Title

Montiel Abello

1.2 Theoretical Background

1.2.1 Rigid Body Dynamics

In practice, robot, sensor, environment exist in 3D Euclidean space - $\mathbb{R}^3$. To model the pose of a rigid body... (something about why Lie groups are necessary to represent pose)

1.2.1.1 Lie Groups

A Lie group $\mathbf{G}$ is a group that is also a differentiable manifold. As a group, $\mathbf{G}$ is a set of elements and a group operation (denoted by multiplication, i.e. AB for $A, B \in \mathbf{G}$) that satisfies the 4 group axioms:

- **Closure:** The group operation $\mathbf{G} \times \mathbf{G} \mapsto \mathbf{G}$ is a function that maps elements of $\mathbf{G}$ onto itself; $\forall A, B \in \mathbf{G}$, $AB \in \mathbf{G}$.
- **Associativity:** Elements of G are associative under the group operation; $\forall A, B, C \in \mathbf{G}$, $(AB)C = A(BC)$.
- **Identity:** There exists an identity element $I \in \mathbf{G}$ such that $\forall A \in \mathbf{G}$, $IA = AI = A$.
- **Inverse:** For all $A \in \mathbf{G}$ there exists an inverse element $A^{-1} \in \mathbf{G}$ such that $AA^{-1} = A^{-1}A = I$.

Because the Lie group $\mathbf{G}$ is a differentiable manifold, it is locally Euclidean. This means that the neighbourhood around every element of $\mathbf{G}$ can be approximated with a tangent plane. This property allows calculus to be performed on elements of $\mathbf{G}$.

Matrix Lie groups
A matrix Lie group is made up of group elements which are $n \times n$ matrices. This work will be focus on matrix Lie groups because the exponential map and Lie bracket functions given below only apply to such Lie groups.

Lie algebra
The tangent space at the identity element of a Lie group is called the Lie algebra $\mathfrak{g}$. It is called the Lie *algebra* because it has a binary operation, known as the Lie bracket $[X, Y]$. For matrix Lie groups the Lie bracket is

$$[A, B] \triangleq AB - BA \tag{1.1}$$

The exponential map and logarithm map
The mapping from the Lie algebra $\mathfrak{g}$ to the Lie group $\mathbf{G}$ is called the expo-

nential map:

$$\exp : \mathfrak{g} \to \mathbf{G} \tag{1.2}$$

Similarly, the logarithm map maps elements from $\mathbf{G}$ to $\mathfrak{g}$:

$$\log : \mathbf{G} \to \mathfrak{g} \tag{1.3}$$

sub-heading?
For an n-dimensional matrix Lie group, the Lie algebra $\mathfrak{g}$ is a vector space isomorphic to $\mathbb{R}^n$. The hat operator $\hat{}$ maps vectors $x \in \mathbb{R}^3$ to elements of $\mathfrak{g}$.

$$\hat{} : x \in \mathbb{R}^n \to \hat{x} \in \mathfrak{g} \tag{1.4}$$

For a matrix Lie group $\mathbf{G}$ whose elements are $n \times n$ matrices, the elements of $\mathfrak{g}$ will also be $n \times n$ matrices. The hat operator is defined

$$\hat{x} = \sum_{i=1}^{n} x_i G^i \tag{1.5}$$

where G^i are $n \times n$ matrices known as the infinitesimal generators of $\mathbf{G}$.

Lie bracket and group operation
For Lie groups endowned with the commutative property $(\forall A, B\mathbf{G}, AB = BA)$, vector addition in the Lie algebra maps to a group operation in the Lie group. For $C = A + B$ where $A, B, C \in \mathfrak{g}$,

$$e^C = e^{A+B} = e^A e^B \tag{1.6}$$

For non-commutative Lie groups, the relationship between the Lie bracket and group operation does not hold. Instead, for $C = \log e^A e^B$, C is calculated with the Baker-Campbell-Hausdorff formula:

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A - B, [A, B]] + \frac{1}{24}[B, [A, [A, B]]] + \dots \tag{1.7}$$

Actions
When a group action for a Lie group G acting on a manifold M is a differentiable map, this is known as a Lie group action. For example, 3D rotations act on 3D points so the Lie group $\mathbf{SO}(3)$ acts on $\mathbb{R}^3$. A left action of $\mathbf{G}$ on M is defined as a differentiable map

$$\Phi : \mathbf{G} \times M \mapsto M \tag{1.8}$$

where

- the identity element I maps M onto itself *(is that the right wording?)

