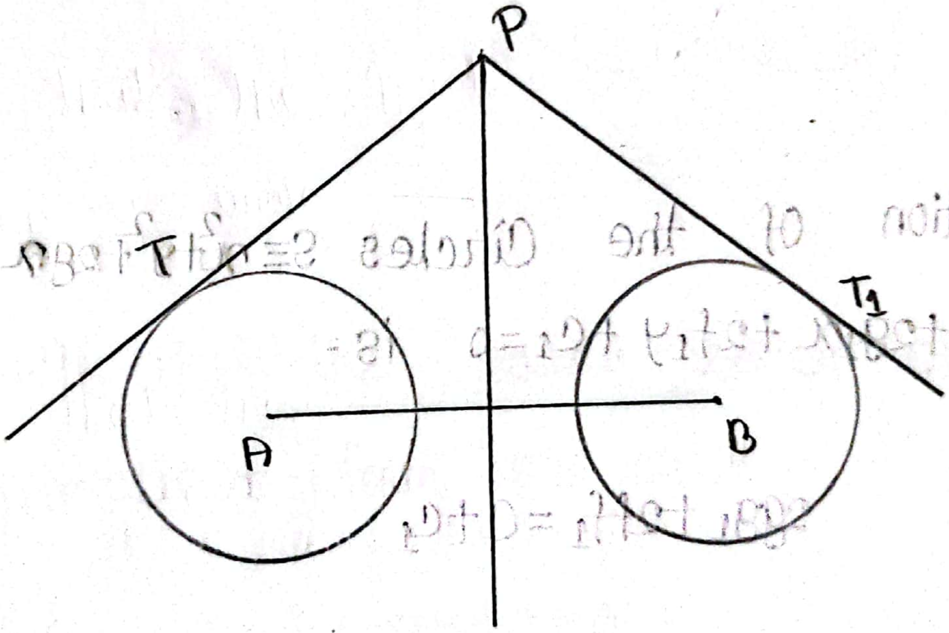
System of Circle

Definition: Radical Axis of two Circle:

The Radical axis of two Circle is defined as the locus of a point such that the lengths of tangents from it to the two Circles are equal.

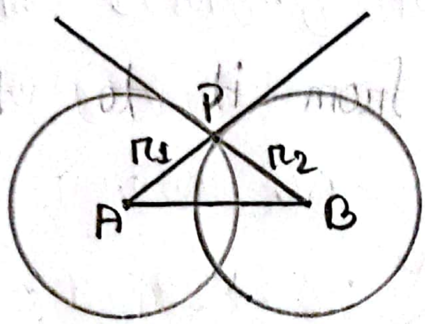


The eqn of the ~~th~~ Radical axis of the two Circles $S \equiv x^2+y^2+2gx+2fy+c=0$ and $S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0$ is

$$2(g-g_1)x_1+2(f-f_1)y_1+c-c_1=0$$

* Orthogonal Circles:

Definition: Two circles are defined to be orthogonal if the tangents at their point of intersection are at right angles.

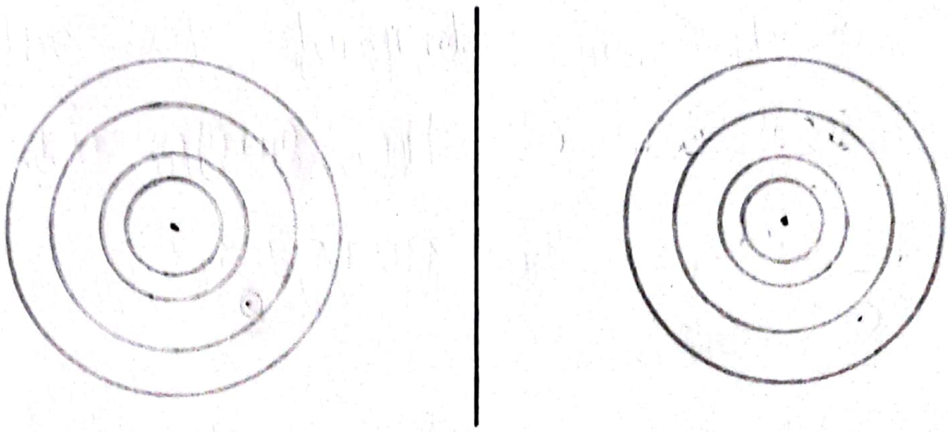


The equation of the circles $S \equiv x^2+y^2+2gx+2fy+c=0$, $S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0$ is -

$$2gg_1+2ff_1=c+c_1$$

Coaxal System

Definition: A System of Circles is said to be coaxal if every pair of System has the Same Radical axis.



