Lecture Notes on Borel Sets

Math 6201 - Topology

AY 2024-2025 Term 1

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(May 2025)

Contents

1 Preliminaries

This section reviews some fundamental concepts needed to understand the basic parts on separation axioms on topology.

1.1 Sets

Definition 1.1 — Subset. (Lipschutz, 1965) A set A is a subset of a set B or, equivalently, B is a superset of A, written $A \subset B$ or $B \supset A$ iff each element in A also belongs to B; that is, if $x \in A$ implies $x \in B$.

Definition 1.2 — Union. (Lipschutz, 1965) The union of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or B, i.e., $A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$

Definition 1.3 — Intersection. (Lipschutz, 1965) The intersection of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B, i.e., $A \cap B = \{x : x \in A \quad \text{and} \quad x \in B)$

Definition 1.4 — Power Sets. (Dugundji, 1966) Let A be any set. Its power set $\mathscr{P}(A)$ is the set of all subsets of A.

Definition 1.5 — Countable. (Dugundji, 1966) A set A is countable if it is finite or equivalent to the set $\mathbb{N}$ of counting numbers. If $A \equiv N$, then A is called countably infinite or denumerable.

1.2 Functions, or Maps

Definition 1.6 — Map. (Dugundji, 1966) Let X and Y be two sets. A *map* $f : X \to Y$ *(or function with domain X and range Y)* is a subset $f \subset X \times Y$ with the property: for each $x \in X$, there is one, and only one, $y \in Y$ satisfying $(x,y) \in f$.To denote $(x,y) \in f$, we write $y = f(x)$ and say that y is the image of x under f.

Definition 1.7 — Surjective function. (Holmes, 2008) Let $f : X \to Y$ be a function. We say that f is surjective if for each $y \in Y$, there exists $x \in X$ such that $f(x) = y$

Definition 1.8 — Injective function. (Holmes, 2008) Let $f : X \to Y$ be a function. We say that f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2 (x_i \in X)$

Definition 1.9 — Bijective function. (Holmes, 2008) Let $f : X \to Y$ be a function. We say that f is bijective if it is both injective and surjective.

1.3 Topology

Definition 1.10 — Topology. (Dugundji, 1966) Let X be a set. A *topology* (or topological structure) in X is a family τ of subsets of X that satisfies:

(1). Each union of members of τ is also a member of τ.

(2). Each *finite* intersection of members of τ is also a member of τ.

(3). $\emptyset$ and X are members of τ.

■ **Example 1.1 — Discrete topology.** Let X be any set and $\tau = \mathscr{P}(X)$. Then τ is a topology on X.

Proof.

1. Clearly, $X, \varnothing \in \tau$
2. Since every possible subset of X is included in τ, the union of any combination of these subsets will always result in another subset of X, which is already in τ.
3. Since all subsets are open, the intersection of any finite number of them will still be a subset of X, hence in τ.

Thus, τ is a topology on X. ■

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■ **Example 1.2 — Indiscrete topology.** Let X be any set and $\tau = \{X, \varnothing\}$. Then τ is a topology on X

Proof.

1. $X \in \tau$ and $\varnothing \in \tau$
2. $X \cup X = X \cup \varnothing = X \in \tau$
   $\varnothing \cup \varnothing = \varnothing \in \tau$
3. $X \cap X = X \in \tau$
   $X \cap \varnothing = \varnothing \cap \varnothing = \varnothing \in \tau$

Thus, τ is a topology on X. ■

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Definition 1.11 — Open sets. (Morris, 2020) Let (X, τ) be a topological space. Then the members of τ are said to be *open sets*.

Definition 1.12 — Closed sets. (Morris, 2020) Let (X, τ) be a topological space. A subset S of X is said to be *closed set* in (X, τ) if its complement in X, namely $X \setminus S$, is open in (X, τ).

Definition 1.13 — Neighborhood. (Holmes, 2008) Let $x \in X$. Any open set containing x is called a neighborhood of x.

1.4 G_δ

Definition 1.14 (Dugundji, 1966) A set G is called G_δ if it is the intersection of at most countably many open sets.

1.5 Relativization

Definition 1.15 — Subspace topology. (Dugundji, 1966) Let (X,τ) be a topological space and $Y \subset X$. The induced topology τ_Y on Y is $\{Y \cap U : U \in \tau\}$. The pair (Y,τ_Y) is called a subspace of (X,τ).

1.6 Continuous maps and Homeomorphisms

Definition 1.16 — Continuous maps. (Belleza, 2025) Let (X,τ_X) and (Y,τ_Y) be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each open set in Y is open in X. That is, f^{-1} maps $\tau_Y \to \tau_X$.

