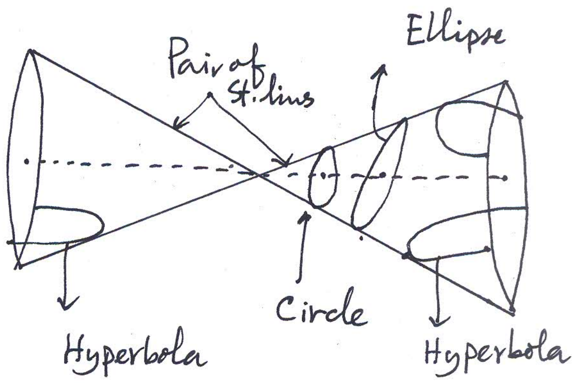
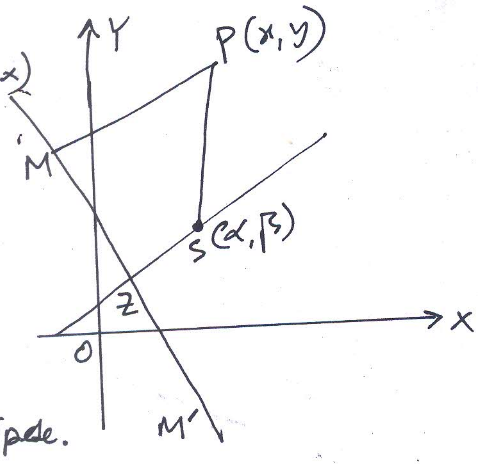
Conic Section and Coniceid in general:

- **Defination of Conic section:** It is the section of a Cone by a plane. In general, it is the curve of projection of the Cone one the Co-ordinate planes.



- **Conic section regarded as locus:**



Let MM′ be a fixed line (directrix) and S is a fixed point (focus). P be a point on the locus such that

$$\frac{PS}{PM} = \text{Constant } (=e)$$

e is known as eccentricity.

If $e<1$ then locus of P is an ellipse.
$e=1 \rightarrow$ Parabola
$e>1 \rightarrow$ Hyperbola.

- In equation $ax^2+2hxy+by^2+2gx+2fy+c=0$

$\Delta \neq 0$, $ab-h^2>0$ ellipse
$\Delta \neq 0$, $ab-h^2<0$ Hyperbola
$\Delta \neq 0$; $ab-h^2=0$ Parabola

1(a)

General Eqn of 2nd degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \longrightarrow (1)$

represents

i) An ellipse if $ab - h^2 > 0$; $\Delta \neq 0$ $(e<1)$

ii) * a Hyperbola if $\Delta \neq 0$, $ab - h^2 < 0$ $(e>1)$

* if $\Delta \neq 0$, $ab - h^2 < 0$ & $a+b=0$

rectangular hyperbola $\boxed{xy = c}$

iii) a parabola if $\Delta \neq 0$, $ab - h^2 = 0$. $(e=1)$

** Centre of the Conic

$$(x_1, y_1) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

$$c_1 = gx_1 + fy_1 + c$$

(1) becomes $ax^2 + 2hxy + by^2 + c_1 = 0$

$$\Rightarrow (-a/c_1)x^2 + 2(-h/c_1)xy + (-b/c_1)y^2 = 1$$

$$Ax^2 + 2Hxy + By^2 = 1$$

// For the length of axis

$$\frac{1}{r^4} - (A+B)\frac{1}{r^2} + AB - H^2 = 0$$

$$\Rightarrow \alpha^2 - (A+B)\alpha + AB - H^2 = 0 \quad [\alpha = 1/r^2]$$

$\downarrow$ solving α_1, α_2

$\therefore$ $r_1 = \alpha_1$, $r_2 = \alpha_2$

// length $2r_1$ & $2r_2$

// Standard form $\frac{x^2}{r_1^2} \pm \frac{y^2}{r_2^2} = 1$.

// $e = 1 - r_2^2/r_1^2$

(2)

Show that the general equation of 2nd degree in x, y represents a conic.

