

SUCCESSIVE DIFFERENTIATION AND LEIBNITZ'S THEOREM

1.1 Introduction

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $f(x)$ be a differentiable function and let its successive derivatives be denoted by $f'(x), f''(x), \dots, f^{(n)}(x)$.

Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2nd Derivative: $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

\vdots

n^{th} Derivative: $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny

1.2 Calculation of n^{th} Derivatives

i. **n^{th} Derivative of e^{ax}**

Let $y = e^{ax}$

$$y_1 = ae^{ax}$$

$$y_2 = a^2e^{ax}$$

\vdots

$$y_n = a^n e^{ax}$$

ii. **n^{th} Derivative of $(ax + b)^m$, m is a +ve integer greater than n**

Let $y = (ax + b)^m$

$$y_1 = m a(ax + b)^{m-1}$$

$$y_2 = m(m-1)a^2(ax + b)^{m-2}$$

\vdots

$$y_n = m(m-1) \dots (m-n+1)a^n(ax + b)^{m-n}$$

$$= \frac{m!}{(m-n)!} a^n (ax + b)^{m-n}$$

iii. **n^{th} Derivative of $y = \log(ax + b)$**

Let $y = \log(ax + b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2! a^3}{(ax+b)^3}$$

\vdots

$$y_n = (-1)^{n-1} \frac{(n-1)! a^n}{(ax+b)^n}$$

iv. **n^{th} Derivative of $y = \sin(ax + b)$**

Let $y = \sin(ax + b)$

$$y_1 = a \cos(ax + b) = a \sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2 \cos\left(ax + b + \frac{\pi}{2}\right) = a^2 \sin\left(ax + b + \frac{2\pi}{2}\right)$$

\vdots

$$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

Similarly if $y = \cos(ax + b)$

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

v. **n^{th} Derivative of $y = e^{ax} \sin(ax + b)$**

Let $y = e^{ax} \sin(bx + c)$

$$y_1 = a e^{ax} \sin(bx + c) + e^{ax} b \cos(bx + c)$$

$$= e^{ax} [a \sin(bx + c) + b \cos(bx + c)]$$

$$= e^{ax} [r \cos\alpha \sin(bx + c) + r \sin\alpha \cos(bx + c)]$$

Putting $a = r \cos\alpha$, $b = r \sin\alpha$

$$= e^{ax} r \sin(bx + c + \alpha)$$

Similarly $y_2 = e^{ax} r^2 \sin(bx + c + 2\alpha)$

\vdots

$$y_n = e^{ax} r^n \sin(bx + c + n\alpha)$$

where $r^2 = a^2 + b^2$ and $\tan\alpha = \frac{b}{a}$

$$\therefore y_n = e^{ax} (a^2 + b^2)^{\frac{n}{2}} \sin\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$$

Similarly if $y = e^{ax} \cos(ax + b)$

$$y_n = e^{ax} r^n \cos(bx + c + n\alpha)$$

$$= e^{ax} (a^2 + b^2)^{\frac{n}{2}} \cos\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$$

Summary of Results

Function	n^{th} Derivative
$y = e^{ax}$	$y_n = a^n e^{ax}$
$y = (ax + b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}, & m > 0, m > n \\ 0, & m > 0, m < n, \\ n! a^n, & m = n \\ \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}, & m = -1 \end{cases}$
$y = \log(ax + b)$	$y_n = (-1)^{n-1} \frac{(n-1)! a^n}{(ax+b)^n}$
$y = \sin(ax + b)$	$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$
$y = \cos(ax + b)$	$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$
$y = e^{ax} \sin(bx + c)$	$y_n = e^{ax} (a^2 + b^2)^{\frac{n}{2}} \sin\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$
$y = e^{ax} \cos(bx + c)$	$y_n = e^{ax} (a^2 + b^2)^{\frac{n}{2}} \cos\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$\begin{aligned}
 y &= \frac{1}{1-5x+6x^2} = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x} \\
 \therefore y_n &= \frac{3(-3)^n(-1)^n n!}{(1-3x)^{n+1}} - \frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}} \\
 \Rightarrow y_n &= (-1)^{n+1} n! \left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1} \right]
 \end{aligned}$$

