Rolles Theorem

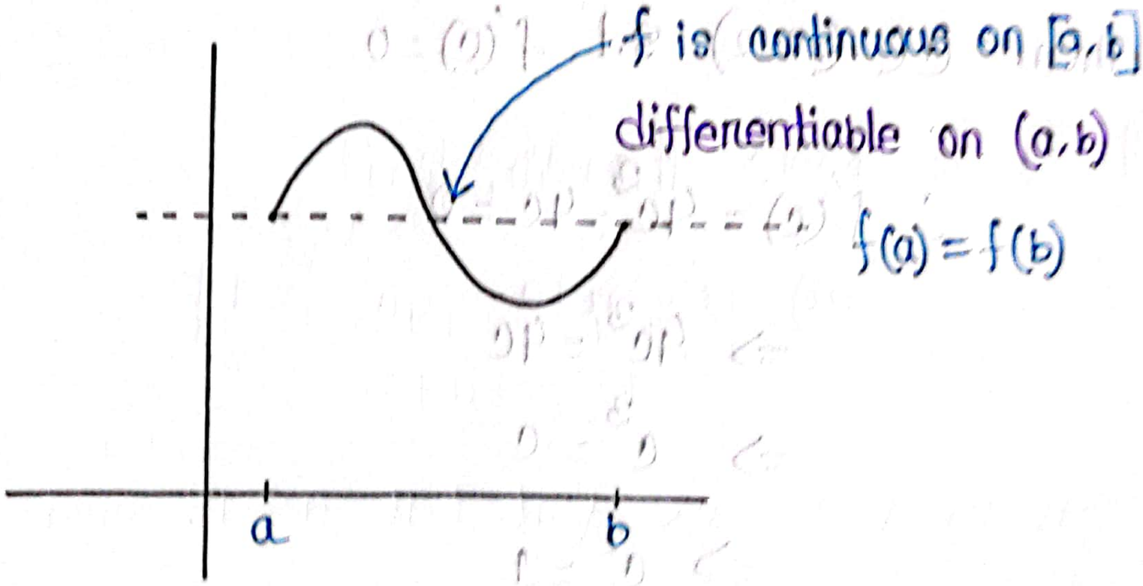
If 1) $f(x)$ is continuous on $[a,b]$

2) $f(x)$ is differentiable on (a,b), and

3) $f(a) = f(b)$

then there is at least one value of x on (a,b). call it c, Such that.

$$f'(c) = 0$$



Eample-1: Verify Rolle's Theorem in [-2, 2] for the function $f(x) = x^4 - 2x^2$.

Soln: We know that $f(x) = x^4 - 2x^2$ is differentiable and continuous on $[-2, 2]$.

and $f(-2) = (-2)^4 - 2\cdot(-2)^2 = 8 = f(2)$

So, now we can apply Rolle's Theorem -

$$f'(x) = 4x^3 - 4x \text{ exists in } [-2, 2]$$

Then, $c \in (-2, 2)$ s.t $f'(c) = 0$

$$\therefore f'(c) = 4c^3 - 4c = 0$$

$$\Rightarrow 4c^3 = 4c$$

$$\Rightarrow c^3 = c$$

$$\Rightarrow c^2 = 1$$

$$\therefore c = \pm 1 \in (-2, 2)$$

Hence, Rolle's Theorem is verified.

Example-2: Verify Rolle's Theorem on $[-2,2]$ for the function $f(x) = 4-x^2$.

Soln: We know that, $f(x) = 4-x^2$ is differentiable and continuous on $[-2,2]$.

and, $f(-2) = 4-(-2)^2 = 4-4 = 0 = f(2)$

So, now we can apply Rolle's Theorem -

$f'(x) = -2x$ exists in $[-2,2]$

Then, $c \in (-2,2)$ s.t $f'(c) = 0$

$\therefore f'(c) = -2c = 0$

$\Rightarrow -2c = 0$

$\therefore c = 0 \in [-2,2]$

Hence, Rolle's ~~Va~~ Theorem is Verified.

Practice-1: Verify Rolle's Theorem,

$$f(x) = x^3 - x \; ; \text{ on } [-1, 1]$$

Practice-2: Verify Rolle's Theorem,

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \text{ on } [-1, 1]$$

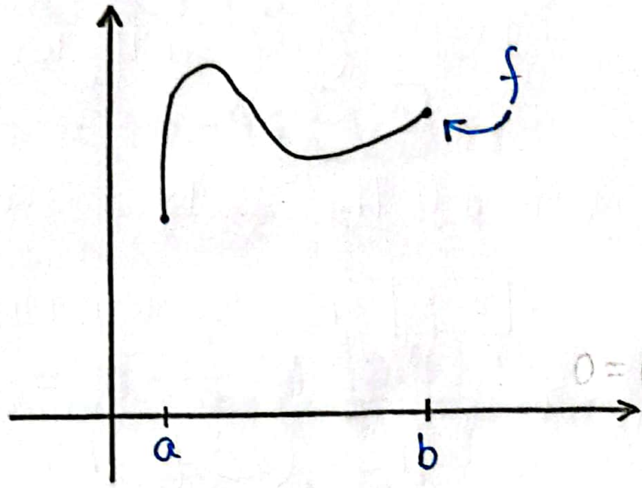
3] Verify Rolle's Theorem,

$$f(x) = \frac{x^2+4}{x^2} \text{ on } [-2, 2]$$

4] Verify Rolle's Theorem,

$$f(x) = x^2 - 6x + 8 \text{ on } [2, 4]$$

Mean Value Theorem - MVT



If: f is continuous on $[a,b]$

differentiable on (a,b)

Then: there is a c in $(a.b)$ Such that -

$$* f(b) \neq f(a); \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

Example-1: Verify MVT for the fuction $f(x) = x^3 - x^2 - 2x$ on $[-1,1]$.

