)}\,dz = \int \frac{dx}{x+1}$$

$$\ln|\sin(z-2)| = \ln|x+1| + \ln C \qquad C>0$$

$$|\sin(z-2)| = C|x+1| \qquad C>0$$

$$\sin(z-2) = C(x+1) \qquad C \neq 0$$

$$\boxed{\sin\frac{y-2x}{x+1} = C(x+1), \quad C \neq 0}$$

U slučaju $tg(z-2) = 0$

$$\Rightarrow \sin(z-2) = 0$$

$$\Rightarrow z-2 = k\pi \qquad k \in \mathbb{Z}$$

$$\Rightarrow \frac{y+2}{x+1} - 2 = k\pi \qquad k \in \mathbb{Z}$$

$$\Rightarrow \boxed{y = (k\pi+2)(x+1) - 2, \ k \in \mathbb{Z}}$$

Uvrštavanjem u početnu jednadžbu vidimo da ovo jesu rješenja.

③ $3xy^2y' + y^3 = x^2 \quad \Big| : 3xy^2$

$y' + \frac{1}{3x} \cdot y = \frac{1}{3} x y^{-2}$ $\rightsquigarrow$ Bernoullijeva jednadžba $\quad \alpha = -2$

Supstitucija: $z = y^{1-\alpha} = y^3$

$\Rightarrow z' = 3y^2y'$

Jednadžba postaje:

$$x \cdot \underbrace{3y^2y'}_{z'} + \underbrace{y^3}_{z} = x^2$$

$$x \cdot z' + z = x^2$$ $\rightsquigarrow$ linearna ODJ 1. reda

1° homogena jednadžba

$x z' + x = 0$

$\frac{dz}{z} = -\frac{dx}{x} \quad \Big/ \int$

Znamo da $z = 0$ ($\Leftrightarrow y = 0$) nije rješenje dobivene nehomogene linearne jednadžbe pa smijemo dijeliti sa z.

$\ln|z| = -\ln|x| + \ln C \qquad C > 0$

$|z| = \frac{C}{|x|} \qquad C > 0$

$z = \frac{C}{x} \qquad C \neq 0$

2° varijacija konstanti

$z(x) = \frac{C(x)}{x} \Rightarrow z'(x) = \frac{C'(x)x - C(x) \cdot 1}{x^2}$

$\Rightarrow x \cdot z' + z = \frac{C'x - \cancel{C}}{x} + \frac{\cancel{C}}{\cancel{x}} = C'$

$\Rightarrow C' = x^2 \quad \Big/ \int dx$

$$\Rightarrow C(x) = \frac{1}{3}x^3 + D \qquad D \in \mathbb{R}$$

$$\Rightarrow z = \frac{\frac{1}{3}x^3 + D}{x} = \frac{1}{3}x^2 + \frac{D}{x} \qquad D \in \mathbb{R}$$

$$\boxed{y^3 = \frac{1}{3}x^2 + \frac{D}{x}, \quad D \in \mathbb{R}}$$

4. $y' - 2xyy' = y^2 - y$

Ova jednadžba nije linearna po y, ali jest po x - zato ćemo x shvatiti kao funkciju od y $(x = x(y))$:

$$(1 - 2xy)\,y' = y^2 - y$$

$$\Rightarrow (1-2xy)\frac{dy}{dx} = y^2 - y$$

$$\Rightarrow 1 - 2xy = (y^2 - y)\frac{dx}{dy}$$

(koristimo teorem o derivaciji inverzne funkcije)

$$\Rightarrow (y^2 - y)x' + 2yx = 1$$

1° homogena jednadžba

$$(y^2 - y)x' + 2yx = 0$$

$$\frac{dx}{x} = -\frac{2}{y-1}dy \quad \Big/ \int$$

Uočimo da $x = 0$ nije rješenje dobivene nehomogene linearne jednadžbe.

$$\ln|x| = -2\ln|y-1| + \ln C \qquad C > 0$$

$$|x| = \frac{C}{|y-1|^2} \qquad C > 0$$

$$x = \frac{C}{(y-1)^2} \qquad C \neq 0$$

2° varijacija konstanti

$$x(y) = \frac{C(y)}{(y-1)^2} \Rightarrow x'(y) = \frac{C'(y)(y-1)^2 - C(y)\cdot 2(y-1)}{(y-1)^4}$$

$$\Rightarrow (y^2-y)x' + 2yx = \frac{C'\cdot y(y-1)^2 - C\cdot 2y(y-1)}{(y-1)^3} + \frac{C\cdot 2y}{(y-1)^2}$$

$$= \frac{C'\cdot y(y-1)^2 - C\cdot \cancel{2y(y-1)} + C\cdot \cancel{2y(y-1)}}{(y-1)^3}$$

$$= \frac{y}{y-1} C'$$

$$\Rightarrow \frac{y}{y-1} C' = 1$$

$$\Rightarrow C' = \frac{y-1}{y} = 1 - \frac{1}{y} \quad / \int dy$$

$$\Rightarrow C(y) = y - \ln|y| - \ln D \qquad D > 0$$

$$\Rightarrow C(y) = y - \ln(Dy) \qquad D \neq 0$$

$$\Rightarrow \boxed{x = \frac{y - \ln(Dy)}{(y-1)^2}, \ D \neq 0}$$

5. Neka je $y=y(x)$ tražena krivulja i (x_0,y_0) neka njena proizvoljna točka. Jednadžba normale na krivulju u toj točki glasi

$$n \ldots \quad y-y_0 = -\frac{1}{y'(x_0)}(x-x_0)$$

Prema uvjetu zadatka mora biti $(1,2)\in n$ pa imamo

$$2-y_0 = -\frac{1}{y'(x_0)}(1-x_0).$$

Budući da je (x_0,y_0) bila proizvoljna točka te krivulje, dobivamo diferencijalnu jednadžbu

$$2-y = -\frac{1}{y'}(1-x)$$

$$(2-y)\,dy = -(1-x)\,dx \qquad /\int$$

$$2y-\frac{1}{2}y^2 = -x+\frac{1}{2}x^2+C \qquad C\in\mathbb{R}$$

Nadalje, prema uvjetu zadatka imamo početni uvjet $y(0)=0$ iz kojeg određujemo konstantu C:

$$2\cdot 0-\frac{1}{2}\cdot 0 = -0+\frac{1}{2}\cdot 0+C \Rightarrow C=0$$

Dakle, jednadžba tražene krivulje je

$$2y-\frac{1}{2}y^2 = -x+\frac{1}{2}x^2$$

$$x^2-2x+y^2-4y=0$$

$$\boxed{(x-1)^2+(y-2)^2=5} \rightsquigarrow \text{kružnica } k((1,2),\sqrt{5})$$

(normala u svakoj točki kružnice prolazi njenim središtem)