

# CURVATURE AND RADIUS OF CURVATURE

## 5.1 Introduction:

Curvature is a numerical measure of bending of the curve. At a particular point on the curve, a tangent can be drawn. Let this line makes an angle  $\Psi$  with positive x- axis. Then curvature is defined as the magnitude of rate of change of  $\Psi$  with respect to the arc length  $s$ .

$$\therefore \text{Curvature at P} = \left| \frac{d\Psi}{ds} \right|$$

It is obvious that smaller circle bends more sharply than larger circle and thus smaller circle has a larger curvature.

Radius of curvature is the reciprocal of curvature and it is denoted by  $\rho$ .

## 5.2

- **Radius of curvature of Cartesian curve:  $y = f(x)$**

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{(1+y_1^2)^{3/2}}{|y_2|} \quad (\text{When tangent is parallel to x - axis})$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|} \quad (\text{When tangent is parallel to y - axis})$$

- **Radius of curvature of parametric curve:**

$$x = f(t), y = g(t)$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|}, \quad \text{where } x' = \frac{dx}{dt} \text{ and } y' = \frac{dy}{dt}$$

**Example 1** Find the radius of curvature at any pt of the cycloid

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta)$$

**Solution:**  $x' = \frac{dx}{d\theta} = a(1 + \cos \theta)$  and  $y' = \frac{dy}{d\theta} = a \sin \theta$

$$x'' = \frac{d^2x}{d\theta^2} = -a \sin \theta \quad \text{and} \quad y'' = \frac{d^2y}{d\theta^2} = a \cos \theta$$

$$\begin{aligned} \text{Now } \rho &= \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|} = \frac{\{a^2(1 + \cos^2 \theta) + a^2 \sin^2 \theta\}^{3/2}}{a^2(1 + \cos^2 \theta) \cos \theta + a^2 \sin^2 \theta} \\ &= \frac{a(1 + \cos^2 \theta + 2 \cos \theta + \sin^2 \theta)^{3/2}}{\cos \theta + \cos^2 \theta + \sin^2 \theta} \\ &= \frac{a(2 + 2 \cos \theta)^{3/2}}{1 + \cos \theta} \\ &= 2 \sqrt{2} a \sqrt{1 + \cos \theta} \\ &= 2 \sqrt{2} a \sqrt{2 \frac{\cos^2 \theta}{2}} = 4a \cos \frac{\theta}{2} \end{aligned}$$

**Example 2** Show that the radius of curvature at any point of the curve  $x^{2/3} + y^{2/3} = a^{2/3}$  ( $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$ ) is equal to three times the length of the perpendicular from the origin to the tangent.

**Solution :**  $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta = x'$

$$\frac{dy}{d\theta} = -3a \sin^2 \theta \cos \theta = y'$$

$$\begin{aligned} x'' &= \frac{d^2x}{d\theta^2} = \frac{d}{d\theta} (-3a \cos^2 \theta \sin \theta) \\ &= -3a [-2 \cos \theta \sin^2 \theta + \cos^3 \theta] \\ &= 6a \cos \theta \sin^2 \theta - 3a \cos^3 \theta \end{aligned}$$

$$\begin{aligned} y'' &= \frac{d^2y}{d\theta^2} = \frac{d}{d\theta} (3a \sin^2 \theta \cos \theta) \\ &= 3a(2 \sin \theta \cos^2 \theta - \sin^3 \theta) \\ &= 6a \sin \theta \cos^2 \theta - 3a \sin^3 \theta \end{aligned}$$

$$\begin{aligned} \text{Now } \rho &= \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|} \\ &= \frac{(9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta)^{3/2}}{|(-3a \cos^2 \theta \sin \theta)(6a \sin \theta \cos^2 \theta - 3a \sin^3 \theta) - 3a \sin^2 \theta \cos \theta(6a \cos \theta \sin^2 \theta - 3a \cos^3 \theta)|} \end{aligned}$$

$$\begin{aligned}
&= \frac{[9a^2 \cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta)]^{3/2}}{|-18a^2 \sin^2 \theta \cos^4 \theta + 9a^2 \cos^2 \theta \sin^4 \theta - 18a^2 \sin^4 \theta \cos^2 \theta + 9a^2 \sin^2 \theta \cos^4 \theta|} \\
&= \frac{9^{3/2} (a \cos \theta \sin \theta)^3}{|-9a^2 \sin^2 \theta \cos^4 \theta - 9a^2 \cos^2 \theta \sin^4 \theta|} \\
&= \frac{(9)^{3/2} (a \cos \theta \sin \theta)^3}{9a^2 \cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta)} \\
\Rightarrow \rho &= 3a \sin \theta \cos \theta \quad \dots\dots\dots(1)
\end{aligned}$$