Example 2 Find the n^{th} derivative of $\sin 6x \cos 4x$

Solution: Let $y = \sin 6x \cos 4x$

$$\begin{aligned}
 &= \frac{1}{2} (\sin 10x + \cos 2x) \\
 \therefore y_n &= \frac{1}{2} \left[10^n \sin\left(10x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right) \right]
 \end{aligned}$$

Example 3 Find n^{th} derivative of $\sin^2 x \cos^3 x$

Solution: Let $y = \sin^2 x \cos^3 x$

$$\begin{aligned}
&= \sin^2 x \cos^2 x \cos x \\
&= \frac{1}{4} \sin^2 2x \cos x = \frac{1}{8} (1 - \cos 4x) \cos x \\
&= \frac{1}{8} \cos x - \frac{1}{8} \cos 4x \cos x \\
&= \frac{1}{8} \cos x - \frac{1}{16} (\cos 3x + \cos 5x) \\
&= \frac{1}{16} (2 \cos x - \cos 3x - \cos 5x) \\
\therefore y_n &= \frac{1}{16} \left[2 \cos \left(x + \frac{n\pi}{2} \right) - 3^n \cos \left(3x + \frac{n\pi}{2} \right) - 5^n \cos \left(5x + \frac{n\pi}{2} \right) \right]
\end{aligned}$$

Example 4 Find the n^{th} derivative of $\sin^4 x$

Solution: Let $y = \sin^4 x = (\sin^2 x)^2$

$$\begin{aligned}
&= \left(\frac{1}{2} 2 \sin^2 x \right)^2 \\
&= \frac{1}{4} ((1 - \cos 2x))^2 \\
&= \frac{1}{4} \left[1 - 2 \cos 2x + \frac{1}{2} (2 \cos^2 2x) \right] \\
&= \frac{1}{4} \left[1 - 2 \cos 2x + \frac{1}{2} (1 + \cos 4x) \right] \\
&= \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \\
\therefore y_n &= -\frac{1}{2} 2^n \cos \left(2x + \frac{n\pi}{2} \right) + \frac{1}{8} 4^n \cos \left(4x + \frac{n\pi}{2} \right)
\end{aligned}$$

Example 5 Find the n^{th} derivative of $e^{3x} \cos x \sin^2 2x$

Solution: Let $y = e^{3x} \cos x \sin^2 2x$

$$\begin{aligned}
\text{Now } \cos x \sin^2 2x &= \frac{1}{2} (\cos x - \cos x \cos 4x) \\
&\therefore \sin^2 2x = \frac{1}{2} (1 - \cos 4x) \\
&= \frac{1}{2} \left(\cos x - \frac{1}{2} (\cos 5x + \cos 3x) \right) \\
\Rightarrow y &= e^{3x} \cos x \sin^2 2x = \frac{1}{2} e^{3x} \cos x - \frac{1}{4} e^{3x} \cos 5x - \frac{1}{4} e^{3x} \cos 3x \\
\therefore y_n &= \frac{1}{2} e^{3x} (9 + 1)^{\frac{n}{2}} \cos \left(x + n \tan^{-1} \frac{1}{3} \right) - \frac{1}{4} e^{3x} (9 + 25)^{\frac{n}{2}} \cos \left(5x + n \tan^{-1} \frac{5}{3} \right) \\
&\quad - \frac{1}{4} e^{3x} (9 + 9)^{\frac{n}{2}} \cos \left(3x + n \tan^{-1} \frac{3}{3} \right) \\
&= \frac{1}{2} e^{3x} 10^{\frac{n}{2}} \cos \left(x + n \tan^{-1} \frac{1}{3} \right) - \frac{1}{4} e^{3x} 34^{\frac{n}{2}} \cos \left(5x + n \tan^{-1} \frac{5}{3} \right) \\
&\quad - \frac{1}{4} e^{3x} 18^{\frac{n}{2}} \cos(3x + n \tan^{-1} 1)
\end{aligned}$$

Example 6 If $y = \sin ax + \cos ax$, prove that $y_n = a^n [1 + (-1)^n \sin 2ax]^{\frac{1}{2}}$

