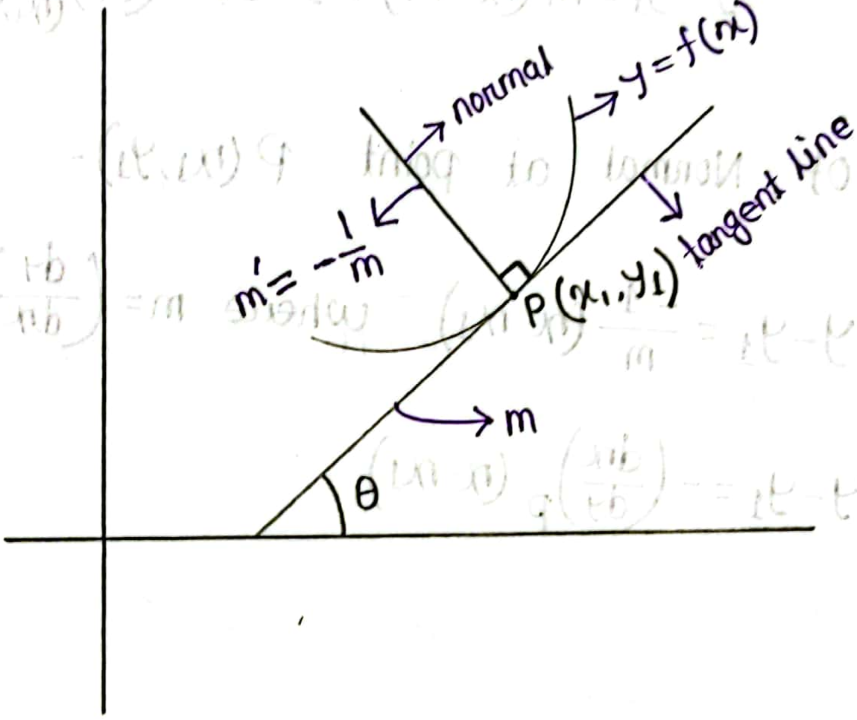
MATH:

Tangent & Normal



Slope of Tangent at point $P(x_1,y_1)$

$$m = \tan\theta = \left(\frac{dy}{dx}\right)_P$$

Slope of Normal at point $P(x_1,y_1)$

$$m' = -\frac{1}{m} = -\frac{1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P$$

where, $m = \left(\frac{dy}{dx}\right)_P \neq 0$

Equation of Tangent at point $P(x_1, y_1)$ -

$$y - y_1 = m(x - x_1) \; ; \; y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Equation of Normal at point $P(x_1, y_1)$ -

$$y - y_1 = \frac{-1}{m}(x - x_1) \quad \text{where } m = \left(\frac{dy}{dx}\right)_P$$

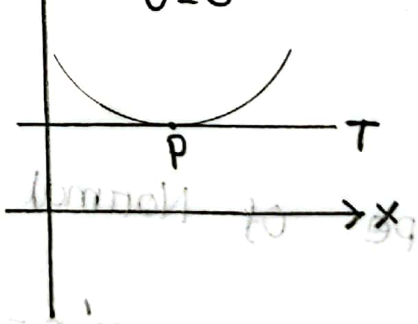
$$\Rightarrow y - y_1 = -\left(\frac{dx}{dy}\right)_P (x - x_1)$$

Some Important Points:

1. If the tangent is parallel to the x-axis / ⊥ Y-axis

Slope of Tangent $\left(\frac{dy}{dx}\right)_P = 0$

$\theta = 0$

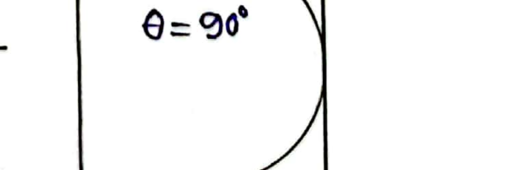


Eqn of tangent, $\boxed{y - y_1 = 0;}$

2. If the Tangent is perpendicular to x-axis / Parallel Y-axis

Slope of Tangent $\left(\frac{dy}{dx}\right)_P = \pm\infty = \frac{1}{0}$

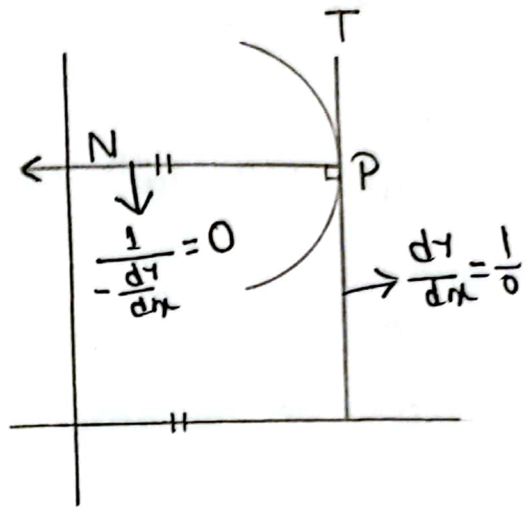
$\theta = 90°$



Eqn of tangent - $\boxed{x - x_1 = 0;}$

3. If Normal is parallel to x-axis:

$$-\frac{1}{\left(\frac{dy}{dx}\right)_P} = 0 \rightarrow \text{Slope of Normal}$$

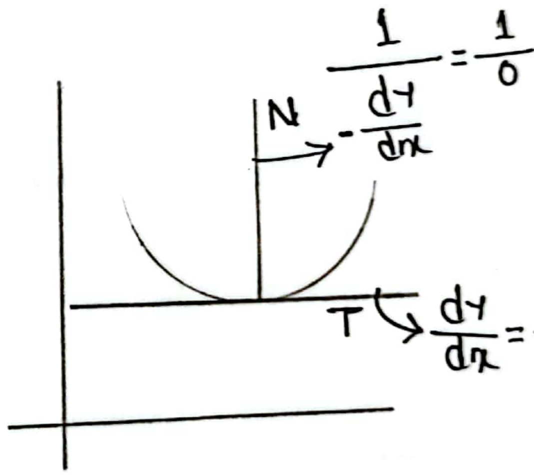


$$-\left(\frac{dx}{dy}\right)_P = 0$$

Eqn of Normal $\boxed{y - y_1 = 0}$

4. If Normal is perpendicular to x-axis:

Slope of Normal $-\left(\frac{dx}{dy}\right)_P = \frac{1}{0}$



Eqn of Normal $\boxed{x - x_1 = 0}$

Curvature and Radius of Curvature

- Radius of Curvature of Cartesian Curve: $y = f(x)$

$$P = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} \Rightarrow \frac{(1+y_1^2)^{3/2}}{|y_2|}$$ (when tangent is parallel to x-axis)

$$P = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|}$$ (when tangent is parallel to y-axis)

Radius of Curvature of parametric curve:

$x = f(t)$, $y = g(t)$

$$P = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|}$$, where $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dt}$

- Radius of curvature of polar curves, $r = f(\theta)$:

$$P = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}$$ (where $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta^2}$)

Partial Derivatives

$f(x,y) = 3x^2y$

$f_x = 6xy$;

$f_{xx} = 6y$; $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

$f_y = 3x^2$; $\frac{\partial f}{\partial y}$

$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6x$

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 6x$

$f_{xx}\big|_{(1,0)} = 0$ $[6 \cdot 0 = 0]$

□ Find the second order derivatives of $f(x,y) = x^2y^3 + x^4y$.