The equation of the tangent at any point on the curve is

$$\begin{aligned}
y - a \sin^3 \theta &= -\tan \theta (x - a \cos^3 \theta) \\
\Rightarrow x \sin \theta + y \cos \theta - a \sin \theta \cos \theta &= 0 \quad \dots\dots\dots(2)
\end{aligned}$$

$\therefore$  The length of the perpendicular from the origin to the tangent (2) is

$$\begin{aligned}
p &= \frac{|0 \cdot \sin \theta + 0 \cdot \cos \theta - a \sin \theta \cos \theta|}{\sqrt{\sin^2 \theta + \cos^2 \theta}} \\
&= a \sin \theta \cos \theta \quad \dots\dots\dots(3)
\end{aligned}$$

Hence from (1) & (3),  $\rho = 3p$

**Example 3** If  $\rho$  &  $\rho'$  are the radii of curvature at the extremities of two conjugate diameters of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  prove that

$$(\rho^{2/3} + \rho'^{2/3})(ab)^{2/3} = a^2 + b^2$$

**Solution:** Parametric equation of the ellipse is

$$x = a \cos \theta, \quad y = b \sin \theta$$

$$x' = -a \sin \theta, \quad y' = b \cos \theta$$

$$x'' = -a \cos \theta, \quad y'' = -b \sin \theta$$

The radius of curvature at any point of the ellipse is given by

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|} = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{|(-a \sin \theta)(-b \sin \theta) - (b \cos \theta)(-a \cos \theta)|}$$

$$= \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} \quad \dots\dots(1)$$

For the radius of curvature at the extremity of other conjugate diameter is obtained by replacing  $\theta$  by  $\theta + \frac{\pi}{2}$  in (1).

Let it be denoted by  $\rho'$ . Then

$$\therefore \rho' = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab}$$

$$\begin{aligned} \therefore \rho^{2/3} + \rho'^{2/3} &= \frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{(ab)^{2/3}} + \frac{a^2 \cos^2 \theta + b^2 \sin^2 \theta}{(ab)^{2/3}} \\ &= \frac{a^2 + b^2}{(ab)^{2/3}} \end{aligned}$$

$$\Rightarrow (ab)^{2/3} (\rho^{2/3} + \rho'^{2/3}) = a^2 + b^2$$

**Example 4** Find the points on the parabola  $y^2 = 8x$  at which the radius of curvature is  $\frac{125}{16}$ .

**Solution:**  $y = 2\sqrt{2} \sqrt{x}$

$$y_1 = \frac{\sqrt{2}}{\sqrt{x}} \quad , \quad y_2 = \frac{-1}{\sqrt{2}x^{3/2}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|} = (1 + \frac{2}{x})^{3/2} \cdot \sqrt{2} x^{3/2} = \sqrt{2} (x+2)^{3/2}$$

$$\text{Given } \rho = \frac{125}{16} \quad \therefore (x+2)^{3/2} = \frac{125}{16\sqrt{2}} = \left(\frac{5}{2\sqrt{2}}\right)^3$$

$$\begin{aligned} \therefore (x+2)^{3/2} &= \frac{5}{2\sqrt{2}} \\ \Rightarrow x+2 &= \frac{25}{8} \quad \Rightarrow x = \frac{9}{8} \end{aligned}$$

$$\Rightarrow y^2 = 8 \left(\frac{9}{8}\right) \text{ i.e. } y = 3, -3$$

Hence the points at which the radius of curvature is  $\frac{125}{16}$  are  $(9, \pm 3)$ .

**Example 5** Find the radius of curvature at any point of the curve

$$y = C \cosh(x/c)$$

**Solution:**  $y_1 = \frac{c}{c} \sinh \frac{x}{c} = \sinh \left( \frac{x}{c} \right)$

$$y_2 = \frac{1}{c} \cosh \frac{x}{c}$$

$$\text{Now, } \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left(1 + \sinh^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c} \cosh \frac{x}{c}}$$

$$= C \cosh^2\left(\frac{x}{c}\right)$$

$$\Rightarrow \rho = \frac{1}{c} y^2$$

**Example 6** For the curve  $y = \frac{ax}{a+x}$ , prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$$

where  $\rho$  is the radius of curvature of the curve at its point  $(x, y)$

**Solution:** Here  $y = \frac{ax}{a+x}$

$$\Rightarrow y_1 = \frac{(a+x)a - ax(1)}{(a+x)^2}$$

$$= \frac{a^2}{(a+x)^2}$$

$$\therefore y_2 = \frac{-2a^2}{(a+x)^3}$$

$$\text{Now, } \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2} \times \frac{(a+x)^3}{(-2a^2)}$$

