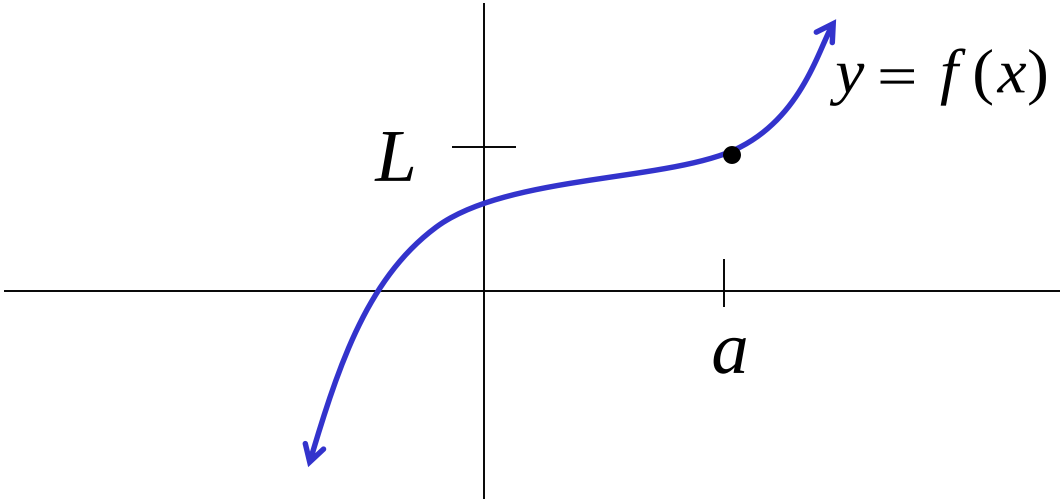
Limit

We say that the limit of $f(x)$ as x approaches a is L and write

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ approach L as x approaches a.



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3) Use your calculator to $\boxed{\text{evaluate}}$ the limits

a. $\lim_{x\to 2}\left(\frac{4(x^2-4)}{x-2}\right)$ Answer : 16

b. $\lim_{x\to 0} g(x)$, where $g(x)=\begin{cases}1, & \text{if } x<0\\ -1, & \text{if } x\geq 0\end{cases}$ Answer : no limit

c. $\lim_{x\to 0} f(x)$, where $\mathrm{f}(x)=\frac{1}{x^2}$ Answer : no limit

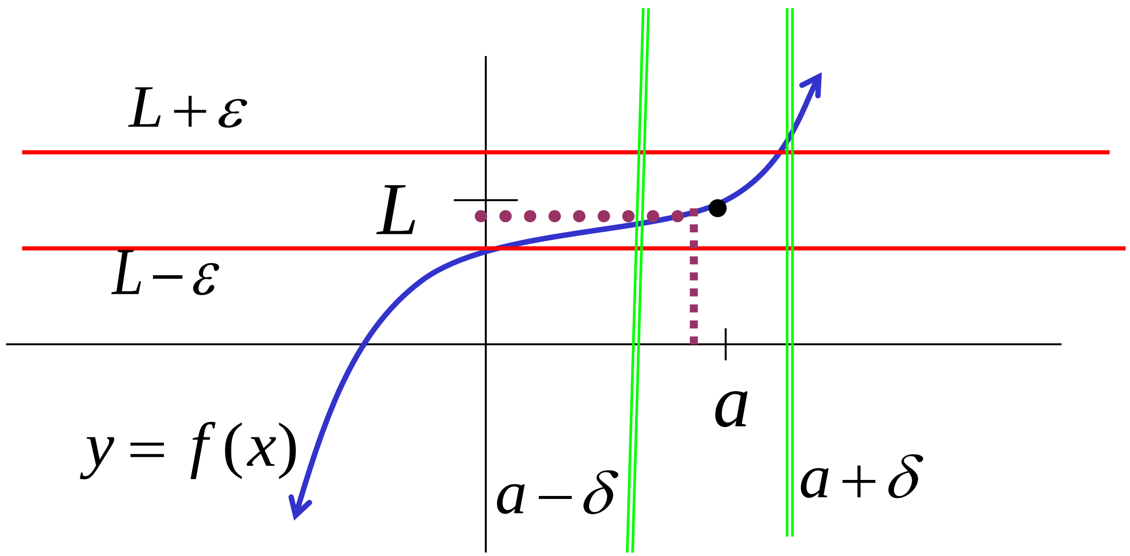
d. $\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)$ Answer : 1/2 one question will be in final

The ε-δ Definition of Limit

We say $\lim_{x \to a} f(x) = L$ if and only if

given a positive number ε, there exists a positive δ such that

if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.



This means that if we are given a

small interval $(L-\varepsilon, L+\varepsilon)$ centered at L,

then we can find a (small) interval $(a-\delta, a+\delta)$

such that for all $x \neq a$ in $(a-\delta, a+\delta)$,

$f(x)$ is in $(L-\varepsilon, L+\varepsilon)$.

Examples

1. Show that $\lim_{x\to 2}(3x+4)=10.$

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

if $|x-2|<\delta,$ then $|(3x+4)-10|<\varepsilon.$

But $|(3x+4)-10|=|3x-6|=3|x-2|<\varepsilon$

if $|x-2|<\dfrac{\varepsilon}{3}$ So we choose $\delta = \dfrac{\varepsilon}{3}.$

2. Show that $\lim_{x\to 1}\dfrac{1}{x}=1.$

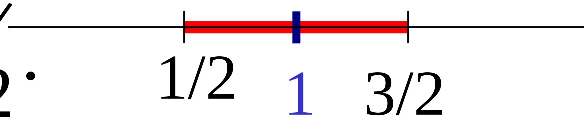
Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

if $|x-1|<\delta,$ then $|{}^{1}\!/_{x}-1|<\varepsilon.$

But $|{}^{1}\!/_{x}-1|=\left|\dfrac{x-1}{x}\right|=\dfrac{1}{x}|x-1|.$

What do we do with the x?

If we decide $|x-1|<\frac{1}{2}$, then $\frac{1}{2}<x<\frac{3}{2}$.



