CHAPTER 5

CURVATURE AND RADIUS OF CURVATURE

5.1 Introduction:

Curvature is a numerical measure of bending of the curve. At a particular point on the curve , a tangent can be drawn. Let this line makes an angle Ψ with positive x- axis. Then curvature is defined as the magnitude of rate of change of Ψ with respect to the arc length s.

∴ Curvature at P = $\left|\frac{d\Psi}{ds}\right|$

It is obvious that smaller circle bends more sharply than larger circle and thus smaller circle has a larger curvature.

Radius of curvature is the reciprocal of curvature and it is denoted by ρ.

5.2

- **Radius of curvature of Cartesian curve: $y = f(x)$**

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{(1+y_1^2)^{3/2}}{|y_2|}$$ (When tangent is parallel to x – axis)

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|}$$ (When tangent is parallel to y – axis)

- **Radius of curvature of parametric curve:**
  $x = f(t), y = g(t)$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|}\ , \quad \text{where } x' = \frac{dx}{dt} \text{ and } y' = \frac{dy}{dt}$$

Example 1 Find the radius of curvature at any pt of the cycloid

$$x = a\,(\theta + \text{Sin}\,\theta),\quad y = a\,(1 - \cos\theta)$$

Solution: $x' = \frac{dx}{d\theta} = a\,(1 + \cos\theta)$ and $y' = \frac{dy}{d\theta} = a \sin\theta$

$x'' = \frac{d^2x}{d\theta^2} = -\ a \sin\theta$ and $y'' = \frac{d^2y}{d\theta^2} = a\cos\theta$

$$\text{Now } \rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|} = \frac{\{a^2(1+Cos\ \theta)^2 + a^2 sin^2\theta\}^{3/2}}{a^2(1+Cos\ \theta)\ Cos\ \theta + a^2\ sin^2\theta}$$

$$= \frac{a\left(1+cos^2\theta+2cos\theta+sin^2\theta\right)^{3/2}}{cos\theta+cos^2\theta+sin^2\theta}$$

$$= \frac{a(2+2cos\theta)^{3/2}}{1+cos\theta}$$

$$= 2\sqrt{2}\ a\sqrt{1+cos\theta}$$

$$= 2\sqrt{2}\ a\sqrt{2\frac{\cos^2\theta}{2}} = 4a\cos\frac{\theta}{2}$$

Example 2 Show that the radius of curvature at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ ($x = a\cos^3\theta$, $y = a\sin^3\theta$) is equal to three times the lenth of the perpendicular from the origin to the tangent.

Solution : $\frac{dx}{d\theta} = -\ 3a\ cos^2\theta\ sin\theta = x'$

$$\frac{dy}{d\theta} = -\ 3a\ sin^2\theta\ cos\theta = y'$$

$$x'' = \frac{d^2x}{d\theta^2} = \frac{d}{d\theta}(-\ 3a\cos^2\theta\sin\theta)$$

$$= -\ 3a\ [-2\cos\theta\ \sin^2\theta + \cos^3\theta]$$

$$= 6\ a\cos\theta\ \sin^2\theta - 3a\cos^3\theta$$

$$y'' = \frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(3a\sin^2\theta\cos\theta)$$

$$= 3a(2\sin\theta\ cos^2\theta - sin^3\theta)$$

$$= 6a\sin\theta\ cos^2\theta - 3a\ sin^3\theta$$

$$\text{Now} \quad \rho = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|}$$

$$= \frac{\left(9a^2cos^4\theta sin^2\theta + 9a^2 sin^4\theta cos^2\theta\right)^{3/2}}{|(-3acos^2\theta\ sin\theta)(6a\ sin\theta cos^2\theta - 3asin^3\theta) - 3asin^2\theta\cos\theta(6a\ cos\theta\ sin^2\theta - 3a\ cos^3\theta)|}$$

$$= \frac{\left[9a^2 cos^2 sin^2\theta\left(cos^2\theta + sin^2\theta\right)\right]^{3/2}}{\left|-18a^2\, sin^2\,\theta cos^4\,\theta + 9a^2\, cos^2\theta\, sin^4\theta - 18a^2\, sin^4\,\theta cos^2\theta + 9a^2\, sin^2\,\theta cos^4\theta\right|}$$

$$= \frac{9^{3/2}(a\, cos\theta\, sin\theta)^3}{\left|-9a^2 sin^2\theta cos^4\theta - 9a^2\, cos^2\theta\, sin^4\theta\right|}$$

$$= \frac{(9)^{3/2}(a\, cos\theta \sin\theta)^3}{9a^2\, cos^2\theta sin^2\theta\, (cos^2\theta + sin^2\theta)}$$

