

# Lecture Notes on Separation Axioms

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## 1 Preliminaries

This section reviews some fundamental concepts needed to understand the basic parts on separation axioms on topology.

### 1.1 Sets

**Definition 1.1 — Subset.** (lipschutz1965) A set  $A$  is a subset of a set  $B$  or, equivalently,  $B$  is a superset of  $A$ , written  $A \subset B$  or  $B \supset A$  iff each element in  $A$  also belongs to  $B$ ; that is, if  $x \in A$  implies  $x \in B$ .

**Definition 1.2 — Union.** (lipschutz1965) The union of two sets  $A$  and  $B$ , denoted by  $A \cup B$ , is the set of all elements which belong to  $A$  or  $B$ , i.e.,  $A \cup B = \{x : x \in A \text{ or } x \in B\}$

**Definition 1.3 — Intersection.** (lipschutz1965) The intersection of two sets  $A$  and  $B$ , denoted by  $A \cap B$ , is the set of elements which belong to both  $A$  and  $B$ , i.e.,  $A \cap B = \{x : x \in A \text{ and } x \in B\}$

**Definition 1.4 — Power Sets.** (Dugundji1966) Let  $A$  be any set. Its power set  $\mathcal{P}(A)$  is the set of all subsets of  $A$ .

**Definition 1.5 — Countable.** (Dugundji1966) A set  $A$  is countable if it is finite or equivalent to the set  $\mathbb{N}$  of counting numbers. If  $A \equiv \mathbb{N}$ , then  $A$  is called countably infinite or denumerable.

### 1.2 Functions, or Maps

**Definition 1.6 — Map.** (Dugundji1966) Let  $X$  and  $Y$  be two sets. A *map*  $f : X \rightarrow Y$  (or *function with domain  $X$  and range  $Y$* ) is a subset  $f \subset X \times Y$  with the property: for each  $x \in X$ , there is one, and only one,  $y \in Y$  satisfying  $(x, y) \in f$ . To denote  $(x, y) \in f$ , we write  $y = f(x)$  and say that  $y$  is the image of  $x$  under  $f$ .

**Definition 1.7 — Surjective function.** (holmes\_topology) Let  $f : X \rightarrow Y$  be a function. We say that  $f$  is surjective if for each  $y \in Y$ , there exists  $x \in X$  such that  $f(x) = y$

**Definition 1.8 — Injective function.** (holmes\_topology) Let  $f : X \rightarrow Y$  be a function. We say that  $f$  is injective if  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$  ( $x_i \in X$ )

**Definition 1.9 — Bijective function.** (holmes\_topology) Let  $f : X \rightarrow Y$  be a function. We say that  $f$  is bijective if it is both injective and surjective.

### 1.3 Topology

**Definition 1.10 — Topology.** (Dugundji1966) Let  $X$  be a set. A *topology* (or topological structure) in  $X$  is a family  $\tau$  of subsets of  $X$  that satisfies:

- (1). Each union of members of  $\tau$  is also a member of  $\tau$ .
- (2). Each *finite* intersection of members of  $\tau$  is also a member of  $\tau$ .

(3).  $\emptyset$  and  $X$  are members of  $\tau$ .

■ **Example 1.1 — Discrete topology.** Let  $X$  be any set and  $\tau = \mathcal{P}(X)$ . Then  $\tau$  is a topology on  $X$ .

*Proof.*

1. Clearly,  $X, \emptyset \in \tau$
2. Since every possible subset of  $X$  is included in  $\tau$ , the union of any combination of these subsets will always result in another subset of  $X$ , which is already in  $\tau$ .
3. Since all subsets are open, the intersection of any finite number of them will still be a subset of  $X$ , hence in  $\tau$ .

