EM-I, CSE, PU

Class Lecture Notes

Basic Differentiation

goodbye

Get Started

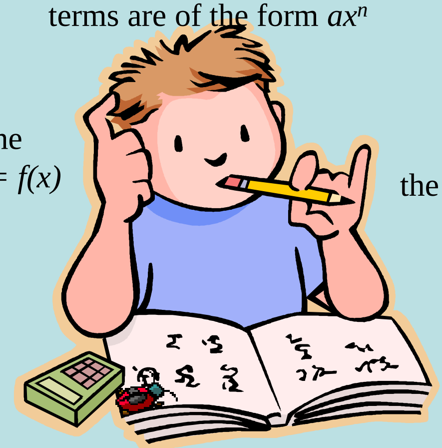
Basic differentiation

You should know the meaning of the terms limit, differentiable at a point,

differentiate, derivative, differentiable over an interval, derived function.

Differentiate a function whose terms are of the form ax^n

find the gradient of the tangent to a curve $y = f(x)$ at $x = a$



the rate of change of f at a

sketch a curve with given equation

Find where the function f is strictly increasing, decreasing or stationary.

determine the greatest/least values of a function on a given interval

solve optimisation problems using calculus

find the stationary point(s) (values) on a curve and determine their nature using differentiation

Rule of thumb:

Multiply by the power and reduce the power by 1.

Examples

$$f(x)=3x^5$$
$$\Rightarrow f'(x)=5\times 3x^{5-1}$$
$$\Rightarrow f'(x)=15x^4$$

$$f(x)=2x^3+4x^2-3x+4$$
$$\Rightarrow f(x)=2x^3+4x^2-3x^1+4x^0$$
$$\Rightarrow f'(x)=6x^2+8x^1-3x^0+0x^{-1}$$
$$\Rightarrow f'(x)=6x^2+8x-3$$

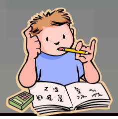
$$f(x)=4\sqrt{x}-5$$
$$\Rightarrow f(x)=4x^{\frac{1}{2}}-5x^0$$
$$\Rightarrow f'(x)=2x^{-\frac{1}{2}}$$
$$\Rightarrow f'(x)=\frac{2}{\sqrt{x}}$$

$$y=f(x)$$
$$\Rightarrow \frac{dy}{dx}=f'(x)$$

$$y=5x^2+\frac{3}{x}$$
$$\Rightarrow y=5x^2+3x^{-1}$$
$$\Rightarrow \frac{dy}{dx}=10x-3x^{-2}$$
$$\Rightarrow \frac{dy}{dx}=10x-\frac{3}{x^2}$$

$$y=\frac{x+5}{\sqrt{x}}$$
$$\Rightarrow y=\frac{x^1}{x^{\frac{1}{2}}}+\frac{5}{x^{\frac{1}{2}}}$$
$$\Rightarrow y=x^{\frac{1}{2}}+5x^{-\frac{1}{2}}$$
$$\Rightarrow \frac{dy}{dx}=\tfrac{1}{2}x^{-\frac{1}{2}}-\tfrac{5}{2}x^{-\frac{3}{2}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{x}}-\frac{5}{2\sqrt{x^3}}$$

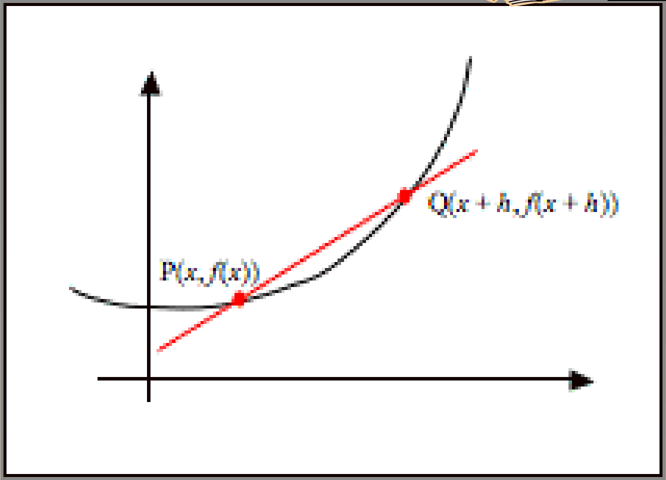
Test Yourself?



A straight line has a gradient $m_{AB} = \dfrac{y_B - y_B}{x_B - x_A}$

The chord PQ has a gradient

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}$$



As h tends to zero, Q tends towards P and the chord PQ becomes the tangent at P.

