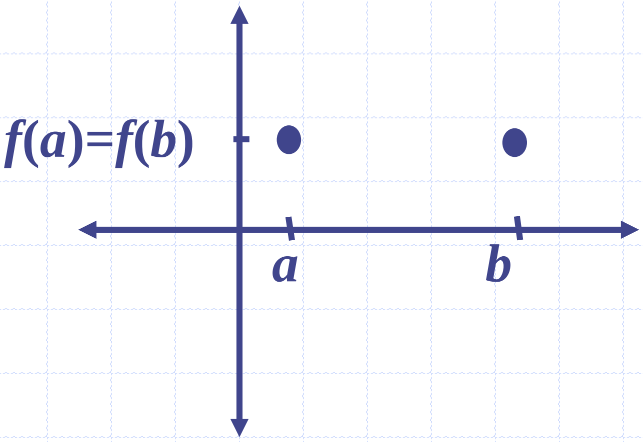
Rolle's Theorem
and
Mean Value Theorem

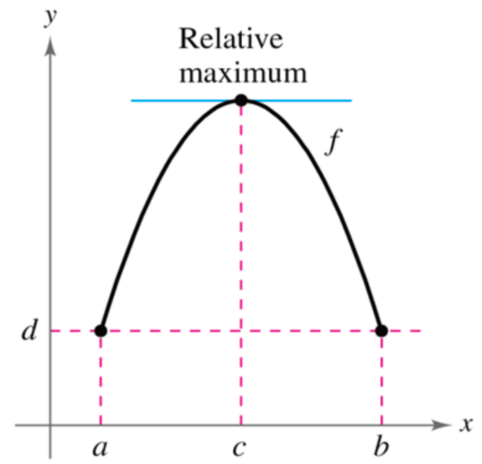
Rolle's Theorem

If you connect from $f\,(a)$ to $f\,(b)$ with a smooth curve, there will be at least one place where $f\,'(c) = 0$

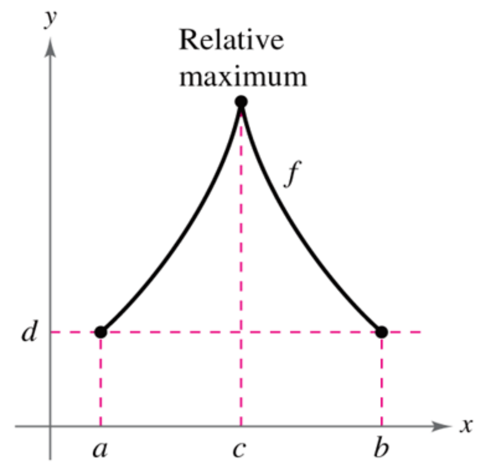


Rolle’s Theorem

Rolle's theorem is an important basic result about differentiable functions. Like many basic results in the calculus it seems very obvious. It just says that between any two points where the graph of the differentiable function $f(x)$ cuts the horizontal line there must be a point where $f'(x) = 0$. The following picture illustrates the theorem.



(a) f is continuous on $[a, b]$ and differentiable on (a, b).



(b) f is continuous on $[a, b]$.