Thus,  $\tau$  is a topology on  $X$ . ■

■ **Example 1.2 — Indiscrete topology.** Let  $X$  be any set and  $\tau = \{X, \emptyset\}$ . Then  $\tau$  is a topology on  $X$

*Proof.*

1.  $X \in \tau$  and  $\emptyset \in \tau$
2.  $X \cup X = X \cup \emptyset = X \in \tau$   
 $\emptyset \cup \emptyset = \emptyset \in \tau$
3.  $X \cap X = X \in \tau$   
 $X \cap \emptyset = \emptyset \cap \emptyset = \emptyset \in \tau$

Thus,  $\tau$  is a topology on  $X$ . ■

■ **Definition 1.11 — Open sets.** (morris2020topology) Let  $(X, \tau)$  be a topological space. Then the members of  $\tau$  are said to be *open sets*.

■ **Definition 1.12 — Closed sets.** (morris2020topology) Let  $(X, \tau)$  be a topological space. A subset  $S$  of  $X$  is said to be *closed set* in  $(X, \tau)$  if its complement in  $X$ , namely  $X \setminus S$ , is open in  $(X, \tau)$ .

■ **Definition 1.13 — Neighborhood.** (holmes\_topology) Let  $x \in X$ . Any open set containing  $x$  is called a neighborhood of  $x$ .

## 1.4 $G_\delta$

■ **Definition 1.14 (Dugundji1966)** A set  $G$  is called  $G_\delta$  if it is the intersection of at most countably many open sets.

## 1.5 Relativization

**Definition 1.15 — Subspace topology.** (Dugundji1966) Let  $(X, \tau)$  be a topological space and  $Y \subset X$ . The induced topology  $\tau_Y$  on  $Y$  is  $\{Y \cap U : U \in \tau\}$ . The pair  $(Y, \tau_Y)$  is called a subspace of  $(X, \tau)$ .

## 1.6 Continuous maps and Homeomorphisms

**Definition 1.16 — Continuous maps.** (BellezaCM2025) Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. A map  $f : X \rightarrow Y$  is called *continuous* if the inverse image of each open set in  $Y$  is open in  $X$ . That is,  $f^{-1}$  maps  $\tau_Y \rightarrow \tau_X$ .

■ **Example 1.3** Let  $(X, \tau)$  be any topological space and  $f : (X, \tau) \rightarrow (X, \tau)$  is defined by  $f(x) = x$  for all  $x \in X$ . Then  $f$  is continuous. ■

*Proof.* To show that  $f$  is continuous, we need to verify that the inverse image of every open set in  $Y = (X, \tau)$  is open in  $X = (X, \tau)$ .

Let  $O$  be an arbitrary open set in  $Y$ . By definition,  $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ . Since  $f(x) = x$ , this simplifies to  $f^{-1}(O) = \{x \in X \mid x \in O\} = O$ . Since  $O$  is open in  $Y$ , and  $Y$  has the same topology as  $X$ ,  $O$  is also open in  $X$ . Thus,  $f^{-1}(O) = O$  is open in  $X$ . Since the inverse image of every open set in  $Y$  is open in  $X$ , the identity map  $f$  is continuous. ■

Ⓡ Not every identity function is continuous. To see this, let  $X = \{1, 2, 3, 4\}$ , and define two topologies on  $X$ ,  $\tau_1 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X\}$ ,  $\tau_2 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{2\}, \{1, 2, 3\}, X\}$ . Clearly,  $\tau_1$  and  $\tau_2$  are topologies on  $X$ . Then  $i : (X, \tau_1) \rightarrow (X, \tau_2)$  defined by  $i(x) = x$  for all  $x \in X$  is not continuous, since there exists  $\{2\} \in \tau_2$  such that  $i^{-1}(\{2\}) = \{2\} \notin \tau_1$ .

**Definition 1.17 — Homeomorphism.** (Dugundji1966) A continuous bijective map  $f : X \rightarrow Y$ , such that  $f^{-1} : Y \rightarrow X$  is also continuous, is called a homeomorphism (or a bicontinuous bijection) and denoted by  $f : X \cong Y$ . Two spaces  $X, Y$  are homeomorphic, written  $X \cong Y$ , if there is a homeomorphism  $f : X \cong Y$ .

