SOLVED PROBLEMS FROM DIFFERENTIATION

- For the students of CSE PU EM-I Math-105
- Do Practice all problems

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a $y = \dfrac{1+3x}{x^2+1}$ is a quotient with $\quad u = 1+3x \quad$ and $\quad v = x^2+1$

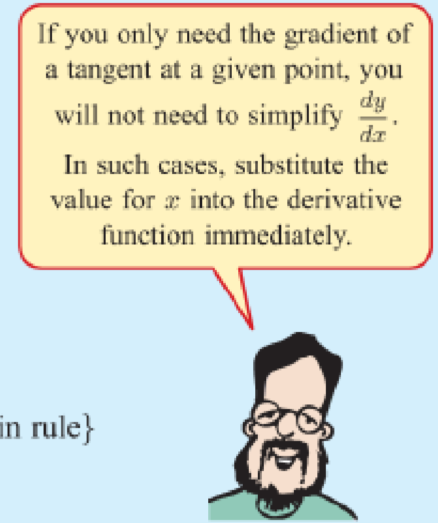
$\therefore \quad u' = 3 \quad$ and $\quad v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ $\quad$ {quotient rule}

$$= \frac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}$$

$$= \frac{3x^2+3-2x-6x^2}{(x^2+1)^2}$$

$$= \frac{3-2x-3x^2}{(x^2+1)^2}$$



b $y = \dfrac{\sqrt{x}}{(1-2x)^2}$ is a quotient with

$u = x^{\frac{1}{2}} \quad$ and $\quad v = (1-2x)^2$

$\therefore \quad u' = \frac{1}{2}x^{-\frac{1}{2}} \quad$ and $\quad v' = 2(1-2x)^1 \times (-2)$ $\quad$ {chain rule}

$\qquad\qquad\qquad\qquad\qquad\quad = -4(1-2x)$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ $\quad$ {quotient rule}

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 - x^{\frac{1}{2}} \times (-4(1-2x))}{(1-2x)^4}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$

$$= \frac{\cancel{(1-2x)}\left[\frac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^{\cancel{4}\,3}}$$

$$= \frac{1-2x+8x}{2\sqrt{x}(1-2x)^3}$$

$$= \frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

Find the gradient function for y equal to:

a $(e^x - 1)^3$

b $\dfrac{1}{\sqrt{2e^{-x}+1}}$

a
$$\begin{aligned} y &= (e^x - 1)^3 \\ &= u^3 \quad \text{where} \quad u = e^x - 1 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\} \\ &= 3u^2 \frac{du}{dx} \\ &= 3(e^x - 1)^2 \times e^x \\ &= 3e^x(e^x - 1)^2 \end{aligned}$$

b
$$\begin{aligned} y &= (2e^{-x} + 1)^{-\frac{1}{2}} \\ &= u^{-\frac{1}{2}} \quad \text{where} \quad u = 2e^{-x} + 1 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\} \\ &= -\tfrac{1}{2}u^{-\frac{3}{2}} \frac{du}{dx} \\ &= -\tfrac{1}{2}(2e^{-x} + 1)^{-\frac{3}{2}} \times 2e^{-x}(-1) \\ &= e^{-x}(2e^{-x} + 1)^{-\frac{3}{2}} \end{aligned}$$

Find the gradient function of:

a $y = \ln(1 - 3x)$

b $y = x^3 \ln x$

a
$$y = \ln(1 - 3x)$$
$$\therefore \quad \frac{dy}{dx} = \frac{-3}{1 - 3x} = \frac{3}{3x - 1}$$

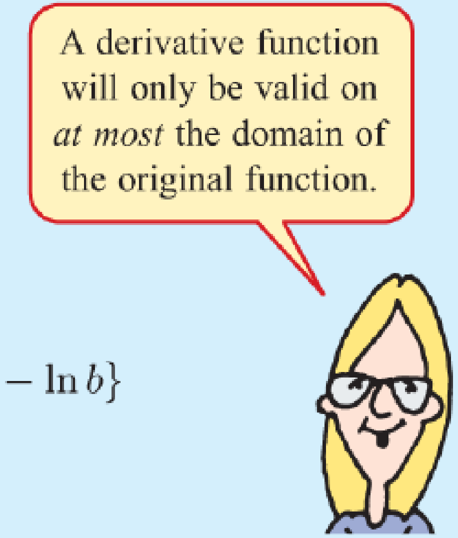
b
$$y = x^3 \ln x$$
$$\therefore \quad \frac{dy}{dx} = 3x^2 \ln x + x^3 \left(\frac{1}{x}\right) \quad \{\text{product rule}\}$$
$$= 3x^2 \ln x + x^2$$
$$= x^2(3 \ln x + 1)$$