And so $\frac{1}{x}<2$.

Thus $\left|\frac{1}{x}-1\right|=\frac{1}{x}|x-1|<2|x-1|$.

Now we choose $\delta=\min\left\{\frac{\varepsilon}{3},\frac{1}{2}\right\}$.

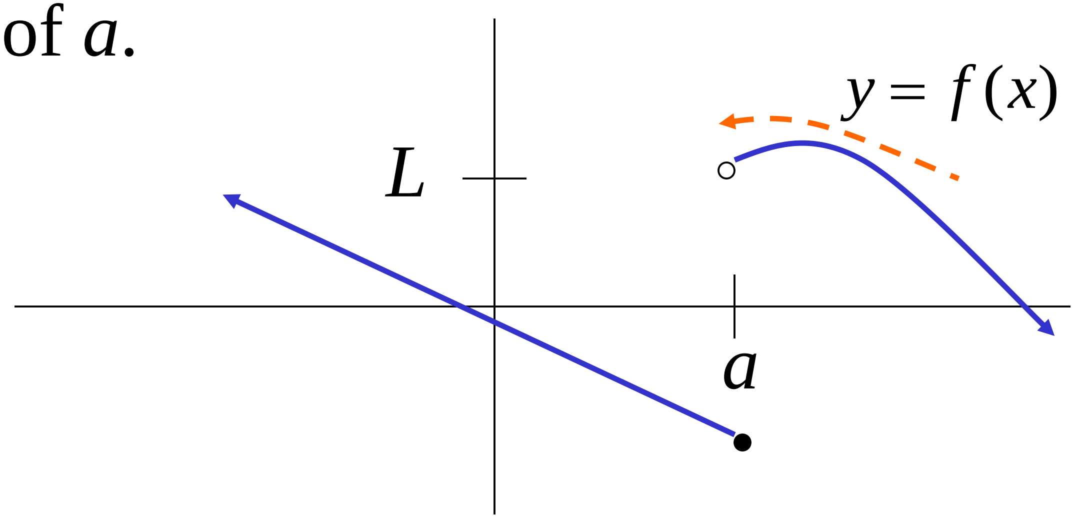
One-Sided Limit

One-Sided Limits

The right-hand limit of $f(x)$, as x approaches a, equals L

written: $\boxed{\lim_{x \to a^+} f(x) = L}$

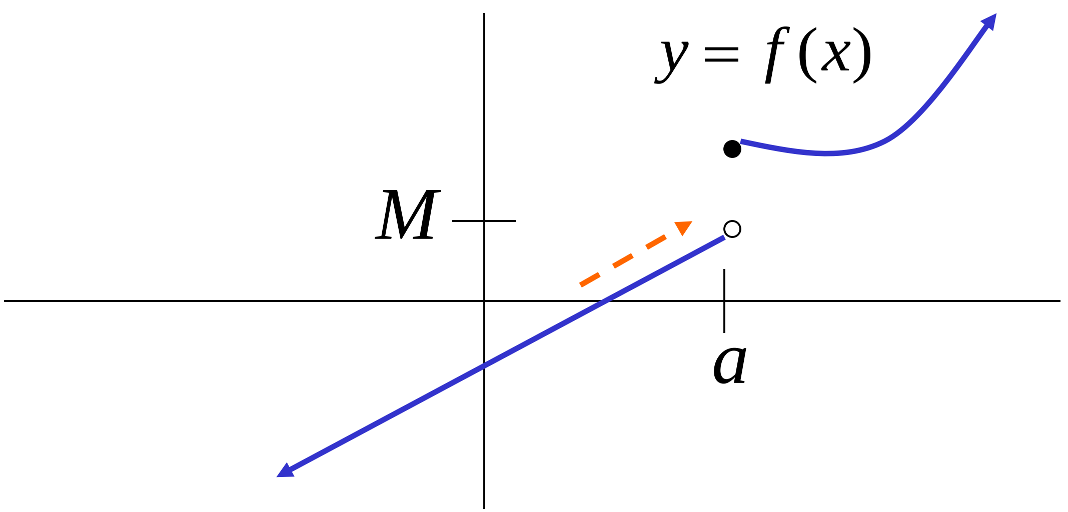
if we can make the value $f(x)$ arbitrarily close to L by taking x to be sufficiently close to the right of a.



The left-hand limit of $f(x)$, as x approaches a, equals M

written: $\boxed{\lim_{x \to a^-} f(x) = M}$

if we can make the value $f(x)$ arbitrarily close to L by taking x to be sufficiently close to the left of a.



Examples

Examples of One-Sided Limit

1. Given $f(x) = \begin{cases} x^2 & \text{if } x \le 3 \\ 2x & \text{if } x > 3 \end{cases}$

Find $\lim_{x \to 3^+} f(x)$

$$\boxed{\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x = 6}$$

Find $\lim_{x \to 3^-} f(x)$

$$\boxed{\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 = 9}$$

More Examples

2. Let $f(x)=\begin{cases} x+1, & \text{if } x>0 \\ x-1, & \text{if } x\le 0. \end{cases}$

Find the limits:

a) $\displaystyle\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 0+1 = 1$

b) $\displaystyle\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x-1) = 0-1 = -1$

c) $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+1) = 1+1 = 2$

d) $\displaystyle\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x+1) = 1+1 = 2$

A Theorem

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L.$

This theorem is used to show a limit does not exist.