Let the general equation of 2nd degree be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \longrightarrow (1)$$

Let the axes be turned through an angle θ. Replacing x and y by $(x\cos\theta - y\sin\theta)$, $(x\sin\theta + y\cos\theta)$ then equation (1) becomes

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0 \longrightarrow (2)$$

where
$$A = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$$
$$B = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$$
$$H = h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta$$
$$G = g\cos\theta + f\sin\theta$$
$$F = f\cos\theta - g\sin\theta$$
$$C = c$$

From the invariant condition we have

$$a+b = A+B, \quad ab-h^2 = AB - H^2$$

Choosing θ such that $H = 0$

ie, $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$

$$\therefore \quad a+b = A+B \text{ and } ab-h^2 = AB \longrightarrow (3)$$

From (2) $\quad Ax^2 + By^2 + 2Gx + 2Fy + C = 0 \longrightarrow (4)$

$$\Rightarrow A\left(x + \frac{G}{A}\right)^2 + B\left(y + F/B\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C$$

Shifting the origin to the point $(-G/A, -F/B)$

$$Ax^2 + By^2 = K$$

$$\therefore \frac{x^2}{K/A} + \frac{y^2}{K/B} = 1 \longrightarrow (5)$$

If $K = 0$ in (5) then it will represent a pair of straight lines.

If $K \neq 0$ and K/A, K/B are both +ve or both -ve then ellipse

$\therefore \frac{K^2}{AB} > 0$ ie, $AB > 0$ ie, $ab - h^2 > 0$

If K/A and K/B are opposite sign then $\frac{K^2}{AB} < 0$ ie, $AB < 0$ 'ab$-h^2 < 0$' then heperbola.

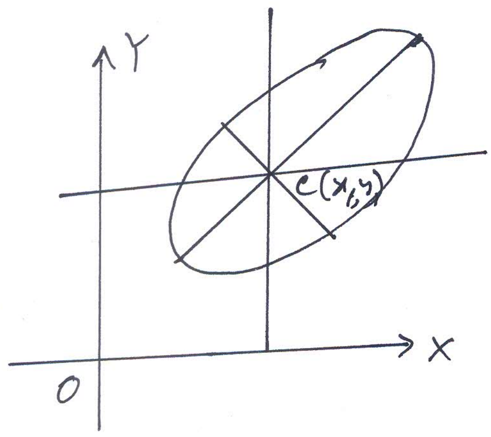
To Find the Co-ordinats of Centre:

Let the Conic be

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \longrightarrow (1)$$

$$\Delta \neq 0,\quad ab - h^2 \neq 0 \quad \text{(For central Conic)}$$

Let $c(x_1, y_1)$ be the co-ordinats of centre. Shifting the origin to the point (x_1, y_1) by replacing x by $(x + x_1)$ and y by $(y + y_1)$ in (1). then (1) becomes



$$ax^2 + 2hxy + by^2 + 2(ax_1 + hy_1 + g)x + 2(hx_1 + by_1 + f)y + S_1 = 0 \longrightarrow (2)$$

Eqn (2) will reduces to $ax^2 + 2hxy + by^2 + c_1 = 0 \longrightarrow (3)$

where

$$S_1 = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

$$c_1 = gx_1 + fy_1 + c$$

ie,

$$ax_1 + hy_1 + g = 0 \longrightarrow (4)$$

$$hx_1 + by_1 + f = 0 \longrightarrow (5)$$

After solving from (4) & (5)

$$x_1 = \frac{hf - bg}{ab - h^2}, \quad y_1 = \frac{gh - af}{ab - h^2}$$

ie, Centre is at $\left(\frac{hf - bg}{ab - h^2}, \frac{gf - af}{ab - h^2}\right)$.

From eqn (3)

$$\left(-\frac{a}{c_1}\right)x^2 + 2\left(-h/c_1\right)xy + \left(-b/c_1\right)y^2 = 1$$

ie,

$$Ax^2 + 2Hxy + By^2 = 1 \longrightarrow (6)$$

Equation (6) is the standard form of eqn (1).