Soln:

$$f(x,y) = x^2y^3 + x^4y$$

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \; ; \; \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6xy^2 + 4x^3$$

and $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 6xy^2 + 4x^3$$

□ For $u = x^3 - 3xy^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Also prove that, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Laplace Eqn:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

* Show that $u = x^3 - 3xy^2$ is a Solution of the laplace eqn.

* If $u = e^{xyz}$, Find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

Soln:

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left(1 + 3xyz + x^2y^2z^2\right)$$

$$* \quad \frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right)$$

* If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$ Show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

* Homogeneous functions:

Definition: A function $z = f(x,y)$ is said to be a homogeneous function of degree n, if $f(tx, ty) = t^n f(x,y)$ for some real number n. Otherwise, f is said to be a nonhomogeneous function.

Euler's Theorem: In general, if $f(x_1, x_2, \dots x_m)$ be a homogeneous function of degree n, then

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_m \frac{\partial f}{\partial x_m} = n.f$$

Verify Euler's Theorem when,

$$f(x,y,z) = axy + byz + czx$$

Soln:

$$\frac{\partial f}{\partial x} = ay + cz \qquad \frac{\partial f}{\partial z} = by + cx$$

$$\frac{\partial f}{\partial y} = ax + bz$$

According to Euler's Theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$$

$$=(ay+cz)x + y(ax+bz) + z(by+cx)$$

$$= axy + cxz + axy + byz + byz + czx$$

$$= 2(axy + byz + czx)$$

$$= 2f(x,y,z)$$

Which verifies Euler's theorem in this case.

If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Solnº

$$\tan u = \frac{x^3+y^3}{x-y}$$

Putting, "$x = xt$ & $y = yt$" we get -

$$f(xt, yt) = \frac{x^3t^3 + y^3t^3}{xt - yt} = \frac{t^3(x^3+y^3)}{t(x-y)} = t^2 \tan u$$

Here u is not a homogeneous function but

$\tan u = \frac{x^3+y^3}{x-y}$ is a homogeneous function of degree 2.

i.e. $x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$

$\Rightarrow \sec^2 u \cdot x\frac{\partial u}{\partial x} + \sec^2 u\, y\frac{\partial u}{\partial x} = 2\tan u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = 2\cdot\frac{\sin u}{\cancel{\cos u}}\cdot\cancel{\cos^2 u}^{\,\cos u}$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \sin 2u.$

$f(x,y) = \mathrm{Tan}\left(\frac{x^3+y^3}{x-y}\right)$

$f(tx,ty) = \mathrm{Tan}\left(\frac{t^3x^3+t^3y^3}{tx-ty}\right)$

$= \mathrm{Tan}\left[\frac{t^3(x^3+y^3)}{t(x-y)}\right]$

$= \mathrm{Tan}\left[t^2\frac{(x^3+y^3)}{(x-y)}\right]$

Here, we can't take out t^2 from the box So it's ~~hom~~ not homogeneous f...

If $u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$ Prove that - $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Solⁿ:

$$u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}.$$

$$\therefore f(x,y) = u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Since u is not a H. f^n but -

$\sin u = z = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a H. f^n of degree '$\frac{1}{2}$'

By E.T -

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2} z$$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y.\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \frac{1}{2}\tan u$$

L' Hospital Rule

* Indeterminate Form:

$$\frac{0}{0},\ 0\cdot\infty,\ \frac{\infty}{\infty},\ (\infty-\infty),\ 1^{\infty},\ 0^{0},\ \infty^{0}$$

Formula:

1) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

2) $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$

3) $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \log a$

4) $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = \log e = 1$

5) $\lim\limits_{x\to 0} (1+x)^{1/x} = e$

6) $\lim\limits_{x\to 0} \frac{\log(1+x)}{x} = \log e = 1$

Problem:

1) Evaluate: $\lim\limits_{x\to 0} \frac{3^x - 2^x}{x}$

$\rightarrow \lim\limits_{x\to 0} \frac{3^x - 2^x}{x} \left[\frac{0}{0}\right] \Rightarrow$ Indeterminate form

According to L'Hospital Rule,

$$= \lim_{x\to 0} \frac{3^x \cdot \log 3 - 2^x \log 2}{1}$$

Applying limits,

$$= 3^0 \log 3 - 2^0 \log 2 = \log 3 - \log 2$$

$$= \log\left(\frac{3}{2}\right)$$

2] Evaluate: $\lim_{x\to 1} \frac{x^x - x}{x-1-\log x}$

$\rightarrow \lim_{x\to 1} \frac{x^x - x}{x-1-\log x} \left[\frac{0}{0}\right]$

$$\frac{d}{dx}(x^x) = x^x(1+\log x)$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \left(\frac{-1}{x^2}\right)$$

Applying L'Hospital Rule,

$$= \lim_{x\to 1} \frac{x^x(1+\log x) - 1}{1 - \frac{1}{x}} \left[\frac{0}{0}\right]$$

Apply L'Hospital Rule Again,

$$= \lim_{x\to 1} \frac{x^x \cdot \frac{1}{x} + x^x(1+\log x)^2}{\frac{1}{x^2}}$$

$$= \frac{1^{1} \cdot \frac{1}{1} + 1^{1}(1+\log 1)^{2}}{\frac{1}{1^{2}}}$$

$$= \frac{1+1}{1} = 2 \text{ (Ans)}$$

◫ L' Hospital Rule for from $\left(\frac{0}{0}\right)$:

Suppose that f and g are differential functions on an open intervel containing $x=a$, except possibly at $x=a$ and that is-

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

If $\lim_{x\to a}\left[f'(x)/g'(x)\right]$ exists, or if this limit is $+\infty$ or $-\infty$ then-

$$\operatorname{limit}_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Practice Problems:

1| $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2}$ 2| $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$

3| $\lim\limits_{x \to \pi/2} \dfrac{1-\sin x}{\cos x}$ 4| $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2}$

* $\lim\limits_{x \to 0} \dfrac{e^x-1}{x^3}$

$= \lim\limits_{x \to 0} \dfrac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(x^3)}$

