Lecture Notes on Separation Axioms

Math 6201 - Topology

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Contents

1 Preliminaries

This section reviews some fundamental concepts needed to understand the basic parts on separation axioms on topology.

1.1 Sets

Definition 1.1 — Subset. (Lipschutz, 1965) A set A is a subset of a set B or, equivalently, B is a superset of A, written $A \subset B$ or $B \supset A$ iff each element in A also belongs to B; that is, if $x \in A$ implies $x \in B$.

Definition 1.2 — Union. (Lipschutz, 1965) The union of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or B, i.e., $A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$

Definition 1.3 — Intersection. (Lipschutz, 1965) The intersection of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B, i.e., $A \cap B = \{x : x \in A \quad \text{and} \quad x \in B)$

Definition 1.4 — Power Sets. (Dugundji, 1966) Let A be any set. Its power set $\mathscr{P}(A)$ is the set of all subsets of A.

Definition 1.5 — Countable. (Dugundji, 1966) A set A is countable if it is finite or equivalent to the set $\mathbb{N}$ of counting numbers. If $A \equiv N$, then A is called countably infinite or denumerable.

1.2 Functions, or Maps

Definition 1.6 — Map. (Dugundji, 1966) Let X and Y be two sets. A *map* $f : X \to Y$ *(or function with domain X and range Y)* is a subset $f \subset X \times Y$ with the property: for each $x \in X$, there is one, and only one, $y \in Y$ satisfying $(x, y) \in f$.To denote $(x, y) \in f$, we write $y = f(x)$ and say that y is the image of x under f.

Definition 1.7 — Surjective function. (Holmes, 2008) Let $f : X \to Y$ be a function. We say that f is surjective if for each $y \in Y$, there exists $x \in X$ such that $f(x) = y$

Definition 1.8 — Injective function. (Holmes, 2008) Let $f : X \to Y$ be a function. We say that f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2 (x_i \in X)$

Definition 1.9 — Bijective function. (Holmes, 2008) Let $f : X \to Y$ be a function. We say that f is bijective if it is both injective and surjective.

1.3 Topology

Definition 1.10 — Topology. (Dugundji, 1966) Let X be a set. A *topology* (or topological structure) in X is a family τ of subsets of X that satisfies:

(1). Each union of members of τ is also a member of τ.
(2). Each *finite* intersection of members of τ is also a member of τ.

(3). $\emptyset$ and X are members of τ.

■ **Example 1.1 — Discrete topology.** Let X be any set and $\tau = \mathscr{P}(X)$. Then τ is a topology on X.

Proof.

1. Clearly, $X, \varnothing \in \tau$
2. Since every possible subset of X is included in τ, the union of any combination of these subsets will always result in another subset of X, which is already in τ.
3. Since all subsets are open, the intersection of any finite number of them will still be a subset of X, hence in τ.

Thus, τ is a topology on X. ■

■

■ **Example 1.2 — Indiscrete topology.** Let X be any set and $\tau = \{X, \varnothing\}$. Then τ is a topology on X

Proof.

1. $X \in \tau$ and $\varnothing \in \tau$
2. $X \cup X = X \cup \varnothing = X \in \tau$
   $\varnothing \cup \varnothing = \varnothing \in \tau$
3. $X \cap X = X \in \tau$
   $X \cap \varnothing = \varnothing \cap \varnothing = \varnothing \in \tau$

Thus, τ is a topology on X. ■

■

Definition 1.11 — Open sets. (Morris, 2020) Let (X, τ) be a topological space. Then the members of τ are said to be *open sets*.

Definition 1.12 — Closed sets. (Morris, 2020) Let (X, τ) be a topological space. A subset S of X is said to be *closed set* in (X, τ) if its complement in X, namely $X \setminus S$, is open in (X, τ).

Definition 1.13 — Neighborhood. (Holmes, 2008) Let $x \in X$. Any open set containing x is called a neighborhood of x.

1.4 G_δ

Definition 1.14 (Dugundji, 1966) A set G is called G_δ if it is the intersection of at most countably many open sets.

1.5 Relativization

Definition 1.15 — Subspace topology. (Dugundji, 1966) Let (X,τ) be a topological space and $Y \subset X$. The induced topology τ_Y on Y is $\{Y \cap U : U \in \tau\}$. The pair (Y,τ_Y) is called a subspace of (X,τ).

