

Title

Montiel Abello

Chapter 1

Introduction

1.1 Literature Review

1.1.1 infinite-dimensional observers

linear:

- observer theory for linear infinite-dimensional systems widely studied
- techniques used typically extensions of luenberger observers used for finite-dimensional systems.
- simplified approach: use spatial discretisation such as finite difference/finite element to reduce infinite-dimensional to finite-dimensional observer. [1, 2]
- better to design infinite-dimensional observer and only discretise for numerical implementation. [3, 4, 5] **TODO:** *describe these design methods.*

nonlinear:

- no universal approach for observer design for infinite-dimensional nonlinear systems
- some methods for special case - infinite dimensional bilinear systems. [6, 7]
- for finite-dimensional nonlinear systems, common design methods are: linearisation (ie EKF), lyapunov method, sliding mode, high gain

1.1.2 symmetry preserving observers

1.1.2.1 early work

1.1.2.2 bonnabel et al

1.1.2.3 trumpf, mahony et al

1.1.2.4 juan's work - in detail

1.2 Theoretical Background

1.2.1 Rigid Body Dynamics

In practice, robot, sensor, environment exist in 3D Euclidean space - \mathbb{R}^3 . To model the pose of a rigid body... (something about why Lie groups are necessary to represent pose)

1.2.1.1 Lie Groups

A Lie group \mathbf{G} is a group that is also a differentiable manifold. As a group, \mathbf{G} is a set of elements and a group operation (denoted by multiplication, i.e. AB for $A, B \in \mathbf{G}$) that satisfies the 4 group axioms:

- **Closure:** The group operation $\mathbf{G} \times \mathbf{G} \mapsto \mathbf{G}$ is a function that maps elements of \mathbf{G} onto itself; $\forall A, B \in \mathbf{G}, AB \in \mathbf{G}$.
- **Associativity:** Elements of \mathbf{G} are associative under the group operation; $\forall A, B, C \in \mathbf{G}, (AB)C = A(BC)$.
- **Identity:** There exists an identity element $I \in \mathbf{G}$ such that $\forall A \in \mathbf{G}, IA = AI = A$.
- **Inverse:** For all $A \in \mathbf{G}$ there exists an inverse element $A^{-1} \in \mathbf{G}$ such that $AA^{-1} = A^{-1}A = I$.

Because the Lie group \mathbf{G} is a differentiable manifold, it is locally Euclidean. This means that the neighbourhood around every element of \mathbf{G} can be approximated with a tangent plane. This property allows calculus to be performed on elements of \mathbf{G} .

Matrix Lie groups

A matrix Lie group is made up of group elements which are $n \times n$ matrices. This work will be focus on matrix Lie groups because the exponential map and Lie bracket functions given below only apply to such Lie groups.

Lie algebra

The tangent space at the identity element of a Lie group is called the Lie algebra \mathfrak{g} . It is called the Lie *algebra* because it has a binary operation, known as the Lie bracket $[X, Y]$. For matrix Lie groups the Lie bracket is

$$[A, B] \triangleq AB - BA \quad (1.1)$$

The exponential map and logarithm map

The mapping from the Lie algebra \mathfrak{g} to the Lie group \mathbf{G} is called the expo-

nential map:

$$\exp : \mathfrak{g} \rightarrow \mathbf{G} \quad (1.2)$$

Similarly, the logarithm map maps elements from \mathbf{G} to \mathfrak{g} :

$$\log : \mathbf{G} \rightarrow \mathfrak{g} \quad (1.3)$$

sub-heading?

For an n -dimensional matrix Lie group, the Lie algebra \mathfrak{g} is a vector space isomorphic to \mathbb{R}^n . The hat operator $\hat{\cdot}$ maps vectors $x \in \mathbb{R}^3$ to elements of \mathfrak{g} .

$$\hat{\cdot} : x \in \mathbb{R}^n \rightarrow \hat{x} \in \mathfrak{g} \quad (1.4)$$

For a matrix Lie group \mathbf{G} whose elements are $n \times n$ matrices, the elements of \mathfrak{g} will also be $n \times n$ matrices. The hat operator is defined

$$\hat{x} = \sum_{i=1}^n x_i G^i \quad (1.5)$$

where G^i are $n \times n$ matrices known as the infinitesimal generators of \mathbf{G} .