$\Longrightarrow$ ρ = 3a sin$\theta cos\theta$ …….(1)

The equation of the tangent at any point on the curve is

y – a sin^3 θ = – tan θ (x – a cos$^3\theta$)

$\Longrightarrow$ x sinθ + y cos θ – a sin θ cos θ = 0 ……..(2)

∴ The length of the perpendicular from the origin to the tangent (2) is

$$p = \frac{|0.sin\theta + 0.cos\theta - a\, sin\theta\, cos\theta|}{\sqrt{sin^2\theta + cos^2\theta}}$$

= a sinθ cosθ ……..(3)

Hence from (1) & (3), ρ = 3p

Example 3 If ρ & ρ ' are the radii of curvature at the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ prove that

$$\left(\rho^{2/3} + \rho'^{2/3}\right)\,(ab)^{2/3} = a^2 + b^2$$

Solution: Parametric equation of the ellipse is

x = a cos θ, y=b sin θ

x'= – a sin θ, y'= b cos θ

x''= – a cos θ, y'' = – b sin θ

The radius of curvature at any point of the ellipse is given by

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|} = \frac{\left(a^2 sin^2\theta + b^2\, cos^2\, b\right)^{3/2}}{\left|(-a\, sin\theta)\,(-bsin\theta) - (bcos\theta)\,(-acos\theta)\right|}$$

$$= \frac{\left(a^2 sin^2\theta + b^2\, cos^2\theta\right)^{3/2}}{ab} \qquad \ldots\ldots(1)$$

For the radius of curvature at the extremity of other conjugate diameter is obtained by replacing θ by $\theta + \frac{\pi}{2}$ in (1).
Let it be denoted by ρ'. Then

$$\therefore\ \rho' = \frac{\left(a^2 sin^2\theta + b^2 sin^2\,\theta\right)^{3/2}}{ab}$$

$$\therefore \rho^{2/3} + \rho'^{2/3} = \frac{a^2 sin^2\theta + b^2\, cos^2\theta}{(ab)^{2/3}} + \frac{a^2 cos^2\theta + b^2\, cos^2\,\theta}{(ab)^{2/3}}$$

$$= \frac{a^2+b^2}{(ab)^{2/3}}$$

$$\Longrightarrow (ab)^{2/3}\left(\rho^{2/3} + \rho'^{2/3}\right) = a^2 + b^2$$

Example 4 Find the points on the parabola y^2 = 8x at which the radius of curvature is $\frac{125}{16}$.

Solution: y = $2\sqrt{2}\,\sqrt{x}$

$$y_1 = \frac{\sqrt{2}}{\sqrt{x}} \qquad , \qquad y_2 = \frac{-1}{\sqrt{2}x^{3/2}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|} = \left(1 + \frac{2}{x}\right)^{3/2}.\sqrt{2}\, x^{3/2} = \sqrt{2}\,(x+2)^{3/2}$$

Given $\rho = \frac{12.5}{16}$ $\therefore (x + 2)^{3/2} = \frac{125}{16\sqrt{2}} = \left(\frac{5}{2\sqrt{2}}\right)^3$

$$\therefore (x + 2)^{3/2} = \frac{5}{2\sqrt{2}}$$

$\Longrightarrow$ x + 2 = $\frac{25}{8}$ $\Longrightarrow$ x = $\frac{9}{8}$

$\Longrightarrow$ y^2 = 8 $\left(\frac{9}{8}\right)$ i.e. y = 3,-3

Hence the points at which the radius of curvature is $\frac{125}{16}$ are (9,$\pm$ 3).