Solution: $y = \sin ax + \cos ax$

$$\therefore y_n = a^n \left[\sin \left(ax + \frac{n\pi}{2} \right) + \cos \left(ax + \frac{n\pi}{2} \right) \right]$$

$$\begin{aligned}
&= a^n \left[\left\{ \sin \left(ax + \frac{n\pi}{2} \right) + \cos \left(ax + \frac{n\pi}{2} \right) \right\}^2 \right]^{\frac{1}{2}} \\
&= a^n \left[\sin^2 \left(ax + \frac{n\pi}{2} \right) + \cos^2 \left(ax + \frac{n\pi}{2} \right) + 2 \sin \left(ax + \frac{n\pi}{2} \right) \cdot \cos \left(ax + \frac{n\pi}{2} \right) \right]^{\frac{1}{2}} \\
&= a^n [1 + \sin(2ax + n\pi)]^{\frac{1}{2}} \\
&= a^n [1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi]^{\frac{1}{2}} \\
&= a^n [1 + (-1)^n \sin 2ax]^{\frac{1}{2}} \quad \because \cos n\pi = (-1)^n \text{ and } \sin n\pi = 0
\end{aligned}$$

Example 7 Find the n^{th} derivative of $\tan^{-1} \frac{x}{a}$

Solution: Let $y = \tan^{-1} \frac{x}{a}$

$$\begin{aligned}
\Rightarrow y_1 &= \frac{dy}{dx} = \frac{1}{a \left(1 + \frac{x^2}{a^2} \right)} = \frac{a}{x^2 + a^2} = \frac{a}{x^2 - (ai)^2} \\
&= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai} \left(\frac{1}{x-ai} - \frac{1}{x+ai} \right) \\
&= \frac{1}{2i} \left(\frac{1}{x-ai} - \frac{1}{x+ai} \right)
\end{aligned}$$

Differentiating above $(n-1)$ times w.r.t. x , we get

$$y_n = \frac{1}{2i} \left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n} \right]$$

Substituting $x = r \cos \theta$, $a = r \sin \theta$ such that $\theta = \tan^{-1} \frac{x}{a}$

$$\begin{aligned}
\Rightarrow y_n &= \frac{(-1)^{n-1}(n-1)!}{2i} \left[\frac{1}{r^n (\cos \theta - i \sin \theta)^n} - \frac{1}{r^n (\cos \theta + i \sin \theta)^n} \right] \\
&= \frac{(-1)^{n-1}(n-1)!}{2ir^n} [(\cos \theta - i \sin \theta)^{-n} - (\cos \theta + i \sin \theta)^{-n}]
\end{aligned}$$

Using De Moivre's theorem, we get

$$\begin{aligned}
y_n &= \frac{(-1)^{n-1}(n-1)!}{2ir^n} [\cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta] \\
&= \frac{(-1)^{n-1}(n-1)!}{r^n} \sin n\theta \\
&= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin \theta} \right)^n} \sin n\theta \quad \because a = r \sin \theta \\
&= \frac{(-1)^{n-1}(n-1)!}{a^n} \sin n\theta \sin^n \theta \quad \text{where } \theta = \tan^{-1} \frac{a}{x}
\end{aligned}$$

Example 8 Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y = \frac{1}{1+x+x^2}$

$$= \frac{1}{(x-w)(x-w^2)} \quad \text{where } w = \frac{-1+i\sqrt{3}}{2} \text{ and } w^2 = \frac{-1-i\sqrt{3}}{2}$$

Resolving into partial fractions

$$y = \frac{1}{w-w^2} \left(\frac{1}{x-w} - \frac{1}{x-w^2} \right)$$

$$= \frac{1}{i\sqrt{3}} \left(\frac{1}{x-w} - \frac{1}{x-w^2} \right) = \frac{-i}{\sqrt{3}} \left(\frac{1}{x-w} - \frac{1}{x-w^2} \right)$$

Differentiating n times w.r.t. x , we get

$$\begin{aligned} y_n &= \frac{-i}{\sqrt{3}} \left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}} \right] \\ &= \frac{-i (-1)^n n!}{\sqrt{3}} \left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}} \right] \\ &= \frac{i (-1)^{n+1} n!}{\sqrt{3}} \left[\frac{1}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{n+1}} \right] \\ &= \frac{i 2^{n+1} (-1)^{n+1} n!}{\sqrt{3}} \left[\frac{1}{(2x+1-i\sqrt{3})^{n+1}} - \frac{1}{(2x+1+i\sqrt{3})^{n+1}} \right] \end{aligned}$$

