Conic Section

General Equation Of Conic Section:

The general equation of Second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

This Eq^n always Represent Conic.

Nature of Conic

Let $ax^2+2hxy+by^2+2gx+2fy+c=0$ be eq^n of conic.

and $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$$\Delta = abc+2fgh-af^2-bg^2-ch^2$$

- In Eq^n $ax^2+2hxy+by^2+2gx+2fy+c=0$

  $\Delta \neq 0,\ ab-h^2>0 \longrightarrow$ ellipse $(e<1)$

  $\Delta \neq 0,\ ab-h^2<0 \longrightarrow$ Hyperbola $(e>1)$

  $\Delta \neq 0,\ ab-h^2=0 \longrightarrow$ Parabola $(e=1)$

Centre of Conic:

$$(x_1, y_1) = \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$$

$$C_1 = gx_1 + fy_1 + c$$

General Eq^n becomes,

$$ax^2 + 2hxy + by^2 + C_1 = 0$$

$$\Rightarrow \left(\frac{-a}{C_1}\right)x^2 + 2\left(\frac{-h}{C_1}\right)xy + \left(\frac{-b}{C_1}\right)y^2 = 1$$

$$Ax^2 + 2Hxy + By^2 = 1$$

For the length of Axes

$$\frac{1}{r^4} - (A+B)\frac{1}{r^2} + AB - H^2 = 0$$

$$\Rightarrow \alpha^2 - (A+B)\alpha + AB - H^2 = 0 \quad \left[\alpha = \frac{1}{r^2}\right]$$

$$\downarrow$$

$$\alpha_1, \alpha_2$$

$$\therefore r_1 = \alpha_1, \quad r_2 = \alpha_2$$

// length $2r_1$ & $2r_2$

Standard Form: $\frac{x^2}{r_1^2} \pm \frac{y^2}{r_2^2} = 1$; $+ \longrightarrow$ ellipse

$- \longrightarrow$ Hyperbola

* Reduce the eqn $x^2+12xy-4y^2-6x+4y+9=0$ to the Standard form.

Soln:

Given, $x^2+12xy-4y^2-6x+4y+9=0$ ——— ①

Here, $ab-h^2 = 1\cdot(-4)-6^2 = -40 < 0$;

and $\Delta = c(ab-h^2)+2fgh-af^2-bg^2$

$= -400 \neq 0$

So, Eqn ① represents a hyperbola.

Centre is at $(x_1, y_1) = \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$

$= \left(0, \frac{1}{2}\right)$

Now, $C_1 = gx_1+fy_1+c = -3\cdot 0+2\cdot\frac{1}{2}+9 = 10$

Eqⁿ ① w.r.t centre $(0, \frac{1}{2})$ as origin is

$$x^2 + 12xy - 4y^2 + 10 = 0$$

$$\Rightarrow -\frac{1}{10}x^2 + \frac{12}{10}xy + \frac{4}{10}y^2 = 1 \quad \text{——— ②}$$

$$A = -\frac{1}{10}, \quad B = \frac{4}{10}, \quad H = \frac{-6}{10}$$

For the length of axes

$$\frac{1}{r^4} - (A+B)\frac{1}{r^2} + AB - H^2 = 0$$

$$\Rightarrow \alpha^2 - \frac{3}{10}\alpha - \frac{4+36}{(10)^2} = 0$$

$$\Rightarrow 10\alpha^2 - 3\alpha - 4 = 0$$

$$\Rightarrow (2\alpha+1)(5\alpha-4) = 0$$

$$\Rightarrow \left(\frac{2}{r_2^2}+1\right)\left(\frac{5}{r_1^2}-4\right) = 0$$

$$\therefore r_1^2 = \frac{5}{4}, \quad r_2^2 = -2$$

$$\therefore r_1 = \frac{\sqrt{5}}{2} \qquad \therefore r_2 = i\sqrt{2}$$

length of the axes are

$$2r_1 = \sqrt{5}, \; 2r_2 = 2\sqrt{2}$$

Standard form of eqn -

$$\frac{x^2}{\left(\frac{\sqrt{5}}{2}\right)^2} - \frac{y^2}{(\sqrt{2})^2} = 1$$

* Reduce the Eqn $x^2+2xy+y^2-6x-2y+4=0$ to the standard form.

Soln: Given. Eqn: $x^2+2xy+y^2-6x-2y+4=0$ —— (i)

$$\therefore \Delta = c(ab-h^2)+2fgh-af^2-bg^2$$

$$= -4 \neq 0$$

$$\therefore ab-h^2 = 1\cdot 1 - 1^2 = 0$$

So, Eqn (1) represents Parabola.

(i) $\Rightarrow (x+y)^2 = 6x+2y-4$

But $(x+y)=0$ and $6x+2y-4=0$ are not at right angle. Introducing co-efficient λ.

$(x+y+\lambda)^2 = (2\lambda+6)x+(2\lambda+2)y+\lambda^2-4$ —— (ii)

Now the lines $x+y+\lambda$ and $(2\lambda+6)x+(2\lambda+2)y+\lambda^2-4=0$ will be at right angle is-

$$(2\lambda+6)\cdot 1+(2\lambda+2)\cdot 1=0$$

$$\Rightarrow 4\lambda=-8$$

$$\therefore \lambda=-2$$

(ii) gives $(x+y-2)^2 = 2x-2y$

$$\Rightarrow \left(\frac{x+y-2}{\sqrt{1^2+1^2}}\right)^2\times 2 = 2\cdot\frac{x-y}{\sqrt{1^2+1^2}}\cdot\sqrt{2}$$

$$\Rightarrow \left(\frac{x+y-2}{\sqrt{2}}\right)^2 = \sqrt{2}\cdot\left(\frac{x-y}{\sqrt{2}}\right)$$

Let, $X=\frac{x-y}{\sqrt{2}}$, $Y=\frac{x+y-2}{\sqrt{2}}$

$$\therefore P=\frac{\sqrt{2}}{4}=\frac{1}{2\sqrt{2}}$$

$\therefore$ Standard form of Eqn (1) is-

$$Y^2=4pX$$

Practice: Reduce the Eqⁿ to its standard form -

1) $8x^2 + 4xy + 5y^2 - 16x - 14y + 13 = 0$

2) $17x^2 + 12xy + 8y^2 - 46x - 28y + 33 = 0$

3) $x^2 - 4xy + 4y^2 + 10x - 8y + 13 = 0$