Lie bracket and group operation

For Lie groups endowed with the commutative property ($\forall A, B \in \mathbf{G}, AB = BA$), vector addition in the Lie algebra maps to a group operation in the Lie group. For $C = A + B$ where $A, B, C \in \mathfrak{g}$,

$$e^C = e^{A+B} = e^A e^B \quad (1.6)$$

For non-commutative Lie groups, the relationship between the Lie bracket and group operation does not hold. Instead, for $C = \log e^A e^B$, C is calculated with the Baker-Campbell-Hausdorff formula:

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A - B, [A, B]] + \frac{1}{24}[B, [A, [A, B]]] + \dots \quad (1.7)$$

Actions

When a group action for a Lie group \mathbf{G} acting on a manifold M is a differentiable map, this is known as a Lie group action. For example, 3D rotations act on 3D points so the Lie group $\mathbf{SO}(3)$ acts on \mathbb{R}^3 . A left action of \mathbf{G} on M is defined as a differentiable map

$$\Phi : \mathbf{G} \times M \mapsto M \quad (1.8)$$

where

- the identity element I maps M onto itself *(is that the right wording?)

$$\Phi(I, m) = m, \forall m \in M \quad (1.9)$$

- Group actions compose according to

$$\Phi(m, \Phi(n, o)) = \Phi(mn, o) \quad (1.10)$$

Adjoint map

EXPLANATION - Sometimes before a function B can be applied on manifold acted on by a group action A , it is necessary to apply the conjugate of A to B ????

For $A \in \mathbf{G}$ define a function Ψ , known as the adjoint map of \mathbf{G} :

$$\Psi_A : \mathbf{G} \rightarrow \mathbf{G}, \Psi_A(B) \triangleq ABA^{-1} \quad (1.11)$$

Taking the derivative:

$$\frac{\partial}{\partial t} \Psi_A(B(t))|_{t=0} = AVA^{-1}, V \triangleq \frac{\partial}{\partial t} B(t)|_{t=0} \quad (1.12)$$

The adjoint representation of \mathbf{G} is given by the mapping

$$\mathbf{Adj}_A : \mathfrak{g} \rightarrow \mathfrak{g}, \mathbf{Adj}_A(V) \triangleq AVA^{-1} \quad (1.13)$$

1.2.1.2 $\mathbf{SO}(3)$

A rotation represents the motion of a point about the origin of a Euclidean space. In \mathbb{R}^3 this is a proper isometry: a transformation that preserves distances between any pair of points and has a determinant of $+1$. The set of all rotations about the origin of \mathbb{R}^3 is known as the *special orthogonal group* $\mathbf{SO}(3)$. This matrix Lie group is a subgroup of the general linear group $\mathbf{GL}(3)$, so its group elements are rotations in \mathbb{R}^3 , and can be represented with 3×3 invertible matrices. These group elements are orthogonal matrices so their columns and rows are orthogonal unit vectors.

Lie algebra

The Lie algebra $\mathfrak{so}(3)$ is vector space of 3×3 skew-symmetric matrices $\hat{\omega}$, where ω is a 3-vector representing an angular velocity. The direction of ω indicates the axis of rotation while its magnitude gives the angular velocity. Elements of $\mathfrak{so}(3)$ are mapped to $\mathbf{SO}(3)$ according to the exponential map:

$$\begin{aligned} \exp : \mathfrak{so}(3) &\rightarrow \mathbf{SO}(3) \\ [\omega]_{\times} &\rightarrow \mathbf{R}_{3 \times 3} \end{aligned} \quad (1.14)$$

i.e. $\forall \omega \in \mathfrak{so}(3), \exp([\omega]_{\times}) \in \mathbf{SO}(3)$

Conversely, the logarithm map maps 3×3 rotation matrices of $\mathbf{SO}(3)$ to elements of $\mathfrak{so}(3)$:

$$\begin{aligned} \log : \mathbf{SO}(3) &\rightarrow \mathfrak{so}(3) \\ \mathbf{R}_{3 \times 3} &\rightarrow [\omega]_{\times} \end{aligned} \quad (1.15)$$

i.e. $\forall \mathbf{R} \in \mathbf{SO}(3), \log(\mathbf{R}) \in \mathfrak{so}(3)$

Actions

By the group action, elements of $\mathbf{SO}(3)$ rotate points in \mathbb{R}^3 about the origin.

$$\Phi : \mathbf{SO}(3) \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \quad (1.16)$$

Adjoint map

$$\Psi_R : \mathbf{SO}(3) \rightarrow \mathbf{SO}(3), \Psi_R(A) \triangleq RAR^{-1} \quad (1.17)$$

Taking the derivative:

$$\frac{\partial}{\partial t} \Psi_R(A(t))|_{t=0} = RBR^{-1}, B \triangleq \frac{\partial}{\partial t} A(t)|_{t=0} \quad (1.18)$$

The adjoint representation of $\mathbf{SO}(3)$ is given by the mapping

$$\mathbf{Adj}_R : \mathfrak{so}(3) \rightarrow \mathfrak{so}(3), \mathbf{Adj}_R(B) \triangleq RBR^{-1} \quad (1.19)$$

???? Hard to explain practical application without discussing reference frames: ie if position defined in body fixed frame but some other transformation defined in inertial frame. First undo rotation to get pose in inertial frame, apply transformation, then re-apply rotation

TODO: more explanation needed here!