■ **Example 1.4** The identity map  $1 : X \rightarrow X$  is a homeomorphism. ■

■ **Example 1.5** Two discrete spaces  $X$  and  $Y$  (similarly, for indiscrete spaces), are homeomorphic if and only if there is a one-to-one function on  $X$  onto  $Y$ . ■

*Proof.* In a discrete space, every subset is open. Let  $f : X \rightarrow Y$  be a bijection (one-to-one and onto). We will show that  $f$  is a homeomorphism. To show  $f$  is continuous, the preimage of any open set in  $Y$  must be open in  $X$ . Since  $Y$  is discrete, every subset of  $Y$  is open. Thus, for any open  $V \subseteq Y$ ,  $f^{-1}(V) \subseteq X$ . Because  $X$  is discrete,  $f^{-1}(V)$  is open in  $X$ . Hence,  $f$  is continuous. Similarly,  $f^{-1} : Y \rightarrow X$  is a bijection. For any open  $U \subseteq X$ ,  $(f^{-1})^{-1}(U) = f(U) \subseteq Y$ . Since  $X$  is discrete,  $U$  is open, and since  $Y$  is discrete,  $f(U)$  is open in  $Y$ . Thus,  $f^{-1}$  is continuous. Hence,  $f$  is a bijection, continuous, and its inverse  $f^{-1}$  is also continuous. Therefore,  $f$  is a homeomorphism, and  $X \cong Y$ . ■

**Definition 1.18 — Topological Invariant.** We call any property of spaces a topological invariant if whenever it is true for one space  $X$ , it is also true for every space homeomorphic to  $X$

## 2 Separation Axioms

Separation axioms are a family of topological invariants that give us new ways of distinguishing between various spaces. The idea is to look how open sets in a space can be used to create “buffer zones” separating pairs of points and closed sets. Separations axioms are denoted by  $T_0, T_1, T_2$ , etc., where  $T$  comes from the German word Trennungsaxiom, which just means “separation axiom” (schechter1996handbook).

### 2.1 $T_0$ and $T_1$ space

**Definition 2.1 —  $T_0$  space or Kolmogorov Space.** (milewski1994topology) The space  $(X, \mathcal{T})$  is said to be a  $T_0$ -space, if for any two distinct  $a, b \in X$ , there is a neighborhood of at least one, which does not contain the other

■ **Example 2.1** Let  $X = \{a, b, c\}$  with topology  $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$  defined on  $X$ , then  $(X, \tau)$  is a  $T_0$  space ■

*Proof.*

1. for  $a$  and  $b$ , there exists an open set  $\{a\}$  such that  $a \in \{a\}$  and  $b \notin \{a\}$
2. for  $a$  and  $c$ , there exists an open set  $\{a\}$  such that  $a \in \{a\}$  and  $c \notin \{a\}$
3. for  $b$  and  $c$ , there exists an open set  $\{b\}$  such that  $b \in \{b\}$  and  $c \notin \{b\}$

■

■ **Example 2.2** The space  $X = \{a, b\}$  with the indiscrete topology is not a  $T_0$  space ■

*Proof.* The two distinct points  $a$  and  $b$  in  $X$  is contained only in the open set  $X$ . Thus, it is not a  $T_0$  space. ■

**Definition 2.2 —  $T_1$ -space or Fréchet Space .** (milewski1994topology) A topological space  $(X, \mathcal{T})$  is called a  $T_1$ -space, if every single element set is closed, that is,  $\forall a \in X, \overline{\{a\}} = \{a\}$ .