Substituting $2x + 1 = r \cos \theta$, $\sqrt{3} = r \sin \theta$ such that $\theta = \tan^{-1} \frac{\sqrt{3}}{2x+1}$

$$y_n = \frac{i 2^{n+1} (-1)^{n+1} n!}{\sqrt{3} r^{n+1}} \left[(\cos \theta - i \sin \theta)^{-(n+1)} - (\cos \theta + i \sin \theta)^{-(n+1)} \right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{i 2^{n+1} (-1)^{n+1} n!}{\sqrt{3} \left(\frac{\sqrt{3}}{\sin \theta}\right)^{n+1}} [\cos(n+1)\theta + i \sin(n+1)\theta - \cos(n+1)\theta + i \sin(n+1)\theta]$$

$$\because \sqrt{3} = r \sin \theta$$

$$= \frac{i 2^{n+1} (-1)^{n+1} n!}{(\sqrt{3})^{n+2}} 2i \sin(n+1)\theta \sin^{n+1}\theta$$

$$= \frac{(-2)^{n+2} n!}{\sqrt{3}^{n+2}} \sin(n+1)\theta \sin^{n+1}\theta \quad \text{where } \theta = \tan^{-1} \frac{\sqrt{3}}{2x+1}$$

Example 9 If $y = x + \tan x$, show that $\cos^2 x \frac{d^2 y}{dx^2} - 2y + 2x = 0$

Solution: $y = x + \tan x$

$$\Rightarrow \frac{dy}{dx} = 1 + \sec^2 x$$

$$\frac{d^2 y}{dx^2} = 2 \sec x (\sec x \tan x) = 2 \sec^2 x \tan x$$

$$\begin{aligned} \therefore \cos^2 x \frac{d^2 y}{dx^2} - 2y + 2x &= 2 \cos^2 x \sec^2 x \tan x - 2(x + \tan x) + 2x \\ &= 2 \tan x - 2x - 2 \tan x + 2x \\ &= 0 \end{aligned}$$

Example 10 If $y = \log(x + \sqrt{x^2 + 1})$, show that $(1 + x^2) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0$

Solution: $y = \log(x + \sqrt{x^2 + 1})$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (\sqrt{1+x^2}) \frac{dy}{dx} = 1$$

Differentiating both sides w.r.t. x , we get

$$(\sqrt{1+x^2}) \frac{d^2y}{dx^2} + \frac{x}{\sqrt{1+x^2}} \frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$$

Exercise 1 A

1. Find the n^{th} derivative of $\frac{x^4}{(x-1)(x-2)}$

$$\text{Ans. } (-1)^n n! \left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right]$$

2. Find the n^{th} derivative of $\cos x \cos 2x \cos 3x$

$$\text{Ans. } \frac{1}{4} \left[2^n \cos \left(2x + \frac{n\pi}{2} \right) + 4^n \cos \left(4x + \frac{n\pi}{2} \right) + 6^n \cos \left(6x + \frac{n\pi}{2} \right) \right]$$

3. If $x = \sin t$, $y = \sin at$, show that $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + a^2 y = 0$

4. If $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, show that $p + \frac{d^2p}{d\theta^2} = \frac{a^2 b^2}{p^3}$

5. If $y = \frac{x}{x^2+a^2}$, find y_n i.e. the n^{th} derivative of y

$$\text{Ans. } \frac{(-1)^n n!}{a^{n+1}} \cos(n+1)\theta \sin^{n+1}\theta \quad \text{where } \theta = \tan^{-1} \frac{x}{a}$$

6. If $y = e^x \sin^2 x$, find y_n i.e. the n^{th} derivative of y

$$\text{Ans. } \frac{1}{2} e^x \left[1 - 16 \left(2x + \frac{n\pi}{2} \right) \right]$$

7. Find n^{th} differential coefficient of $y = \log[(ax+b)(cx+d)]$

$$\text{Ans. } y_n = (-1)^{n-1} (n-1)! \left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n} \right]$$

8. If $y = x \log \frac{x-1}{x+1}$, show that $y_n = (-1)^{n-1} (n-2)! \left[\frac{x-n}{(x-1)^n} + \frac{x+n}{(x+1)^n} \right]$