Rotation representations

There are many conventions by which elements of $\mathbf{SO}(3)$ can be represented. Common representations are: **TODO: go into more detail on below**

Rotation matrix

3×3 matrix where magnitude of each column is 1, columns are orthogonal, determinant is +1. Group operation matrix is multiplication. Left action is left multiplication of point. GROUP OPERATION - matrix multiplication - concatenates rotation

GROUP ACTION - left multiplication with vector - rotates point

Scaled axis

3-vector where direction represents axis of rotation and magnitude represents angle of rotation. Group operation - vector addition? Left action - rodrigues' rotation formula converts scaled axis to rotation matrix which is used to rotate point.

Quaternion

4-vector, same information as axis angle, but different form. Given axis of rotation \mathbf{r} and angle of rotation θ :

$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathbf{r} \end{bmatrix} \quad (1.20)$$

Group operation - quaternion multiplication defined as:

$$\mathbf{q}_1 \mathbf{q}_2 = \begin{bmatrix} w_1 \\ \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix} \quad (1.21)$$

As with rotation matrices, quaternion multiplication is associative but not commutative. GROUP ACTION: formula for rotating vector: $\mathbf{q}^* \mathbf{v}^* \mathbf{q} \text{INV}$

1.2.1.3 SE(3)

The special Euclidean group $\mathbf{SE}(3)$ represents rigid transformation in \mathbb{R}^3 . This is a matrix Lie group whose elements are the set of all rigid transformations in \mathbb{R}^3 and can be represented with 4×4 matrices of the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \quad (1.22)$$

where $\mathbf{R} \in \mathbf{SO}(3)$ and $\mathbf{p} = [p_x \ p_y \ p_z]^\top \in \mathbb{R}^3$.

$\mathbf{SE}(3)$ is a semidirect product of $\mathbf{SO}(3)$ and \mathbb{R}^3 . As its group elements contain a rotation matrix and translation vector, $\mathbf{SE}(3)$ has 6 degrees of freedom and is a 6-dimensional manifold.

Lie algebra

The Lie algebra $\mathfrak{se}(3)$ is a vector space whose elements are 4×4 matrices of the form

$$\begin{bmatrix} [\omega]_\times & \mathbf{v} \\ \mathbf{0}_{1 \times 3} & 0 \end{bmatrix} \quad (1.23)$$

where $\omega = [\omega_x \ \omega_y \ \omega_z]^\top \in \mathfrak{so}(3)$, representing an angular velocity in scaled axis representation, and $\mathbf{v} = [v_x \ v_y \ v_z]^\top \in T_{\mathbf{p}}\mathbb{R}^3$, representing a linear velocity vector.

Elements of $\mathfrak{se}(3)$ are mapped to $\mathbf{SE}(3)$ according to the exponential map:

$$\exp : \mathfrak{se}(3) \rightarrow \mathbf{SE}(3)$$

$$\begin{bmatrix} [\omega]_\times & \mathbf{v} \\ \mathbf{0}_{1 \times 3} & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \quad (1.24)$$

i.e. $\forall \mathbf{T} \in \mathfrak{se}(3), \exp(\mathbf{T}) \in \mathbf{SE}(3)$

Conversely, the logarithm map maps elements of $\mathbf{SE}(3)$ to elements of $\mathfrak{se}(3)$:

$$\begin{aligned} \log : \mathbf{SE}(3) &\rightarrow \mathfrak{se}(3) \\ \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} [\omega]_{\times} & \mathbf{v} \\ \mathbf{0}_{1 \times 3} & 0 \end{bmatrix} \end{aligned} \quad (1.25)$$

i.e. $\forall \mathbf{S} \in \mathbf{SE}(3), \log(\mathbf{S}) \in \mathfrak{se}(3)$