**Theorem 2.1** A topological space  $(X, \tau)$  is a  $T_1$  space if and only if for any pair of distinct points  $a, b \in X$ , the open sets  $G, H \in \tau$  exist, such that  $a \in G, b \notin G$  and  $b \in H, a \notin H$

*Proof.* Suppose  $X$  is a  $T_1$ -space. Then, for any  $x \in X$ ,  $\{x\}$  is a closed set. Let  $a, b \in X$  and  $a \neq b$ . The sets  $X - \{a\}$  and  $X - \{b\}$  are open, and

$$\begin{aligned} a \in X - \{b\} \quad \text{and} \quad b \notin X - \{b\} \\ b \in X - \{a\} \quad \text{and} \quad a \notin X - \{a\}. \end{aligned}$$

Conversely, suppose  $x \in X$ . We shall show that  $\{x\}$  is closed, i.e.,  $X - \{x\}$  is open. Let  $y \in X - \{x\}$ , then  $y \neq x$  and an open set  $H_y$  exists, such that

$$y \in H_y \quad \text{and} \quad x \notin H_y.$$

Thus,

$$y \in H_y \subseteq X - \{x\} \quad \text{and} \quad X - \{x\} = \bigcup_{y \neq x} H_y.$$

Since all  $H_y$  are open sets,  $X - \{x\}$  is open and  $\{x\}$  is closed,  $\{x\} = \overline{\{x\}}$ . ■

■ **Example 2.3** Consider the set  $X = \{a, b, c\}$  with the cofinite topology,

$$\tau = \{X, \emptyset, a, b, c, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

Verify that  $(X, \tau)$  is a  $T_1$  space. ■

*Proof.* The complement of  $\{a\}$  is  $\{b, c\}$ , which is open (its complement  $\{a\}$  is finite). Similarly,  $\{b\}^c = \{a, c\}$  and  $\{c\}^c = \{a, b\}$  are open. By Definition 2.2, since every singleton is closed,  $X$  is  $T_1$ . For  $a$  and  $b$ , there exist open sets  $\{a\}$  and  $\{b\}$  such that  $a \in \{a\}$ ,  $b \notin \{a\}$  and  $b \in \{b\}$ ,  $a \notin \{b\}$ . This satisfies the condition in Theorem 2.1, confirming  $X$  is  $T_1$ .

Thus,  $(X, \tau)$  is a  $T_1$  space. ■

■ **Example 2.4** The Sierpinski Space is  $T_0$  but not  $T_1$  ■

*Proof.* Recall that a *Sierpiński space* is the topological space  $X = \{x, y\}$  with the topology given by  $\{X, \{x\}, \emptyset\}$ . It is  $T_0$  because for  $x$  and  $y$  the open set  $\{x\}$  contains  $x$  but not  $y$ . It is not  $T_1$  because every open set  $U$  containing  $y$  (which is only  $X$ ) contains  $x$ . ■

**Theorem 2.2 (milewski1994topology)** A  $T_1$  space is also a  $T_0$  space.

*Proof.* Let  $X$  be a  $T_1$  space then clearly from its definition it follows that it is also a  $T_0$  space. Since with any pair  $a, b \in X$  there exist an open set  $G$  with  $a \in G$  and  $b \notin G$ . ■

**Theorem 2.3 (milewski1994topology)** If  $(X, \tau)$  and  $(Y, \tau')$  are homeomorphic and  $(X, \tau)$  is a  $T_1$ -space (or  $T_0$ ) then so is  $(Y, \tau')$

*Proof.* Let  $f$  denote a homeomorphism

$$f : X \rightarrow Y$$

and  $X$  be a  $T_1$ -space. A space  $(X, \tau)$  is  $T_1$ , if and only if every one-point subset of  $X$  is closed. Let  $y$  represent any point of  $Y$ ,  $y \in Y$ . The set  $f^{-1}(y)$  is a one-point subset of  $X$  and since  $X$  is  $T_1$ , the set  $\{f^{-1}(y)\}$  is closed.

Since  $f : X \rightarrow Y$  is a homeomorphism, it maps closed sets into closed sets. Therefore, for any  $y \in Y$

$$\{y\} = \overline{\{y\}}.$$

Thus,  $(Y, \tau)$  is a  $T_1$ -space. Similarl proof can be done to when if  $X$  is  $T_0$ , then so is  $Y$ . ■

## 2.2 Hausdorff spaces

**Definition 2.3 —  $T_2$ -space or Hausdorff space.** (Dugundji1966) A space  $X$  is Hausdorff (or separated) if each two distinct points have nonintersecting nbds, that is, whenever,  $p \neq q$  there are nbds  $U(p), V(q)$  such that  $U \cap V = \emptyset$ .