1.6 Continuous maps and Homeomorphisms

Definition 1.16 — Continuous maps. (Belleza, 2025) Let (X,τ_X) and (Y,τ_Y) be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each open set in Y is open in X. That is, f^{-1} maps $\tau_Y \to \tau_X$.

■ **Example 1.3** Let (X,τ) be any topological space and $f : (X,\tau) \to (X,\tau)$ is defined by $f(x) = x$ for all $x \in X$. Then f is continuous. ■

Proof. To show that f is continuous, we need to verify that the inverse image of every open set in $Y = (X,\tau)$ is open in $X = (X,\tau)$.

Let O be an arbitrary open set in Y. By definition, $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$. Since $f(x) = x$, this simplifies to $f^{-1}(O) = \{x \in X \mid x \in O\} = O$. Since O is open in Y, and Y has the same topology as X, O is also open in X. Thus, $f^{-1}(O) = O$ is open in X. Since the inverse image of every open set in Y is open in X, the identity map f is continuous. ■

R Not every identity function is continuous. To see this, let $X = \{1,2,3,4\}$, and define two topologies on X, $\tau_1 = \{\emptyset,\{1\},\{3\},\{1,3\},\{2,3\},\{1,2,3\},X\}$, $\tau_2 = \{\emptyset,\{1\},\{3\},\{1,3\},\{2\},\{1,2,3\},X\}$. Clearly, τ_1 and τ_2 are topologies on X. Then $i : (X,\tau_1) \to (X,\tau_2)$ defined by $i(x) = x$ for all $x \in X$ is not continuous, since there exists $\{2\} \in \tau_2$ such that $i^{-1}(\{2\}) = \{2 \notin \tau_1$.

Definition 1.17 — Homeomorphism. (Dugundji, 1966) A continuous bijective map $f : X \to Y$, such that $f^{-1} : Y \to X$ is also continuous, is called a homeomorphism (or a bicontinuous bijection) and denoted by $f : X \cong Y$. Two spaces X,Y are homeomorphic, written $X \cong Y$, if there is a homeomorphism $f : X \cong Y$.

■ **Example 1.4** The identity map $1 : X \to X$ is a homeomorphism. ■

■ **Example 1.5** Two discrete spaces X and Y (similarly, for indiscrete spaces), are homeomorphic if and only if there is a one-to-one function on X onto Y . ■

Proof. In a discrete space, every subset is open. Let $f : X \to Y$ be a bijection (one-to-one and onto). We will show that f is a homeomorphism. To show f is continuous, the preimage of any open set in Y must be open in X. Since Y is discrete, every subset of Y is open. Thus, for any open $V \subseteq Y$, $f^{-1}(V) \subseteq X$. Because X is discrete, $f^{-1}(V)$ is open in X. Hence, f is continuous. Similarly, $f^{-1} : Y \to X$ is a bijection. For any open $U \subseteq X$, $(f^{-1})^{-1}(U) = f(U) \subseteq Y$. Since X is discrete, U is open, and since Y is discrete, $f(U)$ is open in Y. Thus, f^{-1} is continuous. Hence, f is a bijection, continuous, and its inverse f^{-1} is also continuous. Therefore, f is a homeomorphism, and $X \cong Y$. ■

Definition 1.18 — Topological Invariant. We call any property of spaces a topological invariant if whenever it is true for one space X, it is also true for every space homemorphic to X

2 Separation Axioms

Separation axioms are a family of topological invariants that give us new ways of distinguishing between various spaces. The idea is to look how open sets in a space can be used to create "buffer zones" separating pairs of points and closed sets. Separations axioms are denoted by T_0 T_1, T_2, etc., where T comes from the German word Trennungsaxiom, which just means "separation axiom" (Schechter, 1996).

2.1 T_0 and T_1 space

Definition 2.1 — T_0 space or Kolmogorov Space. (Milewski, 1994) The space $(X, \mathscr{T})$ is said to be a T_0-space, if for any two distinct $a, b \in X$, there is a neighborhood of at least one, which does not contain the other

■ **Example 2.1** Let $X = \{a, b, c\}$ with topology $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ defined on X, then (X, τ) is a T_0 space ■

Proof.

1. for a and b, there exists an open set $\{a\}$ such that $a \in \{a\}$ and $b \notin \{a\}$
2. for a and c, there exists an open set $\{a\}$ such that $a \in \{a\}$ and $c \notin \{a\}$
3. for b and c, there exists an open set $\{b\}$ such that $b \in \{b\}$ and $c \notin \{b\}$

■

■ **Example 2.2** The space $X = \{a, b\}$ with the indiscrete topology is not a T_0 space ■

Proof. The two distinct points a and b in X is contained only in the open set X. Thus, it is not a T_0 space. ■

Definition 2.2 — T_1-space or Fréchet Space . (Milewski, 1994) A topological space $(X, \mathscr{T})$ is called a T_1-space, if every single element set is closed, that is, $\forall a \in X, \{a\} = \overline{\{a\}}$.