The gradient of the tangent at P = $f'(x) = \lim_{h \to 0}\left(\dfrac{f(x+h) - f(x)}{h}\right)$

To find the gradient of the tangent to the curve $y = f(x)$ at $x = a$ we need to evaluate $f'(a)$.

Example

Find the gradient of the tangent to the curve $y = 3x^2 + 2x - 1$ at the point P(1, 4)

$$x = 4$$

$$\frac{dy}{dx} = 6x + 2$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=4} = 6.4 + 2 = 26$$

Thus gradient at P is 26.

Test Yourself?

The Waverley can reach its top speed in 5 minutes.
During that time its distance from the start can be calculated using the formula $D = t + 50t^2$
where t is the time in minutes *and D is measured in metres.*

What is the Waverley's top speed?

Speed, v m/min, is the rate of change of *distance* with time.

$$D = t + 50t^2$$

$$\Rightarrow v = \frac{dD}{dt} = 1 + 100t$$

$$v = 1 + 100 \times 5 = 501 \text{ m/sec}$$

How fast is it accelerating?

Acceleration, a m/min/min, is the rate of change of *speed* with time.

$$v = 1 + 100t$$

$$\Rightarrow a = \frac{dv}{dt} = 100$$

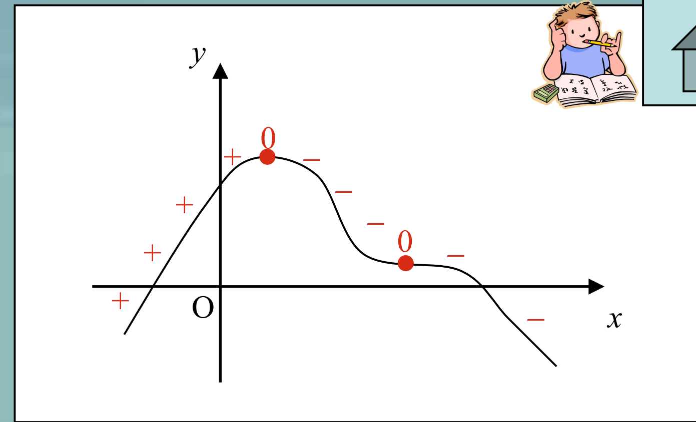
Test Yourself?

The signs indicate where the gradient of the curve is:

positive … the function is increasing

negative … the function is decreasing

zero … the function is stationary



A function is strictly increasing in a region where $f'(x) > 0$

A function is strictly increasing in a region where $f'(x) < 0$

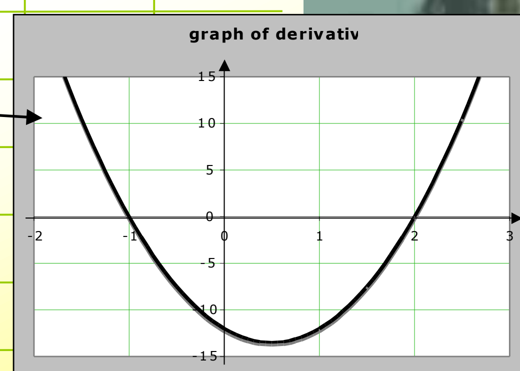
A function is stationary where $f'(x) = 0$

Example

$f(x) = 2x^3 - 3x^2 - 12x + 1$.

Identify where it is (i) increasing (ii) decreasing (iii) stationery

$$f'(x) = 6x^2 - 6x - 12$$



A sketch of the derivative shows us that

$f'(x) < 0$ for $0 < x < 2$ … $f(x)$ decreasing

$f'(x) > 0$ for $x < 0$ or $x > 2$ … $f(x)$ increasing

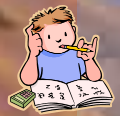
$f'(x) = 0$ for $x = 0$ or $x = 2$ … $f(x)$ stationary

Test Yourself?

To sketch a curve, collect the following information:

- where will it cut the y-axis? [$x = 0$]
- where will it cut the x-axis? [$f(x) = 0$]
- where is it stationary? [$f'(x) = 0$]
- where is $f(x)$ increasing/decreasing [$f'(x) > 0$/ $f'(x) < 0$] especially in the neighbourhood of the stationary points.
- how does $f(x)$ behave as $x \to \pm\infty$

Test Yourself?



When a function is defined on a closed interval, $a \le x \le b$, then it must have a maximum and a minimum value in that interval.