Example 5 Find the radius of curvature at any point of the curve y = C cos h (x/c)

Solution: $y_1 = \frac{c}{c}$ Sin h $\frac{x}{c}$ = Sin h $\left(\frac{x}{c}\right)$

$y_2 = \frac{1}{c}$ cos h $\frac{x}{c}$

Now, $\rho = \frac{\left(1+y_1{}^2\right)^{3/2}}{y_2}$

$$= \frac{\left(1+Sin\ h^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\cos h\,\frac{x}{c}}$$

$$= \text{C cos h}^2\left(\frac{x}{c}\right)$$

$\Longrightarrow \quad \rho = \frac{1}{c}y^2$

Example 6 For the curve y = $\frac{ax}{a+x}$, prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$$

where ρ is the radius of curvature of the curve at its point (x, y)

Solution: Here y = $\frac{ax}{a+x}$

$$\Longrightarrow \quad y_1 = \frac{(a+x)a - ax\ (1)}{(a+x)^2}$$

$$= \frac{a^2}{(a+x)^2}$$

$$\therefore\ y_2 = \frac{-2a^2}{(a+x)^3}$$

Now, $\rho = \frac{\left(1+y^{1^2}\right)^{3/2}}{y_2}$

$$= \left[1+\frac{a^4}{(a+x)^4}\right]^{3/2} \times \frac{(a+x)^3}{\left(-2a^2\right)}$$

$$\therefore\ \rho^{2/3} = \left[1+\frac{a^4}{(a+x)^4}\right] \frac{(a+x)^2}{(-2)^{2/3}\,a^{4/3}}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left[1 + \frac{a^4}{(a+x)^4}\right] \quad \frac{(a+x)^2}{2^{2/3}\, a^{4/3}} \times \frac{2^{2/3}}{a^{2/3}}$$

$$= \frac{1}{a^2}\left[1 + \frac{a^4}{(a+1)^4}\right](a+x)^2$$

$$= \frac{1}{a^2}\left[(a+x)^2 + \frac{a^4}{(a+x)^2}\right]$$

$$= \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$= \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

Example 7 Find the curvature of x = 4 cost, y = 3 sint. At what point on this ellipse does the curvature have the greatest & the least values? What are the magnitudes?

Solution: $\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|}$

Now, $x' = -\ 4 \sin t \quad \Rightarrow \quad x'' = -\ 4 \cos t$

$y' = -3 \cos t \quad \Rightarrow \quad x'' = -\ 3 \sin t$

$$\therefore\ \rho = \frac{\left(16 sin^2\, t + 9\, cost^2\, t\right)^{3/2}}{-4 \sin t\, (-3 sint) - 3\, Cost\, (-4 Cost)}$$

$$= \frac{1}{12}\left(9\, cost^2\, t + 16 sin^2\, t\right)^{3/2}$$

$\Rightarrow (\rho.12)^{2/3} = 9 \cos t^2 t + 16 \sin^2 t$

Now, curvature is the reciprocal of radius of curvature. Curvature is maximum & minimum when ρ is minimum and maximum respectively . For maximum and minimum values;

$$\frac{d}{dt}(16 \sin^2 t + 9 \cos^2 t) = 0$$

$\Rightarrow$ 32 sint cost + 18 cost (–sint) = 0

$\Rightarrow$ 4 sint cost = 0

$\Rightarrow$ t = 0 & $\frac{\pi}{2}$

At t = 0 ie at (4,0)
$(12\,\rho)^{2/3} = 9$
$\Rightarrow$ $12\,\rho = 9^{3/2}$

$\Rightarrow \rho = \frac{9}{4}$ $\therefore \frac{1}{\rho} = \frac{4}{9}$
Similarly, at t = $\frac{\pi}{2}$ ie at (0,3)
$(12\,\rho)^{2/3} = 16$
$12\rho = 4^3$
ρ = 16/3 $\therefore \frac{1}{\rho} = \frac{3}{16}$
Hence, the least value is $\frac{3}{16}$ and the greatest value is $\frac{4}{9}$

Example 8 Find the radius of curvature for $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$ at the points where it touches the coordinate axes.
Solution: On differentiating the given , we get

$$\frac{1}{2\sqrt{ax}} - \frac{1}{2\sqrt{by}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{by}{ax}} \qquad \text{.......(1)}$$

The curve touches the x-axis if $\frac{dy}{dx} = 0$ or y = 0
When y = 0, we have x = a (from the given eq^n)

$\Rightarrow$ given curve touches x – axis at (a,0)

The curve touches y – axis if $\frac{dx}{dy} = 0$ or x = 0
When x = 0, we have y = b

$\Rightarrow$ Given curve touches y-axis at (o, b)

$$\frac{d^2y}{dx} = \sqrt{\frac{b}{a}}\left\{\sqrt{\frac{b}{a}}\cdot\frac{1}{2x} - \frac{1}{2}\sqrt{\frac{y}{x}}\right\} \qquad \{\text{from (1)}\}$$