For the function $$f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x \le 0. \end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist because $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1.$

But

$\lim_{x\to 1} f(x) = 2$ because $\lim_{x\to 1^+} f(x) = 2$ and $\lim_{x\to 1^-} f(x) = 2.$

Limit Theorems

If c is any number, $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then

a) $\lim\limits_{x\to a}\left(f(x)+g(x)\right) = L+M$

b) $\lim\limits_{x\to a}\left(f(x)-g(x)\right) = L-M$

c) $\lim\limits_{x\to a}\left(f(x)\cdot g(x)\right) = L\cdot M$

d) $\lim\limits_{x\to a}\left(f(x)/g(x)\right) = L/M$, $(M \neq 0)$

e) $\lim\limits_{x\to a}\left(c\cdot f(x)\right) = c\cdot L$

f) $\lim\limits_{x\to a}\left(f(x)\right)^n = L^n$

g) $\lim\limits_{x\to a} c = c$

h) $\lim\limits_{x\to a} x = a$

i) $\lim\limits_{x\to a} x^n = a^n$

j) $\lim\limits_{x\to a} \sqrt{f(x)} = \sqrt{L}$, $(L > 0)$

Examples Using Limit Rule

Ex. $$\lim_{x\to 3}\left(x^2+1\right)=\lim_{x\to 3}x^2+\lim_{x\to 3}1$$

$$=\left(\lim_{x\to 3}x\right)^2+\lim_{x\to 3}1$$

$$=3^2+1=10$$

Ex. $$\lim_{x\to 1}\frac{\sqrt{2x-1}}{3x+5}=\frac{\sqrt{\lim_{x\to 1}(2x-1)}}{\lim_{x\to 1}(3x+5)}=\frac{\sqrt{2\lim_{x\to 1}x-\lim_{x\to 1}1}}{3\lim_{x\to 1}x+\lim_{x\to 1}5}$$

$$=\frac{\sqrt{2-1}}{3+5}=\frac{1}{8}$$

More Examples

1. Suppose $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -2$. Find

a) $$\lim_{x\to 3}\big(f(x)+g(x)\big) = \lim_{x\to 3} f(x) + \lim_{x\to 3} g(x)$$
$$= 4 + (-2) = 2$$

b) $$\lim_{x\to 3}\big(f(x)-g(x)\big) = \lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)$$
$$= 4 - (-2) = 6$$

c) $$\lim_{x\to 3}\left(\frac{2f(x)-g(x)}{f(x)g(x)}\right) = \frac{\lim_{x\to 3} 2f(x) - \lim_{x\to 3} g(x)}{\lim_{x\to 3} f(x)\cdot \lim_{x\to 3} g(x)} = \frac{2\cdot 4-(-2)}{4\cdot(-2)} = \frac{-5}{4}$$

Indeterminate Forms

Indeterminate forms occur when substitution in the limit results in 0/0. In such cases either factor or rationalize the expressions.

Ex. $\displaystyle \lim_{x \to -5} \frac{x+5}{x^2-25}$ Notice $\dfrac{0}{0}$ form

$$= \lim_{x \to -5} \frac{x+5}{(x-5)(x+5)}$$

Factor and cancel common factors

$$= \lim_{x \to -5} \frac{1}{(x-5)} = \frac{1}{-10}$$

More Examples

a) $$\lim_{x\to 9}\left(\frac{\sqrt{x}-3}{x-9}\right) = \lim_{x\to 9}\left(\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}\right)$$

$$= \lim_{x\to 9}\left(\frac{x-9}{(x-9)(\sqrt{x}+3)}\right) = \lim_{x\to 9}\left(\frac{1}{\sqrt{x}+3}\right) = \frac{1}{6}$$

b) $$\lim_{x\to -2}\left(\frac{4-x^2}{2x^2+x^3}\right) = \lim_{x\to -2}\left(\frac{(2-x)(2+x)}{x^2(2+x)}\right)$$

$$= \lim_{x\to -2}\left(\frac{2-x}{x^2}\right)$$

$$= \frac{2-(-2)}{(-2)^2} = \frac{4}{4} = 1$$

The Squeezing Theorem

If $f(x) \le g(x) \le h(x)$ when x is near a, and if

$\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$, then $\lim\limits_{x \to a} g(x) = L$

Example: Show that $\lim\limits_{x \to 0} x^2 sin\left(\pi/x\right) = 0.$

Note that we cannot use product rule because $\lim\limits_{x \to 0} sin\left(\pi/x\right)$ DNE!

But $-1 \le \sin\left(\pi/x\right) \le 1$ and so $-x^2 \le x^2 \sin\left(\pi/x\right) \le x^2.$

Since $\lim\limits_{x \to 0} x^2 = \lim\limits_{x \to 0}(-x^2) = 0,$ we use the Squeezing Theorem to conclude

$$\lim_{x \to 0} x^2 sin\left(\pi/x\right) = 0.$$

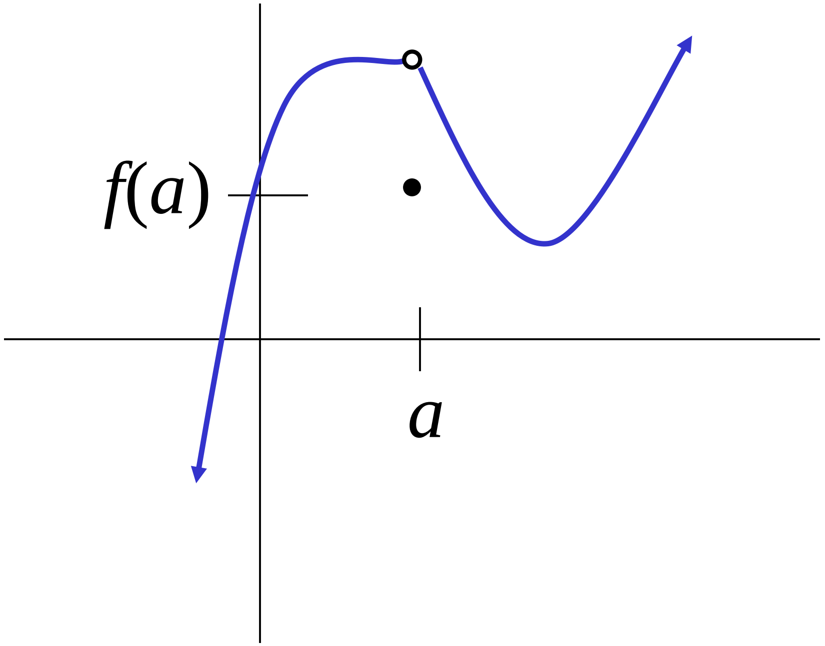
See Graph

Continuity

A function f is ***continuous*** at the point $x = a$ if the following are true:

