1. Partial derivatives

The concept of partial derivative plays a vital role in differential calculus. The different ways of limit discussed in the previous section, yields different type of partial derivatives of a function.

1.1. Definitions. Consider a real valued function $z = f(x, y)$ defined on $E \subset \mathbb{R}^2$ such that E contains a neighbourhood of $(a, b) \in \mathbb{R}^2$. Let Δa be a change in a. If the limit,

$$\lim_{\Delta a \to 0} \frac{f(a + \Delta a, b) - f(a, b)}{\Delta a}$$

exists, then it is called the *partial derivative of f with respect to x at* (a, b) and is denoted by $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ or $f_x(a, b)$ or $z_x(a, b)$. Similarly, let Δb be a change in b. If the limit,

$$\lim_{\Delta b \to 0} \frac{f(a, b + \Delta b) - f(a, b)}{\Delta b}$$

exists, then it is called the *partial derivative of f with respect to y at* (a, b) and is denoted by $\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$ or $f_y(a, b)$ or $z_y(a, b)$.

Notations. If the partial derivatives f_x and f_y exist at each point of E, then they are also the real valued functions on E. Further, we can obtain the partial derivatives of these functions, if they are differentiable. In these cases, we fix up the following notations.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right); \text{ and } f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

The notations of derivatives of order greater than two should be clear from the above pattern.

Proposition. *Consider a real valued function $z = f(x, y)$ defined on $E \subset \mathbb{R}^2$ such that E contains a neighbourhood of $(a, b) \in \mathbb{R}^2$. If f_{xy} and f_{yx} exist and are continuous, then $f_{xy} = f_{yx}$.*

Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Solution. We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

so that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3 \quad \blacktriangleleft$$

Example. For $u = x^3 - 3xy^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Also prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

SOLUTION. Here $u = x^3 - 3xy^2$. Hence,

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2; \quad \frac{\partial u}{\partial y} = -6xy; \quad \frac{\partial^2 u}{\partial x \partial y} = -6y = \frac{\partial^2 u}{\partial y \partial x}.$$

$$\frac{\partial^2 u}{\partial x^2} = 6x; \quad \frac{\partial^2 u}{\partial y^2} = -6x.$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

$\square$

If $u = e^{xyz}$, find the value of $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$

Solution: Let
$$u = e^{xyz}$$

then
$$\frac{\partial u}{\partial z} = xye^{xyz}$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = e^{xyz}(x) + e^{xyz}(xz)(xy)$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = e^{xyz}(x + x^2 yz)$$

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = e^{xyz}(1 + 2xyz) + (x + x^2\ yz)\ yz\,.\,e^{xyz}$$
$$= e^{xyz}(1 + 3xyz + x^2 y^2 z^2)$$

If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Solution: Let $u = \log(x^3 + y^3 + z^3 - 3xyz)$ ...(1)

then from (1), we have

$$\frac{\partial u}{dx} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \cdot \left(3x^2 - 3yz\right)$$

$$= \frac{3\left(x^2 - yz\right)}{x^3 + y^3 + z^3 - 3xyz} \quad ...(2)$$

Similarly, we have

$$\frac{\partial u}{dy} = \frac{3\left(y^2 - xz\right)}{x^3 + y^3 + z^3 - 3xyz} \quad ...(3)$$

and

$$\frac{\partial u}{\partial z} = \frac{3\left(z^2 - xy\right)}{x^3 + y^3 + z^3 - 3xyz} \quad ...(4)$$

By adding (2), (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}$$

Homogeneous functions

Let us observe the following expressions carefully.

(1) $f_1(x,y) = x^2y^4 - x^3y^3 + xy^5$.

(2) $f_2(x,y) = x^4y^4 - x^5y^3 + x^6y^2$.

The combined degree of x and y in each term of the first expression is 6 and that in the second expression is 8. Can we determine whether the combined degree of x and y in each term of the expression $\frac{x}{x^4+y^4}$ is same or not? It seems difficult to determine. Let us develop the following tests.

Test 1: Let us take $t = \frac{y}{x}$. Then

$$x^2y^4 - x^3y^3 + xy^5 = x^6(t^4 - t^3 + t^5) = x^6 f(t)$$

and

$$x^4y^4 - x^5y^3 + x^6y^2 = x^8(t^4 - t^3 + t^2) = x^8 g(t),$$

where f and g are functions of one variable t.

Test 2: Now, let us replace x by tx and y by ty. Then

$$f_1(tx,ty) = (tx)^2(ty)^4 - (tx)^3(ty)^3 + (tx)(ty)^5 = t^6 f_1(x,y)$$

and

$$f_2(tx,ty) = (tx)^4(ty)^4 - (tx)^5(ty)^3 + (tx)^6(ty)^2 = t^8 f_2(x,y).$$

Definitions. A function $z = f(x,y)$ is said to be a *homogeneous function of degree* r, if $f(tx,ty) = t^r f(x,y)$ for some real number r. Otherwise, f is said to be a *nonhomogeneous function*.

In general, if $f(x_1, x_2, \ldots. x_m)$ be a homogeneous function of degree n, then

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_m \frac{\partial f}{\partial x_m} = n \cdot f$$

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Verify Euler's Theorem when

$$f(x, y, z) = axy + byz + czx$$

Solution: Let $f(x, y, z) = axy + byz + czx$ ...(1)

then from (1), we have

$$\frac{\partial f}{\partial x} = ay + cz \Rightarrow x \frac{\partial f}{\partial x} = axy + czx \qquad \ldots(2)$$

Again from (1), we have

$$\frac{\partial f}{\partial y} = ax + bz \Rightarrow y \frac{\partial f}{\partial y} = axy + byz \qquad \ldots(3)$$

and

$$\frac{\partial f}{\partial z} = by + cx \Rightarrow z \frac{\partial f}{\partial y} = bzy + czx \qquad \ldots(4)$$

Then adding (2), (3) and (4), we have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = 2\,(axy + byz + czx)$$

$$= 2f(x, y, z)$$

which verifies Euler's Theorem in this case.

✧ If $u = tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Solution:

Here u is not a homogenous function but $\tan u = \frac{x^3+y^3}{x-y}$ is a homogenous

fucntion of degree 2

i.e., $x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \,.\, \cos^2 u = \sin 2u$