■ **Example 2.5** The real line  $\mathbb{R}$  with the standard topology is a Hausdorff space. ■

*Proof.* For any two distinct points  $x, y \in \mathbb{R}$ , let  $d = |x - y| > 0$ . Then the open intervals

$$U = \left(x - \frac{d}{2}, x + \frac{d}{2}\right) \quad \text{and} \quad V = \left(y - \frac{d}{2}, y + \frac{d}{2}\right)$$

are disjoint neighborhoods of  $x$  and  $y$ , respectively. Thus,  $\mathbb{R}$  is Hausdorff. ■

**Theorem 2.4 (milewski1994topology)** Each Hausdorff space is a  $T_1$  space.

*Proof.* Suppose  $X$  is a Hausdorff space. Then from the definition, for two distinct points  $x$  and  $y$ , there exist two open sets  $U$  and  $V$  such that  $x \in U$ ,  $y \in V$  and  $U \cap V = \emptyset$ . Thus,  $x \notin V$  and  $y \notin U$ , by Theorem 2.1,  $X$  is also a  $T_1$  space. ■

**Theorem 2.5 (milewski1994topology)** Each Hausdorff space is a  $T_0$  space

*Proof.* Suppose  $X$  is a Hausdorff space. Then from Theorem 2.4 and Theorem 2.2, it is also a  $T_0$  space. ■

**Theorem 2.6 (Dugundji1966)** The following three properties are equivalent

1.  $X$  is Hausdorff.
2. Let  $p \in X$ . For each  $q \neq p$ , there is a nbd  $U(p)$  such that  $q \notin \overline{U(p)}$ .
3. For each  $p \in X$ ,  $\bigcap \{\overline{U} \mid U \text{ is a nbd of } p\} = \{p\}$ .

**Theorem 2.7** Every subspace of a Hausdorff space is Hausdorff

**Theorem 2.8** If  $(X, \tau)$  and  $(Y, \tau')$  are homeomorphic and  $(X, \tau)$  is a  $T_2$ -space then so is  $(Y, \tau')$



*Proof.* Let  $f$  denote a homeomorphism,  $f : X \rightarrow Y$  and  $X$  be a  $T_2$ -space. Now  $f$  is a continuous bijective map hence two distinct points  $x_1, x_2$  of  $X$  exist such that  $f^{-1}(y_1) = x_1, f^{-1}(y_2) = x_2$ .  $(X, \tau)$  is a Hausdorff space, therefore, there are two open sets  $U_1, U_2 \subset X$ , such that

$$x_1 \in U_1, \quad x_2 \in U_2, \quad U_1 \cap U_2 = \emptyset.$$

Since  $f$  is bijective,

$$\begin{aligned} f(U_1) &\subset Y, \quad f(U_2) \subset Y \\ f(U_1) \cap f(U_2) &= \emptyset \end{aligned}$$

Now, since  $f^{-1}$  is continuous, the function  $(f^{-1})^{-1} = f$  maps open sets into open sets. Hence,  $f(U_1), f(U_2) \in T'$  are open sets.

$$y_1 \in f(U_1), \quad y_2 \in f(U_2)$$

We conclude that  $(Y, \tau')$  is a  $T_2$ -space. If two spaces are homeomorphic and one of them is a  $T_2$ -space, then so is the other. ■

## 2.3 Regular Spaces

**Definition 2.4 — Regular space. (milewski1994topology)** A topological space  $(X, \mathcal{T})$ , is said to be regular if, given any closed subset  $F \subset X$  and any point  $x \in X$ , such that  $x \notin F$ , there are open sets  $U$  and  $V$ , such that

$$F \subset U, \quad x \in V, \quad \text{and} \quad U \cap V = \emptyset$$

**Definition 2.5 —  $T_3$ -space or Regular Hausdorff space.** A space is a  $T_3$ -space or Regular Hausdorff space if it is both a Hausdorff space and a regular space.