$$\Phi(I, m) = m, \forall m \in M \tag{1.9}$$

- Group actions compose according to

$$\Phi(m, \Phi(n, o)) = \Phi(mn, o) \tag{1.10}$$

Adjoint map
EXPLANATION - Sometimes before a function B can be applied on manifold acted on by a group action A, it is necessary to apply the conjugate of A to B????
For $A \in \mathbf{G}$ define a function Ψ, known as the adjoint map of $\mathbf{G}$:

$$\Psi_A : \mathbf{G} \to \mathbf{G},\ \Psi_A(B) \triangleq ABA^{-1} \tag{1.11}$$

Taking the derivative:

$$\frac{\partial}{\partial t}\Psi_A(B(t))|_{t=0} = AVA^{-1},\ V \triangleq \frac{\partial}{\partial t}B(t)|_{t=0} \tag{1.12}$$

The adjoint representation of $\mathbf{G}$ is given by the mapping

$$\mathbf{Adj}_A : \mathfrak{g} \to \mathfrak{g},\ \mathbf{Adj}_A(V) \triangleq AVA^{-1} \tag{1.13}$$

1.2.1.2 SO(3)

A rotation represents the motion of a point about the origin of a Euclidean space. In $\mathbb{R}^3$ this is a proper isometry: a transformation that preserves distances between any pair of points and has a determinant of $+1$. The set of all rotations about the origin of $\mathbb{R}^3$ is known as the *special orthogonal group* $\mathbf{SO}(3)$. This matrix Lie group is a subgroup of the general linear group $\mathbf{GL}(3)$, so its group elements are 3×3 invertible matrices. These group elements are orthogonal matrices so their columns and rows are orthogonal unit vectors.

Lie algebra
The Lie algebra $\mathfrak{so}(3)$ is vector space of 3×3 skew-symmetric matrices $\hat{\omega}$, where ω is a 3-vector representing an angular velocity. The direction of ω indicates the axis of rotation while its magnitude gives the angular velocity. Elements of $\mathfrak{so}(3)$ are mapped to $\mathbf{SO}(3)$ according to the exponential map:

$$\begin{aligned} \exp : \mathfrak{so}(3) &\to \mathbf{SO}(3) \\ [\omega]_\times &\to \mathbf{R}_{3x3} \end{aligned} \tag{1.14}$$

i.e. $\forall \omega \in \mathfrak{so}(3),\ \exp([\omega]_\times) \in \mathbf{SO}(3)$

Conversely, the logarithm map maps 3×3 rotation matrices of $\mathbf{SO}(3)$ to elements of $\mathfrak{so}(3)$:

$$\begin{aligned} \log : \mathbf{SO}(3) &\to \mathfrak{so}(3) \\ \mathbf{R}_{3x3} &\to [\omega]_\times \end{aligned} \tag{1.15}$$

i.e. $\forall \mathbf{R} \in \mathbf{SO}(3),\ \log(\mathbf{R}) \in \mathfrak{so}(3)$

Actions
By the group action, elements of $\mathbf{SO}(3)$ rotate points in $\mathbb{R}^3$ about the origin.

$$\Phi : \mathbf{SO}(3) \times \mathbf{R}^3 \mapsto \mathbf{R}^3 \tag{1.16}$$

Adjoint map

$$\Psi_R : \mathbf{SO}(3) \to \mathbf{SO}(3),\ \Psi_R(A) \triangleq RAR^{-1} \tag{1.17}$$

Taking the derivative:

$$\frac{\partial}{\partial t}\Psi_R(A(t))|_{t=0} = RBR^{-1},\ B \triangleq \frac{\partial}{\partial t}A(t)|_{t=0} \tag{1.18}$$

The adjoint representation of $\mathbf{SO}(3)$ is given by the mapping

$$\mathbf{Adj}_R : \mathfrak{so}(3) \to \mathfrak{so}(3),\ \mathbf{Adj}_R(B) \triangleq RBR^{-1} \tag{1.19}$$

???? Hard to explain practical application without discussing reference frames: ie if position defined in body fixed frame but some other transformation defined in inertial frame. First undo rotation to get pose in inertial frame, apply transformation, then re-apply rotation

TODO: more explanation needed here!

Rotation representation
There are many conventions by which elements of $\mathbf{SO}(3)$ can be represented. Those A rotation about a point in $\mathbb{R}^3$ can be represented by: **TODO:** *go into more detail on below*

Rotation matrix
3×3 matrix where magnitude of each column is 1, columns are orthogonal, determinant is $+1$. Group operation matrix is multiplication. Left action is left multiplication of point.