Theorem 2.1 A topological space (X, τ) is a T_1 space if and only if for any pair of distinct points $a, b \in X$, the open sets $G, H \in \tau$ exist, such that $a \in G$, $b \notin G$ and $b \in H$, $a \notin H$

Proof. Suppose X is a T_1-space. Then, for any $x \in X$, $\{x\}$ is a closed set. Let $a, b \in X$ and $a \neq b$. The sets $X - \{a\}$ and $X - \{b\}$ are open, and

$$\begin{aligned} a \in X - \{b\} \quad &\text{and} \quad b \notin X - \{b\} \\ b \in X - \{a\} \quad &\text{and} \quad a \notin X - \{a\}. \end{aligned}$$

Conversely, suppose $x \in X$. We shall show that $\{x\}$ is closed, i.e., $X - \{x\}$ is open. Let $y \in X - \{x\}$, then $y \neq x$ and an open set H_y exists, such that

$$y \in H_y \quad \text{and} \quad x \notin H_y.$$

Thus,

$$y \in H_y \subseteq X - \{x\} \quad \text{and} \quad X - \{x\} = \bigcup_{y \neq x} H_y.$$

Since all H_y are open sets, $X - \{x\}$ is open and $\{x\}$ is closed, $\{x\} = \overline{\{x\}}$. ■

■ **Example 2.3** Consider the set $X = \{a,b,c\}$ with the cofinite topology,

$$\tau = \{X, \emptyset, a, b, c, \{a,b\}, \{a,c\}, \{b,c\}\}.$$

Verify that (X,τ) is a T_1 space. ■

Proof. The complement of $\{a\}$ is $\{b,c\}$, which is open (its complement $\{a\}$ is finite). Similarly, $\{b\}^c = \{a,c\}$ and $\{c\}^c = \{a,b\}$ are open. By Definition 2.2, since every singleton is closed, X is T_1. For a and b, there exist open sets $\{a\}$ and $\{b\}$ such that $a \in \{a\}$, $b \notin \{a\}$ and $b \in \{b\}$, $a \notin \{b\}$. This satisfies the condition in Theorem 2.1, confirming X is T_1.

Thus, (X,τ) is a T_1 space. ■

■ **Example 2.4** The Sierpinski Space is T_0 but not T_1 ■

Proof. Recall that a *Sierpiński space* is the topological space $X = \{x,y\}$ with the topology given by $\{X, \{x\}, \emptyset\}$. It is T_0 because for x and y the open set $\{x\}$ contains x but not y. It is not T_1 because every open set U containing y (which is only X) contains x. ■

Theorem 2.2 (Milewski, 1994) A T_1 space is also a T_0 space.

Proof. Let X be a T_1 space then clearly from its definition it follows that it is also a T_0 space. Since with any pair $a,b \in X$ there exist an open set G with $a \in G$ and $b \notin G$. ■

Theorem 2.3 (Milewski, 1994) If (X,τ) and (Y,τ') are homeomorphic and (X,τ) is a T_1-space (or T_0) then so is (Y,τ')

Proof. Let f denote a homeomorphism

$$f : X \to Y$$

and X be a T_1-space. A space (X,τ) is T_1, if and only if every one-point subset of X is closed. Let y represent any point of Y, $y \in Y$. The set $f^{-1}(y)$ is a one-point subset of X and since X is T_1, the set $\{f^{-1}(y)\}$ is closed.

Since $f : X \to Y$ is a homeomorphism, it maps closed sets into closed sets. Therefore, for any $y \in Y$

$$\{y\} = \overline{\{y\}}.$$

Thus, (Y, τ) is a T_1-space. Similarl proof can be done to when if X is T_0, then so is Y. ■

2.2 Hausdorff spaces

Definition 2.3 — T_2-space or Hausdorff space. (Dugundji, 1966) A space X is Hausdorff (or separated) if each two distinct points have nonintersecting nbds, that is, whenever, $p \neq q$ there are nbds $U(p), V(q)$ such that $U \cap V = \varnothing$.