Note that some authors switch the definition of "Regular" and " $T_3$ ". Some also defines them equivalently, such as **Dugundji1966**<empty citation>. Some also defined  $T_3$  by the theorems below. For this lecture notes, we use the definitions that was stated above.

**R** A regular space need not be a  $T_1$ -space.

■ **Example 2.6** Consider the topology  $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$  on the set  $X = \{a, b, c\}$ . Observe that the closed subsets of  $X$  are also  $X, \emptyset, \{a\}$  and  $\{b, c\}$  and that  $(X, \tau)$  does satisfy Definition 2.4. On the other hand,  $(X, \tau)$  is not a  $T_1$ -space since there are single element sets, e.g.  $\{b\}$ , which are not closed. ■

**Theorem 2.9** A space is  $T_3$  if and only if it is both regular and  $T_0$

*Proof.* Suppose a space is  $T_3$  then by the definition, it is both regular and  $T_2$ . By Theorem 2.5, it is also  $T_0$ . For the converse, suppose a space is both a regular and  $T_0$ . Now let  $a, b \in X$  represents distinct points. Since the space is  $T_0$ , there is a neighborhood of at least one, which does not contain the other. Thus, let  $a \notin U(b)$  then  $a \notin \overline{U(b)}$ . By regularity, there

exist open sets  $G$  and  $V$  s.t  $\overline{U(b)} \subset G$  and  $a \in V$  where  $G \cap V = \emptyset$ . This implies that the space is also Hausdorff and hence also a  $T_3$  space. ■

**Theorem 2.10** A space is  $T_3$  if and only if it is both regular and  $T_1$

**Theorem 2.11 (Dugundji1966)** The following three properties are equivalent

1.  $X$  is a  $T_3$  space
2. For each  $x \in X$  and nbd  $U$  of  $x$ , there exists a nbd  $V$  of  $x$  with  $X \in V \subset \bar{V} \subset U$ .
3. For each  $x \in X$  and closed  $A$  not containing  $x$ , there is a nbd  $V$  of  $x$  with  $\bar{V} \cap A = \emptyset$ .

**Theorem 2.12** Every subspace of a regular space is regular

*Proof.* Let  $Y$  be a regular space and  $X \subset Y$  a subspace. Let  $B \subset X$  be closed in  $X$ , and let  $x_0 \in X \setminus B$ . Since  $B$  is closed in  $X$ , there exists a closed set  $A \subset Y$  such that  $B = X \cap A$ . Note that  $x_0 \notin A$  because  $x_0 \in X \setminus B$ . By the regularity of  $Y$ , there exist disjoint open sets  $U$  and  $V$  in  $Y$  such that  $x_0 \in U$  and  $A \subset V$ . The sets  $U \cap X$  and  $V \cap X$  are open in  $X$  (by the definition of the subspace topology),  $x_0 \in U \cap X$ , and  $B = X \cap A \subset V \cap X$ . Since  $U$  and  $V$  are disjoint in  $Y$ , their intersections with  $X$  are also disjoint in  $X$ .

Thus,  $X$  is regular. ■

**Theorem 2.13** Any subspace of a  $T_3$ -space is a  $T_3$ -space

*Proof.* A  $T_3$ -space is a regular Hausdorff space. By Theorem 2.12 and theorem 2.7, its subspace is also a regular Hausdorff space or a  $T_3$ -space. ■

## 2.4 Normal Spaces

**Definition 2.6 — Normal Space.** (milewski1994topology) A topological space  $(X, T)$  is said to be normal if, given any two disjoint closed sets  $F_1$  and  $F_2$  in  $X$ , there are disjoint open sets  $U$  and  $V$ , such that

$$F_1 \subset U \quad \text{and} \quad F_2 \subset V$$

**Definition 2.7 —  $T_4$ -space of Normal Hausdorff space.** A space is a  $T_4$ -space or Normal Hausdorff space if it is both a Hausdorff space and a normal space

Similarly with regular spaces, some authors switch the definition of "Normal" and " $T_4$ ". Some also defines them equivalently, such as **Dugundji1966**<empty citation>. For this lecture notes, we use the definitions that are stated above.

