Measuring Boltzmann's Constant through Brownian Motion of Latex Spheres

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Abstract

We calculated the Boltzmann constant by tracking 1.02 μm latex beads suspended in water exhibiting Brownian motion with a microscope and a video camera. A total of 97 particles were tracked and the slope of their mean squared distance as a function of time was used to derive Boltzmann's constant by the Einstein fluctuation-dissipation relation. We found Boltzmann's constant to be $k_b = 1.37 \pm 0.02 \times 10^{-23}$ J $\cdot$ K^{-1}.

1 Introduction

1.1 Historical Background

Brownian Motion was discovered by the Botanist Robert Brown in 1827 when he noticed that little specs of pollen would exhibit seemingly random motion in a solution, even though he knew pollen to be non-living [3]. In order to prove that the motion could be viewed without life, he extracted a sample of water from a very old quartz (old enough so that there was no possibility of life inside the quartz), and showed that the motion persisted even in this isolated environment[3].

1.2 Theoretical Background

The explanation behind this phenomenon is quite simple, but only after the answer is known. We can start by observing Brownian motion under a microscope by placing 1 micron diameter latex beads into a solution of water. The reason why the latex beads exhibit a random motion is because the much smaller molecules of water are hitting the latex beads roughly 10^{14} times per second [3]. Hence, nothing is special from one second to the next so the motion of the particle is random. There is no formula that could possibly describe the angle at which the particle will move, but we *can* figure out the average distance the particle will move after a certain time interval. This relationship is given as:

$$\langle R^2 \rangle = \frac{4kT}{6\pi\eta a} t \tag{1}$$

where $\langle R^2 \rangle$ is the mean squared distance of a particle as a function of time, and η is the viscosity of the fluid, and a is the radius of the particle.

2 Experiment

2.1 Approach and Apparatus

The setup involves a microscope, a flat microscope slide, a cover slide, double sided tape, nail polish, water, and of course latex beads. The lab has two different sized latex particles to use, and we had more success with the 1.02 micron diameter beads in terms of visibility because they were easier to identify and to dilute properly. The flat microscope slide was first prepared by placing two pieces of double sided tape about a cover slide width apart on one side of the slide. The latex bead solution and water was then mixed in between these pieces of tape and squeezed between the microscope slide and the cover slide. To ensure that none of the solution would escape and cause drifts, we sealed the deal with nail polish. This produced remarkably stable results, and I would like to acknowledge Chris for this idea and for helping us set it up. A visualization of the camera and the microscope is shown in Figure 1.

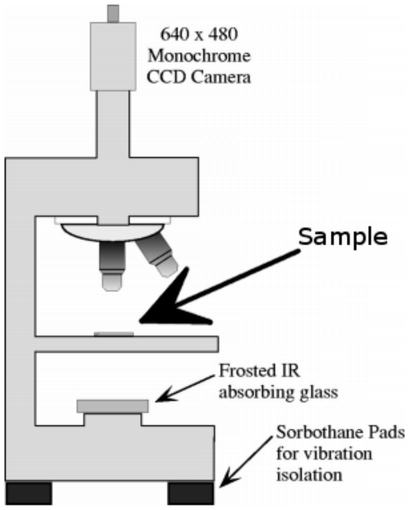

Figure 1: Setup of the experiment, where a Thor Labs camera was placed above the microscope slide, and a 40x magnification lens was used to view the particles. Image source: [4].

2.2 Data Acquisition

In order to create a function of the average squared displacement of multiple beads as a function of time we had to record many particles moving. Data was recorded in segments of about 500 frames, which at 30 frames per second corresponds to roughly 16 seconds. The aim was to track most of the particles for 10 seconds, allowing some room for error, but not too long of a video such that the processing of the video would become a nightmare.

3 Data Analysis

To analyze the particles, I used a variation of ImageJ called Fiji, which comes installed with a plugin called TrackMate, which sufficed for our purposes. First the data was converted to 8-bit format so that we could threshold the particles. Once it was converted, a threshold which was slightly over what it needed to be was applied, as shown in Figure 2. This ensured that each particle was tracked throughout the duration of the video, because often the particle drifts in and out of focus, changing its greyscale value. One side effect of setting the threshold too high is that a lot of noise is generated in the background, but this can easily be removed with a noise filtering algorithm. Most of the particles were also black outlines with white centers, which was not ideal for the tracking algorithm, so to mitigate this problem the "Fill Holes" feature in Fiji was employed to fill the particles. Once this was done, tracking of the particles could commence using the difference of gaussians (DoG) detector[1]. This specific detector was used because it is ideal for tracking small particles. The tracker was parameterized to select far too many particles to track initially and was filtered at many stages. For each video we were able to track on average 10 particles for 10 seconds. This produced a total of 150 raw particle tracks. Figure 2 shows an example image that was processed.