■ **Example 1.3** Let (X,τ) be any topological space and $f : (X,\tau) \to (X,\tau)$ is defined by $f(x) = x$ for all $x \in X$. Then f is continuous. ■

Proof. To show that f is continuous, we need to verify that the inverse image of every open set in $Y = (X,\tau)$ is open in $X = (X,\tau)$.

Let O be an arbitrary open set in Y. By definition, $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$. Since $f(x) = x$, this simplifies to $f^{-1}(O) = \{x \in X \mid x \in O\} = O$. Since O is open in Y, and Y has the same topology as X, O is also open in X. Thus, $f^{-1}(O) = O$ is open in X. Since the inverse image of every open set in Y is open in X, the identity map f is continuous. ■

R Not every identity function is continuous. To see this, let $X = \{1,2,3,4\}$, and define two topologies on X, $\tau_1 = \{\emptyset,\{1\},\{3\},\{1,3\},\{2,3\},\{1,2,3\},X\}$, $\tau_2 = \{\emptyset,\{1\},\{3\},\{1,3\},\{2\},\{1,2,3\},X\}$. Clearly, τ_1 and τ_2 are topologies on X. Then $i : (X,\tau_1) \to (X,\tau_2)$ defined by $i(x) = x$ for all $x \in X$ is not continuous, since there exists $\{2\} \in \tau_2$ such that $i^{-1}(\{2\}) = \{2 \notin \tau_1$.

Definition 1.17 — Homeomorphism. (Dugundji, 1966) A continuous bijective map $f : X \to Y$, such that $f^{-1} : Y \to X$ is also continuous, is called a homeomorphism (or a bicontinuous bijection) and denoted by $f : X \cong Y$. Two spaces X,Y are homeomorphic, written $X \cong Y$, if there is a homeomorphism $f : X \cong Y$.

■ **Example 1.4** The identity map $1 : X \to X$ is a homeomorphism. ■

■ **Example 1.5** Two discrete spaces X and Y (similarly, for indiscrete spaces), are homeomorphic if and only if there is a one-to-one function on X onto Y . ■

Proof. In a discrete space, every subset is open. Let $f : X \to Y$ be a bijection (one-to-one and onto). We will show that f is a homeomorphism. To show f is continuous, the preimage of any open set in Y must be open in X. Since Y is discrete, every subset of Y is open. Thus, for any open $V \subseteq Y$, $f^{-1}(V) \subseteq X$. Because X is discrete, $f^{-1}(V)$ is open in X. Hence, f is continuous. Similarly, $f^{-1} : Y \to X$ is a bijection. For any open $U \subseteq X$, $(f^{-1})^{-1}(U) = f(U) \subseteq Y$. Since X is discrete, U is open, and since Y is discrete, $f(U)$ is open in Y. Thus, f^{-1} is continuous. Hence, f is a bijection, continuous, and its inverse f^{-1} is also continuous. Therefore, f is a homeomorphism, and $X \cong Y$. ■

Definition 1.18 — Topological Invariant. We call any property of spaces a topological invariant if whenever it is true for one space X, it is also true for every space homemorphic to X

1.7 σ-rings

Definition 1.19 A nonempty family $\Sigma \subseteq \mathscr{P}(X)$ is called a σ-ring if

1. $A \in \Sigma \Rightarrow A^C \in \Sigma$,
2. $A_i \in \Sigma$ for $i = 1, 2, \cdots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

2 Borel Sets

A Borel set is any subset of a topological space that can be formed from its open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement. Borel sets are named after Émile Borel.

2.1 Generating a Borel Set

Theorem 2.1 (Dugundji, 1966)

There always exists a unique smallest σ-ring $\mathscr{B}$ containing the topology $\mathscr{T}$ of X. $\mathscr{B}$ is called the family of Borel sets in X, and $\aleph(\mathscr{B}) \leq \aleph(\mathscr{T})^*$. Furthermore:

1. The countable union, countable intersection, and the difference of Borel sets is a Borel set.
2. Each F_σ and each G_δ is a Borel set.

Proof. Observing that $\mathscr{P}(X)$ is a σ-ring containing $\mathscr{T}$, and that the intersection of any family of σ-rings is also a σ-ring, we define $\mathscr{B}$ to be the intersection of all σ-rings containing $\mathscr{T}$. Since only two operations are involved, the estimate of $\aleph(\mathscr{B})$ follows from II, 9.4. To establish (1), we need only verify preservation under intersection, and this follows from

$$\complement \bigcap_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} \complement B_i$$

where $B_i \in \mathscr{B}, i \in \mathbb{Z}^+$. (2) is trivial. ■

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