Actions

$\mathbf{SE}(3)$ group elements acts to perform a rigid transformation on points in \mathbb{R}^3 . This corresponds to a rotation about the origin and a translation. To apply a transformation using the 4×4 matrix elements of $\mathbf{SE}(3)$ to a point $\mathbf{p} = (x, y, z)$ in \mathbb{R}^3 , the point must be represented with homogeneous coordinates: (is p' okay for homogeneous points? \wedge is already used for skew-symmetric matrix)

$$\mathbf{p}' = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad (1.26)$$

The left group action of $\mathbf{SE}(3)$ is now simply a left matrix multiplication of \mathbf{p} :

$$\mathbf{p}'_1 = \mathbf{S}\mathbf{p}'_0 = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}\mathbf{p}_0 + \mathbf{p} \\ 1 \end{bmatrix} \quad (1.27)$$

Adjoint Map

The adjoint map of $\mathbf{SE}(3)$ is

$$\Psi_S : \mathbf{SE}(3) \rightarrow \mathbf{SE}(3), \Psi_S(A) \triangleq SAS^{-1} \quad (1.28)$$

Taking the derivative:

$$\frac{\partial}{\partial t} \Psi_S(A(t))|_{t=0} = SBS^{-1}, B \triangleq \frac{\partial}{\partial t} A(t)|_{t=0} \quad (1.29)$$

The adjoint representation of $\mathbf{SE}(3)$ is given by the mapping

$$\mathbf{Adj}_S : \mathfrak{se}(3) \rightarrow \mathfrak{se}(3), \mathbf{Adj}_S(B) \triangleq SBS^{-1} \quad (1.30)$$

TODO: more explanation needed here!

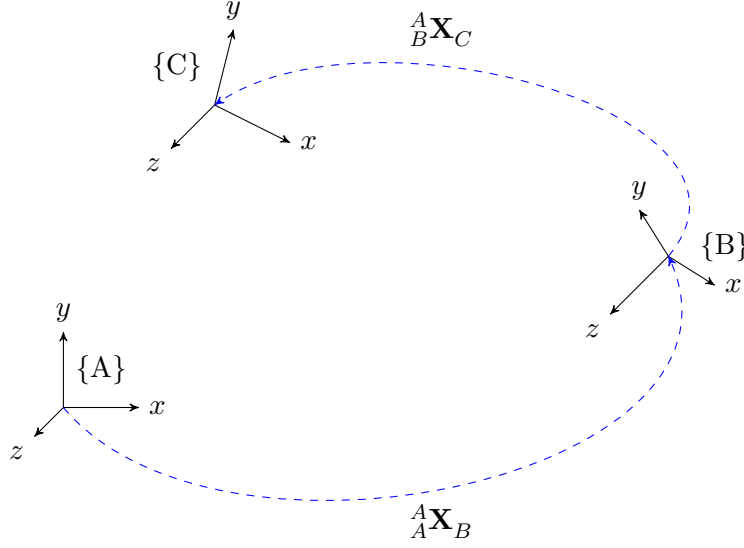


Figure 1.1: what frame should transformations be defined in?

1.2.1.4 Reference Frames

A reference frame is a system of coordinates that is used to uniquely identify points on a manifold. This report will deal with reference frames on \mathbb{R}^3 , that are used both to define the position of a point and the pose of a rigid body in 3D space. Such a reference frame is represented by an element of $\mathbf{SE}(3)$.

Consider three different reference frames, denoted $\{A\}$, $\{B\}$ and $\{C\}$. The notation ${}^A_B \mathbf{X}_C$ defines the transformation in \mathbf{X} of the reference frame $\{C\}$ with respect to the frame $\{B\}$, defined in the frame $\{A\}$.

For example, ${}^A_B \mathbf{R}_C$ defines the rotation of $\{C\}$ with respect to $\{B\}$, defined in $\{A\}$.

The notion of an inertial reference frame is introduced here. This will be defined as a reference frame that is stationary for the purpose of the problem being described.

Figure 1.1 shows...

Pose:

The pose of a rigid body in a given reference frame is defined by its relative position and orientation with respect to the given reference frame and is represented by an element of $\mathbf{SE}(3)$. If a rigid body has orientation aligned with a reference frame $\{C\}$ and position at the origin of $\{C\}$, then the pose

of the rigid body with respect to $\{B\}$ and defined in $\{A\}$ is:

$${}^A\mathbf{S}_C = \begin{bmatrix} {}^A_B\mathbf{R}_C & {}^A_B\mathbf{p}_C \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \quad (1.31)$$

Point:

A point $\mathbf{p} \in \mathbb{R}^3$ in the frame $\{A\}$ is denoted ${}^A\mathbf{p}$ and is expressed as a 3-vector of the weights used to compose it from the basis vectors of $\{A\}$.