$$i)\ \ f(a) \ \text{ is defined}$$

$$ii)\ \ \lim_{x \to a} f(x) \ \text{ exists}$$

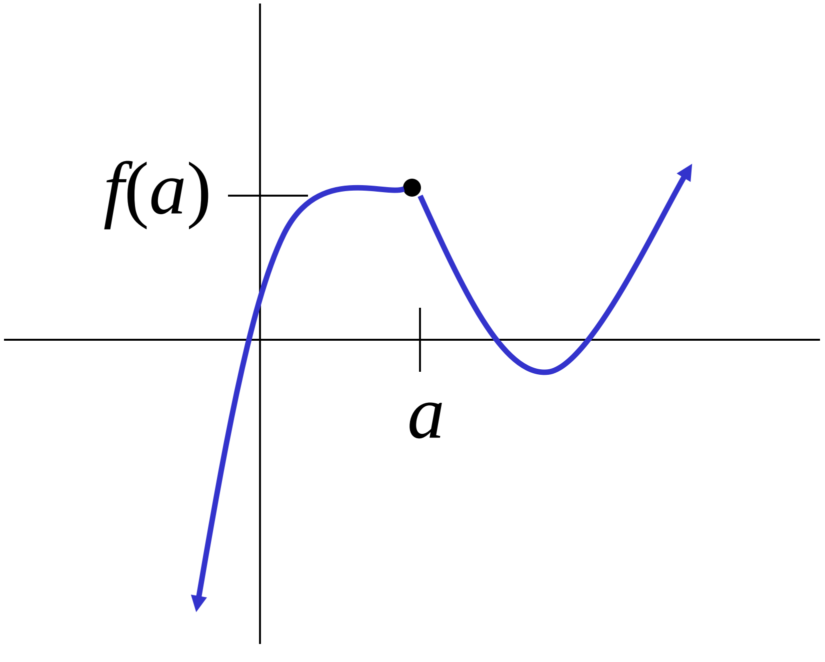


A function f is ***continuous*** at the point $x = a$ if the following are true:

$$i)\ \ f(a)\ \text{ is defined}$$

$$ii)\ \ \lim_{x \to a} f(x)\ \text{ exists}$$

$$iii)\ \lim_{x \to a} f(x) = f(a)$$



Examples

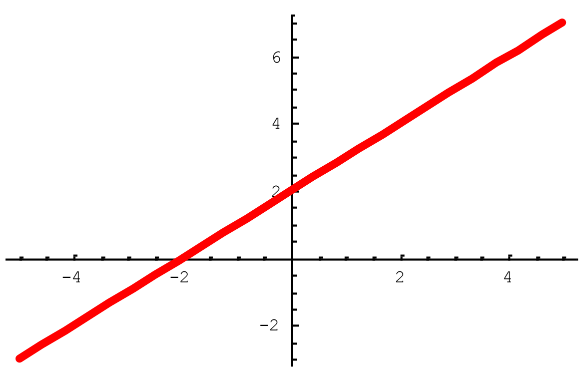
At which value(s) of x is the given function discontinuous?

1. $f(x) = x + 2$

Continuous everywhere

$$\lim_{x \to a}(x+2) = a + 2$$

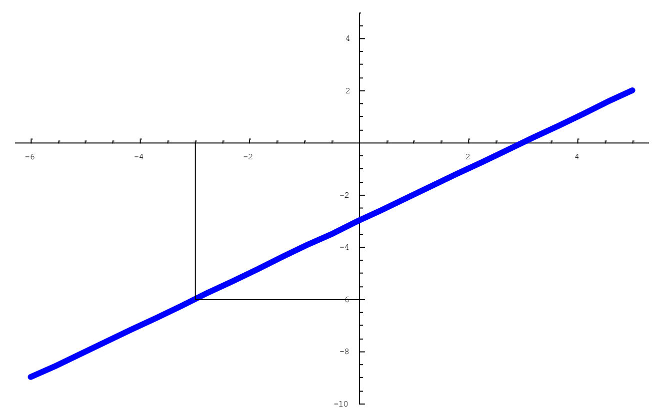
and so $\lim_{x \to a} f(x) = f(a)$



2. $g(x) = \dfrac{x^2 - 9}{x + 3}$

Continuous everywhere except at $x = -3$

$g(-3)$ is undefined

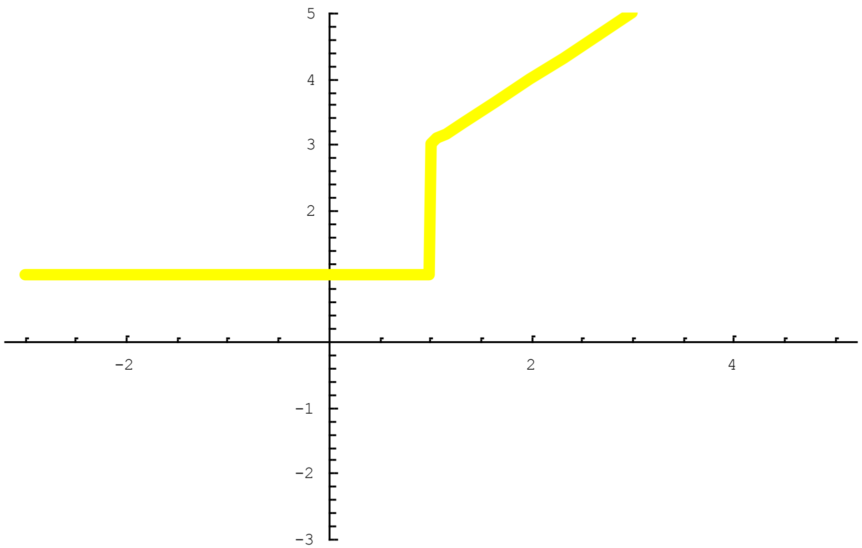


3. $h(x)=\begin{cases} x+2, \text{ if } x>1 \\ 1, \text{ if } x\le 1 \end{cases}$

$\lim_{x\to 1^-} h(x)=1$ and $\lim_{x\to 1^+} h(x)=3$

Thus h is not cont. at $x=1$.