9. If $y = \tan^{-1} \frac{\sqrt{1+x^2}-1}{x}$, show that $y_n = \frac{1}{2} (-1)^{n-1} (n-1)! \sin n\theta \sin^n \theta$

1.2 LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th} derivatives of u and v respectively.

Example11 Find the n^{th} derivative of $x \log x$

Solution: Let $u = \log x$ and $v = x$

$$\text{Then } u_n = (-1)^{n-1} \frac{(n-1)!}{x^n} \text{ and } u_{n-1} = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

By Leibnitz's theorem, we have

$$(u v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\begin{aligned} \Rightarrow (x \log x)_n &= (-1)^{n-1} \frac{(n-1)!}{x^n} x + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} + 0 \\ &= (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} \\ &= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} [-(n-1) + n] \\ &= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} \end{aligned}$$

Example 12 Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$

Solution: Let $u = e^{3x} \sin 4x$ and $v = x^2$

$$\begin{aligned} \text{Then } u_n &= e^{3x} 25^{\frac{n}{2}} \sin \left(4x + n \tan^{-1} \frac{4}{3} \right) \\ &= e^{3x} 5^n \sin \left(4x + n \tan^{-1} \frac{4}{3} \right) \end{aligned}$$

By Leibnitz's theorem, we have

$$(u v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\begin{aligned} \Rightarrow (x^2 e^{3x} \sin 4x)_n &= x^2 e^{3x} 5^n \sin \left(4x + n \tan^{-1} \frac{4}{3} \right) + \\ &\quad 2nx e^{3x} 5^{n-1} \sin \left(4x + (n-1) \tan^{-1} \frac{4}{3} \right) + \\ &\quad n(n-1) e^{3x} 5^{n-2} \sin \left(4x + (n-2) \tan^{-1} \frac{4}{3} \right) + 0 \end{aligned}$$

$$= e^{3x} 5^n \left[x^2 \sin \left(4x + n \tan^{-1} \frac{4}{3} \right) + \frac{2nx}{5} \sin \left(4x + (n-1) \tan^{-1} \frac{4}{3} \right) + \frac{n(n-1)}{25} \sin \left(4x + (n-2) \tan^{-1} \frac{4}{3} \right) \right]$$

Example 13 If $y = a \cos(\log x) + b \sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$$

Solution: Here $y = a \cos(\log x) + b \sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x} \sin(\log x) + \frac{b}{x} \cos(\log x)$$

$$\Rightarrow xy_1 = -a \sin(\log x) + b \cos(\log x)$$

Differentiating both sides w.r.t. x , we get

$$xy_2 + y_1 = -\frac{a}{x} \cos(\log x) + \frac{-b}{x} \sin(\log x)$$

$$\Rightarrow x^2 y_2 + xy_1 = -\{a \cos(\log x) + b \sin(\log x)\}$$

$$= -y$$

$$\Rightarrow x^2 y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{c_1}y_{n+1}2x + n_{c_2}y_n \cdot 2) + (y_{n+1}x + n_{c_1}y_n \cdot 1) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example 14 If $y = \log(x + \sqrt{1+x^2})$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log(x + \sqrt{1+x^2})$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}} \left(1 + \frac{1}{2\sqrt{1+x^2}} 2x \right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2)y_1^2 = 1$$

Differentiating both sides w.r.t. x , we get

$$(1+x^2)2y_1y_2 + 2xy_1^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem

$$[y_{n+2}(1+x^2) + n_{c_1}y_{n+1}2x + n_{c_2}y_n \cdot 2] + (y_{n+1}x + n_{c_1}y_n \cdot 1) = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$$

Example 15 If $y = \sin(m \sin^{-1}x)$, show that

$$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n. \text{ Also find } y_n(0)$$

Solution: Here $y = \sin(m \sin^{-1}x)$ ①

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}} \cos(m \sin^{-1}x) \text{②}$$

$$\Rightarrow (1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2[1 - \sin^2(m \sin^{-1}x)]$$

$$\Rightarrow (1-x^2)y_1^2 = m^2(1-y^2) \text{③}$$

$$\Rightarrow (1-x^2)y_1^2 + m^2y^2 = m^2$$

Differentiating w.r.t. x , we get

$$(1-x^2)2y_1y_2 + y_1^2(-2x) + m^22yy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$$