$${}^A\mathbf{p} = \begin{bmatrix} {}^Ax \\ {}^Ay \\ {}^Az \end{bmatrix} \quad (1.32)$$

Homogeneous coordinates:

To be acted on by an element of $\mathbf{SE}(3)$, a point must be expressed in homogeneous coordinates:

$${}^A\mathbf{p}' = \begin{bmatrix} {}^A\mathbf{p} \\ 1 \end{bmatrix} \quad (1.33)$$

Defining a point in terms of another reference frame:

Consider a point in \mathbb{R}^3 defined as the position of the frame $\{B\}$ with respect to the frame $\{A\}$, defined in terms of the frame $\{B\}$. To redefine the point in terms of $\{A\}$, the left action of ${}^A\mathbf{S}_B \in \mathbf{SE}(3)$ is used:

$${}^A\mathbf{p}' = {}^A\mathbf{S}_B {}^B\mathbf{p}' \quad (1.34)$$

Concatenating poses:

Multiply relative poses.

$${}^A\mathbf{X}_C = {}^A\mathbf{X}_B {}^B\mathbf{X}_C \quad (1.35)$$

Defining a pose in terms of another reference frame:

To define a pose transformation matrix in terms of a different reference frame, a matrix conjugation is used:

$${}^B_C\mathbf{X}_D = ({}^B\mathbf{X}_A) {}^A_C\mathbf{X}_D ({}^B\mathbf{X}_A)^{-1} \quad (1.36)$$

Inverse:

Taking the inverse of a pose transformation matrix has the effect of reversing the transformation, but does not alter the frame that the transformation is defined in terms of.

$$({}^A_B\mathbf{X}_C)^{-1} = {}^A_C\mathbf{X}_B \quad (1.37)$$

1.2.1.5 Rigid Body State Representation

The state of a rigid body moving through 3D space can be represented by its linear and angular position, velocity and acceleration. Higher derivatives could be taken but will be ignored for simplicity. The inertial frame is denoted $\{F\}$ and a frame $\{A\}$ is fixed to the pose of the moving body.

The pose of the body with respect to the inertial frame at time t , defined in the inertial frame is represented by the screw matrix ${}^F\mathbf{S}_A(t) \in \mathbf{SE}(3)$,

$${}^F\mathbf{S}_A(t) = \begin{bmatrix} {}^F\mathbf{R}_A(t) & {}^F\mathbf{p}_A(t) \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \quad (1.38)$$

where ${}^F\mathbf{R}_A(t) \in \mathbf{SO}(3)$ is a rotation matrix, and the position ${}^F\mathbf{p}_A(t) \in \mathbb{R}^3$.

The linear and angular velocity of the body at time t with respect to the inertial frame, defined in the body-fixed frame, is represented by the twist matrix ${}^A\mathbf{T}_A(t) \in \mathfrak{se}(3)$,

$${}^A\mathbf{T}_A(t) = \begin{bmatrix} [{}^A\omega_A(t)]_{\times} & {}^A\mathbf{v}_A(t) \\ \mathbf{0}_{1 \times 3} & 0 \end{bmatrix} \quad (1.39)$$

where ${}^A\omega_A(t) \in \mathfrak{so}(3)$ is an angular velocity in the scaled-axis representation, and the linear velocity is ${}^A\mathbf{v}_A(t) \in T\mathbb{R}^3$.

The linear and angular acceleration of the body at time t with respect to the inertial frame, defined in the body-fixed frame, is represented by the wrench matrix ${}^A\mathbf{W}_A(t) \in T\mathfrak{se}(3)$,

$${}^A\mathbf{W}_A(t) = \begin{bmatrix} [{}^A\alpha_A(t)]_{\times} & {}^A\mathbf{a}_A(t) \\ \mathbf{0}_{1 \times 3} & 0 \end{bmatrix} \quad (1.40)$$

where ${}^A\alpha_A(t) \in T\mathfrak{so}(3)$ is an angular acceleration in the scaled-axis representation, and the linear acceleration is ${}^A\mathbf{a}_A(t) \in T^2\mathbb{R}^3$.

*From now on, will not show frames in notation. This is how S,T,W will be defined unless explicitly stated otherwise.

1.2.1.6 Rigid Body Dynamic Model

The dynamics of the screw, twist and wrench matrices as they are defined in 1.2.1.5 is governed by the following ODEs,

$$\frac{d}{dt}\mathbf{S}(t) = \mathbf{S}(t)\mathbf{T}(t) \quad (1.41)$$

$$\frac{d}{dt}\mathbf{T}(t) = \mathbf{W}(t) \quad (1.42)$$

$$\frac{d}{dt}\mathbf{W}(t) = \mathbf{f}(t) \quad (1.43)$$

where the function $\mathbf{f}(t)$ is known.