■ **Example 2.7** Discrete spaces are  $T_4$  spaces ■

*Proof.* For any two distinct points  $x, y \in X$ , the singleton sets  $\{x\}$  and  $\{y\}$  are open and disjoint. Thus,  $X$  is Hausdorff. Now, Let  $F_1$  and  $F_2$  be disjoint closed sets in  $X$ . In the discrete topology, every set is open, so  $F_1$  and  $F_2$  themselves are open. Thus, we can take

$U = F_1$  and  $V = F_2$  as disjoint open sets containing  $F_1$  and  $F_2$  respectively. This shows that  $X$  is normal. Since  $X$  is both Hausdorff and normal, it is a  $T_4$ -space. ■

■ **Example 2.8** Any space  $(X, \tau)$ , containing more than one point with the indiscrete topology is Normal. ■

*Proof.* The indiscrete topology consists of two sets  $X$  and  $\phi$ .

$$T = \{X, \phi\}.$$

Hence, the only closed sets are  $X$  and  $\phi$  because  $X - \phi = X$  and  $X - X = \phi$ . Thus, there are no non-empty disjoint closed subsets of  $X$ . The space is normal. ■

**Theorem 2.14** Every  $T_4$ -spaces are  $T_3$ -spaces

*Proof.* Let  $(X, T)$  denote a  $T_4$ -space. Hence,  $(X, T)$  is normal and  $T_1$ . Suppose  $F$  is a closed subset of  $X$  and  $a \in F$ . Since  $(X, T)$  is  $T_1$ , the singleton set  $\{a\}$  is closed. Sets  $F$  and  $\{a\}$  are closed and disjoint. Since  $(X, T)$  is normal, the open sets  $U_1$  and  $U_2$  exist, such that

$$\{a\} \subset U_1, \quad F \subset U_2, \quad U_1 \cap U_2 = \emptyset$$

. Therefore  $(X, T)$  is regular and  $T_3$  ■

**Theorem 2.15** The following four properties are equivalent

1.  $X$  is  $T_4$ .
2. For each closed  $A$  and open  $U \supset A$  there is an open  $V$  with  $A \subset V \subset \bar{V} \subset U$ .
3. For each pair of disjoint closed sets  $A, B$ , there is an open  $U$  with  $A \subset U$  and  $\bar{U} \cap B = \emptyset$ .
4. Each pair of disjoint closed sets have nbds whose closures do not intersect.

**Theorem 2.16** A closed subspace of a  $T_4$  space is  $T_4$ .

*Proof.* Let  $X$  be a  $T_4$  space and  $Y$  be a closed subspace of  $X$ . Since every subspace of a  $T_1$ -space is  $T_1$  and  $X$  is  $T_1$  also,  $Y$  is a  $T_1$ -space. Since  $Y$  is closed, a subset  $F$  of  $Y$  is closed in  $Y$ , if and only if  $F$  is closed in  $X$ . Hence, if  $F_1$  and  $F_2$  are disjoint closed subsets of  $Y$ , they are also disjoint closed subsets of  $X$ .

Thus, the open sets  $U_1$  and  $U_2$  exist, such that

$$F_1 \subset U_1, \quad F_2 \subset U_2 \text{ and } U_1 \cap U_2 = \emptyset.$$

Then

$$F_1 \subset U_1 \cap Y, \quad F_2 \subset U_2 \cap Y,$$

and  $U_1 \cap Y$  and  $U_2 \cap Y$  are disjoint subsets of  $Y$ , open in  $Y$ . Since  $(Y, T_Y)$  is  $T_1$  and normal, it is  $T_4$ . ■

**R** A subspace of a normal space need not be normal.

**Definition 2.8 — Completely normal space.** A space  $X$  is completely normal if every pair of sets  $A, B$  satisfying  $\bar{A} \cap B = A \cap \bar{B} = \emptyset$  can be separated. That is there exist disjoint open sets  $U$  and  $V$  such that  $A \subseteq U$  and  $B \subseteq V$ .