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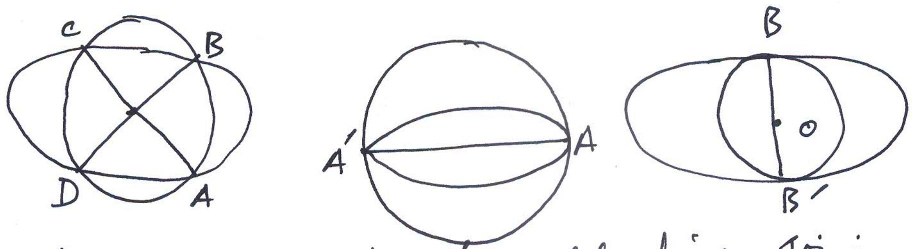
To find the Lengths and position of Central Conic:

(24)

Let the equation of curve be

$$ax^2 + 2hxy + by^2 = 1 \longrightarrow (1)$$

Taking a circle of radius r and centre at $(0,0)$ be $x^2+y^2=r^2 \longrightarrow (2)$



Equation of the pair of straight lines joining the origin to the interception of the curve (1) and the circle (2) is

$$ax^2 + 2hxy + by^2 = \frac{x^2+y^2}{r^2}$$

$$\Rightarrow (a - 1/r^2)x^2 + 2hxy + (b - 1/r^2)y^2 = 0 \longrightarrow (3)$$

If the equation (3) represent two coincident st. lines then

$$(a-1/r^2)(b-1/r^2) = h^2 \qquad \left[\begin{array}{l} b^2-4ac=0 \\ 4h^2=4ac \\ h^2=ac \end{array}\right]$$

$$\Rightarrow 1/r^4 - (a+b)\frac{1}{r^2} + (ab-h^2) = 0 \longrightarrow (4)$$

Equation (4) is a quadratic in r^2 so it gives two values of r^2 say r_1^2 and r_2^2.

Length of axes are $2r_1$ and $2r_2$. Equation of axes are given by from (3) using (4)

$$(a - 1/r^2)x + hy = 0 \longrightarrow (5)$$

$\therefore$ Equation of major axis is $(a - 1/r_1^2)x + hy = 0$

Equation of minor axis is $(a - \frac{1}{r_2^2})x + hy = 0$.

with repect to centre as origin.

Equations w. r. to old origin becomes

$$\left(a - \frac{1}{r_1^2}\right)(x-x_1) + h(y-y_1) = 0$$

and $\left(a - \frac{1}{r_2^2}\right)(x-x_1) + h(y-y_1) = 0$

Standard form of eqn (1) is

$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1$; which is an ellipse.

$\frac{x^2}{r_1^2} - \frac{y^2}{r_2^2} = 1$; which is an hyperbola.

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Ex-10 Reduce the equation $x^2+12xy-4y^2-6x+4y+9=0$ to the standard form. Find also the equations of latus rectum, directrices and axes.

Sol^n: Given $F(x,y) = x^2+12xy-4y^2-6x+4y+9=0 \longrightarrow (1)$

Here, $ab-h^2 = 1(-4)-6^2 = -40<0$

and $\Delta = c(ab-h^2)+2fgh-af^2-bg^2$

$= -400 \neq 0$

Eq^n (1) will represent a hyperbola.

centre is at $(x_1, y_1) = \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$

$= (0, 1/2)$

Now $c_1 = gx_1+fy_1+c = -3.0+2.\frac{1}{2}+9 = 10$

Eq^n (1) w.r.to centre $(0, 1/2)$ as origin is

$x^2+12xy-4y^2+10=0$

$\Rightarrow -\frac{1}{10}x^2 - \frac{12}{10}xy + \frac{4}{10}y^2 = 1 \longrightarrow (2)$

$A = -\frac{1}{10}, \quad B = \frac{4}{10}, \quad H = -\frac{6}{10}$

For the length of axes

$\frac{1}{r^4} - (A+B)\frac{1}{r^2} + AB-H^2 = 0$

$\Rightarrow \alpha^2 - \frac{3}{10}\alpha - \frac{4+36}{(10)^2} = 0 \quad [\alpha = \frac{1}{r^2}]$

$\Rightarrow 10\alpha^2 - 3\alpha - 4 = 0$

$\Rightarrow (2\alpha+1)(5\alpha-4)=0 \quad \Rightarrow \left(\frac{2}{r^2}+1\right)\left(\frac{5}{r^2}-4\right)=0$

$\therefore r_1^2 = 5/4, \quad r_2^2 = -2$

$\therefore r_1 = \frac{\sqrt{5}}{2}, \quad r_2 = i\sqrt{2}$

$\therefore$ Length of the axes are $2r_1 = \sqrt{5}$, $2r_2 = 2\sqrt{2}$

Standard form of eq^n (1) is

$$\frac{x^2}{(\sqrt{5}/2)^2} - \frac{y^2}{(\sqrt{2})^2} = 1 \longrightarrow (3)$$

1) Eccentricity e is given by, $e^2 = 1 - \frac{r_2^2}{r_1^2} = 1 + \frac{2}{5/4} = \frac{13}{5}$

$\therefore e = \sqrt{13/5}$

(b)

2) E_q^n of the transverse axis

$$\left(A-\frac{1}{r_1^2}\right)x+Hy=0 \Rightarrow \left(-\frac{1}{10}-\frac{4}{5}\right)x-\frac{6}{10}y=0$$

$\Rightarrow 3x+2y=0$ w.r.to $(0, \frac{1}{2})$ as origin.