These values can be found either at

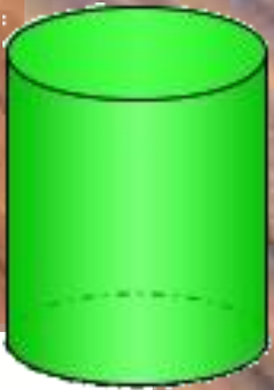
- a stationary point [where $f'(x) = 0$]
- an end-point of the closed interval. [$f(a)$ and $f(b)$]

All you need do is find these values and pick out the greatest and least values.

Example

A manufacturer is making a can to hold 250 ml of juice.
The cost of the can is dependent on its radius, x cm.
For practical reasons the radius must be between 2·5 cm and 4·5 cm.

The cost can be calculated from the formula
$C = x^3 - 5x^2 + 3x + 15, \quad 2{\cdot}5 \le x \le 4{\cdot}5.$
Calculate the maximum and minimum values of the cost function.



$$\frac{dC}{dx} = 3x^2 - 10x + 3$$ … which equals zero at stationary points.

$3x^2 - 10x + 3 = 0$
$\Rightarrow (3x - 1)(x - 3) = 0$
$\Rightarrow x = {}^1/_3$ or $x = 3$

Working to 1 d.p.

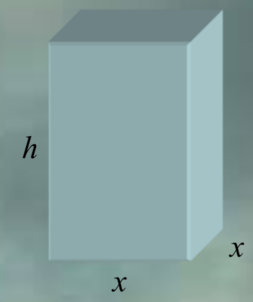
$f({}^1/_3) = 15{\cdot}5$
$f(3) = 6$
$f(2{\cdot}5) = 6{\cdot}9$
$f(4{\cdot}5) = 18{\cdot}4$

By inspection $f_{max} = 18{\cdot}4$ (when $x = 4{\cdot}5$) and $f_{min} = 6$ (when $x = 3$).

Test Yourself?

Example

A box has a square base of side x cm and a height of h cm.
It has a volume of 1 litre (1000 cm^3)
For what value of x will the surface area of the box be minimised?
[… and hence the cost of production be optimised]



To use calculus we must express the *surface area* in terms of x alone … so we must find h in terms of x .

For a cuboid, *volume* = *lbh* … so in this case $1000 = x^2h$.

And so, $h = \dfrac{1000}{x^2} = 1000x^{-2}$

The box is made from 6 rectangles, two of area x^2 cm^2 and four of area xh cm^2

Total Surface Area, $S = 2x^2 + 4xh$

$S = 2x^2 + 4000x^{-1}$

$\Rightarrow \dfrac{dS}{dx} = 4x - 4000x^{-2}$

$= 0$ at stationary points

$\Rightarrow 4x = 4000x^{-2}$

$\Rightarrow x^3 = 1000$

$\Rightarrow x = 10$

x	10-	10	10+
ds/dx	–	0	+
	$\searrow$	$\rightarrow$	$\nearrow$

If $x < 10$, $4x < 40$ and $4000x^{-2} > 40$
So ${}^{dS}/_{dx} < 0$ … a decreasing function

If $x > 10$, $4x > 40$ and $4000x^{-2} < 40$
So ${}^{dS}/_{dx} > 0$ … an increasing function

Decreasing before $x = 10$ and increasing after it
Gives us a minimum turning point.

If $x = 10$ then $h = 10$.

A cube of side 10 cm has a volume of 1000 cm^3 and the smallest possible surface area.

Test Yourself?

Example

Find the stationary points of the function $f(x) = x^5 + 5x^4 - 35x^3 + 1$ and determine their nature.

Differentiate: $f'(x) = 5x^4 + 20x^3 - 105x^2$

Equate to zero: $5x^4 + 20x^3 - 105x^2 = 0$ at stationary points

(factorise) $\Rightarrow\ 5x^2(x^2 + 4x - 21) = 0$

$\Rightarrow\ 5x^2(x - 3)(x + 7) = 0$

$\Rightarrow\ x = 0$ (twice), $x = 3$ or $x = -7$

Make a table of signs:

x	→	-7	→	0	→	3	→
$5x^2$	+	+	+	0	+	+	+
x-3	–	–	–	–	–	0	+
x+7	–	0	+	+	+	+	+
dy/dx	+	0	–	0	–	0	+
profile	/	–	\	–	\	–	/
Nature		max		PI		min	

… scan the critical x's

… examine each factor of dy/dx

… conclusions based on sign of derivative

Stationary points occur at

$x = -7$ (a maximum turning point);

$x = 0$ (a horizontal point of inflexion);

$x = 3$(a minimum turning point.