1.2.1.7 Scanning Laser Rangefinder Dynamic Model

A scanning laser rangefinder fixed to a moving rigid body. State is same as moving rigid body defined above (S,T,W)

+

Unit vector defined in the body fixed frame - ${}^A\mathbf{n}(t) \in T\mathbb{R}^3$.

+

Range $r(t) \in \mathbb{R}^{0+}$, defining range from ${}^F\mathbf{p}_A(t)$ to nearest object in environment in direction ${}^F\mathbf{n}(t) = {}^F\mathbf{R}_A(t) {}^A\mathbf{n}(t)$

Scan direction in sensor frame is vector rotating at constant speed about z-axis, with unit size inside sensor's field-of-view and zero size outside it:

$${}^A\mathbf{n}(t) = \begin{cases} \begin{bmatrix} \cos(\theta_0 + 2\pi t') \\ \sin(\theta_0 + 2\pi t') \\ 0 \end{bmatrix} & \text{if } t' \leq X \\ \mathbf{0} & \text{if } t' > X \end{cases} \quad (1.44)$$

where

$$t' = \text{mod}(t, 1/d\theta) d\theta \quad (1.45)$$

* θ_0 is start of FOV, $t' = X$ at end of FOV

1.2.2 Symmetry Preserving Observers

1.2.2.1 definitions?

1.2.2.2 construction, ie moving frame method etc

1.2.3 Infinite Dimensional Observers

1.2.4 Discretisation Methods?

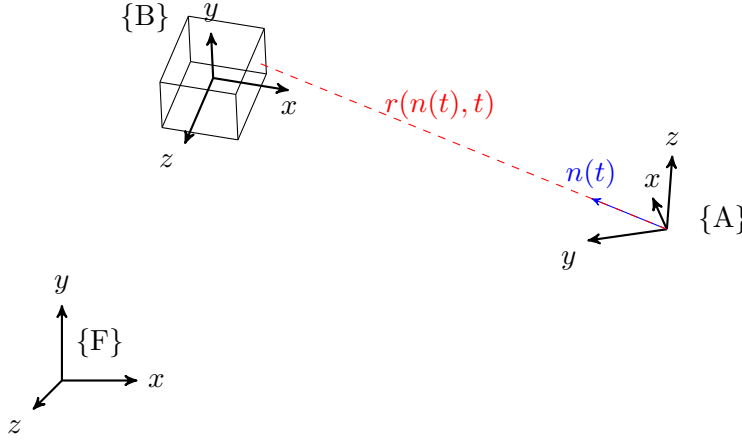


Figure 1.2: caption

1.3 Problem Statement

context etc here...

estimation problem - cube pose & size:

This report will solve a particular estimation problem before generalising the results to a wider class of systems. A situation in which an infinite dimensional observer would be useful is in the estimation of the pose of an object of size moving in an environment of unknown state. For example, consider an autonomous robot deployed in an agricultural survey, which must determine the position and size of a certain crop.

The problem to be ... is shown in Figure 1.2. A 2D scanning range sensor moves through an environment consisting of a target object of known shape, in this case a rigid cube, and an unknown background which may consist of one or more rigid bodies. The state of the sensor is known, but the states of the cube and background environment are unknown. The goal is to use the state of the sensor and the range measurements it provides estimate the state of the target object.

The frames used to describe the motion of the rigid bodies in this problem are:

- $\{F\}$ - the inertial (fixed) frame. For the purposes of this problem, the inertial frame is a frame whose motion is negligible. For the practical experiment this frame will be fixed to the ground.
- $\{A\}$ - the frame fixed to the sensor. The origin of this frame is the centre of rotation of the sensor's scan direction. The x -axis frame $\{A\}$ coincides with the sensor's scan direction \mathbf{n} when \mathbf{n} is at the centre of

the field of view of the sensor. The transformation from $\{F\}$ to $\{A\}$ at time t is defined by the screw matrix of the sensor $\mathbf{S}_s(t)$.

- $\{B\}$ - the frame fixed to the cube. The origin of $\{B\}$ coincides with the centre of the cube and is aligned so that each axis intersects with the centre of a face of the cube. The transformation from $\{F\}$ to $\{B\}$ at time t is defined by the screw matrix of the cube $\mathbf{S}_c(t)$.

The sensor provides measurements of the range r to the nearest object to the sensor (either the cube or the background) in the direction $\mathbf{n}(t)$. The state of the sensor $\mathbf{X}_s(t)$ is defined as:

$$\mathbf{X}_s(t) = \{{}_F^F\mathbf{S}_A(t), {}_F^A\mathbf{T}_A(t), {}_F^A\mathbf{W}_A(t), \mathbf{n}(t)\} \quad (1.46)$$

For simplicity, this will be denoted

$$\mathbf{X}_s(t) = \{\mathbf{S}_s(t), \mathbf{T}_s(t), \mathbf{W}_s(t), \mathbf{n}(t)\} \quad (1.47)$$

*should \mathbf{n} be defined as discrete here? ie $\mathbf{n}(k)$ where $k = \{1\delta t, 2\delta t, 3\delta t \dots\}$

* \mathbf{n} not written as a function of S since it is defined w.r.t. $\{A\}$. what is notation for this?

The direction of measurement $\mathbf{n}(t)$ varies as a rotation about the z-axis of $\{A\}$. This 2D scanning motion depends on the model of the sensor used and is described in more detail in ADD REFERENCE. For simplification, the motion of the sensor itself with respect to the frame $\{F\}$ will be limited to rotation of $\{A\}$ about its y axis.

The state of cube $\mathbf{X}_c(t)$ is defined as

$$\mathbf{X}_c(t) = \{{}_F^F\mathbf{S}_B(t), {}_F^B\mathbf{T}_B(t), {}_F^B\mathbf{W}_B(t), s\} \quad (1.48)$$

For simplicity, this will be denoted

$$\mathbf{X}_c(t) = \{\mathbf{S}_c(t), \mathbf{T}_c(t), \mathbf{W}_c(t), s\} \quad (1.49)$$

The range measurements do not indicate whether the object detected is the cube or a background object. Though the pose of the cube and environment remain unknown, for simplification, it is assumed that either:

- the cube is within X_m from the sensor and background objects are at least X_m away
- these rigid objects do not touch or overlap, and have continuous shape

For simulated data, only the first assumption is necessary. For experimental data sets the environment is more complex so the second assumption is required to identify range measurements corresponding to the cube.

The aim is to design an observer which estimates the state of the cube from the pose of the sensor, as well as \mathbf{n} and r . ie observer function f such that:

$$\hat{\mathbf{X}}_c = f(\mathbf{X}_s, r(t) - \hat{r}(t)) \quad (1.50)$$

*should r be defined as discrete here???

*convergence - how to quantify performance... how to quantify this?

deliverables:

Chapter 2

Theory Results

mathematical description of observer design

orientation update

position update

size update

S,T,W update methods

Chapter 3

Simulation

3.1 Implementation

3.1.1 Rigid Body Motion

The pose of a rigid body is described by its corresponding frame $\{X\}$. The screw, twist and wrench matrices of the body are determined iteratively:

screw matrix: $\mathbf{S}_X(t + \delta t) = \mathbf{S}_X(t) \exp(\delta t \mathbf{T}_X(t))$

twist matrix: $\mathbf{T}_X(t + \delta t) = \mathbf{T}_X(t) + \delta t \mathbf{W}_X(t)$

wrench matrix: $\mathbf{W}_X(t + \delta t) = \mathbf{W}_X(t)$

3.1.2 Sensor modelling

motion:

rigid body motion

scanning: sensor parameters:

- field of view
- angular resolution
- time per scan
- measurement resolution

From these parameters, create vector $\mathbf{n}(t)$. At each time, \mathbf{n} is either a unit vector in direction of scan, in body fixed frame, or returns no value for when sensor is not returning a measurement (outside FOV)

scan direction: ${}^A\mathbf{n}(t)$

$${}^A\mathbf{n}(t) = \begin{cases} \begin{bmatrix} \cos(\theta_0 + 2\pi t') \\ \sin(\theta_0 + 2\pi t') \\ 0 \end{bmatrix} & \text{if } t' \leq X \\ \mathbf{0} & \text{if } t' > X \end{cases} \quad (3.1)$$

where

$$t' = \text{mod}(t, 1/d\theta) d\theta \quad (3.2)$$

* θ_0 is start of FOV, $t' = X$ at end of FOV

3.1.3 Environment Modelling

Environment composed of rectangular prisms. These objects are modelled as an ordered set of 8 points in the inertial reference frame, and 12 triangles. Each triangle is a set of 3 integers, indicating the index of the three points that make up its vertexes. The position of each of these points at each time is determined using the screw matrix and side lengths of the object.