$= \lim\limits_{x \to 0} \dfrac{e^x}{3x^2}$

$= +\infty.$

* Indeterminate Forms of Type $\frac{\infty}{\infty}$:-

1) $\lim\limits_{x \to +\infty} \frac{x}{e^x}$

$= \lim\limits_{x \to +\infty} \frac{1}{e^x}$

$= \frac{1}{\infty} = 0$

* Indeterminate Forms of Type $0.\infty$:-

1) $\lim\limits_{x \to 0^+} x \ln x$

$= \lim\limits_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$

$= \lim\limits_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim\limits_{x \to 0^+} -\frac{1}{\cancel{x}} \times \frac{\cancel{x^2}^{\,x}}{1}$

$= \lim\limits_{x \to 0^+} (-x) = 0$

2) $\lim_{x \to \frac{\pi}{4}} (1-\tan x)\sec 2x$

$= \lim_{x \to \frac{\pi}{4}} \frac{1-\tan x}{\cos 2x}$

$= \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x}$

$= \frac{-(\sqrt{2})^2}{-2 \cdot 1} = \frac{-2}{-2} = 1$ (Ans)

* Indeterminate Forms Of Type $\infty-\infty$:

Example-5: Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

$$\rightarrow \lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right)$$

$$= \lim_{x\to 0^+}\frac{\cos x - 1}{x\cos x + \sin x}$$

$$= \lim_{x\to 0^+}\frac{-\sin x}{\cos x - x\sin x + \cos x}$$

$$= \frac{0}{1-0+1} = \frac{0}{2} = 0$$

* Indeterminate Forms Of Type $0^0, \infty^0, 1^\infty$

$$\lim_{x \to a} [1 + f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

* Find $\lim_{x \to 0} (1 + \sin x)^{\frac{1}{x}}$

$$= e^{\lim_{x \to 0} \frac{\sin x}{x}}$$

$$= e^1$$

$= e$ (Ans)

Practice:

7. $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$

27) $\lim_{x \to +\infty} \left(1 - \frac{3}{x}\right)^x$

16. $\lim_{x \to 0^+} \frac{1 - \ln x}{e^{1/x}}$

29) $\lim_{x \to 0} (e^x + x)^{1/x}$

29] $\lim_{x\to 0} (e^x + x)^{1/x}$

taking the natural logarithm both Side-

$$\ln y = \ln (e^x + x)^{1/x}$$
$$= \frac{1}{x} \ln (e^x + x)$$
$$= \frac{\ln (e^x + x)}{x}$$

Thus, $\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(e^x + x)}{x}$

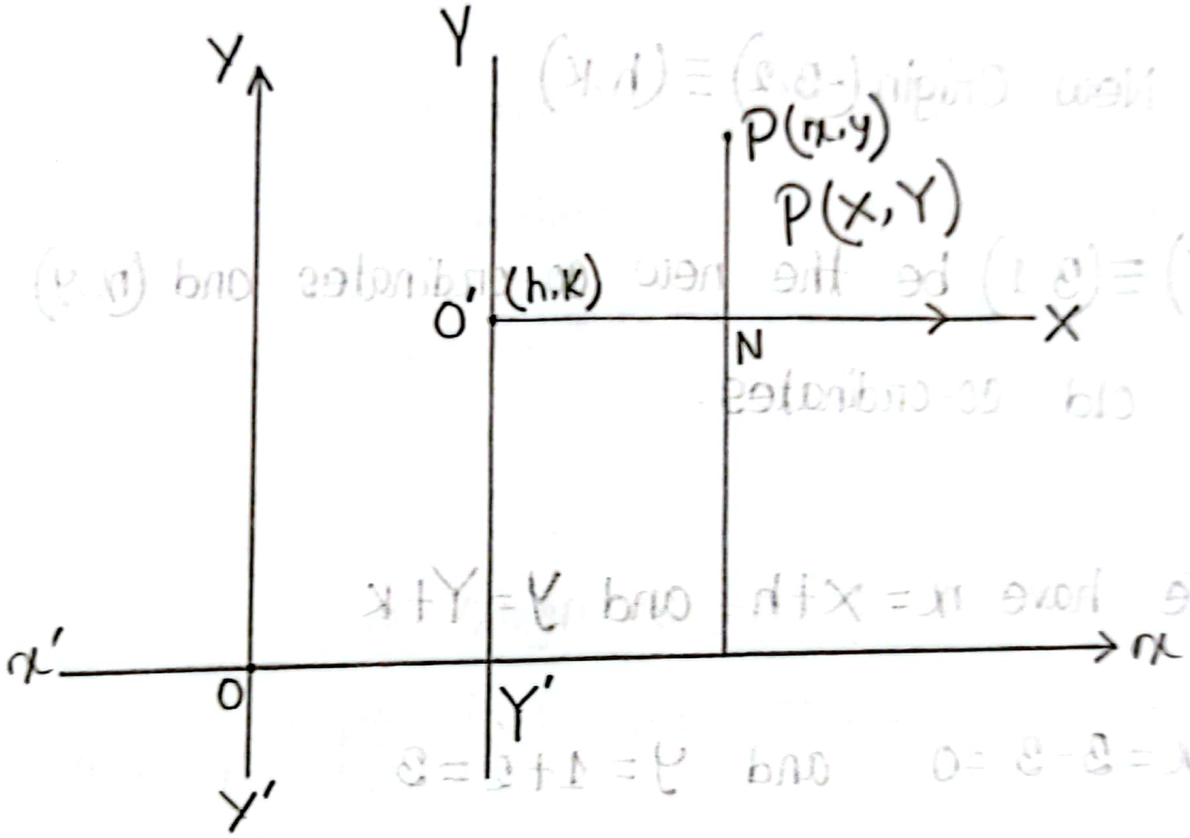
$$= \lim_{x\to 0} \frac{e^x + 1}{e^x + x}$$

$$= \frac{e^0 + 1}{e^0 + 0} = \frac{1+1}{1} = 2$$

Since we have Shown that $\ln y \to 2$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^2$ as ~~θ~~ $x \to 0$ and this implies that $y \to e^2$ as $x \to 0$. Thus, $\lim_{x\to 0} (e^x + x)^{1/x} = e^2$

Chapter 10
Tracing of Curves

* Shift of origin without Changing the Direction of axes:



* New origin = (h,k)

* Old Co-ordinates = (x,y)

* New Co-ordinates = (X,Y)

Formula:

$$\boxed{\begin{aligned} x &= h + X \\ y &= k + Y \end{aligned}}$$

Problem-1: If the origin shifted to a point $(-3, 2)$, axes remaining parallel, the new co-ordinates of the point are $(3, 1)$. Find the old co-ordinates.

Soln:

New Origin $(-3, 2) \equiv (h, k)$

Let $(X, Y) \equiv (3, 1)$ be the new co-ordinates and (x, y) be the old co-ordinates.