■ **Example 2.5** The real line R with the standard topology is a Hausdorff space. ■

Proof. For any two distinct points $x, y \in \mathbb{R}$, let $d = |x - y| > 0$. Then the open intervals

$$U = \left(x - \frac{d}{2}, x + \frac{d}{2}\right) \quad \text{and} \quad V = \left(y - \frac{d}{2}, y + \frac{d}{2}\right)$$

are disjoint neighborhoods of x and y, respectively. Thus, $\mathbb{R}$ is Hausdorff. ■

Theorem 2.4 (Milewski, 1994) Each Hausdorff space is a T_1 space.

Proof. Suppose X is a Hausdorff space. Then from the definition, for two distinct points x and y, there exist two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. Thus, $x \notin V$ and $y \notin U$, by Theorem 2.1, X is also a T_1 space.

■

Theorem 2.5 (Milewski, 1994) Each Hausdorff space is a T_0 space

Proof. Suppose X is a Hausdorff space. Then from Theorem 2.4 and Theorem 2.2, it is also a T_0 space. ■

Theorem 2.6 (Dugundji, 1966) The following three properties are equivalent

1. X is Hausdorff.
2. Let $p \in X$. For each $q \neq p$, there is a nbd $U(p)$ such that $q \notin \overline{U(p)}$.
3. For each $p \in X$, $\bigcap\{\overline{U} \mid U \text{ is a nbd of } p\} = \{p\}$.

Theorem 2.7 Every subspace of a Hausdorff space is Hausdorff

Theorem 2.8 If (X, τ) and (Y, τ') are homeomorphic and (X, τ) is a T_2-space then so is (Y, τ')

Proof. Let f denote a homeomorphism, $f : X \to Y$ and X be a T_2-space. Now f is a continuous bijective map hence two distinct points x_1, x_2 of X exist such that $f^{-1}(y_1) = x_1, f^{-1}(y_2) = x_2$. (X, τ) is a Hausdorff space, therefore, there are two open sets $U_1, U_2 \subset X$, such that

$$x_1 \in U_1, \quad x_2 \in U_2, \quad U_1 \cap U_2 = \emptyset.$$

Since f is bijective,

$$f(U_1) \subset Y, \quad f(U_2) \subset Y$$
$$f(U_1) \cap f(U_2) = \emptyset$$

Now, since f^{-1} is continuous, the function $(f^{-1})^{-1} = f$ maps open sets into open sets. Hence, $f(U_1), f(U_2) \in T'$ are open sets.

$$y_1 \in f(U_1), \quad y_2 \in f(U_2)$$

We conclude that (Y, τ') is a T_2-space. If two spaces are homeomorphic and one of them is a T_2-space, then so is the other. ■

2.3 Regular Spaces

Definition 2.4 — Regular space. (Milewski, 1994) A topological space $(X, \mathscr{T})$, is said to be regular if, given any closed subset $F \subset X$ and any point $x \in X$, such that $x \notin F$, there are open sets U and V, such that

$$F \subset U, \quad x \in V, \quad \text{and} \quad U \cap V = \emptyset$$

Definition 2.5 — T_3-space or Regular Hausdorff space. A space is a T_3-space or Regular Hausdorff space if it is both a Hausdorff space and a regular space.

Note that some authors switch the definition of "Regular" and "T_3". Some also defines them equivalently, such as Dugundji (1966). Some also defined T_3 by the theorems below. For this lecture notes, we use the definitions that was stated above.

R A regular space need not be a T_1-space.

■ **Example 2.6** Consider the topology $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ on the set $X = \{a, b, c\}$. Observe that the closed subsets of X are also X, $\emptyset$, $\{a\}$ and $\{b, c\}$ and that (X, τ) does satisfy Definition 2.4. On the other hand, (X, τ) is not a T_1-space since there are single element sets, e.g. $\{b\}$, which are not closed. ■

Theorem 2.9 A space is T_3 if and only if it is both regular and T_0

Proof. Suppose a space is T_3 then by the definition, it is both regular and T_2. By Theorem 2.5, it is also T_0. For the converse, suppose a space is both a regular and T_0. Now let $a, b \in X$ represents distinct points. Since the space is T_0, there is a neighborhood of at least one, which does not contain the other. Thus, let $a \notin U(b)$ then $a \notin \overline{U(b)}$. By regularity, there exist open sets G and V s.t $\overline{U(b)} \subset G$ and $a \in V$ where $G \cap V = \emptyset$. This implies that the space is also Hausdorff and hence also a T_3 space. ■

Theorem 2.10 A space is T_3 if and only if it is both regular and T_1

Theorem 2.11 (Dugundji, 1966) The following three properties are equivalent

1. X is a T_3 space
2. For each $x \in X$ and nbd U of x, there exists a nbd V of x with $X \in V \subset \bar{V} \subset U$.
3. For each $x \in X$ and closed A not containing x, there is a nbd V of x with $\bar{V} \cap A = \emptyset$.

