# Lecture Notes on Borel Sets

## Math 6201 - Topology

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# Contents

# 1 Preliminaries

This section reviews some fundamental concepts needed to understand the basic parts on separation axioms on topology.

## 1.1 Sets

**Definition 1.1 — Subset.** (Lipschutz, 1965) A set $A$ is a subset of a set $B$ or, equivalently, $B$ is a superset of $A$, written $A \subset B$ or $B \supset A$ iff each element in $A$ also belongs to $B$; that is, if $x \in A$ implies $x \in B$.

**Definition 1.2 — Union.** (Lipschutz, 1965) The union of two sets $A$ and $B$, denoted by $A \cup B$, is the set of all elements which belong to $A$ or $B$, i.e., $A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$

**Definition 1.3 — Intersection.** (Lipschutz, 1965) The intersection of two sets $A$ and $B$, denoted by $A \cap B$, is the set of elements which belong to both $A$ and $B$, i.e., $A \cap B = \{x : x \in A \quad \text{and} \quad x \in B)$

**Definition 1.4 — Power Sets.** (Dugundji, 1966) Let $A$ be any set. Its power set $\mathscr{P}(A)$ is the set of all subsets of $A$.

**Definition 1.5 — Countable.** (Dugundji, 1966) A set $A$ is countable if it is finite or equivalent to the set $\mathbb{N}$ of counting numbers. If $A \equiv N$, then $A$ is called countably infinite or denumerable.

## 1.2 Functions, or Maps

**Definition 1.6 — Map.** (Dugundji, 1966) Let $X$ and $Y$ be two sets. A *map* $f : X \to Y$ *(or function with domain* $X$ *and range* $Y$*)* is a subset $f \subset X \times Y$ with the property: for each $x \in X$, there is one, and only one, $y \in Y$ satisfying $(x,y) \in f$.To denote $(x,y) \in f$, we write $y = f(x)$ and say that $y$ is the image of $x$ under $f$.

**Definition 1.7 — Surjective function.** (Holmes, 2008) Let $f : X \to Y$ be a function. We say that $f$ is surjective if for each $y \in Y$ , there exists $x \in X$ such that $f(x) = y$

**Definition 1.8 — Injective function.** (Holmes, 2008) Let $f : X \to Y$ be a function. We say that $f$ is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2(x_i \in X)$

**Definition 1.9 — Bijective function.** (Holmes, 2008) Let $f : X \to Y$ be a function. We say that $f$ is bijective if it is both injective and surjective.

## 1.3 Topology

**Definition 1.10 — Topology.** (Dugundji, 1966) Let $X$ be a set. A *topology* (or topological structure) in $X$ is a family $\tau$ of subsets of $X$ that satisfies:

(1). Each union of members of $\tau$ is also a member of $\tau$.

(2). Each *finite* intersection of members of $\tau$ is also a member of $\tau$.

(3). $\emptyset$ and $X$ are members of $\tau$.

**Example 1.1 — Discrete topology.** Let $X$ be any set and $\tau = \mathscr{P}(X)$. Then $\tau$ is a topology on $X$.

*Proof.*

1. Clearly, $X, \varnothing \in \tau$
2. Since every possible subset of $X$ is included in $\tau$, the union of any combination of these subsets will always result in another subset of $X$, which is already in $\tau$.
3. Since all subsets are open, the intersection of any finite number of them will still be a subset of $X$, hence in $\tau$.

Thus, $\tau$ is a topology on $X$. ■

■

**Example 1.2 — Indiscrete topology.** Let $X$ be any set and $\tau = \{X, \varnothing\}$. Then $\tau$ is a topology on $X$

*Proof.*

1. $X \in \tau$ and $\varnothing \in \tau$
2. $X \cup X = X \cup \varnothing = X \in \tau$
   $\varnothing \cup \varnothing = \varnothing \in \tau$
3. $X \cap X = X \in \tau$
   $X \cap \varnothing = \varnothing \cap \varnothing = \varnothing \in \tau$

Thus, $\tau$ is a topology on $X$. ■

■

**Definition 1.11 — Open sets.** (Morris, 2020) Let $(X, \tau)$ be a topological space. Then the members of $\tau$ are said to be *open sets*.

**Definition 1.12 — Closed sets.** (Morris, 2020) Let $(X, \tau)$ be a topological space. A subset $S$ of $X$ is said to be *closed set* in $(X, \tau)$ if its complement in $X$, namely $X \setminus S$, is open in $(X, \tau)$.

**Definition 1.13 — Neighborhood.** (Holmes, 2008) Let $x \in X$. Any open set containing $x$ is called a neighborhood of $x$.