$$\therefore \rho^{2/3} = \left[1 + \frac{a^4}{(a+x)^4}\right] \frac{(a+x)^2}{(-2)^{2/3} a^{4/3}}$$

$$\begin{aligned}
\left(\frac{2\rho}{a}\right)^{2/3} &= \left[1 + \frac{a^4}{(a+x)^4}\right] \frac{(a+x)^2}{2^{2/3} a^{4/3}} \times \frac{2^{2/3}}{a^{2/3}} \\
&= \frac{1}{a^2} \left[1 + \frac{a^4}{(a+x)^4}\right] (a+x)^2 \\
&= \frac{1}{a^2} \left[(a+x)^2 + \frac{a^4}{(a+x)^2}\right] \\
&= \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2 \\
&= \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2
\end{aligned}$$

**Example 7** Find the curvature of  $x = 4 \cos t$ ,  $y = 3 \sin t$ . At what point on this ellipse does the curvature have the greatest & the least values? What are the magnitudes?

**Solution:**  $\rho = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|}$

$$\begin{aligned}
\text{Now, } x' &= -4 \sin t & \Rightarrow x'' &= -4 \cos t \\
y' &= 3 \cos t & \Rightarrow y'' &= -3 \sin t
\end{aligned}$$

$$\begin{aligned}
\therefore \rho &= \frac{(16 \sin^2 t + 9 \cos^2 t)^{3/2}}{-4 \sin t (-3 \sin t) - 3 \cos t (-4 \cos t)} \\
&= \frac{1}{12} (9 \cos^2 t + 16 \sin^2 t)^{3/2}
\end{aligned}$$

$$\Rightarrow (\rho \cdot 12)^{2/3} = 9 \cos^2 t + 16 \sin^2 t$$

Now, curvature is the reciprocal of radius of curvature. Curvature is maximum & minimum when  $\rho$  is minimum and maximum respectively. For maximum and minimum values;

$$\frac{d}{dt} (16 \sin^2 t + 9 \cos^2 t) = 0$$

$$\Rightarrow 32 \sin t \cos t + 18 \cos t (-\sin t) = 0$$

$$\Rightarrow 4 \sin t \cos t = 0$$

$$\Rightarrow t = 0 \text{ \& } \frac{\pi}{2}$$

At  $t = 0$  ie at (4,0)

$$(12\rho)^{2/3} = 9$$

$$\Rightarrow 12\rho = 9^{3/2}$$

$$\Rightarrow \rho = \frac{9}{4} \quad \therefore \frac{1}{\rho} = \frac{4}{9}$$

Similarly, at  $t = \frac{\pi}{2}$  ie at (0,3)

$$(12\rho)^{2/3} = 16$$

$$12\rho = 4^3$$

$$\rho = 16/3 \quad \therefore \frac{1}{\rho} = \frac{3}{16}$$

Hence, the least value is  $\frac{3}{16}$  and the greatest value is  $\frac{4}{9}$

**Example 8** Find the radius of curvature for  $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$  at the points where it touches the coordinate axes.

**Solution:** On differentiating the given , we get

$$\frac{1}{2\sqrt{ax}} - \frac{1}{2\sqrt{by}} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{by}{ax}} \quad \dots\dots(1)$$

The curve touches the x-axis if  $\frac{dy}{dx} = 0$  or  $y = 0$

When  $y = 0$ , we have  $x = a$  (from the given eq<sup>n</sup>)

$\Rightarrow$  given curve touches x – axis at (a,0)

The curve touches y – axis if  $\frac{dx}{dy} = 0$  or  $x = 0$

When  $x = 0$ , we have  $y = b$

$\Rightarrow$  Given curve touches y-axis at (o, b)

$$\frac{d^2y}{dx} = \sqrt{\frac{b}{a}} \left\{ \sqrt{\frac{b}{a}} \cdot \frac{1}{2x} - \frac{1}{2} \sqrt{\frac{y}{x}} \right\} \quad \{\text{from (1)}\}$$

$$\text{At } (a,0), \frac{d^2y}{dx^2} = \frac{1}{2a} \frac{b}{a} = \frac{b}{2a^2}$$

$$\therefore \text{At } (a,0), \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = (1+0)^{3/2} \frac{2a^2}{b} = \frac{2a^2}{b}$$

$$\text{At } (0,b), \rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{2b^2}{a}$$