Theorem 2.12 Every subspace of a regular space is regular

Proof. Let Y be a regular space and $X \subset Y$ a subspace. Let $B \subset X$ be closed in X, and let $x_0 \in X \setminus B$. Since B is closed in X, there exists a closed set $A \subset Y$ such that $B = X \cap A$. Note that $x_0 \notin A$ because $x_0 \in X \setminus B$. By the regularity of Y, there exist disjoint open sets U and V in Y such that $x_0 \in U$ and $A \subset V$. The sets $U \cap X$ and $V \cap X$ are open in X (by the definition of the subspace topology), $x_0 \in U \cap X$, and $B = X \cap A \subset V \cap X$. Since U and V are disjoint in Y, their intersections with X are also disjoint in X.

Thus, X is regular. ■

Theorem 2.13 Any subspace of a T_3-space is a T_3-space

Proof. A T_3-space is a regular Hausdorff space. By Theorem 2.12 and theorem 2.7, its subspace is also a regular Hausdorff space or a T_3-space. ■

2.4 Normal Spaces

Definition 2.6 — Normal Space. (Milewski, 1994) A topological space (X, T) is said to be normal if, given any two disjoint closed sets F_1 and F_2 in X, there are disjoint open sets U and V, such that

$$F_1 \subset U \quad \text{and} \quad F_2 \subset V$$

Definition 2.7 — T_4-space of Normal Hausdorff space. A space is a T_4-space or Normal Hausdorff space if it is both a Hausdorff space and a normal space

Similarly with regular spaces, some authors switch the definition of "Normal" and "T_4". Some also defines them equivalently, such as Dugundji (1966). For this lecture notes, we use the definitions that are stated above.

■ **Example 2.7** Discrete spaces are T_4 spaces ■

Proof. For any two distinct points $x, y \in X$, the singleton sets $\{x\}$ and $\{y\}$ are open and disjoint. Thus, X is Hausdorff. Now, Let F_1 and F_2 be disjoint closed sets in X. In the discrete topology, every set is open, so F_1 and F_2 themselves are open. Thus, we can take

$U = F_1$ and $V = F_2$ as disjoint open sets containing F_1 and F_2 respectively. This shows that X is normal. Since X is both Hausdorff and normal, it is a T_4-space. ■

▪ **Example 2.8** Any space (X, τ), containing more than one point with the indiscrete topology is Normal. ▪

Proof. The indiscrete topology consists of two sets X and ϕ.

$$T = \{X, \phi\}.$$

Hence, the only closed sets are X and ϕ because $X - \phi = X$ and $X - X = \phi$. Thus, there are no non-empty disjoint closed subsets of X. The space is normal. ■

Theorem 2.14 Every T_4-spaces are T_3-spaces

Proof. Let (X, T) denote a T_4-space. Hence, (X, T) is normal and T_1. Suppose F is a closed subset of X and $a \in F$. Since (X, T) is T_1, the singleton set $\{a\}$ is closed. Sets F and $\{a\}$ are closed and disjoint. Since (X, T) is normal, the open sets U_1 and U_2 exist, such that

$$\{a\} \subset U_1, \quad F \subset U_2, \quad U_1 \cap U_2$$

. Therefore (X, T) is regular and T_3 ■

Theorem 2.15 The following four properties are equivalent

1. X is T_4.
2. For each closed A and open $U \supset A$ there is an open V with $A \subset V \subset \bar{V} \subset U$.
3. For each pair of disjoint closed sets A, B, there is an open U with $A \subset U$ and $\bar{U} \cap B = \emptyset$.
4. Each pair of disjoint closed sets have nbds whose closures do not intersect.

Theorem 2.16 A closed subspace of a T_4 space is T_4.

Proof. Let X be a T_4 space and Y be a closed subspace of X. Since every subspace of a T_1-space is T_1 and X is T_1 also, Y is a T_1-space. Since Y is closed, a subset F of Y is closed in Y, if and only if F is closed in X. Hence, if F_1 and F_2 are disjoint closed subsets of Y, they are also disjoint closed subsets of X.