h is continuous everywhere else

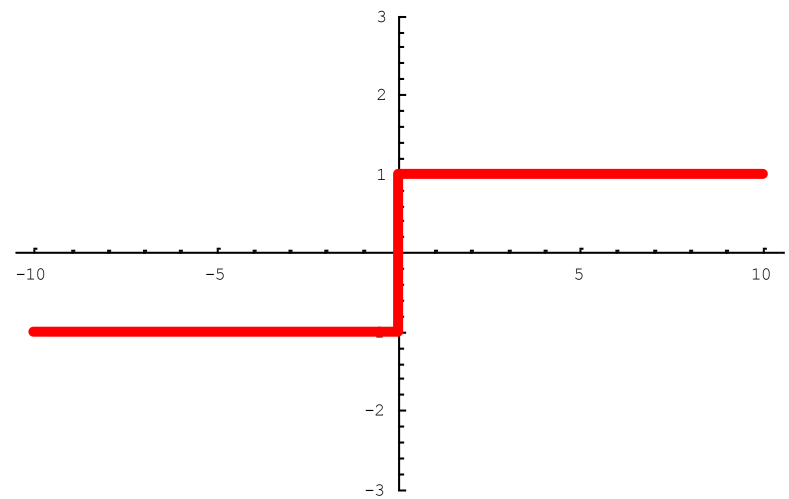


4. $F(x)=\begin{cases} -1, \text{ if } x\le 0 \\ 1, \text{ if } x>0 \end{cases}$

$\lim_{x\to 0^+} F(x)=1$ and $\lim_{x\to 0^-} F(x)=-1$

Thus F is not cont. at $x=0$.

F is continuous everywhere else



Continuous Functions

If f and g are continuous at $x = a$, then

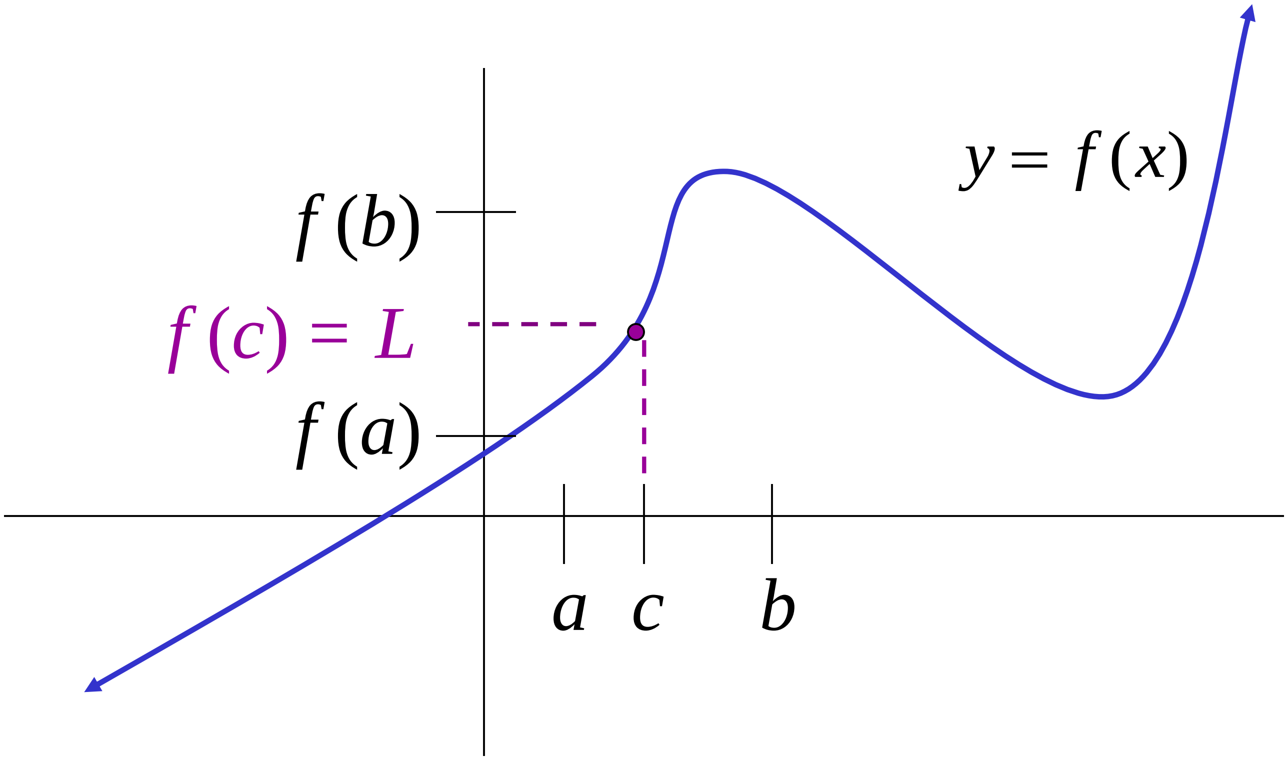
$f \pm g,\ \ fg,\ \ \text{and}\ \ f/g\ \ \left(g(a) \neq 0\right)$ are continuous at $x = a$

A polynomial function $y = P(x)$ is continuous at every point x.

A rational function $R(x) = p(x)/q(x)$ is continuous at every point x in its domain.

Intermediate Value Theorem

If f is a continuous function on a closed interval $[a, b]$ and L is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = L$.



Example

Given $f(x) = 3x^2 - 2x - 5,$

Show that $f(x) = 0$ has a solution on $[1, 2]$.

$$f(1) = -4 < 0$$

$$f(2) = 3 > 0$$

$f(x)$ is continuous (polynomial) and since $f(1) < 0$ and $f(2) > 0$, by the Intermediate Value Theorem there exists a c on $[1, 2]$ such that $f(c) = 0$.

Limits at Infinity

For all $n > 0$, $\boxed{\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0}$

provided that $\frac{1}{x^n}$ is defined.