Differentiate with respect to x:

a $y = \ln(xe^{-x})$ **b** $y = \ln\left[\dfrac{x^2}{(x+2)(x-3)}\right]$

a
$$\begin{aligned} y &= \ln(xe^{-x}) \\ &= \ln x + \ln e^{-x} \qquad \{\ln(ab) = \ln a + \ln b\} \\ &= \ln x - x \qquad \{\ln e^a = a\} \end{aligned}$$
$$\therefore \frac{dy}{dx} = \frac{1}{x} - 1$$

b
$$\begin{aligned} y &= \ln\left[\frac{x^2}{(x+2)(x-3)}\right] \\ &= \ln x^2 - \ln[(x+2)(x-3)] \qquad \left\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right\} \\ &= 2\ln x - [\ln(x+2) + \ln(x-3)] \\ &= 2\ln x - \ln(x+2) - \ln(x-3) \end{aligned}$$
$$\therefore \frac{dy}{dx} = \frac{2}{x} - \frac{1}{x+2} - \frac{1}{x-3}$$



Differentiate with respect to x:

a $x \sin x$

b $4 \tan^2 3x$

a

$$y = x \sin x$$

$$\therefore \quad \frac{dy}{dx} = (1) \sin x + (x) \cos x \qquad \{\text{product rule}\}$$

$$= \sin x + x \cos x$$

b

$$y = 4 \tan^2 3x$$

$$= 4u^2 \quad \text{where} \quad u = \tan 3x$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \qquad \{\text{chain rule}\}$$

$$\therefore \quad \frac{dy}{dx} = 8u \times \frac{du}{dx}$$

$$= 8 \tan 3x \times \frac{3}{\cos^2 3x}$$

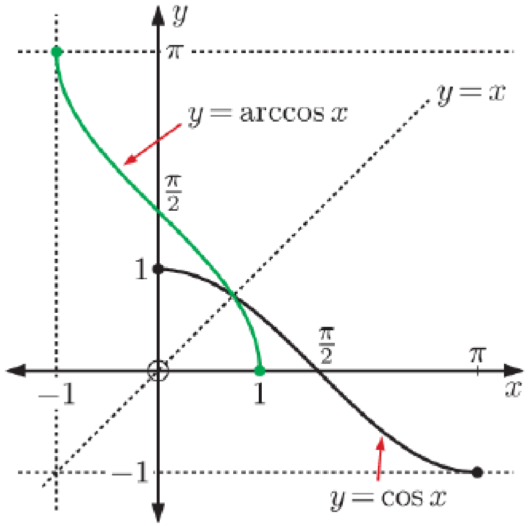
$$= \frac{24 \sin 3x}{\cos^3 3x}$$

Find $f''(x)$ given that $f(x) = x\cos x + \dfrac{1}{x}$.

$$
\begin{aligned}
f(x) &= x\cos x + x^{-1} \\
\therefore\quad f'(x) &= (1)\cos x + (x)(-\sin x) - x^{-2} && \{\text{product rule}\} \\
&= \cos x - x\sin x - x^{-2} \\
\therefore\quad f''(x) &= -\sin x - (1)\sin x - (x)\cos x + 2x^{-3} && \{\text{product rule}\} \\
&= -2\sin x - x\cos x + \frac{2}{x^3}
\end{aligned}
$$

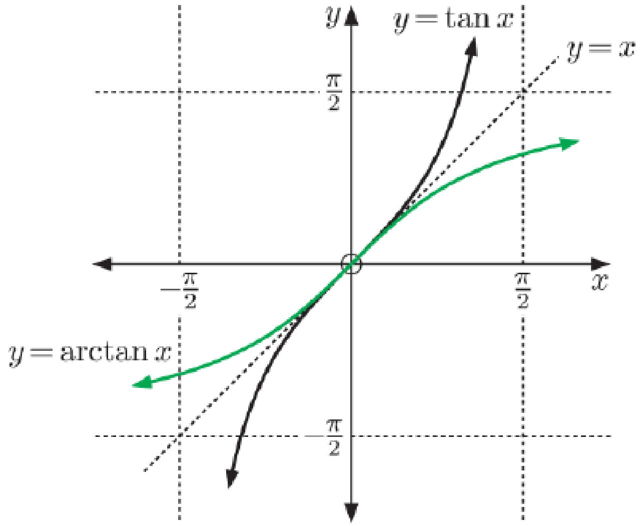
If $y = \arcsin x$, then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$.