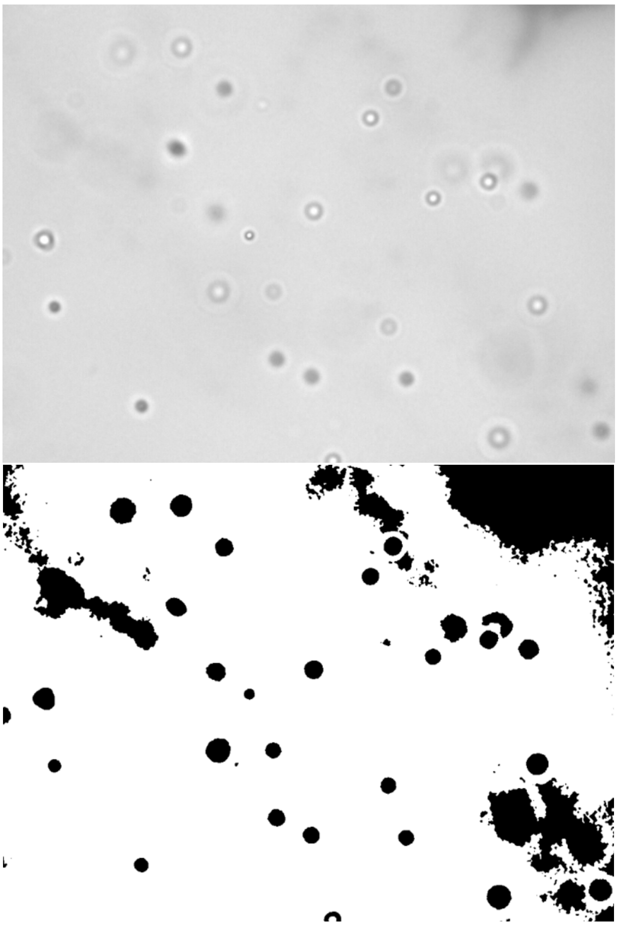

Figure 2: The top image shows the latex beads as recorded by the overhead camera, and the lower image shows the filtered data. Although the lower image is noisy around the edges, it does not affect the tracking algorithm of a particle elsewhere in the video.

The data produced was passed into a comma separated value file, and was analyzed with a python script. The script separated the spreadsheets into each particle's motion, and subtracted the origin of the particle. We are only concerned with how far the particle travels, so we need to get the difference in the distance it travels. This was done for both the x dimension and the y dimension. In the equations below, i represents each unique particle[4].

$$\Delta x_i(t) = x_i(t) - x_i(0) \tag{2}$$

$$\Delta y_i(t) = y_i(t) - y_i(0) \tag{3}$$

Once all of the particles were tracked, I noticed that some of the particles did not move at all. After debugging this issue I found that there were three dots on the microscope that showed up in every video that were sometimes tracked as particles because they looked exactly like a latex bead. I then wrote a program to identify and remove all the particles whose squared distance did not vary any significant amount as time progressed. Once this was done, a final program took all the valid particles and calculated the Pythagorean distance squared traveled for each particle[4].

$$\Delta R_i(t)^2 = [\Delta x_i(t)]^2 + [\Delta y_i(t)]^2 \tag{4}$$

This $\Delta R_i(t)^2$ is inherently noisy for each particle, so they must be summed and averaged to produce any meaningful results.

$$\langle R^2 \rangle = \frac{1}{n}\sum_{i=1}^{n}\left[\Delta R_i(t)^2\right] \tag{5}$$

A conversion from pixels to meters was obtained using a USAF 1951 wheel, which was:

$$1 \text{ pixel} = 1.0079 \pm 0.001 \times 10^{-7} \text{ meters} \tag{6}$$

After this factor was applied to the mean squared distance of all the particles as a function of time, which is shown in Figure 3, a linear regression was fit to the data to derive the slope.

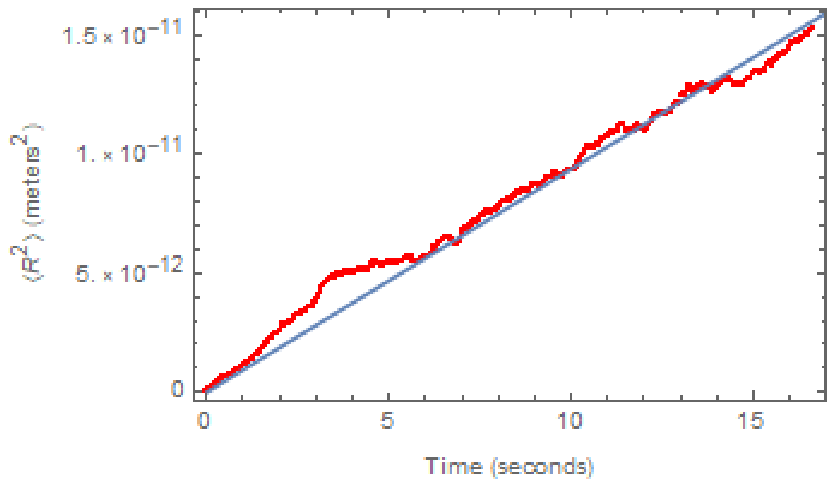

Figure 3: In red is the mean squared distance of 97 latex particles as a function of time, and in blue is a linear regression that is forced to go through the origin. The slope of this line is $9.38 \pm .03 \times 10^{-13}$ m^2/s.