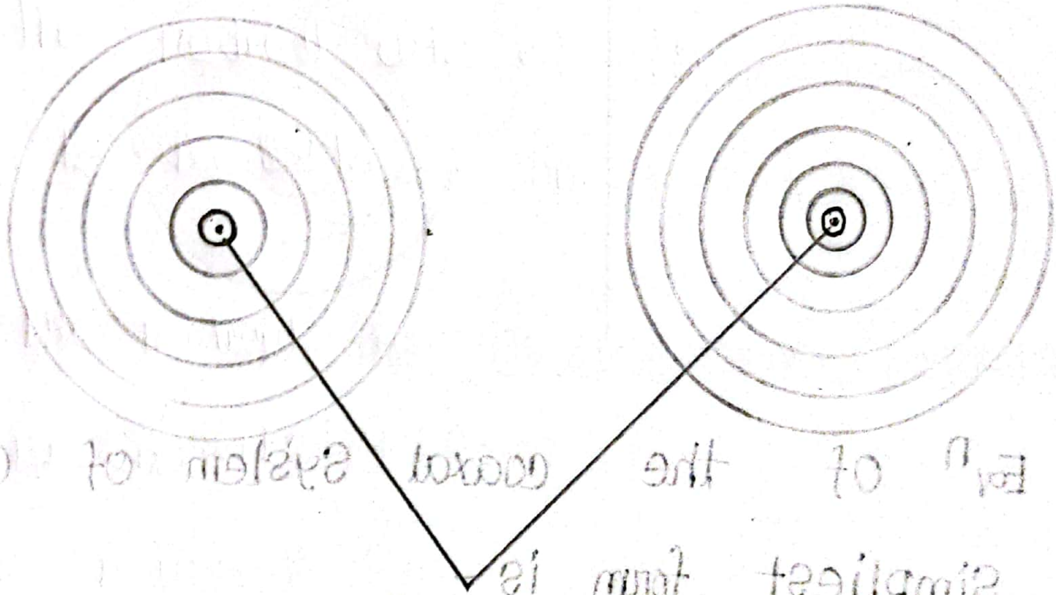
The Eqⁿ of the coaxal System of Circles in the Simpliest form is -

$$x^2+y^2+2\lambda x+c=0$$

Where λ is a Variable and c is a Constant.

Limiting Points

Definition: Limiting Points are defined to be the Centres of point circles belonging to a coaxal system that is they are centres of circles of zero radii belonging to a coaxal system.



Point Circles(Limiting points)

The limiting points of the coaxal system of circles $x^2+y^2+2\lambda x+c=0$ ∴ Centres are $(-\lambda, 0)$ and radii are $\sqrt{\lambda^2-c}$

* For point circles radii are zero.

$$\therefore \sqrt{\lambda^2-c}=0$$

$$\Rightarrow \lambda^2=c$$

$$\therefore \lambda=(\pm\sqrt{c}, 0)$$

* Example 5.5.1: Find the radical axis of the two circles $x^2+y^2+2x+4y-7=0$ and $x^2+y^2-6x+2y-5=0$ and show that it is at right angles to the line of centres of the two circles.

Solⁿ: Here,

$$S \equiv x^2+y^2+2x+4y-7=0$$

and $$S_1 = x^2+y^2-6x+2y-5=0$$

The radical axis of the circles is-

$$S-S_1=0$$

$$\Rightarrow 8x+2y-2=0$$

$$\therefore 4x+y-1=0$$

$$\Rightarrow y=-4x+1$$

The slope of Radical Axis is $m_1=-4$

The centres of the two circles are $(-1,-2)$ and $(3,-1)$

The slope of the line of centres is $m_2=\frac{1}{4}$

$$\therefore m_1 m_2 = (-4)\left(\frac{1}{4}\right) = -1$$

Therefore, the radical axis is perpendicular to the line of centres.