& $3(x-0)+2(y-\frac{1}{2})=0 \Rightarrow 3x+2y-1=0$ w.r.to old origin

3) E_q^n of the Conjugate axes is

$$\left(A-\frac{1}{r_2^2}\right)x+Hy=0$$

$\Rightarrow \left(-\frac{1}{10}+\frac{1}{2}\right)x-\frac{6}{10}y=0 \Rightarrow 2x-3y=0$ w.r.to $(0,\frac{1}{2})$ as origin

& $2(x-0)-3(y-\frac{1}{2})=0$

$2x-3y+\frac{3}{2}=0$ w.r.to old origin.

4) For Foci, S & S', $d=\pm r_1 e = \pm\sqrt{5/2}\cdot\sqrt{\frac{13}{5}} = \pm\sqrt{\frac{13}{2}}$

For foot of directrices Z & Z', $d=\pm r_1/e = \pm\frac{5}{2\sqrt{13}}$

For the vertices A & A', $d=\pm r_1 = \frac{\sqrt{5}}{2}$

For the vertices A & B', $d=\pm r_2 = \pm i\sqrt{2}$.

* Slope of the transverse axis is $\tan\theta=-3/2$ so that $\sin\theta=3/\sqrt{13}$, $\cos\theta=-2/\sqrt{13}$

* Slope of the Conjugate axis is $\tan\theta=2/3$ so that ~~$\tan\theta$~~ $\sin\theta=2/\sqrt{13}$ & $\cos\theta=\frac{3}{\sqrt{13}}$ $(x_1\pm d\cos\theta)\ (x_2\pm d\sin\theta)$

ie, Coordinates of foci S & S' are $\left(0\pm\sqrt{\frac{13}{2}}\left(-\frac{2}{\sqrt{13}}\right), \frac{1}{2}\pm\sqrt{\frac{13}{2}}\times\frac{3}{\sqrt{13}}\right)$

ie $(\mp1, \frac{1}{2}\pm\frac{3}{2})$

$S(-1,2)$ & $S'(+1,-1)$

co-ordinats of Z and Z' are

$$\left(0\pm\frac{5}{2\sqrt{13}}\cdot\frac{-2}{\sqrt{13}}, \frac{1}{2}\pm\frac{5}{2\sqrt{13}}\cdot\frac{3}{\sqrt{13}}\right)$$

$$Z\left(-\frac{5}{13}, \frac{14}{13}\right) \ \& \ Z'\left(\frac{5}{13}, -\frac{1}{13}\right)$$

Vertex A & A'

$\left(0\pm\frac{\sqrt{5}}{2}\cdot\frac{-2}{\sqrt{13}}, \frac{1}{2}\pm\frac{\sqrt{5}}{2}\cdot\frac{3}{\sqrt{13}}\right)$, B & B' $\left(0\pm i\sqrt{2}\cdot\frac{3}{\sqrt{13}}, \frac{1}{2}\pm i\sqrt{2}\cdot\frac{2}{\sqrt{13}}\right)$

E_q^n of latus recta are $\parallel$ to Conjugate axis $2x-3y+k=0$ passes through S, S' $\therefore k=8,-5$

NO-22
PK-119

$$F(x,y) = x^2 + 2xy + y^2 - 6x - 2y + 4 = 0 \longrightarrow (1)$$

abs $\Delta = c(ab-h^2) + 2fgh - af^2 - bg^2$

$= -4 \neq 0$

$ab - h^2 = 1.1 - 1^2 = 0$

(1) represent parabola.