The corresponding stationary values are

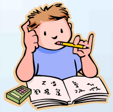
$f(-7) = 7204$

$f(0) = 1$

$f(3) = -296$

The stationary points are: (–7, 7204) a max TP; (0, 1) a horizontal PI; (3, –296) a min TP

Test Yourself?



Differentiate

(a) $3x^5 + 4x^3 - x - 3$

(b) $3x^2 + 2\sqrt{x}$

(c) $4 + \dfrac{3}{x}$

(d) $\dfrac{2x + \sqrt{x}}{x^2}$

reveal

Differentiate

(a) $3x^5 + 4x^3 - x - 3$

(b) $3x^2 + 2\sqrt{x}$

(c) $4 + \dfrac{3}{x}$

(d) $\dfrac{2x + \sqrt{x}}{x^2}$

(a) $15x^4 + 12x^2 - 1$

(b) $3x^2 + 2\sqrt{x} = 3x^2 + 2x^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = 6x + x^{-\frac{1}{2}} = 6x + \frac{1}{\sqrt{x}}$$

(c) $4 + \dfrac{3}{x} = 4 + 3x^{-1}$

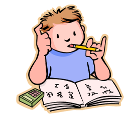
$$\Rightarrow \frac{dy}{dx} = -3x^{-2} = -\frac{3}{x^2}$$

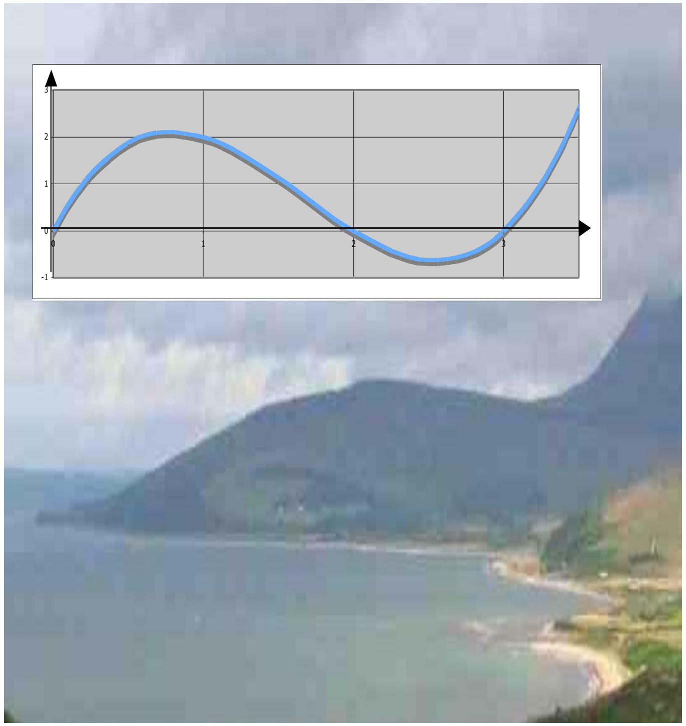
(d) $\dfrac{2x + \sqrt{x}}{x^2} = \dfrac{2x}{x^2} + \dfrac{x^{\frac{1}{2}}}{x^2} = 2x^{-1} + x^{-\frac{3}{2}}$

$$\Rightarrow \frac{dy}{dx} = -2x^{-2} - \tfrac{3}{2}x^{-\frac{5}{2}}$$

You must make each term take the shape ax^n

Gradient at a Point





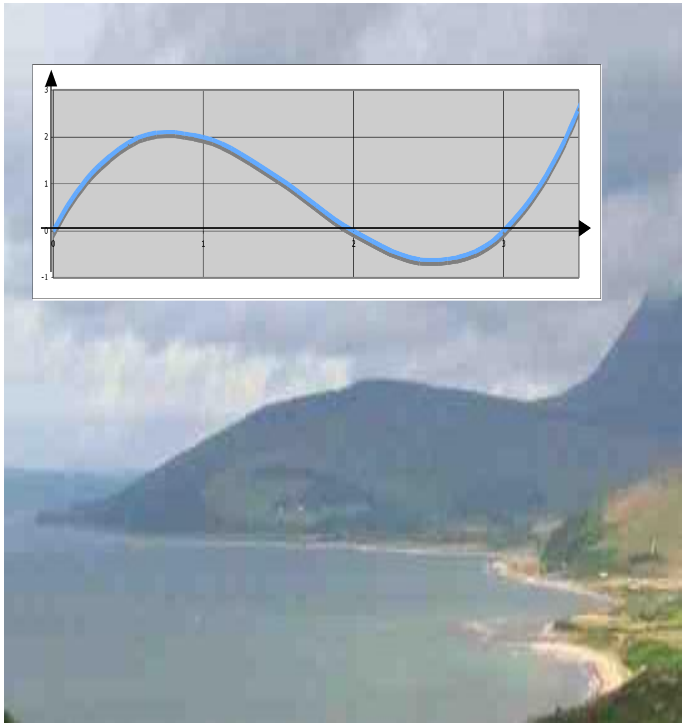
reveal

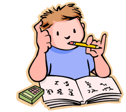
Using convenient units and axes, the profile of a hill has been modelled by