We have $x = X + h$ and $y = Y + k$

$x = 3 - 3 = 0$ and $y = 1 + 2 = 3$

$\therefore (x, y) \equiv (0, 3)$

Ans: The Old Co-ordinates of point $(3, 1)$ are $(0, 3)$

Practice Problem:

1. If the origin shifted to a point $(1,2)$, axes remaining parallel, find the new co-ordinates of the point $(2,-3)$. Ans: $(1,-5)$

2. The point $(3,8)$ becomes $(-2,1)$ after the shift of origin. Find the coordinates of the point, where the origin is shifted. Ans: $(5,7)$

Problem-2: When the origin is shifted to $(-1,2)$ by the translation of axes, find the transformer equation $x^2+y^2+2x-4y+1=0$

Soln: Given equation is $x^2+y^2+2x-4y+1=0 \longrightarrow (1)$

We take new origin $(h,k)=(-1,2)$, then

$x=X+h \Rightarrow x=X-1$

$y=Y+K \Rightarrow y=Y+2$

Put the value of x & y in eqn ① $\Rightarrow$

$$(X-1)^2+(Y+2)^2+2(X-1)-4(Y+2)+1=0$$

$$\Rightarrow (X^2-2X+1)+(Y^2+4Y+4)+2X-2-4Y-8+1=0$$

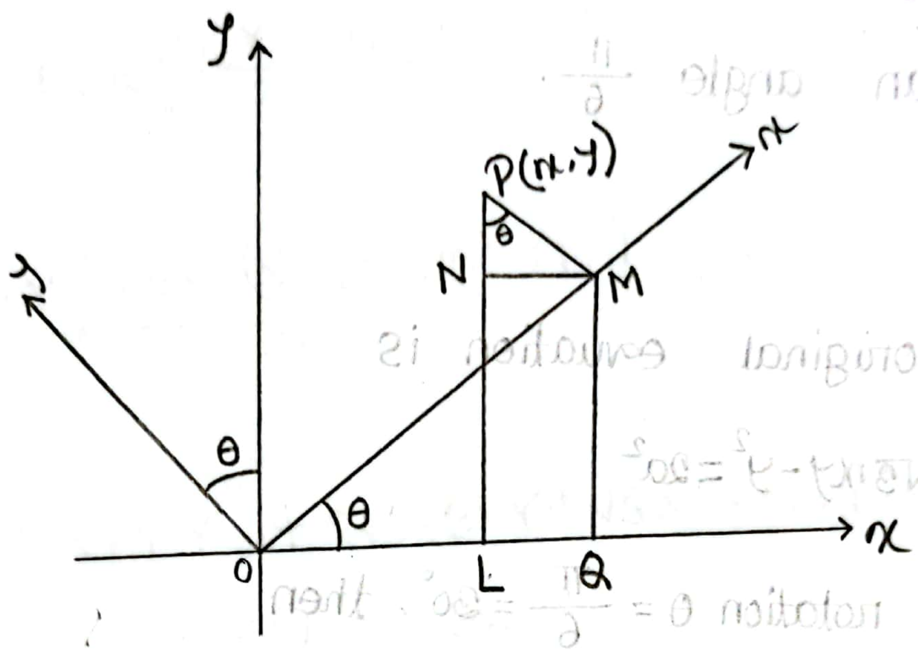
$$\Rightarrow X^2+Y^2-4=0$$

$$\therefore X^2+Y^2=4$$

So, the required transformed eqn is, $X^2+Y^2-4=0$

Practice: When the origin is shifted to the point (2, 3) the transformed equation of a curve is $X^2+3XY-2Y^2+17X-7Y-11=0$. Find the original Equation of the curve.

Rotation of Axes Without Changing the Origin :



$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$

Co-ordinates of the center:

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$$

Removal of XY- Term:

$$\tan 2\theta = \frac{2h}{a-b} \quad \text{or,} \quad \theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$

Invariants

i) $a+b = A+B$

ii) $ab-h^2 = AB-H^2$

Problem: Find the transformed equation of $x^2+2\sqrt{3}xy-y^2=2a^2$, when the axes are rotated through an angle $\frac{\pi}{6}$.

Soln:

The given original equation is

$$x^2+2\sqrt{3}xy-y^2=2a^2$$

Angle of rotation $\theta = \frac{\pi}{6} = 30^\circ$, then

$$x = X\cos\theta - Y\sin\theta$$

$$\Rightarrow x = X\cos 30^\circ - Y\sin 30^\circ$$

$$\Rightarrow x = X\cdot\frac{\sqrt{3}}{2} - Y\cdot\frac{1}{2}$$

$$\therefore x = \frac{\sqrt{3}X - Y}{2}$$

$$y = Y\cos\theta + X\sin\theta$$

$$\Rightarrow y = Y\cos 30^\circ + X\sin 30^\circ$$

$$\therefore y = \frac{\sqrt{3}Y + X}{2}$$

From original equation,

the transformed equation is,

$$\left(\frac{\sqrt{3}X-Y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}X-Y}{2}\right)\left(\frac{\sqrt{3}Y+X}{2}\right) - \left(\frac{\sqrt{3}Y+X}{2}\right)^2 = 2a^2$$

$$\Rightarrow \frac{(\sqrt{3}X-Y)^2 + 2\sqrt{3}(\sqrt{3}X-Y)(\sqrt{3}Y+X) - (\sqrt{3}Y+X)^2}{4} = 2a^2$$

$$\Rightarrow 3X^2 - 2\sqrt{3}XY + Y^2 + 2\sqrt{3}(3XY + \sqrt{3}X^2 - \sqrt{3}Y^2 - XY) - (3Y^2 + 2\sqrt{3}XY + X^2) = 4\times 2a^2$$

$$\Rightarrow 3X^2 - 2\sqrt{3}XY + Y^2 + 6\sqrt{3}XY + 6X^2 - 6Y^2 - 2\sqrt{3}XY - 3Y^2 - 2\sqrt{3}XY - X^2 = 8a^2$$

$$\Rightarrow 8X^2 - \cancel{6\sqrt{3}XY} - 8Y^2 + \cancel{6\sqrt{3}XY} = 8a^2$$

$$\Rightarrow 8(X^2 - Y^2) = 8a^2$$

$$\therefore X^2 - Y^2 = a^2$$

The Required transformed equation is $X^2 - Y^2 = a^2$

Find the point to which the origin is to be shifted so as to remove the first degree terms from the equation $4x^2+9y^2-8x+36y+4=0$

Soln:

The given equation is $4x^2+9y^2-8x+36y+4=0$

Comparing the equation with

$ax^2+2hxy+by^2+2gx+2fy+c=0$

Then, $a=4, b=9, c=4, h=0, g=-4, f=18$

$\therefore$ The required point $= \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$