Scaled axis
3-vector where direction represents axis of rotation and magnitude represents angle of rotation. Group operation - vector addition? Left action - rodrigues' rotation formula converts scaled axis to rotation matrix which is used to rotate point.

Quaternion
4-vector, same information as axis angle, but different form. Given axis of rotation $\mathbf{r}$ and angle of rotation θ:

$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathbf{r} \end{bmatrix} \tag{1.20}$$

Group operation - quaternion multiplication defined as:

$$\mathbf{q}_1\mathbf{q}_2 = \begin{bmatrix} w_1 \\ \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix} \tag{1.21}$$

As with rotation matrices, quaternion multiplication is associative but not commutative.

1.2.1.3 SE(3)

The special Euclidean group $\mathbf{SE}(3)$ represents rigid transformation in $\mathbb{R}^3$. This is a matrix Lie group whose elements are (or should it be "can be represented by"?) 4×4 matrices of the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{1.22}$$

where $\mathbf{R} \in \mathbf{SO}(3)$ and $\mathbf{p} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^\top \in \mathbb{R}^3$.

$\mathbf{SE}(3)$ is a semidirect product of $\mathbf{SO}(3)$ and $\mathbb{R}^3$. As its group elements contain a rotation matrix and translation vector, $\mathbf{SE}(3)$ has 6 degrees of freedom and is a 6-dimensional manifold.

Lie algebra
The Lie algebra $\mathfrak{se}(3)$ is a vector space whose elements are 4×4 matrices of the form

$$\begin{bmatrix} [\omega]_\times & \mathbf{v} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \tag{1.23}$$

where $\omega = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^\top \in \mathbf{so}(3)$, representing an angular velocity in scaled axis representation, and $\mathbf{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^\top \in T_\mathbf{p}\mathbb{R}^3$, representing a linear velocity vector.

Elements of $\mathfrak{se}(3)$ are mapped to $\mathbf{SE}(3)$ according to the exponential map:

$$\begin{gathered} \exp : \mathfrak{se}(3) \to \mathbf{SE}(3) \\ \begin{bmatrix} [\omega]_\times & \mathbf{v} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \to \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \end{gathered} \tag{1.24}$$

i.e. $\forall \mathbf{T} \in \mathfrak{se}(3),\ \exp(\mathbf{T}) \in \mathbf{SE}(3)$

Conversely, the logarithm map maps elements of $\mathbf{SE}(3)$ to elements of $\mathfrak{se}(3)$:

$$\begin{gathered} \log : \mathbf{SE}(3) \to \mathfrak{se}(3) \\ \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \to \begin{bmatrix} [\omega]_\times & \mathbf{v} \\ \mathbf{0}_{1\times3} & 0 \end{bmatrix} \end{gathered} \tag{1.25}$$

i.e. $\forall \mathbf{S} \in \mathbf{SE}(3),\ \log(\mathbf{S}) \in \mathfrak{se}(3)$

Actions

$\mathbf{SE}(3)$ group elements acts to perform a rigid transformation on points in $\mathbb{R}^3$. This corresponds to a rotation about the origin and a translation. To apply a transformation using the 4×4 matrix elements of $\mathbf{SE}(3)$ to a point $\mathbf{p} = (x, y, z)$ in $\mathbb{R}^3$, the point must be represented with homogeneous coordinates: (is p' okay for homogeneous points? $\hat{}$ is already used for skew-symmetric matrix)

$$\mathbf{p}' = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{1.26}$$

The left group action of $\mathbf{SE}(3)$ is now simply a left matrix multiplication of $\mathbf{p}$:

$$\mathbf{p'}_1 = \mathbf{S}\mathbf{p'}_0 = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}\mathbf{p}_0 + \mathbf{p} \\ 1 \end{bmatrix} \tag{1.27}$$

Adjoint Map

The adjoint map of $\mathbf{SE}(3)$ is

$$\Psi_S : \mathbf{SE}(3) \to \mathbf{SE}(3),\ \Psi_S(A) \triangleq SAS^{-1} \tag{1.28}$$

Taking the derivative:

$$\frac{\partial}{\partial t}\Psi_S(A(t))|_{t=0} = SBS^{-1},\ B \triangleq \frac{\partial}{\partial t}A(t)|_{t=0} \tag{1.29}$$

The adjoint representation of $\mathbf{SE}(3)$ is given by the mapping

$$\mathbf{Adj}_S : \mathfrak{se}(3) \to \mathfrak{se}(3),\ \mathbf{Adj}_S(B) \triangleq SBS^{-1} \tag{1.30}$$

TODO: more explanation needed here!