At (a,0), $\frac{d^2y}{dx^2} = \frac{1}{2a}\,\frac{b}{a} = \frac{b}{2a^2}$

∴ At (a,o), $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = (1+\ 0)^{3/2}\ \frac{2a^2}{b} = \frac{2a^2}{b}$

At (o,b), $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{2b^2}{a}$

5.3 Radius of curvature of Polar curves r = f (θ):

$\rho = \frac{\left(r^2+r_1^2\right)^{3/2}}{2r_1^2+r^2-rr_2}$ $\left(where\ r_1 = \frac{dr}{d\theta},\ r_2 = \frac{d^2r}{d\theta^2}\right)$

Example 9 Prove that for the cardioide r = a (1 + cos θ),

$\frac{\rho^2}{r}$ is const.

Solution: Here r = a (1+ cos θ)

$\Rightarrow r_1 = -$ a Sin θ and $r_2 = -$ a cos θ

$\therefore\ r^2 + r_1^2 = a^2\left[(1+\cos\theta)^2 + sin^2\theta\right] = 2a^2\,(1 + \cos\theta)$

$r^2 + 2r_1^2 - rr^2 = a^2[(1+\cos\theta)^2 + 2sin^2\theta + \cos\theta(1+\cos\theta)]$

$= 3a^2\,(1+\cos\theta)$

$\therefore\ \rho^2 = \frac{\left(r^2+r_1^2\right)^3}{\left(r^2+2r_1^2-rr_2\right)^2} = \frac{8a^6(1+cos\theta)^3}{9a^4\,(1+\cos\theta)^2} = \frac{8}{9}a^2$ (1+ cos θ)

$\Rightarrow\ \rho^2 = \frac{8a}{9}$ r

$\therefore\ \frac{\rho^2}{r} = \frac{8a}{9}$ which is a constant.

Example 10 Show that at the point of intersection of the curves r = a θ and r θ = a, the curvatures are in the ratio 3:1 (0 < θ< 2π)

Solution: The points of intersection of curves r = a θ & r θ = a are given by a θ^2 = a or $\theta = \pm 1$
Now for the curve r=a θ we have r_1 = a and r_2 = 0

$$\therefore \text{ At } \theta = \pm 1,\ \rho = \left[\frac{(r^2+r_1^2)^{3/2}}{2a^2+a^2\theta^2-0}\right]_{\theta=\pm 1} = \frac{a\,(2\sqrt{2})}{3} = \rho_1$$

For the curve r θ = a,

$$r_1 = \frac{-a}{\theta^2} \quad \text{and} \quad r_2 = \frac{2a}{\theta^3}$$

$$\text{At } \theta = \pm 1,\ \rho = \left[\frac{\left(\frac{a^2}{\theta^2}+\frac{a^2}{\theta^4}\right)^{3/2}}{\frac{2a^2}{\theta^4}+\frac{a^2}{\theta^2}-\frac{2a^2}{\theta^4}}\right]_{\theta=\pm 1,} = \left[a\,\frac{(1+\theta^2)^{3/2}}{\theta^4}\right]_{\theta=\pm 1}$$

$$= 2a\sqrt{2} = \rho_2$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{2a\sqrt{2}}{2a\sqrt{2/3}} = \frac{3}{1}$$

$$\therefore\ \rho_2 : \rho_1 = 3 : 1$$

Example 11 Find the radius of curvature at any point (r, θ) of the curve $r^m = a^m \cos m\theta$

Solution: $r^m = a^m \cos m\theta$

$$\Rightarrow m\log r = m\log a + \log\cos m\theta$$

$$\Rightarrow \frac{m}{r} r_1 = -m\,\frac{\sin m\theta}{\cos m\theta} \quad \text{(on differentiating w.r.t. } \theta\text{)}$$

$$\Rightarrow r_1 = -r\tan m\theta \qquad \ldots\ldots.(1)$$

Now $r_2 = -(r_1 \tan m\theta + rm\sec^2 m\theta)$

$$= r\tan^2 m\theta - rm\sec^2 m\theta \quad \text{(from (1))}$$

$$\therefore \rho = \frac{\left(r^2+r^2\tan^2 m\theta\right)^{3/2}}{r^2+2r^2\tan^2 m\theta - r^2\tan^2 m\theta + r^2 m\sec^2 m\theta}$$

$$= \frac{r^3\sec^3 m\theta}{r^2\sec^2 m\theta + r^2 m\sec^2 m\theta} \qquad = \frac{r}{m+1}\sec m\theta$$