Ex. $\lim_{x\to\infty} \frac{3x^2+5x+1}{2-4x^2} = \lim_{x\to\infty} \frac{3+\frac{5}{x}+\frac{1}{x^2}}{\frac{2}{x^2}-4}$

$\boxed{\text{Divide by } x^2}$

$$= \frac{\lim_{x\to\infty} 3 + \lim_{x\to\infty}\left(\frac{5}{x}\right) + \lim_{x\to\infty}\left(\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(\frac{2}{x^2}\right) - \lim_{x\to\infty} 4} = \frac{3+0+0}{0-4} = -\frac{3}{4}$$

More Examples

1. $$\lim_{x\to\infty}\left(\frac{2x^3-3x^2+2}{x^3-x^2-100x+1}\right)=\lim_{x\to\infty}\left(\frac{\frac{2x^3}{x^3}-\frac{3x^2}{x^3}+\frac{2}{x^3}}{\frac{x^3}{x^3}-\frac{x^2}{x^3}-\frac{100x}{x^3}+\frac{1}{x^3}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{2-\frac{3}{x}+\frac{2}{x^3}}{1-\frac{1}{x}-\frac{100}{x^2}+\frac{1}{x^3}}\right)$$

$$=\frac{2}{1}=2$$

2. $$\lim_{x\to\infty}\left(\frac{4x^2-5x+21}{7x^3+5x^2-10x+1}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4x^2}{x^3}-\frac{5x}{x^3}+\frac{21}{x^3}}{\frac{7x^3}{x^3}+\frac{5x^2}{x^3}-\frac{10x}{x^3}+\frac{1}{x^3}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4}{x}-\frac{5}{x^2}+\frac{21}{x^3}}{7+\frac{5}{x}-\frac{10}{x^2}+\frac{1}{x^3}}\right)$$

$$=\frac{0}{7}$$

$$=0$$

3. $$\lim_{x\to\infty}\left(\frac{x^2+2x-4}{12x+31}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{x^2}{x}+\frac{2x}{x}-\frac{4}{x}}{\frac{12x}{x}+\frac{31}{x}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x+2-\frac{4}{x}}{12+\frac{31}{x}}\right)$$

$$=\frac{\infty+2}{12}$$

$$=\infty$$

4. $$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

$$= \lim_{x\to\infty}\left(\frac{\left(\sqrt{x^2+1}-x\right)}{1}\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right)$$

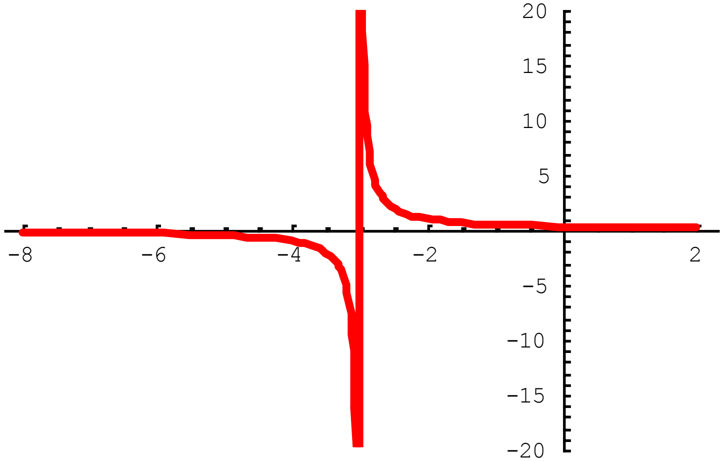
$$= \lim_{x\to\infty}\left(\frac{1}{\sqrt{x^2+1}+x}\right)$$

$$=\frac{1}{\infty+\infty}=\frac{1}{\infty}=0$$

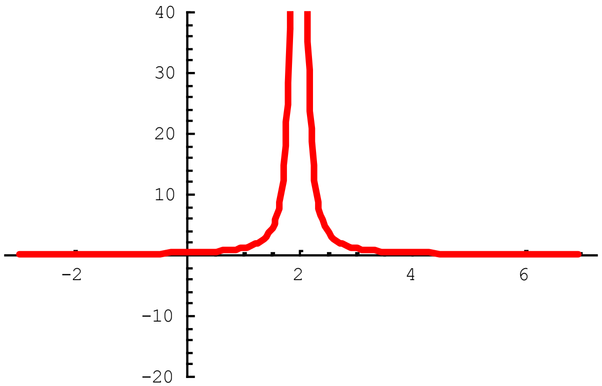
Infinite Limits

For all $n > 0$,

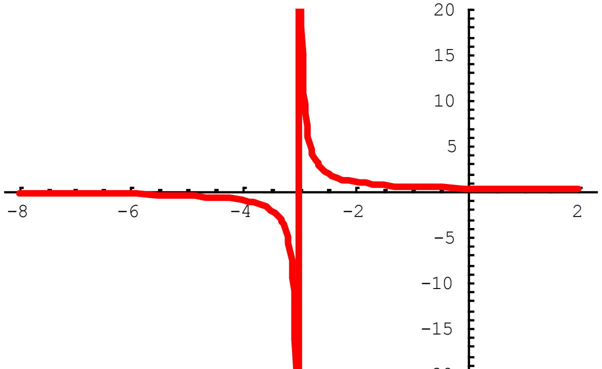
$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = \infty$$



$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = \infty \text{ if } n \text{ is even}$$



$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty \text{ if } n \text{ is odd}$$



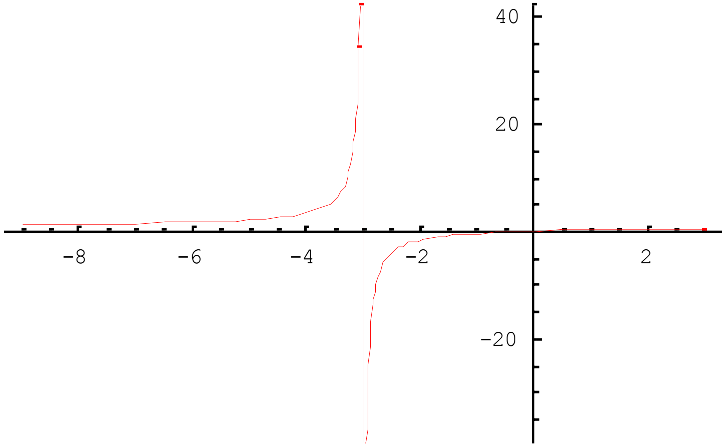
More Graphs

Examples

Find the limits

1. $$\lim_{x\to 0^+}\left(\frac{3x^2+2x+1}{2x^2}\right) = \lim_{x\to 0^+}\left(\frac{3+\frac{2}{x}+\frac{1}{x^2}}{2}\right) = \frac{3+\infty+\infty}{2} = \infty$$