(1) $\Rightarrow$ $(x+y)^2 = 6x + 2y - 4$

But $x+y=0$ and $6x+2y-4=0$ are not at rt. angles. introducing a constant λ

$$(x+y+\lambda)^2 = (2\lambda+6)x + (2\lambda+2)y + \lambda^2 - 4 \longrightarrow (2)$$

Now the lines $x+y+\lambda=0$ and

$(2\lambda+6)x + (2\lambda+2)y + \lambda^2 - 4 = 0$ will be

at right angles if

$(2\lambda+6)\cdot 1 + (2\lambda+2)\cdot 1 = 0$

$\Rightarrow 4\lambda = -8$,

$\therefore \lambda = -2$

(2) gives $(x+y-2)^2 = 2x - 2y$

$$\Rightarrow \left(\frac{x+y-2}{\sqrt{1^2+1^2}}\right)^2 \times 2 = 2\left(\frac{x-y}{\sqrt{1^2+(-1)^2}}\right) \times \sqrt{2}$$

$$\Rightarrow \left(\frac{x+y-2}{\sqrt{2}}\right)^2 = \sqrt{2}\left(\frac{x-y}{\sqrt{2}}\right) \longrightarrow (3)$$

Let $X = \frac{x-y}{\sqrt{2}}$, $Y = \frac{x+y-2}{\sqrt{2}}$

(3) $\Rightarrow$ $p = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

Standard form of (1) is

$$Y^2 = 4pX.$$

Reduce the equation to its standard form (8)

1. $F(x,y) \equiv 8x^2 + 4xy + 5y^2 - 16x - 14y + 13 = 0$

- $\Delta \neq 0$
  $\Rightarrow -20 \neq 0$
- $ab - h^2 = 36 > 0$

$\therefore$ $\Delta \neq 0$
$ab - h^2 > 0$
ellipse.

- centre $\left(\frac{13}{18}, \frac{10}{9}\right)$
- $c_1 = gx_1 + fy_1 + c$
  $= -5/9$
- $\frac{72}{5}x^2 + \frac{36}{5}xy + 9y^2 = 1$
- $r_1 = \frac{\sqrt{5}}{6}$, $r_2 = \frac{\sqrt{5}}{9}$
- $\frac{x^2}{(\sqrt{5}/6)^2} + \frac{y^2}{\left(\frac{\sqrt{5}}{9}\right)^2} = 1$

2. $F(x,y) \equiv 17x^2 + 12xy + 8y^2 - 46x - 28y + 33 = 0$

- $\Delta = -400 \neq 0$
- $ab - h^2 = 100 > 0$
- Centre $(1,1)$, $c_1 = gx_1 + fy_1 + c = -4$
- $\frac{17}{4}x^2 + 3xy + 2y^2 = 1$
- $r_1 = 2/\sqrt{5}$, $r_2 = 1/\sqrt{5}$
- $\frac{x^2}{(2/\sqrt{5})^2} + \frac{y^2}{(1/\sqrt{5})^2} = 1$
- Length of the axes
  $2r_1 = 4/\sqrt{5}$, $2r_2 = 2/\sqrt{5}$
- Equation of the axes
  // $2x + y = 0$ w.r.to centre as origin
  $2x + y - 3 = 0$ w.r.to old origin.
  // $x - 2y = 0$ w.r.to $(1,1)$ as origin
  $x - 2y + 1 = 0$ w.r.to old origin.

HW:

$$x^2 - 4xy + 4y^2 + 10x - 8y + 13 = 0$$

$\Delta = -36 \neq 0$

$ab - h^2 = 0$

Parabola

$$(x - 2y + \lambda)^2 = (2\lambda - 10) + (8 - 4\lambda) + \lambda^2 - 13$$

$$\lambda = 13/5$$

$$X = \frac{2x + y + 13/5}{\sqrt{5}}, \quad Y = \frac{x - 2y + 13/5}{\sqrt{5}}$$

" $$8x^2 - 4xy + 5y^2 - 16x - 14y + 17 = 0$$

$\Delta = -324 \neq 0$

$ab - h^2 = 36 > 0$

Ellipse

$$(x_1, y_1) = (3/2, 2)$$

$$c_1 = -9$$

$$8x^2 - 4xy + 5y^2 - 9 = 0$$

$$A = 8/9, \quad B = 5/9, \quad H = -2/9$$

$$\frac{1}{r^4} - (A+B)\frac{1}{r^2} + AB - H^2 = 0$$

$$r_1^2 = 9/4, \quad r_2^2 = 1$$

$$\boxed{\frac{x^2}{(3/2)^2} + \frac{y^2}{1^2} = 1}$$