### 5.3 Radius of curvature of Polar curves $r = f(\theta)$ :

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} \quad \left(\text{where } r_1 = \frac{dr}{d\theta}, r_2 = \frac{d^2r}{d\theta^2}\right)$$

**Example 9** Prove that for the cardioide  $r = a(1 + \cos \theta)$ ,

$$\frac{\rho^2}{r} \text{ is const.}$$

**Solution:** Here  $r = a(1 + \cos \theta)$

$$\Rightarrow r_1 = -a \sin \theta \text{ and } r_2 = -a \cos \theta$$

$$\therefore r^2 + r_1^2 = a^2 [(1 + \cos \theta)^2 + \sin^2 \theta] = 2a^2 (1 + \cos \theta)$$

$$r^2 + 2r_1^2 - rr_2 = a^2 [(1 + \cos \theta)^2 + 2\sin^2 \theta + \cos \theta(1 + \cos \theta)]$$

$$= 3a^2 (1 + \cos \theta)$$

$$\therefore \rho^2 = \frac{(r^2 + r_1^2)^3}{(r^2 + 2r_1^2 - rr_2)^2} = \frac{8a^6(1 + \cos \theta)^3}{9a^4(1 + \cos \theta)^2} = \frac{8}{9} a^2 (1 + \cos \theta)$$

$$\Rightarrow \rho^2 = \frac{8a}{9} r$$

$$\therefore \frac{\rho^2}{r} = \frac{8a}{9} \text{ which is a constant.}$$

**Example 10** Show that at the point of intersection of the curves  $r = a \theta$  and  $r \theta = a$ , the curvatures are in the ratio 3:1 ( $0 < \theta < 2\pi$ )



**Solution:** The points of intersection of curves  $r = a \theta$  &  $r \theta = a$  are given by  $a \theta^2 = a$  or  $\theta = \pm 1$

Now for the curve  $r = a \theta$  we have  $r_1 = a$  and  $r_2 = 0$

$$\therefore \text{At } \theta = \pm 1, \rho = \left[ \frac{(r^2 + r_1^2)^{3/2}}{2a^2 + a^2 \theta^2 - 0} \right]_{\theta=\pm 1} = \frac{a(2\sqrt{2})}{3} = \rho_1$$

For the curve  $r \theta = a$ ,

$$r_1 = \frac{-a}{\theta^2} \quad \text{and} \quad r_2 = \frac{2a}{\theta^3}$$

$$\begin{aligned} \text{At } \theta = \pm 1, \rho &= \left[ \frac{\left( \frac{a^2}{\theta^2} + \frac{a^2}{\theta^4} \right)^{3/2}}{\frac{2a^2}{\theta^4} + \frac{a^2}{\theta^2} - \frac{2a^2}{\theta^4}} \right]_{\theta=\pm 1} = \left[ a \frac{(1+\theta^2)^{3/2}}{\theta^4} \right]_{\theta=\pm 1} \\ &= 2a \sqrt{2} = \rho_2 \end{aligned}$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{2a \sqrt{2}}{2a \sqrt{2/3}} = \frac{3}{1}$$

$$\therefore \rho_2 : \rho_1 = 3 : 1$$

**Example 11** Find the radius of curvature at any point  $(r, \theta)$  of the curve  $r^m = a^m \cos m \theta$

**Solution:**  $r^m = a^m \cos m \theta$

$$\Rightarrow m \log r = m \log a + \log \cos m \theta$$

$$\Rightarrow \frac{m}{r} r_1 = -m \frac{\sin m \theta}{\cos m \theta} \quad (\text{on differentiating w.r.t. } \theta)$$

$$\Rightarrow r_1 = -r \tan m \theta \quad \dots\dots(1)$$

$$\text{Now } r_2 = -(r_1 \tan m \theta + r m \sec^2 m \theta)$$

$$= r \tan^2 m \theta - r m \sec^2 m \theta \quad (\text{from (1)})$$

$$\begin{aligned}\therefore \rho &= \frac{(r^2 + r^2 \tan^2 m\theta)^{3/2}}{r^2 + 2r^2 \tan^2 m\theta - r^2 \tan^2 m\theta + r^2 m \sec^2 m\theta} \\ &= \frac{r^3 \sec^3 m\theta}{r^2 \sec^2 m\theta + r^2 m \sec^2 m\theta} = \frac{r}{m+1} \sec m\theta\end{aligned}$$