Practices

1. Find the radical axis of the circles $x^2+y^2+2x+4y=0$ and $2x^2+2y^2-7x-8y+1=0$; **Ans:** $11x+16y-1=0$

2. Find the radical axis of the circles $x^2+y^2-4x-2y-11=0$ and $x^2+y^2-2x-6y+1=0$ and show that the radical axis is perpendicular to the line of centres.

Ans: $x-2y+6=0$

3. Show that the circles $x^2+y^2-6x-9y+13=0$ and $x^2+y^2-2x-16y=0$ touch each other. Find the co-ordinates of point of contact.

Soln: $S_1 \equiv x^2+y^2-6x-9y+13=0$; Here, $r_1 = \frac{\sqrt{65}}{2}$

$(-g,-f) \equiv \left(3, \frac{9}{2}\right) C_1$

$S_2 \equiv x^2+y^2-2x-16y=0$; Here, $r_2 = \sqrt{65}$

$(-g_1,-f_1) \equiv (1,8) C_2$

Distance between the centres of two circle-

$$C_1C_2 = \sqrt{4+\frac{49}{4}} = \frac{\sqrt{65}}{2}$$

$$|r_2-r_1| = \left|\sqrt{65} - \frac{\sqrt{65}}{2}\right| = \frac{\sqrt{65}}{2}$$

$$\because C_1C_2 = |r_2-r_1|$$

Thus, the two circles touches each other internally.

Since the circle touches each other internally.

The point of contact P divides C_1C_2 externally in the ratio $r_1 : r_2$ i.e. $\frac{\sqrt{65}}{2} : \sqrt{65} = 1:2$

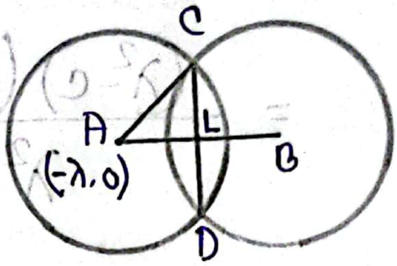
Therefore, Co-ordinates of P are -

$$\left[\frac{1(1) - 2(3)}{1-2}, \frac{1(8) - 2\cdot\left(\frac{9}{2}\right)}{1-2}\right] = (5, 1)$$

(Ans)

Example 5.5.9: Prove that the length of common chord of the two circles $x^2+y^2+2\lambda x+c=0$ and $x^2+y^2+2\mu y-c=0$ is $2\sqrt{\dfrac{(\lambda^2-c)(\mu^2+c)}{\lambda^2+\mu^2}}$.

Soln: The two given circles are -



$$x^2+y^2+2\lambda x+c=0$$

$$x^2+y^2+2\mu y-c=0$$

Centres are $A(-\lambda,0)$ and $B(-\mu,0)$

radii are $AC=\sqrt{\lambda^2-c}$ and $BC=\sqrt{\mu^2+c}$

The radical axis is $\lambda x-\mu y+c=0$. The perpendicular distance from A on CD is AL.

$$\therefore AL=\frac{|\lambda^2-c|}{\sqrt{\lambda^2+\mu^2}}$$

$$\therefore CL^2=AC^2-AL^2=(\lambda^2-c)-\frac{(\lambda^2-c)^2}{\lambda^2+\mu^2}$$

$$= \frac{(\lambda^2-c)(\lambda^2+\mu^2)-(\lambda^2-c)^2}{\lambda^2+\mu^2}$$

$$= \frac{(\lambda^2-c)[(\lambda^2+\mu^2)-(\lambda^2-c)]}{\lambda^2+\mu^2}$$

$$= \frac{(\lambda^2-c)(\lambda^2+\mu^2-\lambda^2+c)}{\lambda^2+\mu^2}$$

$$= \frac{(\lambda^2-c)(\mu^2+c)}{\lambda^2+\mu^2}$$

Therefore the length of common chord

$$= CD = 2CL = 2\sqrt{\frac{(\lambda^2-c)(\mu^2+c)}{\lambda^2+\mu^2}}$$

Example 5.5.10: Show that the circle $x^2+y^2-8x-6y+21=0$ is orthogonal to the circle $x^2+y^2-2y-15=0$. Find the common chord and the equation of the circle passing through the centres and intersecting points of the circles.