Rolle's Theorem

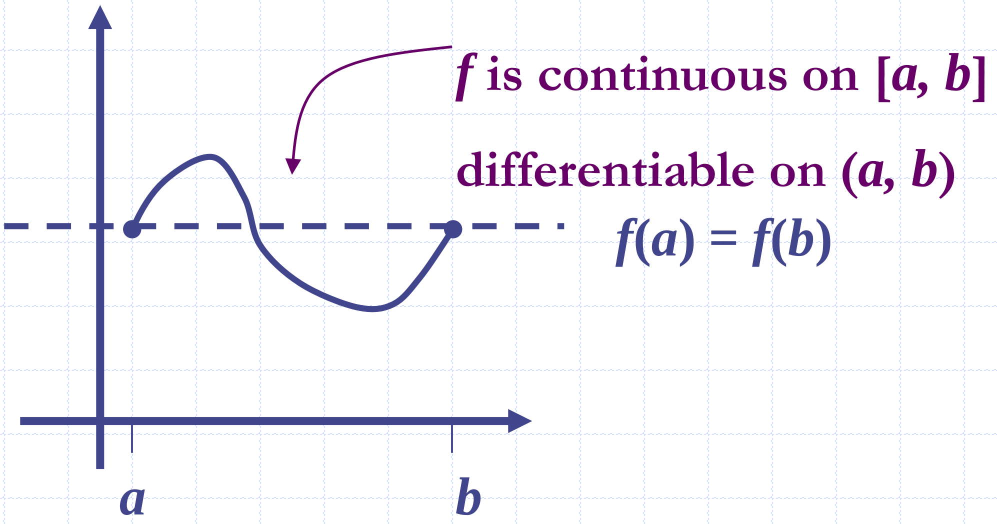
If 1) $f(x)$ is continuous on $[a, b]$,

2) $f(x)$ is differentiable on (a, b), and

3) $f(a) = f(b)$

then there is at least one value of x on (a, b), call it c, such that

$f'(c) = 0$.



Example

<u>**Example 1**</u> $f(x) = x^4 - 2x^2$ on $[-2, 2]$

(f is continuous and differentiable)

$$f(-2) = 8 = f(2)$$

Since , then Rolle's Theorem applies…

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0$$

then, $x = -1$, $x = 0$, and $x = 1$

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of *c*.

Example 2 $f(x) = 4 - x^2$ $[-2, 2]$

Rolle’s Theorem

Does Rolle’s Theorem apply?

If not, why not?

If so, find the value of *c*.

Example 3 $f(x) = x^3 - x$ $[-1, 1]$

Example

Example 4

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \quad \text{on } [-1, 1]$$

(Graph the function over the interval on your calculator)

continuous on [-1, 1]

not differentiable at 0

not differentiable on (-1, 1)

***f*(-1) = 1 = *f*(1)**

Rolle's Theorem Does NOT apply since

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of *c*.

Example 5 $f(x) = \dfrac{x^2+4}{x^2}$ $[-2, 2]$

Note

When working with Rolle's make sure you

1. State $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).
2. Show that $f(a) = f(b)$.
3. State that there exists at least one $x = c$ in (a, b) such that $f'(c) = 0$.

This theorem only guarantees the existence of an extrema in an open interval. It does not tell you how to find them or how many to expect. If YOU can not find such extrema, it does not mean that it can not be found. In most of cases, it is enough to know the existence of such extrema.

Mean Value Theorem- MVT

The Mean Value Theorem is one of the most important theoretical tools in Calculus. It states that if *f*(*x*) is defined and continuous on the interval [*a*,*b*] and differentiable on (*a*,*b*), then there is at least one number *c* in the interval (*a*,*b*) (that is *a*<*c*<*b*) such that