The slope, which we will call s, was found to be $9.38 \pm .03 \times 10^{-13}$ m^2/s. The error in this slope was calculated by the standard error in the linear regression fit. To obtain a value for the viscosity of water, an exponential was fitted to an online database of the viscosity of water[6]. This equation had the form:

$$\mathrm{V}(T) = \exp\left[a + \frac{b}{T-c}\right] \tag{7}$$

which produces the following fit shown in Figure 4.

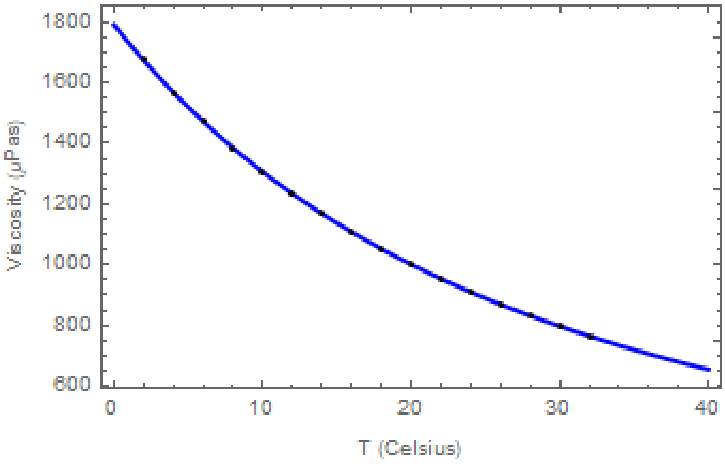

Figure 4: The viscosity of water as a function of temperature. The viscosity of water at our temperature was interpolated from this equation, which was $909 \pm 6\ \mu Pas$.

Multiple samples of the temperature were measured with a thermistor and a multimeter. The average temperature was recorded, as well as the standard deviation in this measurement.

From the slope of Figure 3, we can attain a value for Boltzmann's constant with the following equation and parameters[4].

$$k = \frac{6\pi\eta a s}{4T} \tag{8}$$

$$T = 297.3 \pm 0.3\ K$$
$$\eta = 909 \pm 6\ \mu\mathrm{Pas}$$
$$a = 1.02 \pm 0.01\ \mu\mathrm{m}$$
$$s = 9.38 \pm 0.03 \times 10^{-13}\ \mathrm{m}^2/\mathrm{s}$$

Solving for k and propagating our error gives:

$$k = 1.37 \pm 0.02 \times 10^{-23}\ \mathrm{J} \cdot \mathrm{K}^{-1} \tag{9}$$

The error for Boltzmann's constant was calculated as follows [5].

$$\delta k = |k| \sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta \eta}{\eta}\right)^2 + \left(\frac{\delta T}{T}\right)^2} \tag{10}$$

Compared with the NIST standard for the Boltzmann constant which is

$$k_b = 1.38064852 \pm 0.00000079 \times 10^{23}\ \mathrm{J} \cdot \mathrm{K}^{-1} \tag{11}$$

we have excellent agreement.

4 Error Analysis

In order to reduce vibrational noise in the particles, (that is vibration besides the Brownian motion) we made sure the table on which the microscope was sitting was stable and was not disturbed during recording. We also reduced any drift in the particles by ensuring there were no currents in the sample, which was done by sealing the sample and providing ample water so that there were no zones of low pressure. Although the drift was not formally analyzed, no noticeable drift was visible during the tracking of all the particles.

One error which was tough to solve for was the conversion factor between one pixel and a physical distance. The image captured from the video camera was noisy when zoomed into and made it hard to determine the width of a line to the exact pixel. The width for this experiment was taken by picking a point between pure black the background (which was grey) for multiple lines and to average them together. A better solution would be to analyze one row from the image of which the pixel color would vary as a function of it's x position. A program would then sweep over these pixels and fit a repeating step function to when the pixels turn black. This function would be fit by multiplying the pixel intensity by the step function and then integrating over the row, the result of which would be minimized. Due to time constraints, I was not able to implement this solution, but had I, it would have provided a much cleaner width for a line with a hopefully tighter error.

5 Summary and Conclusion

By tracking the average squared distance (in a 2D plane) of latex particles when submerged in water we were able to derive a rough estimate of Boltzmann's constant. This analysis was first made possible by Einstein in his 1905 paper on Brownian motion[2], which we are able to realize in an undergraduate lab today. The advancement in technology over the last 100 years has been remarkable, and this lab certainly takes advantage of it. Using the equipment we had, we were able to determine the Boltzmann constant to within 1% of the NIST standard, for which we found $k_b = 1.37 \pm 0.02 \times 10^{-23}$ J $\cdot$ K^{-1}.

References

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