Soln: $f(x) = x^3 - x^2 - 2x$ is continuous and differentiable.

Then, $f'(x) = 3x^2 - 2x - 2$ ——— (i)

and $\dfrac{f(b)-f(a)}{b-a} = \dfrac{-2-0}{1+1} = \dfrac{-2}{2} = -1$

$\therefore f'(c) = 3c^2 - 2c - 2$

$\therefore 3c^2 - 2c - 2 = -1$

$\Rightarrow 3c^2 - 2c - 1 = 0$

$\Rightarrow 3c^2 - 3c + c - 1 = 0$

$\Rightarrow 3c(c-1) + 1(c-1) = 0$

$\Rightarrow (3c+1)(c-1) = 0$

$3c+1 = 0$ and $c - 1 = 0$

$\Rightarrow c = -\frac{1}{3}$ and $c = 1$

But, $-\frac{1}{3} \in (-1, 1)$

So, MVT is verified.

Practice: Verify MVT for these function

1) $f(x) = x^3 - x$; $[-3, 5]$

2) $f(x) = x^3 + x - 4$; $[-1, 2]$

3) $f(x) = \sqrt{25 - x^2}$; $[-5, 3]$

4) $f(x) = x - \frac{1}{x}$; $[3, 4]$

5) $f(x) = \sqrt{x^2 - 4}$; $[2, 4]$

Taylors Theorem | Series: Polynomials

Definition: If f can be differentiated n times at x_0, then we define the nth Taylor Polynomial for f about $x = x_0$ to be-

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

$$+ \cdots\cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Example 4: Find the first four Taylor Polynomials for $\ln x$ about $x = 2$;

Soln: Let, $f(x) = \ln x$. Thus,

$f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{4}$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(2) = \frac{1}{4}$$

Substituting with $x_0 = 2$, yields

$$P_0(x) = f(2) = \ln 2$$

$$P_1(x) = f(2) + f'(2)(x-2)$$

$$= \ln 2 + \frac{1}{2}(x-2)$$

$$P_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$$

$$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3$$

17-24: Find the Taylor Polynomials of orders $n=0,1,2,3$ and 4 about $x=x_0$. and then find the nth Taylor Polynomial for the function in Sigma notation.

24| $\ln x$; $x_0=e$

Soln: Let $f(x)=\ln x$. Thus, and $x_0=e$

$f(x)=\ln x$ $\qquad$ $f(e)=\ln e=1$

$f'(x)=\frac{1}{x}$ $\qquad$ $f'(e)=\frac{1}{e}$

$f''(x)=-\frac{1}{x^2}$ $\qquad$ $f''(e)=-\frac{1}{e^2}$

$f'''(x)=\frac{2}{x^3}$ $\qquad$ $f'''(e)=\frac{2}{e^3}$

$f''''(x)=-\frac{6}{x^4}$ $\qquad$ $f''''(e)=-\frac{6}{e^4}$

Substituting with $x_0=e$ yields,

$P_0(x)=f(e)=\ln e=1$

$P_1(x)=f(e)+f'(e)(x-e)=1+\frac{1}{e}(x-e)$

$$P_2(x) = f(e) + f'(e)(x-e) + \frac{f''(e)}{2!}(x-e)^2$$

$$= 1 + \frac{x-e}{e} + \frac{-\frac{1}{e^2}}{2}(x-e)^2$$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2$$

$$P_3(x) = f(e) + f'(e)(x-e) + f''(e)/_{2!}(x-e)^2 + \frac{f'''(e)}{3!}(x-e)^3$$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3$$

$$P_4(x) = f(e) + f'(e)(x-e) + f''(e)/_{2!}(x-e)^2 + \frac{f'''(e)}{3!}(x-e)^3$$

$$+ \frac{f''''(e)}{4!}(x-e)^4$$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 - \frac{1}{4e^4}(x-e)^4$$

20) $\dfrac{1}{x+2}$; $x_0 = 3$;

Soln:

$$P_0(x) = \frac{1}{5}$$

$$P_1(x) = \frac{1}{5} - \frac{1}{25}(x-3)$$

$$P_2(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2$$

$$P_3(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{625}(x-3)^2$$

$$P_4(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{625}(x-3)^2 + \frac{1}{3125}(x-3)^4$$

☐ **Maclaurin Polynomial:** If f can be differentiated n times at 0, then we define the nth Maclaurin polynomial for f to be -

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots\cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Example-2: Find the Maclaurin polynomials P_0, P_1, P_2, P_3 and P_n for e^x

Soln: Let $f(x) = e^x$. Thus,

$$f'(x) = f''(x) = f'''(x) = \cdots\cdots = f^{(n)}(x) = e^x$$

and

$$f(0) = f''(0) = f'''(0) = \cdots\cdots = f^{(n)}(0) = e^0 = 1$$

Therefore,

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)x = 1 + x$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$$

$$P_n(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots\cdots+\frac{f^{(n)}(0)}{n!}x^n$$

$$= 1+x+\frac{x^2}{2!}+\cdots\cdots+\frac{x^n}{n!}$$

Example-5: Find the nth Maclaurin polynomial for

$$\frac{1}{1-x}$$

Ans: $P_n(x) = 1+x+x^2+\cdots\cdots+x^n \quad (n=0,1,2\ldots)$

(7–16) Find the nth Maclaurin polynomials for the function-

11) $\ln(1+x)$ 12) $\frac{1}{1+x}$