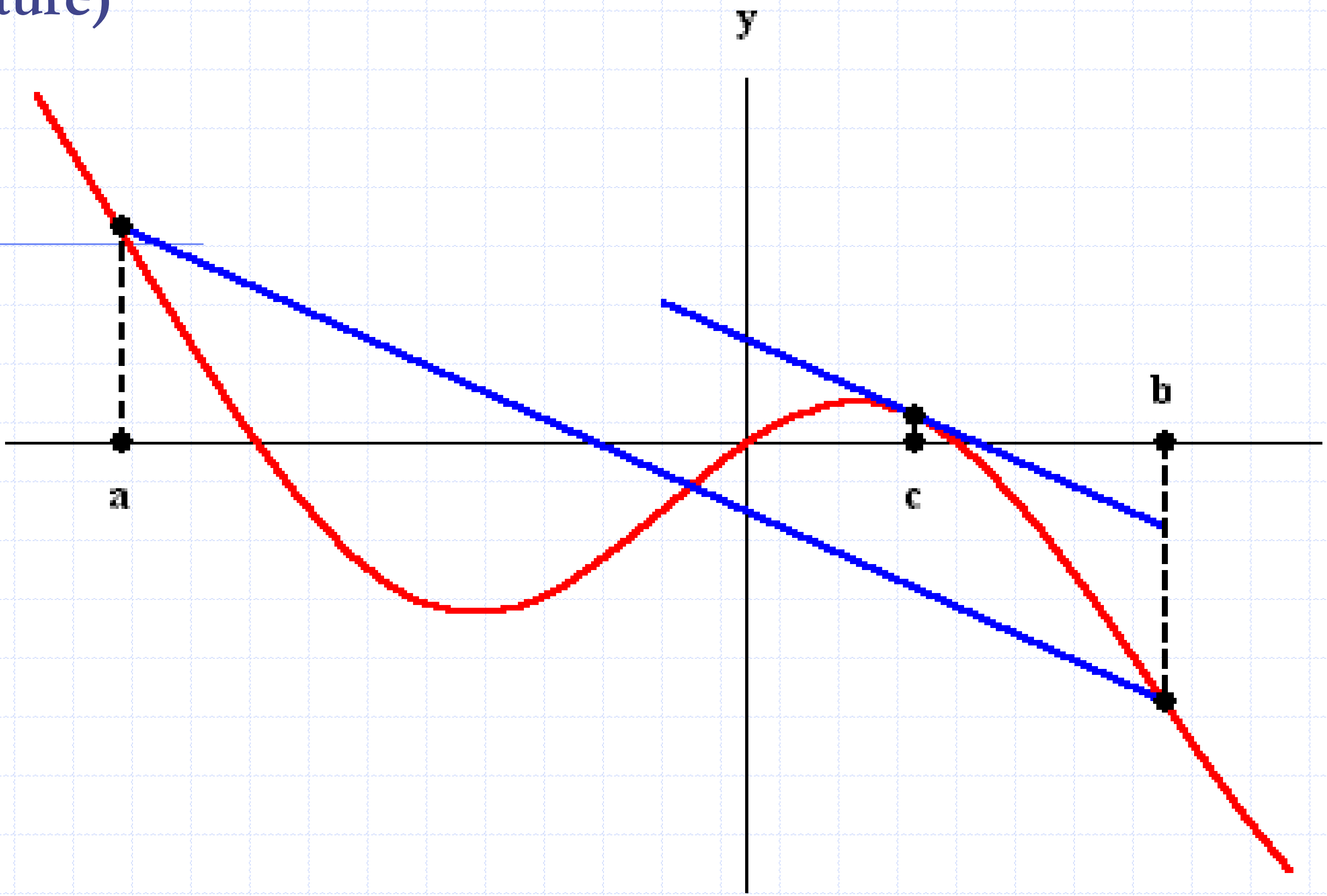
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, there exists a point in the interval (*a*,*b*) which has a horizontal tangent. In fact, the Mean Value Theorem can be stated also in terms of slopes. Indeed, the number

$$\frac{f(b) - f(a)}{b - a}$$

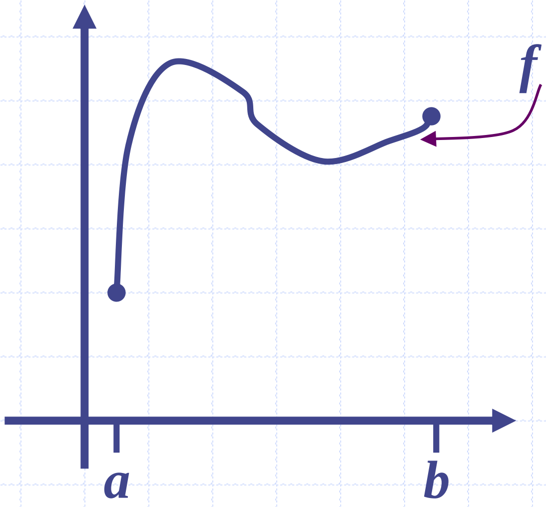
is the slope of the line passing through (*a*, *f*(*a*)) and (*b*, *f*(*b*)). So the conclusion of the Mean Value Theorem states that there exists a point such that the tangent line is parallel to the line passing through (*a*, *f*(*a*)) and (*b*, *f*(*b*)).

(see Picture)



The special case, when $f(a) = f(b)$ is known as <u>Rolle's Theorem</u>. In this case, we have $f\,'(c) = 0$.

Mean Value Theorem- MVT



If: *f* is continuous on [*a*, *b*],

differentiable on (*a*, *b*)

Then: there is a *c* in (*a*, *b*) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example

Example 6 $f(x) = x^3 - x^2 - 2x$ on [-1,1]

(f is continuous and differentiable)