$H = 0{\cdot}1(x^3 - 5x^2 + 6x)$ where H is the height and x is the distance from the origin.

What is the gradient of the curve when $x = 2$?

Gradient at a Point





$$H = 0\cdot 1x^3 - 0\cdot 5x^2 + 0\cdot 6x$$

$$\frac{dH}{dx} = 0\cdot 3x^2 - x + 0\cdot 6$$

$$\Rightarrow \left.\frac{dH}{dx}\right|_{x=2} = 0\cdot 3\times 2^2 - 2 + 0\cdot 6$$

$$= 1\cdot 2 - 2 + 0\cdot 6$$

$$= -0\cdot 2$$

Using convenient units and axes, the profile of a hill has been modelled by

$H = 0\cdot 1(x^3 - 5x^2 + 6x)$ where H is the height and x is the distance from the origin.

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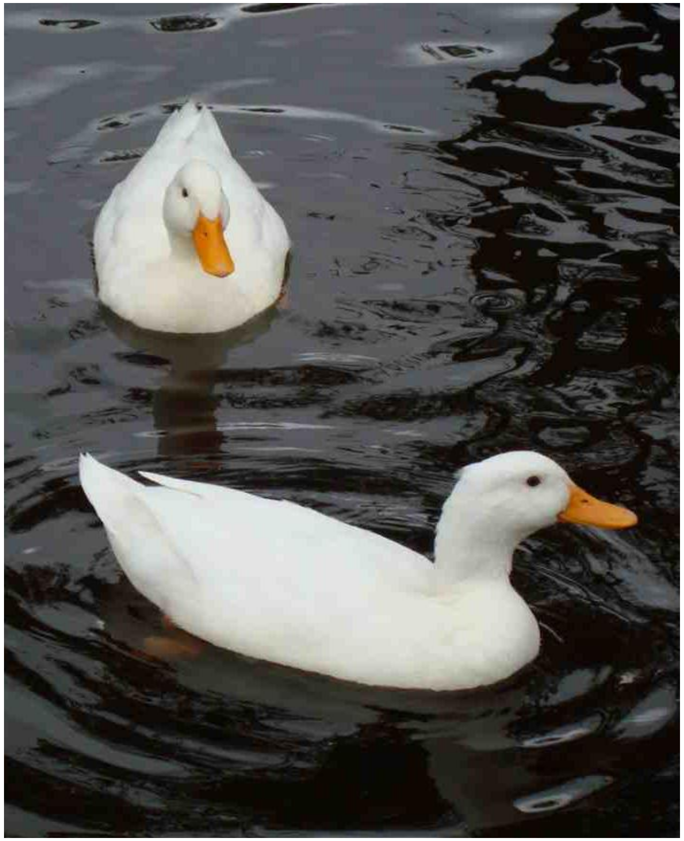
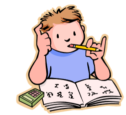
Rates of change