Using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{c_1}y_{n+1}(-2x) + n_{c_2}y_n(-2)] - (y_{n+1}x + n_{c_1}y_n \cdot 1) + m^2y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n \text{④}$$

Putting $x = 0$ in ①, ② and ③

$$y(0) = 0, y_1(0) = m \text{ and } y_2(0) = 0$$

Putting $x = 0$ in ④

$$y_{n+2}(0) = (n^2-m^2)y_n(0)$$

Putting $n = 1, 2, 3 \dots$ in the above equation, we get

$$y_3(0) = (1^2-m^2)y_1(0)$$

$$= (1^2-m^2)m \quad \because y_1(0) = m$$

$$y_4(0) = (2^2 - m^2)y_2(0)$$

$$= 0 \quad \because y_2(0) = 0$$

$$y_5(0) = (3^2 - m^2)y_3(0)$$

$$= m(1^2 - m^2)(3^2 - m^2)$$

\vdots

$$\Rightarrow y_n(0) = \begin{cases} 0, & \text{if } n \text{ is even} \\ m(1^2 - m^2)(3^2 - m^2) \dots [(n-2)^2 - m^2], & \text{if } n \text{ is odd} \end{cases}$$

Example 16 If $y = e^{m \sin^{-1} x}$, show that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0$. Also find $y_n(0)$.

Solution: Here $y = e^{m \sin^{-1} x} \dots \textcircled{1}$

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}} e^{m \sin^{-1} x}$$

$$= \frac{my}{\sqrt{1-x^2}} \dots \textcircled{2}$$

$$\Rightarrow (1 - x^2)y_1^2 = m^2 y^2$$

Differentiating above equation w.r.t. x , we get

$$(1 - x^2)2y_1 y_2 + y_1^2(-2x) = m^2 2y y_1$$

$$\Rightarrow (1 - x^2)y_2 - xy_1 - m^2 y = 0 \dots \textcircled{3}$$

Differentiating above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1 - x^2) + n_{C_1} y_{n+1}(-2x) + n_{C_2} y_n(-2)] - (y_{n+1}x + n_{C_1} y_n) - m^2 y_n = 0$$

$$\Rightarrow y_{n+2}(1 - x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) - m^2 y_n = 0$$

$$\Rightarrow (1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0 \dots \textcircled{4}$$

To find $y_n(0)$: Putting $x = 0$ in $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$

$$y(0) = 1, y_1(0) = m \text{ and } y_2(0) = m^2$$

Also putting $x = 0$ in, we get

$$y_{n+2}(0) = (n^2 + m^2)y_n(0)$$

Putting $n = 1, 2, 3 \dots$ in the above equation, we get

$$y_3(0) = (1^2 + m^2)y_1(0)$$

$$= (1^2 + m^2)m$$

$$\because y_1(0) = m$$

$$y_4(0) = (2^2 + m^2)y_2(0)$$

$$= m^2(2^2 + m^2)$$

$$\because y_2(0) = m^2$$

$$y_5(0) = (3^2 + m^2)y_3(0)$$

$$= m(1^2 + m^2)(3^2 + m^2)$$

\vdots

$$\Rightarrow y_n(0) = \begin{cases} m^2(2^2 + m^2) \dots [(n-2)^2 + m^2], & \text{if } n \text{ is even} \\ m(1^2 + m^2)(3^2 + m^2) \dots [(n-2)^2 + m^2], & \text{if } n \text{ is odd} \end{cases}$$

Example 17 If $y = \tan^{-1}x$, show that

$$(1 - x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0. \text{ Also find } y_n(0)$$

Solution: Here $y = \tan^{-1}x \dots \dots \textcircled{1}$

$$\Rightarrow y_1 = \frac{1}{1+x^2} \dots \dots \textcircled{2}$$

$$y_2 = \frac{-2x}{1+x^2}$$

$$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0 \dots \dots \textcircled{3}$$

Differentiating equation $\textcircled{3}$ n times w.r.t. x using Leibnitz's theorem

$$[y_{n+2}(1+x^2) + n_{c_1}y_{n+1}(2x) + n_{c_2}y_n(2)] + 2(y_{n+1}x + n_{c_1}y_n1) = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + 2(y_{n+1}x + ny_n) = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0 \dots \dots \textcircled{4}$$