$= \left(\frac{0+36}{36-0^2}, \frac{0-72}{36-0^2}\right)$

$= \left(\frac{36}{36}, \frac{-72}{36}\right)$

$= (1,-2)$

$\therefore$ The required point is $(1,-2)$

Pair of Straight Lines

* General Eq^n of 2^{nd} degree-

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{—(i)}$$

Consider two Straight Lines represented by the following ~~eques~~ equation-

$$l_1x+m_1y+n_1=0 \quad \text{—(ii)}$$

$$l_2x+m_2y+n_2=0 \quad \text{—(iii)}$$

By Multiplying (ii) & (iii) we get-

$$(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)=0$$

$$\Rightarrow l_1l_2x^2+(l_1m_2+l_2m_1)xy+m_1m_2y^2+(l_1n_2+l_2n_1)x+(m_1n_2+m_2n_1)y+n_1n_2=0 \quad \text{—(iv)}$$

By Comparing (i) and (iv) $\Rightarrow$

$$l_1l_2=a;\quad m_1m_2=b;\quad n_1n_2=c$$

$$l_1m_2+l_2m_1=2h;\quad l_1n_2+l_2n_1=2g;\quad m_1n_2+m_2n_1=2f.$$

Homogeneous Equation of 2nd degree In x and y:

$$ax^2 + 2hxy + by^2 = 0$$

$\downarrow$

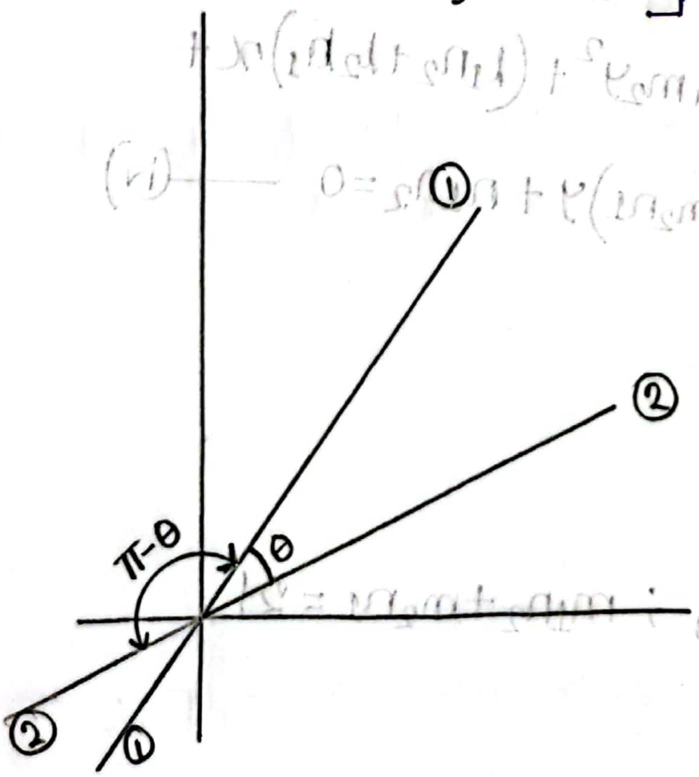
* $b\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) = 0$

* (i.e) $b(y - m_1x)(y - m_2x) = 0$

* $m_1 + m_2 = \frac{-2h}{b}$

* $m_1 m_2 = \frac{a}{b}$

$$\left.\begin{array}{l} y = m_1x \\ y = m_2x \end{array}\right] \rightarrow \text{two Straight lines}$$



* $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left(\frac{\pm 2\sqrt{h^2 - ab}}{a+b}\right)$$

$$\therefore \theta = \tan^{-1}\left(\frac{\pm 2\sqrt{h^2 - ab}}{a+b}\right)$$

* If the lines are parallel or coincident, then $\theta = 0^\circ$.

$$\therefore \tan\theta = \tan 0^\circ = 0$$

$$\Rightarrow \frac{\pm 2\sqrt{h^2-ab}}{a+b} = 0$$

$$\Rightarrow \sqrt{h^2-ab} = 0$$

$$\Rightarrow h^2 - ab = 0$$

$$\therefore h^2 = ab;$$

* If the lines are Perpendicular then $\theta = 90^\circ$;

$$\tan\theta = \tan 90^\circ = \infty = \frac{1}{0}$$

$$\therefore \frac{2\sqrt{h^2-ab}}{a+b} = \frac{1}{0}$$

$$\Rightarrow a+b = 0$$

$$\therefore a+b=0;$$

☐ Condition For General Equation of a 2nd Degree Equation to represent a pair of Straight lines:

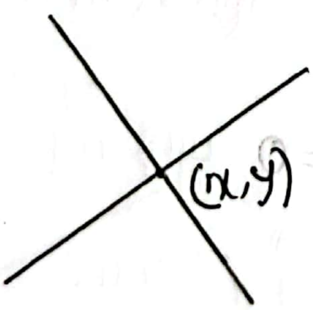
$ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent a pair of Straight lines is-

$$abc+2fgh-af^2-bg^2-ch^2=0$$

or

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

☐ Point of of intersection-



$$(x,y) \equiv \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2} \right)$$

* $x^2+2xy+y^2+2x+2y+1=0$

Here, $a=1, h=1, b=1, g=1, f=1, c=1$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

So, it will represent a Straight line.

Practice: $x^2+2xy+y^2+2x+2y+\lambda=0$; $\lambda=?$

Example-3.1: The gradient of one of the lines $ax^2+2hxy+by^2=0$ is twice that of the other. Show that $8h^2=9ab$.

Soln:

Here, $y=m_1x$

$y=m_2x$

it has been given that, $m_2=2m_1$

we know, $m_1+m_2=\frac{-2h}{b}$

$$\Rightarrow m_1+2m_1=\frac{-2h}{b}$$

$$\Rightarrow 3m_1=\frac{-2h}{b}$$

$$\therefore m_1=\frac{-2h}{3b}$$

again, $m_1m_2=\frac{a}{b}$

$$\Rightarrow m_1\cdot 2m_1=\frac{a}{b}$$

$$\Rightarrow 2m_1^2=\frac{a}{b}$$

$$\Rightarrow 2\cdot\frac{4h^2}{9b^2}=\frac{a}{b}$$

$$\Rightarrow 8h^2\cdot\cancel{b}=9b^{\cancel{2}}\cdot a$$

$$\therefore 8h^2=9ab$$

(Shown)

Practice: If Slope of the given lines by $ax^2+2hxy+by^2=0$ is the square of the other, Show that

$ab(a+h)-6ahb+8h^3=0$

Example 3.14: Find λ so that the equation $x^2+5xy+4y^2+3x+2y+\lambda=0$ represents a pair of line.

Find also their point of intersection and the angle between them.