**Example 12** Show that the radius of curvature at the point  $(r, \theta)$

of the curve  $r^2 \cos 2\theta = a^2$  is  $\frac{r^3}{a^2}$

**Solution:**  $r^2 = a^2 \sec 2\theta$

$$\Rightarrow 2rr_1 = 2a^2 \sec 2\theta \tan 2\theta$$

$$\Rightarrow r_1 = r \tan 2\theta$$

$$\text{and } r_2 = 2r \sec^2 \theta + r_1 \tan 2\theta$$

$$= 2r \sec^2 2\theta + r \tan^2 2\theta \quad (\because r = r \tan 2\theta)$$

$$\begin{aligned}\text{Now } \rho &= \frac{(r_1^2 + r_2^2)^{3/2}}{2r_1^2 + r^2 - rr_2} \Rightarrow \rho = \frac{((r^2 + r^2 \tan^2 2\theta))^{3/2}}{2r^2 \tan^2 2\theta + r^2 - r^2 (2\sec^2 2\theta + \tan^2 2\theta)} \\ &= \frac{(r^2 \sec^2 2\theta)^{3/2}}{r^2 (2 \tan^2 2\theta + 1 - 2\sec^2 2\theta - \tan^2 2\theta)} \\ &= \frac{r^3 \sec^3 2\theta}{r^2 \sec^2 2\theta} \\ &= r \sec 2\theta \\ &= r \cdot \frac{r^2}{a^2} = \frac{r^3}{a^2}\end{aligned}$$

## 5.4 Radius of curvature at the origin by Newton's method

It is applicable only when the curve passes through the origin and has x-axis or y-axis as the tangent there.

When x-axis is the tangent, then

$$\rho = \lim_{x \rightarrow 0} \frac{x^2}{2y}$$

When y- axis is the tangent, then

$$\rho = \lim_{x \rightarrow 0} \frac{y^2}{2x}$$

**Example13** Find the radius of curvature at the origin of the curve

$$x^3y - xy^3 + 2x^2y + xy - y^2 + 2x = 0$$

**Solution:** Tangent is  $x = 0$  ie y-axis,

$$\rho = \lim_{y \rightarrow 0} \frac{y^2}{2x}$$

Dividing the given equation by  $2x$ , we get

$$\frac{x^3y}{2x} - \frac{xy^3}{2x} + \frac{2x^2y}{2x} + \frac{xy}{2x} - \frac{y^2}{2x} + \frac{2x}{2x} = 0$$

$$x^3 \left( \frac{y}{2x} \right) - xy \left( \frac{y^2}{2x} \right) + xy + x \left( \frac{y}{2x} \right) - \left( \frac{y^2}{2x} \right) + 1 = 0$$

Taking limit  $y \rightarrow 0$  on both the sides , we get  $\rho = 1$

### Exercise 5A

1. Find the radius of curvatures at any point the curve

$$y = 4 \sin x - \sin 2x \text{ at } x = \frac{\pi}{2}$$

$$\text{Ans } \rho = \frac{1}{4} (5)^{3/2}$$

2. If  $\rho_1, \rho_2$  are the radii of curvature at the extremes of any chord of the cardioide  $r = a(1 + \cos \theta)$  which passes through the pole, then

$$\rho_1^1 + \rho_2^2 = \frac{16a^2}{9}$$

- 3 Find the radius of curvature of  $y^2 = x^2(a+x)(a-x)$  at the origin

$$\text{Ans. } a\sqrt{2}$$

4. Find the radius of curvature at any point 't' of the curve

$$x = a(\cos t + \log \tan t/2), y = a \sin t$$

$$\text{Ans. } a \cos t$$

5. Find the radius of curvature at the origin, for the curve

$$2x^3 - 3x^2y + 4y^3 + y^2 - 3x = 0$$

Ans.  $\rho = 3/2$

6. Find the radius of curvature of  $y^2 = \frac{4a^2(2a-x)}{x}$  at a point where the curve meets x – axis

Ans.  $\rho = a$

7. Prove that if  $\rho_1, \rho_2$  are the radii of curvature at the extremities of a focal chord of a parabola whose semi latus rectum is  $l$  then

$$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (l)^{-2/3}$$

8. Find the radius of curvature to the curve  $r = a(1 + \cos \theta)$  at the point where the tangent is parallel to the initial line.

Ans.  $\rho = \frac{2}{\sqrt{3}} \cdot a$

9. For the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , prove that  $\rho = \frac{a^2b^2}{p^3}$  where  $p$  is the perpendicular distance from the centre on the tangent at  $(x,y)$ .