Simple harmonic motion

Average value of Kinetic and Potential energies of a harmonic Oscillator:

The Potential energy (P.E.) of the Particle at a displacement y is given by

$$= \frac{1}{2} m \omega^2 y^2$$ [If, m is the mass of Particle]

$$= \frac{1}{2} m \cdot \omega^2 a^2 \sin^2 (\omega t + \phi)$$

So, the average P.E. of the particle over a complete cycle or a whole time Period T.

$$= \frac{1}{T} \int_0^T \frac{1}{2} m \omega^2 a^2 \sin^2 (\omega t + \phi)\, dt.$$

$$= \frac{1}{T} \cdot \frac{m\omega^2 a^2}{4} \int_0^T 2 \sin^2 (\omega t + \phi)\, dt.$$

$$= \frac{m\omega^2 a^2}{4T} \int_0^T \left[1 - \cos 2 (\omega t + \phi)\right] dt.$$

$$= \frac{m\omega^2 a^2}{4T} \left[\int_0^T dt - \int_0^T \cos 2 (\omega t + \phi)\, dt\right].$$

The average value of both a sine and a cosine function for a complete cycle or a whole time period T is zero. We therefore, have

average P.E of the Particle

$$= \frac{1}{4T} m\omega^2 a^2 [t]_0^T - 0$$

$$= \frac{1}{4T} m\omega^2 a^2 T$$

$$= \frac{1}{4} m\omega^2 a^2$$

$$= \frac{1}{4} Ka^2 \quad | \because m\omega^2 = K \longrightarrow (1)$$

$$= \frac{1}{2} \cdot \frac{1}{2} Ka^2$$

The kinetic energy (K.E.) of the particle at displacement y is given by,

$$= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 \quad \left[= \frac{1}{2} mv^2\right]$$

$$= \frac{1}{2} m \left[\frac{d}{dt} a \sin(\omega t + \phi)\right]^2$$

$$= \frac{1}{2} m \omega^2 a^2 \cos^2(\omega t + \phi).$$

The average K.E. of the particle over a complete cycle or a whole time period T, as in the case of P.E. is given by,

$$= \frac{1}{T} \int_0^T \frac{1}{2} m\omega^2 a^2 \cos^2(\omega t + \phi) dt.$$

$$= \frac{m\omega^2 a^2}{4T} \int_0^T 2\cos^2(\omega t + \phi)\, dt.$$

$$= \frac{m\omega^2 a^2}{4T} \int_0^T \left[1 + \cos 2(\omega t + \phi)\right] dt.$$

$$= \frac{m\omega^2 a^2}{4T} \left[\int_0^T dt + \int_0^T \cos 2(\omega t + \phi)\, dt \right]$$

Again, the average value of a sine or cosine function over a complete cycle or a whole time period is zero. Hence,

average K.E of the particle.

$$= \frac{m\omega^2 a^2}{4T} \left[t \right]_0^T + 0$$

$$= \frac{m\omega^2 a^2}{4T} \cdot T$$

$$= \frac{1}{4} m\omega^2 a^2 = \frac{1}{4} k a^2 = \frac{1}{2} \cdot \frac{1}{2} k a^2$$

Thus,

average value of P.E. of the Particle = average value of K.E of the Particle.

$$= \frac{1}{4} m\omega^2 a^2 = \frac{1}{4} k a^2$$

= half the total energy.

Composition of simple harmonic motion

Lissajous Figures:

When a particle is influenced simultaneously by two simple harmonic motions at right angles to each other, the resultant motion of the particle traces a curve. These curves are called Lissajous's figures. The shape of the curve depends on the time period, phase difference and the amplitude of the two constituent vibrations.

Lissajous figure's are helpful in determining the ratio of the time periods of two vibrations and to compare the frequencies of two tunning forks.

1. Composition of two simple harmonic motion in a straight line.

2. Composition of two simple harmonic vibrations of equal time periods acting at right angles.

3. Composition of two simple harmonic motions at right angles to each other and having time periods in the ratio 1:2.

---o---

Composition of two simple harmonic vibrations of equal time periods acting at right angles:

Let,

$$x = a\sin(\omega t + \alpha) \quad \text{--- (1)}$$

and,

$$y = b\sin\omega t \quad \text{--- (2)}$$

represent the displacements of a particle along the X and Y axis due to the influence of two simple harmonic vibrations acting simultaneously on a particle in perpendicular directions. Here, the two vibrations are of the same time period but are of different amplitudes and different phase angles.