$f(x) = \cos x$, $0 \leqslant x \leqslant \pi$ has inverse function $f^{-1}(x) = \arccos x$.



If $y = \arccos x$, then
$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

$f(x) = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ has inverse function $f^{-1}(x) = \arctan x$.



If $y = \arctan x$, then
$$\frac{dy}{dx} = \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

Find $\dfrac{dy}{dx}$ if: **a** $x^2 + y^3 = 8$ **b** $x + x^2y + y^3 = 100$

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a

$$x^2 + y^3 = 8$$

$$\therefore \quad \frac{d}{dx}(x^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(8)$$

$$\therefore \quad 2x + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2x}{3y^2}$$

b

$$x + x^2y + y^3 = 100$$

$$\therefore \quad \frac{d}{dx}(x) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(100)$$

$$\therefore \quad 1 + \left[2xy + x^2 \frac{dy}{dx}\right] + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \quad (x^2 + 3y^2) \frac{dy}{dx} = -1 - 2xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1 - 2xy}{x^2 + 3y^2}$$

Find the gradient of the tangent to $x^2 + y^3 = 5$ at the point where $x = 2$.

We first find $\dfrac{dy}{dx}$: $\quad 2x + 3y^2 \dfrac{dy}{dx} = 0$

{implicit differentiation}

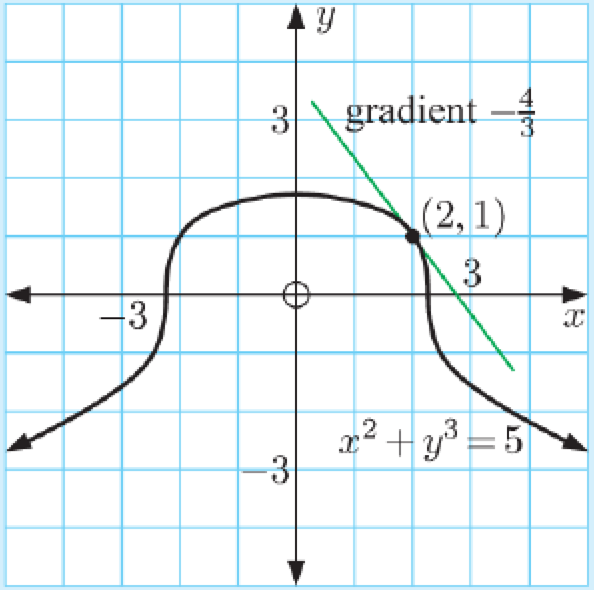
$$\therefore \quad \frac{dy}{dx} = \frac{-2x}{3y^2}$$

When $x = 2$, $\quad 4 + y^3 = 5$

$$\therefore \quad y = 1$$

$\therefore$ at the point (2, 1), $\quad \dfrac{dy}{dx} = \dfrac{-2(2)}{3(1)^2} = -\frac{4}{3}$

$\therefore$ the gradient of the tangent at $x = 2$ is $-\frac{4}{3}$.



Example

If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution

$$x^y = e^{x-y}$$

Taking logarithm on both sides,

$$y \log x = (x-y)$$

$$y(1+\log x) = x$$

$$y = \frac{x}{1+\log x}$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{(1+\log x)(1) - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

1. Consider the relation $xy^2 - 3x - y = 0$.

a Find $\frac{dy}{dx}$.

b Hence find the gradient of the tangent to the relation at:

i $(0, 0)$ **ii** $(2, 2)$

2. Consider $y = \sqrt{x}(3 - x)^2$.

a Show that $\frac{dy}{dx} = \frac{(3 - x)(3 - 5x)}{2\sqrt{x}}$.

b Find the x-coordinates of all points on $y = \sqrt{x}(3 - x)^2$ where the tangent is horizontal.

c State the domain of $\frac{dy}{dx}$. Discuss how it differs from the domain of the original function.

Derivatives of Composite and Parametric Function

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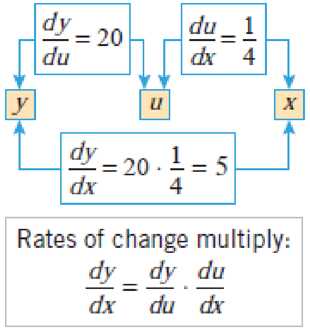
The cost of a car trip is a combination of fuel efficiency and the cost of gasoline.



DERIVATIVES OF COMPOSITIONS

Suppose you are traveling to school in your car, which gets 20 miles per gallon of gasoline. The number of miles you can travel in your car without refueling is a function of the number of gallons of gas you have in the gas tank. In symbols, if y is the number of miles you can travel and u is the number of gallons of gas you have initially, then y is a function of u, or $y = f(u)$. As you continue your travels, you note that your local service station is selling gasoline for \$4 per gallon. The number of gallons of gas you have initially is a function of the amount of money you spend for that gas. If x is the number of dollars you spend on gas, then $u = g(x)$. Now 20 miles per gallon is the rate at which your mileage changes with respect to the amount of gasoline you use, so

$$f'(u) = \frac{dy}{du} = 20 \text{ miles per gallon}$$

Similarly, since gasoline costs \$4 per gallon, each dollar you spend will give you 1/4 of a gallon of gas, and

$$g'(x) = \frac{du}{dx} = \frac{1}{4} \text{ gallons per dollar}$$

Notice that the number of miles you can travel is also a function of the number of dollars you spend on gasoline. This fact is expressible as the composition of functions

$$y = f(u) = f(g(x))$$

You might be interested in how many miles you can travel per dollar, which is dy/dx. Intuition suggests that rates of change multiply in this case (see Figure 2.6.1), so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{20 \text{ miles}}{1 \text{ gallon}} \cdot \frac{1 \text{ gallon}}{4 \text{ dollars}} = \frac{20 \text{ miles}}{4 \text{ dollars}} = 5 \text{ miles per dollar}$$

Composite ….Chain rule

▶ **Example** Find dw/dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution. In this case the chain rule computations take the form

$$\begin{aligned} \frac{dw}{dt} &= \frac{dw}{dx} \cdot \frac{dx}{dt} \\ &= \frac{d}{dx}[\tan x] \cdot \frac{d}{dt}[4t^3 + t] \\ &= (\sec^2 x) \cdot (12t^2 + 1) \\ &= [\sec^2(4t^3 + t)] \cdot (12t^2 + 1) = (12t^2 + 1)\sec^2(4t^3 + t) \end{aligned}$$ ◀

Differentiation of Parametric Equation

$$\frac{dy}{dx} = \frac{dy/dt}{\underbrace{dx/dt}_{\neq 0}}$$

$$x = f(t)$$
$$y = g(t)$$

SECOND DERIVATIVES OF PARAMETRIC EQUATIONS

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\underbrace{\left(\frac{dx}{dt}\right)}_{\neq 0}}$$

find $\frac{dy}{dx}$ when $x = t^2$, $y = 2t$

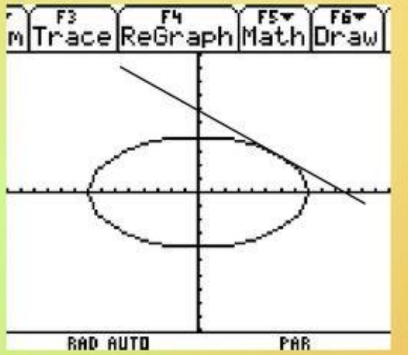
$$x = t^2 \qquad y = 2t$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx} = 2 . \frac{1}{2t}$$

$$\underline{\frac{dy}{dx} = \frac{1}{t}}$$

Derivative of Parametric Equations



- Consider the graph of
  x = 2 sin t, y = cos t
- We seek the slope, that is $\frac{dy}{dx}$
- For parametric equations $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
  - For our example $\frac{dy}{dx} = \frac{-\sin t}{2\cos t} = \frac{-\tan t}{2}$

By using parametric differentiation, determine the derivative of $5x^3 + x^2 - 2$ with respect to $4x^2 + 8$.

If $y = f(x)$ and $z = g(x)$ then:

$g^{-1}(z) = x$

$$\frac{dx}{dz} = \frac{1}{\frac{dz}{dx}}$$

$$\frac{dy}{dz} = \frac{dy}{dx} \times \frac{dx}{dz}$$

$$= \frac{dy}{dx} \div \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(5x^3 + x^2 - 2\right) = 15x^2 + 2x$$

$$\frac{dz}{dx} = \frac{d}{dx}\left(4x^2 + 8\right) = 8x$$

$y = 5x^3 + x^2 - 2$

$z = 4x^2 + 8$

$$\frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx}$$

$$= \boxed{\frac{15x^2 + 2x}{8x}}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}},x\neq\pm1$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}},x\neq\pm1$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}},x\neq\pm1,0$$

$$\frac{d}{dx}\left(\csc^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}},x\neq\pm1,0$$

If $x = e^{\sin^{-1}\theta}$ and $y = e^{-\cos^{-1}\theta}$, then find the value of $\frac{dy}{dx}$.