MVT applies

$$f'(x) = 3x^2 - 2x - 2$$

$$f'(c) = \frac{-2-0}{1-(-1)} = -1$$

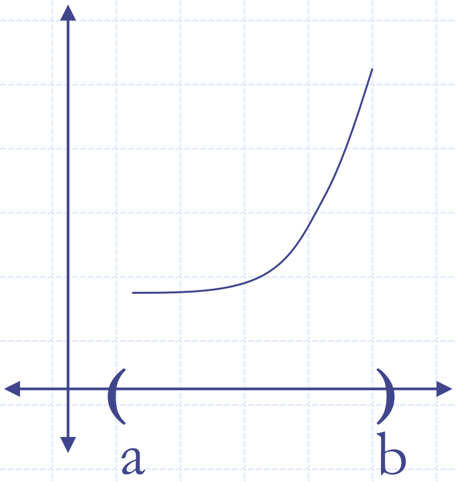
$$3c^2 - 2c - 2 = -1$$

$$(3c+1)(c-1) = 0$$

$$c = -\frac{1}{3}, \quad c = 1$$

Mean Value Theorem- MVT

Note:



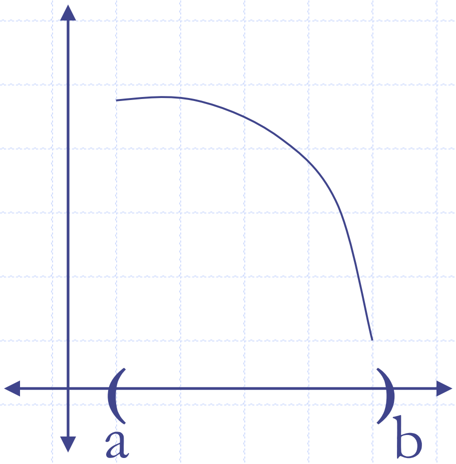
$$f'(x) > 0 \text{ on } (a,b) \quad \Rightarrow$$

$$f \text{ is increasing on } (a,b)$$

The graph of f is rising

Mean Value Theorem- MVT

Note:



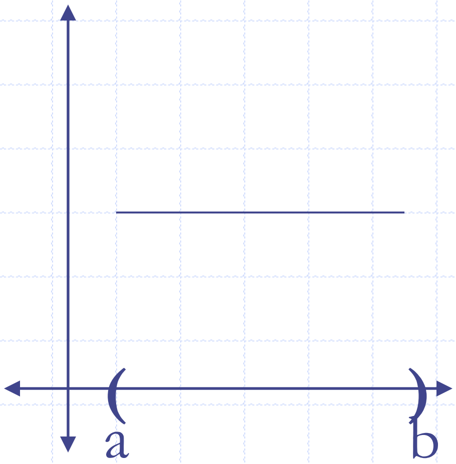
$$f\,'(x) < 0 \text{ on } (a,b) \quad \Rightarrow$$

$$f \text{ is decreasing on } (a,b)$$

The graph of f is falling

Mean Value Theorem- MVT

Note:



f is constant on (a,b)

The graph of f is level

Example

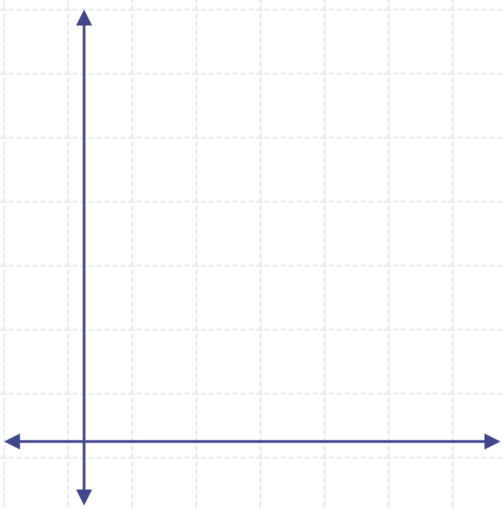
Example 7

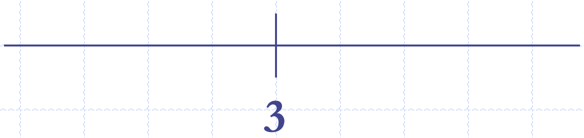
$$f(x) = x^2 - 6x + 12$$

$$f'(x) = 2x - 6$$

$$= 2(x-3)$$

$$= 0 \text{ iff } x = 3$$





Finding a Tangent Line

<u>Example 8</u> Find all values of c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(x) = \frac{x+1}{x},\ [\tfrac{1}{2}, 2]$$

$$\frac{f(2) - f(1/2)}{2 - 1/2} = \frac{3/2 - 3}{3/2} = -1$$

$$f'(x) = \frac{d}{dx}\left(1 + \frac{1}{x}\right) = -\frac{1}{x^2}$$

$$f'(c) = -\frac{1}{c^2} = -1$$

$c = 1$