Soln: Given, $x^2+5xy+4y^2+3x+2y+\lambda=0$

$$\text{or, } x^2+2\cdot\frac{5}{2}xy+4y^2+2\cdot\frac{3}{2}x+2\cdot1\cdot y+\lambda=0$$

Here, $a=1$, $h=\frac{5}{2}$, $b=4$, $g=\frac{3}{2}$, $f=1$, $c=\lambda$

we know, $abc+2fgh-af^2-bg^2-ch^2=0$

$$\Rightarrow 4\lambda+\cancel{2}\cdot1\cdot\frac{3}{\cancel{2}}\cdot\frac{5}{2}-1-\cancel{4}\cdot\frac{9}{\cancel{4}}-\lambda\cdot\frac{25}{4}=0$$

$$\Rightarrow 4\lambda+\frac{15}{2}-1-9-\frac{25\lambda}{4}=0$$

$$\Rightarrow 4\lambda-\frac{25\lambda}{4}=\frac{5}{2}$$

$$\Rightarrow \frac{16\lambda - 25\lambda}{\cancel{4}\,2} = \frac{5}{\cancel{2}}$$

$$\Rightarrow -9\lambda = 10$$

$$\therefore \lambda = -\frac{10}{9}$$

Point of intersection -

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$$

$$= \left(\frac{\frac{5}{2}-6}{-\frac{9}{4}}\right), \left(\frac{\frac{15-4}{\cancel{4}}}{-\frac{9}{\cancel{4}}}\right)$$

$$= \left(\frac{\frac{5-12}{2}}{-\frac{9}{4}}, -\frac{11}{9}\right)$$

$$= \left(\frac{-\frac{7}{2}}{-\frac{9}{4}}, -\frac{11}{9}\right)$$

$$= \left(\frac{14}{9}, -\frac{11}{9}\right)$$

Angle between them-

$$\theta = \tan^{-1}\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

$$= \tan^{-1}\left|\frac{2\sqrt{\frac{25}{4}-4}}{1+4}\right|$$

$$= \tan^{-1}\left|\frac{2\sqrt{\frac{9}{4}}}{5}\right|$$

$$= \tan^{-1}\left|\frac{\cancel{2}\cdot\frac{3}{\cancel{2}}}{5}\right|$$

$$= \tan^{-1}\left|\frac{3}{5}\right|$$

Ex-3.15: Find the value of λ so that the equation $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines. Find also point of intersection. $\left(\lambda = 2,\ \left(4, \frac{7}{2}\right)\right)$

Ex-3.16: Find the value of λ so that the equation $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ represents a pair of straight line. $\left(\lambda = 3, \frac{9}{2}\right)$

Example 3.17: Prove that the general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents parallel straight lines if $h^2=ab$ and $bg^2=af^2$. Prove that the distance between the two straight lines is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$.

Soln: Let the parallel lines be $lx+my+n=0$

and $lx+my+n_1=0$

Then, $ax^2+2hxy+by^2+2gx+2fy+c=(lx+my+n)(lx+my+n_1)$

By comparing, we get-

$l^2=a$ $\qquad$ $ln_1+nl=2g$; $n+n_1=\frac{2g}{l}$

$m^2=b$ $\qquad$ $mn_1+nm=2f$; $f=\frac{m(n+n_1)}{2}$

$nn_1=c$ $\qquad$ $h^2=ab=l^2m^2$

$$bg^2=m^2.\left[\frac{l(n+n_1)}{2}\right]^2$$

$$=m^2.l^2\left[\frac{(n+n_1)}{2}\right]^2$$

$$\Rightarrow bg^2=l^2f^2=af^2 \qquad =l^2\left[\frac{m(n+n_1)}{2}\right]^2$$

$$=l^2f^2$$

Also the distance between the lines $lx+my+n=0$ and $lx+my+n_1=0$ is -

$$\left|\frac{n-n_1}{\sqrt{l^2+m^2}}\right| = \frac{\sqrt{(n-n_1)^2}}{\sqrt{l^2+m^2}}$$

$$= \frac{\sqrt{(n+n_1)^2-4nn_1}}{\sqrt{l^2+m^2}}$$

$$= \frac{\sqrt{\left(\frac{2g}{l}\right)^2-4c}}{\sqrt{a+b}}$$

$$* \quad \frac{2\sqrt{g^2-ac}}{\sqrt{a(a+b)}}$$

$$= \frac{\sqrt{\frac{4g^2}{l^2}-4c}}{\sqrt{a+b}}$$

$$= \frac{\sqrt{\frac{4g^2}{a}-4c}}{\sqrt{a+b}} = \frac{\frac{\sqrt{4(g^2-ac)}}{\sqrt{a}}}{\sqrt{a+b}}$$

$$= \frac{\sqrt{\frac{4g^2-4ac}{a}}}{\sqrt{a+b}} = \frac{2\sqrt{g^2-ac}}{\sqrt{a}\sqrt{a+b}}$$

$$= \sqrt{\frac{4(g^2-ac)}{a(a+b)}}$$

Ex-3.18: If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines equidistant from the origin, show that $f^4-g^4=c(bf^2-ag^2)$

Soln:

$$\left.\begin{array}{l} lx+my+n=0 \\ l_1x+m_1y+n_1=0 \end{array}\right\rangle \text{not Parallel}$$

Here, $ll_1=a$ $\qquad lm_1+l_1m=2h$

$mm_1=b$ $\qquad ln_1+l_1n=2g$

$nn_1=c$ $\qquad mn_1+m_1n=2f$

Here, $d_1=d_2$

$$\left|\frac{n}{\sqrt{l^2+m^2}}\right|=\left|\frac{n_1}{\sqrt{l_1^2+m_1^2}}\right|$$

$$\Rightarrow \frac{n^2}{l^2+m^2}=\frac{n_1^2}{l_1^2+m_1^2}$$

$$\Rightarrow n^2(l_1^2+m_1^2)=n_1^2(l^2+m^2)$$

$$\Rightarrow n^2l_1^2-n_1^2l^2=n_1^2m^2-n^2m_1^2$$

$$\Rightarrow (nl_1+n_1l)(nl_1-n_1l)=(n_1m+nm_1)(n_1m-nm_1)$$

$$\Rightarrow (nl_1+n_1l)^2\left[(nl_1+n_1l)^2-4n_1nll_1\right]=(n_1m+nm_1)^2\left[(n_1m+nm_1)^2-4mm_1nn_1\right.$$

$$\Rightarrow (2g)^2\left[4g^2-4ac\right]=4f^2\left[4f^2-4bc\right]$$

$$\Rightarrow 4g^2(4g^2-4ac)=4f^2(4f^2-4bc)$$

$$\Rightarrow g^2(g^2-ac)=f^2(f^2-bc)$$

$$\Rightarrow g^4-g^2ac=f^4-f^2bc$$

$$\Rightarrow g^4-f^4=g^2ac-f^2bc$$

$$\Rightarrow f^4-g^4=f^2bc-g^2ac$$

$$\therefore f^4-g^4=c(bf^2-ag^2)$$

2. Prove that the equations $8x^2+8xy+2y^2+26x+13y+15=0$ represents two parallel straight lines and find the distance between them.