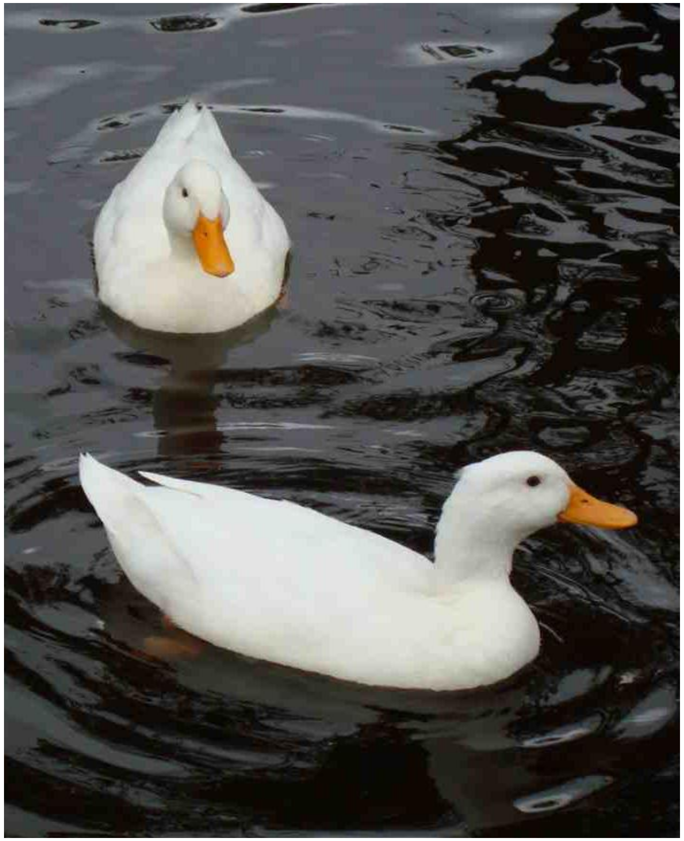
reveal

The radius, r cm, of a particular circular ripple is related to the time, t, in seconds since the photo was taken.

$$r = 4t + 3$$

How fast is the area of the circle growing when the radius is 10 cm?

Rates of change



The radius, r cm, of a particular circular ripple is related to the time, t, in seconds since the photo was taken.

$$r = 4t + 3$$

How fast is the area of the circle growing when the radius is 11 cm?

Since $r = 4t + 3$

then $r = 11 \Rightarrow t = 2$.

and the area of the circular ripple, A cm^2, is $\pi(4t + 3)^2$

$$A = \pi(4t + 3)^2$$

$$\Rightarrow \quad A = 16\pi t^2 + 24\pi t + 9\pi$$

$$\Rightarrow \frac{dA}{dt} = 32\pi t + 24\pi$$

$$\Rightarrow \left.\frac{dA}{dt}\right|_{t=2} = 32\pi \times 2 + 24\pi = 88\pi$$

When the radius is 11 cm, the area is increasing at the rate of $276{\cdot}5$ cm^2 per second correct to 1 d.p.

Increasing/decreasing functions

reveal

During one study of red squirrels the number in one area was modelled by the function

$N(x) = x^3 - 15x^2 + 63x - 10, \quad 1 < x < 12$

Where x is the number of years since the study started.

During what years was this a decreasing function?

Increasing/decreasing functions

During one study of red squirrels
the number in one area was modelled
by the function
$N(x) = x^3 - 15x^2 + 63x - 10, \quad 1 < x < 12$
Where x is the number of years since the
study started.

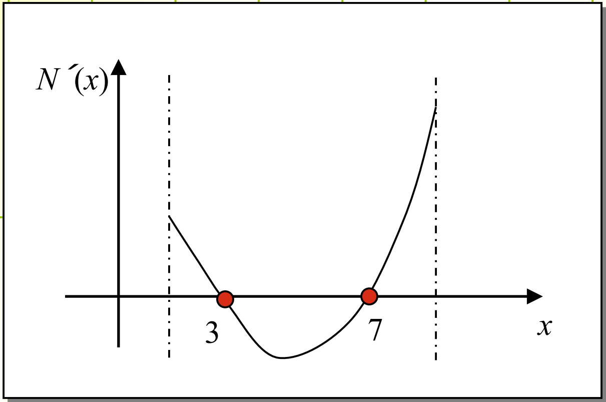
During what years was this a decreasing function?

$N(x) = x^3 - 15x^2 + 63x - 10, \quad 1 < x < 12$

$\Rightarrow N'(x) = 3x^2 - 30x + 63$

$N'(x) = 0$ at stationary points

$3x^2 - 30x + 63 = 0 \Rightarrow x = 3 \text{ or } 7$



The sketch shows that $N'(x) < 0$ for $3 < x < 7$.

The population was on the decrease between the 3rd and 7th years of the survey.

Curve sketching

Sketch the curve with equation

$y = x^3 - 5x^2 - 8x + 12$

Identifying

(i) where will it cut the y-axis?

(ii) where will it cut the x-axis?

(iii) where is it stationary?

(iv) where it is increasing/decreasing especially in the neighbourhood of the stationary points.

(v) how it behave as $x \to \pm\infty$

reveal

Sketch the curve with equation

$y = x^3 - 5x^2 - 8x + 12$

Identifying

(i) where will it cut the y-axis?

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(iii) where is it stationary?

(iv) where it is increasing/decreasing especially in the neighbourhood of the stationary points.