1.2.1.4 Reference Frames

A reference frame is a system of coordinates that is used to uniquely identify points on a manifold. This report will deal with reference frames on $\mathbb{R}^3$, that are used both to define the position of a point and the pose of a rigid body in 3D space. Such a reference frame is represented by an element of $\mathbf{SE}(3)$.

Consider three different reference frames, denoted {A},{B} and {C}. The notation ${}^{A}_{B}\mathbf{X}_{C}$ defines the difference in $\mathbf{X}$ of the reference frame {C} with respect to the frame {B}, defined in the frame {A}.

For example, ${}^{A}_{B}\mathbf{R}_{C}$ defines the rotation of {C} with respect to {B}, defined in {A}.

The notion of an inertial reference frame is introduced here. This will be defined as a reference frame that is stationary for the purpose of the problem being described.

Pose:
The pose of a rigid body in a given reference frame is defined by its relative position and orientation with respect to the given reference frame and is represented by an element of $\mathbf{SE}(3)$. If a rigid body has orientation aligned with a reference frame {C} and position at the origin of {C}, then the pose of the rigid body with respect to {B} and defined in {A} is:

$$ {}^{A}_{B}\mathbf{S}_{C} = \begin{bmatrix} {}^{A}_{B}\mathbf{R}_{C} & {}^{A}_{B}\mathbf{p}_{C} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{1.31} $$

TODO: DIAGRAM HERE

Point:
To represent the position of a point in $\mathbb{R}^3$, the convention used will be to first define a reference frame with origin located as this point. The position of a point at the origin of {C} with respect to another point {B}, defined in terms of the reference frame {A} is a 4-vector in homogeneous coordinates:

$$ {}^{A}_{B}\mathbf{p'}_{C} = \begin{bmatrix} {}^{A}_{B}\mathbf{p}_{C} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}x_{C} \\ {}^{A}_{B}y_{C} \\ {}^{A}_{B}z_{C} \\ 1 \end{bmatrix} \tag{1.32} $$

TODO: DIAGRAM HERE

Defining a point in terms of another reference frame:
Consider a point in $\mathbb{R}^3$ defined as the position of the frame {B} with respect to the frame {A}, defined in terms of the frame {B}. To redefine the point in terms of {A}, the left action of ${}^{A}_{A}\mathbf{S}_{B} \in \mathbf{SE}(3)$ is used:

$$ {}^{A}_{A}\mathbf{p'}_{B} = {}^{A}_{A}\mathbf{S}_{B}\,{}^{B}_{A}\mathbf{p'}_{B} \tag{1.33} $$

Defining a pose in terms of another reference frame:
To define a pose transformation matrix in terms of a different reference frame, a matrix conjugation is used:

$$ {}^{B}_{C}\mathbf{X}_{D} = ({}^{B}_{B}\mathbf{X}_{A}){}^{A}_{C}\mathbf{X}_{D}({}^{B}_{B}\mathbf{X}_{A})^{-1} \tag{1.34} $$

Inverse:
Taking the inverse of a pose transformation matrix has the effect of reversing the transformation, but does not alter the frame that the transformation is defined in terms of.

$$ ({}^{A}_{B}\mathbf{X}_{C})^{-1} = {}^{A}_{C}\mathbf{X}_{B} \tag{1.35} $$

1.2.1.5 Rigid Body State Representation

The state of a rigid body moving through 3D space can be represented by its linear and angular position, velocity and acceleration. Higher derivatives could be taken but will be ignored for simplicity. The inertial frame is denoted {A} and a frame {B} is fixed to the pose of the moving body.

The pose of the body with respect to the inertial frame at time t, defined in the inertial frame is represented by the screw matrix ${}^{A}_{A}\mathbf{S}_{B}(t) \in \mathbf{SE}(3)$,

$$ {}^{A}_{A}\mathbf{S}_{B}(t) = \begin{bmatrix} {}^{A}_{A}\mathbf{R}_{B}(t) & {}^{A}_{A}\mathbf{p}_{B}(t) \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{1.36} $$

where ${}^{A}_{A}\mathbf{R}_{B}(t) \in \mathbf{SO}(3)$ is a rotation matrix, and the position ${}^{A}_{A}\mathbf{p}_{B}(t) \in \mathbb{R}^3$.