Example 12 Show that the radius of curvature at the point (r, θ)

of the curve $r^2 \cos 2\theta = a^2$ is $\frac{r^3}{a^2}$

Solution: $r^2 = a^2 \sec 2\theta$

$$\Rightarrow 2rr_1 = 2a^2 \sec 2\theta \, tan2\theta$$
$$\Rightarrow \quad r_1 = r\, tan2\theta$$

and $r_2 = 2r\, sec^2\theta + r_1 tan2\theta$

$= 2r \sec^2 2\theta + r \tan^2 2\theta \quad (\because r = r \tan 2\theta)$

Now $\rho = \frac{\left(r^1+r_1^2\right)^{3/2}}{2r_1^2 + r^2 - rr_2} \Rightarrow \rho = \frac{\left(\left(r^2+r^2\tan^2 2\theta\right)\right)^{3/2}}{2r^2\, tan^2 2\,\theta + r^2 - r^2\,(2sec^2 2\theta + tan^2 2\theta)}$

$$= \frac{\left(r^2 sec^2 2\theta\right)^{3/2}}{r^2\,(2\,tan^2 2\theta + 1 - 2sec^2 2\theta - tan^2 2\theta)}$$

$$= \frac{r^3 sec^3 2\theta}{r^2 sec^2 2\theta}$$

$$= r \sec 2\theta$$

$$= r\,.\frac{r^2}{a^2} = \frac{r^3}{a^2}$$

5.4 Radius of curvature at the origin by Newton's method

It is applicable only when the curve passes through the origin and has x-axis or y-axis as the tangent there.

When x-axis is the tangent, then

$$\rho = \lim_{x\to 0} \frac{x^2}{2y}$$

When y- axis is the tangent, then

$$\rho = \lim_{x\to 0}\frac{y^2}{2x}$$

Example13 Find the radius of curvature at the origin of the curve

$$x^3y - xy^3 + 2x^2y + xy - y^2 + 2x = 0$$

Solution: Tangent is x = 0 ie y–axis,

$$\rho \quad = \lim_{y\to 0}\frac{y^2}{2x}$$

Dividing the given equation by 2x, we get

$$\frac{x^3y}{2x} - \frac{xy^3}{2x} + \frac{2x^2y}{2x} + \frac{+xy}{2x} \frac{-y^2}{2x} + \frac{2x}{2x} = 0$$

$$x^3\left(\frac{y}{2x}\right) - xy\left(\frac{y^2}{2x}\right) + xy + x\left(\frac{y}{2x}\right) - \left(\frac{y^2}{2x}\right) + 1 = 0$$

Taking limit $y \to 0$ on both the sides , we get $\rho = 1$

Exercise 5A

1. Find the radius of curvatures at any point the curve

y = 4 sin x – sin2x at $x = \frac{\pi}{2}$

Ans $\rho = \frac{1}{4}(5)^{3/2}$

2. If ρ_1 ,ρ_2 are the radii of curvature at the extremes of any chord of the cardioide r = a (1 + cos θ) which passes through the pole, then

$\rho_1^1 + \rho_2^2 = \frac{16a^2}{9}$

3 Find the radius of curvature of $y^2 = x^2$ (a+x) (a –x) at the origin

Ans. a$\sqrt{2}$

4. Find the radius of curvature at any point 't' of the curve

x = a (cost + log tan t/2), y = a sin$t$

Ans. a cos$t$

5. Find the radius of curvature at the origin, for the curve $2x^3 - 3x^2y + 4y^3 + y^2 - 3x = 0$

Ans. $\rho = 3/2$

6. Find the radius of curvature of $y^2 = \frac{4a^2(2a-x)}{x}$ at a point where the curve meets x – axis

Ans. $\rho = a$

7. Prove the if ρ_1, ρ_2 are the radii of curvature at the extremities of a focal chord of a parabola whose semi latus rectum is l then $(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (l)^{-2/3}$

8. Find the radius of curvature to the curve r = a (1+ cos θ) at the point where the tangent is parallel to the initial line.

Ans. $\rho = \frac{2}{\sqrt{3}} . a$

9. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $\rho = \frac{a^2b^2}{p^3}$ where p is the perpendicular distance from the centre on the tangent at (x,y).