(v) how it behave as $x \to \pm\infty$

(i) $x = 0 \Rightarrow y = 12$ … (0, 12)

(ii) $y = 0 \Rightarrow x^3 - 5x^2 - 8x + 12 = 0$

$\Rightarrow (x - 1)(x + 2)(x - 6) = 0$

$x = 1$ or $x = -2$ or $x = 6$ … (1, 0), (–2,0),(6, 0)

(iii) $^{dy}/_{dx} = 3x^2 - 10x - 8$

$= 0$ at S.P.s

$3x^2 - 10x - 8 = 0 \Rightarrow (3x + 2)(x - 4) = 0$

$\Rightarrow x = -^2/_3$ or $x = 4$ … a max TP at $(-^2/_3, {}^{400}/_{27})$

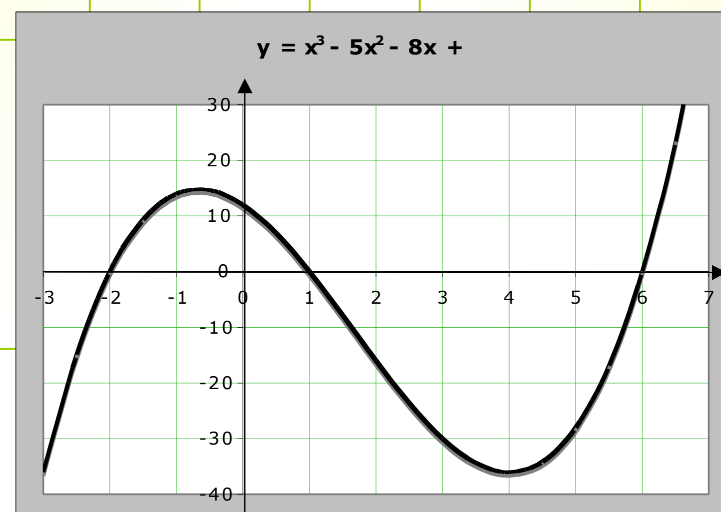
$\Rightarrow y = {}^{400}/_{27}$ or $y = -36$ … a min at $(4, -36)$

(iv)

x	→	$-^2/_3$	→	4	→
3x + 2	–	0	+	+	+
x – 4	–	–	–	0	+
dy/dx	+	0	–	0	+
inc/dec	/	-	\	-	/
nature		max		min	

(v) When x is large and positive, y is large and positive (1st quad)

When x is large and negative, y is large and negative (3rd quad)



reveal

A shop takes x deliveries a year of cereal.

The suppliers are willing to make between 20 and 200 deliveries a year.

The annual cost of these deliveries can be calculated from the formula:

$$C(x) = 4x + \frac{10000}{x} + 1000 \quad ; 20 \leq x \leq 200$$

Calculate the number of deliveries that will minimise the costs.

A shop takes x deliveries a year of cereal.

The suppliers are willing to make between 20 and 200 deliveries a year.

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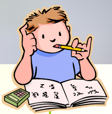
$$C(x) = 4x + \frac{10000}{x} + 1000 \quad ; 20 \le x \le 200$$

What is the manager's best strategy to minimise the costs.



Find stationary point(s)

$$C(x) = 4x + \frac{10000}{x} + 1000$$

$$\Rightarrow C'(x) = 4 - \frac{10000}{x^2}$$

At stationary points $C'(x) = 0$

$$4 - \frac{10000}{x^2} = 0$$

$$\Rightarrow x^2 = \frac{10000}{4} = 2500$$

$$\Rightarrow x = 50$$

Examine end-points and stationary point(s)

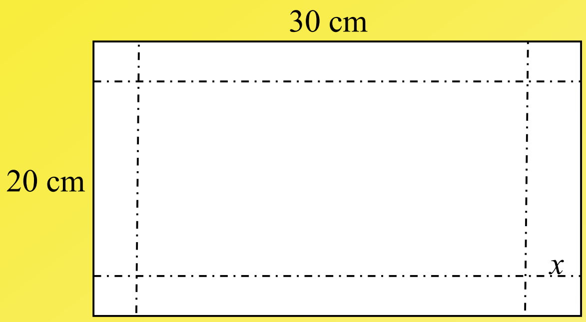
$$C(20) = 4 \times 20 + \frac{10000}{20} = 580$$

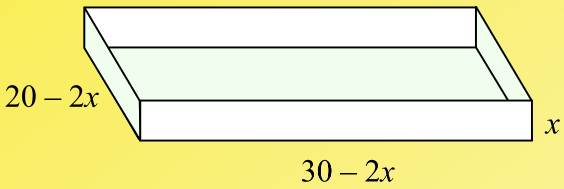
$$C(200) = 4 \times 200 + \frac{10000}{200} = 850$$

$$C(50) = 4 \times 50 + \frac{10000}{50} = 400$$

Optimum strategy: Order 50 times a year. This will minimise costs at £400.