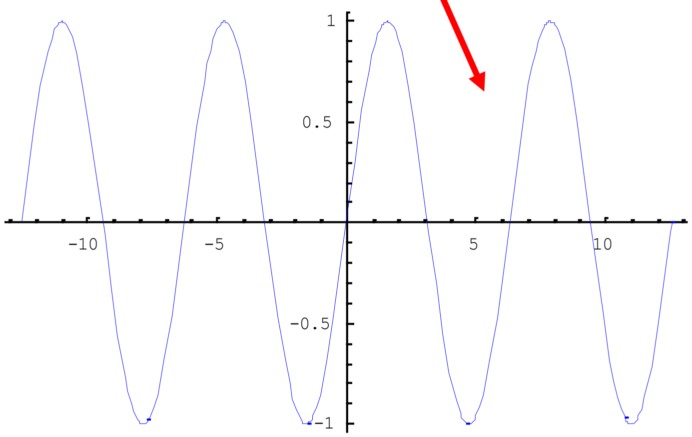
2. $$\lim_{x\to -3^+}\left(\frac{2x+1}{2x+6}\right) = \lim_{x\to -3^+}\left(\frac{2x+1}{2(x+3)}\right) = -\infty$$

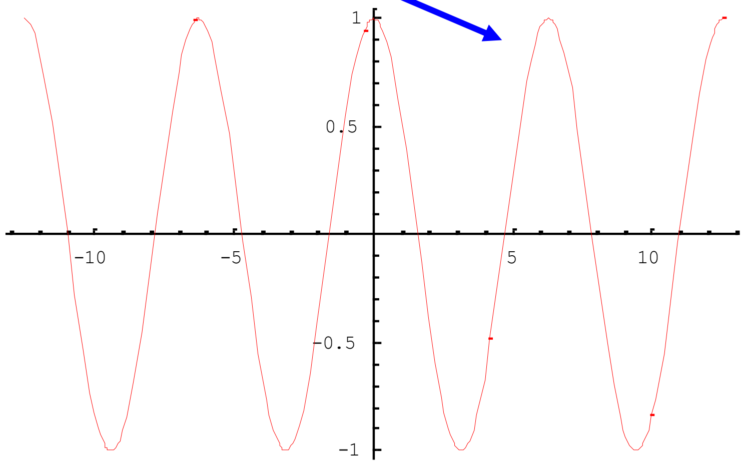


Limit and Trig Functions

From the graph of trigs functions

$$f(x) = \sin x \text{ and } g(x) = \cos x$$





we conclude that they are continuous everywhere

$$\lim_{x \to c} \sin x = \sin c \text{ and } \lim_{x \to c} \cos x = \cos c$$

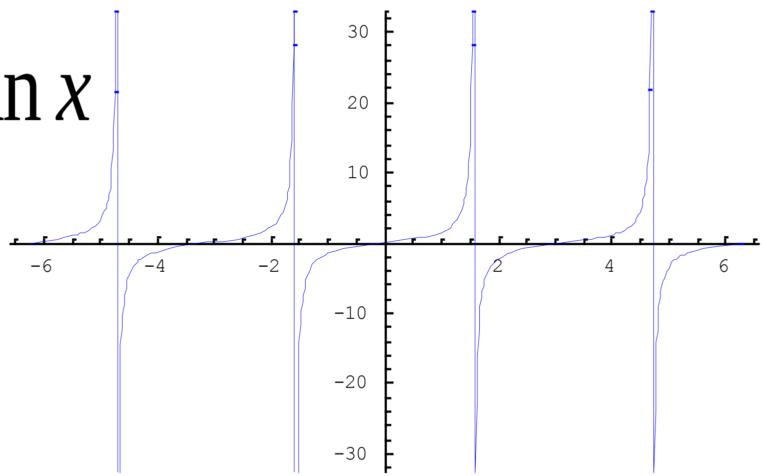
Tangent and Secant

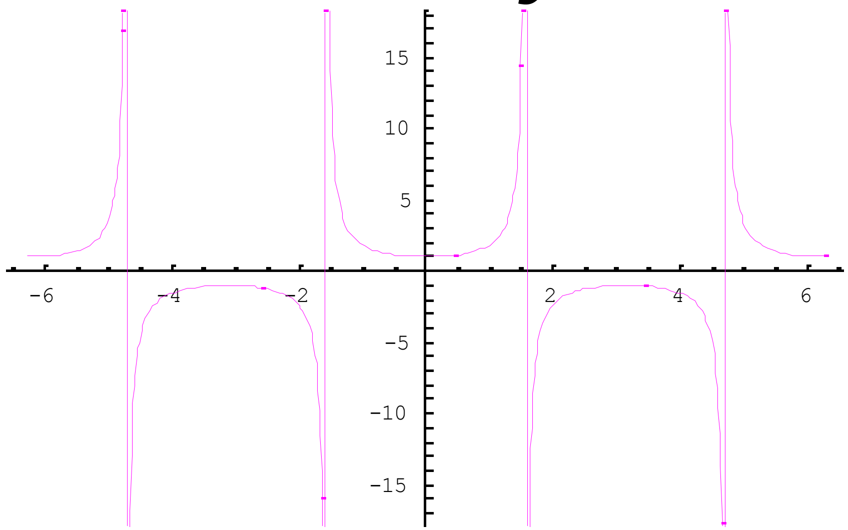
Tangent and secant are continuous everywhere in their domain, which is the set of all real numbers

$$x \neq \pm \tfrac{\pi}{2},\ \pm \tfrac{3\pi}{2},\ \pm \tfrac{5\pi}{2},\ \pm \tfrac{7\pi}{2}, \cdots$$