Thus, the open sets U_1 and U_2 exist, such that

$$F_1 \subset U_1, \quad F_2 \subset U_2 \text{ and } U_1 \cap U_2 = \emptyset.$$

Then

$$F_1 \subset U_1 \cap Y, \quad F_2 \subset U_2 \cap Y,$$

and $U_1 \cap Y$ and $U_2 \cap Y$ are disjoint subsets of Y, open in Y. Since (Y, T_Y) is T_1 and normal, it is T_4. ■

R A subspace of a normal space need not be normal.

Definition 2.8 — Completely normal space. A space X is completely normal if every pair of sets A, B satisfying $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ can be separated. That is there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Definition 2.9 — T_5-spaces or completely normal Hausdorff spaces. A space that is both Hausdorff and completely normal is a T_5 space.

2.5 Urysohn's Characterization of Normality

Theorem 2.17 — Urysohn Lemma. (Dugundji, 1966) The following two properties are equivalent:

1. X is T_4.
2. For each pair of disjoint closed sets, A, B in X, there exists a continuous $f : X \to \mathbb{R}$, called a Urysohn function for A, B, such that:
   (a) $0 \leq f(x) \leq 1$ for all $x \in X$
   (b) $f(a) = 0$ for all $a \in A$.
   (c) $f(b) = 0$ for all $b \in B$.

Corollary 2.18 (Dugundji, 1966) A necessary and sufficient condition for the existence of a Urysohn function satisfying $A = f^{-1}(0)$ is that A be a G_δ.

Corollary 2.19 (Dugundji, 1966) A necessary and suffiecient condition that there be a Urysohn function f with $A = f^{-1}(0), B = f^{-1}(1)$ is that both A and B be G_δ.

Definition 2.10 — T_6-spaces or perfectly normal Hausdorff spaces. A T_4 space in which each closed set is a G_δ is a T_6 space.

Theorem 2.20 Every T_6 space is a T_5 space.

2.6 Tietze's Characterization of Normality

Theorem 2.21 — H. Tietze Theorem. (Dugundji, 1966) The following two properties are equivalent:

1. X is a T_4-space
2. For every closed $A \subset X$, each continuous $f : A \to \mathbb{R}$ has a continuous $f : X \to \mathbb{R}$. Furthermore, if $|f(a)| < c$ on A, then F can be chosen so that $|F(x)| < c$ on X.

2.7 Completely Regular Spaces

Definition 2.11 — Completely regular space. (Dugundji, 1966) A space is completely regular if for each point $p \in X$ and closed A not containing p, there is a continuous $\varphi : X \to [0,1]$ such that $\varphi(p) = 1$ and $\varphi(a) = 0$ for each $a \in A$

Theorem 2.22 Every completely regular space is regular

Proof. Let F represent a closed subset of X and $a \in X$ a point which does not belong to F. By hypothesis, a continuous function

$$f : X \to [0,1]$$

exists, such that $f(F) = \{1\}$ and $f(a) = 0$. An interval $[0,1]$ is a Hausdorff space. Hence, two open disjoint subsets U_1 and U_2 of $[0,1]$ exists, such that

$$0 \in U_1 \quad \text{and} \quad 1 \in U_2.$$

Since f is continuous, $f^-1(U_1)$ and $f^-1(U_2)$ are open. These subsets are disjoint such that

$$a \in f^-1(U_1), \quad F \subset f^-1(U_2).$$

Hence, (X,T) is regular. ■

Definition 2.12 — Tychonoff space. A completely regular Hausdorff space is a Tychonoff space.

Theorem 2.23 (Dugundji, 1966) Every subspace of a Tychonoff space is Tychonoff.

References

Belleza, K. (2025). Continuous maps (general topology). [Last accessed 19 May 2025]. https://drive.google.com/file/d/1C4QofGsse0Id7WWCgmd424qTMKC51sfJ/view

Dugundji, J. (1966). *Topology.* Allyn; Bacon, Inc.

Holmes, R. R. (2008). *Introduction to topology*. https://api.semanticscholar.org/CorpusID:13883851

Lipschutz, S. (1965). *Schaum's outline of general topology*. McGraw-Hill.

Milewski, E. G. (1994). *Topology problem solver*. Research & Education Assoc.

Morris, S. A. (2020). *Topology without tears* [Version of June 28, 2020]. Self-published. http://www.topologywithouttears.net

Schechter, E. (1996). *Handbook of analysis and its foundations*. Academic Press.