Sol^n:

we know, $d = 2\sqrt{\dfrac{g^2-ac}{a(a+b)}}$

Here, $8x^2+2.4.xy+2y^2+2.13.x+2.\frac{13}{2}.y+15=0$

So, $a=8, h=4, b=2, g=13, f=\frac{13}{2}, c=15$

$\therefore h^2=4^2=16$ and $ab=8.2=16$

$\therefore h^2=ab$

again, $bg^2=2.13^2=338$ and $af^2=8.\frac{169}{4}=338$

$\therefore bg^2=af^2$

$\therefore d=2\sqrt{\dfrac{13^2-8.15}{8(8+2)}}$

$=2\sqrt{\dfrac{49}{80}}=2.\dfrac{7}{\frac{4\sqrt{5}}{2}}=\dfrac{7}{2\sqrt{5}}$

Ex 3.27: If the straight lines joining the origin to the point of intersection of $3x^2-xy+3y^2+2x-3y+4=0$ and $2x+3y=K$ are at right angles prove that $6K^2-5K+52=0$

Soln: Let, $3x^2-xy+3y^2+2x-3y+4=0$

$$2x+3y = K$$

$$\Rightarrow \frac{2x+3y}{K}=1$$

$$\therefore 3x^2-xy+3y^2+(2x-3y)\frac{(2x+3y)}{K}+4\left(\frac{2x+3y}{K}\right)^2=0$$

$$\Rightarrow K^2(3x^2-xy+3y^2)+K(4x^2-9y^2)+4(4x^2+12xy+9y^2)=0$$

$$\Rightarrow 3K^2x^2-K^2xy+3K^2y^2+4Kx^2-9Ky^2+16x^2+48xy+36y^2=0$$

$$\Rightarrow x^2(3K^2+4K+16)+y^2(3K^2-9K+36)-xy(K^2-48)=0$$

Since the two straight lines are at right angles-

$a+b=0 \rightarrow$ we know

$$\Rightarrow 3K^2+4K+16+3K^2-9K+36=0$$

$$\therefore 6K^2-5K+52=0$$

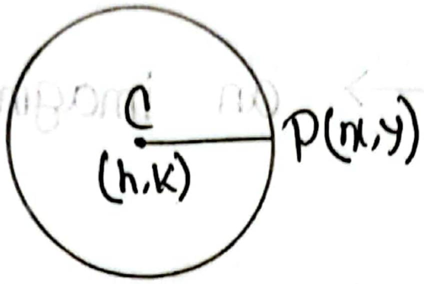
Practice 3.28: Show that the pair of straight lines joining the origin to the point of intersection of the straight lines $y = mx + c$ and the circle $x^2 + y^2 = a^2$ are at right angles $2c^2 = a^2(1 + m^2)$

Exercise

1) Show that the line joining the origin to the points common to $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x - 2y = 1$ are at right angles.

Chapter-4

Circle



$$CP^2 = r^2 = (x-h)^2 + (y-k)^2;$$

If the centre of the circle is at the origin, then the equation will be-

$$x^2+y^2=r^2$$

Another equation of Circle

$$x^2+y^2+2gx+2fy+c=0$$

Centre$(-g,-f)$ and Radius; $r=\sqrt{g^2+f^2-c}$

Note: No xy term will be present in a circle equation.

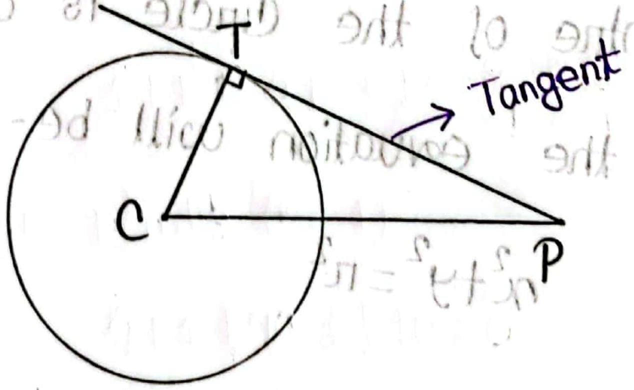
Note:

1. $g^2+f^2-c \longrightarrow '+' \longrightarrow$ Real Circle.

2. $g^2+f^2-c \longrightarrow '0' \longrightarrow$ a point circle.

3. $g^2+f^2-c \longrightarrow '-' \longrightarrow$ an imaginary circle.

$\boxplus$ Length of TANGENT From Point $P(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$



$C(-g,-f)$ & $r=\sqrt{g^2+f^2-c}$

Tangent line $\longrightarrow$ PT

$* \; PT^2 = PC^2 - r^2 = x_1^2+y_1^2+2gx_1+2fy_1+c$

$\boxplus \; PT = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

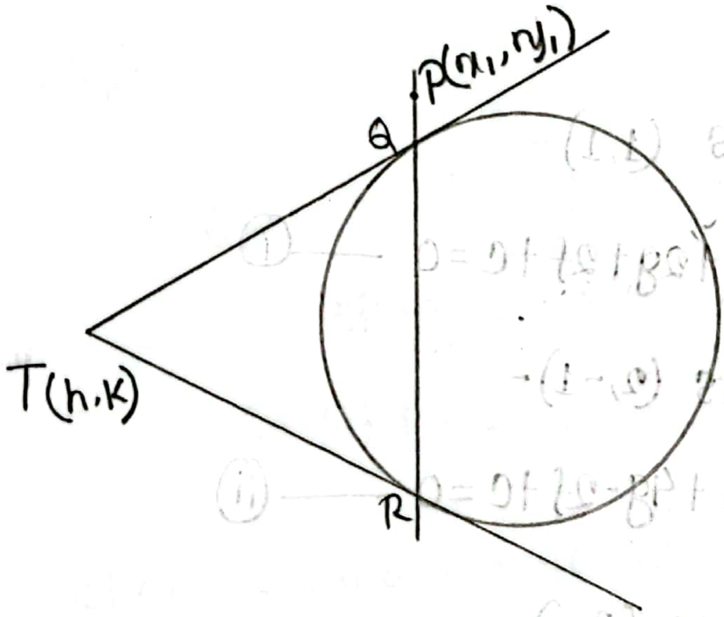
Note:

1. If $PT^2 > 0 \longrightarrow P(x_1, y_1)$ lies outside the circle

2. If $PT^2 = 0 \longrightarrow P(x_1, y_1)$ lies on the circle

3. If $PT^2 < 0 \longrightarrow P(x_1, y_1)$ lies inside the circle

4.10 POLE and POLAR



QR = Variable Chord

The Polar of (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Example 4.8: Find the equation of the circle passing through the points (1,1), (2,-1) and (3,2).

Solⁿ: Let the eqⁿ of the circle,

$$x^2+y^2+2gx+2fy+c=0.$$

The circle passes through the points (1,1), (2,-1) and (3,2).

For points (1,1) -

$$1+1+2g+2f+c=0 \quad \text{—— (i)}$$

For points (2,-1) -

$$4+1+4g-2f+c=0 \quad \text{—— (ii)}$$

For points (3,2) -

$$9+4+6g+4f+c=0 \quad \text{—— (iii)}$$

If we solve i, ii & iii =>

$$f=-\frac{1}{2},\ g=-\frac{5}{2}\ \&\ c=4$$

∴ Eqⁿ, $x^2+y^2-5x-y+4=0$

Example-4.9: Show that the points (3,4), (0,5), (-3,-4) and (-5,0) are concyclic and find the radius of the circle.

lies on a common circle.

Soln: Let the equation of the circle be,

$$x^2+y^2+2gx+2fy+c=0.$$

This passes through the points (3,4), (0,5) and (-3,-4).

Therefore,

$$6g+8f+c=-25 \quad \text{—— (i)}$$

$$10f+c=-25 \quad \text{—— (ii)}$$

$$-6g-8f+c=-25 \quad \text{—— (iii)}$$

By solving i, ii & iii - we got,

$$g=0,\ f=0,\ c=25$$

So. the equation is, $x^2+y^2=25$ —(iv)

By putting $x=-5$ & $y=0$ on eqn iv-

$$(-5)^2+0=25$$

Therefore (-5,0) lies on the circle.

The Centre of the circle is (0,0), radius = 5 unit

Example-4.15: Find the length of the tangent from $P(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$.

Soln: $P(2, 3)$ $x^2+y^2+8x+4y+8=0$

The length of the tangent, $g=4, f=2, c=8$

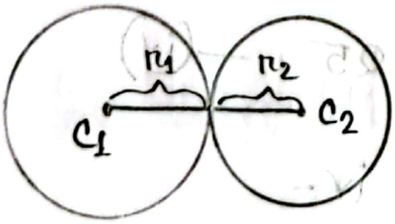
$$PT = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

$$= \sqrt{4+9+2\cdot4\cdot2+2\cdot2\cdot3+8}$$

$$= \sqrt{49}$$

$$= 7 \text{ units}$$

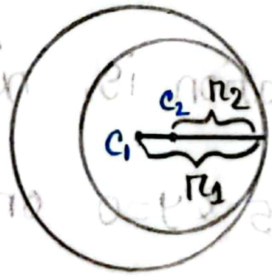
* Note:



$C_1C_2 = r_1 + r_2$

↓

for externally touching



$C_1C_2 = r_1 \sim r_2$

↓

for internally touching

Example 4.16: Determine whether the following points lie outside, on, or inside the circle $x^2+y^2-4x+4y-8=0$; $A(0,1)$, $B(5,9)$, $C(-2,3)$.

Soln: Eqn; $x^2+y^2-4x+4y-8=0$

$$AT_1^2 = 0+1-0+4-8 = -3 < 0$$

$\therefore$ point A lies inside

$$BT_1^2 = 25+81-20+36-8 = 114 > 0$$

$\therefore$ point B lies outside.

$$CT_1^2 = 4+9+8+12-8 = 25 > 0$$

$\therefore$ point C lies outside.

Example 4.19: Show that the circles $x^2+y^2-2x+6y+6=0$ and $x^2+y^2-5x+6y+15=0$ touch each other internally.

Soln: For the circle, $x^2+y^2-2x+6y+6=0$

centre is $A(1,-3)$ and radius, $r_1=\sqrt{1^2+(-3)^2-6}$

$= 2$ units

For the circle, $x^2+y^2-5x+6y+15=0$

centre is $B(\frac{5}{2},-3)$ and radius, $r_2=\sqrt{(\frac{5}{2})^2+(-3)^2-15}$

$= \frac{1}{2}$ units

Distance between, $AB = \sqrt{(1-\frac{5}{2})^2+(-3+3)^2} = \frac{3}{2}$

$\therefore r_1 - r_2 = 2 - \frac{1}{2} = \frac{3}{2}$

So, the two circles touches internally.

Exercise

7) Find the equation of the circle passing through the following points:

i) $(2,1), (1,2), (8,9)$ Ans: $x^2+y^2-10x-10y-25=0$

ii) $(0,1), (2,3), (-2,5)$ Ans: $3x^2+3y^2+2x-20y+17=0$

iii) $(5,2), (2,1), (1,4)$ Ans: $x^2+y^2-6x-6y+13=0$

12) Show that the points $(-1,2), (-2,4), (-1,3)$ and $(2,0)$ are on a circle and find its equation.

16) Prove that the tangents from $(0,5)$ to the circles $x^2+y^2+2x-4=0$ and $x^2+y^2-y+1=0$ are equal.

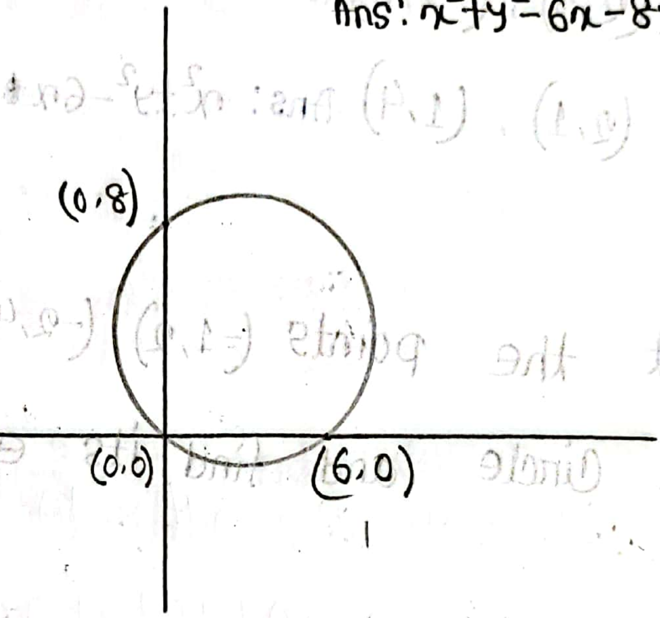
22) Does the point $(2,1)$ lie (i) on, (ii) inside or (iii) outside the circle $x^2+y^2-4x-6y+9=0$?

23) Show that the circles $x^2+y^2-2x+2y+1=0$ and $x^2+y^2+6x-4y-3=0$ touch each other externally.

Pole and Polar

20| A Circle passes through the origin and the points $(6,0)$ and $(0,8)$. Find its equation and also the equation of the tangent to the circle at the origin.

Ans: $x^2+y^2-6x-8y=0$, $3x+4y=0$



26| Show that the circles $x^2+y^2-4x+2y+1=0$ and $x^2+y^2-12x+8y+43=0$ touch each other externally.

28| Find the the length of the tangent from the origin to the circle $4x^2+4y^2+6x+7y+1=0$

38| Show that the points $\left(1, \frac{\sqrt{11}+6}{6}\right)$ lies outside the circle $3x^2+3y^2-5x-6y+4=0$