$$y = \sec x$$

$$y = \tan x$$





Examples

a) $\lim_{x\to\left(\pi/2\right)^+} \sec x = -\infty$

b) $\lim_{x\to\left(\pi/2\right)^-} \sec x = \infty$

c) $\lim_{x\to\left(-3\pi/2\right)^+} \tan x = -\infty$

d) $\lim_{x\to\left(-3\pi/2\right)^-} \tan x = \infty$

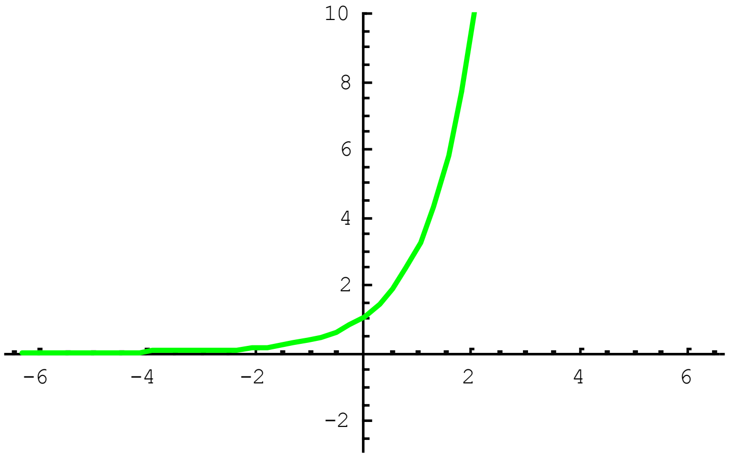
e) $\lim_{x\to\pi^-} \cot x = -\infty$

f) $\lim_{x\to\pi/4} \tan x = 1$

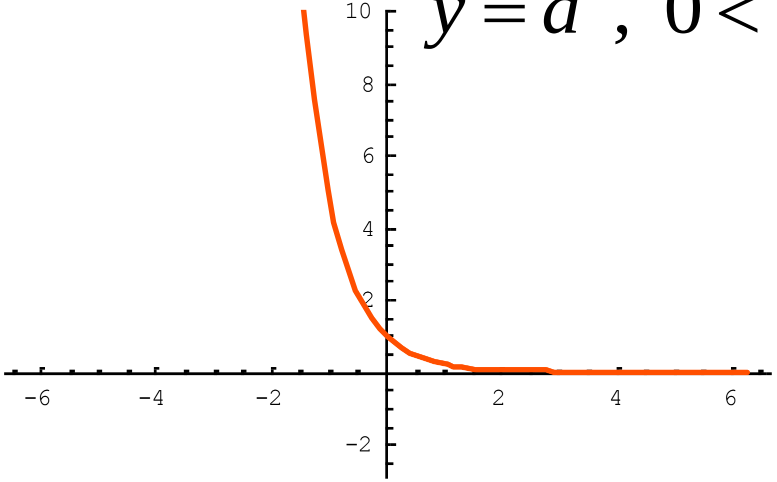
g) $\lim_{x\to\left(-3\pi/2\right)} \cot x = \lim_{x\to\left(-3\pi/2\right)} \dfrac{\cos x}{\sin x} = \dfrac{0}{1} = 0$

Limit and Exponential Functions

$$y = a^x,\ a > 1$$



$$y = a^x,\ 0 < a < 1$$



The above graph confirm that exponential functions are continuous everywhere.

$$\lim_{x \to c} a^x = a^c$$

Asymptotes

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if eihter

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L.$$

The line $x = c$ is called a vertical asymptote of the curve $y = f(x)$ if eihter

$$\lim_{x\to c^-} f(x) = \pm\infty \text{ or } \lim_{x\to c^+} f(x) = \pm\infty.$$

Examples

Find the asymptotes of the graphs of the functions

1. $f(x) = \dfrac{x^2+1}{x^2-1}$

(i) $\lim_{x \to 1^-} f(x) = -\infty$

Therefore the line $x = 1$

is a vertical asymptote.

(ii) $\lim_{x \to -1^-} f(x) = +\infty.$

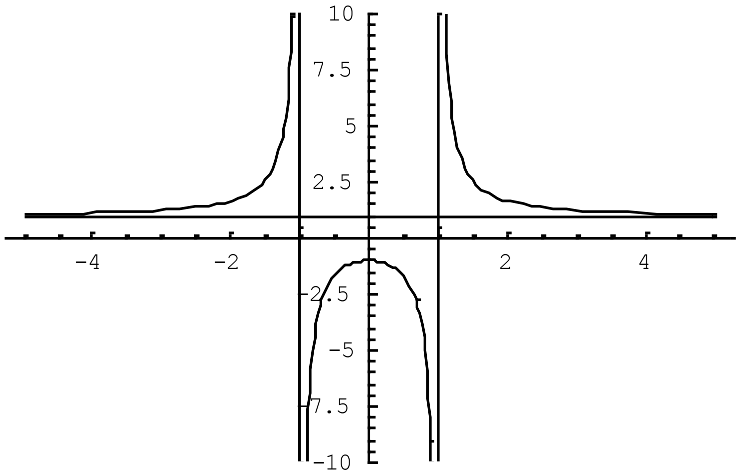
Therefore the line $x = -1$

is a vertical asymptote.

(iii) $\lim_{x \to \infty} f(x) = 1.$

Therefore the line $y = 1$

is a horizonatl asymptote.



2. $f(x) = \dfrac{x-1}{x^2-1}$

(i) $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \left(\dfrac{x-1}{x^2-1} \right)$

$= \lim\limits_{x \to 1} \left(\dfrac{x-1}{(x-1)(x+1)} \right) = \lim\limits_{x \to 1} \left(\dfrac{1}{x+1} \right) = \dfrac{1}{2}.$

Therefore the line $x = 1$

is NOT a vertical asymptote.

(ii) $\lim\limits_{x \to -1^+} f(x) = +\infty.$

Therefore the line $x = -1$

is a vertical asymptote.

(iii) $\lim\limits_{x \to \infty} f(x) = 0.$

Therefore the line $y = 0$

is a horizonatl asymptote.