Optimisation



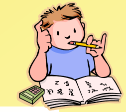


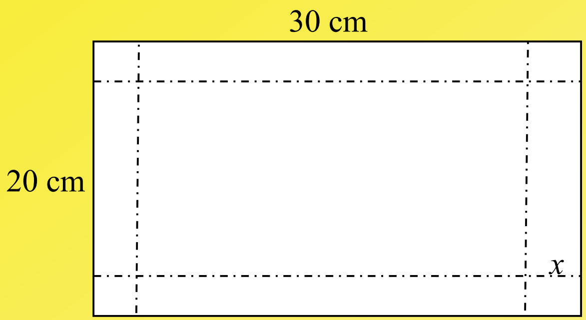
reveal

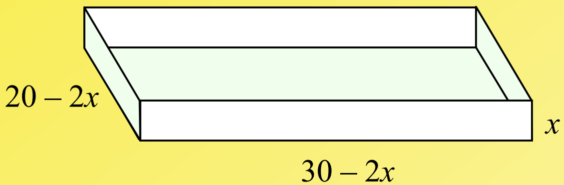
An A4 sheet of paper, roughly 20 by 30 cm has its edges folded up to create a tray. Each crease is x cm from the edge.

What size should x be in order to maximise the volume of the tray?

Optimisation







An A4 sheet of paper, roughly 20 by 30 cm has its edges folded up to create a tray. Each crease is x cm from the edge.

What size should x be in order to maximise the volume of the tray?

Express volume in terms of x:

$V = x(20 - 2x)(30 - 2x)$

$= 4x^3 - 100x^2 + 600x$

Differentiate

$$\frac{dV}{dx} = 12x^2 - 200x + 600$$

$= 0$ at S.P.s

$$12x^2 - 200x + 600 = 0$$

$$\Rightarrow x = \frac{200 \pm \sqrt{40000 - 4.12.600}}{24}$$

$$\Rightarrow x = 12\cdot 7 \text{ or } 3\cdot 9 \text{ (to 1 d.p.)}$$

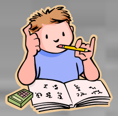
Check nature

x	→	3·9	→	12·7	→
dV/dx	–	0	+	0	–
inc/dec	\		/		\
Nature		mim		max	

Conclusion

When $x = 3\cdot 9$, the volume is at a maximum of 1056 cm^3.

Stationary points and nature



A function is defined by

$$f(x)=\frac{x^5}{5}+\frac{x^4}{4}-\frac{x^3}{3}-\frac{x^2}{2}$$

reveal

(a) Show that its derivative has factors x, $(x+1)$ and $(x-1)$

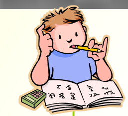
(b) Find the stationary points of the function and determine their nature.

A function is defined by

$$f(x) = \frac{x^5}{5} + \frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2}$$

(a) Show that its derivative has factors x, $(x + 1)$ and $(x - 1)$

(b) Find the stationary points of the function and determine their nature.



differentiate

$$f'(x) = x^4 + x^3 - x^2 - x$$
$$= x(x^3 + x^2 - x - 1)$$
$$= x[x^2(x+1) - (x+1)]$$
$$= x(x+1)(x^2 - 1)$$
$$= x(x+1)(x-1)(x+1)$$

Equate to zero

At S.Ps $f'(x) = 0$

i.e. $x = 0$ or $x = -1$ or $x = 1$

Make nature table

x	→	-1	→	0	→	1	→
$(x+1)^2$	+	0	+	+	+	+	+
x	–	–	–	0	+	+	+
(x-1)	–	–	–	–	–	0	+
f'(x)	+	0	+	0	–	0	+
inc/dec	/	-	/	-	\	-	/
nature		PI		max		min	

Find corresponding y-values.

$$f(-1) = -\tfrac{1}{5} + \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{2} = -\tfrac{7}{60}$$
$$f(1) = \tfrac{1}{5} + \tfrac{1}{4} - \tfrac{1}{3} - \tfrac{1}{2} = -\tfrac{23}{60}$$
$$f(0) = 0$$

Summarise findings

$(-1, -{}^{7}/_{60})$ point of inflexion

$(1, -{}^{23}/_{60})$ minimum turning point

(0, 0) minimum turning point