The linear and angular velocity of the body at time t with respect to the inertial frame, defined in the body-fixed frame, is represented by the twist matrix ${}^{B}_{A}\mathbf{T}_{B}(t) \in \mathfrak{se}(3)$,

$$ {}^{B}_{A}\mathbf{T}_{B}(t) = \begin{bmatrix} [{}^{B}_{A}\omega_{B}(t)]_{\times} & {}^{B}_{A}\mathbf{v}_{B}(t) \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \tag{1.37} $$

where ${}^{B}_{A}\omega_{B}(t) \in \mathfrak{so}(3)$ is an angular velocity in the scaled-axis representation, and the linear velocity is ${}^{B}_{A}\mathbf{v}_{B}(t) \in T_{\mathbf{p}}\mathbb{R}^3$.

The linear and angular acceleration of the body at time t with respect to the inertial frame, defined in the body-fixed frame, is represented by the wrench matrix,

$$ {}^{B}_{A}\mathbf{W}_{B}(t) = \begin{bmatrix} [{}^{B}_{A}\alpha_{B}(t)]_{\times} & {}^{B}_{A}\mathbf{a}_{B}(t) \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \tag{1.38} $$

where ${}^{B}_{A}\alpha_{B}(t) \in ???$ is an angular acceleration in the scaled-axis representation, and the linear acceleration is ${}^{B}_{A}\mathbf{a}_{B}(t) \in T_{\mathbf{v}}\mathbb{R}^3$.

1.2.1.6 Rigid Body Dynamic Model

The dynamics of the screw, twist and wrench matrices as they are defined in 1.2.1.5 is governed by the following ODEs,

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{A}_{A}\mathbf{S}_{B}(t) = {}^{A}_{A}\mathbf{S}_{B}(t){}^{B}_{A}\mathbf{T}_{B}(t) \tag{1.39}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{B}_{A}\mathbf{T}_{B}(t) = {}^{B}_{A}\mathbf{W}_{B}(t) \tag{1.40}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{B}_{A}\mathbf{W}_{B}(t) = \mathbf{f}(t) \tag{1.41}$$

where the function $\mathbf{f}(t)$ is known.

1.2.1.7 Scanning Laser Rangefinder Dynamic Model

A scanning laser rangefinder fixed to a moving rigid body. State is same as moving rigid body defined above (S,T,W)
+
Unit vector defined in the body fixed frame - ${}^{B}_{B}\mathbf{n}_{?}(t)??? \in T_{\mathbf{p}}\mathbb{R}^{3}$.
+
Range $r(t) \in \mathbb{R}^{0+}$, defining range from ${}^{A}_{A}\mathbf{p}_{B}(t)$ to nearest object in environment in direction ${}^{A}_{A}\mathbf{R}_{B}(t)\mathbf{n}$?

1.2.2 Symmetry Preserving Observers

1.2.2.1 definitions?

1.2.2.2 construction, ie moving frame method etc

1.2.3 Infinite Dimensional Observers

1.2.4 Discretisation Methods?

Bibliography

[1] C. Harkort and J. Deutscher, "Finite-dimensional observer-based control of linear distributed parameter systems using cascaded output observers," *International Journal of Control*, vol. 84, no. 1, pp. 107–122, 2011.

[2] L. Meirovitch and H. Baruh, "On the problem of observation spillover in self-adjoint distributed-parameter systems," *Journal of Optimization Theory and Applications*, vol. 39, no. 2, pp. 269–291, 1983.

[3] G. Haine, "Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint generator," *Mathematics of Control, Signals, and Systems*, vol. 26, no. 3, pp. 435–462, 2014.

[4] J. W. Helton, "Systems with infinite-dimensional state space: the hilbert space approach," *Proceedings of the IEEE*, vol. 64, no. 1, pp. 145–160, 1976.

[5] K. Ramdani, M. Tucsnak, and G. Weiss, "Recovering the initial state of an infinite-dimensional system using observers," *Automatica*, vol. 46, no. 10, pp. 1616–1625, 2010.

[6] C.-Z. Xu, P. Ligarius, and J.-P. Gauthier, "An observer for infinite-dimensional dissipative bilinear systems," *Computers & Mathematics with Applications*, vol. 29, no. 7, pp. 13–21, 1995.

[7] H. Bounit and H. Hammouri, "Observers for infinite dimensional bilin ear, systems," *European journal of control*, vol. 3, no. 4, pp. 325–339, 1997.