**Definition 2.9 —  $T_5$ -spaces or completely normal Hausdorff spaces.** A space that is both Hausdorff and completely normal is a  $T_5$  space.

## 2.5 Urysohn's Characterization of Normality

**Theorem 2.17 — Urysohn Lemma. (Dugundji1966)** The following two properties are equivalent:

1.  $X$  is  $T_4$ .
2. For each pair of disjoint closed sets,  $A, B$  in  $X$ , there exists a continuous  $f : X \rightarrow \mathbb{R}$ , called a Urysohn function for  $A, B$ , such that:
  - (a)  $0 \leq f(x) \leq 1$  for all  $x \in X$
  - (b)  $f(a) = 0$  for all  $a \in A$ .
  - (c)  $f(b) = 1$  for all  $b \in B$ .

**Corollary 2.18 (Dugundji1966)** A necessary and sufficient condition for the existence of a Urysohn function satisfying  $A = f^{-1}(0)$  is that  $A$  be a  $G_\delta$ .

**Corollary 2.19 (Dugundji1966)** A necessary and sufficient condition that there be a Urysohn function  $f$  with  $A = f^{-1}(0), B = f^{-1}(1)$  is that both  $A$  and  $B$  be  $G_\delta$ .

**Definition 2.10 —  $T_6$ -spaces or perfectly normal Hausdorff spaces.** A  $T_4$  space in which each closed set is a  $G_\delta$  is a  $T_6$  space.

**Theorem 2.20** Every  $T_6$  space is a  $T_5$  space.

## 2.6 Tietze's Characterization of Normality

**Theorem 2.21 — H. Tietze Theorem. (Dugundji1966)** The following two properties are equivalent:

1.  $X$  is a  $T_4$ -space
2. For every closed  $A \subset X$ , each continuous  $f : A \rightarrow \mathbb{R}$  has a continuous  $f : X \rightarrow \mathbb{R}$ . Furthermore, if  $|f(a)| < c$  on  $A$ , then  $F$  can be chosen so that  $|F(x)| < c$  on  $X$ .

## 2.7 Completely Regular Spaces

**Definition 2.11 — Completely regular space.** (Dugundji1966) A space is completely regular if for each point  $p \in X$  and closed  $A$  not containing  $p$ , there is a continuous  $\varphi : X \rightarrow [0, 1]$  such that  $\varphi(p) = 1$  and  $\varphi(a) = 0$  for each  $a \in A$

**Theorem 2.22** Every completely regular space is regular

*Proof.* Let  $F$  represent a closed subset of  $X$  and  $a \in X$  a point which does not belong to  $F$ . By hypothesis, a continuous function

$$f : X \rightarrow [0, 1]$$

exists, such that  $f(F) = \{1\}$  and  $f(a) = 0$ . An interval  $[0, 1]$  is a Hausdorff space. Hence, two open disjoint subsets  $U_1$  and  $U_2$  of  $[0, 1]$  exists, such that

$$0 \in U_1 \quad \text{and} \quad 1 \in U_2.$$

Since  $f$  is continuous,  $f^{-1}(U_1)$  and  $f^{-1}(U_2)$  are open. These subsets are disjoint such that

$$a \in f^{-1}(U_1), \quad F \subset f^{-1}(U_2).$$

Hence,  $(X, T)$  is regular. ■

**Definition 2.12 — Tychonoff space.** A completely regular Hausdorff space is a Tychonoff space.

**Theorem 2.23** (Dugundji1966) Every subspace of a Tychonoff space is Tychonoff.