Soln:

$$x^2+y^2-8x-6y+21=0$$

$$x^2+y^2-2y-15=0$$

The condition for orthogonality is $2gg_1+2ff_1=c+c_1$

(i.e) $2(-4)(0)+2(-3)(-1)=21-15$

$\Rightarrow 0+6=6$ which is true.

Therefore, the two circles cut each other orthogonally.

The eqn of the common chord is $S-S_1=0$

$$-8x-4y+36=0$$

$$\Rightarrow 2x+y-9=0$$

Any circle passing through the ~~centre~~ intersection of the circles is $S+\lambda L=0$ // Note

This passes through the centre (4,3) of the first circle.

$$16+9-32-18+21+\lambda(8+3-9)=0$$

$$\Rightarrow -4+\lambda(2)=0$$

$$\therefore \lambda=2$$

Therefore, the eqn of the required circle is -

$$x^2+y^2-8x-6y+21+2(2x+y-9)=0$$

$$\text{(i.e) } x^2+y^2-4x-4y+3=0$$

Imp.

Example 5.6.8: Find the limiting points of the coaxal system determined by the circle $x^2+y^2+2x+4y+7=0$ and $x^2+y^2+4x+2y+5=0$

Soln: Given that,

$$x^2+y^2+2x+4y+7=0$$

$$x^2+y^2+4x+2y+5=0$$

The radical axis of these two circle is-

$2x-2y-2=0$. Any circle of the coaxal System is $x^2+y^2+2x+4y+7+\lambda(2x-2y-2)=0$

Centre is $(-1-\lambda, -2+\lambda)$

Radius is $\sqrt{(1+\lambda)^2+(2-\lambda)^2+2\lambda-7} = \sqrt{2\lambda^2-2}$

Limiting points are the centres of circles of radii zero.

Therefore limiting points are $(-2,-1)$ and $(0,-3)$

Example 5.6.9: The point $(2,1)$ is a limiting point of a system of coaxal circles of which $x^2+y^2-6x-4y-3=0$ is a member. Find the equation the radial axis and the co-ordinates of the other limiting point.

Soln: Given that,

$$x^2+y^2-6x-4y-3=0$$

Since, $(2,1)$ is a limit point, the point circle corresponding to the coaxal system is-

$$(x-2)^2+(y-1)^2=0$$

$$\Rightarrow x^2+y^2-4x-2y+5=0$$

The Radical axis of the system is,

$$S-S_1=0 \quad \text{or}, \quad 2x+2y+8=0$$

$$\Rightarrow x+y+4=0$$

Any Circle of the coaxal System is $S + \lambda L = 0$

$$x^2+y^2-6x-4y-3+\lambda(2x+2y+8)=0$$

Centre is $(3-\lambda, 2-\lambda)$

$$\text{Radius} = \sqrt{(3-\lambda)^2+(2-\lambda)^2+(3-8\lambda)}$$

For point Circle, radius $=0$

$$\Rightarrow (3-\lambda)^2+(2-\lambda)^2+3-8\lambda=0$$

$$\Rightarrow 2\lambda^2-18\lambda+16=0$$

$$\Rightarrow \lambda^2-9\lambda+8=0$$

$$\therefore \lambda = 1, 8$$

$\therefore$ limiting points are, $(2,1)$ and $(-5,-6)$

Practice:

5| Find the co-ordinates of limiting point of the coaxal circles determined by the two circles $x^2+y^2-4x-6y-3=0$ and $x^2+y^2-24x-26y+277=0$.

Ans: $\cancel{(1,2)}$, $(3,4)$ $(10,11)$

6| Find the co-ordinates of the limiting points of the coaxal system of circles of which two members are $x^2+y^2+2x-6y=0$ and $2x^2+2y^2-10y+5=0$

Ans: $(1,2)$, $(3,1)$

7| Find the co-axal system of circles if one of whose member is $x^2+y^2+2x+3y-7=0$ and a limiting point is $(1,-2)$.