points in body frame: `initialPoints = s*0.5*[];`

$$P = \frac{1}{2}s \begin{bmatrix} -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \quad (3.3)$$

triangles:

$$Tri = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 3 & 7 \\ 4 & 8 & 7 \\ 5 & 6 & 7 \\ 8 & 6 & 7 \\ 2 & 6 & 5 \\ 2 & 1 & 5 \\ 2 & 6 & 8 \\ 2 & 4 & 8 \\ 1 & 5 & 7 \\ 1 & 3 & 7 \end{bmatrix} \quad (3.4)$$

DIAGRAM HERE

3.1.4 Measurement Modelling

3.1.4.1 Range Computation

Given the screw matrix (in the inertial frame) and scan direction (in the body fixed frame) of the sensor, the position and scan direction in the inertial frame are determined. The distance to the nearest environment object from this point, along the scan direction is determined with the Möller-Trumbore ray-triangle intersection algorithm.

Input: (O,D,pts,tri), where O and D are origin and direction of ray, pts are points in \mathbb{R}^3 , tri are indexes of points making up vertexes of triangles

Output: (intersect, range, angle, closest), where intersect is binary - collision or none. range is distance to intersection point, angle is incidence angle with surface hit, closest in index of triangle hit

Algorithm 1: Möller-Trumbore ray-triangle intersection algorithm

3.1.4.2 Noise Modelling

Noise model depends on sensor used.

$$\hat{r}(t) = f_s(r(t), \theta(t), \phi(k)) \quad (3.5)$$

where $\theta(t)$ is incidence angle of measurement, ϕ is surface properties of object k that was measured, f is some function for noise model of particular sensor.

For Hokuyo UBG-04LX-F01 used, noise model was measured experimentally:

$$f_{UBG} = \quad (3.6)$$

3.1.5 Observer implementation

3.1.5.1 Estimate: internal model

3.1.5.2 Identifying object/background

The variable *observingObject* indicates whether the range measurement is of the target object or the background. It is assumed that initially the sensor will be observing background, so $observingObject_0 = FALSE$

Range assumption: Assuming that the object is within X, background outside PSEUDOCODE if $range \leq X$ $observingObject = 1$ else $observingObject = 0$ end

Continuity assumption: if $abs(range_{i+1} - range_i) > X$ observingObject
 $= \text{mod}(\text{observingObject}+1,2)$ end

3.1.5.3 Update

Orientation update:

at least 3 points, find planes and normal, rotate about cross of normals

- \hat{z} gives rotation axis

Position update:

find mean of intersection points of predicted, measured, vector subtraction

- \hat{z} translation vector

Size update:

if different pattern, $\Delta s = \text{dot}(\text{translation vector}, \text{scan direction})$

if same pattern, $\Delta s = \text{vector}(\text{mean measured points} - \text{mean predicted points})$

Update scheme:

Can update S,T,W of state (or perhaps some combination)

Scale rotation axis, translation vector, Δs depending on if update is via S,T,W

3.2 Results

Chapter 4

Experiment

intro: high level description of what was done, experimental setup

components: arm, cube, sensor

cube motion: different motions used

sensor calibration:

sensor motion:

experimental results: observer performance

Chapter 5

Conclusion

Bibliography

- [1] C. Harkort and J. Deutscher, “Finite-dimensional observer-based control of linear distributed parameter systems using cascaded output observers,” *International Journal of Control*, vol. 84, no. 1, pp. 107–122, 2011.
- [2] L. Meirovitch and H. Baruh, “On the problem of observation spillover in self-adjoint distributed-parameter systems,” *Journal of Optimization Theory and Applications*, vol. 39, no. 2, pp. 269–291, 1983.
- [3] G. Haine, “Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint generator,” *Mathematics of Control, Signals, and Systems*, vol. 26, no. 3, pp. 435–462, 2014.
- [4] J. W. Helton, “Systems with infinite-dimensional state space: the hilbert space approach,” *Proceedings of the IEEE*, vol. 64, no. 1, pp. 145–160, 1976.
- [5] K. Ramdani, M. Tucsnak, and G. Weiss, “Recovering the initial state of an infinite-dimensional system using observers,” *Automatica*, vol. 46, no. 10, pp. 1616–1625, 2010.
- [6] C.-Z. Xu, P. Ligarius, and J.-P. Gauthier, “An observer for infinite-dimensional dissipative bilinear systems,” *Computers & Mathematics with Applications*, vol. 29, no. 7, pp. 13–21, 1995.
- [7] H. Bounit and H. Hammouri, “Observers for infinite dimensional bilinear systems,” *European journal of control*, vol. 3, no. 4, pp. 325–339, 1997.