## 1.4 $G_\delta$

**Definition 1.14** (Dugundji, 1966) A set $G$ is called $G_\delta$ if it is the intersection of at most countably many open sets.

## 1.5 Relativization

**Definition 1.15 — Subspace topology.** (Dugundji, 1966) Let $(X, \tau)$ be a topological space and $Y \subset X$. The induced topology $\tau_Y$ on $Y$ is $\{Y \cap U : U \in \tau\}$. The pair $(Y, \tau_Y)$ is called a subspace of $(X, \tau)$.

## 1.6 Continuous maps and Homeomorphisms

**Definition 1.16 — Continuous maps.** (Belleza, 2025) Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each open set in $Y$ is open in $X$. That is, $f^{-1}$ maps $\tau_Y \to \tau_X$.

■ **Example 1.3** Let $(X, \tau)$ be any topological space and $f : (X, \tau) \to (X, \tau)$ is defined by $f(x) = x$ for all $x \in X$. Then f is continuous. ■

*Proof.* To show that $f$ is continuous, we need to verify that the inverse image of every open set in $Y = (X, \tau)$ is open in $X = (X, \tau)$.

Let $O$ be an arbitrary open set in $Y$. By definition, $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$. Since $f(x) = x$, this simplifies to $f^{-1}(O) = \{x \in X \mid x \in O\} = O$. Since $O$ is open in $Y$, and $Y$ has the same topology as $X$, $O$ is also open in $X$. Thus, $f^{-1}(O) = O$ is open in $X$. Since the inverse image of every open set in $Y$ is open in $X$, the identity map $f$ is continuous. ■

**R** Not every identity function is continuous. To see this, let $X = \{1, 2, 3, 4\}$, and define two topologies on $X$, $\tau_1 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X\}$, $\tau_2 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{2\}, \{1, 2, 3\}, X\}$. Clearly, $\tau_1$ and $\tau_2$ are topologies on $X$. Then $i : (X, \tau_1) \to (X, \tau_2)$ defined by $i(x) = x$ for all $x \in X$ is not continuous, since there exists $\{2\} \in \tau_2$ such that $i^{-1}(\{2\}) = \{2 \notin \tau_1$.

**Definition 1.17 — Homeomorphism.** (Dugundji, 1966) A continuous bijective map $f : X \to Y$, such that $f^{-1} : Y \to X$ is also continuous, is called a homeomorphism (or a bicontinuous bijection) and denoted by $f : X \cong Y$. Two spaces $X, Y$ are homeomorphic, written $X \cong Y$, if there is a homeomorphism $f : X \cong Y$.

■ **Example 1.4** The identity map $1 : X \to X$ is a homeomorphism. ■

■ **Example 1.5** Two discrete spaces $X$ and $Y$ (similarly, for indiscrete spaces), are homeomorphic if and only if there is a one-to-one function on $X$ onto $Y$ . ■

*Proof.* In a discrete space, every subset is open. Let $f : X \to Y$ be a bijection (one-to-one and onto). We will show that $f$ is a homeomorphism. To show $f$ is continuous, the preimage of any open set in $Y$ must be open in $X$. Since $Y$ is discrete, every subset of $Y$ is open. Thus, for any open $V \subseteq Y$, $f^{-1}(V) \subseteq X$. Because $X$ is discrete, $f^{-1}(V)$ is open in $X$. Hence, $f$ is continuous. Similarly, $f^{-1} : Y \to X$ is a bijection. For any open $U \subseteq X$, $(f^{-1})^{-1}(U) = f(U) \subseteq Y$. Since $X$ is discrete, $U$ is open, and since $Y$ is discrete, $f(U)$ is open in $Y$. Thus, $f^{-1}$ is continuous. Hence, $f$ is a bijection, continuous, and its inverse $f^{-1}$ is also continuous. Therefore, $f$ is a homeomorphism, and $X \cong Y$. ■

**Definition 1.18 — Topological Invariant.** We call any property of spaces a topological invariant if whenever it is true for one space $X$, it is also true for every space homemorphic to $X$

## 1.7 $\sigma$-rings

**Definition 1.19** A nonempty family $\Sigma \subseteq \mathscr{P}(X)$ is called a $\sigma$-ring if

1. $A \in \Sigma \Rightarrow A^C \in \Sigma$,
2. $A_i \in \Sigma$ for $i = 1, 2, \cdots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

## 1.8 Projection Map

*This definition is found on page 11 of dugundji*

**Definition 1.20 — Projection map.** (Dugundji, 1966) For any sets $X$, $Y$, the map $p_1 : X \times Y \to X$ determined by $(x, y) \to x$ is called "projection onto the first coordinate"; $p_2 : X \times Y \to Y$, where $p_2(x, y) = y$ is "projection onto the second coordinate."