From eqⁿ (2), $\sin\omega t = \frac{y}{b}$

$$\cos\omega t = \sqrt{1 - \frac{y^2}{b^2}} \quad [\sin^2\theta + $$

From eqⁿ (1),

$$\frac{x}{a} = [\sin\omega t \cos\alpha + \cos\omega t \sin\alpha] \quad \text{--- (3)}$$

substituting the values of $\sin\omega t$ and co

in eqⁿ (3) ⇒

$$\frac{x}{a} = \left[\frac{y}{b}\cos\alpha + \sqrt{1 - \frac{y^2}{b^2}} \cdot \sin\alpha\right]$$

$$\text{or, } \frac{x}{a} - \frac{y}{b}\cos\alpha = \sqrt{1 - \frac{y^2}{b^2}}\sin\alpha.$$

or, squaring,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\alpha - \frac{2xy}{ab}\cos\alpha = (1 -$$

$$\text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2}[\sin^2\alpha + \cos^2\alpha] - \frac{2xy}{ab}\cos$$

$= \sin$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha.$$

This represents the general equation of ellipse. Thus, due to the ~~Suprimen~~ superimposition of two simple harmonic vibrations, the displacement of the particle will be along a curve (fig – 1) given by eqn (4).

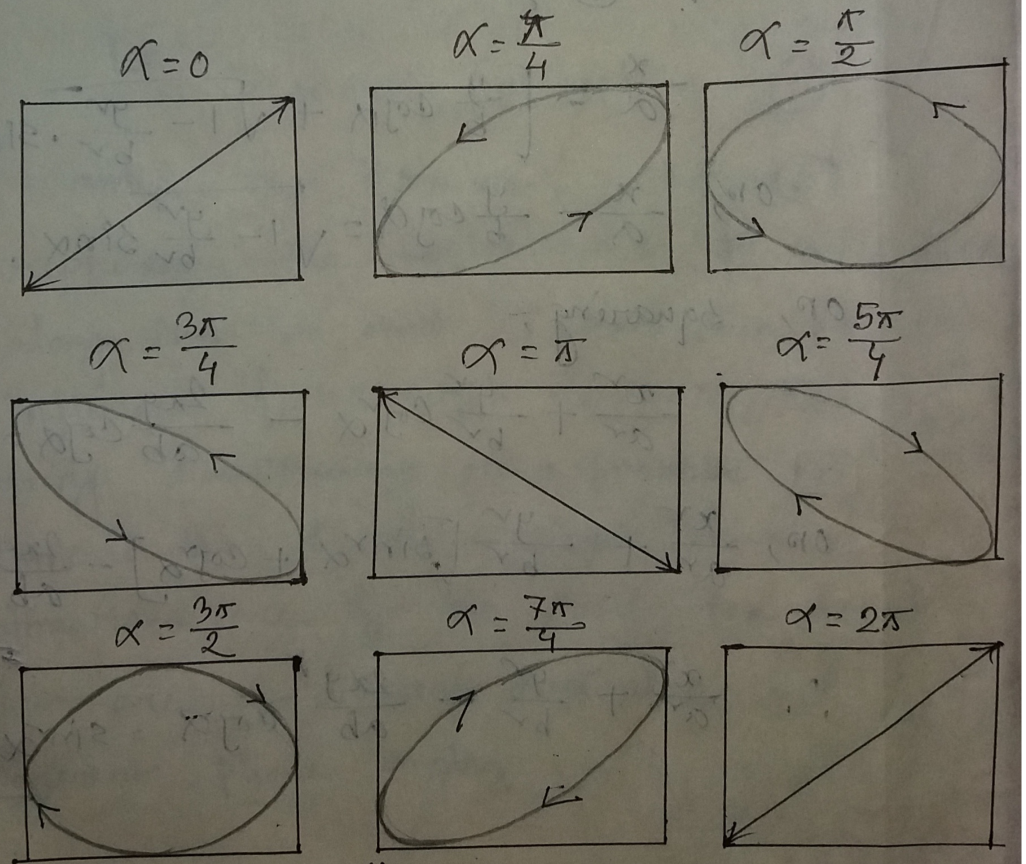

fig: