Lecture no. 5: Simple Harmonic Motion: Mathematical Problems

Example 1[illegible]. *Show that for a particle executing simple harmonic motion, its velocity at any instant is*

$$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

The displacement,

$$y = a \sin \omega t \qquad \ldots (1)$$

The velocity at any instant is,

$$\frac{dy}{dt} = a\omega \cos \omega t \qquad \ldots (2)$$

From equation (1)

$$\sin \omega t = \frac{y}{a}$$

$$\cos \omega t = \sqrt{1 - \sin^2 \omega t}$$

$$\cos \omega t = \sqrt{1 - \frac{y^2}{a^2}}$$

$$\frac{dy}{dt} = a\omega\sqrt{1 - \frac{y^2}{a^2}}$$

or

$$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2}.$$

Example 1[illegible]. *For a particle vibrating simple harmonically the displacement is 8 cm at the instant the velocity is 6 cm/s and the displacement is 6 cm at the instant the velocity is 8 cm/s. Calculate (i) amplitude, (ii) frequency and (iii) time period.*

The velocity of a particle executing SHM,

$$v = \frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

In the first case,

$$v_1 = \omega\sqrt{a^2 - y_1^2}$$

Here $v_1 = 6$ cm/s, $y_1 = 8$ cm

$$\therefore \quad 6 = \omega\sqrt{a^2 - 64} \qquad \ldots (1)$$

In the second case,

$$v_2 = \omega\sqrt{a^2 - y_2^2}$$

Here $v_2 = 8$ cm/s, $y_2 = 6$ cm

$$\therefore \quad 8 = \omega\sqrt{a^2 - 36} \qquad \ldots (2)$$

Dividing (2) by (1) and squaring

$$\frac{64}{36} = \frac{a^2 - 36}{a^2 - 64}$$

$$a = 10 \text{ cm.}$$

The amplitude of vibration $= 10$ **cm.**

Substituting the value of

$$a = 10 \text{ cm in equation (1)}$$

$$6 = \omega \sqrt{100 - 64}$$

$$\omega = 1 \text{ radian/s}$$

Frequency $n = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}$ **hertz** 0.159 Hz

Time period $T = \dfrac{1}{n} = 2\pi$ **seconds.** 6.285 Sec

Example 1.3. *Show that for a particle executing SHM, the instantaneous velocity is* $\omega\sqrt{a^2 - y^2}$ *and instantaneous acceleration is* $-\omega^2 y$.

For a particle executing SHM,

$$y = a \sin(\omega t + \alpha) \qquad \ldots (1)$$

The instantaneous velocity,

$$v = \frac{dy}{dt} = + a\omega \cos(\omega t + \alpha) \qquad \ldots (2)$$

From equation (1)

$$\sin(\omega t + \alpha) = \frac{y}{a}$$

$$\cos(\omega t + \alpha) = \sqrt{1 - \sin^2(\omega t + \alpha)}$$

$$= \sqrt{1 - \frac{y^2}{a^2}}$$

$$v = a\omega \sqrt{1 - \frac{y^2}{a^2}}$$

$$v = \omega \sqrt{a^2 - y^2} \qquad \ldots (3)$$

The instantaneous acceleration,

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = -a\omega^2 \sin(\omega t + \alpha)$$

$$= -\omega^2 [a \sin(\omega t + \alpha)]$$

$$= -\omega^2 y \qquad \ldots(4)$$

Example [redacted] *The motion of a particle in SHM is given by*

$$x = a \sin \omega t$$

If it has a speed u when the displacement is x_1 and speed v when the displacement is x_2. Show that the amplitude of vibration is

$$a = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{\frac{1}{2}}$$

(Utkal, 1989)

Here

$$x = a \sin \omega t$$

$$u = \frac{dx_1}{dt} = \omega \sqrt{a^2 - x_1^2} \qquad \ldots(i)$$

and

$$v = \frac{dx_2}{dt} = \omega \sqrt{a^2 - x_2^2} \qquad \ldots(ii)$$

Squaring and dividing

$$\frac{u^2}{v^2} = \frac{a^2 - x_1^2}{a^2 - x_2^2}$$

$$u^2 a^2 - u^2 x_2^2 = v^2 a^2 - v^2 x_1^2$$

$$a^2 [v^2 - u^2] = v^2 x_1^2 - u^2 x_2^2$$

$$a = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{\frac{1}{2}} \qquad \ldots(iii)$$

Example [redacted] *A particle performs simple harmonic motion given by the equation*

$$y = 20 \sin [\omega t + \alpha]$$

If the time period is 30 seconds and the particle has a displacement of 10 cm at t=0, find (i) epoch ; (ii) the phase angle at t=5 seconds, and (iii) the phase difference between two positions of the particle 15 seconds apart.

Here

$$y = 20 \sin (\omega t + \alpha)$$

$$T = 30 \text{ s}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15} \text{ radians/s}$$

(*i*) At $t = 0,\quad y = 10$ cm

$$\therefore \quad 10 = 20 \sin\left(\frac{\pi}{15} \times 0 + \alpha\right)$$

or

$$\sin \alpha = 0{\cdot}5$$

or $\alpha = \dfrac{\pi}{6}$ radian

(*ii*) At $t = 5$ s.

The phase angle $= (\omega t + \alpha)$

$$= \left(\frac{\pi}{15} \times 5 + \frac{\pi}{6}\right)$$

$$= \frac{\pi}{2}$$

(*iii*) At $t = 0$

the phase angle $\theta_1 = \dfrac{\pi}{6}$

At $t = 15$

the phase angle $\theta_2 = (\omega t + \alpha)$

$$\theta_2 = \left(\frac{\pi}{15} \times 15 + \frac{\pi}{6}\right)$$

$$\theta_2 = \frac{7\pi}{6}$$

The phase difference $\theta_2 - \theta_1 = \dfrac{7\pi}{6} - \dfrac{\pi}{6} = \pi$ radians.

Example 16. *A particle executes simple harmonic motion given by the equation*

$$y = 12 \sin\left(\frac{2\pi t}{10} + \frac{\pi}{4}\right)$$

Calculate (i) amplitude; (ii) frequency; (iii) epoch; (iv) displacement at $t = 1{\cdot}25$ s, (v) velocity at $t = 2{\cdot}5$ s and (vi) acceleration at $t=5$ s.

Here $$y = 12 \sin\left(\frac{2\pi t}{10} + \frac{\pi}{4}\right) \quad \ldots (1)$$

The displacement equation is

$$y = a \sin(\omega t + \alpha) \quad \ldots (2)$$

Comparing equations (1) and (2)

(*i*) Amplitude $a = 12$ units

(*ii*) $\omega = \dfrac{2\pi}{10}$

Frequency $n = \dfrac{\omega}{2\pi} = \dfrac{1}{10} = 0{\cdot}1$ hertz

(*iii*) Epoch. $\alpha = \dfrac{\pi}{4}$

(*iv*) When $t = 1{\cdot}25$ s

$$y = 12 \sin\left(\frac{2\pi \times 1\cdot25}{10} + \frac{\pi}{4}\right)$$

$$y = 12 \sin \pi/2$$

or $$y = \mathbf{12\ units}$$

(v) At $t = 2{\cdot}5$ s

$$\text{Velocity} = \frac{dy}{dt} = a\omega \cos(\omega t + \alpha)$$

$$\frac{dy}{dt} = 12 \times \frac{2\pi}{10} \cos\left[\frac{2\pi}{10} \times 2{\cdot}5 + \frac{\pi}{4}\right]$$

$$\frac{dy}{dt} = -\mathbf{5{\cdot}552\ units.}$$

The – ve sign shows that the velocity is directed towards the mean position.

(vi) At $t = 5$ s

$$\text{Acceleration} = \frac{d^2y}{dt^2} = -a\omega^2 \sin(\omega t + \alpha)$$

$$\frac{d^2y}{dt^2} = -12 \times \left(\frac{2\pi}{10}\right)^2 \sin\left(\frac{2\pi}{10} \times 5 + \frac{\pi}{4}\right)$$

$$= -0{\cdot}48\,\pi^2 \sin\left(\pi + \frac{\pi}{4}\right) = \mathbf{3{\cdot}35\ units.}$$

Example 1. *A simple harmonic motion is represented by*

$$y = 10 \sin\left(10\,t - \frac{\pi}{6}\right)$$

where y is measured in metres, *t in* seconds *and the phase angle in* radians. *Calculate :*

(i) the frequency;

(ii) the time period;

(iii) the maximum displacement;

(iv) the maximum velocity;

(v) the maximum acceleration, and;

(vi) displacement, velocity and acceleration at time, t=0 and t=1 second.

Here $$y = 10 \sin\left(10\,t - \frac{\pi}{6}\right) \qquad \ldots(1)$$

The displacement equation is

$$y = a \sin(\omega t + \alpha) \qquad \ldots(2)$$

(*i*) From (1) and (2)

$$\omega = 10$$

But $\omega = 2\pi n$

$\therefore$ $2\pi n = 10$

or $n = \dfrac{10}{2\pi}$ hertz

$n = 1.6$ **hertz.**

(*ii*) Time period,

$$T = \frac{1}{n} = \frac{2\pi}{10}$$

or $T = 0.63$ s

(*iii*) Maximum displacement,

$a = 10$ m

(*iv*) Velocity, $\dfrac{dy}{dt} = a\omega \cos(\omega t + \alpha)$

Therefore, maximum velocity

$$\frac{dy}{dt} = a\omega$$

But $a = 10$ m

and $\omega = 10$

$\therefore$ $\dfrac{dy}{dt} = 10 \times 10 = 100$ m/s

(*v*) Acceleration, $\dfrac{d^2y}{dt^2} = -a\omega^2 \sin(\omega t + \alpha)$

Maximum acceleration

$$\frac{d^2y}{dt^2} = -a\omega^2$$

or $\dfrac{d^2y}{dt^2} = -10 \times (10)^2 = -1000$ m/s^2

– ve sign shows that the acceleration is directed towards the mean position.

(*vi*) From equation (1)

(*a*) At $t = 0$

$$y = 10 \sin\left(-\frac{\pi}{6}\right)$$

$$y = -5 \text{ m}$$

$$\frac{dy}{dt} = a\omega \cos(\omega t + \alpha)$$

At $\quad t = 0$

$$\frac{dy}{dt} = a\omega \cos \alpha$$

$$= 10 \times 10 \cos\left(-\frac{\pi}{6}\right)$$

$$= 100 \times 0{\cdot}866 = 86{\cdot}6 \text{ m/s}$$

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin \alpha$$

or
$$\frac{d^2y}{dt^2} = -a\omega^2 \sin\left(-\frac{\pi}{6}\right)$$

or
$$\frac{d^2y}{dt^2} = a\omega^2 \sin\left(\frac{\pi}{6}\right)$$

$$= 10 \times 100 \times 0{\cdot}5 = 500 \text{ m/s}^2.$$

(*b*) From equation (1)

At $\quad t = 1,$

Displacement

$$y = 10 \sin\left(10 - \frac{\pi}{6}\right)$$

$$y = 10 \sin\left(\frac{60 - 3{\cdot}142}{6}\right)$$

$$y = 10 \sin\left(\frac{56{\cdot}858}{6}\right)$$

$$y = 10 \sin(3\pi) \quad \text{(approximately)}$$

or
$$y = 10 \sin \pi = 0.$$

Velocity

$$\frac{dy}{dt} = a\omega \cos\left(10 - \frac{\pi}{6}\right)$$

or
$$\frac{dy}{dt} = a\omega \cos \pi \text{ (approximately)}$$

$$= 10 \times 10 \times (-1) = -100 \text{ m/s}.$$

Acceleration

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin\left(10 - \frac{\pi}{6}\right)$$

$$= -a\omega^2 \sin \pi \text{ (approximately)} = 0.$$

$$\frac{a}{2} = a \sin (10 \sec^{-1}) . t$$

$$\sin (10 \sec^{-1}) . t = \frac{1}{2}$$

$$10 \sec^{-1} . t = \sin^{-1} \frac{1}{2} = \frac{\pi}{3}$$

$$\therefore \ t = \frac{\pi}{3 \times 10} \ \sec$$

(x) *what is the displacement of the body as a function of time ?*

The general equation for displacement of a body executing simple harmonic motion is

$$y = a \sin (wt + \phi)$$

The value of w, as already obtained, is

$$w = \frac{2\pi}{T} = 10 \text{ radian}/sec.$$

$$\therefore \ y = a \ \sin \ (10t + \phi)$$

At t = 0, y = a = 0.04 m, so that at that instant

$$y = 0.04 \sin \phi = 0.04$$

$$\therefore \ \sin \phi = 1$$

$$\text{or, } \phi = \sin^{-1} 1 = \frac{\pi}{2} \text{ radian.}$$

Therefore, with a = 0.04 m, w = 10 rad./sec and $\phi = \frac{\pi}{2}$ radian, we get

$$y = 0.04 \sin (10\,t + \frac{\pi}{2})$$

$$= \mathbf{0.04 \sin 10t.}$$

Example 1.2. *A body is vibrating with simple harmonic motion of amplitude 15 cm and frequency 4 Hz. Compute (a) the maximum values of the acceleration and velocity and (b) the acceleration and velocity when the displacement is 9 cm.*

Soln. (a) $v_{max} = w.a$

$a = 15$ cm

$n = 4$ Hz

$\therefore \quad w = 2\pi n = 2 \times 3.14 \times 4$

$= 25.12$ rad/sec.

$\therefore \quad v_{max} = 25.12 \times 15 = $ **376.8 cm/sec.**

$(accln)_{max} = -w^2.a$

$= -(25.12)^2 \times 15$

$=$ **− 9470 cm/sec².**

(b) when y = 9 cm

$v = w.\sqrt{a^2 - y^2} = 25.12 \sqrt{(15)^2 - 9^2}$

$=$ **300 cm/sec**

accln. $= -w^2.y = -(25.12)^2 \times 9$

$=$ **− 5680 cm/sec².**

Example 1.3. *A particle executes linear harmonic motion about the point x = 0. At t = 0, it has displacement y = 0.37 cm and zero velocity. If the frequency of the motion is 0.25/sec, determine (a) the period, (b) the amplitude, (c) the maximum speed and (d) the maximum acceleration.*

Soln.

(a) $T = \frac{1}{n} = \frac{1}{0.25} = $ **4 sec.**

(b) $y = a \sin (wt + \delta)$

at t = 0, y = 0.37 cm.

$\therefore a \sin \delta = 0.37$ (i)

Again $v = \frac{dy}{dt} = wa \cos (wt + \delta)$

at t = 0, v = 0

$\therefore wa \cos \delta = 0$

or, $a \cos \delta = 0$ (ii)

from (i) and (ii),

$a^2 = (0.37)^2$

or, $\mathbf{a = \pm 0.37}$ **cm.**

(c) $v = w.\sqrt{a^2 - y^2}$

v is maximum when $y = 0$.

$\therefore \quad v_{max} = w.a = a \times 2\pi n$

$= 0.37 \times 2 \times 3.14 \times 0.25$

$= \mathbf{0.5809}$ **cm/sec.**

(d) $(accln)_{max} = -w^2.a$

$= -(2\pi n)^2 . a$

$= -(2 \times 3.14 \times 0.25)^2 \times 0.37$

$= -\mathbf{0.912013}$ **cm/sec^2.**

Example 1.4. *The displacement of an oscillating particle at an instant t is given by*

$$y = a \cos wt + b \sin wt.$$

Show that it is executing a simple harmonic motion.

If a = 5 cm, b = 12 cm and w = 4 radian/sec, calculate (i) the amplitude, (ii) the time period, (iii) the maximum velocity and (iv) the maximum acceleration of the particle.

Soln.

$$y = a \cos wt + b \sin wt$$

or, $\dfrac{dy}{dt} = -aw \sin wt + bw \cos wt$

or, $\dfrac{d^2y}{dt^2} = -w^2.a \cos wt - w^2b \sin wt$

$= -w^2 (a \cos wt + b \sin wt)$

$= -w^2y$

or, $$\frac{d^2y}{dt^2} + w^2y = 0$$

Hence the motion is simple harmonic.

(i) Let $a = A \sin \alpha$ and $b = A \cos \alpha$.

Then

$$y = A \sin\alpha \cos wt + A \cos \alpha \sin wt$$

$$= A \sin (wt + \alpha)$$

This represents a simple harmonic motion with amplitude A.

$\therefore \quad A^2 \sin^2 \alpha + A^2 \cos^2 \alpha = a^2 + b^2$

or, $A = \sqrt{a^2+b^2}$

$a = 5$ cm

$b = 12$ cm

$\therefore \quad A = \sqrt{5^2+12^2}$

$= 13$ cm

(ii) $T = \frac{2\pi}{w} = \frac{2\pi}{4} = \frac{\pi}{2}$ sec.

(iii) $v_{max} = w.A = 13 \times 4$

$= 52$ cm/sec

(iv) $(\text{accln})_{max} = -w^2.A = -(4)^2 \times 13$

$= -208$ **cm/sec^2.**

Example 1.5. *The positions of a particle executing simple harmonic motion along the x-axis are x = A and x = B at time t an 2t respectively. Show that its period of oscillation is given by*

$$T = (2\pi t) / \cos^{-1} (B/2A)$$

Soln.

$$A = a \sin wt$$

$$B = a \sin (w.\ 2t)$$

$$= a \sin 2wt$$

$$= a\ 2 \sin wt \cos wt$$

$$\therefore \quad \frac{A}{B} = \frac{a \sin wt}{a\ 2 \sin wt \cos wt}$$

$$= \frac{1}{2 \cos wt}$$

$$\text{or,} \quad \cos wt = \frac{B}{2A}$$

$$\text{or,} \quad wt = \cos^{-1}\left(\frac{B}{2A}\right)$$

$$\text{or,} \quad w = \frac{1}{t} \cos^{-1}\left(\frac{B}{2A}\right)$$

$$\text{or,} \quad \frac{2\pi}{T} = \frac{\cos^{-1}\left(\frac{B}{2A}\right)}{t}$$

$$\therefore \quad T = \frac{2\pi t}{\cos^{-1}\left(\frac{B}{2A}\right)}$$

Example 1.6. *For a particle executing simple harmonic motion the displacement is 8 cm at the instant the velocity is 6 cm/sec and the displacement is 6 cm at the instant the velocity is 8 cm/sec. Calculate (i) amplitude, (ii) frequency and (iii) time period.*

Soln.

Velocity of a particle executing simple harmonic motion,

$$v = \frac{dy}{dt} = w\sqrt{a^2 - y^2}$$

Now v = 6 cm/sec when y = 8 cm.

$$\therefore \quad 6 = w\sqrt{a^2 - 64} \qquad \text{(i)}$$

Again v = 8 cm/sec when y = 6 cm

$$\therefore \quad 8 = w\sqrt{a^2 - 64} \qquad \text{(ii)}$$

Dividing (ii) by (i) and squaring

$$\frac{64}{36} = \frac{a^2 - 36}{a^2 - 64}$$

or, a = **10 cm.**

Substituting a = 10 cm in eqn. (i)

$$\therefore \quad 6 = w\sqrt{100 - 64}$$

or, w = **1 rad/sec.**

Hence frequency,

$$n = \frac{w}{2\pi} = \frac{1}{2\pi} \text{ Hz.}$$

time period

$$T = \frac{1}{n} = \mathbf{2\pi \text{ seconds.}}$$

Example *1.7. A simple harmonic motion is represented by*

$$y = 10 \sin\left(10t - \frac{\pi}{6}\right)$$

where y is measured in metres, t in seconds and the phase angle in radians. Calculate (i) the frequency, (ii) the time period, (iii) the maximum displacement, (iv) the maximum, velocity and (v) the maximum acceleration and (vi) displacement, velocity and acceleration at time t = 0 and t = 1 second.

Soln.

Here $y = 10 \sin\left(10t - \frac{\pi}{6}\right)$ (1)

Comparing with the displacement equation

$y = a \sin(wt + \delta)$ (2)

we get, (i) $w = 2\pi n = 10$

or, $n = \dfrac{10}{2\pi}$ = **1.6 Hz.**

(ii) time period, $T = \dfrac{1}{n} = \dfrac{2\pi}{10}$ = **0.63 sec.**

(iii) maximum displacement (amplitude)
a = **10 m.**

(iv) maximum velocity,

$v_{max} = w.a = 10 \times 10$ = **100 m/sec.**

(v) $(\text{accln.})_{max} = -w^2a = -(10)2 \times 10$

$= -$ **1000 m/sec^2.**

minus sign shows that the acceleration is directed towards the mean position.

(vi) From eqn. (1)

(a) at t = 0

$y = 10 \sin (-\dfrac{\pi}{6}) = -5\ m.$

velocity, $\dfrac{dy}{dt} = a\,w \cos \delta$

$= 10 \times 10 \cos (-\dfrac{\pi}{6})$

$= 100 \times 0.866$ = **86.6 m/sec.**

Acceleration, $\dfrac{d^2y}{dt^2} = -aw^2 \sin \delta$

$= -10 \times 10^2 \times \sin (-\dfrac{\pi}{6})$

$= -10 \times 100 \times 0.5$

$= -$ **500 m/sec^2.**

(b) From eqn. (1), at t = 1,

displacement

$y = 10 \sin (10 - \dfrac{\pi}{6})$