# 2 Cartesian product topology

*We form a product topology based on subbasis???*

**Definition 2.1 — Cylinder set.**

**Definition 2.2** (Dugundji, 1966) Let $\{Y_\alpha \mid \alpha \in \mathscr{A}\}$ be any family of topological spaces. For each $\alpha \in \mathscr{A}$, let $\mathscr{T}_\alpha$ be the topology for $Y_\alpha$. The cartesian product topology in $\prod_\alpha Y_\alpha$ is that having for subbasis all sets

$$\langle U_\beta \rangle = p_\beta^{-1}(U_\beta),$$

where $U_\beta$ ranges over all members of $\mathscr{T}_\beta$ and $\beta$ over all elements of $\mathscr{A}$.

If any one factor $Y_\alpha = \emptyset$, then the cartesian product is also empty and the topologies of the non-empty factors then play no role in determining that of the product; we wish to exclude such behavior, so we shall assume throughout this book that each factor of any cartesian product of topological spaces is non-empty.

*This comes from Ai but it checks out*

■ **Example 2.1** Subbasis for the Product Topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ Let's illustrate the subbasis for the product topology using the familiar example of $\mathbb{R}^2$.

Let $Y_1 = \mathbb{R}$ and $Y_2 = \mathbb{R}$, where both are equipped with the standard topology. In the standard topology on $\mathbb{R}$, open sets are unions of open intervals $(a, b)$.

The product space is $\mathbb{R}^2$, which is the Cartesian plane. Our index set here is $\mathscr{A} = \{1, 2\}$.

According to the definition, the subbasis for the product topology on $\prod_{\alpha \in \mathscr{A}} Y_\alpha$ (which is $\mathbb{R}^2$ in this case) consists of all sets of the form $p_\beta^{-1}(U_\beta)$, where $U_\beta$ is an open set in $Y_\beta$ and $\beta \in \mathscr{A}$.

Let's look at what these sets represent geometrically.

First, consider subbasis elements from $p_1^{-1}(U_1)$. Let $U_1$ be an open interval in $\mathbb{R}$, for example, $U_1 = (a,b)$ for some $a,b \in \mathbb{R}$ with $a < b$. The set $p_1^{-1}((a,b))$ consists of all points $(x,y) \in \mathbb{R}^2$ such that their first coordinate $x$ is in the interval $(a,b)$.

$$p_1^{-1}((a,b)) = \{(x,y) \in \mathbb{R}^2 \mid a < x < b\}$$

Geometrically, this represents an **infinite vertical strip** in the Cartesian plane, extending infinitely in the positive and negative $y$-directions, bounded by vertical lines $x = a$ and $x = b$.

Similarly, consider subbasis elements from $p_2^{-1}(U_2)$. Let $U_2$ be an open interval in $\mathbb{R}$, for example, $U_2 = (c,d)$ for some $c,d \in \mathbb{R}$ with $c < d$. The set $p_2^{-1}((c,d))$ consists of all points $(x,y) \in \mathbb{R}^2$ such that their second coordinate $y$ is in the interval $(c,d)$.

$$p_2^{-1}((c,d)) = \{(x,y) \in \mathbb{R}^2 \mid c < y < d\}$$

Geometrically, this represents an **infinite horizontal strip** in the Cartesian plane, extending infinitely in the positive and negative $x$-directions, bounded by horizontal lines $y = c$ and $y = d$.

Therefore, the subbasis $\mathscr{S}$ for the product topology on $\mathbb{R}^2$ is the collection of all such open vertical strips and all such open horizontal strips:

$$\mathscr{S} = \{p_1^{-1}(U_1) \mid U_1 \text{ is open in } \mathbb{R}\} \cup \{p_2^{-1}(U_2) \mid U_2 \text{ is open in } \mathbb{R}\}$$

**Forming a Basis from the Subbasis** The product topology is defined by taking the collection of all *finite intersections* of these subbasis elements to form a basis. Consider the intersection of a typical open vertical strip and a typical open horizontal strip:

$$p_1^{-1}((a,b)) \cap p_2^{-1}((c,d)) = \{(x,y) \in \mathbb{R}^2 \mid a < x < b \text{ and } c < y < d\}$$

This precise set is an **open rectangle** in $\mathbb{R}^2$, often denoted as $(a,b) \times (c,d)$.

Since the collection of all open rectangles forms a well-known basis for the standard topology on $\mathbb{R}^2$, this example clearly demonstrates how the subbasis for the product topology correctly generates the familiar "standard" or "Euclidean" topology on $\mathbb{R}^2$. This approach generalizes to any product of topological spaces, ensuring that the projection maps are continuous, which is a fundamental property of the product topology. ∎

## References

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