To find $y_n(0)$: Putting $x = 0$ in $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$, we get

$$y(0) = 0, y_1(0) = 1 \text{ and } y_2(0) = 0$$

Also putting $x = 0$ in $\textcircled{4}$, we get

$$y_{n+2}(0) = -n(n+1)y_n(0)$$

Putting $n = 1, 2, 3 \dots$ in the above equation, we get

$$\begin{aligned}
y_3(0) &= -1(2)y_1(0) \\
&= -2 \quad \because y_1(0) = 1 \\
y_4(0) &= -2(3)y_2(0) \\
&= 0 \quad \because y_2(0) = 0 \\
y_5(0) &= -3(4)y_3(0) \\
&= -3(4)(-2) = 4! \\
y_6(0) &= -4(5)y_4(0) = 0 \\
y_7(0) &= -5(6)y_5(0) = -5(6)4! = -(6!) \\
&\vdots \\
\Rightarrow y_{2n+1}(0) &= (-1)^n(2n)! \text{ and } y_{2n}(0) = 0
\end{aligned}$$

Example 18 If $y = (\sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Also find $y_n(0)$

Solution: Here $y = (\sin^{-1}x)^2$①

$$\Rightarrow y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}} \quad \dots\dots\dots ②$$

Squaring both the sides, we get

$$\begin{aligned}
(1-x^2)y_1^2 &= 4(\sin^{-1}x)^2 \\
\Rightarrow (1-x^2)y_1^2 &= 4(y)^2
\end{aligned}$$

Differentiating the above equation w.r.t. x , we get

$$\begin{aligned}
(1-x^2)2y_1y_2 + y_1^2(-2x) - 4y_1 &= 0 \\
\Rightarrow (1-x^2)y_2 + y_1(-x) - 2 &= 0 \quad \dots\dots\dots ③
\end{aligned}$$

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$$\begin{aligned}
[y_{n+2}(1-x^2) + n_{c_1}y_{n+1}(-2x) + n_{c_2}y_n(-2)] - (y_{n+1}x + n_{c_1}y_n) &= 0 \\
\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) &= 0 \\
\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_nn^2 &= 0 \dots\dots\dots ④
\end{aligned}$$

To find $y_n(0)$: Putting $x = 0$ in ①, ② and ③, we get

$$y(0) = 0, y_1(0) = 0 \text{ and } y_2(0) = 2$$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = n^2 y_n(0)$$

Putting $n = 1, 2, 3 \dots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0)$$

$$= 0 \quad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2(0)$$

$$= 2^2 2 \quad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

\vdots

$$\Rightarrow y_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2 \cdot 2^2 \cdot 4^2 \dots \dots \dots (n-2)^2, & \text{if } n \text{ is even} \end{cases}$$

Exercise 1 B

1. Find y_n , if $y = x^3 \cos x$

$$\text{Ans. } x^3 \cos \left(x + \frac{n\pi}{2} \right) + 3nx^2 \cos \left[x + \frac{1}{2}(n-1)\pi \right] + 3n(n-1)x \cos \left[x + \frac{1}{2}(n-2)\pi \right] + n(n-1)(n-2) \cos \left[x + \frac{1}{2}(n-3)\pi \right]$$

2. Find y_n , if $y = x^2 e^x \cos x$

$$\text{Ans. } 2^{\frac{n}{2}} e^x \cos \left(x + \frac{n\pi}{4} \right)$$

3. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

4. If $y\sqrt{1+x^2} = \log(x + \sqrt{1+x^2})$, prove that

$$(1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0$$

5. If $y = [x + \sqrt{1+x^2}]^m$, prove that $(x^2 + 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

6. If $y = (\sinh^{-1} x)^2$, show that

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0. \text{ Also find } y_n(0).$$

$$\text{Ans. } y_{2n+1}(0) = 0 \text{ and } y_{2n}(0) = (-1)^{n-1} 2 \cdot 2^2 \cdot 4^2 \dots \dots \dots (2n-2)^2$$

7. If $y = \cos(m \sin^{-1} x)$, show that

$$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n. \text{ Also find } y_n(0).$$

8. If $f(x) = \tan x$, prove that $f^n(0) - n_{c_2} f^{n-2}(0) + n_{c_4} f^{n-4}(0) - \dots = \sin \frac{n\pi}{2}$